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HORSESHOE-TYPE DIFFEOMORPHISMS WITH A HOMOCLINIC TANGENCY AT THE BOUNDARY OF HYPERBOLICITY

By

Ulrich A. Hoensch

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ABSTRACT

HORSESHOE-TYPE DIFFEOMORPHISMS WITH A HOMOCLINIC TANGENCY AT THE BOUNDARY OF HYPERBOLICITY

By

Ulrich A. Hoensch

In 1979 Devaney and Nitecki showed that for certain parameters in the real Henon family, the set of points with bounded orbits is hyperbolic, and the dynamics are topologically equivalent to those of the full shift on two symbols. It was long known that this set of parameters could be enlarged by considering the geometry given by the invariant manifolds of one or both of the (hyperbolic) fixed points. In this paper we use this approach to extend Devaney and Nitecki's results, and also to illustrate some methods and assumptions that are used in the process.

In Chapter 2, we give results concerning the geometry and position of these invariant manifolds, in particular we investigate the situation before and at the first homoclinic tangency, and establish some sufficient conditions for quadratic contact.

In Chapter 3, we illustrate the symbolic dynamics associated with the existence of a topological "horseshoe"; this is the first part on symbolic dynamics. The second part is given in Chapter 5, where we use a hyperbolicity condition to establish topological equivalence of the dynamics to the full shift on two symbols.

Chapter 4 introduces an abstract class of maps - a class of maps that satisfy certain geometric and hyperbolicity conditions. Here we give the main definitions and technical conditions needed; the strongest result in this chapter is that of proving hyperbolicity of a return map.

Finally, Chapter 6 is devoted to applying the results of the previous chapters to the Henon map. We state our main results in this chapter.

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Contents

1 Introduction **1**

2 The Henon Map **5**

2.1 Introduction 5

2.2 Fixed Points and Images of Curves 5

2.3 Invariant Manifolds 11

2.4 Quadratic Contact at the First Tangency 16

3 Symbolic Dynamics (Part I) **19**

3.1 Orientation-Preserving "Horseshoe" Maps before the First Tangency . 19

3.2 A Coding for $\Lambda = \bigcap_{n \in \mathbf{Z}} F^n(Q)$ before the First Tangency. 24

3.3 Orientation-Preserving "Horseshoe" Maps at the First Tangency . . . 25

3.4 A Coding for $\tilde{\Lambda} = \bigcap_{n \in \mathbf{Z}} F^n(Q)$ at the First Tangency. 27

4 The Abstract Model **30**

4.1 Basic Definitions and Assumptions 30

4.2 Hyperbolicity Results for $|F_{2y}|, |F_{1y}| \ll |F_{1x}|$ 37

5 Symbolic Dynamics (Part II) **46**

5.1 Assumptions and Definitions 46

5.2 Stable and Unstable Curves 47

5.3 Topological Equivalence 48

6 Application to the Henon Map **49**

6.1 Geometric Conditions 49

6.2 The Region of (R, α)-Hyperbolicity 50

6.3 Concavity Conditions 52

6.4 Main Results for Henon Maps 55

1 Introduction

In their 1979 paper, R. Devaney and Z. Nitecki proved that the set Λ of points with bounded orbits of the Henon map $H(x,y) = (rx(1-x) - by, x)$, $b \neq 0$, is a hyperbolic set, and that $H|_\Lambda$ is conjugate to the full shift (σ, Σ) on two symbols (cf. sections 3 and 5 for an explanation of these terms), *provided* the parameters are chosen so that $r > (2+\sqrt{5})(1+b)$ (cf. [**DN**]). Devaney and Nitecki's method of proof uses real geometry and extends to include C^2-perturbations of the Henon maps considered. These results involve only geometric estimates on the relative position of $Q := [0,1]^2$, $H(Q)$ and $H^{-1}(Q)$ - it turns out that Λ is actually equal to the largest H-invariant set containing Q; i.e., $\Lambda = \bigcap_{n \in \mathbb{Z}} H^n(Q)$.

On the other hand, for a diffeomorphism F, Smale's homoclinic point theorem gives a result about the existence of a (possibly small) hyperbolic set associated with the occurrence of a transverse homoclinic point q of a hyperbolic saddle point p. The hyperbolic set is the maximal F^N-invariant set (for some possibly large N) of a tubular neighbourhood R about part of the unstable manifold of p, and R contains both p and q. Smale's homoclinic point theorem relies on the geometry of the stable and unstable manifolds of the saddle point p. It is important to note that if the angle of intersection of the stable and unstable manifolds at q is small (q makes the transition from a transverse homoclinic point to a homoclinic tangency), R must be chosen to be very narrow.

Relating [**DN**]'s result to the geometry of the invariant manifolds, we note that the main requirement would be that the homoclinic contact be quadratic, and - more restrictvely - that the distance d between unstable and stable manifolds between the two associated homoclinic intersections has to be rather large.

We introduce an abstract class of maps that bridges the gap between having a large hyperbolic invariant set, but requiring that d be large, and allowing d to be small, with the trade-off that the hyperbolic invariant set then shrinks to consist simply of the hyperbolic saddle point and the orbit of the homoclinic point q. The abstract class contains the Henon family $H(x, y) = (rx(1-x) - by, x)$, for $0 < b \ll 1$, and C^2-perturbations. For a map F in this class, we denote by Λ the set of points with bounded orbits. We obtain hyperbolicity of a "return map" on Λ, which is a (possibly high) iterate of H, depending on in which region of Λ the initial point lies. This allows us to establish symbolic dynamics of $F|_\Lambda$ before and at the first tangency. The main technical assumption is on the relative concavity of the stable and unstable manifolds, related to their distance d.

We devote the rest of this section to introduce some of the concepts just mentioned. We limit outselves to diffeomorphisms of $\mathbb{R}^2$ - the definitions and results can be naturally extended to general euclidian spaces, and finite dimensional manifolds.

Hyperbolic saddle points, invariant manifolds, and homoclinic intersections

Let F be a C^r-diffeomorphism ($r \geq 1$) of an open set $U \subset \mathbb{R}^2$ onto $V = F(U) \subset \mathbb{R}^2$. A *fixed point* is a point $p \in U$ such that $F(p) = p$. We say that the fixed point p is *hyperbolic* if none of the eigenvalues λ_1, λ_2 of the differential DF_p has modulus 1; if $0 < |\lambda_1| < 1 < |\lambda_2|$, then the fixed point is called a *hyperbolic saddle point.*

Given a hyperbolic saddle point p, we consider the sets

$$W^u(p) = \left\{q : \left|p - F^{-n}(q)\right| \to 0 \text{ as } n \to \infty\right\}$$

and

$$W^s(p) = \{q : |p - F^n(q)| \to 0 \text{ as } n \to \infty\}.$$

Then $W^u(p)$ and $W^s(p)$ are injectively immersed C^r-curves containing p (cf. e.g. [**HK**]). $W^u(p)$ is called the *unstable manifold* of the hyperbolic saddle point p, and $W^s(p)$ is called the *stable manifold* of the hyperbolic saddle point p.

A *homoclinic point* is a point $q \neq p$ in the intersection of $W^u(p)$ and $W^s(p)$. If the angle of intersection is not zero, then q is a *transverse homoclinic point*; otherwise q is called a *homoclinic tangency*.

Hyperbolic sets and the cone criterion

Let Λ be a compact F-invariant set; i.e., $F(\Lambda) = \Lambda$. Then Λ is called (uniformly) hyperbolic, if there exist $\lambda > 1$, $C > 0$, such that for each $p \in \Lambda$, there is a splitting $T_p\mathbb{R}^2 = E_p^u \oplus E_p^s$ such that:

- the splitting is DF-invariant: $DF_p(E_p^u) = E_{F(p)}^u$ and $DF_p(E_p^s) = E_{F(p)}^s$,
- the splitting depends continuously on $p \in \Lambda$,
- if $v \in E_p^u$, then $|DF_p^n(v)| \geq C \cdot \lambda^n \cdot |v|$ for all $n > 0$,
- if $v \in E_p^s$, then $|DF_p^{-n}(v)| \geq C \cdot \lambda^n \cdot |v|$ for all $n > 0$.

If p is a hyperbolic fixed point, then $\Lambda = \{p\}$ is a hyperbolic set. We also have invariant manifolds for hyperbolic sets. Assume for instance that Λ is a hyperbolic set of *saddle type*; i.e., $\dim(E_p^u) = \dim(E_p^s) = 1$ for all $p \in \Lambda$.

Now we consider the sets

$$W^u(p) = \left\{q : \left|F^{-n}(p) - F^{-n}(q)\right| \to 0 \text{ as } n \to \infty\right\}$$

and

$$W^s(p) = \{q : |F^n(p) - F^n(q)| \to 0 \text{ as } n \to \infty\}.$$

Then $W^u(p)$ and $W^s(p)$ are again injectively immersed C^r-curves ([**HK**]). $W^u(p)$ is called the *unstable manifold* of the point $p \in \Lambda$, and $W^s(p)$ is called the *stable manifold* of the point $p \in \Lambda$.

In order to show that a given compact F-invariant set is a hyperbolic set, one can use the following *cone criterion.*

A *cone* in $\mathbb{R}^2$ (or in $T_p\mathbb{R}^2$) is a set of the form

$$C = C(u, v) = \{\alpha u + \beta v : \alpha\beta \geq 0\},$$

where $u, v \in \mathbb{R}^2$ (or $u, v \in T_p\mathbb{R}^2$).

Cone Criterion

Suppose Λ is a compact, F-invariant set, and suppose there exists a $\lambda > 1$, and for each $p \in \Lambda$ there exists an unstable cone C_p^u in $T_p\mathbb{R}^2$ and a stable cone C_p^s in $T_p\mathbb{R}^2$ satisfying the conditions:

- $C_p^u \cap C_p^s = \{0\}$,
- the unstable cones are DF-invariant: $DF_p(C_p^u) \subset C_{F(p)}^u$,
- the stable cones are DF^{-1}-invariant: $DF_{F(p)}^{-1}(C_{F(p)}^s) \subset C_p^s$,
- the cones depend continuously on $p \in \Lambda$,
- if $v \in C_p^u$, then $|DF_p(v)| \geq \lambda \cdot |v|$,
- if $v \in C_{F(p)}^s$, then $|DF_{F(p)}^{-1}(v)| \geq \lambda \cdot |v|$.

Then Λ is a hyperbolic set.

2 The Henon Map

2.1 Introduction

The Henon map we consider is given as

$$H_{b,r}(x,y) = (rx(1-x) - by, x),$$

For $b \neq 0$, this is a diffeomorphism of the plane $\mathbb{R}^2$, and for $b = 0$, we have the logistic map $H_{0,r}(x,y) = (rx(1-x), x)$.

In [**DN**], R. Devaney and Z. Nitecki use the following form for the Henon map:

$$h_{A,B}(x,y) = (1 + y - Ax^2, Bx),$$

whereas in [**NY**], H. Nusse and J. Yorke use the form

$$\mathcal{H}_{\rho,c}(x,y) = (\rho - x^2 + cy, x).$$

All these maps are conjugate via affine coordinate changes; for $A, B \neq 0$, let

$$T_{A,B}(x,y) = \frac{r}{2A}\left(2x - 1, 2By - B\right)), \text{ and } S_{A,B}(x,y) = \left(Ax, \frac{A}{B}y\right).$$

Then for $r \neq 2 + 2b$, $b \neq 0$, and $A, B \neq 0$, we have

$$T_{\frac{r(r-2-2b)}{4}, -b} \circ H_{b,r} = h_{\frac{r(r-2-2b)}{4}, -b} \circ T_{\frac{r(r-2-2b)}{4}, -b}, \text{ and } S_{A,B} \circ h_{A,B} = \mathcal{H}_{A,B} \circ S_{A,B}.$$

We want to investigate the dynamics of the map $H(x,y)$ under iterates.

2.2 Fixed Points and Images of Curves

We note that for $b \neq 0$, the inverse of the Henon map is

$$H_{b,r}^{-1}(x,y) = \left(y, \frac{r}{b}y(1-y) - \frac{x}{b}\right).$$

Where convenient, we write $H(x,y) = (H_1(x,y), H_2(x,y)) = (rx(1-x) - by, x)$ and thus omit the dependence on the parameters b and r.

Also,

$$DH_{(x,y)} = \begin{pmatrix} r(1-2x) & -b \\ 1 & 0 \end{pmatrix},$$

and

$$DH^{-1}_{H(x,y)} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{b} & \frac{r}{b}(1-2x) \end{pmatrix}.$$

Note that the Jacobian determinant $\det DH_{b,r} = b$. Throughout this paper, we assume $0 < b < 1$; i.e., that the Henon map is *orientation-preserving* and *dissipative.*

The following result is easily verified.

Proposition 2.1

(1) The Henon map $H_{b,r}$ has exactly two fixed points; namely, $p_0 = (0,0)$ and $p_1 = \left(1 - \frac{b+1}{r}, 1 - \frac{b+1}{r}\right)$.

(2) $DH_{b,r}|_{p_0}$ has the two eigenvectors $(\lambda_1, 1)$ and $(\lambda_2, 1)$, where

$$\lambda_{1,2} = \frac{r \pm \sqrt{r^2 - 4b}}{2}$$

are the respective eigenvalues.

(3) $DH_{b,r}|_{p_1}$ has the two eigenvectors $(\mu_1, 1)$ and $(\mu_2, 1)$, where

$$\mu_{1,2} = \frac{\tilde{r} \pm \sqrt{\tilde{r}^2 - 4b}}{2}$$

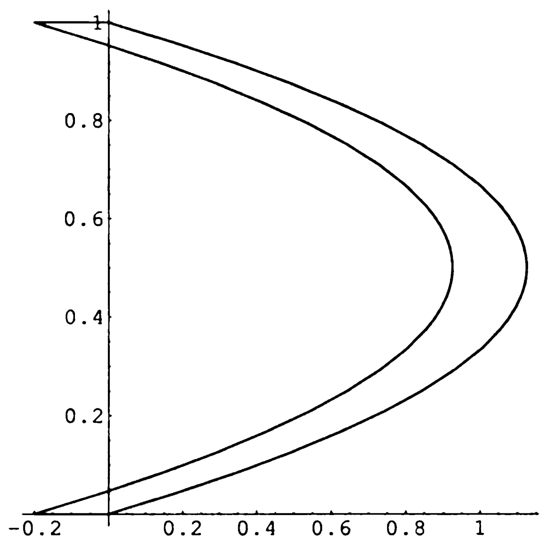
are the respective eigenvalues, and $\tilde{r} := 2(b+1) - r$.

(4) If $r > 1 + b$, then $p_0 = (0,0)$ is a hyperbolic saddle point with $0 < \lambda_2 < 1 < \lambda_1$.

(5) If $r > 3(1+b)$, then $p_1 = \left(1 - \frac{b+1}{r}, 1 - \frac{b+1}{r}\right)$ is a hyperbolic saddle point with $\mu_2 < -1 < \mu_1 < 0$.

Note that the images (under $H_{b,r}$) of vertical lines are horizontal lines, and that the images (under $H_{b,r}$) of horizontal lines are parabolas of the form $t \mapsto (rt(1-t)+D, t)$.

Let $I = [0,1]$, and $Q = I^2$. Then the image of Q is a "horseshoe", with the left and right boundaries of Q being mapped to the bottom and top horizontal bounding lines of $H_{b,r}(Q)$ (with length b), and the bottom and top boundaries of Q being mapped to the left and right bounding parabolas of $H_{b,r}(Q)$ (whose horizontal distance is b). A picture of $H_{b,r}(Q)$ with $r = 4.5$ and $b = 0.2$ is given below.



Picture 2.1

The next two results show that there are certain invariant classes of curves.

Proposition 2.2 *Suppose $\gamma(t) = (rt(1-t) + g(t), t)$ is a curve in $\mathbb{R}^2$ such that $2r\left|t - \frac{1}{2}\right| \geq 1 + b$ for all t, and such that $|g'(t)| \leq b$ and $|g''(t)| \leq \frac{2br}{1-b}$. Then $H_{b,r}(\gamma(t))$ can be written in the form $(rs(1-s) + h(s), s)$, where $|h'(s)| \leq b$ and $|h''(s)| \leq \frac{2br}{1-b}$.*

Proof: We have that for $s =: s(t) := rt(1-t) + g(t)$, $H_{b,r}(\gamma(t)) = H_{b,r}(s(t), t) = (rs(1-s) - bt, s)$, and

$$\left|\frac{ds}{dt}\right| = |r(1-2t) + g'(t)| \geq 2r\left|t - \frac{1}{2}\right| - |g'(t)| \geq 1 + b - b = 1.$$

This means that $s(t)$ has an inverse $t(s)$. Letting $h(s) = -b \cdot t(s)$, we get $H_{b,r}(\gamma(t)) = (rs(1-s) + h(s), s)$, and $\left|\frac{dh}{ds}\right| = b \cdot \left|\frac{dt}{ds}\right| \leq b \cdot 1 = b$.

Also,

$$\left|\frac{d^2h}{ds^2}\right| = b \cdot \left|\frac{d^2t}{ds^2}\right| = b \cdot \left|\frac{d^2s}{dt^2}\right| \cdot \left|\frac{dt}{ds}\right|^3 \leq b \cdot |-2r + g''(t)| \cdot 1^3 \leq b \cdot \left(2r + \frac{2br}{1-b}\right) = \frac{2br}{1-b}. \quad \square$$

Proposition 2.3 *Suppose $\gamma(t) = (g(t), t)$ is a curve in $\mathbb{R}^2$ such that $2r\left|t - \frac{1}{2}\right| \geq 1+b$ for all t, and such that $|g'(t)| \leq b$ and $|g''(t)| \leq \frac{2b^2r}{1-b^2}$. Then $H_{b,r}^{-1}(\gamma(t))$ can be written in the form $(h(s), s)$, where $|h'(s)| \leq b$ and if $0 < b \leq \frac{1}{\sqrt{2}}$, $|h''(s)| \leq \frac{2b^2r}{1-b^2}$. Furthermore, the signs of $h'(s)$ and $h''(s)$ are equal to the sign of $\frac{1}{2} - t$.*

Proof: We have that for $s := s(t) := \frac{r}{b}t(1-t) - \frac{g(t)}{b}$, $H_{b,r}^{-1}(\gamma(t)) = (t, s(t))$, and

$$\frac{ds}{dt} = \frac{1}{b} \cdot (r(1-2t) - g'(t)) \geq \frac{2r}{b}\left(\frac{1}{2} - t\right) - \frac{1}{b}(b) \geq \frac{1+b}{b} - 1 = \frac{1}{b},$$

if $2r\left(\frac{1}{2} - t\right) \geq 1 + b$,

$$-\frac{ds}{dt} = \frac{1}{b} \cdot (r(2t-1) + g'(t)) \geq \frac{2r}{b}\left(t - \frac{1}{2}\right) + \frac{1}{b}(-b) \geq \frac{1+b}{b} - 1 = \frac{1}{b},$$

if $2r\left(t - \frac{1}{2}\right) \geq 1 + b$.

In any case, $\left|\frac{ds}{dt}\right| \geq \frac{1}{b}$, and this means that $s(t)$ has an inverse $t(s)$. Letting $h(s) = t(s)$, we get $H_{b,r}^{-1}(\gamma(t)) = (h(s), s)$, and $\frac{dh}{ds} = \frac{dt}{ds}$ gives $\left|\frac{dh}{ds}\right| \leq b$ and the statements about the sign of $h'(s)$.

The condition $0 < b \leq \frac{1}{\sqrt{2}}$ guarantees that $\frac{d^2s}{dt^2} \leq 0$; the formula $\frac{d^2t}{ds^2} = -\frac{d^2s}{dt^2} \cdot \left(\frac{dt}{ds}\right)^3$ gives

$$\left|\frac{d^2h}{ds^2}\right| = \left|\frac{d^2t}{ds^2}\right| = \left|\frac{d^2s}{dt^2}\right| \cdot \left|\frac{dt}{ds}\right|^3 \leq \frac{1}{b} \cdot |-2r + g''(t)| \cdot b^3 \leq b^2 \cdot \left(2r + \frac{2b^2r}{1-b^2}\right) = \frac{2b^2r}{1-b^2},$$

and the statements about the sign of $h''(s)$. $\square$

We define the following sets:

$$\mathcal{E} = \mathcal{E}_{b,r} = \left\{(x,y) : 2r\left|x - \frac{1}{2}\right| \geq 1 + b\right\},$$

$$\mathcal{E}' = \mathcal{E}'_{b,r} = \left\{(x,y) : 2r\left|y - \frac{1}{2}\right| \geq 1 + b\right\},$$

$$\mathcal{S} = \mathcal{S}_{b,r} = \left\{(x,y) : 2r\left|x - \frac{1}{2}\right| \leq 1 + b\right\},$$

and

$$\mathcal{S}' = \mathcal{S}'_{b,r} = \left\{(x,y) : 2r\left|y - \frac{1}{2}\right| \leq 1 + b\right\}.$$

Note that $H(\mathcal{E}) = \mathcal{E}'$ and $H(\mathcal{S}) = \mathcal{S}'$. Note also that $\mathcal{S}_{b,r}$ is a closed vertical strip about the line $x = \frac{1}{2}$, and that that $\mathcal{S}'_{b,r}$ is a closed horizontal strip about the line $y = \frac{1}{2}$.

We will be interested in the invariant set $\Lambda = \bigcap_{n \in \mathbf{Z}} H^n(Q)$. We make the following observation regarding the relative positions of Q, $H(Q)$, $\mathcal{S}$ and $\mathcal{S}'$.

Lemma 2.1 *Suppose* $0 < b \leq 1$.

(1) $H(Q) \cap Q$ *has two connected components if and only if* $r > 4 \cdot (1 + b)$; *i.e.,* Λ *is a "topological horseshoe".*

(2) Let $\mathcal{S}'_Q = \mathcal{S}' \cap Q$. *Then* $H(Q) \cap \mathcal{S}'_Q = \emptyset$ *if and only if* $r > \left(2 + \sqrt{5}\right) \cdot (1 + b)$.

Proof: The left boundary of $H(Q)$ is given by the parabola

$$\Gamma := H(\{(x,1) : 0 \leq x \leq 1\}) = \{(rx(1-x) - b, x) : 0 \leq x \leq 1\}.$$

Γ intersects the right boundary of Q precisely when $r \geq 4(1+b)$, and Γ intersects $\mathcal{S}'_Q$ precisely when $r \leq (2 + \sqrt{5})(1 + b)$. $\square$

We define:

$$\mathcal{E}_{left} = \left\{ (x,y) : 2r\left(\frac{1}{2} - x\right) \geq 1 + b \right\},$$

$$\mathcal{E}_{right} = \left\{ (x,y) : 2r\left(x - \frac{1}{2}\right) \geq 1 + b \right\},$$

$$\mathcal{E}'_{bottom} = \left\{ (x,y) : 2r\left(\frac{1}{2} - y\right) \geq 1 + b \right\},$$

$$\mathcal{E}'_{top} = \left\{ (x,y) : 2r\left(y - \frac{1}{2}\right) \geq 1 + b \right\}.$$

We also let $Q_{bottom,left} = Q \cap \mathcal{E}'_{bottom} \cap \mathcal{E}_{left}$, and $Q_{top,left}$, $Q_{bottom,right}$, and $Q_{top,right}$ along the same lines. Then we have the following lemma.

Lemma 2.2 *Suppose* $0 < b \leq 1$, *and* $r \geq 2(1+2b)$, *then we have the following:*

(a) $H^{-1}(\mathcal{E}_{left} \cap Q) \cap Q$ *consists of two connected components* C_{left} *and* C_{right}.

(b) C_{left} *is full-height in* $\mathcal{E}_{left} \cap Q$, *and* $C_{left} = H^{-1}(Q_{bottom,left}) \cap Q$.

(c) C_{right} *is full-height in* $\mathcal{E}_{right} \cap Q$, *and* $C_{right} = H^{-1}(Q_{top,left}) \cap Q$.

Proof: H^{-1} maps the left boundary of $\mathcal{E}_{left} \cap Q$ to the parabola $y \mapsto \left(y, \frac{r}{b}y(1-y)\right)$, the bottom boundary of $\mathcal{E}_{left} \cap Q$ to a vertical line $\{0\} \times [0, -D]$, and the top boundary of $\mathcal{E}_{left} \cap Q$ to a vertical line $\{1\} \times [0, -D]$, for some $D > 0$. It remains to be checked whether the pre-image $H^{-1}(l)$ of the right boundary l of $\mathcal{E}_{left} \cap Q$ avoids the region $\left\{ (x,y) : 0 \leq y \leq 1,\ 2r\left|\frac{1}{2} - x\right| < 1 + b \right\}$. Let $x^\star$ be such that $2r\left(\frac{1}{2} - x^\star\right) = 1 + b$. Then $l = (x^\star, t)$, $0 \leq t \leq 1$, and $H^{-1}(l) = \left(t, \frac{r}{b}t(1-t) - \frac{x^\star}{b}\right)$.

Suppose that t is such that $2r\left|\frac{1}{2} - t\right| \leq 1 + b$. Then we need to show that $\frac{r}{b}t(1-t) - \frac{x^\star}{b} \geq 1$. Using that $t(1-t) = \frac{1}{4} - \left(t - \frac{1}{2}\right)^2$, we get $\frac{r}{b}t(1-t) \geq \frac{r^2 - (1+b)^2}{4br}$, and consequently

$$\frac{r}{b}t(1-t) - \frac{x^\star}{b} \geq \frac{r^2 - (1+b)^2}{4br} - \frac{r-(1+b)}{2br} = \frac{r^2 - 2r - b^2 + 1}{4br}.$$

We need $r^2 - 2r - b^2 + 1 \geq 4br$. Since $r \geq 2(1+2b)$, we have $r - (1+2b) \geq 1 + 2b$, and then $[r-(1+2b)]^2 \geq (1+2b)^2$. This means

$$r^2 - 2r(1+2b) + (1+2b)^2 \geq (1+2b)^2 = 1 + 4b + 4b^2 \geq 5b^2 + 4b,$$

because $b \leq 1$. This gives $r^2 - 2r - b^2 + 1 \geq 4br$, as required. □

2.3 Invariant Manifolds

The two results that follow indicate the position of the stable and unstable manifolds, given certain condtions on b and r. Let $W^s(p_i)$ denote the stable manifold of the fixed point p_i, and let $W^u(p_i)$ denote the unstable manifold of the fixed point p_i, $i = 0, 1$. Recall that $Q = I^2 = [0,1]^2$, and let $l^s_{1,1}$ and $l^s_{1,2}$ be the first and second connected component (resp.) of $W^s(p_0) \cap Q$; let $l^s_{2,1}$ and $l^s_{2,2}$ be the first and second connected component (resp.) of $W^s(p_1) \cap Q$.

Proposition 2.4 *Suppose* $0 < b \leq 1$, *and* $r \geq 3(1+b)$. *Then we can write*

(1) $l^s_{1,1} : [0,1] \to Q, \quad y \mapsto (f^s_{1,1}(y), y)$, *where:*

(1a) $f^s_{1,1}(0) = 0$, $0 \leq f^s_{1,1}(y)$, *and* $2r\left(\frac{1}{2} - f^s_{1,1}(y)\right) \geq 1 + b$,

(1b) $0 \leq (f^s_{1,1})'(y) \leq b$, *and*

(1c) if $0 < b \leq \frac{1}{\sqrt{2}}$, *then* $0 \leq (f^s_{1,1})''(y) \leq \frac{2b^2 r}{1-b^2}$.

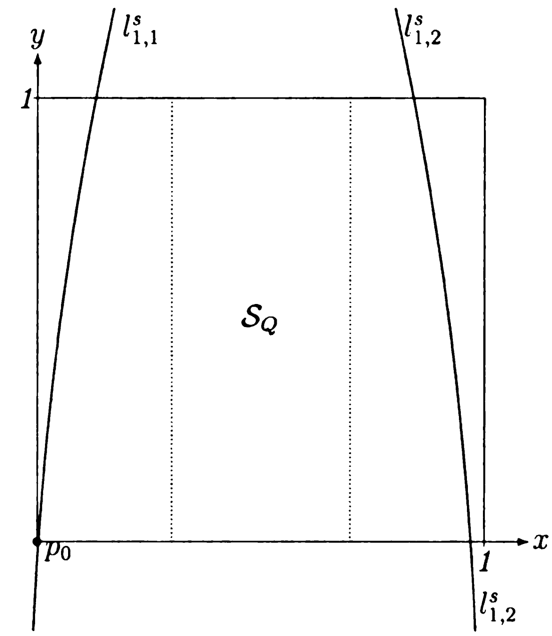
(2) $l^s_{1,2} : [0,1] \to Q, \quad y \mapsto (f^s_{1,2}(y), y)$, *where:*

(2a) $f^s_{1,2}(y) \leq 1$, *and* $2r\left(f^s_{1,2}(y) - \frac{1}{2}\right) \geq 1 + b$,

(2b) $-b \leq (f^s_{1,2})'(y) \leq 0$, *and*

(2c) if $0 < b \leq \frac{1}{\sqrt{2}}$, *then* $-\frac{2b^2 r}{1-b^2} \leq (f^s_{1,2})''(y) \leq 0$.

The following picture illustrates the general position of the first two connected components of $W^s(p_0)$ relative to the region $\mathcal{S}_Q = \left\{(x,y) \in Q : 2r\left|x - \frac{1}{2}\right| \leq 1 + b\right\}$.



Picture 2.2

Proof: For a fixed small $\delta > 0$, consider the curve $\gamma(t) = (g(t), t) = (\lambda_2 \cdot t, t)$, where $0 \leq t < \delta$ and $\lambda_2 = \dfrac{r - \sqrt{r^2 - 4b}}{2}$ is the contracting eigenvector of DH_{p_0}. If $r > 1 + b$, then we have that $0 < \lambda_2 = \dfrac{2b}{r + \sqrt{r^2 - 4b}} < b \leq 1$. We note that if $0 < b \leq 1$ and $r \geq 3(1 + b)$, then $r \geq 2(1 + 2b)$. Thus, Proposition 2.3 and Lemma 2.2 give that the first two connected components in Q of $H^{-1}(\gamma(t))$ and of all subsequent pre-images have the properties listed.

It is well known in the theory of invariant manifolds (cf. for example [**S1**]) that for some small $\delta > 0$, $H^{-n}(\Gamma) \to W^s(p_0)$ as $n \to \infty$, where $\Gamma = \{\gamma(t) : -\delta < t < \delta\}$. It is easy to check that if $q = \gamma(t)$ for $t < 0$, $H^{-n}(q)$ will not return to Q. $\square$

We also have results on parts of the unstable manifold of $p_0 = (0, 0)$. First, we establish a set $\mathcal{P}$ of (b, r)-parameter values for which we have control over the unstable manifold.

Lemma 2.3 *Let* $t \mapsto (rt(1-t)+g(t), t)$ *be a curve such that* $g(0) = 0$ *and* $|g'(t)| \leq b$. *Let* $t^\star$ *be such that* $2r\left(\dfrac{1}{2} - t^\star\right) = 1 + b$, *and let* $x^\star = rt^\star(1-t^\star) + g(t^\star)$.

Let

$$\mathcal{P} = \{(b,r) : r\left(rt^\star(1-t^\star) + bt^\star\right)\left(1 + bt^\star - rt^\star(1-t^\star)\right) \leq (1+b)t^\star\}.$$

Then for every pair of parameters $(b,r) \in \mathcal{P}$, *we have that*

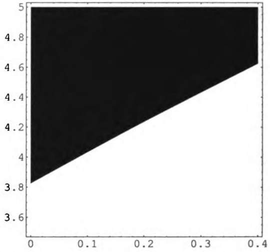
$$(x^\star, t^\star) \in \left\{(x,y) : 2r\left(x - \frac{1}{2}\right) \geq 1 + b\right\}$$

and

$$H(x^\star, t^\star) \in \left\{(x,y) : 2r\left(\frac{1}{2} - x\right) \geq 1 + b\right\}.$$

Proof: This is an elementary argument using that if $(b,r) \in \mathcal{P}$, then $(x^\star, t^\star)$ is not to the left of the image of the right boundary of $\left\{(x,y) : 2r\left(\frac{1}{2} - x\right) \geq 1 + b\right\}$. $\square$

The following is a picture of the (global) region of control $\mathcal{P}$.



Picture 2.3

Let l_1^u be the first connected component of $W^u(p_0) \cap \left\{(x,y) : 2r\left(y - \frac{1}{2}\right) \leq 1 + b\right\} \cap Q$ and let l_2^u be the second connected component of $W^u(p_0) \cap \left\{(x,y) : 2r\left|y - \frac{1}{2}\right| \leq 1 + b\right\} \cap Q$.

Proposition 2.5 *Suppose* $0 < b < 1$, *and* $(b, r) \in \mathcal{P}$. *Let* $y_1^\star$ *be such that* $2r\left(y_1^\star - \frac{1}{2}\right) = 1 + b$ *and let* $y_2^\star$ *be such that* $2r\left(\frac{1}{2} - y_2^\star\right) = 1 + b$. *Then we can write*

(1) $l_1^u : [0, y_1^\star] \to Q, \quad y \mapsto (ry(1-y) + f_1^u(y), y)$, *where:*

(1a) $f_1^u(0) = 0$,

(1b) $|(f_1^u)'(y)| \le b$, *and*

(1c) $|(f_1^u)''(y)| \le \dfrac{2br}{1-b}$.

(2) $l_2^u : [y_2^\star, y_1^\star] \to Q, \quad y \mapsto (ry(1-y) + f_2^u(y), y)$, *where:*

(2a) $f_2^u(y) < f_1^u(y)$,

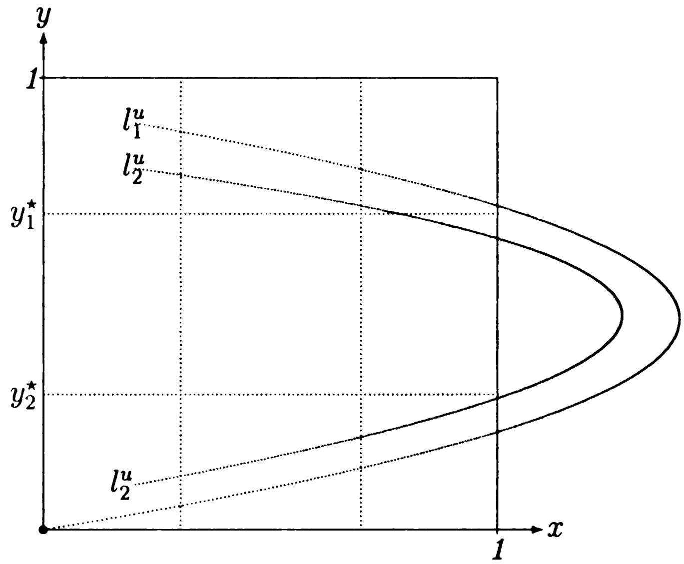
(2b) $|(f_2^u)'(y)| \le b$, *and*

(2c) $|(f_2^u)''(y)| \le \dfrac{2br}{1-b}$.

Proof: For a fixed small $\delta > 0$, consider the curve $\gamma(t) = \left(t, \dfrac{t}{\lambda_1}\right)$, where $0 \le t < \delta$ and $\lambda_1 = \dfrac{r + \sqrt{r^2 - 4b}}{2}$ is the expanding eigenvector of DH_{p_0}. The first image of $\gamma(t)$ is $H(\gamma(t)) = \left(rt(1-t) - b \cdot \dfrac{t}{\lambda_1}, t\right)$. Let $g(t) = -b \cdot \dfrac{t}{\lambda_1}$. If $r \ge 1 + b$, then we have that $|g'(t)| = \dfrac{b}{\lambda_1} = \dfrac{2b}{r + \sqrt{r^2 - 4b}} \le b < 1$.

It follows from Proposition 2.2 that for $n \ge 1$, $H^n(\gamma(t))$ has properties (1a)-(1c), at least as long as the y-range is within $[0, y_2^\star]$. If $H^n(\gamma(t))$ has y-range within $[0, y_2^\star)$, the y-range will strictly increase under iterates ($\frac{ds}{dt} > 1$ in the proof of Proposition 2.2). Using Lemma 2.3, we may assume that for some $n \ge 1$, $H^n(\gamma(t))$ has x-range $[0, y_1^\star]$, and hence $H^{n+1}(\gamma(t))$ has y-range $[0, y_1^\star]$. This proves part (1), since $H^n(\gamma(t)) \to W^u(p_0)$ as $n \to \infty$.

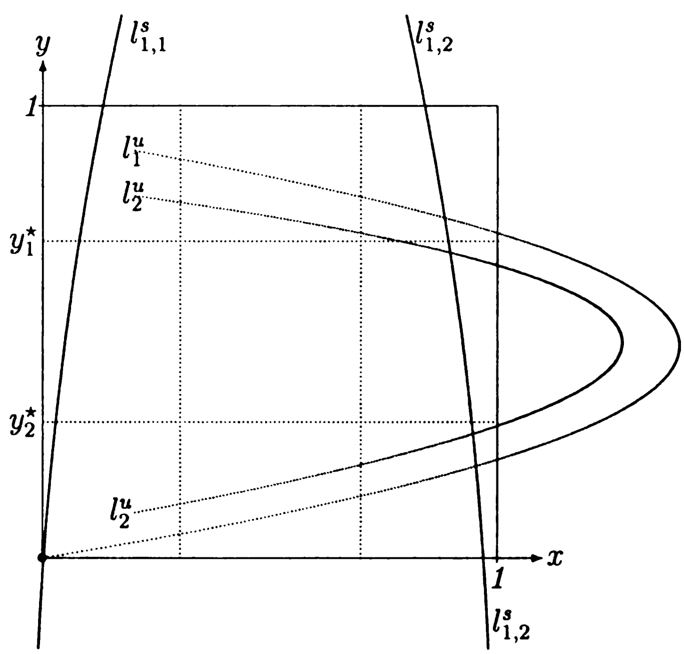
Also, Lemma 2.3 and Lemma 2.2 give that $H^{n+2}(\gamma(t))$ has y-range contained in $[y_2^\star, y_1^\star]$, which proves part (2). Finally, it is again easy to check that if $q = \gamma(t)$ for $t < 0$, $H^n(q)$ will not return to Q. $\square$



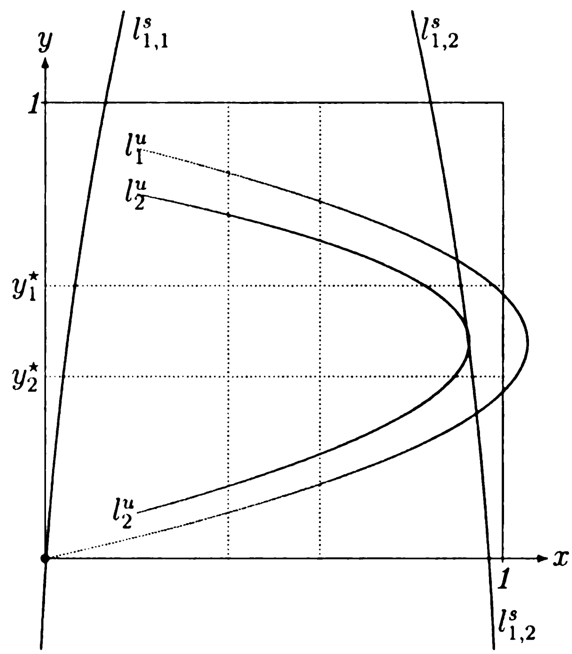
Picture 2.4

It follows from Propositions 2.4 and 2.5 that for each there exists a curve $b \mapsto r(b)$ with $(b, r(b)) \in \mathcal{P}$ such that if $(b, r) \in \mathcal{P}$ and $r > r(b)$, the curves l_2^u and $l_{1,2}^s$ have two transverse intersections, and for $r = r(b)$, l_2^u and $l_{1,2}^s$ are tangent.

The following pictures illustrate the previous results for $r > r(b)$ and $r = r(b)$.



Picture 2.5



Picture 2.6

2.4 Quadratic Contact at the First Tangency

If we restrict the extent of the region $\mathcal{P}$, we can verify that the contact between l^u_2 and $l^s_{1,2}$ is quadratic.

Definition 2.1 *Let $\mathcal{P}$ be as in Lemma 2.3, except that additionally* $0 < b < \dfrac{\sqrt{13}-1}{6}$.

Proposition 2.6 *Let $\gamma^u : t \mapsto (g^u(t), t)$ be a curve whose concavity $K^u := (g^u(t))''$ satisfies $|K^u + 2r| \leq \dfrac{2br}{1-b}$, and let $\gamma^s : t \mapsto (g^s(t), t)$ be a curve whose concavity $K^s := (g^s(t))''$ satisfies $|K^s| \leq \dfrac{2b^2 r}{1-b^2}$. Let $0 < b < \dfrac{\sqrt{13}-1}{6}$. Then $K^s > K^u$ for all t in the common domain of γ^u and γ^s.*

The proof of the above Proposition is elementary. In particular, it gives us the following.

Corollary 2.1 *If $(b, r(b)) \in \mathcal{P}$ ($\mathcal{P}$ as defined in Definition 2.1), then the tangency at $(b, r(b))$ is quadratic.*

We now investigate consequences of having a homoclinic tangency.

Suppose $\gamma^u(t) = (rt(1-t) + g^u(t), t)$ and $\gamma^s(t) = (g^s(t), t)$ are two curves with $|(g^u)'| \leq b$ and $-b \leq (g^s)' \leq 0$ such that γ^u, γ^s have a tangency at t_0; i.e.,

$$\textbf{(*)} \qquad rt_0(1-t_0) + g^u(t_0) = g^s(t_0)$$

and

$$\textbf{(**)} \qquad r(1-2t_0) + (g^u)'(t_0) = (g^s)'(t_0).$$

Let $\Delta(t) = g^s(t) - g^u(t)$, then $|\Delta'(t)| \leq 2b$, and **(**)** implies $2r\left|t_0 - \frac{1}{2}\right| = |\Delta'(t_0)| \leq 2b$, in particular, since we assume $0 < b \leq 1$, we have $2r\left|t_0 - \frac{1}{2}\right| \leq 1 + b$. This gives the following result.

Proposition 2.7 *If $(b, r(b)) \in \mathcal{P}$, then the tangency between l_2^u and $l_{1,2}^s$ occurs in the region*

$$Q_{center,right} = \left\{(x,y) \in Q : 2r\left|y - \frac{1}{2}\right| \leq 1 + b, \quad 2r\left(x - \frac{1}{2}\right) \geq 1 + b\right\}.$$

Now, we want to give estimates for the parameter $r(b)$. Since $rt_0(1-t_0) = \frac{r}{4} - r\left(t_0 - \frac{1}{2}\right)^2$, **(*)** and the equation $2r\left|t_0 - \frac{1}{2}\right| = |\Delta'(t_0)|$ give

$$r^2 - 4r\Delta(t_0) = [\Delta'(t_0)]^2$$

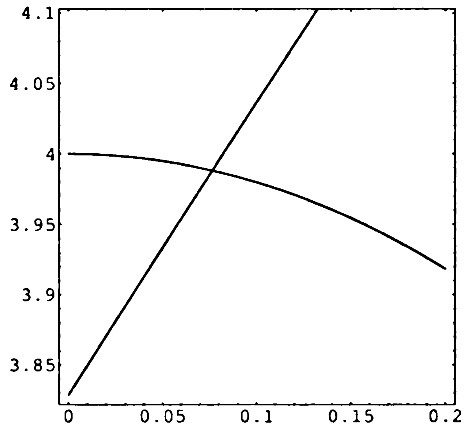
where $r = r(b)$ is understood to depend on t_0, the y-coordinate of the tangency. Note that we know that $\Delta(1/2) = 1$. Hence we must solve the initial value problem

$$[r(t)]^2 - 4r(t)\Delta(t) = [\Delta'(t)]^2 \qquad \Delta(1/2) = 1.$$

(Now r depends on t; note that using this notation, $r(1/2) = 4$.) Using the estimates $[\Delta'(t)]^2 \geq 0$, $\Delta(t) \geq 1 + \int_{1/2}^{t} \Delta' \geq 1 - 2b\left|t - \frac{1}{2}\right|$, and $\left|t - \frac{1}{2}\right| \leq \frac{b}{r}$, we get

$$r \geq 4 - 8\frac{b^2}{r} \quad , \text{or} \quad r \geq 2 + 2\sqrt{1 - 2b^2}.$$

The following picture shows the curve $b \mapsto (b, 2 + 2\sqrt{1-2b^2})$ vs. the lower boundary of $\mathcal{P}$.



Picture 2.7

This justifies the following proposition.

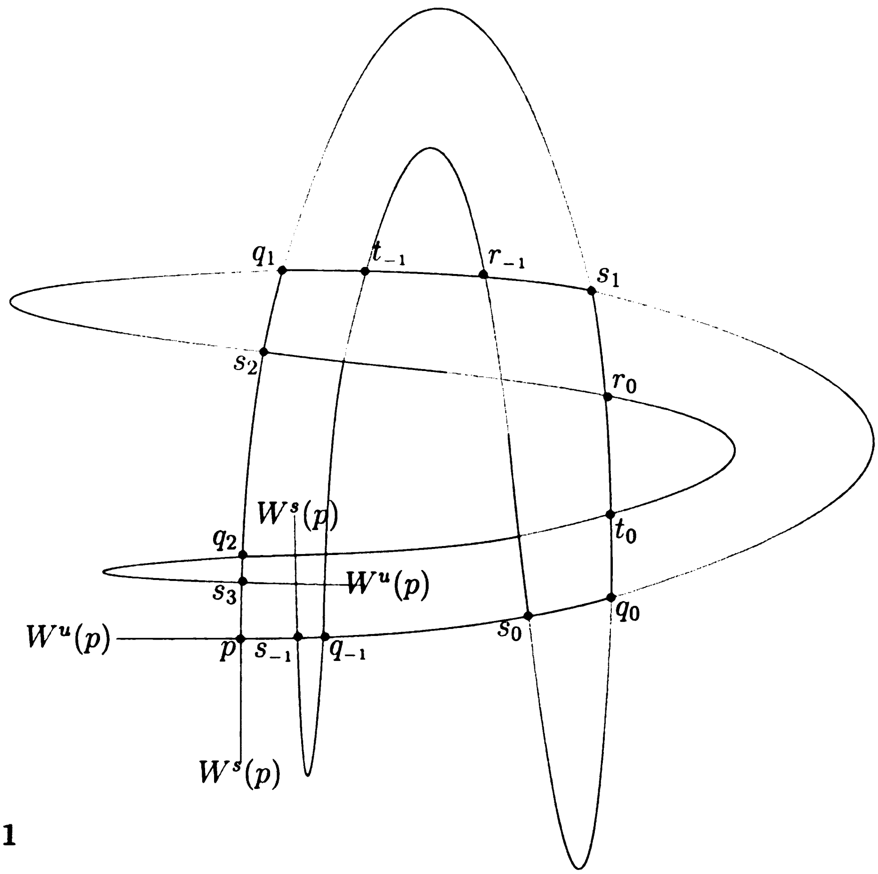
Proposition 2.8 *If* $0 < b < 0.07$*, then the tangency between* l_2^u *and* $l_{1,2}^s$ *is quadratic.*

3 Symbolic Dynamics (Part I)

We have established that for $(b, r) \in \mathcal{P}$, and $r > r(b)$, the Henon map exhibits a topological horseshoe, and for $r = r(b)$, there is a first tangency between the stable and unstable manifolds of the fixed point $(0, 0)$. We will consider such maps in their own right.

3.1 Orientation-Preserving "Horseshoe" Maps before the First Tangency

Let F be a diffeomorphism of $\mathbb{R}^2$, and let $p \in \mathbb{R}^2$ be a fixed hyperbolic saddle point of F. Suppose the stable and unstable manifold $W^s(p)$ and $W^u(p)$ of F at p have transverse homoclinic intersections only. Then F exhibits a "topological horseshoe" which can be illustrated as follows (for orientation-preserving F).



Picture 3.1

Note that the dynamics of the points q_i, s_i, r_i and t_i in the picture above are given by $F(q_i) = q_{i+1}$, $F(s_i) = s_{i+1}$, etc.

We want to define certain regions bounded by parts of the stable and unstable manifolds. We let Q be the region enclosed by the part of $W^u(p)$ connecting p and q_0, the part of $W^s(p)$ connecting q_0 and s_1, the part of $W^u(p)$ connecting s_1 and q_1, and the part of $W^s(p)$ connecting q_1 and p. We use the notation

$$Q = p \xrightarrow{u} q_0 \xrightarrow{s} s_1 \xrightarrow{u} q_1 \xrightarrow{s} p.$$

We also define the regions

$$\underline{1} = p \xrightarrow{u} q_{-1} \xrightarrow{s} t_{-1} \xrightarrow{u} q_1 \xrightarrow{s} p$$

and

$$\underline{2} = s_0 \xrightarrow{u} q_0 \xrightarrow{s} s_1 \xrightarrow{u} r_{-1} \xrightarrow{s} s_0.$$

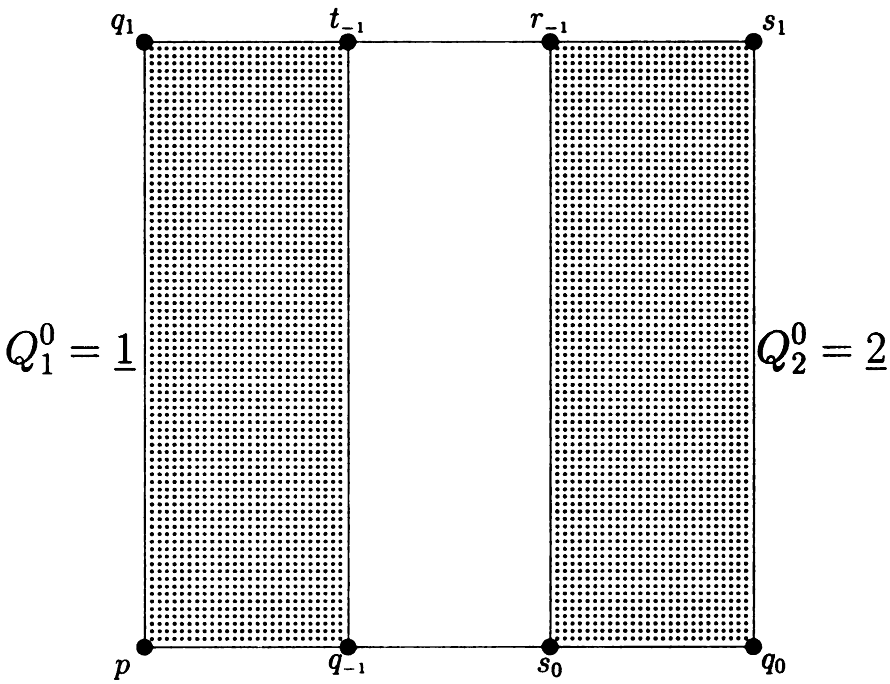
Note that $\underline{1} \cup \underline{2} = Q \cap F^{-1}(Q)$. We define the following regions (also called *blocks*):

For $i_1, i_2, \ldots, i_k \in \{1, 2\}$ and $n_1, n_2, \ldots, n_k \in \mathbb{Z}$, let

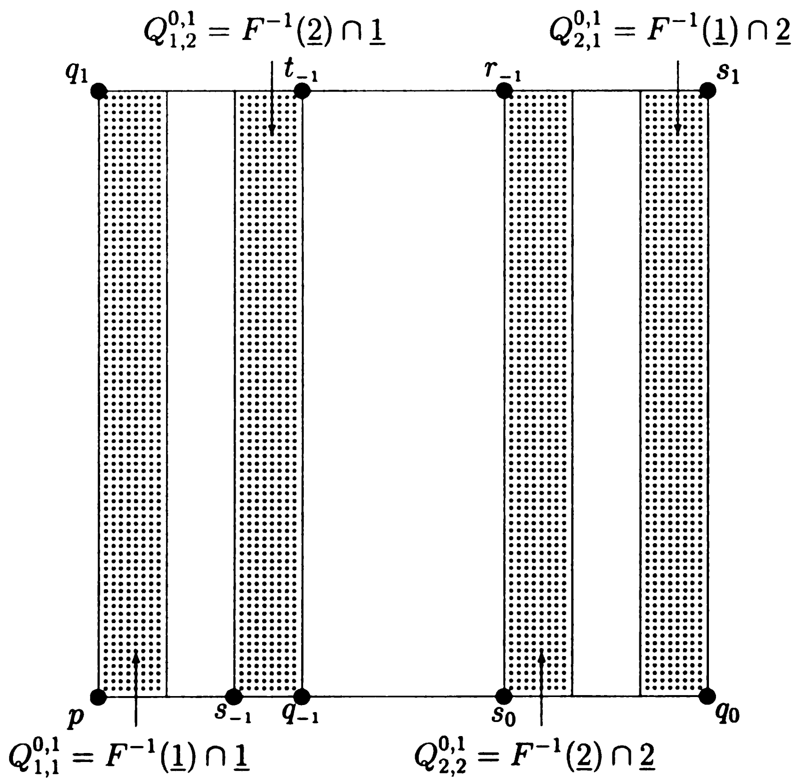
$$Q_{i_1,\ldots,i_k}^{n_1,\ldots,n_k} := \left\{ x \in Q : F^{n_j}(x) \in \underline{i_j}, 1 \leq j \leq k \right\} = \bigcap_{1 \leq j \leq k} F^{-n_j}(\underline{i_j})$$

Then we get the following schematical pictures for certain blocks:

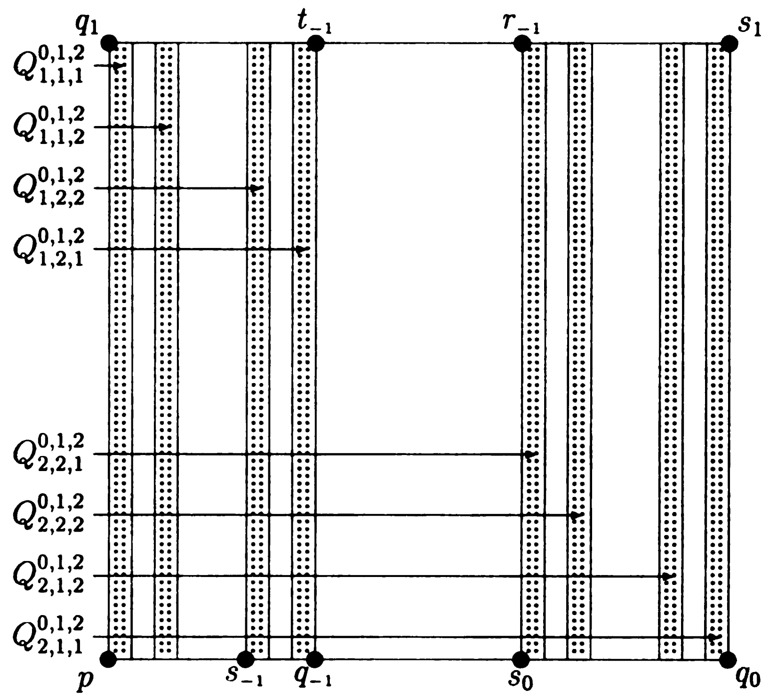
Picture 3.2 *The blocks $Q_1^0 = \underline{1}$ and $Q_2^0 = \underline{2}$.*



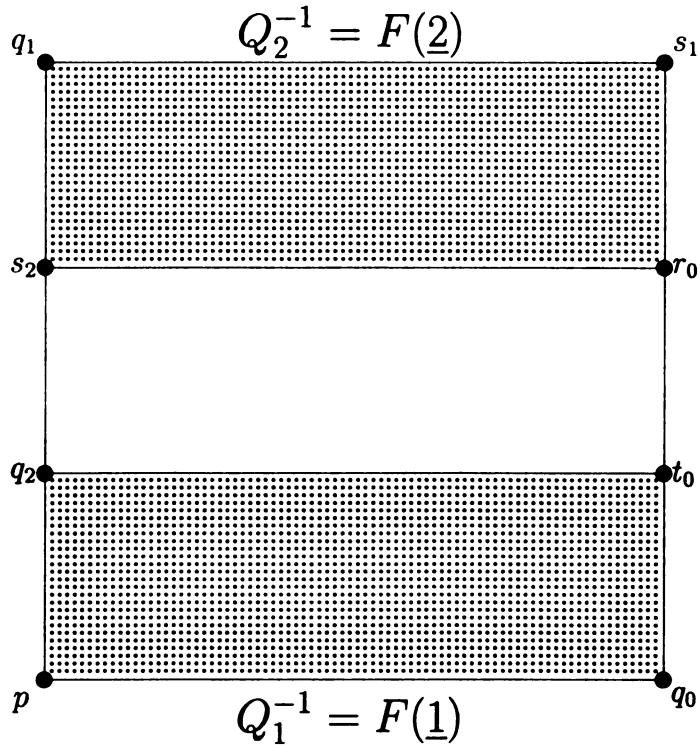
Picture 3.3 *The blocks $Q_{i,j}^{0,1} = F^{-1}(\underline{j}) \cap \underline{i}$.*



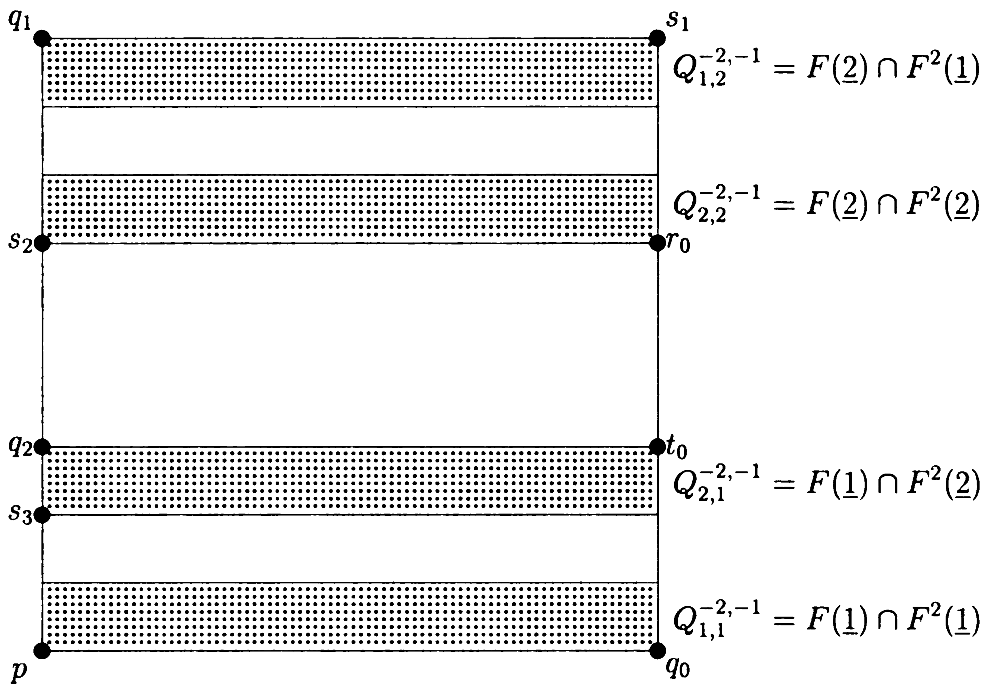
Picture 3.4 *The blocks* $Q^{0,1,2}_{i,j,k} = F^{-2}(\underline{k}) \cap F^{-1}(\underline{j}) \cap \underline{i}$.



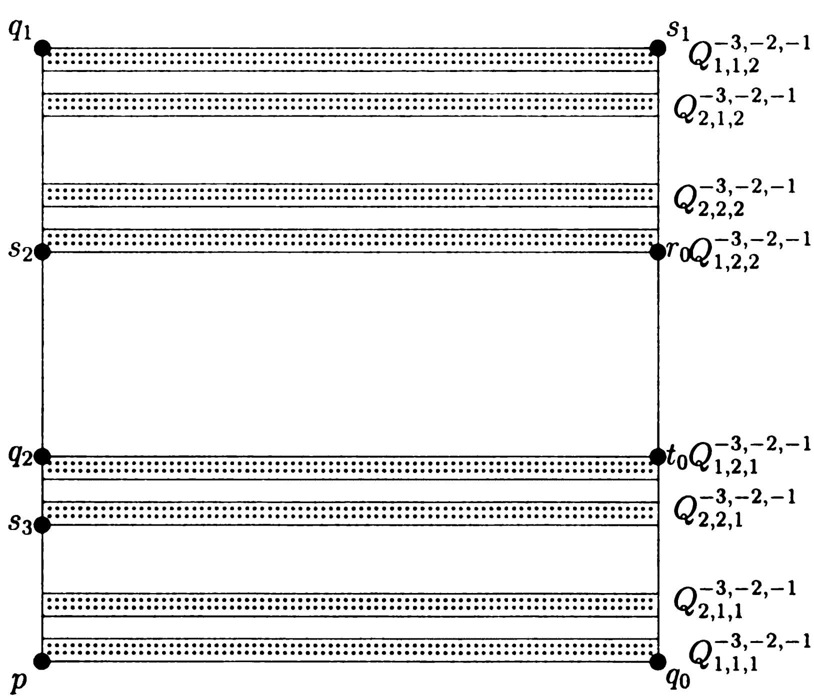
Picture 3.5 *The blocks* $Q^{-1}_1 = F(\underline{1})$ *and* $Q^{-1}_2 = F(\underline{2})$.



Picture 3.6 *The blocks* $Q^{-2,-1}_{i,j} = F(\underline{j}) \cap F^2(\underline{i})$.



Picture 3.7 *The blocks* $Q^{-3,-2,-1}_{i,j,k} = F(\underline{k}) \cap F^2(\underline{j}) \cap F^3(\underline{i})$.



We note the following.

Proposition 3.1 *Let $i_{-k}, \ldots, i_{-1}, i_0, i_1, \ldots, i_k \in \{1, 2\}$. Then:*

(1) Each block $R = Q^{0,1,\ldots,k}_{i_0,i_1\ldots,i_k}$ is a full-height sub-rectangle of $Q^{0,1,\ldots,k-1}_{i_0,i_1\ldots,i_{k-1}}$. In particular R is a full-height sub-rectangle of Q.

(2) Each block $R = Q^{-k,-k+1\ldots,-1}_{i_{-k},i_{-k+1}\ldots,i_{-1}}$ is a full-width sub-rectangle of $Q^{-k+1,\ldots,-1}_{i_{-k+1},\ldots,i_{-1}}$. In particular R is a full-width sub-rectangle of Q.

(3) Consequently, each block $Q_k := Q^{-k,\ldots,k}_{i_{-k},\ldots,i_k}$ is non-empty.

(4) Since every block Q_k is compact and non-empty, we have that given any sequence $(a_n) \in \{1,2\}^{\mathbb{Z}}$, the set $\bigcap\limits_{n\in\mathbb{Z}} F^{-n}\left(\underline{a_n}\right) = \bigcap\limits_{n\in\mathbb{Z}} Q^{-n,\ldots,n}_{a_{-n},\ldots,a_n}$ is non-empty.

3.2 A Coding for $\Lambda = \bigcap\limits_{n\in\mathbb{Z}} F^n(Q)$ before the First Tangency.

We now consider the set $\Lambda = \bigcap\limits_{n\in\mathbb{Z}} F^n(Q)$. This is a non-empty F-invariant set. For each $x \in \Lambda$, we define a sequence $\mathbf{a} = (a_n)_{n\in\mathbb{Z}}$, $a_n = 1$ or $a_n = 2$, via

$$a_n = \begin{cases} 1 & \text{if} \quad F^n(x) \in \underline{1} \\ 2 & \text{if} \quad F^n(x) \in \underline{2} \end{cases}.$$

Let Σ denote the space of bi-infinite sequences of 1's or 2's; i.e., $\Sigma = \{1,2\}^{\mathbb{Z}}$. Then we have defined a map $\psi : \Lambda \to \Sigma$, $\psi(x) = (a_n)$. We define a metric d on Σ as follows. If $\mathbf{a} = (a_n), \mathbf{b} = (b_n) \in \Sigma$, then if $\mathbf{a} \neq \mathbf{b}$, we let $d(\mathbf{a},\mathbf{b}) = \left(\frac{1}{2}\right)^N$, where N is such that $a_n = b_n$ for $|n| < N$, and $a_n \neq b_n$ for $n = N$ or $n = -N$. For $\mathbf{a} = \mathbf{b}$, we let $d(\mathbf{a},\mathbf{b}) = 0$. It is easy to check that this is a metric on Σ. We have that two sequences are close if they agree on a *cylinder set* $C_N := C^{-N,\ldots,N}_{i_{-N},\ldots,i_N} := \{(a_n) \in \Sigma : a_n = i_n \text{ for } -N \leq n \leq N\}$ for large N. The map ψ is called the *coding map* or simply the *coding* of Λ. The next two results show that this map is continuous and onto.

Lemma 3.1 *The coding map $\psi : \Lambda \to \Sigma$ is continuous.*

Proof: Suppose $N \geq 0$ and $x \in \Lambda$ are given. Let $(a_n) = \psi(x)$. For each $n = -N, \ldots N$, there exists a $\delta_n > 0$ so that the δ_n-ball $B_{\delta_n}(F^n(x))$ around $F^n(x)$ satisfies $\underline{a_n} \cap B_{\delta_n}(F^n(x)) \subset \underline{a_n}$. Let $B(x) := \bigcap_{|n| \leq N} F^{-n}(B_{\delta_n}(F^n(x)))$. This is a non-empty open set containing x. Now, if $y \in B(x)$, $(\psi(x))_n = (\psi(y))_n$ for $|n| \leq N$. $\square$

Lemma 3.2 *The coding map $\psi : \Lambda \to \Sigma$ is onto.*

Proof: This follows immediately from Proposition 3.1, part (4). $\square$

We now define the *left shift* $\sigma : \Sigma \to \Sigma$, $(\sigma(a_n))_k = a_{k+1}$. It is easy to see that σ is a homeomorphism of Σ, and that $\sigma \circ \psi = \psi \circ F$.

We have therefore established the following.

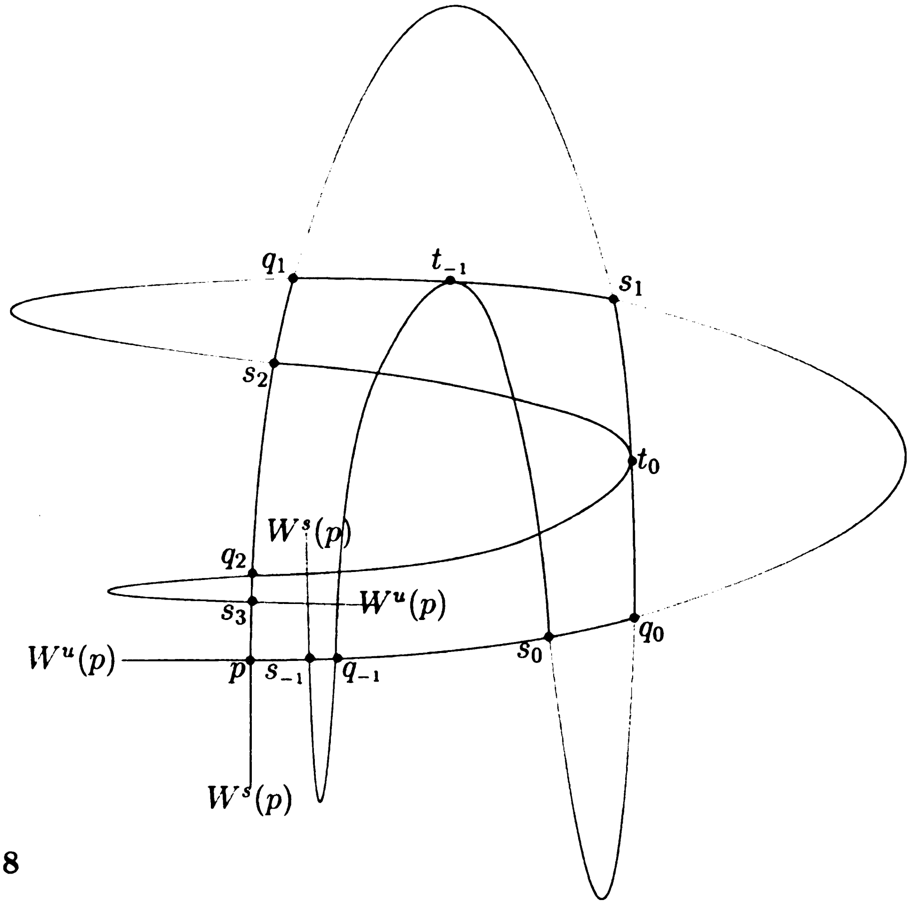
Proposition 3.2 *The map $\psi : \Lambda \to \Sigma$ is a* semi-conjugacy *between the map $F : \Lambda \to \Lambda$ and $\sigma : \Sigma \to \Sigma$. This means that the diagram*

$$\begin{array}{ccc} \Lambda & \xrightarrow{F} & \Lambda \\ \psi \downarrow & & \psi \downarrow \\ \Sigma & \xrightarrow{\sigma} & \Sigma \end{array}$$

commutes.

3.3 Orientation-Preserving "Horseshoe" Maps at the First Tangency

We consider the same situation as in 3.1, except now the stable and unstable manifold $W^s(p)$ and $W^u(p)$ of F at p have a homoclinic tangency. Then F also exhibits a "degenerate topological horseshoe" which can be understood as follows (for orientation-preserving F).



Picture 3.8

Again, the dynamics of the points q_i, s_i and t_i in the picture above are given by $F(q_i) = q_{i+1}$, $F(s_i) = s_{i+1}$, etc.

Similar to what is done in 3.1, we define the regions

$$Q = p \xrightarrow{u} q_0 \xrightarrow{s} s_1 \xrightarrow{u} q_1 \xrightarrow{s} p,$$

$$\underline{1} = p \xrightarrow{u} q_{-1} \xrightarrow{s} t_{-1} \xrightarrow{u} q_1 \xrightarrow{s} p$$

and

$$\underline{2} = s_0 \xrightarrow{u} q_0 \xrightarrow{s} s_1 \xrightarrow{u} t_{-1} \xrightarrow{s} s_0.$$

We still have that $\underline{1} \cup \underline{2} = Q \cap F^{-1}(Q)$. What is different from the situation in 3.1 is that $\underline{1} \cap \underline{2} = \{t_{-1}\}$. We again define blocks exactly as before.

For $i_1, i_2, \ldots, i_k \in \{1, 2\}$ and $n_1, n_2, \ldots, n_k \in \mathbb{Z}$, let

$$Q^{n_1,\dots,n_k}_{i_1,\dots,i_k} := \left\{x \in Q : F^{n_j}(x) \in \underline{i_j}, 1 \leq j \leq k\right\} = \bigcap_{1\leq j\leq k} F^{-n_j}(\underline{i_j}).$$

These blocks look like the ones in 3.1, only with the points $F^n(r_0)$ and $F^n(t_0)$ collapsed to $F^n(t_0)$, for each $n \in \mathbb{Z}$.

Proposition 3.1 holds verbatim in the present situation.

Proposition 3.3 *Let $i_{-k}, \dots, i_{-1}, i_0, i_1, \dots, i_k \in \{1, 2\}$. Then:*

(1) Each block $R = Q^{0,1,\dots,k}_{i_0,i_1\dots,i_k}$ is a full-height sub-rectangle of $Q^{0,1,\dots,k-1}_{i_0,i_1\dots,i_{k-1}}$. In particular R is a full-height sub-rectangle of Q.

(2) Each block $R = Q^{-k,-k+1\dots,-1}_{i_{-k},i_{-k+1}\dots,i_{-1}}$ is a full-width sub-rectangle of $Q^{-k+1,\dots,-1}_{i_{-k+1},\dots,i_{-1}}$. In particular R is a full-width sub-rectangle of Q.

(3) Consequently, each block $Q_k := Q^{-k,\dots,k}_{i_{-k},\dots,i_k}$ is non-empty.

(4) Since every block Q_k is compact and non-empty, we have that given any sequence $(a_n) \in \{1,2\}^{\mathbf{Z}}$, the set $\bigcap_{n\in\mathbb{Z}} F^{-n}\left(\underline{a_n}\right) = \bigcap_{n\in\mathbb{Z}} Q^{-n,\dots,n}_{a_{-n},\dots,a_n}$ is non-empty.

3.4 A Coding for $\tilde{\Lambda} = \bigcap_{n\in\mathbb{Z}} F^n(Q)$ at the First Tangency.

Let Q be as in section 3.3. We let $\tilde{\Lambda} = \bigcap_{n\in\mathbf{Z}} F^n(Q)$, a non-empty F-invariant set. Our first objective is to define a coding for $\tilde{\Lambda}$. Let $\Sigma = \{1,2\}^{\mathbf{Z}}$. We define the equivalence relation $\sim$ on Σ. We let $\mathbf{t} = (t_n) \in \Sigma$ be the sequence such that $t_{-3}, t_{-4}, \dots = 1$, $t_{-2} = 2$, $t_{-1} = 1$, $t_0 = 2$, $t_1, t_2, \dots = 1$; i.e.,

$$\mathbf{t} = \left(\dots, 1, 1, 2, 1, \overset{\bullet}{2}, 1, 1, \dots\right)$$

(the dot $\bullet$ denotes the 0th position). We also let

$$\mathbf{r} = \left(\dots, 1, 1, 2, 2, \overset{\bullet}{2}, 1, 1, \dots\right).$$

Now, we define that $\sigma^n(\mathbf{t}) \sim \sigma^n(\mathbf{r})$ and $\sigma^n(\mathbf{r}) \sim \sigma^n(\mathbf{t})$ for all $n \in \mathbb{Z}$, and $\mathbf{a} \sim \mathbf{a}$ for all $\mathbf{a} \in \Sigma$. This is an equivalence relation. We denote by $\tilde{\Sigma}$ the set of equivalence classes of $\sim$, and we let $\pi : \Sigma \to \tilde{\Sigma}$, $\mathbf{a} \mapsto \tilde{\mathbf{a}}$ be the canonical projection onto $\tilde{\Sigma}$. Next, we let $\mathcal{O}(t_0) = \{F^n(t_0) : n \in \mathbb{Z}\}$, and we define the map $\tilde{\psi} : \tilde{\Lambda} \to \tilde{\Sigma}$ as follows.

- If $x \in \tilde{\Lambda} \setminus \mathcal{O}(t_0)$, then define the sequence $\mathbf{a} = (a_n) \in \Sigma$ by

$$a_n = \begin{cases} 1 & \text{if} \quad F^n(x) \in \underline{1} \\ 2 & \text{if} \quad F^n(x) \in \underline{2} \end{cases},$$

  and then let $\tilde{\psi}(x) = \tilde{\mathbf{a}}$.

- If $x = F^n(t_0)$ for some $n \in \mathbb{Z}$, then let $\tilde{\psi}(x) = \widetilde{\sigma^n(\mathbf{t})}$.

Proposition 3.3 shows that $\tilde{\psi} : \tilde{\Lambda} \to \tilde{\Sigma}$ is onto. To show the continuity of the map $\tilde{\psi}$, we make the assumption that there is a continuous transition from the situation before the first tangency to the situation at the first tangency; more precisely, we assume

(C) there exists a continuous, open and onto map $\tau : \Lambda \to \tilde{\Lambda}$ such that the diagram

$$\begin{array}{ccc} \Lambda & \xrightarrow{\psi} & \Sigma \\ \tau \downarrow & & \pi \downarrow \\ \tilde{\Lambda} & \xrightarrow{\tilde{\psi}} & \tilde{\Sigma} \end{array}$$

commutes.

Using the quotient topology on $\tilde{\Sigma}$ (this means that a set G is open in $\tilde{\Sigma}$ iff $\pi^{-1}(G)$ is open in Σ), we see that then $\tilde{\psi} : \tilde{\Lambda} \to \tilde{\Sigma}$ is continuous; namely, if G is open in $\tilde{\Sigma}$, then $\pi^{-1}(G)$ is open in Σ, and hence $\psi^{-1} \circ \pi^{-1}(G) = \tau^{-1} \circ \tilde{\psi}^{-1}(G)$ is open in Λ. Applying τ to the left side of this equality gives that $\tilde{\psi}^{-1}(G)$ is open in $\tilde{\Lambda}$.

We define the left shift $\tilde{\sigma}$ on the quotient space $\tilde{\Sigma}$ simply by $\tilde{\sigma}(\tilde{\mathbf{a}}) = \widetilde{\sigma(\mathbf{a})}$. It is easy to check that σ is well-defined, a homeomorphism of $\tilde{\Sigma}$, and that $\tilde{\sigma} \circ \tilde{\psi} = \tilde{\psi} \circ F$. We have the following version of Proposition 3.2

Proposition 3.4 *Under the assumption (C), the map $\tilde{\psi} : \tilde{\Lambda} \to \tilde{\Sigma}$ is a semi-conjugacy between the map $F : \tilde{\Lambda} \to \tilde{\Lambda}$ and $\tilde{\sigma} : \tilde{\Sigma} \to \tilde{\Sigma}$. This means that the diagram*

$$\begin{array}{ccc} \tilde{\Lambda} & \xrightarrow{F} & \tilde{\Lambda} \\ \tilde{\psi} \downarrow & & \tilde{\psi} \downarrow \\ \tilde{\Sigma} & \xrightarrow{\tilde{\sigma}} & \tilde{\Sigma} \end{array}$$

commutes.

4 The Abstract Model

4.1 Basic Definitions and Assumptions

Suppose that $F(x,y) = (F_1(x,y), F_2(x,y))$ is a C^2 diffeomorphism of $\mathbb{R}^2$ onto its image. For $(x,y) \in \mathbb{R}^2$, the differential map $DF_{(x,y)} : T_{(x,y)}\mathbb{R}^2 \to T_{F(x,y)}\mathbb{R}^2$ is

$$DF_{(x,y)} = \begin{pmatrix} F_{1x}(x,y) & F_{1y}(x,y) \\ F_{2x}(x,y) & F_{2y}(x,y) \end{pmatrix}.$$

Then the inverse of $DF_{(x,y)}$ is given by

$$\left(DF_{(x,y)}\right)^{-1} = DF^{-1}_{F(x,y)} = \frac{1}{J_F(x,y)} \cdot \begin{pmatrix} F_{2y}(x,y) & -F_{1y}(x,y) \\ -F_{2x}(x,y) & F_{1x}(x,y) \end{pmatrix},$$

where $J_F(x,y) = \det DF_{(x,y)} = F_{1x}(x,y) \cdot F_{2y}(x,y) - F_{2x}(x,y) \cdot F_{1y}(x,y)$. We use the maximum norm $|(v_1, v_2)| = \max\{|v_1|, |v_2|\}$ for $(v_1, v_2) \in \mathbb{R}^2$ or $(v_1, v_2) \in T_p\mathbb{R}^2$. Then we have

$$\left|DF_{(x,y)}\right| = \max\left\{|F_{1x}(x,y)| + |F_{1y}(x,y)|, |F_{2x}(x,y)| + |F_{2y}(x,y)|\right\}.$$

Note that for $v \in T_{(x,y)}\mathbb{R}^2$, $|DF_{(x,y)}(v)| \leq |DF_{(x,y)}| \cdot |v|$, and also that for $w \in T_{F(x,y)}\mathbb{R}^2$, $|DF^{-1}_{F(x,y)}(w)| \geq \dfrac{1}{|DF_{(x,y)}|} \cdot |w|$.

Definition 4.1 *Suppose $\alpha \geq 0$ and $p \in \mathbb{R}^2$. We define*

(a) the unstable α-cone *at p to be $K^u(\alpha, p) = \{(v_1, v_2) \in T_p\mathbb{R}^2 : |v_2| \leq \alpha|v_1|\}$;*

(b) the stable α-cone *at p to be $K^s(\alpha, p) = \{(v_1, v_2) \in T_p\mathbb{R}^2 : |v_1| \leq \alpha|v_2|\}$;*

(c) a $K^u(\alpha)$-curve *is a curve $\gamma(t)$ in such that $\dot\gamma(t) \in K^u(\alpha, \gamma(t))$ for all t;*

(d) a $K^s(\alpha)$-curve *is a curve $\gamma(t)$ in such that $\dot\gamma(t) \in K^s(\alpha, \gamma(t))$ for all t;*

(e) a $K^u(\alpha)$-line *is a $K^u(\alpha)$-curve $\gamma(t)$ such that $curv(\gamma)(t) = 0$ for all t;*

(f) a $K^s(\alpha)$-line *is a $K^s(\alpha)$-curve $\gamma(t)$ such that $curv(\gamma)(t) = 0$ for all t.*

Definition 4.2 *Let* $I = [0,1] \subset \mathbb{R}$, $I = (0,1] \subset \mathbb{R}$, $I = [0,1) \subset \mathbb{R}$, *or* $I = (0,1) \subset \mathbb{R}$ *and let* $I^2 = I \times I \subset \mathbb{R}^2$ *(i.e. there are* $4 \times 4 = 16$ *choices for* I^2*). A* C^2-rectangle Q *is the image of* I^2 *under a* C^2*-diffeomorphism* Ψ*. We define* bottom, top, left and right boundaries *of* Q *by*

$$\partial_{bottom} Q = \Psi(I \times \{0\}), \qquad \partial_{top} Q = \Psi(I \times \{1\}),$$

$$\partial_{left} Q = \Psi(\{0\} \times I), \qquad \partial_{right} Q = \Psi(\{1\} \times I).$$

If Q *is a* C^2*-rectangle, then we say that* R *is a* C^2-subrectangle *of* Q *if* R *is itself a* C^2*-rectangle, and if* $R \subset Q$*. Moreover, we say that* R *is a* full-height subrectangle *of* Q *if* $\partial_{bottom} R \subset \partial_{bottom} Q$ *and* $\partial_{top} R \subset \partial_{top} Q$*;* R *is a* full-width subrectangle *of* Q *if* $\partial_{left} R \subset \partial_{left} Q$ *and* $\partial_{right} R \subset \partial_{right} Q$*.*

A curve γ *is a* full-height curve *in* Q *if* $\gamma \subset Q$ *and* γ *connects* $\partial_{bottom} Q$ *and* $\partial_{top} Q$*; a curve* γ *is a* full-width curve *in* Q *if* $\gamma \subset Q$ *and* γ *connects* $\partial_{left} Q$ *and* $\partial_{right} Q$*.*

Let Q be a C^2-rectangle in $\mathbb{R}^2$, and suppose that Q can be written as the union $E_1 \cup Q_0 \cup E_2$, where E_1, Q_0, E_2 are closed, full-height C^2-subrectangles of Q with disjoint interiors, and such that $\partial_{right} E_1 = \partial_{left} Q_0$, $\partial_{right} Q_0 = \partial_{left} E_2$.

In all that follows, $0 < \alpha < 1$, $R > 1$ and $K > \epsilon > 0$ are fixed constants. We assume the following geometric conditions for the map F.

(G1) Both $F(E_1)$ and $F(E_2)$ are full-width C^2-subrectangles of $E_1 \cup Q_0$ such that

(a) $F(\partial_{bottom} E_1) = \partial_{bottom}(E_1 \cup Q_0)$,

(b) $F(\partial_{bottom} E_2) = \partial_{top}(E_1 \cup Q_0)$,

(c) $F(\partial_{left} E_1) \subset \partial_{left} E_1$,

(d) $F(\partial_{right} E_2) \subset \partial_{left} E_1$.

(G2) *F maps Q_0 parabolically across E_2.* This means that the set $F(Q_0) \cap E_2$ consists of two connected components that are full-width subrectangles of E_2 (this is the situation "before the first tangency"), or $F(Q_0) \cap E_2$ consists of two full-width subrectangles of E_2 that intersect in one single point, which we denote by p_t (this is the situation "at the first tangency"). Furthermore, there exists a full-height curve γ in Q_0 such that F maps γ outside of E_2 (i.e., $F(\gamma) \cap E_2 = \emptyset$), or - in the situation at the first tangency - we have $F(\gamma) \cap E_2 = \{p_t\}$.

We call such a curve γ a *critical curve.* We also assume that

(a) $F\left(\partial_{bottom} Q_0\right) \supset \partial_{bottom} E_2 \cup \partial_{top} E_2$,

(b) $F\left(\partial_{left} Q_0\right) \subset \partial_{left} E_2$,

(c) $F\left(\partial_{right} Q_0\right) \subset \partial_{left} E_2 = \partial_{right} Q_0$.

Definition 4.3 *Suppose* $0 < \alpha < 1$ *and* $R > 1$. *We say a diffeomorphism* F *is* (R, α)-hyperbolic *on a set* E, *if for every* $p \in E$, *we have:*

(1) if $v \in K^u(\alpha, p)$, *then* $DF_p(v) \in K^u(\alpha, F(p))$ *and* $|DF_p(v)| \geq R\,|v|$;

(2) if $v \in K^s(\alpha, F(p))$, *then* $DF^{-1}_{F(p)}(v) \in K^s(\alpha, p)$ *and* $\left|DF^{-1}_{F(p)}(v)\right| \geq R\,|v|$.

The following lemma gives necessary conditions for (R, α)-hyperbolicity.

Lemma 4.1 *If* $F(x, y) = (F_1(x, y), F_2(x, y))$ *is* (R, α)*-hyperbolic on* E, *then we have the following estimates on* E:

$$|F_{1x}| \geq R, \quad \frac{|F_{1y}|}{|F_{1x}|} \leq \alpha, \quad \frac{|F_{2x}|}{|F_{1x}|} \leq \alpha, \quad \textit{and } |F_{1x}| \geq R \cdot |J_F|.$$

Proof: Let $p \in E$. Since $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in K^u(\alpha, p)$ and hence $DF_p \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} F_{1x}(p) \\ F_{2x}(p) \end{pmatrix} \in K^u(\alpha, F(p))$, we get $|F_{2x}(p)| \leq \alpha |F_{1x}(p)|$. Also, $|F_{1x}(p)| = \left| DF_p \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right| \geq R \cdot 1$.

Since $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in K^s(\alpha, F(p))$ and hence $DF^{-1}_{F(p)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{J_F(p)} \cdot \begin{pmatrix} -F_{1y}(p) \\ F_{1x}(p) \end{pmatrix} \in K^s(\alpha, p)$, we get $|F_{1y}(p)| \leq \alpha |F_{1x}(p)|$. Also, $\frac{|F_{1x}(p)|}{|J_F(p)|} = \left| DF^{-1}_{F(p)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right| \geq R \cdot 1$. $\square$

We also have sufficient conditions. The proof of the first lemma is elementary; the second lemma comes from [**JN**].

Lemma 4.2 *Suppose* $0 < \alpha < 1$ *and* $R > 1$. *Suppose also that for all* $p \in E$, *the diffeomorphism* F *satisfies the conditions:*

(1) if $v \notin K^s(\alpha, p)$, *then* $|DF_p(v)| \geq R\,|v|$; *and*

(2) if $v \notin K^u(\alpha, F(p))$, *then* $\left|DF^{-1}_{F(p)}(v)\right| \geq R\,|v|$.

Then F *is* (R, α)*-hyperbolic on* E.

Lemma 4.3 *Suppose* $0 < \alpha < 1$ *and* $R > 1$. *Suppose also that for all* $p \in E$, *the diffeomorphism* F *satisfies the conditions:*

(1) $|F_{2x}(p)| + \alpha|F_{2y}(p)| + \alpha^2|F_{1y}(p)| \leq \alpha|F_{1x}(p)|$,

(2) $|F_{1x}(p)| - \alpha|F_{1y}(p)| \geq R$,

(3) $|F_{1y}(p)| + \alpha|F_{2y}(p)| + \alpha^2|F_{2x}(p)| \leq \alpha|F_{1x}(p)|$,

(4) $|F_{1x}(p)| - \alpha|F_{2x}(p)| \geq J_F(p)R$.

Then F *is* (R, α)*-hyperbolic on* E.

We suppose that the following hyperbolicity condition holds.

(H1) F is (R, α)-hyperbolic on $E_1 \cup E_2$.

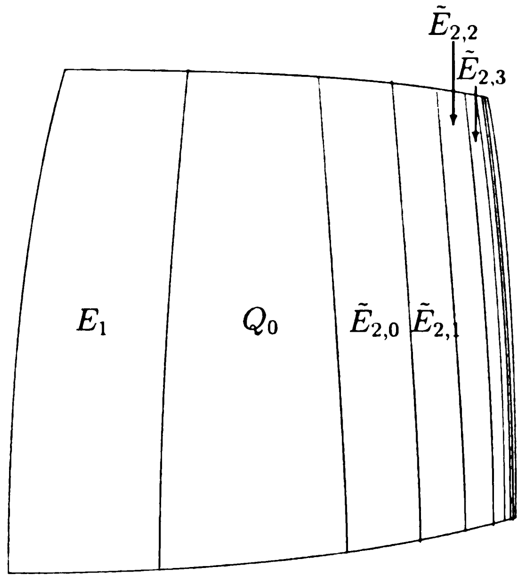
We define the sets

$$E_{2,0} = E_2$$

$$E_{2,1} = E_2 \cap F^{-1}(E_1)$$

$$E_{2,k} = E_2 \cap F^{-1}(E_1) \cap \ldots \cap F^{-k}(E_1).$$

Then each $E_{2,k}$ is a full-height C^2-subrectangle of E_2. Also, each $\tilde{E}_{2,k} := E_{2,k} \setminus E_{2,k+1}$ is full-height in E_2, and we have $E_2 = \bigcup_{k=0}^{\infty} \tilde{E}_{2,k} \cup \partial_{right} E_2$.

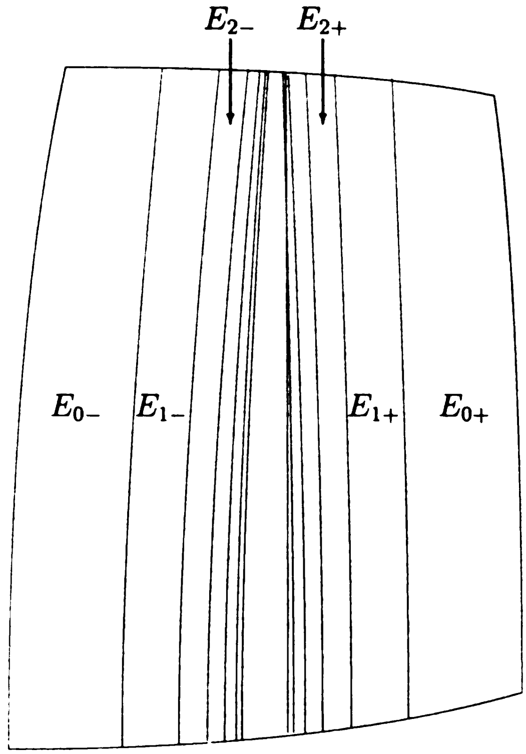


Picture 4.1

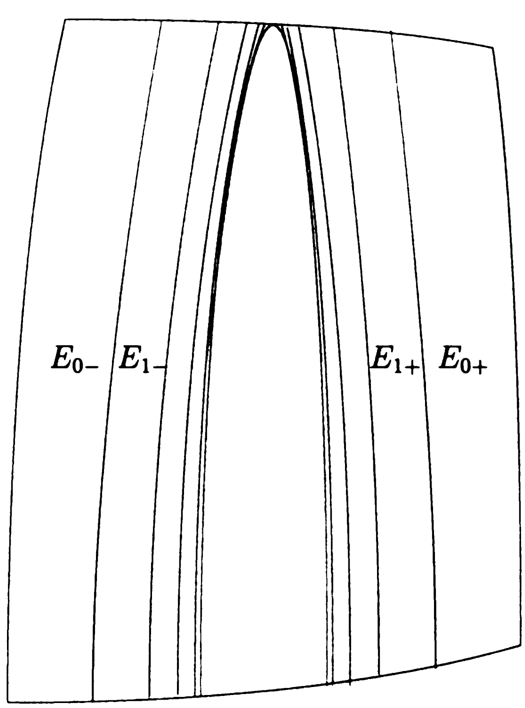
We have that for each $k \geq 0$, each of the two connected components of $F^{-1}(\tilde{E}_{2,k}) \cap Q_0$ is full-height in Q_0. We denote these components by $E_{k\pm}$.

The following two pictures illustrate the geometry of these components.

Picture 4.2 *The region Q_0 before the first tangency*



Picture 4.3 *The region Q_0 at the first tangency*



We make the following assumption on how certain curves intersect, and their concavity.

(K1) If $p \in E_{k\pm}$, let $q = F^2(p) \in F(\tilde{E}_{2,k})$. Then for every $K^u(1/\alpha)$-line l through p there exists a $K^s(\alpha)$-line κ through q such that $l' = F(l)$ and $\kappa' = F^{-1}(\kappa)$ intersect in exactly two points (one of them being $F(p) = F^{-1}(q)$).

Furthermore, between these two points of intersection,

(a) l' can be parametrized as a curve $(x(t), t)$, and $-2K - \epsilon \leq \ddot{x}(t) \leq -2K + \epsilon$ for all t;

(b) κ' can be parametrized as a curve $(y(t), t)$, and $-\epsilon \leq \ddot{y}(t) \leq \epsilon$ for all t.

The maximal distance of l' and κ' between these points of intersection is denoted by $d_p(\kappa, l)$. Let

$$d_p(l) = \max_{\kappa} d_p(\kappa, l) \quad \text{and} \quad d_p = \min_{l} d_p(l).$$

We also let

$$C_\epsilon = \frac{2K - 2\epsilon}{\sqrt{K + \epsilon}}$$

$$R_k = \inf\left\{\left|DF_p^{k+1}\begin{pmatrix}1\\0\end{pmatrix}\right| \cdot \sqrt{d_p} : p \in \tilde{E}_{2,k}\right\}$$

$$\beta = \inf\left\{\frac{|DF_p(v)|}{|v|} : v \neq 0, v \in K^u(1/\alpha, p), p \in Q_0\right\}$$

We now assume

(H2) $\inf_{p \in E_1 \cup E_2} |F_{1x}(p)| \cdot \beta > 1$ and $\inf_{k \geq 0} R_k \cdot C_\epsilon \cdot \beta > 1$;

4.2 Hyperbolicity Results for $|F_{2y}|, |F_{1y}| \ll |F_{1x}|$

We will be concerned with the situation when $|F_{2y}|, |F_{1y}|$ are small when compared with $|F_{1x}|$. Then we have the following four results.

Proposition 4.1 *Suppose **(H1)**, $\alpha, \beta > 0$ and $\alpha\beta < 1$. Given $\epsilon > 0$, there exists a $\delta > 0$ such that if $\frac{|F_{2y}|}{|F_{1x}|}, \frac{|F_{1y}|}{|F_{1x}|} < \delta$ on $E_1 \cup E_2$, then we have that if $p \in E_1 \cup E_2$ and $v \in K^s(\beta, F(p))$, then $DF^{-1}_{F(p)}(v) \in K^s(\epsilon, p)$.*

Proof: Consider $v = (v_1, v_2) \in K^s(\beta, F(p))$, and let $v' = (v'_1, v'_2) = DF^{-1}_{F(p)}(v)$. Then

$$v'_1 = \frac{1}{J_F} \cdot (F_{2y}v_1 - F_{1y}v_2) \text{ and } v'_2 = \frac{1}{J_F} \cdot (-F_{2x}v_1 + F_{1x}v_2).$$

Hence

$$\frac{|v'_1|}{|v'_2|} \leq \frac{|F_{2y}||v_1| + |F_{1y}||v_2|}{|F_{1x}||v_2| - |F_{2x}||v_1|} \leq \frac{|F_{2y}|\beta + |F_{1y}|}{|F_{1x}| - |F_{2x}|\beta}.$$

The (R, α)-hyperbolicity on $E_1 \cup E_2$ implies $\frac{|F_{2x}|}{|F_{1x}|} \leq \alpha$. This means

$$\frac{|v'_1|}{|v'_2|} \leq \frac{|F_{2y}|\beta + |F_{1y}|}{|F_{1x}|(1 - \alpha\beta)} \leq \frac{\delta(\beta + 1)}{1 - \alpha\beta} \leq \epsilon. \quad \square$$

Remark 4.1 *Proposition 1 asserts that if $\frac{|F_{2y}|}{|F_{1x}|}$ and $\frac{|F_{1y}|}{|F_{1x}|}$ are small, then the left and right boundaries of $E_1 \cup E_2$ and the left and right boundaries of each $\tilde{E}_{2,k}$ are C^1-close to vertical lines.*

Proposition 4.2 *Suppose $0 \leq \epsilon \leq 1$, and $\delta > 0$. If $\frac{|F_{2y}|}{|F_{1x}|}, \frac{|F_{1y}|}{|F_{1x}|} < \delta$ on $E_1 \cup E_2$, then for each $p \in E_1 \cup E_2$ and $v \in K^s(\epsilon, p)$, we have that $|DF_p(v)| \leq (\epsilon + \delta) \cdot \sup_{E_1 \cup E_2} \{|F_{1x}|, |F_{2x}|\} \cdot |v|$.*

Proof: For $v = (v_1, v_2) \in K^s(\epsilon, p)$ (i.e., $|v_1| \leq \epsilon|v_2|$), let $v' = (v'_1, v'_2) = DF_p(v)$. Then

$$v'_1 = F_{1x}v_1 + F_{1y}v_2 \qquad v'_2 = F_{2x}v_1 + F_{2y}v_2.$$

Hence

$$|v_1'| \leq |F_{1x}|(\epsilon + \delta)|v_2| \qquad\qquad |v_2'| \leq (|F_{2x}|\epsilon + |F_{1x}|\delta))\,|v_2|.$$

Finally, note that $|v| = |v_2|$. $\square$

Proposition 4.3 *Suppose **(H1)**, $\alpha, \beta > 0$ and $\alpha\beta < 1$. Given $M > 0$, there exists a $\delta > 0$ such that if $\dfrac{|F_{2y}|}{|F_{1x}|}, \dfrac{|F_{1y}|}{|F_{1x}|} < \delta$ on $E_1 \cup E_2$, then we have that if $p \in E_1 \cup E_2$ and $v \in K^s(\beta, F(p))$, then $\left|DF^{-1}_{F(p)}(v)\right| \geq M \cdot |v|$.*

Proof: For $v = (v_1, v_2) \in K^s(\beta, F(p))$ (i.e., $|v_1| \leq \beta|v_2|$), let $v' = (v_1', v_2') = DF^{-1}_{F(p)}(v)$. We have that

$$|v'| \geq |v_2'| \geq \frac{1}{|J_F|} \cdot (|F_{1x}||v_2| - |F_{2x}||v_1|)\,,$$

where F_{1x}, F_{2x} and J_F are evaluated at $p \in E_1 \cup E_2$.

So we can estimate

$$|F_{1x}||v_2| - |F_{2x}||v_1| \geq |F_{1x}||v_2| - |F_{2x}| \cdot \beta \cdot |v_2| \geq |F_{1x}| \cdot (1 - \alpha\beta) \cdot |v_2|$$

and

$$|F_{1x}||v_2| - |F_{2x}||v_1| \geq |F_{1x}| \cdot \frac{1}{\beta} \cdot |v_1| - |F_{2x}||v_1| \geq |F_{1x}| \cdot \frac{1}{\beta} \cdot (1 - \alpha\beta) \cdot |v_1|.$$

In both estimates we used $|F_{2x}| \leq \alpha|F_{1x}|$ (cf. Lemma 4.1).

Now, $|J_F| \leq \delta \cdot |F_{1x}|^2 + \alpha\delta|F_{1x}|^2$. Hence,

$$|v'| \geq \frac{\min(1, 1/\beta) \cdot (1 - \alpha\beta)}{\delta \cdot (1 + \alpha) \cdot |F_{1x}|} \cdot |v|.$$

Using that $|F_{1x}|$ is bounded on $E_1 \cup E_2$, we get $\dfrac{\min(1, 1/\beta) \cdot (1 - \alpha\beta)}{\delta \cdot (1 + \alpha) \cdot |F_{1x}|} \geq M$ if δ is small enough. $\square$

Proposition 4.4 *Suppose **(H1)**. There exists a $\delta_0 > 0$ such that if $0 < \delta \leq \delta_0$ and $\dfrac{|F_{2y}|}{|F_{1x}|}, \dfrac{|F_{1y}|}{|F_{1x}|} < \delta$ on $E_1 \cup E_2$, then we have that if $p \in E_1 \cup E_2$ and $v \in K^u(1, p)$, then $DF_p(v) \in K^u(1, F(p))$ and $|DF_p(v)| \geq (1 - \delta) \cdot |F_{1x}(p)| \cdot |v|$.*

Proof: If $v = (v_1, v_2)$ is such that $|v_1| \geq |v_2|$ and $v' = (v_1', v_2') := DF_p(v)$, then

$$|v'| \geq |v_1'| \geq (|F_{1x}| - |F_{1y}|) \cdot |v_1| \geq (1-\delta) \cdot |F_{1x}| \cdot |v_1| = (1-\delta) \cdot |F_{1x}| \cdot |v|,$$

and

$$|v_2'| \leq (|F_{2x}| + |F_{2y}|) \cdot |v_1| \leq (\alpha + \delta) \cdot |F_{1x}| \cdot |v_1|.$$

Hence $\dfrac{|v_1'|}{|v_2'|} \geq \dfrac{1-\delta}{\alpha+\delta}$. Since $0 < \alpha < 1$, we can find a $\delta_0 > 0$ with $\dfrac{1-\delta}{\alpha+\delta} \geq 1$ for all $0 < \delta \leq \delta_0$. □

We want to study the return map on Q_0. We make the following definition:

Definition 4.4 *Suppose $p \in E_{k\pm}$. Then the* return time of p to Q_0 *is* $N(p) := k+2$, *and $\Phi := F^{k+2}$ is the return map on $E_{k\pm}$. This defines the return map $\Phi : \bigcup_{k=0}^{\infty} E_{k\pm} \longrightarrow Q_0$.*

For the next result we assume **(G1)**, **(G2)**, **(K1)** and **(H1)**, **(H2)**.

Theorem 4.1 *There exists an $\tilde{\alpha}$ with $1 > \tilde{\alpha} > \alpha$, an $\tilde{R} > 1$, and there exists a $\delta > 0$ such that if $\dfrac{|F_{2y}|}{|F_{1x}|}, \dfrac{|F_{1y}|}{|F_{1x}|} < \delta$ on $E_1 \cup E_2$, then the map Φ is $(\tilde{R}, \tilde{\alpha})$-hyperbolic.*

Proof: For $\tilde{\alpha}$, we may choose any number between 1 and α. We want to verify the conditions (1) and (2) in Lemma 4.2 for Φ.

(1) Let $p \in E_{k\pm}$, and suppose $v \notin K^s(\tilde{\alpha}, p)$. We want to show that $|D\Phi_p(v)| \geq \tilde{R}|v|$ for some $\tilde{R} > 1$.

    Since $v \notin K^s(\tilde{\alpha}, p)$, we have that $v \in K^u(1/\tilde{\alpha}, p) \subset K^u(1/\alpha, p)$. Let $p' = F(p) \in \tilde{E}_{2,k}$ and let $v' = DF_p(v) = (v_1', v_2')$. By the definition of β, we have that $|v'| \geq \beta|v|$. If $\beta > 1$, then we are done; so we assume $0 < \beta \leq 1$. We consider the two cases:

- $|v'_1| \geq |v'_2|$.

Using **(H2)**, we can choose $\lambda_1 > \lambda_2 > 1$ such that $\inf_{p' \in E_1 \cup E_2} |F_{1x}(p')| \cdot \beta \geq \lambda_1$.

Since $v^1 := v' \in K^u(1, p')$, Proposition 4.4 allows us to assume that $v^j := DF^j_p(v) \in K^u(1, F^j(p))$ for $2 \leq j \leq k+2$. Furthermore, for $\delta > 0$ sufficiently small, $|v^{j+1}| \geq (1-\delta) \cdot |F_{1x}(F^j(p))| \cdot |v^j|$ for $1 \leq j \leq k+1$.

If $\delta \leq 1 - \dfrac{\lambda_2}{\lambda_1}$, we have

$$|v^2| \geq (1-\delta) \cdot |F_{1x}(p')| \cdot |v'| \geq \frac{\lambda_2}{\lambda_1} \cdot |F_{1x}(p')| \cdot \beta \cdot |v| \geq \lambda_2 \cdot |v|,$$

and also

$$|v^{j+1}| \geq \frac{\lambda_2}{\lambda_1} \cdot |F_{1x}(F^j(p))| \cdot |v^j| \geq \frac{\lambda_2}{\lambda_1 \beta} \cdot |F_{1x}(F^j(p))| \cdot \beta \cdot |v^j| \geq \frac{\lambda_2}{\beta} \cdot |v^j|$$

for $2 \leq j \leq k+1$.

This means that $|D\Phi_p(v)| = |v^{k+2}| \geq \left(\dfrac{\lambda_2}{\beta}\right)^k \cdot \lambda_2 \cdot |v| \geq \tilde{R} \cdot |v|$.

- $|v'_1| < |v'_2|$.

Let $\epsilon_1 > 0$, and let $q = F^2(p) = F(p')$. Let $u \in K^s(\alpha, q)$ be such that for the curves $l : t \mapsto p + tv$ and $\kappa : t \mapsto q + tu$, we have $d_p(\kappa, l) \geq d_p$. Also, let $\tilde{v} = DF^{-1}_q(u)$. If $\delta > 0$ is chosen to be small, **(H1)** and Proposition 4.1 give that $\tilde{v} \in K^s(\epsilon_1, p')$. We can write $v' = w_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + w_2 \dfrac{\tilde{v}}{|\tilde{v}|}$. Note that $|w_2| = |v'_2| = |v'|$. Using **(K1)**, we can write $\dfrac{\tilde{v}}{|\tilde{v}|} = \begin{pmatrix} \dot{y}(t^\star) \\ 1 \end{pmatrix}$ and $v' = w_2 \cdot \begin{pmatrix} \dot{x}(t^\star) \\ 1 \end{pmatrix}$, where $x(t), y(t)$ are as in **(K1)**, and $p' = \begin{pmatrix} x(t^\star) \\ t^\star \end{pmatrix} = \begin{pmatrix} y(t^\star) \\ t^\star \end{pmatrix}$. Thus we have $w_2 \cdot \dot{x}(t^\star) = w_1 + w_2 \cdot \dot{y}(t^\star)$.

The functions $x(t)$ and $y(t)$ satisfy the hypothesis of Lemma 4.4 below, so we get that $|\dot{x}(t^\star) - \dot{y}(t^\star)| \geq C_\epsilon \cdot \sqrt{d_{p'}}$. Since $|\dot{x}(t^\star) - \dot{y}(t^\star)| = \frac{|w_1|}{|w_2|}$, we have $|w_1| \geq C_\epsilon \cdot \sqrt{d_{p'}} \cdot |w_2| = C_\epsilon \cdot \sqrt{d_{p'}} \cdot |v'|$. We have

$$\left|DF_{p'}^{k+1}(v')\right| \geq |w_1| \cdot \left|DF_{p'}^{k+1}\begin{pmatrix}1\\0\end{pmatrix}\right| - |w_2| \cdot \left|DF_{p'}^{k+1}\left(\frac{\tilde{v}}{|\tilde{v}|}\right)\right|.$$

Using **(H2)**, we choose $\lambda_1 > \lambda_2 > 1$ such that $\inf_{k \geq 0} R_k \cdot C_\epsilon \cdot \beta \geq \lambda_1$.

Proposition 4.2 asserts that $\left|DF_{p'}^{k+1}\left(\frac{\tilde{v}}{|\tilde{v}|}\right)\right|$ may be chosen arbitrarily small (if ϵ_1, δ are chosen small). Hence we choose ϵ_1, δ such that

$$\left|DF_{p'}^{k+1}\left(\frac{\tilde{v}}{|\tilde{v}|}\right)\right| \leq \frac{\lambda_1 - \lambda_2}{2\beta}.$$

Now,

$$\left|DF_{p'}^{k+1}(v')\right| \geq C_\epsilon \cdot \sqrt{d_{p'}} \cdot |v'| \cdot \left|DF_{p'}^{k+1}\begin{pmatrix}1\\0\end{pmatrix}\right| - |v'| \cdot \frac{\lambda_1 - \lambda_2}{2\beta}$$

$$\geq \left[R_k \cdot C_\epsilon - \frac{\lambda_1 - \lambda_2}{2\beta}\right] \cdot |v'|$$

$$\geq \left[R_k \cdot C_\epsilon \cdot \beta - \frac{\lambda_1 - \lambda_2}{2}\right] \cdot |v|$$

$$\geq \left[\frac{\lambda_1 + \lambda_2}{2}\right] \cdot |v| \geq \tilde{R} \cdot |v|.$$

(2) Let $p \in E_{k\pm}$, and suppose $v \notin K^u(\tilde{\alpha}, F^{k+2}(p))$, i.e. $v \in K^s(1/\tilde{\alpha}, F^{k+2}(p))$. We want to show that $|D\Phi_{\Phi(p)}^{-1}(v)| \geq \tilde{R}|v|$. Let $w = DF_{F^{k+2}(p)}^{-(k+1)}(v) \in T_{F(p)}\mathbb{R}^2$. Note that we have that

$$|D\Phi_{\Phi(p)}^{-1}(v)| = |DF_{F(p)}^{-1}(w)| \geq \frac{1}{|DF_p|} \cdot |w| \geq \frac{1}{\tilde{M}} \cdot |w|,$$

where $\tilde{M} := \sup_{Q_0} \{|F_{1x}| + |F_{1y}|, |F_{2x}| + |F_{2y}|\}$. Let $M > \tilde{M}$. Proposition 4.3 allows us to assume that $\left|DF^{-1}_{F^{k+2}(p)}(v)\right| \geq M \cdot |v|$. Proposition 4.1 (with $\beta = \frac{1}{\tilde{\alpha}}$) gives that for $\delta > 0$ small, $v^1 := DF^{-1}_{F^{k+2}(p)}(v) \in K^s(\alpha, F^{k+1}(p))$ and consequently $v^j := DF^{-j}_{F^{k+2}(p)}(v) \in K^s(\alpha, F^{k-j+2}(p))$ for $2 \leq j \leq k+1$.

(R, α)-hyperbolicity on $E_1 \cup E_2$ gives $|v^{j+1}| \geq R \cdot |v^j|$ for $1 \leq j \leq k$.

Combining these results, we get

$$|D\Phi^{-1}_{\Phi(p)}(v)| \geq \frac{1}{\tilde{M}} \cdot |w| = \frac{1}{\tilde{M}} \cdot |v^{k+1}| \geq \frac{1}{\tilde{M}} \cdot R^k \cdot |v^1| \geq \frac{M}{\tilde{M}} \cdot R^k \cdot |v| \geq \tilde{R} \cdot |v|. \quad \square$$

The following lemma gives an estimate for the angle between curves with certain curvatures. This lemma is used in the proof of Theorem 4.1.

Lemma 4.4 *Let $2K > \epsilon > 0$, and let $x(t), y(t)$ be a C^2 functions on some interval $[a, b]$ such that $-2K - \epsilon \leq \ddot{x}(t) \leq -2K + \epsilon$ and $-\epsilon \leq \ddot{y}(t) \leq \epsilon$ for all t. Let t_0 be a t-value with $\dot{x}(t_0) = \dot{y}(t_0)$, $d := x(t_0) - y(t_0) \geq 0$, and $x(t^\star) = y(t^\star)$ for some $t^\star \in [a, b]$. Then*

$$|\dot{x}(t^\star) - \dot{y}(t^\star)| \geq \frac{2K - 2\epsilon}{\sqrt{K + \epsilon}} \cdot \sqrt{d}.$$

Proof: If $t_0 = t^\star$, then $d = 0$; so we may assume $t_0 \neq t^\star$. We have

$$0 = x(t^\star) - y(t^\star) = x(t_0) - y(t_0) + \frac{\ddot{x}(\tau) - \ddot{y}(\tau)}{2} \cdot (t^\star - t_0)^2$$

for some τ between t_0 and $t^\star$, or equivalently, $d = -\frac{\ddot{x}(\tau) - \ddot{y}(\tau)}{2} \cdot (t^\star - t_0)^2$. This means $|t^\star - t_0| \geq \frac{\sqrt{d}}{\sqrt{K + \epsilon}}$.

On the other hand, $\dot{x}(t^\star) - \dot{y}(t^\star) = (\ddot{x}(\tau) - \ddot{y}(\tau)) \cdot (t^\star - t_0)$ for some other τ between t_0 and $t^\star$, i.e. $|\dot{x}(t^\star) - \dot{y}(t^\star)| \geq (2K - 2\epsilon) \cdot |t^\star - t_0| \geq \frac{2K - 2\epsilon}{\sqrt{K + \epsilon}} \cdot \sqrt{d}$. $\square$

Next, we want to give sufficient conditions for **(K1)**, conditions (a) and (b) to hold. Concerning **(K1)** (a) we have the following result:

Proposition 4.5 *Suppose $0 < \alpha < 1$. Let D be a bounded open subset of $\mathbb{R}^2$ and let $F(x,y)$ be a C^2-diffeomorphism of $\mathbb{R}^2$. Suppose that*

(a) $|F_{1x}| > 0$ on $F^{-1}(D)$, and

(b) $|F_{2x}| \leq \alpha|F_{1x}|$ on $F^{-1}(D)$.

Then for any $\epsilon > 0$ there is a $\delta > 0$ such that if

(c) $|F_{2y}| < \delta|F_{1x}|$ and $|F_{1y}| < \delta|F_{1x}|$ on $F^{-1}(D)$, and

(d) $|F_{1yy}| < \delta$ and $|F_{2yy}| < \delta$ on $F^{-1}(D)$,

then the pre-image $F^{-1}(\kappa)$ of every $K^s(\alpha)$-line $\kappa \subset D$ can be parametrized as a curve $(y(s), s)$ with $-\epsilon \leq \ddot{y}(s) \leq \epsilon$.

Proof: Let $q = (x,y)$ and let $\kappa \subset D$ be a $K^s(\alpha)$-line through q; we may parametrize κ as $\kappa(t) = (x + tu_1, y + t)$, where $|u_1| \leq \alpha$.

Let $g(t) = \left(F^{-1}\right)_2(\kappa(t)) = \left(F^{-1}\right)_2(x + tu_1, y + t)$.

Then

$$\dot{g}(t) = \left(F^{-1}\right)_{2x}(\kappa(t)) \cdot u_1 + \left(F^{-1}\right)_{2y}(\kappa(t)) = \frac{1}{J_F(\kappa'(t))} \cdot \left[-F_{2x}(\kappa'(t)) \cdot u_1 + F_{1x}(\kappa'(t))\right],$$

where $\kappa'(t) = F^{-1}(\kappa(t))$.

Hence,

$$|\dot{g}(t)| \geq \frac{1}{|J_F(\kappa'(t))|} \cdot \left[|F_{1x}(\kappa'(t))| - \alpha|F_{2x}(\kappa'(t))|\right].$$

Conditions (a) and (b) imply that $m := \inf\left\{|F_{1x}(z)| - \alpha|F_{2x}(z)| : z \in F^{-1}(D)\right\} > 0$. So the function $s = g(t)$ is invertible, and we can write $F^{-1}(\kappa(t))$ as $(y(s), s)$, where $y(s) = \left(F^{-1}\right)_1(\kappa(g^{-1}(s))) = \left(F^{-1}\right)_1(x + u_1 g^{-1}(s), y + g^{-1}(s))$.

Since $\dfrac{d}{ds}\left(g^{-1}\right)(s) = \dfrac{1}{\dot{g}(t)}$, we have that

$$\dot{y}(s) = \left(F^{-1}\right)_{1x}(\kappa(g^{-1}(s))) \cdot \frac{u_1}{\dot{g}(t)} + \left(F^{-1}\right)_{1y}(\kappa(g^{-1}(s))) \cdot \frac{1}{\dot{g}(t)}$$

$$= \frac{1}{\dot{g}(t)} \cdot \frac{1}{J_F} \cdot [F_{2y} \cdot u_1 - F_{1y}] = \frac{F_{2y} \cdot u_1 - F_{1y}}{F_{1x} - F_{2x} \cdot u_1},$$

where J_F and the partial derivatives of F are evaluated at $F^{-1}(\kappa(g^{-1}(s))) = (y(s), s) \in F^{-1}(D)$.

At this point, it is good to note that conditions (b) and (c) imply that

$$|\dot{y}(s)| \leq \frac{|F_{2y}| \cdot \alpha + |F_{1y}|}{|F_{1x}| - |F_{2x}| \cdot \alpha} \leq \frac{\alpha + 1}{1 - \alpha^2} \cdot \delta.$$

Now, we want to investigate $\ddot{y}(s)$:

$$\ddot{y}(s) = \frac{F_{2yx}\dot{y}(s)u_1 + F_{2yy}u_1 - F_{1yx}\dot{y}(s) - F_{1yy}}{F_{1x} - F_{2x}u_1}$$

$$-\frac{(F_{2y}u_1 - F_{1y}) \cdot (F_{1xx}\dot{y}(s) + F_{1xy} - F_{2xx}\dot{y}(s)u_1 - F_{2xy}u_1)}{(F_{1x} - F_{2x}u_1)^2}.$$

Using that $|F_{1x}| - |F_{2x}|\alpha \geq m$ on $F^{-1}(D)$, we have

$$|\ddot{y}(s)| \leq \frac{|F_{2yx}||\dot{y}(s)|\alpha + |F_{2yy}|\alpha + |F_{1yx}||\dot{y}(s)| + |F_{1yy}|}{m}$$

$$+\frac{(|F_{2y}|\alpha + |F_{1y}|) \cdot (|F_{1xx}||\dot{y}(s)| + |F_{1xy}| + |F_{2xx}||\dot{y}(s)|\alpha + |F_{2xy}|\alpha)}{m^2}.$$

Since $|\dot{y}(s)| \leq \dfrac{\alpha+1}{1-\alpha^2} \cdot \delta$, and using conditions (a)-(d), we can find a $\delta > 0$ such that $|\ddot{y}(s)| \leq \epsilon$. □

Concerning **(K1)** (b) we have the following result:

Proposition 4.6 *Suppose* $0 < \beta$. *Let* D *be a bounded open subset of* $\mathbb{R}^2$ *and let* $F(x, y)$ *be a* C^2*-diffeomorphism of* $\mathbb{R}^2$. *Suppose that*

(a) $|F_{2x}| - \beta|F_{2y}| > 0$ *on* D.

Then the image $F(l)$ of every $K^u(\beta)$-line $l \subset D$ can be parametrized as a curve $(x(s), s)$, and furthermore,

$$\ddot{x}(s) = \frac{F_{1xx} + 2F_{1xy}v_2 + F_{1yy}v_2^2}{(F_{2x} + F_{2y}v_2)^2} - \frac{(F_{1x} + F_{1y}v_2) \cdot (F_{2xx} + 2F_{2xy}v_2 + F_{2yy}v_2^2)}{(F_{2x} + F_{2y}v_2)^3},$$

where v_2 is the slope of the line l.

Proof: Let $p = (x, y)$ and let $l \subset D$ be a $K^u(\beta)$-line through p; we may parametrize l as $l(t) = (x + t, y + tv_2)$, where $|v_2| \leq \beta$.

Let $g(t) = F_2(l(t)) = F_2(x + t, y + tv_2)$. Then $\dot{g}(t) = F_{2x}(l(t)) + F_{2y}(l(t)) \cdot v_2$, and also $|\dot{g}(t)| \geq |F_{2x}(l(t))| - \beta \cdot |F_{2x}(l(t))|$.

Condition (a) implies that the function $s = g(t)$ is invertible, and we can write $F(l(t))$ as $(x(s), s)$, where $x(s) = F_1(l(g^{-1}(s))) = F_1(x + g^{-1}(s), y + v_2 g^{-1}(s))$.

Since $\frac{d}{ds}\left(g^{-1}\right)(s) = \frac{1}{\dot{g}(t)}$, we have that

$$\dot{x}(s) = F_{1x}(l(g^{-1}(s))) \cdot \frac{1}{\dot{g}(t)} + F_{1y}(l(g^{-1}(s))) \cdot \frac{v_2}{\dot{g}(t)} = \frac{F_{1x} + F_{1y} \cdot v_2}{F_{2x} + F_{2y} \cdot v_2},$$

where the partial derivatives of F are evaluated at $l(g^{-1}(s)) \in D$.

Now,

$$\ddot{x}(s) = \frac{F_{1xx} + 2F_{1xy}v_2 + F_{1yy}v_2^2}{(F_{2x} + F_{2y}v_2)^2} - \frac{(F_{1x} + F_{1y}v_2) \cdot (F_{2xx} + 2F_{2xy}v_2 + F_{2yy}v_2^2)}{(F_{2x} + F_{2y}v_2)^3}. \quad \square$$

5 Symbolic Dynamics (Part II)

5.1 Assumptions and Definitions

We look at the diffeomorphisms $F : \mathbb{R}^2 \to \mathbb{R}^2$ as considered in section 3, only now we will assume certain hyperbolicity conditions that assure that the coding maps $\psi : \bigcap_{n\in\mathbf{Z}} F^n(Q) =: \Lambda \to \Sigma$ (before the first tangency) and $\tilde{\psi} : \bigcap_{n\in\mathbf{Z}} F^n(Q) =: \tilde{\Lambda} \to \tilde{\Sigma}$ (at the first tangency) are actually homeomorphisms.

We assume the the geometric conditions **(G1)**, **(G2)** and the hyperbolicity condition **(H1)**, as formulated in section 4. Recall that there we had two "hyperbolic regions" E_1 and E_2, and a "parabolic region" Q_0. We defined the sets

$$E_{2,k} = E_2 \cap F^{-1}(E_1) \cap \ldots \cap F^{-k}(E_1) \quad \text{and} \quad \tilde{E}_{2,k} := E_{2,k} \setminus E_{2,k+1}$$

($k \geq 0$), which are full-height rectangles in Q, and for each $k \geq 0$, we let

E_{k-} denote the left component of $F^{-1}(\tilde{E}_{2,k}) \cap Q_0$, and

E_{k+} denote the right component of $F^{-1}(\tilde{E}_{2,k}) \cap Q_0$.

Furthermore, we let $E_{\infty-}$ denote the left component of $F^{-1}(\partial_{right}E_2) \cap Q_0$, and $E_{\infty+}$ denote the right component of $F^{-1}(\partial_{right}E_2) \cap Q_0$.

Then we have that $Q_0 \cap F^{-1}(E_2)$ is "stratified" by the full-height (in Q) rectangles $E_{k\pm}$; i.e.,

$$Q_0 \cap F^{-1}(E_2) = \bigcup_{k\geq 0} E_{k\pm} \cup E_{\infty\pm}$$

We have the return map $\Phi = F^{k+2} : E_{k\pm} \to Q_0$. We now assume that this return map is uniformly hyperbolic; i.e., Φ is (R,α)-hyperbolic, with the same $R > 1$ and $0 < \alpha < 1$ on each "stratum" $E_{k\pm}$:

(H3) For all $k \geq 0$, the map $F^{k+2} : E_{k\pm} \to Q_0$ is (R,α)-hyperbolic.

Note that we can write

$$Q_1 = E_1 \cup \bigcup_{k\geq 0} E_{k-} \cup E_{\infty -} \quad \text{and} \quad Q_2 = E_2 \cup \bigcup_{k\geq 0} E_{k+} \cup E_{\infty +},$$

and that in combination with **(H1)**, (an appropriate power of) F can be thought as being uniformly hyperbolic on each "stratum" of $Q_1 \cup Q_2$. More precisely, by additionally letting $\Phi = F$ on $E_1 \cup E_2$ we have that Φ is (R, α)-hyperbolic on $(Q_1 \cup Q_2) \setminus E_{\infty\pm}$.

We refer to the collection $\mathcal{S} := \{E_1, E_2, E_{k\pm} : k \geq 0\}$ as "strata". For $x \in Q_1 \cup Q_2$, let S_x denote the $S \in \mathcal{S}$ with $x \in S$.

5.2 Stable and Unstable Curves

For $x \in \Lambda = \bigcap_{n\in\mathbf{Z}} F^n(Q)$, let

$$W^s(x) = \left\{ y \in Q_1 \cup Q_2 : S_{F^n(x)} = S_{F^n(y)} \text{ for all } n \geq 0 \right\}$$

and

$$W^u(x) = \left\{ y \in Q_1 \cup Q_2 : S_{F^{-n}(x)} = S_{F^{-n}(y)} \text{ for all } n \geq 0 \right\}.$$

Let $Q_U = E_1 \cup (Q_0 \cap F(E_1)) \cup E_2$ and $Q_C = E_1 \cup (Q_0 \cap F(E_1 \cup E_2))$. Then the hyperbolicity assumptions **(H1)** and **(H3)** give us that for each $x \in \Lambda$,

- We have $F(W^s(x)) \subset W^s(F(x))$ and $W^s(x)$ is a continuous, full-height curve in Q, containing x, and it is a $K^s(\alpha)$-curve in Q_U. If $y \in W^s(x)$, then $|\Phi^n(x) - \Phi^n(y)| \to 0$ as $n \to \infty$.
- We have $F^{-1}(W^u(F(x))) \subset W^u(x)$ and $W^u(x)$ is a continuous, full-width curve in Q, containing x, and it is a $K^u(\alpha)$-curve in Q_C. If $y \in W^u(x)$, then $\left|\Phi^{-n}(x) - \Phi^{-n}(y)\right| \to 0$ as $n \to \infty$.

We are therefore justified in calling $W^s(x)$ the *stable* curve of $x \in \Lambda$, and $W^u(x)$ the *unstable* curve of $x \in \Lambda$.

Furthermore, on $Q_U \cap Q_C$, $W^s(x) \cap W^u(x) = \{x\}$. By applying F, we get this property on all of Q.

Now if $y \in \Lambda$ has the same coding as x; i.e., $\psi(x) = \psi(y)$, this last property gives that $x = y$. In other words, we now have that the coding map $\psi : \Lambda \to \Sigma$ is one-to-one. This lets us improve upon the results in section 3.

5.3 Topological Equivalence

We consider a family of C^1-diffeomorphisms $F_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$, $\alpha \geq \alpha_0$ with the following properties:

- the family depends continuously on the parameter α,
- each F_α, $\alpha > \alpha_0$, satisfies the geometric conditions **(G1)**, **(G2)** "before the first tangency",
- F_{α_0} satisfies the geometric conditions **(G1)**, **(G2)** "at the first tangency",
- each F_α, $\alpha \geq \alpha_0$, satisfies the hyperbolicity conditions **(H1)**, **(H3)**.

Then we have the following resulting concerning the topological dynamics of F:

Theorem 5.1 *Let* $\Lambda_\alpha = \bigcap_{n \in \mathbb{Z}} (F_\alpha)^n(Q)$. *($Q$ is defined in **(G1)**, **(G2)**).*

(1) *If $\alpha > \alpha_0$, then there exists a homeomorphism $\psi_\alpha : \Lambda_\alpha \to \Sigma$ such that*

$$(\psi_\alpha \circ F_\alpha)(x) = (\sigma \circ \psi_\alpha)(x) \text{ for all } x \in \Lambda_\alpha.$$

(Σ, σ) is the left-shift on two symbols, as described in section 3. ψ_α is the coding map. Also, the set Λ_α is hyperbolic.

(2) *There exists a homeomorphism $\tilde{\psi} : \Lambda_{\alpha_0} \to \tilde{\Sigma}$ such that*

$$\left(\tilde{\psi} \circ F_{\alpha_0}\right)(x) = \left(\tilde{\sigma} \circ \tilde{\psi}\right)(x) \text{ for all } x \in \Lambda_{\alpha_0}.$$

$(\tilde{\Sigma}, \tilde{\sigma})$ is the factor of the left-shift on two symbols, obtained by identifying the two possible codings for homoclinic tangencies. Refer to section 3 for full details.

6 Application to the Henon Map

Recall that the Henon map is given by

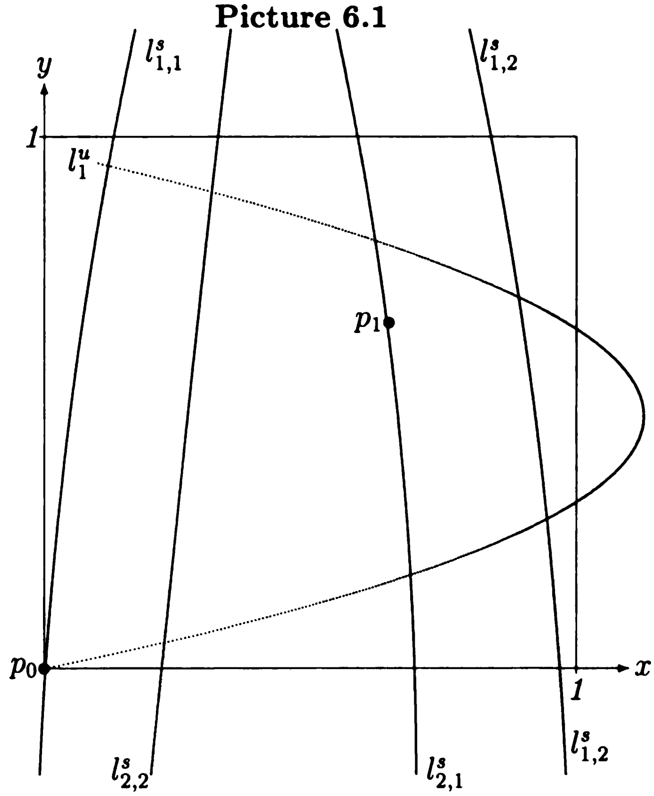
$$H_{b,r}(x,y) = H(x,y) = (H_1(x,y), H_2(x,y)) = (rx(1-x) - by, x).$$

We want to show that $H(x,y)$ satisfies the assumptions used for proving Theorem 4.1.

6.1 Geometric Conditions

We observe that regarding the conditions **(G1)** and **(G2)**, we need the following geometry for the invariant manifolds $W^u(p_0)$, $W^s(p_0)$ and $W^s(p_1)$:

Picture 6.1



Recall that l^u_1, $l^s_{1,1}$, $l^s_{1,2}$ are parts of $W^u(p_0)$, $W^s(p_0)$, and $l^s_{2,1}$, $l^s_{2,2}$ are parts of $W^s(p_1)$. The geometry we need is present when these invariant manifolds are defined; i.e., when both fixed points are hyperbolic saddles.

To make things precise, we let

$$E_1 = \left\{(x,y) : f^s_{1,1}(y) \le x \le f^s_{2,2}(y), x \le ry(1-y) + f^u_1(y), y \in [0,1]\right\},$$

$$Q_0 = \left\{(x,y) : f^s_{2,2}(y) \le x \le f^s_{2,1}(y), x \le ry(1-y) + f^u_1(y), y \in [0,1]\right\},$$

$$E_2 = \left\{(x,y) : f^s_{2,1}(y) \le x \le f^s_{1,2}(y), x \le ry(1-y) + f^u_1(y), y \in [0,1]\right\}.$$

($f^s_{i,j}(y)$ parametrizes $l^s_{i,j}$, $ry(1-y) + f^u_i(y)$ parametrizes l^u_i; cf. section 2 .)

Then

Proposition 6.1 *If $r > 3(1+b)$, and $r > r(b)$, then the map $F = H_{b,r}$ satisfies the conditions **(G1)** and **(G2)**.*

In the next sections, we proceed to verify the conditons **(H1)**, **(H2)** and **(K1)**.

6.2 The Region of (R,α)-Hyperbolicity

We use the sufficient conditions given in Lemma 4.3 to determine a region where $H(x,y) = H_{b,r}(x,y)$ will be (R,α)-hyperbolic, for some $R > 1$, and some $0 < \alpha < 1$.

Note that

$$H_{1x}(x,y) = r(1-2x) \qquad H_{1y}(x,y) = -b \qquad H_{2x}(x,y) = 1 \qquad H_{2y}(x,y) = 0.$$

Then the conditions (1)-(4) in Lemma 4.3 become:

(1) $1 + \alpha^2 \cdot b \le \alpha \cdot 2r \left| x - \frac{1}{2} \right|$,

(2) $2r \left| x - \frac{1}{2} \right| - \alpha \cdot b \ge R$,

(3) $b + \alpha^2 \le \alpha \cdot 2r \left| x - \frac{1}{2} \right|$,

(4) $2r\left|x-\frac{1}{2}\right| - \alpha \geq b \cdot R.$

With the objective of choosing $\alpha \approx 1$ and $R \approx 1$, we recall the definition of the closed region

$$\mathcal{E} = \mathcal{E}_{b,r} = \left\{(x,y) : 2r\left|x-\frac{1}{2}\right| \geq 1+b\right\}.$$

The interior of $\mathcal{E}$ is the complement of the closed vertical strip

$$\mathcal{S} = \mathcal{S}_{b,r} = \left\{(x,y) : 2r\left|x-\frac{1}{2}\right| \leq 1+b\right\}.$$

If $p = (x,y) \in \mathcal{E}$, we see that we can choose R and α close to 1 so that the conditions (1)-(4) above hold. This gives the following result:

Proposition 6.2 *If $\mathcal{R}$ is any (possibly disconnected) closed region such that $\mathcal{R} \cap \mathcal{S}_{b,r} = \emptyset$, then there exist $R > 1$ and $0 < \alpha < 1$ such that $H_{b,r}$ is (R,α)-hyperbolic on $\mathcal{R}$.*

In conjunction with Lemma 2.1, part (2), this proposition gives an easy proof of [**DN**]'s results for the orientation-preserving case ($b > 0$). It is actually not difficult to obtain the result for $|b|$ instead of b, using the same simple geometric arguments. We have:

Corollary 6.1 *If $r > \left(2+\sqrt{5}\right)(1+b)$, then there exists $R > 1$ and $0 < \alpha < 1$ so that $H = H_{b,r}$ is (R,α)-hyperbolic on $Q \cap H^{-1}(Q)$. In particular, the set $\Lambda = \bigcap_{n\in\mathbf{Z}} H^n(Q)$ (which is also the set of points with bounded orbits) is a hyperbolic set and $H|_\Lambda$ is topologically equivalent to the two-shift (Σ,σ).*

Remark 6.1 *This result uses the fact that for $r > \left(2+\sqrt{5}\right)(1+b)$, the image $H(l)$ of the line $\{(x,1) : 0 \leq x \leq 1\}$ is to the right of the region $\{(x,y) : 2r|y-\frac{1}{2}| \leq 1+b,\ 0 \leq x \leq 1\}$. It can be improved upon by considering the upper component l' of $H(l) \cap Q$, and then estimating when $H(l')$ is to the left of this region. We omit the calculations and state only that by proceeding in this way, a better lower bound on r, valid for all $b > 0$, than the one in the previous corollary can be obtained.*

We also have the following corollary:

Corollary 6.2 *There exists a $b_0 > 0$ an $R > 1$, and a $0 < \alpha < 1$ such that if $0 < b \leq b_0$, and $r > 3(1+b)$, then $H_{b,r}$ is (R, α)-hyperbolic on $E_1 \cup E_2$.*

6.3 Concavity Conditions

We verify condition **(K1)**. Note that for $H(x,y) = (rx(1-x) - by, x)$, we have that on $\mathcal{E} = \{(x,y) : 2r|x - \frac{1}{2}| \geq 1 + b\}$,

$$|H_{1x}| = 2r|x - \frac{1}{2}| \geq 1 + b > 0, \quad \frac{|H_{2x}|}{|H_{1x}|} \leq \frac{1}{1+b}, \quad \frac{|H_{2x}|}{|H_{1x}|} = 0, \text{and} \quad \frac{|H_{1y}|}{|H_{1x}|} \leq \frac{b}{1+b}.$$

Furthermore,

$$H_{2x} = 1, \quad H_{2y} = 0, \quad H_{1yy} = H_{2yy} = 0, \quad H_{1xy} = H_{1yy} = H_{2xx} = H_{2xy} = 0$$

on all of $\mathbb{R}^2$.

Now, we apply Proposition 4.5 with D a small open neighbourhood of $F(\mathcal{E}) \cap [0,1]^2$, and Proposition 4.6 with D a small open neighbourhood of $\mathcal{S} \cap [0,1]^2$ to get statement (b) and (a) (with $K = r$), respectively, of **(K1)**, provided the lines l' and κ' intersect as in **(K1)** for $b > 0$ small.

To see this intersection property, we make the following argument: as $b \to 0$, the map $H(x,y) = H_{b,r}(x,y) = (rx(1-x) - by, x)$ limits to the logistic map $\tilde{H}(x,y) = \tilde{H}_r(x,y) = (rx(1-x), x)$. Also, as $b \to 0$, we see from Proposition 1 that the pre-image of any $K^s(\alpha)$-line κ will become a vertical line, whereas the image of any $K^u(\beta)$-line l will be a parabola $s \mapsto (rs(1-s), s)$. So the intersection property holds for $\tilde{H}$, and since we think of H as a C^2-perturbation of $\tilde{H}$, we have that this property also holds for $b > 0$ small.

Hence, we have so far established that for $b > 0$ small, and $r > 3(1+b)$, the Henon map satisfies conditions **(G1)**, **(G2)**, **(H1)** and **(K1)** of the Abstract Model. We define the sets $E_{2,k}$, $\tilde{E}_{2,k}$ and $E_{k\pm}$, $k = 0, 1, 2, \ldots$, as in the abstract case. Also, $\Phi : \bigcup_{k=0}^{\infty} E_{k\pm} \longrightarrow Q_0$ will be the first-return map as in Definiton 4.

Concerning **(H2)**, note first that for the Henon map $H(x, y)$, $|DH_p(v_1, v_2)| \geq |v_1|$, hence if $v \in K^u(1/\alpha, p)$, $0 < \alpha < 1$, we have that $|DH_p(v_1, v_2)| \geq \alpha \cdot |v|$. This means that

$$\beta = \inf \left\{ \frac{|DF_p(v)|}{|v|} : v \neq 0, v \in K^u(1/\alpha, p), p \in Q_0 \right\} \geq \alpha,$$

and, since $|H_{1x}| \geq r - 2$ on $E = \{(x, y) : |x - \frac{1}{2}| \geq \frac{1}{2} - \frac{1}{r}\}$, we have $\inf_{p \in E_1 \cup E_2} |F_{1x}(p)| \cdot \beta > 1$.

To prove the second part of **(H2)**, we need the following lemma to estmate the return time to Q_0:

Lemma 6.1 *Let $l_1^s = l_{1,1}^s$ be the left and $l_2^s = l_{1,2}^s$ be the right branch of the stable manifold of the fixed point $(0, 0)$. Then for $p \in \tilde{E}_{2,N}$ we have*

$$dist(p, l_2^s) \geq dist(H^{N+1}(p), l_1^s) \cdot \left(\frac{1}{r + b} \right)^{N+1}$$

Proof: Let $v \in T_{(x,y)}\mathbb{R}^2$. Then $|DH_{(x,y)}| = \max\{1, r|1 - 2x| + b\}$ and we have the estimate $\left|DH_{(x,y)}(v)\right| \leq |DH_{(x,y)}| \cdot |v|$, and hence $\left|DH_{(x,y)}(v)\right| \leq (r + b) \cdot |v|$.

The last inequality gives the following result:

$$dist(H^{i+1}(p), l_1^s) \leq dist(H^{i+1}(p), H(l_1^s)) \leq (r + b) \cdot dist(H^i(p), l_1^s)$$

for $i = 1, \ldots, N$, and

$$dist(H(p), l_1^s) \leq (r + b) \cdot dist(p, H^{-1}(l_1^s)) \leq (r + b) \cdot dist(p, l_2^s).$$

Hence $dist(H^{N+1}(p), l_1^s) \leq (r + b)^{N+1} \cdot dist(p, l_2^s)$. $\square$

Now, let us complete the proof of **(H2)**; we assume $b > 0$ small, $r > r(b) \approx 4$, and we can choose $0 < \alpha < 1$ as close to 1 as necessary.

Suppose $p = (x, y) \in \tilde{E}_{2,k}$; i.e., $p \in E_2, H(p) \in E_1, \ldots, H^k(p) \in E_1, H^{k+1}(p) \in Q_0$ for some $k \geq 0$. Let $p_0 = p = (x_0, y_0)$ and $p_i = H^i(p) = (x_i, y_i)$ for $i = 1, \ldots, k+1$.

We may make the following estimates:

$$x_0 + d_p \geq 0.99, \quad 2r\left|x_0 - \frac{1}{2}\right| - b \cdot \alpha \geq 7.8 \cdot \left(\frac{1}{2} - d_p\right),$$

and for $i = 1, \ldots, k$,

$$x_i \leq (4.1)^i \cdot d_p, \quad 2r\left|x_i - \frac{1}{2}\right| - b \cdot \alpha \geq 7.8 \cdot \left(\frac{1}{2} - (4.1)^i \cdot d_p\right).$$

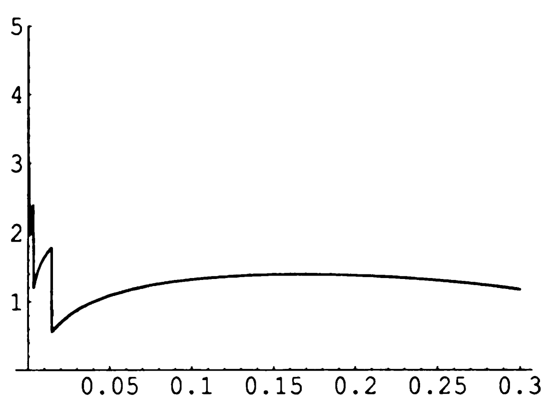
Using Lemma 4 with $N = k$, we get that $d_p \geq \frac{1}{4} \cdot \left(\frac{1}{4.1}\right)^{k+1} \geq \left(\frac{1}{4.1}\right)^{k+2}$, or $k \geq -\dfrac{\log(d_p)}{\log(4.1)} - 2$.

Let $m(d_p) = Floor\left[-\dfrac{\log(d_p)}{\log(4.1)} - 2\right]$, and let

$$\rho(d_p) = \log(7.8(0.5 - d_p)) + \sum_{i=1}^{m(d_p)} \left[\log(7.8(0.5 - (4.1)^i \cdot d_p))\right] + \log(\sqrt{d_p}).$$

Letting $\zeta(x) = \rho(x) + \log(2 \cdot 1.9)$, we need only show that $\inf\{\zeta(x) : 0 \leq x \leq 0.3\} > 0$, to show **(H2)**.

The graph of $\zeta(x)$ (for $0 \leq x \leq 0.3$) is shown below:



Summarizing the results of this section gives:

Proposition 6.3 *There exists an $0 < \alpha < 1$, an $R > 1$, and there exists a $b_0 > 0$ such that if $0 < b < b_0$ and $r \geq r(b)$, then the return map Φ to Q_0 of the Henon map $H_{b,r}$ is (R, α)-hyperbolic.*

6.4 Main Results for Henon Maps

For each $b > 0$, there exists a unique value of the parameter r, denoted by $r(b)$, such that for $r > r(b)$, the invariant curves of $p = (0,0)$ intersect transversely, whereas for $r = r(b)$, they have their first homoclinic tangency.

Let $\Lambda = \bigcap_{n \in \mathbb{Z}} F^n(Q)$ denote the set of $(x, y) \in \mathbb{R}^2$ with bounded orbits.

Now we state our results:

Theorem 6.1 *Let $H(x,y) = (rx(1-x) - by, x)$ be the Henon map.*

Then there exists a $b_0 > 0$ such that for all $0 < b \leq b_0$, we have the following:

(1) *If $r > r(b)$, then there exists a homeomorphism $\psi : \Lambda \to \Sigma$ such that the diagram*

$$\begin{array}{ccc} \Lambda & \xrightarrow{H} & \Lambda \\ \psi \downarrow & & \psi \downarrow \\ \Sigma & \xrightarrow{\sigma} & \Sigma \end{array}$$

commutes. Futhermore, the set Λ is hyperbolic.

(2) *If $r = r(b)$, then there exists a homeomorphism $\tilde{\psi} : \Lambda \to \tilde{\Sigma}$ such that the diagram*

$$\begin{array}{ccc} \Lambda & \xrightarrow{H} & \Lambda \\ \tilde{\psi} \downarrow & & \tilde{\psi} \downarrow \\ \tilde{\Sigma} & \xrightarrow{\tilde{\sigma}} & \tilde{\Sigma} \end{array}$$

commutes.

Where:

- (σ, Σ): the full shift on two symbols;
- $\psi : \Lambda \to \Sigma$: the coding map of points $x \in \Lambda$;
- $(\tilde{\sigma}, \tilde{\Sigma})$: the quotient of (σ, Σ), obtained by identifying the two ambiguous codings for homoclinic tangencies;
- $\tilde{\psi} : \Lambda \to \tilde{\Sigma}$: the coding map of points $x \in \Lambda$ - sending each x to its equivalence class in $\tilde{\Sigma}$.

Theorem 6.2 *The results in the previous theorem also hold for C^2-perturbations of the Henon maps considered.*

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