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R. M. de Jong
Major Professor's Signature

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Chien-Ho Wang

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ABSTRACT

Essays on nonlinear transformations of nonstationary time series

By

Chien-Ho Wang

My dissertation consists of four essays on nonlinear transformations of nonstationary time series. The dissertation has five chapters. The first chapter gives the motivation for each of the four essays on nonlinear transformations of nonstationary time series.

In Chapter 2, we consider periodic transformations of nonstationary time series. We will use the scaled I(1) process $n^{-\alpha}x_t$, where $\alpha \in (0, 1/4)$. It is shown that a result of de Jong (2001) can be extended to

$$n^{-\alpha-1/2}\sum_{t=1}^{n}(T(n^{-\alpha}x_t) - \mu) \xrightarrow{d} N(0, V)$$

where $\mu = (2\pi)^{-1}\int_{-\pi}^{\pi} T(x)dx$, V is the covariance matrix, and x_t is a so-called unit root process.

In Chapter 3, we extend the asymptotic results for nonlinear transformations of integrated time series of Park and Phillips (1999). We use less restrictions than Park and Phillips to derive the improved results for integrable functions and asymptotically homogeneous functions. In addition to the improved results, we propose a

new asymptotic result for non-integrable functions. This new result can extend the original Park and Phillips results to some functions that are not locally integrable.

In Chapter 4, we investigate the question as to what happens to Dickey-Fuller tests when the data under consideration is a trigonometric transformation of an I(1) process. We use analytical tools provided by de Jong (2001) to establish that for the Dickey-Fuller t-test, we have

$$n^{-1/2}\hat{t}_{\mu} \xrightarrow{p} (E\cos(\varepsilon_t) - 1)(1 - (E\cos(\varepsilon_t))^2)^{-1/2}$$

where $\hat{t}_{\mu}$ is Dickey-Fuller t-test under trigonometric transformations of I(1) processes with intercept. The above result implies that the periodic transformation of integrated process will asymptotically indicate stationarity.

In Chapter 5, we consider a different approach for threshold unit root model. We consider the Dickey-Fuller unit root test of the threshold unit root model

$$\Delta y_t = \begin{cases} \varepsilon_t & \text{if } |y_{t-1}| \leq C \\ \mu + \varphi y_{t-1} + \varepsilon_t & \text{if } |y_{t-1}| > C \end{cases},$$

where $-2 < \varphi < 0$. We will relax the assumption that threshold value,C,is known. We derive the asymptotic results that can be used to establish the asymptotic distribution of the Dickey-Fuller unit root test in a regression of Δy_t on a constant and $y_{t-1}I(|y_{t-1}| > C)$ that has been optimized over the parameter C that is unidentified under the null hypothesis.

For my father, Pei-Chen Wang and my mother, Shu-Ying Wang

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CHAPTER 1

Introduction

This thesis consists on four essays of nonlinear transformations on nonstationary time series. Before we discuss nonlinear transformation on nonstationary time series, we will first introduce some basic concepts about nonstationary time series and provide a limited overview of the relevant literature.

1.1 Basic properties of I(1) processes

In this section, we will introduce the concepts about nonstationary time series regression. I will introduce the cointegration and unit root models separately.

1.1.1 Concepts of unit root models

First, we introduce the basic linear time series regression model,

$$x_t = \rho x_{t-1} + w_t \quad t = 1, 2, \ldots, T \tag{1.1}$$

where w_t is a stationary process. In general, we need that the coefficient, ρ, must satisfy the condition that $|\rho| < 1$ to ensure stationarity. In two breakthrough articles by Dickey and Fuller (1979, 1981), the case of $\rho = 1$ was first investigated. They find that the t-test will converge to a non-standard distribution under $\rho = 1$. They also simulate this unit root distribution. After Dickey and Fuller's articles, Said and Dickey (1984) and Phillips and Perron (1988) extended the unit root limit theory to serially correlated errors. The limit theory of linear unit root models is by now well developed.

1.1.2 The concept of cointegration

In international economics and macroeconomics, there exist long run relationships between nonstationary variables. One example is the Purchasing Power parity (PPP) research in international finance. PPP states that in the long run the exchange rate adjusted price levels in two countries should be the same. The empirical models about PPP is

$$p_i = \rho p_j + r_{ij} + u_{ij}$$

where p_i (p_j) is the price level in country i (j), r_{ij} is exchange rate between country i and j, and u_{ij} is a stationary series. Because p_i and p_j are I(1) processes, we cannot use the traditional ordinary least squares method to obtain the limit properties of the estimated coefficients. In a breakthrough paper, Engle and Granger (1987) developed the linear cointegration regression model. They considered the time series regression

$$y_t = \beta x_t + u_t \tag{1.2}$$

where x_t and y_t are two different I(1) processes and u_t is a white noise process. We can investigate long run relationships between some economic and financial time series using the cointegration concept. Since cointegration was proposed, it has become mainstream in econometric research.

1.1.3 Scope of this dissertation

There are a lot of nonlinear relationships between economic variables in economic theory. Using linear times series models for all economic times series has a lot of restrictions. However, if we only transform the I(1) processes and directly use transformed variables to regress two transformed variables for cointegration or unit root models, we will have some problems. The main problem about nonlinear transformations of nonstationary time series is that transformed I(1) series may not keep the same nonstationary properties as before. Granger and Hallman (1989, 1991) first discussed these possible problems. They used nine kinds of functional forms to investigate whether the transformed I(1) series still keep the nonstationary properties. They found whether the integrated process keep its nonstationary characteristic after transformed will depend on functional forms. The other problem about nonlinear transformations of nonstationary time series is the use of unit root tests. Because the properties of transformed I(1) series change, we may misjudge the properties of transformed I(1) series when we use the Dickey-Fuller unit root tests. In Granger and Hallman's research, they simulated Dickey-Fuller test critical values under different transformations. They found the critical values will change depending on functional forms. Granger and Hallman's papers investigated about these problems in detail, but they only used simulations to investigate these problems. They did not derive any formal limit theory for transformed I(1) series. After Granger and Hallman's papers, Park and Phillips (1999) extended the existing limit theory for integrated processes to nonlinear models. They considered three classes of functional forms: integrable functions, asymptotically homogeneous functions and explosive functions. They use the concept of local time to derive asymptotic results for nonlinear regression models. Although Park and Phillips' results are remarkable, some functional forms cannot be considered in their results, and their results are still restrictive in terms of the necessary conditions. In this dissertation, we will investigate these questions.

1.2 Thesis Structure

In Chapter 2, we consider periodic transformations of nonstationary time series. In Park and Phillips' paper (Park and Phillips (1999)), the authors derive asymptotic properties of nonlinear transformations of I(1) series for three classes of functional forms: integrable functions, asymptotically homogeneous functions, and explosive functions. The key element here is that the I(1) process was not rescaled by the square root of sample size. After Park and Phillips' work, de Jong (2002) established the asymptotics for periodic transformations of I(1) processes. In that paper, it is proven that for periodic function $T(.)$ and for I(1) processes x_t,

$$n^{-1/2}\sum_{t=1}^{n}(T(x_t)-\mu) \xrightarrow{d} N(0,\sigma^2)$$

where $\mu = (2\pi)^{-1}\int_{-\pi}^{\pi} T(x)dx$. De Jong derived this result for periodic functions of I(1) processes that have not been scaled. In Chapter 2, we will use the scaled I(1) process $n^{-\alpha}x_t$ instead of x_t in de Jong's paper, where $\alpha \in (0, 1/4)$. It is shown that de Jong's original result can be extended to

$$n^{-\alpha-1/2}\sum_{t=1}^{n}(T(n^{-\alpha}x_t)-\mu) \xrightarrow{d} N(0, 2\sigma^{-2}\sum_{j=1}^{\infty} j^{-2}(a_j^2+b_j^2))$$

where $\mu = (2\pi)^{-1}\int_{-\pi}^{\pi} T(x)dx$, $a_j = \pi^{-1}\int_{-\pi}^{\pi}\cos(jx_t)T(x)dx$ and $b_j = \pi^{-1}\int_{-\pi}^{\pi}\sin(jx_t)T(x)dx$. When $\alpha = 0$, this new result will specialize to Theorem 1 of de Jong (2001). In this chapter, we therefore extend the results for periodic transformations of I(1) processes in de Jong's original paper.

In Chapter 3, we extend the asymptotic results for nonlinear transformations of integrated time series of Park and Phillips (1999). In Park and Phillips, they prove that for I(1) processes x_t and integrable function $T(.)$,

$$n^{-1/2}\sum_{t=1}^{n}T(x_t)\xrightarrow{d}(\int_{-\infty}^{\infty}T(s)ds)L(1,0)$$

where $L(t,s)$ is a two-parameter stochastic process called "Brownian local time". They established the above result for I(1) processes that have not been scaled by $n^{-\alpha}x_t$. In this chapter, we use the scaled I(1) processes $n^{-\alpha}x_t$ instead of x_t. We will use the results of de Jong (2001) to extend their result for integrable functions to

$$n^{-1/2-\alpha}\sum_{t=1}^{n}T(n^{-\alpha}x_t)\xrightarrow{d}(\int_{-\infty}^{\infty}T(s)ds)L(1,0),$$

where $\alpha\in[0,1/2)$. The asymptotics for integrable functions is derived under less strict conditions than in Park and Phillips (1999).

The asymptotically homogeneous functions as defined in Park and Phillips (1999) are assumed to satisfy

$$T(\lambda x)=\nu(\lambda)H(x)+R(x,\lambda).$$

For the remainder function $R(.,.)$, Park and Phillips ensured asymptotically neglibility of

$$n^{-1}\sum_{t=1}^{n}R(x_t,n^{1/2}).$$

Their result for asymptotically homogeneous functions is then

$$\nu(n^{1/2})^{-1}n^{-1}\sum_{t=1}^{n}T(x_t)\xrightarrow{d}\int_0^1 H(\sigma W(r))dr,$$

where $\sigma^2=\lim_{n\to\infty}n^{-1}Ex_n^2$. We use the scaled I(1) processes $n^{-\alpha}x_t$ instead of x_t, as in Park and Phillips (1999). We generalized the original definition of Park and Phillips of asymptotically homogeneous functions. For functions $H(.)$ and $\nu(.)$, we assume that for all $K>0$,

$$\int_{-K}^{K}|\nu(\lambda)^{-1}T(\lambda x)-H(x)|dx\to 0,$$

when $\lambda \to \infty$. Under regularity conditions, we derive the result

$$\nu(n^{-1/2-\alpha})^{-1} n^{-1} \sum_{t=1}^{n} T(n^{-\alpha} x_t) \xrightarrow{d} \int_0^1 H(\sigma W(r)) dr$$

where $\alpha \in [0, 1/2)$. This result extends the limit theory for the asymptotically homogeneous functions as derived by Park and Phillips.

In Chapter 4, we investigate the question as to what happens to Dickey-Fuller tests when the data under consideration is a trigonometric transformation of an I(1) process. Granger and Hallman (1991) investigated this question by simulations. They concluded that the I(1) series will change its properties after periodic transformations. We use analytical tools provided by de Jong (2001) to establish that for the Dickey-Fuller t-test, we have

$$n^{-1/2} \hat{t}_\mu \xrightarrow{p} (E\cos(\varepsilon_t) - 1)(1 - (E\cos(\varepsilon_t))^2)^{-1/2}$$

where $\hat{t}_\mu$ is Dickey-Fuller t-test under trigonometric transformations of I(1) processes with intercept. Otherwise, for the coefficient $\hat{\rho}$ we show that

$$n^{1/2}(\hat{\rho} - E\cos(\varepsilon_t)) \xrightarrow{d} N(0, V)$$

where $V = (3/8)E(\cos(\varepsilon_t) - E\cos(\varepsilon_t))^2 + (1/8)E(\sin(\varepsilon_t))^2$. Because Dickey-Fuller t-tests diverge at rate $\sqrt{n}$. The above result implies that the periodic transformation of integrated process will asymptotically indicate stationarity. These theoretical results are supported by the simulations in Granger and Hallman (1991).

In Chapter 5, we consider a different approach for threshold unit root model. In González and Gonzalo's paper (González and Gonzalo (1997)), they used the threshold unit root model:

$$y_t = \begin{cases} \varphi_1 y_{t-1} + \varepsilon_t & \text{if } y_{t-1} \le C \\ \varphi_2 y_{t-1} + \varepsilon_t & \text{if } y_{t-1} > C \end{cases}$$

They derive the asymptotic properties of Dickey-Fuller unit root tests that the null hypothesis of unit root exists in at least one regime against stationary threshold autoregressive model. But their model has a main drawback. In González and Gonzalo's TUR model, they only allow the case that all regimes are stationary in alternative hypothesis. In this chapter, we consider the Dickey-Fuller unit root test of the threshold unit root model

$$\Delta y_t = \begin{cases} \varepsilon_t & \text{if } |y_{t-1}| \le C \\ \varphi y_{t-1} + \varepsilon_t & \text{if } |y_{t-1}| > C \end{cases},$$

where $-2 < \varphi < 0$. We will relax the assumption that threshold value,C,is known. We derive the asymptotic results that can be used to establish the limit distribution of the Dickey-Fuller t-test for $H_0 : \varphi = 0$ against the alternative of $H_1 : -2 < \varphi < 0$ that has been optimized over the parameter C that is unidentified under the null hypothesis.

CHAPTER 2

Asymptotics for scaled periodic transformations of integrated time series

2.1 Introduction

Nonstationary time series have been attractive for recent research in econometrics. The applications of nonlinear transformation of nonstationary time series have been of major interest in international economics and macroeconomics. Although a lot of macroeconomic models had used nonlinear transformations for some time series data, the transformed data properties do not totally understand by econometricians. The first paper to investigate this question was Granger and Hallman (1991). Granger and Hallman (1991) used the Monte Carlo method to investigate the relationship in nonlinear transformations of nonstationary time series. They concluded that the stationarity of nonlinear transformation depends on functional forms. After Granger and Hallman's breakthrough research, Ermini and Granger (1993) investigate the variances, covariances and high monent conditions under transformed data series with

Gaussianity, but they did not build the limit theory under nonlinear transformations of I(1) processes. In recent paper, Park and Phillips(1999) established the limit distribution of the form.

$$a_n \sum_{t=1}^{n} T(x_t)$$

where $x_t = x_0 + \sum_{j=1}^{t} \varepsilon_j$, x_0 is an arbitrary random variable that is independent of all other ε_t, the ε_j satisfy a weak dependence condition, $x_t \in \mathbb{R}$, a_n is a proper scaling factor such that $a_n \to 0$ as $n \to \infty$, and $T(.)$ is a transformation of the integrated process x_t that is allowed to be within one of three function classes: integrable functions, the asymptotically homogeneous functions and the explosive functions. After Park and Phillips, de Jong (2000) extended Park and Phillips original results to periodic transformations on nonstationary time series. De Jong considered continuously differentiable periodic functions and concluded that the periodic nature of the trigonometric functions effectively "reduces" the dependence in the integrated process to a point at which a central limit theorem holds.

In this chapter, we extend the result in de Jong (2001). We use a scaled integrated process $n^{-\alpha} x_t$ instead of x_t in de Jong. We use a martingale approximation and a Fourier series expansion result to obtain the main theorem about periodic transformations for scaled integrated process. Compared with the main results in de Jong (2001), we can find that Theorem 1 of de Jong (2001) is a special case of our general results.

2.2 Assumptions and main result

We consider a time series x_t generated by

$$x_t = x_{t-1} + \varepsilon_t \tag{2.1}$$

in which ε_t is a sequence of independent and identical distributed random variables with mean zero and variance σ^2. $F_t = \Omega(\varepsilon_t, \varepsilon_{t-1}, \ldots, \varepsilon_1, x_0)$ is a sigma field including the information until time period t. Other assumptions will be made throughout this paper.

Assumption 2.1 *ε_t has a symmetric distribution with $E(\varepsilon_t) = 0$ and $Var(\varepsilon_t) = \sigma^2$.*

Assumption 2.2 $E|\varepsilon_t|^5 < \infty$.

Using these assumptions, we can obtain the useful lemmas as below.

Lemma 2.1 *For the process x_t defined before, if ε_t satisfies Assumption 2.1 and 2.2 with $0 < \alpha < 1/4$ and for any $\zeta \in \mathbb{R}, \zeta \neq 0$, then*

$$\left| n^{-\alpha-1/2}\sum_{t=1}^{n} \sin(n^{-\alpha}\zeta x_t) - (2\zeta^{-2}\sigma^{-2})n^{\alpha-1/2}\sum_{t=1}^{n} \left(\sin(n^{-\alpha}\zeta x_t) - E(\sin(n^{-\alpha}\zeta x_t)|F_{t-1})\right)\right|$$
$$= o_p(1).$$

and

$$\left| n^{-\alpha-1/2}\sum_{t=1}^{n} \cos(n^{-\alpha}\zeta x_t) - (2\zeta^{-2}\sigma^{-2})n^{\alpha-1/2}\sum_{t=1}^{n} \left(\cos(n^{-\alpha}\zeta x_t) - E(\cos(n^{-\alpha}\zeta x_t)|F_{t-1})\right)\right|$$
$$= o_p(1).$$

Lemma 2.2 *For the process x_t defined before, if ε_t satisfies Assumption 2.1 and 2.2 with $0 < \alpha < 1/4$ and for any $\zeta, \gamma \in \mathbb{R}, \zeta, \gamma \neq 0$, then for $\gamma = \zeta$*

$$n^{2\alpha-1}\sum_{t=1}^{n} E\{[\sin(n^{-\alpha}\zeta x_t) - E(\sin(n^{-\alpha}\zeta x_t)|F_{t-1})]^2|F_{t-1}\} \xrightarrow{p} (1/2)(\zeta\sigma)^2,$$

and for $\gamma \neq \zeta$

$$n^{2\alpha-1}\sum_{t=1}^{n} E\{[\sin(n^{-\alpha}\gamma x_t) - E(\sin(n^{-\alpha}\gamma x_t)|F_{t-1})]$$
$$\times[\sin(n^{-\alpha}\zeta x_t) - E(\sin(n^{-\alpha}\zeta x_t)|F_{t-1})]|F_{t-1}\} \xrightarrow{p} 0.$$

With the same method, we can also obtain a lemma about the cosine function.

Lemma 2.3 *For the process x_t as defined before, if ε_t satisfies Assumption 2.1 and 2.2 with $0 < \alpha < 1/4$ and for any $\zeta, \gamma \in \mathbb{R}, \zeta, \gamma \neq 0$, then for $\gamma = \zeta$*

$$n^{2\alpha-1}\sum_{t=1}^{n} E\{[\cos(n^{-\alpha}\zeta x_t) - E(\cos(n^{-\alpha}\zeta x_t)|F_{t-1})]^2|F_{t-1}\}\xrightarrow{p}(1/2)(\zeta\sigma)^2,$$

and for $\gamma \neq \zeta$

$$n^{2\alpha-1}\sum_{t=1}^{n} E\{[\cos(n^{-\alpha}\gamma x_t) - E(\cos(n^{-\alpha}\gamma x_t)|F_{t-1})]$$

$$[\cos(n^{-\alpha}\zeta x_t) - E(\cos(n^{-\alpha}\zeta x_t)|F_{t-1})]|F_{t-1}\}\xrightarrow{p}0.$$

From these three lemmas, we can build the limit distribution of the periodic transformation of rescaled integrated process.

Theorem 2.1 *For the process x_t defined before, if ε_t satisfies Assumption 2.1, 2.2 and $0 < \alpha < 1/4$, then*

$$(n^{-\alpha-\frac{1}{2}}\sum_{t=1}^{n}\sin(\frac{\zeta_1 x_t}{n^\alpha}), \ldots, n^{-\alpha-\frac{1}{2}}\sum_{t=1}^{n}\sin(\frac{\zeta_m x_t}{n^\alpha}), n^{-\alpha-\frac{1}{2}}\sum_{t=1}^{n}\cos(\frac{\zeta_1 x_t}{n^\alpha}),$$

$$\ldots, n^{-\alpha-\frac{1}{2}}\sum_{t=1}^{n}\cos(\frac{\zeta_m x_t}{n^\alpha}))' \xrightarrow{d} N(0, A)$$

where A is a $2m \times 2m$ matrix with diagonal elements $(2(\zeta_1\sigma)^{-2}, \ldots 2(\zeta_m\sigma)^{-2}, 2(\zeta_1\sigma)^{-2} \ldots 2(\zeta_m\sigma)^{-2})$. The other elements are zero.

From Theorem 2.1, we obtain two main results. First, we can find that the limit variances depend on the square of rescaled parameter ζ_j. When ζ_j is large, the limit variance is small. Second, when $\alpha = 0$, the result of Theorem 2.1 will be equal with Theorem 1 of de Jong (2001). From our result, we can find Theorem 2.1 extends the result obtained from de Jong. From Theorem 2.1, we can obtain the following corollary.

Corollary 2.1 *For the process x_t defined before, if ε_t satisfies Assumption 2.1 and 2.2 and $0 < \alpha < 1/4$, then*

$$n^{-\alpha-1/2}\sum_{t=1}^{n}\sin(n^{-\alpha}\zeta x_t)\xrightarrow{d}N(0,2(\zeta\sigma)^{-2}) \quad \textit{and}$$

$$n^{-\alpha-1/2}\sum_{t=1}^{n}\cos(n^{-\alpha}\zeta x_t)\xrightarrow{d}N(0,2(\zeta\sigma)^{-2}).$$

From these results, we can find periodic transformations decrease the dependence in scaled nonstationary time series. In fact, this result support Granger and Hallman's conclusion that periodic transformation of I(1) process is stationary series. But the variance forms of the limit distribution will depend on scaled factors ζ_i. Using these results with Fourier series concepts, we can extend our results to more general result about periodic transformations of scaled integrated process.

Theorem 2.2 *For the process x_t defined before, assuming that ε_t is an i.i.d. sequence of random variables satisfied Assumption 2.1, 2.2, and assuming that $T(.)$ is continuously differentiable and periodic on $[-\pi,\pi]$ and $0 \le \alpha < 1/4$, we have*

$$n^{-\alpha-1/2}\sum_{t=1}^{n}(T(n^{-\alpha}x_t)-\mu)\xrightarrow{d}N(0,2\sigma^{-2}\sum_{j=1}^{\infty}(j^{-2})(a_j^2+b_j^2)). \quad j \ge 0$$

where

$$\mu = 2\pi^{-2}\int_{-\pi}^{\pi}T(x)dx \quad a_j = \pi^{-1}\int_{-\pi}^{\pi}\cos(jx_t)T(x)dx \quad \textit{and}$$

$$b_j = \pi^{-1}\int_{-\pi}^{\pi}\sin(jx_t)T(x)dx.$$

An possible extension of the above results is to the case of asymptotic distribution of ε_t. But from the present proof, it is far from clear how to go about to estabilish such the results.

2.3 Conclusions and possible extensions

In this chapter, we established the limit distribution for summations of continuously differentiable periodic functions of scaled integrated process. We use scaled I(1) processes $n^{-\alpha}x_t$ instead of x_t in de Jong (2000). We can obtain more general result for limit theory of periodic transformations of integrated time series under $0 < \alpha < 1/4$ Even though we obtain more general result, but these results still depend on ε_t must be an i.i.d. and symmetric distribution. From these results, we can build the limit behavior of regression under periodic transformations of I(1) processes. For example, we can establish the behavior of least squares estimator $\hat{b}$ without intercept in the model

$$y_t = bT(n^{-\alpha}x_t) + u_t.$$

Where u_t is a martingale difference sequence of random variable with respect to the sigma field and $T(.)$ is a periodic function. The least square estimator $\hat{b}$ is equal to

$$\hat{b} = (\sum_{t=1}^{n} T(n^{-\alpha}x_t)^2)^{-1}(\sum_{t=1}^{n} T(n^{-\alpha}x_t)y_t) = b + (\sum_{t=1}^{n} T(n^{-\alpha}x_t)^2)^{-1}(\sum_{t=1}^{n} T(n^{-\alpha}x_t)u_t).$$

For the denominator of the least square estimator is the periodic function. We can use the theorem we developed to build the asymptotic properties. About the numerator of least square estimator, we need to analyze the property of $\sum_{t=1}^{n} T(n^{-\alpha}x_t)u_t$. This term is a summation of martingale difference equation. If $E(u_t^2|F_{t-1}) = Eu_t^2$, the asymptotic normality holds for $\sqrt{n}(\hat{b} - b)$. This result enlarge the original result from de Jong.

2.4 Mathematical Appendix

For the proofs of Lemma 2.1 and 2.2, we need the following lemmas.

Lemma 2.4 *For the process x_t as defined above, under Assumption 2.1 and 2.2,*

$$\left|E\left(\sin\left(n^{-\alpha}\zeta x_t\right)|F_{t-1}\right) - (1-(1/2)n^{-2\alpha}(\zeta\sigma)^2)\sin(n^{-\alpha}\zeta x_{t-1})\right| \leq (1/24)\zeta^4 E|\varepsilon_t|^4 n^{-4\alpha}.$$

and

$$\left|E\left(\cos\left(n^{-\alpha}\zeta x_t\right)|F_{t-1}\right) - (1-(1/2)n^{-2\alpha}(\zeta\sigma)^2)\cos(n^{-\alpha}\zeta x_{t-1})\right| \leq (1/24)\zeta^4 E|\varepsilon_t|^4 n^{-4\alpha}.$$

Proof of Lemma 2.4:

From conditional expectation definition and the identity $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$,

$$\left|E\left(\sin\left(n^{-\alpha}\zeta x_t\right)|F_{t-1}\right) - (1-(1/2)n^{-2\alpha}(\zeta\sigma)^2)\sin(n^{-\alpha}\zeta x_{t-1})\right|$$

$$= |\sin(n^{-\alpha}\zeta x_{t-1})E\cos(n^{-\alpha}\zeta\varepsilon_t) + \cos(n^{-\alpha}\zeta x_{t-1})E\sin(n^{-\alpha}\zeta\varepsilon_t)$$

$$-(1-(1/2)n^{-2\alpha}(\zeta\sigma)^2)\sin(n^{-\alpha}\zeta x_{t-1})|$$

$$= \left|\sin(n^{-\alpha}\zeta x_{t-1})(E\cos(n^{-\alpha}\zeta\varepsilon_t) - 1 + (1/2)n^{-2\alpha}(\zeta\sigma)^2) + \cos(n^{-\alpha}\zeta x_{t-1})E\sin(n^{-\alpha}\zeta\varepsilon_t)\right|$$

$$\leq \left|\sin(n^{-\alpha}\zeta x_{t-1})(E\cos(n^{-\alpha}\zeta\varepsilon_t) - 1 + (1/2)n^{-2\alpha}(\zeta\sigma)^2)\right| + \left|\cos(n^{-\alpha}\zeta x_{t-1})E\sin(n^{-\alpha}\zeta\varepsilon_t)\right|$$

$$\leq |\sin(n^{-\alpha}\zeta x_{t-1})||E\cos(n^{-\alpha}\zeta\varepsilon_t) - 1 + (1/2)n^{-2\alpha}(\zeta\varepsilon_t)^2| + |\cos(n^{-\alpha}\zeta x_{t-1}||E\sin(n^{-\alpha}\zeta\varepsilon_t)|.$$

By the inequality $|\cos(x) - 1 - x^2| \leq (1/24)x^4$ and $|\sin(x) - x| \leq (1/6)x^3$ with Assumption 2.1, it follows that

$$\left|E\cos(n^{-\alpha}\zeta\varepsilon_t) - (1-(1/2)n^{-2\alpha}(\zeta\sigma)^2)\right| \leq (1/24)\zeta^4 E|\varepsilon_t|^4 n^{-4\alpha} \tag{2.2}$$

Because $0 < \sin(n^{-\alpha}\zeta x_t) < 1$ and $0 < \cos(n^{-\alpha}\zeta x_t) < 1$, it follows that

$$\left|E\left(\sin\left(n^{-\alpha}\zeta x_t\right)|F_{t-1}\right) - (1-(1/2)n^{-2\alpha}(\zeta\sigma)^2)\sin(n^{-\alpha}\zeta x_{t-1})\right|$$

$$\leq \left|E\cos(n^{-\alpha}\zeta\varepsilon_t) - 1 + (1/2)n^{-2\alpha}(\zeta\sigma)^2\right| + \left|E\sin(n^{-\alpha}\zeta\varepsilon_t)\right| \leq (1/24)\zeta^4 E\left|\varepsilon_t\right|^4 n^{-4\alpha}.$$

where the last inequality uses the following Equation (2.2). Using the same terminology, we can obtain another result.

$$\left|E\left(\cos\left(n^{-\alpha}\zeta x_t\right)|F_{t-1}\right) - (1-(1/2)n^{-2\alpha}(\zeta\sigma)^2)\cos(n^{-\alpha}\zeta x_{t-1})\right|$$

$$\le \left|E\cos(n^{-\alpha}\zeta\varepsilon_t) - 1 + (1/2)n^{-2\alpha}(\zeta\sigma)^2\right| + \left|E\sin(n^{-\alpha}\zeta\varepsilon_t)\right| \le (1/24)\zeta^4 E|\varepsilon_t|^4 n^{-4\alpha}.$$

□

Lemma 2.5 *Let ε_t satisfy Assumption 2.1 and 2.2 with $0 < \alpha < 1/4$ Then*

$$E\left[n^{2\alpha-1}\sum_{t=1}^{n}(\sin(n^{-\alpha}\gamma x_t)\sin(n^{-\alpha}\zeta x_t) - E(\sin(n^{-\alpha}\gamma x_t)\sin(n^{-\alpha}\zeta x_t)|F_{t-1}))\right]^2$$
$$= o(1)$$

and

$$E\left[n^{2\alpha-1}\sum_{t=1}^{n}(\cos(n^{-\alpha}\gamma x_t)\cos(n^{-\alpha}\zeta x_t) - E(\cos(n^{-\alpha}\gamma x_t)\cos(n^{-\alpha}\zeta x_t)|F_{t-1}))\right]^2$$
$$= o(1).$$

Proof of Lemma 2.5:

First, we note that by the martingale difference property of the summands,

$$E\left[n^{2\alpha-1}\sum_{t=1}^{n}(\sin(n^{-\alpha}\gamma x_t)\sin(n^{-\alpha}\zeta x_t) - E(\sin(n^{-\alpha}\gamma x_t)\sin(n^{-\alpha}\zeta x_t)|F_{t-1}))\right]^2$$

$$= n^{4\alpha-2}\sum_{t=1}^{n}E\left[(\sin(n^{-\alpha}\gamma x_t)\sin(n^{-\alpha}\zeta x_t) - E(\sin(n^{-\alpha}\gamma x_t)\sin(n^{-\alpha}\zeta x_t)|F_{t-1}))\right]^2. \quad (2.3)$$

Because $\sin(n^{-\alpha}\gamma x_t)\sin(n^{-\alpha}\zeta x_t)$ and $E(\sin(n^{-\alpha}\gamma x_t)\sin(n^{-\alpha}\zeta x_t)|F_{t-1})$ take their values in [0,1], it follows that

$$|\sin(n^{-\alpha}\gamma x_t)\sin(n^{-\alpha}\zeta x_t) - E(\sin(n^{-\alpha}\gamma x_t)\sin(n^{-\alpha}\zeta x_t)|F_{t-1})| \le 2. \quad (2.4)$$

From Equation (2.3) and (2.4), it now follows that.

$$E\left[n^{2\alpha-1}\sum_{t=1}^{n}(\sin(n^{-\alpha}\gamma x_t)\sin(n^{-\alpha}\zeta x_t) - E(\sin(n^{-\alpha}\gamma x_t)\sin(n^{-\alpha}\zeta x_t)|F_{t-1}))\right]^2$$

$$\leq 4n^{4\alpha-1}. \qquad (2.5)$$

From the assumption $0 < \alpha < 1/4$ and Equation (2.5), when $n \to \infty$, the result follows. Similarly,

$$E\left[n^{2\alpha-1}\sum_{t=1}^{n}(\cos(n^{-\alpha}\gamma x_t)\cos(n^{-\alpha}\zeta x_t) - E(\cos(n^{-\alpha}\gamma x_t)\cos(n^{-\alpha}\zeta x_t)|F_{t-1}))\right]^2 \to 0.$$

□

Lemma 2.6 *Under Assumption 2.1 and 2.2, if* $0 < \alpha < 1/4$ *and for any* $\zeta, \gamma \in \mathbb{R}, \zeta, \gamma \neq 0$, *then for* $\gamma = \zeta$ $\gamma, \zeta \neq 0$

$$(1/n)\sum_{t=1}^{n}\sin^2(n^{-\alpha}\zeta x_t)\xrightarrow{p}(1/2) \quad \textit{and} \quad (1/n)\sum_{t=1}^{n}\cos^2(n^{-\alpha}\zeta x_t)\xrightarrow{p}(1/2).$$

and for $\gamma \neq \zeta$ $\gamma, \zeta \neq 0$

$$(1/n)\sum_{t=1}^{n}\sin(n^{-\alpha}\gamma x_t)\sin(n^{-\alpha}\zeta x_t)\xrightarrow{p}0 \quad \textit{and} \quad (1/n)\sum_{t=1}^{n}\cos(n^{-\alpha}\gamma x_t)\cos(n^{-\alpha}\zeta x_t)\xrightarrow{p}0.$$

Proof of Lemma 2.6:

1. First, from Lemma 2.5, it follows that

$$E\left[n^{2\alpha-1}\sum_{t=1}^{n}(\sin^2(n^{-\alpha}\zeta x_t) - E(\sin^2(n^{-\alpha}\zeta x_t)|F_{t-1}))\right]^2 = o(1).$$

implying that:

$$\left|n^{2\alpha-1}\sum_{t=1}^{n}\left[\sin^2(n^{-\alpha}\zeta x_t) - E(\sin^2(n^{-\alpha}\zeta x_t)|F_{t-1})\right]\right| = o_p(1). \qquad (2.6)$$

Second, from the definition of x_t, we can write $\sin^2(n^{-\alpha}\zeta x_t)$ as below:

$$\sin^2(n^{-\alpha}\zeta x_t) = \left(\sin(n^{-\alpha}\zeta x_{t-1})\cos(n^{-\alpha}\zeta\varepsilon_t) + \cos(n^{-\alpha}\zeta x_{t-1})\sin(n^{-\alpha}\zeta\varepsilon_t)\right)^2$$

$$= \sin^2(n^{-\alpha}\zeta x_{t-1})\cos^2(n^{-\alpha}\zeta\varepsilon_t) + \cos^2(n^{-\alpha}\zeta x_{t-1})\sin^2(n^{-\alpha}\zeta\varepsilon_t)$$

$$+2\sin(n^{-\alpha}\zeta x_{t-1})\cos(n^{-\alpha}\zeta x_{t-1})\sin(n^{-\alpha}\zeta\varepsilon_t)\cos(n^{-\alpha}\zeta\varepsilon_t). \tag{2.7}$$

Under Assumption 2.1, using independence and the symmetry of the distribution of ε_t, the conditional expectation from Equation (2.7), $E(\sin^2(n^{-\alpha}\zeta x_t)|F_{t-1})$, is

$$E(\sin^2(n^{-\alpha}\zeta x_t)|F_{t-1})$$

$$= \sin^2(n^{-\alpha}\zeta x_{t-1})E\cos^2(n^{-\alpha}\zeta\varepsilon_t) + \cos^2(n^{-\alpha}\zeta x_{t-1})E\sin^2(n^{-\alpha}\zeta\varepsilon_t). \tag{2.8}$$

Substituting (2.8) into (2.6), we can obtain result as below:

$$|n^{2\alpha-1}\sum_{t=1}^{n}\sin^2(n^{-\alpha}\zeta x_t)$$

$$-n^{2\alpha-1}\sum_{t=1}^{n}\left(\sin^2(n^{-\alpha}\zeta x_{t-1})E\cos^2(n^{-\alpha}\zeta\varepsilon_t) + \cos^2(n^{-\alpha}\zeta x_{t-1})E\sin^2(n^{-\alpha}\zeta\varepsilon_t)\right)|$$

$$= o_p(1). \tag{2.9}$$

Also note that

$$\cos^2(n^{-\alpha}\zeta x_t) = 1 - \sin^2(n^{-\alpha}\zeta x_t). \tag{2.10}$$

Combining (2.9) and (2.10), we can obtain the equation

$$|n^{2\alpha-1}\sum_{t=1}^{n}\sin^2(n^{-\alpha}\zeta x_t)$$

$$-n^{2\alpha-1}\sum_{t=1}^{n}\left((E\cos^2(n^{-\alpha}\zeta\varepsilon_t) - E\sin^2(n^{-\alpha}\zeta\varepsilon_t))\sin^2(n^{-\alpha}\zeta x_{t-1}) + E\sin^2(n^{-\alpha}\zeta\varepsilon_t)\right)|$$

$$= o_p(1). \tag{2.11}$$

From Equation (2.11), it implies

$$n^{2\alpha-1}\sum_{t=1}^{n}\sin^2(n^{-\alpha}\zeta x_t)$$

$$= n^{2\alpha-1}\sum_{t=1}^{n}\left((E\cos^2(n^{-\alpha}\zeta\varepsilon_t) - E\sin^2(n^{-\alpha}\zeta\varepsilon_t))\sin^2(n^{-\alpha}\zeta x_{t-1}) + E\sin^2(n^{-\alpha}\zeta\varepsilon_t)\right)$$

$$+o_p(1). \tag{2.12}$$

Also, we have equality:

$$\sum_{t=1}^{n}\sin^2(n^{-\alpha}\zeta x_{t-1}) = \sum_{t=1}^{n}\sin^2(n^{-\alpha}\zeta x_t)+\sin^2(n^{-\alpha}\zeta x_0)-\sin^2(n^{-\alpha}\zeta x_n). \tag{2.13}$$

and

$$E\sin^2(n^{-\alpha}\zeta\varepsilon_t) + E\cos^2(n^{-\alpha}\zeta\varepsilon_t) = 1. \tag{2.14}$$

Substituting (2.13) and (2.14) into (2.12), we can rewrite Equation (2.12) as below:

$$\left[2E\sin^2(n^{-\alpha}\zeta\varepsilon_t)\right]n^{2\alpha-1}\sum_{t=1}^{n}\sin^2(n^{-\alpha}\zeta x_t) = n^{2\alpha}\left(E\sin^2(n^{-\alpha}\zeta\varepsilon_t)\right) + o_p(1).$$

By Taylor expansion, we can obtain the inequality

$$\left|n^{2\alpha}E\sin^2(n^{-\alpha}\zeta\varepsilon_t) - \zeta^2\sigma^2\right| \le (1/3)\zeta^4 E\left|\varepsilon_t\right|^4 n^{-2\alpha}.$$

By this inequality, it suffices to show

$$\left[2(\zeta^2\sigma^2 + O(n^{-2\alpha}))\right]n^{-1}\sum_{t=1}^{n}\sin^2(n^{-\alpha}\zeta x_t) = \zeta^2\sigma^2 + O(n^{-2\alpha}) + o_p(1).$$

After some algebra, we can obtain

$$n^{-1}\sum_{t=1}^{n}\sin^2(n^{-\alpha}\zeta x_t)\xrightarrow{p}1/2.$$

2. For $\gamma \neq \zeta$ $\gamma, \zeta \neq 0$

First, from Lemma 2.5, we obtain that

$$E\left[n^{2\alpha-1}\sum_{t=1}^{n}\left[\sin(n^{-\alpha}\gamma x_t)\sin(n^{-\alpha}\zeta x_t) - E(\sin(n^{-\alpha}\gamma x_t)\sin(n^{-\alpha}\zeta x_t)|F_{t-1})\right]\right]^2$$

$$= o(1).$$

It follows that

$$\left|n^{2\alpha-1}\sum_{t=1}^{n}\left[\sin(n^{-\alpha}\gamma x_t)\sin(n^{-\alpha}\zeta x_t) - E(\sin(n^{-\alpha}\gamma x_t)\sin(n^{-\alpha}\zeta x_t)|F_{t-1})\right]\right|$$

$$= o_p(1). \tag{2.15}$$

Second, from the definition of x_t, we can write $\sin(n^{-\alpha}\gamma x_t)\sin(n^{-\alpha}\zeta x_t)$ as below:

$$\sin(n^{-\alpha}\gamma x_t)\sin(n^{-\alpha}\zeta x_t)$$

$$= \sin(n^{-\alpha}\gamma x_{t-1})\sin(n^{-\alpha}\zeta x_{t-1})\cos(n^{-\alpha}\gamma\varepsilon_t)\cos(n^{-\alpha}\zeta\varepsilon_t)$$

$$+\cos(n^{-\alpha}\gamma x_{t-1})\sin(n^{-\alpha}\zeta x_{t-1})\sin(n^{-\alpha}\gamma\varepsilon_t)\cos(n^{-\alpha}\zeta\varepsilon_t)$$

$$+\sin(n^{-\alpha}\gamma x_{t-1})\cos(n^{-\alpha}\zeta x_{t-1})\cos(n^{-\alpha}\gamma\varepsilon_t)\sin(n^{-\alpha}\zeta\varepsilon_t)$$

$$+\cos(n^{-\alpha}\gamma x_{t-1})\cos(n^{-\alpha}\zeta x_{t-1})\sin(n^{-\alpha}\gamma\varepsilon_t)\sin(n^{-\alpha}\zeta\varepsilon_t). \tag{2.16}$$

Under Assumption 2.1, the conditional from Equation (2.16) is,

$$E(\sin(n^{-\alpha}\gamma x_t)\sin(n^{-\alpha}\zeta x_t)|F_{t-1})$$

$$= \sin(n^{-\alpha}\gamma x_{t-1})\sin(n^{-\alpha}\zeta x_{t-1})E\cos(n^{-\alpha}\gamma\varepsilon_t)\cos(n^{-\alpha}\zeta\varepsilon_t)$$

$$+ \cos(n^{-\alpha}\gamma x_{t-1})\cos(n^{-\alpha}\zeta x_{t-1})E\sin(n^{-\alpha}\gamma\varepsilon_t)\sin(n^{-\alpha}\zeta\varepsilon_t). \quad (2.17)$$

With the same method as first proof, we can obtain the equality as below:

$$n^{2\alpha-1}\sum_{t=1}^{n}\sin(n^{-\alpha}\gamma x_t)\sin(n^{-\alpha}\zeta x_t)$$

$$= E\cos(n^{-\alpha}\gamma\varepsilon_t)\cos(n^{-\alpha}\zeta\varepsilon_t)n^{2\alpha-1}\sum_{t=1}^{n}\sin(n^{-\alpha}\gamma x_{t-1})\sin(n^{-\alpha}\zeta x_{t-1})$$

$$+E\sin(n^{-\alpha}\gamma\varepsilon_t)\sin(n^{-\alpha}\zeta\varepsilon_t)n^{2\alpha-1}\sum_{t=1}^{n}\cos(n^{-\alpha}\gamma x_{t-1})\cos(n^{-\alpha}\zeta x_{t-1})+o_p(1). \quad (2.18)$$

We move first item of the right side in Equation (2.18) to the left side. We obtain the following Equation:

$$n^{2\alpha-1}\sum_{t=1}^{n}\sin(n^{-\alpha}\gamma x_t)\sin(n^{-\alpha}\zeta x_t)$$

$$-E\cos(n^{-\alpha}\gamma\varepsilon_t)\cos(n^{-\alpha}\zeta\varepsilon_t)n^{2\alpha-1}\sum_{t=1}^{n}\sin(n^{-\alpha}\gamma x_{t-1})\sin(n^{-\alpha}\zeta x_{t-1})$$

$$= E\sin(n^{-\alpha}\gamma\varepsilon_t)\sin(n^{-\alpha}\zeta\varepsilon_t)n^{2\alpha-1}\sum_{t=1}^{n}\cos(n^{-\alpha}\gamma x_{t-1})\cos(n^{-\alpha}\zeta x_{t-1}). \quad (2.19)$$

Also, we have equalities:

$$\sum_{t=1}^{n}\sin(n^{-\alpha}\gamma x_{t-1})\sin(n^{-\alpha}\zeta x_{t-1})$$

$$= \sum_{t=1}^{n}\sin(n^{-\alpha}\gamma x_t)\sin(n^{-\alpha}\zeta x_t) + \sin(n^{-\alpha}\gamma x_0)\sin(n^{-\alpha}\zeta x_0)$$

$$- \sin(n^{-\alpha}\gamma x_n)\sin(n^{-\alpha}\zeta x_n). \quad (2.20)$$

$$\sum_{t=1}^{n} \cos(n^{-\alpha}\gamma x_{t-1}) \cos(n^{-\alpha}\zeta x_{t-1})$$

$$= \sum_{t=1}^{n} \cos(n^{-\alpha}\gamma x_t) \cos(n^{-\alpha}\zeta x_t) + \cos(n^{-\alpha}\gamma x_0) \cos(n^{-\alpha}\zeta x_0)$$

$$- \cos(n^{-\alpha}\gamma x_n) \cos(n^{-\alpha}\zeta x_n). \tag{2.21}$$

Substitute (2.20) and (2.21) into (2.19), we can rewrite Equation (2.19) as below:

$$[1 - E\cos(n^{-\alpha}\gamma\varepsilon_t) \cos(n^{-\alpha}\zeta\varepsilon_t)] n^{2\alpha-1} \sum_{t=1}^{n} \sin(n^{-\alpha}\gamma x_t) \sin(n^{-\alpha}\zeta x_t)$$

$$= [E\sin(n^{-\alpha}\gamma\varepsilon_t) \sin(n^{-\alpha}\zeta\varepsilon_t)] n^{2\alpha-1} \sum_{t=1}^{n} \cos(n^{-\alpha}\gamma x_t) \cos(n^{-\alpha}\zeta x_t) + o_p(1). \tag{2.22}$$

By Taylor expansion, we have two inequalities

$$\left| n^{2\alpha-1}[1 - E\cos(n^{-\alpha}\gamma\varepsilon_t) \cos(n^{-\alpha}\zeta\varepsilon_t)] - (\gamma^2 + \zeta^2)\sigma^2 \right| \le (1/4)(\gamma\zeta)^2 E\left|\varepsilon_t^4\right| n^{-2\alpha}.$$

and

$$\left| n^{2\alpha-1}[E\sin(n^{-\alpha}\gamma\varepsilon_t) \sin(n^{-\alpha}\zeta\varepsilon_t)] - \gamma\zeta\sigma^2 \right| \le (1/6)E\left|\gamma\zeta(\gamma^2 + \zeta^2)\varepsilon_t^4\right| n^{-2\alpha}$$

By these inequalities, they suffice to show:

$$[(\gamma^2 + \zeta^2)\sigma^2 + O(n^{-2\alpha})] n^{2\alpha-1} \sum_{t=1}^{n} \sin(n^{-\alpha}\gamma x_t) \sin(n^{-\alpha}\zeta x_t)$$

$$= [\gamma\zeta\sigma^2 + O(n^{-2\alpha})] n^{2\alpha-1} \sum_{t=1}^{n} \cos(n^{-\alpha}\gamma x_t) \cos(n^{-\alpha}\zeta x_t) + o_p(1). \tag{2.23}$$

Using the same trick, we can obtain another equation about $n^{2\alpha-1}\sum_{t=1}^{n} \cos(n^{-\alpha}\gamma x_t) \cos(n^{-\alpha}\zeta x_t)$

$$[(\gamma^2 + \zeta^2)\sigma^2 + O(n^{-2\alpha})] n^{2\alpha-1} \sum_{t=1}^{n} \cos(n^{-\alpha}\gamma x_t) \cos(n^{-\alpha}\zeta x_t)$$

$$= [\gamma\zeta\sigma^2 + O(n^{-2\alpha})]n^{2\alpha-1}\sum_{t=1}^{n}\sin(n^{-\alpha}\gamma x_t)\sin(n^{-\alpha}\zeta x_t) + o_p(1). \quad (2.24)$$

Solving Equations (2.23) and (2.24) simultaneously, we can obtain

$$n^{-1}\sum_{t=1}^{n}\sin(n^{-\alpha}\gamma x_t)\sin(n^{-\alpha}\zeta x_t)\xrightarrow{p}0 \quad \textit{and} \quad n^{-1}\sum_{t=1}^{n}\cos(n^{-\alpha}\gamma x_t)\cos(n^{-\alpha}\zeta x_t)\xrightarrow{p}0.$$

□

Proof of Lemma 2.1:

First, we note that the following identity holds:

$$n^{\alpha-1/2}\sum_{t=1}^{n}\sin(n^{-\alpha}\zeta x_t) = n^{\alpha-1/2}\sum_{t=1}^{n}\big(\sin(n^{-\alpha}\zeta x_t) - E(\sin(n^{-\alpha}\zeta x_t)|F_{t-1})\big)$$

$$+n^{\alpha-1/2}\sum_{t=1}^{n}E(\sin(n^{-\alpha}\zeta x_t)|F_{t-1}). \quad (2.25)$$

By Lemma 2.4, we can rewrite Equation (2.25) as below:

$$n^{\alpha-1/2}\sum_{t=1}^{n}\sin(n^{-\alpha}\zeta x_t) = n^{\alpha-1/2}\sum_{t=1}^{n}\big(\sin(n^{-\alpha}\zeta x_t) - E(\sin(n^{-\alpha}\zeta x_t)|F_{t-1})\big)$$

$$+n^{\alpha-1/2}\sum_{t=1}^{n}\sin(n^{-\alpha}\zeta x_{t-1})E\cos(n^{-\alpha}\zeta\varepsilon_t).$$

We move the second item on the right hand side of the equality to the left and leave the summation of differences on the left side. We obtain,

$$n^{\alpha-1/2}\sum_{t=1}^{n}\sin(n^{-\alpha}\zeta x_t) - n^{\alpha-1/2}\sum_{t=1}^{n}\sin(n^{-\alpha}\zeta x_{t-1})E\cos(n^{-\alpha}\zeta\varepsilon_t)$$

$$= n^{\alpha-1/2}\sum_{t=1}^{n}\big(\sin(n^{-\alpha}\zeta x_t) - E(\sin(n^{-\alpha}\zeta x_t)|F_{t-1})\big) \quad (2.26)$$

Also we have following equality:

$$\sum_{t=1}^{n}\sin(n^{-\alpha}\zeta x_{t-1}) = \sum_{t=1}^{n}\sin(n^{-\alpha}\zeta x_t) + \sin(n^{-\alpha}\zeta x_0) - \sin(n^{-\alpha}\zeta x_n). \quad (2.27)$$

Now plugging in the result of Equation (2.27) into Equation (2.26) gives

$$n^{\alpha-1/2}\sum_{t=1}^{n}\sin(n^{-\alpha}\zeta x_t) - n^{\alpha-1/2}E\cos(n^{-\alpha}\zeta\varepsilon_t)\sum_{t=1}^{n}\sin(n^{-\alpha}\zeta x_t)$$

$$-n^{\alpha-1/2}(E\cos(n^{-\alpha}\zeta\varepsilon_t))(\sin(n^{-\alpha}\zeta x_0) - \sin(n^{-\alpha}\zeta x_n))$$

$$= n^{\alpha-1/2}\sum_{t=1}^{n}\big(\sin(n^{-\alpha}\zeta x_t) - E(\sin(n^{-\alpha}\zeta x_t)|F_{t-1})\big). \tag{2.28}$$

After combination the left side of Equation (2.28), we can obtain

$$n^{\alpha-1/2}(1 - E\cos(n^{-\alpha}\zeta\varepsilon_t))\sum_{t=1}^{n}\sin(n^{-\alpha}\zeta x_t)$$

$$-n^{\alpha-1/2}(E\cos(n^{-\alpha}\zeta\varepsilon_t))(\sin(n^{-\alpha}\zeta x_0) - \sin(n^{-\alpha}\zeta x_n))$$

$$= n^{\alpha-1/2}\sum_{t=1}^{n}\big(\sin(n^{-\alpha}\zeta x_t) - E(\sin(n^{-\alpha}\zeta x_t)|F_{t-1})\big). \tag{2.29}$$

We move the last item on the left side of Equation (2.29) to the right side. After some algebra, we obtain.

$$n^{\alpha-1/2}(1 - E\cos(n^{-\alpha}\zeta\varepsilon_t))\sum_{t=1}^{n}\sin(n^{-\alpha}\zeta x_t)$$

$$= n^{\alpha-1/2}\sum_{t=1}^{n}\big(\sin(n^{-\alpha}\zeta x_t) - E(\sin(n^{-\alpha}\zeta x_t)|F_{t-1})\big)$$

$$+n^{\alpha-1/2}(E\cos(n^{-\alpha}\zeta\varepsilon_t))(\sin(n^{-\alpha}\zeta x_0) - \sin(n^{-\alpha}\zeta x_n)).$$

By Taylor expansion, we can obtain the following inequality.

$$\left|n^{2\alpha}(1 - E\cos(n^{-\alpha}\zeta\varepsilon_t)) - (1/2)(\zeta\sigma)^2\right| \le (1/24)\zeta^4 E|\varepsilon_t|^4 n^{-2\alpha}.$$

By this inequality, it suffices to show.

$$((1/2)(\zeta\sigma)^2 + O(n^{-2\alpha}))n^{-\alpha-1/2}\sum_{t=1}^{n}\sin(n^{-\alpha}\zeta x_t)$$

$$= n^{\alpha-1/2}\sum_{t=1}^{n}\big(\sin(n^{-\alpha}\zeta x_t) - E(\sin(n^{-\alpha}\zeta x_t)|F_{t-1})\big)$$

$$+n^{\alpha-1/2}(1-(1/2)n^{-2\alpha}(\zeta\sigma)^2+O(n^{-4\alpha}))(\sin(n^{-\alpha}\zeta x_0)-\sin(n^{-\alpha}\zeta x_n)).$$

By rearranging terms and boundedness of summands,

$$n^{-\alpha-1/2}\sum_{t=1}^{n}\sin(n^{-\alpha}\zeta x_t)=(2(\zeta\sigma)^{-2})n^{\alpha-1/2}\sum_{t=1}^{n}\big(\sin(n^{-\alpha}\zeta x_t)-E(\sin(n^{-\alpha}\zeta x_t)|F_{t-1})\big)$$

$$+o_p(1).$$

□

Proof of Lemma 2.2:

1. First, by the law of iterated expectation, it follows that

$$n^{2\alpha-1}\sum_{t=1}^{n}E\{[\sin(n^{-\alpha}\zeta x_t)-(E\sin(n^{-\alpha}\zeta x_t)|F_{t-1})]^2|F_{t-1}\}$$

$$=n^{2\alpha-1}\sum_{t=1}^{n}\{E(\sin^2(n^{-\alpha}\zeta x_t)|F_{t-1})-E(\sin(n^{-\alpha}\zeta x_t)|F_{t-1})^2\}.$$

From the definitions and assumptions of x_t, this statistic can be rewritten as

$$n^{2\alpha-1}\sum_{t=1}^{n}\{E(\sin^2(n^{-\alpha}\zeta x_t)|F_{t-1})-(E\sin(n^{-\alpha}\zeta x_t)|F_{t-1})^2\}$$

$$=n^{2\alpha-1}\sum_{t=1}^{n}E[\sin^2(n^{-\alpha}\zeta x_{t-1})\cos^2(n^{-\alpha}\zeta\varepsilon_t)+\cos^2(n^{-\alpha}\zeta x_{t-1})\sin^2(n^{-\alpha}\zeta\varepsilon_t)$$

$$+2\sin(n^{-\alpha}\zeta x_{t-1})\cos(n^{-\alpha}\zeta x_{t-1})\sin(n^{-\alpha}\zeta\varepsilon_t)\cos(n^{-\alpha}\zeta\varepsilon_t)|F_{t-1}]$$

$$-[\sin^2(n^{-\alpha}\zeta x_{t-1})(E\cos(n^{-\alpha}\zeta\varepsilon_t))^2+\cos^2(n^{-\alpha}\zeta x_{t-1})(E\sin(n^{-\alpha}\zeta\varepsilon_t))^2$$

$$+2\sin(n^{-\alpha}\zeta x_{t-1})\cos(n^{-\alpha}\zeta x_{t-1})E\sin(n^{-\alpha}\zeta\varepsilon_t)E\cos(n^{-\alpha}\zeta\varepsilon_t)]\}. \qquad (2.30)$$

The conditional expectation of Equation (2.30) is:

$$n^{2\alpha-1}\sum_{t=1}^{n}\{E(\sin^2(n^{-\alpha}\zeta x_t)|F_{t-1})-(E\sin(n^{-\alpha}\zeta x_t)|F_{t-1})^2\}$$

$$= n^{2\alpha-1}\sum_{t=1}^{n}\{[\sin^2(n^{-\alpha}\zeta x_{t-1})E\cos^2(n^{-\alpha}\zeta\varepsilon_t) + \cos^2(n^{-\alpha}\zeta x_{t-1})E\sin^2(n^{-\alpha}\zeta\varepsilon_t)$$

$$+2\sin(n^{-\alpha}\zeta x_{t-1})\cos(n^{-\alpha}\zeta x_{t-1})E\sin(n^{-\alpha}\zeta\varepsilon_t)\cos(n^{-\alpha}\zeta\varepsilon_t)]$$

$$-[\sin^2(n^{-\alpha}\zeta x_{t-1})(E\cos(n^{-\alpha}\zeta\varepsilon_t))^2 + \cos^2(n^{-\alpha}\zeta x_{t-1})(E\sin(n^{-\alpha}\zeta\varepsilon_t))^2$$

$$+2\sin(n^{-\alpha}\zeta x_{t-1})\cos(n^{-\alpha}\zeta x_{t-1})E\sin(n^{-\alpha}\zeta\varepsilon_t)E\cos(n^{-\alpha}\zeta\varepsilon_t)]\}.$$

From Assumption 2.1 and 2.2, the odd moments of ε_t are equal to zero. We can rewrite original equation as below:

$$n^{2\alpha-1}\sum_{t=1}^{n}\{E(\sin^2(n^{-\alpha}\zeta x_t)|F_{t-1}) - (E(\sin(n^{-\alpha}\zeta x_t)|F_{t-1})^2\}$$

$$= n^{2\alpha-1}\sum_{t=1}^{n}\cos^2(n^{-\alpha}\zeta x_{t-1})E\sin^2(n^{-\alpha}\zeta\varepsilon_t) + o_p(1)$$

$$= n^{-1}\sum_{t=1}^{n}\cos^2(n^{-\alpha}\zeta x_{t-1})(\zeta^2\sigma^2 + O(n^{-2\alpha})) + o_p(1)$$

$$= n^{-1}(\zeta\sigma)^2\sum_{t=1}^{n}\cos^2(n^{-\alpha}\zeta x_{t-1}) + o_p(1).$$

By Lemma 2.6,

$$n^{2\alpha-1}\sum_{t=1}^{n}\{E(\sin^2(n^{-\alpha}\zeta x_t)|F_{t-1}) - (E(\sin(n^{-\alpha}\zeta x_t)|F_{t-1})^2\}$$

$$= n^{-1}(\zeta\sigma)^2\sum_{t=1}^{n}\cos^2(n^{-\alpha}\zeta x_{t-1}) + o_p(1) = (1/2)(\zeta\sigma)^2 + o_p(1).$$

2. For $\gamma \neq \zeta$

First, by the law of iterated expectation, it follows that

$$n^{2\alpha-1}\sum_{t=1}^{n}\{[\sin(n^{-\alpha}\gamma x_t)\sin(n^{-\alpha}\zeta x_t) - E(\sin(n^{-\alpha}\gamma x_t)|F_{t-1})E(\sin(n^{-\alpha}\zeta x_t)|F_{t-1})]|F_{t-1}\}$$

$$= n^{2\alpha-1}\sum_{t=1}^{n}\{E(\sin(n^{-\alpha}\gamma x_t)\sin(n^{-\alpha}\zeta x_t)|F_{t-1})$$

$$-E(\sin(n^{-\alpha}\gamma x_t)|F_{t-1})E(\sin(n^{-\alpha}\zeta x_t)|F_{t-1})\}$$

From the definition of x_t, this statistic can be rewritten as

$$n^{2\alpha-1}\sum_{t=1}^{n}\{E(\sin(n^{-\alpha}\gamma x_t)\sin(n^{-\alpha}\zeta x_t)|F_{t-1})$$

$$-E(\sin(n^{-\alpha}\gamma x_t)|F_{t-1})E(\sin(n^{-\alpha}\zeta x_t)|F_{t-1})\}$$

$$= n^{2\alpha-1}\sum_{t=1}^{n}\sin(n^{-\alpha}\gamma x_{t-1})\sin(n^{-\alpha}\zeta x_{t-1})E\cos(n^{-\alpha}\gamma\varepsilon_t)\cos(n^{-\alpha}\zeta\varepsilon_t)$$

$$+\sin(n^{-\alpha}\gamma x_{t-1})\cos(n^{-\alpha}\zeta x_{t-1})E\cos(n^{-\alpha}\gamma\varepsilon_t)\sin(n^{-\alpha}\zeta\varepsilon_t)$$

$$+\cos(n^{-\alpha}\gamma x_{t-1})\sin(n^{-\alpha}\zeta x_{t-1})E\sin(n^{-\alpha}\gamma\varepsilon_t)\cos(n^{-\alpha}\zeta\varepsilon_t)$$

$$+\cos(n^{-\alpha}\gamma x_{t-1})\cos(n^{-\alpha}\zeta x_{t-1})E\sin(n^{-\alpha}\gamma\varepsilon_t)\sin(n^{-\alpha}\zeta\varepsilon_t)$$

$$-[\sin(n^{-\alpha}\gamma x_{t-1})\sin(n^{-\alpha}\zeta x_{t-1})E\cos(n^{-\alpha}\gamma\varepsilon_t)E\cos(n^{-\alpha}\zeta\varepsilon_t)$$

$$+\sin(n^{-\alpha}\gamma x_{t-1})\cos(n^{-\alpha}\zeta x_{t-1})E\cos(n^{-\alpha}\gamma\varepsilon_t)E\sin(n^{-\alpha}\zeta\varepsilon_t)$$

$$+\cos(n^{-\alpha}\gamma x_{t-1})\sin(n^{-\alpha}\zeta x_{t-1})E\sin(n^{-\alpha}\gamma\varepsilon_t)E\cos(n^{-\alpha}\zeta\varepsilon_t)$$

$$+\cos(n^{-\alpha}\gamma x_{t-1})\cos(n^{-\alpha}\zeta x_{t-1})E\sin(n^{-\alpha}\gamma\varepsilon_t)E\sin(n^{-\alpha}\zeta\varepsilon_t)]\}$$

From Assumption 2.1 and 2.2, the odd moments of ε_t are equal to zero. We can obtain:

$$n^{2\alpha-1}\sum_{t=1}^{n}\{E(\sin(n^{-\alpha}\gamma x_t)\sin(n^{-\alpha}\zeta x_t)|F_{t-1}) - E(\sin(n^{-\alpha}\gamma x_t)|F_{t-1})E(\sin(n^{-\alpha}\zeta x_t)|F_{t-1})\}$$

$$= n^{2\alpha-1}\sum_{t=1}^{n}\{\cos(n^{-\alpha}\gamma x_{t-1})\cos(n^{-\alpha}\zeta x_{t-1})E\sin(n^{-\alpha}\gamma\varepsilon_t)E\sin(n^{-\alpha}\zeta\varepsilon_t)\} + o_p(1)$$

Because $-1 \leq \cos(n^{-\alpha}\gamma x_{t-1})\cos(n^{-\alpha}\zeta x_{t-1}) \leq 1$, it implies

$$n^{2\alpha-1}\sum_{t=1}^{n}\{E(\sin(n^{-\alpha}\gamma x_t)\sin(n^{-\alpha}\zeta x_t)|F_{t-1})$$

$$-E(\sin(n^{-\alpha}\gamma x_t)|F_{t-1})E(\sin(n^{-\alpha}\zeta x_t)|F_{t-1})\}$$

$$= (\zeta\sigma^2 + O(n^{-2\alpha}))n^{-1}\sum_{t=1}^{n}\{\cos(n^{-\alpha}\gamma x_{t-1})\cos(n^{-\alpha}\zeta x_{t-1})\} + o_p(1).$$

By the Lemma 2.6, we know

$$n^{-1}\sum_{t=1}^{n}\cos(n^{-\alpha}\gamma x_{t-1})\cos(n^{-\alpha}\zeta x_{t-1}) = 0 \quad \text{under } \gamma \neq \zeta \text{ and } \gamma, \zeta \geq 0$$

We can obtain the result as below:

$$n^{2\alpha-1}\sum_{t=1}^{n}\{[\sin(n^{-\alpha}\gamma x_t)\sin(n^{-\alpha}\zeta x_t)$$

$$-E(\sin(n^{-\alpha}\gamma x_t)|F_{t-1})E(\sin(n^{-\alpha}\zeta x_t)|F_{t-1})]|F_{t-1}\}$$

$$= (\zeta\sigma^2 + O(n^{-2\alpha}))n^{-1}\sum_{t=1}^{n}\{\cos(n^{-\alpha}\gamma x_{t-1})\cos(n^{-\alpha}\zeta x_{t-1})\} + o_p(1) = 0.$$

□

Proof of Lemma 2.3:

1. Using the law of iterated expectation and Assumption 2.1, we can obtain the equation as below:

$$n^{2\alpha-1}\sum_{t=1}^{n}E\{[\cos(n^{-\alpha}\zeta x_t) - (E\cos(n^{-\alpha}\zeta x_t)|F_{t-1})]^2|F_{t-1}\}$$

$$= n^{2\alpha-1}\sum_{t=1}^{n}\sin^2(n^{-\alpha}\zeta x_{t-1})E\sin^2(n^{-\alpha}\zeta\varepsilon_t) + o_p(1)$$

$$= n^{-1}\sum_{t=1}^{n}\sin^2(n^{-\alpha}\zeta x_{t-1})((\zeta\sigma)^2 + O(n^{-2\alpha})) + o_p(1)$$

$$= n^{-1}(\zeta\sigma)^2\sum_{t=1}^{n}\sin^2(n^{-\alpha}\zeta x_{t-1}) + o_p(1).$$

By Lemma 2.6,

$$n^{2\alpha-1}\sum_{t=1}^{n}\{E(\cos^2(n^{-\alpha}\zeta x_t)|F_{t-1}) - (E(\cos(n^{-\alpha}\zeta x_t)|F_{t-1})^2\}$$

$$= n^{-1}(\zeta\sigma)^2\sum_{t=1}^{n}\sin^2(n^{-\alpha}\zeta x_{t-1}) + o_p(1)$$

$$= (1/2)(\zeta\sigma)^2 + o_p(1).$$

2. $\gamma \neq \zeta$

By the law of iterated expectation and Assumption 2.1, we can use similarly way as proof of Lemma 2.2 to obtain the equation.

$$n^{2\alpha-1}\sum_{t=1}^{n}\{[\cos(n^{-\alpha}\gamma x_t)\cos(n^{-\alpha}\zeta x_t)$$

$$-E(\cos(n^{-\alpha}\gamma x_t)|F_{t-1})E(\cos(n^{-\alpha}\zeta x_t)|F_{t-1})]|F_{t-1}\}$$

$$= (\zeta\sigma^2 + O(n^{-2\alpha}))n^{-1}\sum_{t=1}^{n}\{\sin(n^{-\alpha}\gamma x_{t-1})\sin(n^{-\alpha}\zeta x_{t-1})\} + o_p(1).$$

By the Lemma 2.6, we know

$$n^{-1}\sum_{t=1}^{n}\sin(n^{-\alpha}\gamma x_{t-1})\sin(n^{-\alpha}\zeta x_{t-1}) = 0 \quad \text{under } \gamma \neq \zeta \text{ and } \gamma, \zeta \geq 0$$

We can obtain the result as below:

$$n^{2\alpha-1}\sum_{t=1}^{n}\{[\cos(n^{-\alpha}\gamma x_t)\cos(n^{-\alpha}\zeta x_t)$$

$$-E(\cos(n^{-\alpha}\gamma x_t)|F_{t-1})E(\cos(n^{-\alpha}\zeta x_t)|F_{t-1})]|F_{t-1}\}$$

$$=(\zeta\sigma^2+O(n^{-2\alpha}))n^{-1}\sum_{t=1}^{n}\{\sin(n^{-\alpha}\gamma x_{t-1})\sin(n^{-\alpha}\zeta x_{t-1})\}+o_p(1)=0.$$

□

For proof Theorem 2.1, we need another lemma as below.

Lemma 2.7 *For the process x_t defined before, if ε_t satisfies Assumption 2.1, 2.2 with $0<\alpha<1/4$ for any $\zeta_m\in\mathbb{R}$, then*

$$\begin{pmatrix} n^{-\alpha-\frac{1}{2}}\sum_{t=1}^{n}\sin(\frac{\zeta_1 x_t}{n^\alpha})-(\frac{2}{\sigma^2})n^{\alpha-\frac{1}{2}}\sum_{t=1}^{n}(\sin(\frac{\zeta_1 x_t}{n^\alpha})-E(\sin(\frac{\zeta_1 x_{t-1}}{n^\alpha})|F_{t-1}) \\ \vdots \\ n^{-\alpha-\frac{1}{2}}\sum_{t=1}^{n}\sin(\frac{\zeta_m x_t}{n^\alpha})-(\frac{2}{\sigma^2})n^{\alpha-\frac{1}{2}}\sum_{t=1}^{n}(\sin(\frac{\zeta_m x_t}{n^\alpha})-E(\sin(\frac{\zeta_m x_{t-1}}{n^\alpha})|F_{t-1}) \\ n^{-\alpha-\frac{1}{2}}\sum_{t=1}^{n}\cos(\frac{\zeta_1 x_t}{n^\alpha})-(\frac{2}{\sigma^2})n^{\alpha-\frac{1}{2}}\sum_{t=1}^{n}(\cos(\frac{\zeta_1 x_t}{n^\alpha})-E(\cos(\frac{\zeta_1 x_{t-1}}{n^\alpha})|F_{t-1}) \\ \vdots \\ n^{-\alpha-\frac{1}{2}}\sum_{t=1}^{n}\cos(\frac{\zeta_m x_t}{n^\alpha})-(\frac{2}{\sigma^2})n^{\alpha-\frac{1}{2}}\sum_{t=1}^{n}(\cos(\frac{\zeta_m x_t}{n^\alpha})-E(\cos(\frac{\zeta_m x_{t-1}}{n^\alpha})|F_{t-1}) \end{pmatrix}=o_p(1).$$

for $m=1,2,\ldots$

Proof of Lemma 2.7:

By Lemma 2.1 and Taylor expansion, we can obtain the following equalities.

$$\left|n^{-\alpha-\frac{1}{2}}\sum_{t=1}^{n}\sin(\frac{\zeta x_t}{n^\alpha})-(\frac{2}{\zeta^2\sigma^2})n^{\alpha-\frac{1}{2}}\sum_{t=1}^{n}\left(\sin(\frac{\zeta x_t}{n^\alpha})-E(\sin(\frac{\zeta x_t}{n^\alpha})|F_{t-1})\right)\right|=o_p(1)$$

$$\left|n^{-\alpha-\frac{1}{2}}\sum_{t=1}^{n}\cos(\frac{\zeta x_t}{n^\alpha})-(\frac{2}{\zeta^2\sigma^2})n^{\alpha-\frac{1}{2}}\sum_{t=1}^{n}\left(\cos(\frac{\zeta x_t}{n^\alpha})-E(\cos(\frac{\zeta x_t}{n^\alpha})|F_{t-1})\right)\right|=o_p(1)$$

So they suffice to show.

$$\begin{pmatrix} n^{-\alpha-\frac{1}{2}}\sum_{t=1}^{n}\sin(\frac{\varsigma_1 x_t}{n^\alpha}) - (\frac{2}{\sigma^2})n^{\alpha-\frac{1}{2}}\sum_{t=1}^{n}(\sin(\frac{\varsigma_1 x_t}{n^\alpha}) - E(\sin(\frac{\varsigma_1 x_{t-1}}{n^\alpha})|F_{t-1}) \\ \vdots \\ n^{-\alpha-\frac{1}{2}}\sum_{t=1}^{n}\sin(\frac{\varsigma_m x_t}{n^\alpha}) - (\frac{2}{\sigma^2})n^{\alpha-\frac{1}{2}}\sum_{t=1}^{n}(\sin(\frac{\varsigma_m x_t}{n^\alpha}) - E(\sin(\frac{\varsigma_m x_{t-1}}{n^\alpha})|F_{t-1}) \\ n^{-\alpha-\frac{1}{2}}\sum_{t=1}^{n}\cos(\frac{\varsigma_1 x_t}{n^\alpha}) - (\frac{2}{\sigma^2})n^{\alpha-\frac{1}{2}}\sum_{t=1}^{n}(\cos(\frac{\varsigma_1 x_t}{n^\alpha}) - E(\cos(\frac{\varsigma_1 x_{t-1}}{n^\alpha})|F_{t-1}) \\ \vdots \\ n^{-\alpha-\frac{1}{2}}\sum_{t=1}^{n}\cos(\frac{\varsigma_m x_t}{n^\alpha}) - (\frac{2}{\sigma^2})n^{\alpha-\frac{1}{2}}\sum_{t=1}^{n}(\cos(\frac{\varsigma_m x_t}{n^\alpha}) - E(\cos(\frac{\varsigma_m x_{t-1}}{n^\alpha})|F_{t-1}) \end{pmatrix} = o_p(1).$$

□

Proof of theorem 2.1:

By Lemma 2.1 and 2.7, $\forall\varsigma \in \mathbb{R}$

$$\left|n^{-\alpha-\frac{1}{2}}\sum_{t=1}^{n}\sin(\frac{\varsigma x_t}{n^\alpha}) - (\frac{2}{\varsigma^2\sigma^2})n^{\alpha-\frac{1}{2}}\sum_{t=1}^{n}\left(\sin(\frac{\varsigma x_t}{n^\alpha}) - E(\sin(\frac{\varsigma x_t}{n^\alpha})|F_{t-1})\right)\right| = o_p(1).$$

$$\left|n^{-\alpha-\frac{1}{2}}\sum_{t=1}^{n}\cos(\frac{\varsigma x_t}{n^\alpha}) - (\frac{2}{\varsigma^2\sigma^2})n^{\alpha-\frac{1}{2}}\sum_{t=1}^{n}\left(\cos(\frac{\varsigma x_t}{n^\alpha}) - E(\cos(\frac{\varsigma x_t}{n^\alpha})|F_{t-1})\right)\right| = o_p(1).$$

So these suffice to show that:

$$\frac{2}{\varsigma^2\sigma^2}n^{\alpha-\frac{1}{2}}\sum_{t=1}^{n}\left(\sin(\frac{\varsigma x_t}{n^\alpha}) - E(\sin(\frac{\varsigma x_t}{n^\alpha})|F_{t-1})\right) \xrightarrow{d} N(0, \frac{2}{\varsigma^2\sigma^2}).$$

$$\frac{2}{\varsigma^2\sigma^2}n^{\alpha-\frac{1}{2}}\sum_{t=1}^{n}\left(\cos(\frac{\varsigma x_t}{n^\alpha}) - E(\cos(\frac{\varsigma x_t}{n^\alpha})|F_{t-1})\right) \xrightarrow{d} N(0, \frac{2}{\varsigma^2\sigma^2}). \quad \forall\varsigma \in \mathbb{R}$$

By the martingale difference central limit theorem (Hamilton (1994) p.193-195) and Lemma 2.2 and 2.4, it follows that.

$$(n^{-\alpha-\frac{1}{2}}\sum_{t=1}^{n}\sin(\frac{\varsigma_1 x_t}{n^\alpha}), \ldots, n^{-\alpha-\frac{1}{2}}\sum_{t=1}^{n}\sin(\frac{\varsigma_m x_t}{n^\alpha}), n^{-\alpha-\frac{1}{2}}\sum_{t=1}^{n}\cos(\frac{\varsigma_1 x_t}{n^\alpha}),$$
$$\ldots, n^{-\alpha-\frac{1}{2}}\sum_{t=1}^{n}\cos(\frac{\varsigma_m x_t}{n^\alpha}))' \xrightarrow{d} N(0, AI)$$

where A is a $2m \times 2m$ matrix that diagonal elements is $(\frac{2}{\varsigma_1^2\sigma^2}, \ldots \frac{2}{\varsigma_m^2\sigma^2}, \frac{2}{\varsigma_1^2\sigma^2} \cdots \frac{2}{\varsigma_m^2\sigma^2})$. The other elements are zero. □

Proof of Theorem 2.2:

First we know that if for all k, $X_{nk} \xrightarrow{p} X$ as $n \rightarrow \infty$ and $\lim_{k\to\infty} \limsup_{n\to\infty} E|X_n - X_{nk}| = 0$, then $X_n \xrightarrow{p} X$. This is , for all ζ,

$$\lim_{n\to\infty} E\exp(i\zeta X_n) = \lim_{k\to\infty}\lim_{n\to\infty}[E\exp(i\zeta X_{nk}) - E\exp(i\zeta X_n)] + \lim_{k\to\infty}\lim_{n\to\infty} E\exp(i\zeta X_{nk})$$

$$= E\exp(i\zeta X) + o(1)$$

Because $\lim_{k\to\infty}\lim_{n\to\infty}|E\exp(i\zeta X_{nk}) - E\exp(i\zeta X_n)| \leq |\zeta|\lim_{k\to\infty}\lim_{n\to\infty} E|X_{nk} - X_n|$.

Second, we have Fourier series

$$T(x) = (a_0/2) + \sum_{j=1}^{\infty}(a_j\cos(jx_t) + b_j\sin(jx_t)).$$

where $a_j = \pi^{-1}\int_{-\pi}^{\pi}\cos(jx_t)T(x)dx$ and $b_j = \pi^{-1}\int_{-\pi}^{\pi}\sin(jx_t)T(x)dx$ for $j \geq 0$. Noting that $\mu = 2\pi^{-1}\int_{-\pi}^{\pi}T(x)dx = a_0/2$,

$$n^{-\alpha-1/2}\sum_{t=1}^{n}(T(n^{-\alpha}x_t) - \mu) = \sum_{j=1}^{\infty}[a_j n^{-\alpha-1/2}\sum_{t=1}^{n}\cos(n^{-\alpha}jx_t) + b_j n^{-\alpha-1/2}\sum_{t=1}^{n}\sin(n^{-\alpha}jx_t)]$$

Now set

$$X_{nk} = ((\sum_{j=1}^{k}j^{-2}(a_j + b_j))^{-1}\sum_{j=1}^{\infty}j^{-2}(a_j + b_j))^{1/2}$$

$$\times\sum_{j=1}^{k}(a_j n^{-\alpha-1/2}\sum_{t=1}^{n}\cos(n^{-\alpha}jx_t) + b_j n^{-\alpha-1/2}\sum_{t=1}^{n}\sin(n^{-\alpha}jx_t))$$

and

$$X_n = \sum_{j=1}^{k}(a_j n^{-\alpha-1/2}\sum_{t=1}^{n}\cos(n^{-\alpha}jx_t) + b_j n^{-\alpha-1/2}\sum_{t=1}^{n}\sin(n^{-\alpha}jx_t))$$

From Corollary 2.1, we know.

$$X_{nk} \xrightarrow{d} N(0, 2\sigma^{-2}\sum_{j=1}^{\infty}(j^{-2})(a_j^2 + b_j^2)).$$

Also by the Fatou lemma (see Chung (2001, p.45)) and the inequality $(a+b)^2 \le 2a^2 + 2b^2$,

$$\lim_{k\to\infty} \limsup_{n\to\infty} (E|X_n - X_{nk}|)^2$$

$$= \lim_{k\to\infty} \limsup_{n\to\infty} E(\sum_{j=k}^{\infty} n^{-\alpha-1/2} \sum_{t=1}^{n} (a_j \cos(n^{-\alpha} j x_t) + b_j \sin(n^{-\alpha} j x_t)))^2$$

$$\le 2 \lim_{k\to\infty} \limsup_{n\to\infty} \sum_{j=k}^{\infty} \sum_{l=k}^{\infty} a_j a_l E n^{-1-2\alpha} \sum_{t=1}^{n} \sum_{s=1}^{n} \cos(n^{-\alpha} j x_t) \cos(n^{-\alpha} l x_s)$$

$$+2 \lim_{k\to\infty} \limsup_{n\to\infty} \sum_{j=k}^{\infty} \sum_{l=k}^{\infty} b_j b_l E n^{-1-2\alpha} \sum_{t=1}^{n} \sum_{s=1}^{n} \sin(n^{-\alpha} j x_t) \sin(n^{-\alpha} l x_s)$$

We will only consider the first term, since the second can be dealt with analogously. First, we consider the case that for $t < s$ and $j \neq l$.

$$\left|E \cos(n^{-\alpha} j x_t) \cos(n^{-\alpha} l x_s)\right|$$

$$= \left|(1/4) E(\exp(n^{-\alpha} i j x_t) + \exp(-n^{-\alpha} i j x_t))(\exp(n^{-\alpha} i l x_s) + \exp(-n^{-\alpha} i l x_t)\right|$$

$$\le (1/4) \left| E \exp(n^{-\alpha} i (j-l) x_0) E \prod_{p=1}^{t} \exp(n^{-\alpha} i (j-l) \varepsilon_p) \prod_{q=t+1}^{s} \exp(-n^{-\alpha} i l \varepsilon_q) \right|$$

$$+(1/4) \left| E \exp(n^{-\alpha} i (j+l) x_0) E \prod_{p=1}^{t} \exp(n^{-\alpha} i (j+l) \varepsilon_p) \prod_{q=t+1}^{s} \exp(n^{-\alpha} i l \varepsilon_q) \right|$$

$$+(1/4) \left| E \exp(n^{-\alpha} i (l-j) x_0) E \prod_{p=1}^{t} \exp(n^{-\alpha} i (l-j) \varepsilon_p) \prod_{q=t+1}^{s} \exp(-n^{-\alpha} i l \varepsilon_q) \right|$$

$$+(1/4) \left| E \exp(n^{-\alpha} i (-j-l) x_0) E \prod_{p=1}^{t} \exp(n^{-\alpha} i (-j-l) \varepsilon_p) \prod_{q=t+1}^{s} \exp(-n^{-\alpha} i l \varepsilon_q) \right|. \tag{2.31}$$

Because $E \exp(n^{-\alpha} i (j+l) x_0)$ is bounded by 1, We can rewrite equation (2.31):

$$\left|E \cos(n^{-\alpha} j x_t) \cos(n^{-\alpha} l x_s)\right| \le \left| E \prod_{p=1}^{t} \exp(n^{-\alpha} i (j+l) \varepsilon_p) \prod_{q=t+1}^{s} \exp(n^{-\alpha} i l \varepsilon_q) \right| \tag{2.32}$$

For Equation (2.32), we separate four cases to discuss.

1. $0 < n^{-\alpha}(j+l) \le (1/2)$ and $0 < n^{-\alpha}l \le (1/2)$

By Taylor expansion and de Jong ((2001) P.6-7), we can obtain the inequality as below:

$$\left|E\exp(n^{-\alpha}i(j+l)\varepsilon_p)\right| \le 1-(1/6)n^{-2\alpha}((j+l)\sigma)^2 \tag{2.33}$$

Using the same method, we can obtain another inequality:

$$\left|E\exp(n^{-\alpha}il\varepsilon_q)\right| \le 1-(1/6)n^{-2\alpha}(l\sigma)^2 \tag{2.34}$$

We can rewrite Equation (2.32) as below:

$$\begin{aligned}
&\left|E\prod_{p=1}^{t}\exp(n^{-\alpha}i(j+l)\varepsilon_p)\prod_{q=t+1}^{s}\exp(n^{-\alpha}il\varepsilon_q)\right| \\
&\le \left|\prod_{p=1}^{t}(1-(1/6)n^{-2\alpha}((j+l)\sigma)^2)\prod_{q=t+1}^{s}(1-(1/6)n^{-2\alpha}(l\sigma)^2)\right| \\
&= \left|\exp(\sum_{p=1}^{t}\log(1-(1/6)n^{-2\alpha}((j+l)\sigma)^2))\exp(\sum_{q=t+1}^{s}\log(1-(1/6)n^{-2\alpha}(l\sigma)^2))\right| \\
&\le \exp(\sum_{p=1}^{t}(-(1/6)n^{-2\alpha}((j+l)\sigma)^2))\exp(\sum_{q=t+1}^{s}(-(1/6)n^{-2\alpha}(l\sigma)^2)) \\
&= \exp(-(1/6)n^{-2\alpha}t((j+l)\sigma)^2)\exp(-(1/6)n^{-2\alpha}(s-t)(l\sigma)^2)
\end{aligned} \tag{2.35}$$

and the same inequality as (2.35) will also hold for $t \ge s$. For $j = l$, assuming again that $t \le s$,

$$\left|E\cos(n^{-\alpha}jx_t)\cos(n^{-\alpha}jx_s)\right|$$

$$= (1/4)\left|E(\exp(n^{-\alpha}ijx_t)+\exp(-n^{-\alpha}ijx_t))(\exp(n^{-\alpha}ijx_s)+\exp(-n^{-\alpha}ijx_s))\right|$$

$$\leq (1/4)\big|E(\exp(n^{-\alpha}ij(x_t + x_s))\big| + (1/4)\big|E(\exp(n^{-\alpha}ij(x_t - x_s))\big|$$

$$+(1/4)\big|E(\exp(n^{-\alpha}ij(-x_t + x_s))\big| + (1/4)\big|E(\exp(n^{-\alpha}ij(-x_t - x_s))\big|$$

$$\leq \exp(-(1/6)n^{-2\alpha}t(2j)^2\sigma^2)\exp(-(1/6)n^{-2\alpha}(s-t)j^2\sigma^2)$$

and again the same inequalities holds for $t \geq s$. Therefore,

$$\left|\lim_{k\to\infty}\lim_{n\to\infty}\sup\sum_{j=k}^{\infty}\sum_{l=k}^{\infty}a_j a_l E n^{-2\alpha-1}\sum_{t=1}^{n}\sum_{s=1}^{n}\cos(n^{-\alpha}jx_t)\cos(n^{-\alpha}lx_s)\right|$$

$$\leq \lim_{k\to\infty}\lim_{n\to\infty}\sum_{j=k}^{\infty}\sum_{l=k}^{\infty}|a_j||a_l|$$

$$\times n^{-2\alpha-1}\sum_{t=1}^{n}\sum_{s=1}^{n}\exp(-(1/6)n^{-2\alpha}t(j+l)^2\sigma^2)\exp(-(1/6)n^{-2\alpha}(s-t)(l\sigma)^2) \quad (2.36)$$

Because the last item of Equation (2.36) is independent of t, under $s > t$ we can rewrite Equation (2.36) as below:

$$\left|\lim_{k\to\infty}\limsup_{n\to\infty}\sum_{j=k}^{\infty}\sum_{l=k}^{\infty}a_j a_l E n^{-2\alpha-1}\sum_{t=1}^{n}\sum_{s=1}^{n}\cos(n^{-\alpha}jx_t)\cos(n^{-\alpha}lx_s)\right|$$

$$\leq |\lim_{k\to\infty}\limsup_{n\to\infty}\sum_{j=k}^{\infty}\sum_{l=k}^{\infty}a_j a_l$$

$$\times E n^{-2\alpha-1}\sum_{t=1}^{n}\sum_{s-t=1}^{n}\exp(-(1/6)n^{-2\alpha}t(j+l)^2\sigma^2)\exp(-(1/6)n^{-2\alpha}(s-t)(l\sigma)^2)|$$

$$\leq \lim_{k\to\infty}\limsup_{n\to\infty}\sum_{j=k}^{\infty}\sum_{l=k}^{\infty}|a_j||a_l|$$

$$\times n^{-2\alpha-1}\sum_{t=1}^{n}\exp(-(1/6)n^{-2\alpha}t(j+l)^2\sigma^2)\sum_{s-t=1}^{n}\exp(-(1/6)n^{-2\alpha}(s-t)(l\sigma)^2)$$

$$(2.37)$$

The last item of Equation (2.37) can be calculate as following:

$$\lim_{n\to\infty}\sum_{s-t=1}^{n}\exp(-(1/6)n^{-2\alpha}(s-t)(l\sigma)^2)$$

$$=\sum_{s-t=1}^{\infty}\exp(-(1/6)n^{-2\alpha}(s-t)(l\sigma)^2)$$

$$=\exp(-(1/6)n^{-2\alpha}(l\sigma)^2)(1-\exp(-(1/6)n^{-2\alpha}(l\sigma)^2))^{-1}$$

$$=(\exp((1/6)n^{-2\alpha}(l\sigma)^2)-1)^{-1} \tag{2.38}$$

Under $0 < n^{-\alpha}l \leq 1/2$, we have the following inequality by Taylor expansion.

$$\exp((1/6)n^{-2\alpha}(l\sigma)^2)-1 \geq (1/6)n^{-2\alpha}(l\sigma)^2$$

From this inequality, we can obtain the following relationship by inverse this inequality.

$$(\exp((1/6)n^{-2\alpha}(l\sigma)^2)-1)^{-1} \leq 6n^{2\alpha}(l\sigma)^{-2} \tag{2.39}$$

Combining (2.38) with (2.39), we can obtain the result as below:

$$\lim_{n\to\infty}\sum_{s-t=1}^{n}\exp(-(1/6)n^{-2\alpha}(s-t)(l\sigma)^2)$$

$$=(\exp((1/6)n^{-2\alpha}(l\sigma)^2)-1)^{-1} \leq (1+(1/6)n^{-2\alpha}(l\sigma)^2-1)^{-1} \leq 6n^{2\alpha}(l\sigma)^{-2} \tag{2.40}$$

Using the same method, we can obtain the other inequality:

$$\lim_{n\to\infty}\sum_{t=1}^{n}\exp(-(1/6)n^{-2\alpha}t((j+l)\sigma)^2) \leq 6n^{2\alpha}((j+l)\sigma)^{-2} \tag{2.41}$$

We substitute (2.40) and (2.41) into (2.37). The Equation (2.37) can be rewritten as below:

$$(2.37) \le \lim_{k\to\infty}\lim_{n\to\infty}\sup\sum_{j=k}^{\infty}\sum_{l=k}^{\infty}|a_j||a_l|n^{-1}(6(l\sigma)^{-2})\sum_{t=1}^{n}\exp(-(1/6)n^{-2\alpha}t(j+l)^2\sigma^2)$$

$$\le \lim_{k\to\infty}\lim_{n\to\infty}\sum_{j=k}^{\infty}\sum_{l=k}^{\infty}|a_j||a_l|n^{-1}(6(l\sigma)^{-2})(6n^{2\alpha}((j+l)\sigma)^{-2})$$

$$= \lim_{k\to\infty}\lim_{n\to\infty}\sum_{j=k}^{\infty}\sum_{l=k}^{\infty}|a_j||a_l|n^{2\alpha-1}(6(l\sigma)^{-2})(6((j+l)\sigma)^{-2})$$

$$\le \lim_{k\to\infty}\lim_{n\to\infty}\sum_{j=k}^{\infty}a_j^2n^{2\alpha-1}(6(j\sigma)^{-2})(6((2j)\sigma)^{-2})$$

$$+\lim_{k\to\infty}\lim_{n\to\infty}\sum_{j=k}^{\infty}\sum_{l=k,l\neq j}^{\infty}|a_j||a_l|n^{2\alpha-1}(6(l\sigma)^{-2})(6((j+l)\sigma)^{-2})$$

$$\le C_1\lim_{k\to\infty}(\sum_{j=k}^{\infty}a_j^2+(\sum_{j=k}^{\infty}|a_j|)^2) \quad \text{for some constant } C_1$$

2. $n^{-\alpha}(j+l) > (1/2)$ and $n^{-\alpha}l > (1/2)$

According to Theorem 2.1.4 of Lukacs (1970, p18) $E\exp(n^{-\alpha}i(j+l)\varepsilon_p) < 1$ if $n^{-\alpha}(j+l) \in \mathbb{R}\backslash\{0\}$ and $E\exp(n^{-\alpha}il\varepsilon_q) < 1$ if $n^{-\alpha}l \in \mathbb{R}\backslash\{0\}$. We assume $\alpha = \max\{\sup_{|n^{-\alpha}(j+l)|>1/2}|E\exp(n^{-\alpha}i(j+l)\varepsilon_p)|, \sup_{|n^{-\alpha}l|>1/2}|E\exp(n^{-\alpha}il\varepsilon_q)|\} < 1$. We can rewrite Equation (2.31) as below:

$$\left|E\cos(n^{-\alpha}jx_t)\cos(n^{-\alpha}lx_s)\right| \le \left|\prod_{p=1}^{t}E\exp(n^{-\alpha}i(j+l)\varepsilon_p)\prod_{q=t+1}^{s}E\exp(n^{-\alpha}il\varepsilon_q)\right|$$

$$\le \alpha^t\alpha^{s-t} \le \alpha^{\max(t,s)} \tag{2.42}$$

and the same inequality as (2.41) will also hold for $t \ge s$. For $j = l$, assuming again that $t \le s$,

$$\left|E\cos(n^{-\alpha}jx_t)\cos(n^{-\alpha}jx_s)\right|$$

$$= (1/4)\left|E(\exp(n^{-\alpha}ijx_t) + \exp(-n^{-\alpha}ijx_t))(\exp(n^{-\alpha}ijx_s) + \exp(-n^{-\alpha}ijx_s))\right|$$

$$\leq (1/4)\left|E(\exp(n^{-\alpha}ij(x_t + x_s))\right| + (1/4)\left|E(\exp(n^{-\alpha}ij(x_t - x_s))\right|$$

$$+(1/4)\left|E(\exp(n^{-\alpha}ij(-x_t + x_s))\right| + (1/4)\left|E(\exp(n^{-\alpha}ij(-x_t - x_s))\right| \leq \alpha^{\max(t,s)}$$

and again the same inequalities holds for $t \geq s$. Therefore,

$$\left|\lim_{k\to\infty} \limsup_{n\to\infty} \sum_{j=k}^{\infty}\sum_{l=k}^{\infty} a_j a_l E n^{-2\alpha-1} \sum_{t=1}^{n}\sum_{s=1}^{n} \cos(n^{-\alpha}jx_t)\cos(n^{-\alpha}lx_s)\right|$$

$$\leq \lim_{k\to\infty}\lim_{n\to\infty} \sum_{j=k}^{\infty}\sum_{l=k}^{\infty} |a_j||a_l| n^{-2\alpha-1} \sum_{t=1}^{n}\sum_{s=1}^{n} \alpha^{\max(t,s)}$$

$$\leq \lim_{k\to\infty}\lim_{n\to\infty} \sum_{j=k}^{\infty} a_j^2 n^{-2\alpha-1} \sum_{t=1}^{n}\sum_{s=1}^{\infty} \alpha^s + \lim_{k\to\infty}\lim_{n\to\infty} \sum_{j=k}^{\infty}\sum_{l=k,l\neq j}^{\infty} |a_j||a_l| n^{-2\alpha-1} \sum_{t=1}^{n}\sum_{s=1}^{\infty} \alpha^s$$

$$\leq \lim_{k\to\infty}\lim_{n\to\infty} \sum_{j=k}^{\infty} a_j^2 n^{-2\alpha-1}((1-\alpha)^{-1} n\alpha)$$

$$+\lim_{k\to\infty}\lim_{n\to\infty} \sum_{j=k}^{\infty}\sum_{l=k,l\neq j}^{\infty} |a_j||a_l| n^{-2\alpha-1}((1-\alpha)^{-1} n\alpha)$$

$$\leq C_2 \lim_{k\to\infty} (\sum_{j=k}^{\infty} a_j^2 + (\sum_{j=k}^{\infty} |a_j|)^2). \quad \text{for some constant } C_2$$

3. $0 < n^{-\alpha}(j+l) \leq 1/2$ and $n^{-\alpha}l > 1/2$

First, we assume $\alpha = \sup_{|n^{-\alpha}(j+l)|>1/2} |E\exp(n^{-\alpha}i(j+l)\varepsilon_q)| < 1$. We can rewrite Equation (2.32) with Equation (2.34) as below:

$$\left|E\prod_{p=1}^{t} \exp(n^{-\alpha}i(j+l)\varepsilon_p) \prod_{q=t+1}^{s} \exp(n^{-\alpha}il\varepsilon_q)\right|$$

$$\leq \left|\prod_{p=1}^{t} (1 - (1/6)n^{-2\alpha}((j+l)\sigma)^2) \prod_{q=t+1}^{s} E\exp(n^{-\alpha}il\varepsilon_q)\right|$$

$$= \left|\exp(\sum_{p=1}^{t} \log(1 - (1/6)n^{-2\alpha}((j+l)\sigma)^2)) \prod_{q=t+1}^{s} E\exp(n^{-\alpha} il\varepsilon_q)\right|$$

$$\leq \exp(\sum_{p=1}^{t} (-(1/6)n^{-2\alpha}((j+l)\sigma)^2))\alpha^{s-t}$$

$$= \exp(-(1/6)n^{-2\alpha}(t(j+l)\sigma)^2)\alpha^{s-t} \tag{2.43}$$

and the same inequality as (2.43) will also hold for $t \geq s$. For $j = l$, assuming again that $t \leq s$,

$$\left|E\cos(n^{-\alpha}jx_t)\cos(n^{-\alpha}jx_s)\right|$$

$$= (1/4)\left|E(\exp(n^{-\alpha}ijx_t) + \exp(-n^{-\alpha}ijx_t))(\exp(n^{-\alpha}ijx_s) + \exp(-n^{-\alpha}ijx_s))\right|$$

$$\leq (1/4)\left|E(\exp(n^{-\alpha}ij(x_t + x_s))\right| + (1/4)\left|E(\exp(n^{-\alpha}ij(x_t - x_s))\right|$$

$$+(1/4)\left|E(\exp(n^{-\alpha}ij(-x_t + x_s))\right| + (1/4)\left|E(\exp(n^{-\alpha}ij(-x_t - x_s))\right|$$

$$\leq \exp(-(1/6)n^{-2\alpha}t(2j\sigma^2)\alpha^{s-t}$$

and again the same inequalities holds for $t \geq s$. Therefore,

$$\left|\lim_{k\to\infty}\lim_{n\to\infty}\sup\sum_{j=k}^{\infty}\sum_{l=k}^{\infty} a_j a_l E n^{-2\alpha-1}\sum_{t=1}^{n}\sum_{s=1}^{n}\cos(n^{-\alpha}jx_t)\cos(n^{-\alpha}lx_s)\right|$$

$$\leq \lim_{k\to\infty}\lim_{n\to\infty}\sum_{j=k}^{\infty}\sum_{l=k}^{\infty}|a_j||a_l|n^{-2\alpha-1}\sum_{t=1}^{n}\exp(-(1/6)n^{-2\alpha}t((j+l)\sigma)^2)\sum_{s-t=1}^{n}\alpha^{s-t}$$

$$\leq \lim_{k\to\infty}\lim_{n\to\infty}\sum_{j=k}^{\infty} a_j^2 n^{-2\alpha-1}\sum_{t=1}^{n}\exp(-(1/6)n^{-2\alpha}t(2j\sigma)^2)\sum_{s-t=1}^{\infty}\alpha^{s-t}$$

$$+\lim_{k\to\infty}\lim_{n\to\infty}\sum_{j=k}^{\infty}\sum_{l=k,l\neq j}^{\infty}|a_j||a_l|n^{-2\alpha-1}\sum_{t=1}^{n}\exp(-(1/6)n^{-2\alpha}t((j+l)\sigma)^2)\sum_{s-t=1}^{\infty}\alpha^{s-t}$$

$$\leq \lim_{k\to\infty}\lim_{n\to\infty}\sum_{j=k}^{\infty} a_j^2 n^{-1}(6((2j)\sigma)^{-2})((1-\alpha)^{-1}\alpha)$$

$$+\lim_{k\to\infty}\lim_{n\to\infty}\sum_{j=k}^{\infty}\sum_{l=k,l\neq j}^{\infty} |a_j||a_l|n^{-1}(6((j+l)\sigma)^{-2})((1-\alpha)^{-1}\alpha)$$

$$\leq C_3\lim_{k\to\infty}(\sum_{j=k}^{\infty} a_j^2 + (\sum_{j=k}^{\infty}|a_j|)^2). \quad \text{for some constant } C_3$$

4. $n^{-\alpha}(j+l) > 1/2$ and $0 < n^{-\alpha}l \leq 1/2$

   First, we assume $\alpha = \sup_{|n^{-\alpha}(j+l)|>1/2} |E\exp(n^{-\alpha}i(j+l)\varepsilon_q)| < 1$. We can rewrite Equation (2.32) with Equation (2.33) as below:

$$\left|E\prod_{p=1}^{t}\exp(n^{-\alpha}i(j+l)\varepsilon_p)\prod_{q=t+1}^{s}\exp(n^{-\alpha}il\varepsilon_q)\right|$$

$$\leq \left|\prod_{p=1}^{t}E\exp(n^{-\alpha}i(j+l)\varepsilon_p)\prod_{q=t+1}^{s}(1-(1/6)n^{-2\alpha}(l\sigma)^2)\right|$$

$$= \left|\prod_{p=1}^{t}\alpha\exp(\sum_{q=t+1}^{s}\log(1-(1/6)n^{-2\alpha}(l\sigma)^2))\right| \leq \alpha^t\exp(\sum_{q=t+1}^{s}(-(1/6)n^{-\alpha}(l\sigma)^2))$$

$$= \alpha^t\exp(-(1/6)n^{-2\alpha}(s-t)(l\sigma)^{-2}) \tag{2.44}$$

   and the same inequality as (2.44) will also hold for $t \geq s$. For $j = l$, assuming again that $t \leq s$,

$$\left|E\cos(n^{-\alpha}jx_t)\cos(n^{-\alpha}jx_s)\right|$$

$$= (1/4)\left|E(\exp(n^{-\alpha}ijx_t)+\exp(-n^{-\alpha}ijx_t))(\exp(n^{-\alpha}ijx_s)+\exp(-n^{-\alpha}ijx_s))\right|$$

$$\leq (1/4)\left|E(\exp(n^{-\alpha}ij(x_t+x_s))\right| + (1/4)\left|E(\exp(n^{-\alpha}ij(x_t-x_s))\right|$$

$$+(1/4)\left|E(\exp(n^{-\alpha}ij(-x_t+x_s))\right| + (1/4)\left|E(\exp(n^{-\alpha}ij(-x_t-x_s))\right|$$

$$\leq \alpha^t \exp(-(1/6)n^{-2\alpha}(s-t)(l\sigma)^2)$$

and again the same inequalities holds for $t \geq s$. Therefore,

$$\left| \lim_{k\to\infty} \lim_{n\to\infty} \sup \sum_{j=k}^{\infty} \sum_{l=k}^{\infty} a_j a_l E n^{-2\alpha-1} \sum_{t=1}^{n} \sum_{s=1}^{n} \cos(n^{-\alpha} j x_t) \cos(n^{-\alpha} l x_s) \right|$$

$$\leq \lim_{k\to\infty} \lim_{n\to\infty} \sum_{j=k}^{\infty} \sum_{l=k}^{\infty} |a_j||a_l| n^{-2\alpha-1} \sum_{t=1}^{n} \alpha^t \sum_{s-t=1}^{n} \exp(-(1/6)n^{-2\alpha}(s-t)(l\sigma)^2)$$

$$\leq \lim_{k\to\infty} \lim_{n\to\infty} \sum_{j=k}^{\infty} a_j^2 n^{-2\alpha-1} \sum_{t=1}^{n} \alpha^t \sum_{s-t}^{\infty} \exp(-(1/6)n^{-2\alpha}(s-t)(l\sigma)^2)$$

$$+ \lim_{k\to\infty} \lim_{n\to\infty} \sum_{j=k}^{\infty} \sum_{l=k, l\neq j}^{\infty} |a_j||a_l| n^{-2\alpha-1} \sum_{t=1}^{n} \alpha^t \sum_{s-t=1}^{\infty} \exp(-(1/6)n^{-2\alpha}(s-t)(l\sigma)^2)$$

$$\leq \lim_{k\to\infty} \lim_{n\to\infty} \sum_{j=k}^{\infty} a_j^2 n^{-1}((1-\alpha)^{-1}\alpha)(6(l\sigma)^{-2})$$

$$+ \lim_{k\to\infty} \lim_{n\to\infty} \sum_{j=k}^{\infty} \sum_{l=k, l\neq j}^{\infty} |a_j||a_l| n^{-1}((1-\alpha)^{-1}\alpha)(6(l\sigma)^{-2})$$

$$\leq C_4 \lim_{k\to\infty} (\sum_{j=k}^{\infty} a_j^2 + (\sum_{j=k}^{\infty} |a_j|)^2). \quad \text{for some constant } C_4$$

From four cases before, if it can be shown that $\sum_{j=0}^{\infty} |a_j| < \infty$ *and* $\sum_{j=0}^{\infty} a_j^2 < \infty$, this proof will be complete. These conditions hold because $\sum_{j=0}^{\infty} (a_j^2 + b_j^2) < \infty$ and $\sum_{j=0}^{\infty} (|a_j| + |b_j|) < \infty$ by the assumptions on T(.). (See Apostol (1971) p340.)

□

CHAPTER 3

Further results on the asymptotics for nonlinear transformations of integrated time series

3.1 Introduction

This chapter proves three results about functions of integrated processes. Our first result is an extension of a result in Park and Phillips (1999), where it is proven that for integrable functions $T(.)$ and for I(1) processes x_t,

$$n^{-1/2}\sum_{t=1}^{n} T(x_t) \xrightarrow{d} (\int_{-\infty}^{\infty} T(s)ds)L(1,0), \tag{3.1}$$

where $L(t,s)$ is a two-parameter stochastic process called (Brownian) *local time*. The remarkable thing about this result is that it establishes limit theory for a function of an I(1) process that has not been rescaled by $n^{-1/2}$. Park and Phillips establish the above result under some regularity conditions on the I(1) process x_t and the integrable function $T(.)$. In this paper, we show that Park and Phillips' regularity conditions for the above result can be relaxed and also that their result can be extended to yield, for $0 \leq \alpha < 1/2$,

$$n^{-1/2-\alpha} \sum_{t=1}^{n} T(n^{-\alpha} x_t) \xrightarrow{d} (\int_{-\infty}^{\infty} T(s) ds) L(1,0). \tag{3.2}$$

A central tool for the proof of this first result is a lemma that was recently established in de Jong (2001). Also in Park and Phillips (1999), it is shown that for functions $T(.)$ that satisfy

$$T(\lambda x) = \nu(\lambda) H(x) + R(x, \lambda) \tag{3.3}$$

under conditions on $R(.,.)$ that basically serve to ensure asymptotic negligibility of

$$\nu(n^{1/2})^{-1} n^{-1} \sum_{t=1}^{n} R(x_t, n^{1/2}), \tag{3.4}$$

we have

$$\nu(n^{1/2})^{-1} n^{-1} \sum_{t=1}^{n} T(x_t) \xrightarrow{d} \int_0^1 H(\sigma W(r)) dr, \tag{3.5}$$

where $\sigma^2 = \lim_{n \to \infty} n^{-1} E x_n^2$. Again the interesting aspect of the above result is the fact that it considers integrated processes that have not been rescaled by $n^{-1/2}$. Functions $T(.)$ that satisfy the appropriate condition are coined *asymptotically homogeneous* by Park and Phillips. The asymptotically homogeneous condition is trivially satisfied for $T(x) = |x|^a$ for $a \geq 0$, but is general enough to also deal with functions such as $T(x) = |x|^a \log |x|$ for all $a \geq 0$. In this paper, we show the more general result that whenever for functions $H(.)$ and $\nu(.)$ we have

$$\nu(\lambda)^{-1} T(\lambda x) \to H(x) \quad \text{as } \lambda \to \infty \tag{3.6}$$

in L_1 sense, we have for $0 \leq \alpha < 1/2$, under regularity conditions,

$$\nu(n^{1/2-\alpha})^{-1} n^{-1} \sum_{t=1}^{n} T(n^{-\alpha} x_t) \xrightarrow{d} \int_0^1 H(\sigma W(r)) dr. \tag{3.7}$$

Therefore, we show that Park and Phillips' class of asymptotically homogeneous functions can be extended, and we consider $n^{-\alpha} x_t$ for $0 \leq \alpha < 1/2$ instead of x_t as the argument for $T(.)$.

A third result that is proven in this chapter concerns averages of the type

$$n^{-1}\sum_{t=1}^{n}|n^{-1/2}x_t|^{-m}I(n^{-1/2}x_t > c_n) \tag{3.8}$$

and

$$n^{-1}\sum_{t=1}^{n}|n^{-1/2}x_t|^{-m}I(|n^{-1/2}x_t| > c_n), \tag{3.9}$$

where $m > 1$. While it has been shown in de Jong (2001) and Pötscher (2001) that under regularity conditions for locally integrable functions $T(.)$ we have

$$n^{-1}\sum_{t=1}^{n}T(n^{-1/2}x_t) \xrightarrow{d} \int_0^1 T(\sigma W(r))dr, \tag{3.10}$$

it is yet unknown what happens to functions $T(.)$ that are not integrable. Using a "clipping device" involving a deterministic sequence c_n that converges to 0 with n, it will be proven that for $m > 1$,

$$(m-1)c_n^{1-m}n^{-1}\sum_{t=1}^{n}|\sigma^{-1}n^{-1/2}x_t|^{-m}I(\sigma^{-1}n^{-1/2}x_t > c_n) \xrightarrow{d} L(1,0), \tag{3.11}$$

and also that

$$(1/2)(m-1)c_n^{1-m}n^{-1}\sum_{t=1}^{n}|\sigma^{-1}n^{-1/2}x_t|^{-m}I(|\sigma^{-1}n^{-1/2}x_t| > c_n) \xrightarrow{d} L(1,0). \tag{3.12}$$

3.2 Assumptions and result for integrable functions

Identically to Park and Phillips (1999), linear process conditions for x_t are assumed

$$x_t = x_{t-1} + w_t, \tag{3.13}$$

where w_t is generated according to

$$w_t = \sum_{k=0}^{\infty}\phi_k\varepsilon_{t-k} \tag{3.14}$$

where ε_t is assumed to be a sequence of i.i.d. random variables with mean zero, and where it is assumed that $\sum_{k=0}^{\infty} \phi_k \neq 0$. In addition, we will assume that x_0 is an arbitrary random variable that is independent of all w_t, $t \geq 1$. The main assumptions used in this paper are Assumption 2.1 and 2.2 from Park and Phillips (1999):

Assumption 3.1 *$\sum_{k=0}^{\infty} k^{1/2}\phi(k) < \infty$ and $E\varepsilon_t^2 < \infty$.*

Assumption 3.2

(a) *$\sum_{k=0}^{\infty} k|\phi_k| < \infty$ and $E|\varepsilon_t|^p < \infty$ for some $p > 2$.*

(b) *The distribution of ε_t is absolutely continuous with respect to the Lebesgue measure and has characteristic function $\psi(s)$ for which $\lim_{s\to\infty} s^\eta \psi(s) = 0$ for some $\eta > 0$.*

Assumption 3.1 guarantees that $n^{-1/2}x_{[rn]} \Rightarrow \sigma W(r)$ where "$\Rightarrow$" denotes weak convergence in $C[0,1]$, i.e. the space of functions that are continuous on $[0,1]$, while Assumption 3.2 in addition also guarantees a convergence rate for a Skorokhod representation of $n^{-1/2}x_{[rn]}$. Several of the manipulations in the proofs of the results in this paper require the use of local time $L(.,.)$. Local time is a random function satisfying

$$L(t,s) = \lim_{\varepsilon\to 0}(2\varepsilon)^{-1}\int_0^t I(|W(r)-s| < \varepsilon)dr. \tag{3.15}$$

See Park and Phillips (1999, p. 271-272) and Chung and Williams (1990, Ch. 7) for more details regarding local time.

Park and Phillips (1999) establish the following result for integrable functions of integrated random variables:

Theorem 3.1 *Suppose that $T(.)$ is integrable and Assumption 3.2 holds with $p > 4$. If $T(.)$ is square integrable and satisfies the Lipschitz condition*

$$|T(x) - T(y)| \leq c|x-y|^l \tag{3.16}$$

over its support for some constants c and $l > 6/(p-2)$, then

$$n^{-1/2}\sum_{t=1}^{n} T(x_t) \xrightarrow{d} (\int_{-\infty}^{\infty} T(s)ds)L(1,0). \tag{3.17}$$

For differentiable functions $T(.)$, we need to set $l = 1$, implying that we need $p > 8$ in order for the theorem to work. In order to improve the above result, we needed the following useful lemma, that was established in de Jong (2001):

Lemma 3.1 *Under Assumption 3.2, for all $y \in \mathbb{R}$, $\delta > 0$, and $n \geq M$ for some value of M,*

$$P(y \leq n^{-1/2}x_n \leq y + \delta) \leq C\delta, \tag{3.18}$$

where C and M do not depend on y, δ, or n.

Using this lemma, we were able to improve Park and Phillips' result and show the following quite general result:

Theorem 3.2 *Suppose Assumption 3.2 holds. Also assume that $|T(x)| \leq R(x)$, and assume that $R(.)$ is integrable, continuous on $\mathbb{R}$, and monotone on $(0,\infty)$ and $(-\infty,0)$. If $T(.)$ is continuous, then for $0 \leq \alpha < 1/2$,*

$$n^{-1/2-\alpha}\sum_{t=1}^{n} T(n^{-\alpha}x_t) \xrightarrow{d} (\int_{-\infty}^{\infty} T(s)ds)L(1,0). \tag{3.19}$$

Compared to Park and Phillips' theorem, we have completely removed their Lipschitz-continuity condition and weakened it to continuity, and in addition, their requirement on p has been removed. Also, weights $n^{-\alpha}$ for $0 \leq \alpha < 1/2$ are allowed for. While no $R(.)$ function such as present in Theorem 3.2 is explicitly used in their Theorem 3.1, from Park and Phillips' proof it is clear that existence of such a function is implied. Therefore, Theorem 3.2 is a "clean" improvement to Park and Phillips' Theorem 3.1.

3.3 Asymptotically homogeneous functions

In this section, we improve Park and Phillips' (1999) result for asymptotically homogeneous functions. Park and Phillips assume that

$$T(\lambda x) = \nu(\lambda)H(x) + R(x, \lambda) \tag{3.20}$$

and they show that

$$\nu(n^{1/2})^{-1}n^{-1}\sum_{t=1}^{n} T(x_t) \xrightarrow{d} \int_0^1 H(\sigma W(r))dr \tag{3.21}$$

if either

a. $|R(x,\lambda)| \leq a(\lambda)P(x)$, where $\limsup_{\lambda\to\infty} a(\lambda)/\nu(\lambda) = 0$ and P is locally integrable,

   or

b. $|R(x,\lambda)| \leq b(\lambda)Q(\lambda x)$, where $\limsup_{\lambda\to\infty} b(\lambda)/\nu(\lambda) < \infty$ and Q is locally integrable and vanishes at infinity, i.e. $Q(x) \to 0$ as $|x| \to \infty$.

In this paper, we redefine their notion of an asymptotically homogeneous function, as follows:

Definition 3.1 *A function $T(.)$ is called* asymptotically homogeneous *if for all $K > 0$ and some function $H(.)$,*

$$\lim_{\lambda\to\infty} \int_{-K}^{K} |\nu(\lambda)^{-1}T(\lambda x) - H(x)|dx = 0. \tag{3.22}$$

Obviously from the dominated convergence theorem it follows that if for some $\nu(.)$ and $H(.)$, pointwise in x,

$$\nu(\lambda)^{-1}T(\lambda x) \to H(x) \quad \text{as } \lambda \to \infty \tag{3.23}$$

and $|\nu(\lambda)^{-1}T(\lambda x)| \leq G(x)$ for a locally integrable function $G(.)$, then $T(.)$ is asymptotically homogeneous. Below, we will call a function *monotone regular* if for some

$\{a_1, \ldots, a_q\}$, $T(.)$ is monotone on (a_j, a_{j+1}) for $j = 0, \ldots, q$ (setting $a_0 = -\infty$ and $a_{q+1} = \infty$).

The main result of this section is the following:

Theorem 3.3 *Suppose Assumption 3.1 holds. Also assume that $T(.)$ is asymptotically homogeneous. In addition, assume that $H(.)$ is continuous and $T(.)$ is monotone regular. Then, for $0 \le \alpha < 1/2$,*

$$\nu(n^{1/2-\alpha})^{-1} n^{-1} \sum_{t=1}^{n} T(n^{-\alpha} x_t) \xrightarrow{d} \int_0^1 H(\sigma W(r)) dr = \int_{-\infty}^{\infty} H(\sigma s) L(1, s) ds. \quad (3.24)$$

It is also possible to show that our definition of an asymptotically homogeneous function is more general than Park and Phillips'. Under Assumption a. above,

$$\int_{-K}^{K} |\nu(\lambda)^{-1} T(\lambda x) - H(x)| dx = \nu(\lambda)^{-1} \int_{-K}^{K} |R(x, \lambda)| dx$$

$$\le a(\lambda)\nu(\lambda)^{-1} \int_{-K}^{K} P(x) dx \to 0 \quad (3.25)$$

as $\lambda \to \infty$ if $P(.)$ is locally integrable. Under Assumption b. above,

$$\int_{-K}^{K} |\nu(\lambda)^{-1} T(\lambda x) - H(x)| dx = \nu(\lambda)^{-1} \int_{-K}^{K} |R(x, \lambda)| dx$$

$$\le b(\lambda)\nu(\lambda)^{-1} \int_{-K}^{K} Q(\lambda x) dx \to 0 \quad (3.26)$$

as $\lambda \to \infty$, because $\limsup_{\lambda \to \infty} b(\lambda)\nu(\lambda)^{-1} < \infty$ and $\lim_{\lambda \to \infty} \int_{-K}^{K} Q(\lambda x) dx = 0$ by boundedness of $Q(.)$ (which is also assumed in Park and Phillips (1999)). Therefore, obviously the set of functions that is "asymptotically homogeneous" in this paper is wider than in Park and Phillips (1999). But clearly, most functions that one may expect to be useful for applications should be expected to already be in Park and Phillips' class of asymptotically homogeneous functions, and the main achievement of our analysis is the redefinition of the class of asymptotically homogeneous functions to as large as possible a collection of functions. It appears to us that the above result should be close to the limits of what should be possible in this setting, and for the authors of this paper, it is hard to see how the above definition of the class of asymptotically homogeneous functions can be relaxed further to yield an even larger function class that generates similar behavior.

3.4 Nonintegrable functions

In de Jong (2001) and Pötscher (2001) it is proven that under regularity conditions, in spite of possible poles in $T(.)$, as long as $\int_{-K}^{K} |T(x)|dx < \infty$ for all $K > 0$, we have

$$n^{-1}\sum_{t=1}^{n} T(n^{-1/2}x_t) \xrightarrow{d} \int_0^1 T(\sigma W(r))dr. \tag{3.27}$$

These results raise the question as to what will happen if a nonintegrable function of an integrated process is used for $T(.)$ in statistics of the form

$$n^{-1}\sum_{t=1}^{n} T(n^{-1/2}x_t). \tag{3.28}$$

This issue appears to have never been tackled before in either the statistics or the econometrics literature. This section explores this issue for functions

$$T(x) = |x|^{-m} I(x > 0) \tag{3.29}$$

and

$$T(x) = |x|^{-m}, \tag{3.30}$$

for $m > 1$. As it turns out and is perhaps to be expected, the observations "close to zero" take over the limit behavior of the statistic in this case. We will need a "clipping device" and we construct statistics similar to those constructed in Park and Phillips (1999) for integrable functions. Our first result is the following:

Theorem 3.4 *Let* $c_n = n^{-(2p+1)/3p+\eta}$ *for some* $\eta > 0$ *such that* $-(2p+1)/3p+\eta < 0$. *In addition, assume that*

$$T(x) = |x|^{-m} \tag{3.31}$$

for some $m > 1$. *Let* $d_n = \int_{c_n}^{1} T(x)dx$. *Then under Assumption 3.2,*

$$d_n^{-1} n^{-1} \sum_{t=1}^{n} T(\sigma^{-1} n^{-1/2} x_t) I(\sigma^{-1} n^{-1/2} x_t > c_n) \xrightarrow{d} L(1,0). \tag{3.32}$$

Clearly, in the above theorem $d_n = (m-1)^{-1}(c_n^{1-m} - 1)$, but we choose the above formulation to bring out better where our rescaling factor d_n originates from.

The proof of the following "two-sided" version of the above theorem is analogous and therefore omitted:

Theorem 3.5 *Let* $c_n = n^{-(2p+1)/3p+\eta}$ *for some* $\eta > 0$ *such that* $-(2p+1)/3p+\eta < 0$. *Assume that*

$$T(x) = |x|^{-m} \tag{3.33}$$

for some $m > 1$. *Let* $d_n = 2\int_{c_n}^{1} T(x)dx$. *Then under Assumption 3.2,*

$$d_n^{-1} n^{-1} \sum_{t=1}^{n} T(\sigma^{-1} n^{-1/2} x_t) I(|\sigma^{-1} n^{-1/2} x_t| > c_n) \xrightarrow{d} L(1,0). \tag{3.34}$$

The above theorems leave the issue wide open to what function class the above theorem can be extended. The line of proof employed in the Appendix may allow for some generalization, but it is not clear to the authors what the outer limits are for which a result as the above might hold. Furthermore, the clipping device is intriguing, and one could conjecture that for the above definitions the theorem will remain true if c_n in the theorem and in the definition of d_n were to be replaced by $\min_{1\le t\le n} n^{-1/2} x_t I(x_t > 0)$ and $\min_{1\le t\le n} n^{-1/2}|x_t|$ respectively.

Proofs

Throughout this section, to improve readability, we will assume for every proof that $\sigma^2 = 1$.

Below we use the following definitions, which are identically to Park and Phillips (1999):

$$N_n(\nu_n; a, b) = \int_0^1 I(a \leq \nu_n n^{-1/2} x_{[rn]} \leq b) dr = n^{-1} \sum_{t=1}^{n} I(a \leq \nu_n n^{-1/2} x_t \leq b), \quad (3.35)$$

and

$$N(\nu_n; a, b) = \int_0^1 I(a \leq \nu_n n^{-1/2} W(r) \leq b) dr. \quad (3.36)$$

In the proofs below, M and C are the constants from Lemma 3.1. The following lemma from Park and Phillips (1999) was needed in order to prove our results.

Lemma 3.2 *Under Assumption 3.2, as* $n \to \infty$,

$$E(N_n(\nu_n; 0, \delta) - N_n(\nu_n; k\delta, (k+1)\delta))^2 \leq c(\delta n^{-1} \nu_n^{-1})(1 + k\delta^2 n \log(n) \nu_n^{-2}) \quad (3.37)$$

and

$$N_n(\nu_n; 0, \pi_n) = N(\nu_n; 0, \pi_n) + o_p(n^{-(2p-1)/3p+\varepsilon}) \quad (3.38)$$

for $\pi_n \geq \nu_n n^{-2(p+1)/3p}$ *and any* $\varepsilon > 0$.

Proof:

See Park and Phillips (1999). □

We are now in a position to prove the main theorems of this paper.

Proof of Theorem 3.2:

Define $T_K(x) = T(x)I(|x| \leq K)$, $T'_K(x) = T(x)I(x > K)$, and $T''_K(x) = T(x)I(x < -K)$. We will show that

$$\lim_{K\to\infty} \limsup_{n\to\infty} E|n^{-1/2-\alpha}\sum_{t=1}^{n} T'_K(n^{-\alpha}x_t)| = 0 \tag{3.39}$$

and the same argument, mutatis mutandis, will hold for $n^{-1/2-\alpha}\sum_{t=1}^{n} T''_K(n^{-\alpha}x_t)$. Then, we will show that for all $K > 0$,

$$n^{-1/2-\alpha}\sum_{t=1}^{n} T_K(n^{-\alpha}x_t) \xrightarrow{d} (\int_{-K}^{K} T(s)ds)L(1,0), \tag{3.40}$$

and the result then follows (for a formal proof that this is sufficient, see for example the start of the proof of Theorem 1 of de Jong (2001)). To show the result of Equation (3.39), note that for all $K > 0$,

$$|n^{-1/2-\alpha}\sum_{t=1}^{M} T(n^{-\alpha}x_t)I(n^{-\alpha}x_t > K)| \leq Mn^{-1/2}R(K) \to 0 \tag{3.41}$$

as $n \to \infty$, and

$$E|n^{-1/2-\alpha}\sum_{t=M+1}^{n} T(n^{-\alpha}x_t)I(n^{-\alpha}x_t > K)|$$

$$= E|\sum_{j=1}^{\infty} n^{-1/2-\alpha}\sum_{t=M+1}^{n} T(n^{-\alpha}x_t)I(Kj < n^{-\alpha}x_t \leq K(j+1))|$$

$$\leq E\sum_{j=1}^{\infty} n^{-1/2-\alpha}\sum_{t=M+1}^{n} R(Kj)I(Kjt^{-1/2}n^{\alpha} < t^{-1/2}x_t \leq K(j+1)t^{-1/2}n^{\alpha})$$

$$\leq \sum_{j=1}^{\infty} n^{-1/2}\sum_{t=1}^{n} R(Kj)CKt^{-1/2}$$

$$\leq C(\sup_{n\geq 1} n^{-1/2}\sum_{t=1}^{n} t^{-1/2})K\sum_{j=1}^{\infty} R(Kj)$$

$$= C'\int_{1}^{\infty} R(K[j])d(Kj)$$

$$= C' \int_K^\infty R(K[x/K])dx = C' \int_K^{2K} R(K[x/K])dx + C' \int_{2K}^\infty R(K[x/K])dx$$

$$\leq C'(KR(K) + \int_K^\infty R(x)dx) \to 0 \qquad (3.42)$$

as $K \to \infty$, where $C' = C \sup_{n \geq 1} n^{-1/2} \sum_{t=1}^{n} t^{-1/2}$, and $KR(K) \to 0$ under the assumptions of the theorem because

$$R(2K)K \leq \int_K^{2K} R(x)dx \leq \int_K^\infty R(x)dx \to 0 \qquad (3.43)$$

as $K \to \infty$. The first inequality follows from the assumed boundedness of $|T(.)|$ by $R(.)$ and the assumed monotonicity of $R(.)$, and the second is an application of Lemma 3.1. This completes the proof of the result of Equation (3.39). The remainder of the proof follows the line of proof of Park and Phillips (1999, proof of Theorem 5.1), but some modifications will be made. In order to show the result of Equation (3.40) and thereby make the proof of Theorem 3.2 complete, define for $\delta > 0$

$$T^\delta(x) = \int_{-K/\delta}^{K/\delta - 1} T(j\delta) I(j\delta \leq n^{-\alpha} x_t \leq (j+1)\delta) dj, \qquad (3.44)$$

and note that for all $K > 0$, $\int_{-K/\delta}^{K/\delta-1} I(j\delta \leq n^{-\alpha}x_t \leq (j+1)\delta)dj = I(|n^{-\alpha}x_t| \leq K)$, and therefore

$$E|n^{-1/2-\alpha} \sum_{t=1}^{n} (T_K(n^{-\alpha}x_t) - T^\delta(n^{-\alpha}x_t))|$$

$$= E| \int_{-K/\delta}^{K/\delta-1} n^{-1/2-\alpha} \sum_{t=1}^{n} (T(j\delta) - T(n^{-\alpha}x_t)) I(j\delta \leq n^{-\alpha}x_t \leq (j+1)\delta)dj|$$

$$\leq \sup_{x \in [-K,K]} \sup_{x' \in [-K,K]: |x-x'| \leq \delta} |T(x) - T(x')| E \int_{-K/\delta}^{K/\delta-1} n^{-1/2-\alpha} \sum_{t=1}^{n} I(j\delta \leq n^{-\alpha}x_t \leq (j+1)\delta)dj$$

$$= \sup_{x \in [-K,K]} \sup_{x' \in [-K,K]: |x-x'| \leq \delta} |T(x) - T(x')| n^{-1/2-\alpha} \sum_{t=1}^{n} P(-n^\alpha t^{-1/2} K \leq t^{-1/2} x_t \leq n^\alpha t^{-1/2} K)$$

$$\leq \sup_{x \in [-K,K]} \sup_{x' \in [-K,K]: |x-x'| \leq \delta} |T(x) - T(x')| n^{-1/2} \sum_{t=1}^{n} 2CKt^{-1/2}$$

$$\leq 2C'K \sup_{x \in [-K,K]} \sup_{x' \in [-K,K]: |x-x'| \leq \delta} |T(x) - T(x')| \to 0 \tag{3.45}$$

as $\delta \to 0$ by continuity of $T(.)$, where the second inequality is Lemma 3.1. Therefore, we can consider $n^{-1/2-\alpha} \sum_{t=1}^{n} T^{\delta}(n^{-\alpha}x_t)$ instead of $n^{-1/2-\alpha} \sum_{t=1}^{n} T_K(n^{-\alpha}x_t)$. Now

$$n^{-1/2-\alpha} \sum_{t=1}^{n} T^{\delta}(n^{-\alpha}x_t)$$

$$= \int_{K/\delta}^{K/\delta-1} T(j\delta) n^{-1/2-\alpha} \sum_{t=1}^{n} I(j\delta \leq n^{-\alpha}x_t \leq (j+1)\delta) dj$$

$$= \sum_{-K/\delta}^{K/\delta-1} T(j\delta) n^{1/2-\alpha} N_n(n^{1/2-\alpha}; j\delta, (j+1)\delta), \tag{3.46}$$

and

$$| \int_{-K/\delta}^{K/\delta-1} T(j\delta) n^{1/2-\alpha} N_n(n^{1/2-\alpha}; j\delta, (j+1)\delta) dj$$

$$- \int_{-K/\delta}^{K/\delta-1} T(j\delta) dj n^{1/2-\alpha} N_n(n^{1/2-\alpha}; 0, \delta)| = o_p(1) \tag{3.47}$$

because by the Cauchy-Schwartz inequality,

$$E(\int_{-K/\delta}^{K/\delta-1} T(j\delta) n^{1/2-\alpha} N_n(n^{1/2-\alpha}; j\delta, (j+1)\delta) dj - \int_{-K/\delta}^{K/\delta-1} T(j\delta) dj n^{1/2-\alpha} N_n(n^{1/2-\alpha}; 0, \delta))^2$$

$$\leq n^{1-2\alpha} \int_{-K/\delta}^{K/\delta} R(j\delta)^2 dj \int_{-K/\delta}^{K/\delta} E(N_n(n^{1/2-\alpha}; j\delta, (j+1)\delta) - N_n(n^{1/2-\alpha}; 0, \delta))^2 dj$$

$$\leq n^{1-2\alpha} \int_{-K/\delta}^{K/\delta} R(j\delta)^2 dj \int_{-K/\delta}^{K/\delta} c(\delta n^{-3/2+\alpha})(1 + |j|\delta^2 \log(n) n^{2\alpha}) dj$$

$$\leq n^{-1/2-\alpha}(1/\delta)(\int_{-K}^{K} R(s)^2 ds) c2K(1 + K\delta n^{2\alpha} \log(n)) = o(1), \tag{3.48}$$

where the second inequality is Lemma 3.2. Therefore, it suffices to consider

$$\int_{-K/\delta}^{K/\delta-1} T(j\delta) dj n^{1/2-\alpha} N_n(n^{1/2-\alpha}; 0, \delta) = \delta^{-1} \int_{-K}^{K-\delta} T(s) ds n^{1/2-\alpha} N_n(n^{1/2-\alpha}; 0, \delta).$$

Now note that

$$|n^{1/2-\alpha} N_n(n^{1/2-\alpha}; 0, \delta) - n^{1/2-\alpha} N(n^{1/2-\alpha}; 0, \delta)| = o_p(n^{1/2-\alpha} n^{-(2p-1)/3p})$$

$$= o_p(n^{(1-p/2)/(3p)}) = o_p(1) \tag{3.49}$$

by the second part of Lemma 3.2. Therefore,

$$|\int_{-K}^{K-\delta} T(s)dsn^{1/2-\alpha}N_n(n^{1/2-\alpha};0,\delta) - \int_{-K}^{K-\delta} T(s)dsn^{1/2-\alpha}N(n^{1/2-\alpha};0,\delta)| = o_p(1), \tag{3.50}$$

implying that it suffices to analyze

$$(\int_{-K}^{K-\delta} T(s)ds)(\delta^{-1}n^{1/2-\alpha}N(n^{1/2-\alpha};0,\delta)). \tag{3.51}$$

As $n \to \infty$,

$$\delta^{-1}n^{1/2-\alpha}N(n^{1/2-\alpha};0,\delta) \to L(1,0) \quad \text{almost surely}, \tag{3.52}$$

as explained in the text following Lemma 2.5 of Park and Phillips (1999). In addition, as $\delta \to 0$, by continuity of $T(.)$,

$$\int_{-K}^{K-\delta} T(s)ds \to \int_{-K}^{K} T(s)ds. \tag{3.53}$$

Therefore,

$$n^{-1/2-\alpha}\sum_{t=1}^{n} T_K(n^{-\alpha}x_t) \xrightarrow{d} (\int_{-K}^{K} T(s)ds)L(1,0), \tag{3.54}$$

implying that the condition of Equation (3.40) is now verified. This completes the proof. □

For the proof of Theorem 3.3, we need the following lemma:

Lemma 3.3 *Under Assumption 3.1, for any $K > 0$,*

$$n^{-1}\sum_{t=1}^{n} I(n^{-1/2}x_t \leq x) \Rightarrow \int_0^1 I(W(r) \leq x)dr, \tag{3.55}$$

where "$\Rightarrow$" denotes weak convergence in $D[-K,K]$ (i.e. the space of functions that are continuous on $[0,1]$ except for a finite number of discontinuities).

Proof of Lemma 3.3:

Pointwise in x, the result follows from Theorem 3.2 of Park and Phillips (1999), and therefore it suffices to show stochastic equicontinuity of $n^{-1}\sum_{t=1}^{n} I(n^{-1/2}x_t \leq x)$. By the Skorokhod representation, we can assume that $\sup_{r\in[0,1]}|n^{-1/2}x_{[rn]} - W(r)| \xrightarrow{as} 0$. Then for n large enough, $\sup_{r\in[0,1]}|n^{-1/2}x_{[rn]} - W(r)| \leq \delta$ almost surely, implying that for n large enough

$$\sup_{|x|\leq K}\sup_{x':x<x'<x+\delta}|n^{-1}\sum_{t=1}^{n}(I(n^{-1/2}x_t \leq x) - I(n^{-1/2}x_t \leq x'))|$$

$$\leq \sup_{|x|\leq K} n^{-1}\sum_{t=1}^{n} I(x \leq n^{-1/2}x_t \leq x+\delta)$$

$$\leq \sup_{|x|\leq K}\int_0^1 I(x-\delta \leq W(r) \leq x+2\delta)dr$$

$$= \sup_{|x|\leq K}\int_{x-\delta}^{x+2\delta} L(1,s)ds \leq 3\delta \sup_{|s|\leq K}|L(1,s)| \tag{3.56}$$

where the equality follows from the occupation times formula (see Park and Phillips (1999, Lemma 2.4)) and because $\sup_{|s|\leq K}|L(1,s)|$ is a well-defined random variable. The above chain of inequalities establishes stochastic equicontinuity of $n^{-1}\sum_{t=1}^{n} I(n^{-1/2}x_t \leq x)$, which completes the proof. □

Proof of Theorem 3.3:

Because $\sup_{1\leq t\leq n} n^{-1/2}|x_t| = O_p(1)$, it now suffices to show that for any $K > 0$,

$$\nu(n^{1/2-\alpha})^{-1}n^{-1}\sum_{t=1}^{n} T(n^{-\alpha}x_t)I(|n^{-1/2}x_t| \leq K) \xrightarrow{d} \int_0^1 H(W(r))I(|W(r)| \leq K)dr$$

$$= \int_{-K}^{K} H(s)L(1,s)ds. \tag{3.57}$$

Now, by Lemma 3.3, $n^{-1}\sum_{t=1}^{n} I(n^{-1/2}x_t \leq x) \Rightarrow \int_0^1 I(W(r) \leq x)dr$. By the Skorokhod Representation Theorem, we can assume without loss of generality that $|n^{-1}\sum_{t=1}^{n} I(n^{-1/2}x_t \leq x) - \int_0^1 I(W(r) \leq x)dr| = c_n \xrightarrow{as} 0$. Now for all $\delta > 0$, let

$$S_{1n\delta} = S_{1n} = \nu(n^{1/2-\alpha})^{-1} n^{-1} \sum_{t=1}^{n} T(n^{-\alpha}x_t) I(|n^{-1/2}x_t| \leq K)$$

$$= \nu(n^{1/2-\alpha})^{-1} \int_{-K/\delta}^{K/\delta - 1} n^{-1} \sum_{t=1}^{n} T(n^{-\alpha}x_t) I(j\delta \leq n^{-1/2}x_t \leq (j+1)\delta) dj, \quad (3.58)$$

$$S_{2n\delta} = \nu(n^{1/2-\alpha})^{-1} \int_{-K/\delta}^{K/\delta - 1} T(n^{1/2-\alpha}j\delta) n^{-1} \sum_{t=1}^{n} I(j\delta \leq n^{-1/2}x_t \leq (j+1)\delta) dj, \quad (3.59)$$

$$S_{3n\delta} = \nu(n^{1/2-\alpha})^{-1} \int_{-K/\delta}^{K/\delta - 1} T(n^{1/2-\alpha}j\delta) \int_0^1 I(j\delta \leq W(r) \leq (j+1)\delta) dr\, dj, \quad (3.60)$$

$$S_{4n\delta} = \nu(n^{1/2-\alpha})^{-1} \int_{-K/\delta}^{K/\delta - 1} T(n^{1/2-\alpha}j\delta)\delta L(1, j\delta) dj$$

$$= \nu(n^{1/2-\alpha})^{-1} \int_{-K}^{K-\delta} T(n^{1/2-\alpha}s) L(1, s) ds, \quad (3.61)$$

$$S_{5n\delta} = S_5 = \int_{-K}^{K} H(s) L(1, s) ds = \int_0^1 H(W(r)) I(|W(r)| \leq K) dr. \quad (3.62)$$

We will show that $\lim_{\delta \to 0} \limsup_{n \to \infty} |S_{jn\delta} - S_{j+1,n\delta}| = 0$ almost surely for $j = 1, \ldots, 4$. By the monotone regular condition, we can act as if $T(.)$ is monotone without loss of generality. For $|S_1 - S_{2n\delta}|$ we then have

$$\limsup_{n \to \infty} |S_1 - S_{2n\delta}|$$

$$\leq \limsup_{n \to \infty} \nu(n^{1/2-\alpha})^{-1} \int_{-K/\delta}^{K/\delta - 1} n^{-1} \sum_{t=1}^{n} |T(n^{-\alpha}x_t) - T(n^{1/2-\alpha}j\delta)| I(j\delta \leq n^{-1/2}x_t \leq (j+1)\delta)$$

$$\leq \limsup_{n \to \infty} \nu(n^{1/2-\alpha})^{-1} \int_{-K/\delta}^{K/\delta - 1} n^{-1} \sum_{t=1}^{n} |T(n^{1/2-\alpha}(j+1)\delta) - T(n^{1/2-\alpha}j\delta)| I(j\delta \leq n^{-1/2}x_t \leq (j+1$$

$$\leq \limsup_{n \to \infty} \int_{-K/\delta}^{K/\delta - 1} |\nu(n^{1/2-\alpha})^{-1} T(n^{1/2-\alpha}(j+1)\delta)$$

$$-\nu(n^{1/2-\alpha})^{-1}T(n^{1/2-\alpha}j\delta) - H((j+1)\delta) + H(j\delta)|dj$$

$$+\int_{-K/\delta}^{K/\delta-1} |H((j+1)\delta) - H(j\delta)|dj = \int_{-K}^{K-\delta} |H(x+\delta) - H(x)|dx, \quad (3.63)$$

and as $\delta \to 0$, the last term disappears because of continuity of $H(.)$, the second inequality follows from monotonicity of $T(.)$, and the third by our definition of an asymptotically homogeneous function. To show that $\lim_{\delta\to 0}\limsup_{n\to\infty}|S_{2n\delta}-S_{3n\delta}| = 0$ almost surely, note that

$$|\nu(n^{1/2-\alpha})^{-1}\int_{-K/\delta}^{K/\delta-1} T(n^{1/2-\alpha}j\delta)(n^{-1}\sum_{t=1}^{n} I(j\delta \le n^{-1/2}x_t \le (j+1)\delta)$$

$$-\int_0^1 I(j\delta \le W(r) \le (j+1)\delta)dr)dj|$$

$$\le 2c_n\nu(n^{1/2-\alpha})^{-1}\int_{-K/\delta}^{K/\delta-1} |T(n^{1/2-\alpha}j\delta)|dj$$

$$\le 2c_n\delta^{-1}\int_{-K}^{K} |\nu(n^{1/2-\alpha})^{-1}T(n^{1/2-\alpha}x)-H(x)|dx+2c_n\delta^{-1}\int_{-K}^{K}|H(x)|dx = o(1)(3.64)$$

almost surely under our assumptions and by the definition of c_n. For $|S_{3n\delta} - S_{4n\delta}|$ we have

$$|S_{3n\delta} - S_{4n\delta}|$$

$$\le \nu(n^{1/2-\alpha})^{-1}\int_{-K/\delta}^{K/\delta} \delta|T(n^{1/2-\alpha}j\delta)|(\delta^{-1}\int_0^1 I(j\delta \le W(r) \le (j+1)\delta)dr - L(1,j\delta))dj$$

$$\le \nu(n^{1/2-\alpha})^{-1}\int_{-K/\delta}^{K/\delta} \delta|T(n^{1/2-\alpha}j\delta)|dj \sup_{|x|\le K} |\delta^{-1}\int_0^1 I(x \le W(r) \le x+\delta)dr-L(1,x)|.(3.65)$$

By the earlier argument,

$$\sup_{n\ge 1}\sup_{\delta>0}\nu(n^{1/2-\alpha})^{-1}\int_{-K/\delta}^{K/\delta} \delta|T(n^{1/2-\alpha}j\delta)|dj < \infty, \quad (3.66)$$

and therefore it suffices to show that as $\delta \to 0$,

$$\sup_{|x|\le K} |\delta^{-1}\int_0^1 I(x \le W(r) \le x+\delta)dr - L(1,x)| \to 0. \quad (3.67)$$

By the occupation times formula, the above expression satisfies

$$\sup_{|x|\le K} |\delta^{-1}\int_x^{x+\delta} L(1,s)ds - L(1,x)| = \sup_{|x|\le K} |\delta^{-1}\int_x^{x+\delta} (L(1,s) - L(1,x))ds|$$

$$\le \sup_{|x|\le K} \sup_{s\in[x,x+\delta]} |L(1,s) - L(1,x)| \to 0 \quad \text{as } \delta \to 0 \tag{3.68}$$

by uniform continuity of $L(1,.)$ on $[-K, K]$. Finally, for $|S_{4n\delta} - S_5|$, we have

$$\lim_{n\to\infty} |\int_{-K}^{K} (\nu(n^{1/2-\alpha})^{-1}T(n^{1/2-\alpha}s) - H(s))L(1,s)ds|$$

$$\le \sup_{|s|\le K} |L(1,s)| \lim_{n\to\infty} \int_{-K}^{K} |\nu(n^{1/2-\alpha})^{-1}T(n^{1/2-\alpha}s) - H(s)|ds = 0 \tag{3.69}$$

by the definition of an asymptotically homogeneous function, which completes the proof. □

The following lemma is needed for the proof of Theorem 3.4.

Lemma 3.4 *For any sequence b_n such that $c_n = o(b_n)$, under the assumptions of Theorem 3.4,*

$$\lim_{\delta\to 0}\lim_{n\to\infty} \sum_{j=0}^{\infty} T(j\delta c_n)I((j+1)\delta c_n > c_n)I(j\delta c_n \le b_n)d_n^{-1}\delta c_n = 1. \tag{3.70}$$

Proof of Lemma 3.4:

This result follows because

$$\sum_{j=0}^{\infty} T(j\delta c_n)I((j+1)\delta c_n > c_n)I(j\delta c_n \le b_n)d_n^{-1}\delta c_n$$

$$= \int_{j=0}^{\infty} T([j]\delta c_n)I(([j]+1)\delta c_n > c_n)I([j]\delta c_n \le b_n)d_n^{-1}\delta c_n dj$$

$$\le \int_{j=0}^{\infty} T((j-1)\delta c_n)I((j+1)\delta c_n > c_n)I((j-1)\delta c_n \le b_n)d_n^{-1}\delta c_n dj$$

$$= \int_{x=0}^{\infty} T(x)I(x + 2\delta c_n > c_n)I(x \leq b_n)d_n^{-1}dx$$

$$= (\int_{x=c_n}^{1} T(x)dx)^{-1} \int_{x=c_n(1-2\delta)}^{b_n} T(x)dx. \tag{3.71}$$

Now because $T(x) = |x|^{-m}I(x > 0)$, the last expression equals

$$(m-1)(c_n^{1-m} - 1)^{-1}(m-1)^{-1}((c_n(1-2\delta))^{1-m} - b_n^{1-m})$$

$$= (c_n^{1-m} - 1)^{-1}((c_n(1-2\delta))^{1-m} - b_n^{1-m}), \tag{3.72}$$

and because $m > 1$ and $c_n = o(b_n)$, the result now follows. A similar argument will hold for a lower bound, which then completes the proof of the lemma. □

Proof of Theorem 3.4:

Note that, for $b_n = c_n^{1-1/m-\alpha}$ for some $\alpha > 0$ small enough that $b_n \to 0$ and $d_n^{-1}T(b_n) \to 0$ as $n \to \infty$,

$$d_n^{-1}n^{-1}\sum_{t=1}^{n} T(n^{-1/2}x_t)I(n^{-1/2}x_t > c_n)$$

$$= d_n^{-1}n^{-1}\sum_{t=1}^{n} T(n^{-1/2}x_t)I(n^{-1/2}x_t > c_n)I(n^{-1/2}x_t \leq b_n)$$

$$+d_n^{-1}n^{-1}\sum_{t=1}^{n} T(n^{-1/2}x_t)I(n^{-1/2}x_t > b_n), \tag{3.73}$$

and the second term is $o_p(1)$ because

$$d_n^{-1}n^{-1}\sum_{t=1}^{n} T(n^{-1/2}x_t)I(n^{-1/2}x_t > b_n)$$

$$\leq d_n^{-1}T(b_n) \to 0 \tag{3.74}$$

by assumption. Now note that trivially, for all $\delta > 0$, defining $W_n(r) = n^{-1/2}x_{[nr]}$,

$$d_n^{-1}n^{-1}\sum_{t=1}^{n} T(n^{-1/2}x_t)I(n^{-1/2}x_t > c_n)I(n^{-1/2}x_t \leq b_n)$$

$$= \sum_{j=0}^{\infty} d_n^{-1} \int_0^1 T(W_n(r)) I(W_n(r) > c_n) I(W_n(r) \leq b_n) I(j\delta c_n \leq W_n(r) < (j+1)\delta c_n) dr. \quad (3.75)$$

An upper bound for the last term is

$$\sum_{j=0}^{\infty} T(j\delta c_n) d_n^{-1} \int_0^1 I(W_n(r) > c_n) I(W_n(r) \leq b_n) I(j\delta c_n \leq W_n(r) < (j+1)\delta c_n) dr$$

$$\leq \sum_{j=0}^{\infty} T(j\delta c_n) I((j+1)\delta c_n > c_n) I(j\delta c_n \leq b_n) d_n^{-1} \int_0^1 I(j\delta c_n \leq W_n(r) < (j+1)\delta c_n) dr$$

$$= \sum_{j=0}^{\infty} T(j\delta c_n) I((j+1)\delta c_n > c_n) I(j\delta c_n \leq b_n) d_n^{-1} N_n(1; j\delta c_n, (j+1)\delta c_n). \quad (3.76)$$

Similarly, a lower bound is

$$\sum_{j=0}^{\infty} T((j+1)\delta c_n) I(j\delta c_n > c_n) I((j+1)\delta c_n \leq b_n) d_n^{-1} N_n(1; j\delta c_n, (j+1)\delta c_n). \quad (3.77)$$

We will only consider the upper bound and determine its limit, but the argument for the lower bound is identical and renders the same limit. By Lemma 3.2,

$$E \sum_{j=0}^{\infty} T(j\delta c_n) I((j+1)\delta c_n > c_n) I(j\delta c_n \leq b_n) d_n^{-1} |N_n(1; j\delta c_n, (j+1)\delta c_n) - N_n(1; 0, \delta c_n)|$$

$$\leq \sum_{j=0}^{\infty} T(j\delta c_n) I((j+1)\delta c_n > c_n) I(j\delta c_n \leq b_n) d_n^{-1} (c(\delta c_n/n)(1 + (j(\delta c_n)^2 n \log(n))))^{1/2}$$

$$\leq (d_n^{-1} \delta c_n \sum_{j=0}^{\infty} T(j\delta c_n) I((j+1)\delta c_n > c_n) I(j\delta c_n \leq b_n))$$

$$\times \delta^{-1} c_n^{-1} (c(\delta c_n/n)(1 + ((b_n/(\delta c_n))(\delta c_n)^2 n \log(n))))^{1/2}. \quad (3.78)$$

Now, by Lemma 3.4,

$$\lim_{\delta \to 0} \limsup_{n \to \infty} d_n^{-1} \delta c_n \sum_{j=0}^{\infty} T(j\delta c_n) I((j+1)\delta c_n > c_n) I(j\delta c_n \leq b_n) = 1, \quad (3.79)$$

and therefore the expression of Equation (3.78) converges to zero in probability if

$$c_n^{-2}((c_n/n) + (c_n/n)((b_n/(c_n))(c_n)^2 n \log(n))) \to 0. \quad (3.80)$$

First, note that by assumption $c_n^{-1}n^{-1} \to 0$, and that the second part of the above expression is

$$O(b_n \log(n)) = o(1) \tag{3.81}$$

by assumption. Therefore, it suffices to consider

$$\sum_{j=0}^{\infty} T(j\delta c_n)I((j+1)\delta c_n > c_n)I(j\delta c_n \leq b_n)d_n^{-1}\delta c_n(N_n(1;0,\delta c_n)/(\delta c_n)). \tag{3.82}$$

Now by the comment following Lemma 2.5 in Park and Phillips (1999),

$$N_n(1;0,\delta c_n)/(\delta c_n) = L(1,0) + o_p(1) \tag{3.83}$$

if $\delta c_n \geq n^{-(2p-1)/3p+\eta}$ for some $\eta > 0$, which is the case by assumption for n large enough. Therefore, we only need consider

$$L(1,0)d_n^{-1}\delta c_n \sum_{j=0}^{\infty} T(j\delta c_n)I((j+1)\delta c_n > c_n)I(j\delta c_n \leq b_n). \tag{3.84}$$

Now by Lemma 3.4, it follows that by choosing δ arbitrarily small, the limit distribution will be arbitrarily close to $L(1,0)$; and noting that the same argument will work for the lower bound, this suffices to prove the result. □

CHAPTER 4

Unit root tests when the data are a trigonometric transformation of an integrated process

4.1 Introduction

Unit root tests were first studied by Dickey and Fuller (1979) with proof and simulations. From this beginning paper, unit root testing became mainstream research in time series econometrics. It is a widely believed that many time series in macroeconomics are I(1) processes, as argued by Nelson and Plosser (1982). Economists have concentrated on how to test for a possible unit root in data series. After the Dickey-Fuller unit root test, Said and Dickey (1984) and Phillips and Perron (1988) proposed revised unit root tests to take into account the possible autoregressive-moving average in errors. Their papers corrected the drawbacks of the Dickey-Fuller unit root test. For the development of unit root tests in econometric time series, see Phillips and Xiao (1998).

In international finance and macroeconomics, there are a lot of nonlinear models for time series, for example used for modelling the real exchange rate. For empirical reasons, researchers often use nonlinear transformations to transform integrated time series. One important question is whether the unit root phenomena still exist after transformation. The first paper to discuss this question is Granger and Hallman (1988, 1991). They used simulation to analyze the characteristics of unit root tests when the data is a function of an integrated process. After Granger and Hallman's paper, Ermini and Granger (1993) established some asymptotic properties for transformations of I(1) processes under normality assumptions. Following these three papers, Franses and Koop (1998), Franses and McAleer (1998), and Kobayashi and McAleer (1999) analyzed unit root tests when the data are functions of an integrated process. They find that Dickey-Fuller tests are sensitive to nonlinear transformations; for example, it can happen that a variable is found to be nonstationary in level, but stationary after transformations. They consider the logarithm transformation of integrated time series and propose the revision for sensitive problem when we use augment Dickey-Fuller unit root tests under transformed integrated process. But all these papers only study the logarithm transformation. About other functional forms, they do not establish theoretical results.

This chapter establishes analytically what the asymptotic behavior of the Dickey-Fuller unit root tests will be when the true data-generating process is a trigonometric function of an integrated process. For example, the data could be generated as $\sin(x_t)$, where x_t is an integrated series. This problem has been analyzed mainly through simulations in Granger and Hallman (1991), and this chapter gives the mathematical underpinning for their conclusions. Another paper that is related is Ermini and Granger (1993); in that paper, various moments and covariances are calculated that involve functions of integrated processes. Ermini and Granger's (1993) results are obtained by strongly relying on a normality assumption. In this chapter, we try to relax the normality assumptions under Ermini and Granger. We only keep the symmetric distribution of residual item,ε_t, and obtain the asymptotical distribution of Dickey-Fuller unit root tests under periodic transformation of integrated process.

One important tool for the analysis of this chapter is provided in de Jong (2001). In that paper, it is established that for functions $T(.)$ that are periodic on $[-\pi, \pi]$, for an integrated process x_t that satisfies some regularity conditions,

$$n^{-1/2} \sum_{t=1}^{n} (T(x_t) - (2\pi)^{-1} \int_{-\pi}^{\pi} T(x)dx) \xrightarrow{d} N(0, \sigma^2), \tag{4.1}$$

where "$\xrightarrow{d}$" denotes convergence in distribution. This paper extends the tools developed in de Jong (2001) somewhat in order to arrive at a complete asymptotic analysis of the problem under consideration.

4.2 Assumption and main results

We consider a time series x_t generated by

$$x_t = x_{t-1} + \varepsilon_t \tag{4.2}$$

where ε_t is a sequence of independent and identical distributed mean zero random variables with a continuous distribution, a mean of zero, and a variance σ^2. $\mathcal{F}_t = \sigma(\varepsilon_t, \varepsilon_{t-1}, \ldots, \varepsilon_1, x_0)$ is the sigma field that includes all the information in ε_t until time period t. Below, let $\hat{\rho}$ denote the regression coefficient resulting from a regression of y_t on y_{t-1}, and let $\hat{\rho}_\mu$ denote the regression coefficient resulting from a regression of y_t on y_{t-1} and a constant. In all results below, we will allow for both $y_t = \sin(x_t)$ and $y_t = \cos(x_t)$, but as intuition suggests, for both choices of y_t the asymptotic results are identical.

For the convergence behavior of $\hat{\rho}$ and $\hat{\rho}_\mu$, the following result can be established:

Theorem 4.1 *For the process x_t as defined before, for $y_t = \sin(x_t)$,*

$$\hat{\rho} \xrightarrow{p} E\cos(\varepsilon_t) \quad \text{and} \quad \hat{\rho}_\mu \xrightarrow{p} E\cos(\varepsilon_t), \tag{4.3}$$

and similarly for $y_t = \cos(x_t)$,

$$\hat{\rho} \xrightarrow{p} E\cos(\varepsilon_t) \quad \text{and} \quad \hat{\rho}_\mu \xrightarrow{p} E\cos(\varepsilon_t). \tag{4.4}$$

In the theorem above and elsewhere in this chapter, " $\xrightarrow{p}$ " denotes convergence in probability. All proofs for this chapter are deferred to the Mathematical Appendix. For the regression coefficients $\hat{\rho}$ and $\hat{\rho}_\mu$, the following theorem establishes root-n consistency and asymptotic normality under the additional assumption that the distribution of ε_t is symmetric:

Theorem 4.2 *For the processes x_t defined before, if ε_t has a symmetric distribution, for $y_t = \sin(x_t)$,*

$$n^{1/2}(\hat{\rho} - E\cos(\varepsilon_t)) \xrightarrow{d} N(0, V) \quad \text{and} \quad n^{1/2}(\hat{\rho}_\mu - E\cos(\varepsilon_t)) \xrightarrow{d} N(0, V), \tag{4.5}$$

and similarly for $y_t = \cos(x_t)$,

$$n^{1/2}(\hat{\rho} - E\cos(\varepsilon_t)) \xrightarrow{d} N(0, V) \quad \text{and} \quad n^{1/2}(\hat{\rho}_\mu - E\cos(\varepsilon_t)) \xrightarrow{d} N(0, V), \tag{4.6}$$

where

$$V = (3/8)E(\cos(\varepsilon_t) - E\cos(\varepsilon_t))^2 + (1/8)E(\sin(\varepsilon_t))^2. \tag{4.7}$$

The above theorem implies that the Dickey-Fuller coefficient tests will go off to $-\infty$ at rate n - the same rate as would apply for stationary processes y_t - and therefore the Dickey-Fuller coefficient tests will asymptotically indicate stationarity. Finally, we establish the asymptotic behavior of the Dickey-Fuller t-statistics $\hat{t}$ and $\hat{t}_\mu$ for the coefficients of y_{t-1} resulting respectively from a regression of Δy_t on y_{t-1} and a regression of Δy_t on y_{t-1} and a constant:

Theorem 4.3 *For the processes x_t defined before, for $y_t = \sin(x_t)$, defining*

$$c = (E\cos(\varepsilon_t) - 1)(1 - (E\cos(\varepsilon_t))^2)^{-1/2}, \tag{4.8}$$

we have

$$n^{-1/2}\hat{t} \xrightarrow{p} c \quad \text{and} \quad n^{-1/2}\hat{t}_\mu \xrightarrow{p} c, \tag{4.9}$$

and similarly for $y_t = \cos(x_t)$,

$$n^{-1/2}\hat{t} \xrightarrow{p} c \quad \text{and} \quad n^{-1/2}\hat{t}_\mu \xrightarrow{p} c. \tag{4.10}$$

Unlike Theorem 4.2, the result of Theorem 4.3 does not rely on a symmetry assumption for the distribution of ε_t.

From our results, it is clear that the asymptotic behavior of the Dickey-Fuller coefficient and t-tests in terms of convergence rates is identical to that of the case of stationary random variables, and that the t-test will asymptotically indicate stationarity. This conclusion was also obtained through simulation in Granger and Hallman (1991).

4.3 Conclusion

In this chapter, we introduced the unit root test under trigonometric transformations. As is shown in the preceding theorems, the trigonometric transformation of an integrated process will result in a stationary process. When we use the Dickey-Fuller unit root test under trigonometric transformation, the test will diverge to $-\infty$. These results support the Monte Carlo simulation of Granger and Hallman. Compared with the Ermini and Granger paper, we only keep symmetric distribution and relax all normality assumptions. From our proof, we can obtain more generalized results about trigonometric transformation of an integrated process.

4.4 Mathematical Appendix

Proof of Theorem 4.1:

We will consider $y_t = \sin(x_t)$; the case $y_t = \cos(x_t)$ is analogous. By the law of large numbers for bounded martingale difference sequences,

$$n^{-1}\sum_{t=2}^{n}(\sin(x_t)\sin(x_{t-1}) - E(\sin(x_t)\sin(x_{t-1})|\mathcal{F}_{t-1})) \xrightarrow{p} 0, \tag{4.11}$$

and therefore in order to find the probability limit of $(n-1)^{-1}\sum_{t=2}^{n} y_t y_{t-1}$, it suffices to consider

$$n^{-1}\sum_{t=2}^{n} E(\sin(x_t)\sin(x_{t-1})|\mathcal{F}_{t-1})$$

$$= n^{-1}\sum_{t=2}^{n}(\sin(x_{t-1})^2 E\cos(\varepsilon_t) + \sin(x_{t-1})\cos(x_{t-1})E\sin(\varepsilon_t))$$

$$= E\cos(\varepsilon_t)n^{-1}\sum_{t=2}^{n}\sin(x_{t-1})^2 + E\sin(\varepsilon_t)n^{-1}\sum_{t=2}^{n}\sin(x_{t-1})\cos(x_{t-1}). \tag{4.12}$$

From Theorem 2 of de Jong (2001), i.e. the result of Equation (4.1), it follows that

$$n^{-1}\sum_{t=2}^{n}\sin(x_{t-1})^2 \xrightarrow{p} 1/2 \quad \text{and} \quad n^{-1}\sum_{t=2}^{n}\sin(x_{t-1})\cos(x_{t-1}) \xrightarrow{p} 0, \tag{4.13}$$

and therefore

$$n^{-1}\sum_{t=2}^{n} E(\sin(x_t)\sin(x_{t-1})|\mathcal{F}_{t-1}) \xrightarrow{p} (1/2)E\cos(\varepsilon_t). \tag{4.14}$$

It now follows that

$$\hat{\rho} = \frac{n^{-1}\sum_{t=2}^{n} y_t y_{t-1}}{n^{-1}\sum_{t=2}^{n} y_{t-1}^2} \xrightarrow{p} \frac{(1/2)E\cos(\varepsilon_t)}{(1/2)} = E\cos(\varepsilon_t). \tag{4.15}$$

For $\hat{\rho}_\mu$, the same result follows by noting that

$$\hat{\rho}_\mu = \frac{n^{-1}\sum_{t=2}^{n}(y_t - \bar{y})(y_{t-1} - \bar{y})}{n^{-1}\sum_{t=2}^{n}(y_{t-1} - \bar{y})^2}, \tag{4.16}$$

and by noting that again by the result of Equation (4.1), $\bar{y} \xrightarrow{p} 0$. □

Proof of Theorem 4.2:

Again, we will consider $y_t = \sin(x_t)$, and note that the case $y_t = \cos(x_t)$ is analogous. For such y_t,

$$n^{1/2}(\hat{\rho} - E\cos(\varepsilon_t)) = \frac{n^{-1/2}\sum_{t=2}^{n}(\sin(x_t)\sin(x_{t-1}) - \sin^2(x_{t-1})E\cos(\varepsilon_t))}{n^{-1}\sum_{t=2}^{n}\sin^2(x_{t-1})}$$

$$= \frac{n^{-1/2}\sum_{t=2}^{n}(\sin(x_{t-1})\cos(x_{t-1})\sin(\varepsilon_t) + \sin(x_{t-1})^2\cos(\varepsilon_t) - \sin(x_{t-1})^2 E\cos(\varepsilon_t))}{n^{-1}\sum_{t=2}^{n}\sin^2(x_t)}. \quad (4.17)$$

Now, noting that the denominator converges in probability to $1/2$ as before, and in addition note that the summands g_t in the numerator are martingale differences with respect to $\mathcal{F}_t$ by symmetry of the distribution of ε_t, which implies that $E\sin(\varepsilon_t) = 0$. We now apply the martingale difference central limit theorem; see e.g. Theorem 3.2 of Hall and Heyde (1980). To verify the conditions of this theorem, it now only remains to be shown that $n^{-1}\sum_{t=2}^{n} g_t^2 \xrightarrow{p} V \in (0, \infty)$. This will be true because

$$n^{-1}\sum_{t=2}^{n} g_t^2$$

$$= n^{-1}\sum_{t=2}^{n}\sin(x_{t-1})^2(\sin(x_{t-1})(\cos(\varepsilon_t) - E\cos(\varepsilon_t)) + \cos(x_{t-1})\sin(\varepsilon_t))^2$$

$$= n^{-1}\sum_{t=2}^{n}\sin(x_{t-1})^4(\cos(\varepsilon_t) - E\cos(\varepsilon_t))^2$$

$$+ n^{-1}\sum_{t=2}^{n}\sin(x_{t-1})^2\cos(x_{t-1})^2\sin(\varepsilon_t)^2$$

$$+ n^{-1}\sum_{t=2}^{n} 2\sin(x_{t-1})^3\cos(x_{t-1})(\cos(\varepsilon_t) - E\cos(\varepsilon_t))\sin(\varepsilon_t). \quad (4.18)$$

By the martingale difference law of large numbers, the last expression equals

$$o_P(1) + n^{-1}\sum_{t=1}^{n}\sin(x_{t-1})^4 E(\cos(\varepsilon_t) - E\cos(\varepsilon_t))^2$$

$$+ n^{-1}\sum_{t=1}^{n}\sin(x_{t-1})^2\cos(x_{t-1})^2 E(\sin(\varepsilon_t))^2$$

$$+n^{-1}\sum_{t=1}^{n} 2\sin(x_{t-1})^3\cos(x_{t-1})E((\cos(\varepsilon_t) - E\cos(\varepsilon_t))\sin(\varepsilon_t)). \tag{4.19}$$

By Theorem 2 of de Jong (2001) as quoted in Equation (4.1), we know that

$$n^{-1}\sum_{t=1}^{n} \sin(x_{t-1})^4 \xrightarrow{p} (2\pi)^{-1}\int_{-\pi}^{\pi} \sin^4(x)dx = 3/8, \tag{4.20}$$

$$n^{-1}\sum_{t=1}^{n} \sin(x_{t-1})^2\cos(x_{t-1})^2 \xrightarrow{p} (2\pi)^{-1}\int_{-\pi}^{\pi} \sin^2(x)\cos^2(x)dx = 1/8, \tag{4.21}$$

and

$$n^{-1}\sum_{t=1}^{n} 2\sin(x_{t-1})^3\cos(x_{t-1}) \xrightarrow{p} (2\pi)^{-1}\int_{-\pi}^{\pi} 2\sin(x)^3\cos(x)dx = 0. \tag{4.22}$$

Therefore, it follows that

$$n^{-1}\sum_{t=1}^{n} g_t^2 \xrightarrow{p} (3/8)E(\cos(\varepsilon_t) - E\cos(\varepsilon_t))^2 + (1/8)E(\sin(\varepsilon_t))^2 = V. \tag{4.23}$$

For $\hat{\rho}_\mu$, note that

$$|n^{1/2}(\hat{\rho}_\mu - \hat{\rho})| \leq o_P(1) + 2n^{1/2}\bar{y}^2 = O_P(n^{-1/2}), \tag{4.24}$$

implying that the same limit as for $n^{1/2}(\hat{\rho} - E\cos(\varepsilon_t))$ results for $n^{1/2}(\hat{\rho}_\mu - E\cos(\varepsilon_t))$ as well, and this observation completes the proof of Theorem 4.2. □

Proof of Theorem 4.3:

First note that, for both the cases $y_t = \sin(x_t)$ and $y_t = \cos(x_t)$, using the results obtained in the proof of Theorem 4.1,

$$s^2 = (n-1)^{-1}\sum_{t=2}^{n}(y_t - \hat{\rho}y_{t-1})^2$$

$$= o_P(1) + n^{-1}\sum_{t=2}^{n} y_t^2 - 2\hat{\rho}n^{-1}\sum_{t=2}^{n} y_t y_{t-1} + \hat{\rho}^2 n^{-1}\sum_{t=2}^{n} y_t^2$$

$$\xrightarrow{p} (1/2) - 2E\cos(\varepsilon_t)(1/2)E\cos(\varepsilon_t) + (E\cos(\varepsilon_t))^2(1/2) = (1/2) - (1/2)(E\cos(\varepsilon_t))^2. \quad (4.25)$$

Therefore, it now follows that

$$n^{-1/2}\hat{t} = n^{-1/2}\frac{(n-1)^{1/2}(\hat{\rho}-1)}{(s^2/((n-1)^{-1}\sum_{t=2}^{n} y_{t-1}^2))^{1/2}}$$

$$= n^{-1/2}\frac{(n-1)^{1/2}(\hat{\rho}-E\cos(\varepsilon_t))}{(s^2/((n-1)^{-1}\sum_{t=2}^{n} y_{t-1}^2))^{1/2}} + n^{-1/2}\frac{(n-1)^{1/2}(E\cos(\varepsilon_t)-1)}{(s^2/((n-1)^{-1}\sum_{t=2}^{n} y_{t-1}^2))^{1/2}}$$

$$\xrightarrow{p} (E\cos(\varepsilon_t)-1)(1-(E\cos(\varepsilon_t))^2)^{-1/2}. \quad (4.26)$$

For $n^{-1/2}\hat{t}_\mu$, the same result holds, because the $\bar{y}$ that would appear in the expression for $\hat{t}_\mu$ converges to 0 in probability, and therefore the difference between $n^{-1/2}\hat{t}$ and $n^{-1/2}\hat{t}_\mu$ converges to zero in probability asymptotically. □

CHAPTER 5

Some results on the asymptotics for threshold unit root test

5.1 Introduction

Economic time series data often show some sudden changes, as a result of an external shock. It is generally believed that linear time series models cannot capture such a structural change. One statistical model that attempts to capture such a sudden structural change in different regimes is the threshold autoregressive model developed by Tong (1990). The threshold autoregressive model captures regime switching based on the lagged values of the variables. This is a very attractive property for economists, but the threshold autoregressive model still has some drawbacks. One of main drawback is that for inference in threshold autoregressive models, there is limited theory for testing null hypotheses that imply a unit root. The first paper to investigate unit root structure in threshold autoregressive model is González and Gonzalo (1997). They present a threshold unit root (TUR) model that has either stable roots existing in all regimes or unit roots in at least one regime. In the context of their threshold unit root model, they derive the asymptotic distribution of a Dickey-Fuller t-test. However, their analysis has some problems. First, in González and Gonzalo's threshold unit root model, they consider the threshold value to be known and fixed. But generally in economics time series, threshold values can be unknown. Second, the threshold unit root test in González and Gonzalo consider unit root exists in one regime threshold autoregressive model. They test one of all regimes existing unit root against alternative hypothesis that threshold autoregressive model does not have unit root in any regimes. But their model does not consider the case that unit root exists in one regime of TUR model in advance and test the null hypothesis of a pure I(1) process against the alternative hypothesis of a TUR model that has one regime with unit root. For improvement of these drawbacks, we establish an asymptotic result that can be used for testing the null of a unit root ($\varphi = 0$) against the alternative of a threshold unit root model:

$$\Delta y_t = \begin{cases} \mu + \varepsilon_t & \text{if } |y_{t-1}| \leq C \\ \mu + \varphi y_{t-1} + \varepsilon_t & \text{if } |y_{t-1}| > C, \end{cases} \tag{5.1}$$

where $-2 < \varphi < 0$. We will relax the assumption that threshold value, C, is known, as in González and Gonzalo's TUR model, and we consider tests that have been optimized over the unidentified parameter,C.

This chapter is organized as follow. In Section 5.2, we will derive the appropriate asymptotic results. With results, we can establish the asymptotic distributions of Dickey-Fuller t-test in regression Δy_t on constant and $y_{t-1}I(y_{t-1} > C)$, optimized over a set of possible value of C. In section 5.3, we will explore the possible further extension with the asymptotics. The conclusion will be found in Section 5.4. All proofs are in Mathematical Appendix.

5.2 Main results

For developing the asymptotic distribution of the threshold unit root test, we will use results involving the *Brownian local time* and a result by Perkins (1982) involving convergence to Brownian local time. In Perkins' Theorem 1.1, it is shown that

$$(1/2)L_n(1,\pi) = \sum_{t=1}^{n-1} |n^{-1/2}y_t - \pi|(I(n^{-1/2}y_{t-1} \le \pi) - I(n^{-1/2}y_t \le \pi))$$

$$= \sum_{t=1}^{n-1} |n^{-1/2}y_t - \pi|(I(n^{-1/2}y_{t-1} \le \pi)I(n^{-1/2}y_t > \pi) + I(n^{-1/2}y_{t-1} > \pi)I(n^{-1/2}y_t \le \pi))$$

$$= \sum_{t=1}^{n-1} |n^{-1/2}y_t - \pi|I(\min(n^{-1/2}y_t, n^{-1/2}y_{t-1}) \le \pi \le \max(n^{-1/2}y_t, n^{-1/2}y_{t-1}))$$

$$\Rightarrow (1/2)L(1,\pi), \tag{5.2}$$

where $L(t,s)$ is the two-parameter stochastic process called "Brownian local time" and " $\Rightarrow$" denote weak convergence. In order to establish our results, we need the following assumption for ε_t.

Assumption 5.1 *ε_t is an i.i.d. sequence random variables with mean zero, variance σ^2 and $E|\varepsilon_t|^4 < \infty$. The distribution of ε_t is absolutely continuous with respect to the Lebesgue measure and has characteristic function $\psi(s)$ for which $\lim_{s\to\infty} s^\eta\psi(s) = 0$ for some $\eta > 0$.*

Assumption 5.1 implies Assumptions 1 and 2 of Park and Phillips (1999) and the assumptions of Theorem 1.2 from Perkins (1982), implying that we can combine results from both papers here. The following results now follow relatively easily from Perkins (1982):

Theorem 5.1 *Assume ε_t satisfies Assumption 5.1, and assume that $\Delta y_t = \varepsilon_t$ and $y_0 = 0$. Then,*

$$n^{-1/2}\sum_{t=1}^{n} \varepsilon_t I(n^{-1/2}y_{t-1} \le \pi) \Rightarrow \sigma \int_0^1 I(\sigma W(r) \le \pi)dW(r). \tag{5.3}$$

Theorem 5.2 *Assume ε_t satisfies Assumption 5.1, and assume that $\Delta y_t = \varepsilon_t$ and $y_0 = 0$. Then*

$$n^{-1}\sum_{t=1}^{n}\varepsilon_t y_{t-1} I(n^{-1/2}y_{t-1} \leq \pi) \Rightarrow \sigma^2 \int_0^1 (W(r))I(\sigma W(r) \leq \pi)dW(r). \tag{5.4}$$

The proofs of Theorems 5.1 and 5.2 can be found in the Mathematical Appendix.

5.3 Applications

Consider the threshold unit root model

$$\Delta y_t = \begin{cases} \mu + \varepsilon_t & \text{if } |y_{t-1}| \leq C \\ \mu + \varphi y_{t-1} + \varepsilon_t & \text{if } |y_{t-1}| > C. \end{cases} \tag{5.5}$$

From Chan, Petrucelli, Tong and Woolford (1985), it is known that under regularity conditions, if ε_t is an i.i.d. error and $-2 < \varphi < 0$, then y_t will be ergodic, and the usual law of large numbers will be hold for y_t and y_t^2. With the results we establish in Section 5.2, we can construct tests for the unit root hypothesis $H_0 : \varphi = 0$ against the alternative of an ergodic TUR model, i.e. $-2 < \varphi < 0$. If the threshold value were known, we could obtain an estimator $\hat{\varphi}$ of φ by a regression of Δy_t on constant and $y_{t-1}I(y_{t-1} > C)$. However, if the threshold value is a priori unknown, the problem arises that under H_0, the threshold value is unidentified. One solution for this problem is to use the smallest possible t-value over the space of relevant values for threshold value as our test statistics. Assuming $C = n^{1/2}\pi$ is given, define $\hat{\varphi}_\mu$ as the least square regression coefficient from a regression of Δy_t on $y_{t-1}I(y_{t-1} > n^{1/2}\pi)$ with intercept, and similarity define $\hat{t}^{\mu}_{\varphi=0}$ as the usual t-test for $H_0 : \varphi = 0$ from the regression with intercept. Under the null hypothesis of $\varphi = 0$ (and assuming that $y_0 = 0$), the $\hat{t}^{\mu}_{\varphi=0}$ statistics can be written as

$$(1/s_1)(\sum_{t=1}^{n} y_{t-1}^2 I(y_{t-1} > n^{1/2}\pi))^{-1/2}(\sum_{t=1}^{n}\varepsilon_t y_{t-1} I(y_{t-1} > n^{1/2}\pi)), \tag{5.6}$$

where s_1^2 is the usual error variance estimator. For numerator of (5.6), we can use the results that we establish in this chapter to obtain

$$n^{-1}\sum_{t=1}^{n}\varepsilon_t y_{t-1} I(n^{-1/2}y_{t-1} \le \pi) \Rightarrow \sigma^2 \int_0^1 (W(r)) I(\sigma W(r) \le \pi) dW(r). \tag{5.7}$$

For the denominator, Park and Phillips (2001) established that for a compact subset of Π of $\mathbb{R}$,

$$n^{-1}\sum_{t=1}^{n}(n^{-1/2}y_t)^2 I(y_{t-1} > n^{1/2}\pi) \Rightarrow \sigma^2 \int_0^1 W(r)^2 I(\sigma|W(r)| > \pi) dr. \tag{5.8}$$

Combining these two results, we can conjecture the possible asymptotic distribution of the statistic under the null to be,

$$\inf_{\pi\in\Pi} \hat{t}^{\mu}_{\varphi=0} \xrightarrow{d} \inf_{\pi\in\Pi} \frac{\int_0^1 W(r)I(|W(r)| > \pi/\sigma)dW(r) - W(1)\int_0^1 W(r)I(|W(r)| > \pi/\sigma)dr}{(\int_0^1 W(r)^2 I(|W(r)| > \pi/\sigma)dr - (\int_0^1 W(r)I(|W(r)| > \pi/\sigma)dr)^2)^{1/2}}. \tag{5.9}$$

The problem with this conjecture is that the denominator equals zero for $\pi > \sigma \sup_{r\in[0,1]}|W(r)|$, and therefore the above result does not follow straightforwardly from the continuous mapping theorem. The application of our theorems towards the problem of testing for a threshold unit root will be part of the future research.

5.4 Conclusion and further research

In this chapter, we derived two theorems involving the product of an error and an indictor function. With the two results we established, we can consider Dickey-Fuller t-tests that detect the null hypothesis of a unit root against alternative of a threshold unit root model. With regard to further research, we can derive the Dickey-Fuller unit root test for TUR model under $\pi \in \Pi$ with our asymptotic results. In addition to obtain asymptotics of the Dickey-Fuller unit root tests under our TUR model, we can relax the assumptions about residuals, ε_t. We can use the stationary ARMA processes instead of white noises in residual series of TUR model. For this improvement, the asymptotic properties of augment Dickey-Fuller unit root tests can be derived.

Mathematical Appendix

Proof of Theorem 5.1:

First note that,

$$n^{-1/2}\sum_{t=1}^{n}\varepsilon_t I(n^{-1/2}y_{t-1}\le\pi)$$

$$= n^{-1/2}\sum_{t=1}^{n}(y_t - n^{1/2}\pi)I(n^{-1/2}y_{t-1}\le\pi) - n^{-1/2}\sum_{t=1}^{n}(y_{t-1} - n^{1/2}\pi)I(n^{-1/2}y_{t-1}\le\pi)$$

$$= n^{-1/2}\sum_{t=1}^{n-1}(y_t - n^{1/2}\pi)I(n^{-1/2}y_{t-1}\le\pi) + n^{-1/2}(y_n - n^{1/2}\pi)I(n^{-1/2}y_{n-1}\le\pi)$$

$$-n^{-1/2}\sum_{t=1}^{n-1}(y_t - n^{1/2}\pi)I(n^{-1/2}y_t\le\pi) - n^{-1/2}(y_0 - n^{1/2}\pi)I(n^{-1/2}y_0\le\pi)$$

$$= n^{-1/2}(y_n - n^{1/2}\pi)I(n^{-1/2}y_{n-1}\le\pi) - n^{-1/2}(y_0 - n^{1/2}\pi)I(n^{-1/2}y_0\le\pi)$$

$$+n^{-1/2}\sum_{t=1}^{n}(y_t - n^{1/2}\pi)I(n^{-1/2}y_{t-1}\le\pi) - n^{-1/2}\sum_{t=1}^{n}(y_t - n^{1/2}\pi)I(n^{-1/2}y_t\le\pi)$$

$$= n^{-1/2}(y_n - n^{1/2}\pi)I(n^{-1/2}y_{n-1}\le\pi) - n^{-1/2}(y_0 - n^{1/2}\pi)I(n^{-1/2}y_0\le\pi)$$

$$+n^{-1/2}\sum_{t=1}^{n}(y_t - n^{1/2}\pi)[I(n^{-1/2}y_{t-1}\le\pi) - I(n^{-1/2}y_t\le\pi)]$$

$$= n^{-1/2}(y_n - n^{1/2}\pi)I(n^{-1/2}y_{n-1}\le\pi) - n^{-1/2}(y_0 - n^{1/2}\pi)I(n^{-1/2}y_0\le\pi)$$

$$+n^{-1/2}\sum_{t=1}^{n}(y_t - n^{1/2}\pi)I(\min(n^{-1/2}y_t, n^{-1/2}y_{t-1})\le\pi\le\max(n^{-1/2}y_t, n^{-1/2}y_{t-1})). \tag{5.10}$$

For the first term of the last formula, we have

$$(n^{-1/2}y_n - \pi)I(n^{-1/2}y_{n-1}\le\pi) = (n^{-1/2}y_{n-1} - \pi + n^{-1/2}\varepsilon_n)I(n^{-1/2}y_{n-1}\le\pi).$$

By Chebyshev's inequality,

$$E \sup_{\pi \in \mathbf{R}} |n^{-1/2}\varepsilon_t| I(n^{-1/2}y_{n-1} \leq \pi) \leq n^{-1/2}E|\varepsilon_t| \rightarrow 0$$

as $n \rightarrow \infty$. Also,

$$|(n^{-1/2}y_{n-1})I(n^{-1/2}y_{n-1} \leq \pi)| \Rightarrow (\sigma W(1) - \pi)I(|\sigma W(1)| \leq \pi). \tag{5.11}$$

For the second term, we can obtain

$$(n^{-1/2}y_0 - \pi)I(n^{-1/2}y_0 \leq \pi) \Rightarrow (-\pi)I(0 \leq \pi). \tag{5.12}$$

For the third term, from Perkins (1982), it follows that

$$n^{-1/2}\sum_{t=1}^{n}(y_t - n^{1/2}\pi)I(\min(n^{-1/2}y_t, n^{-1/2}y_{t-1}) \leq \pi \leq \max(n^{-1/2}y_t, n^{-1/2}y_{t-1}))$$

$$\Rightarrow (1/2)L(1,\pi). \tag{5.13}$$

The results of Equation (5.11), (5.12) and (5.13) now imply that the statistic of Equation (5.10) converges weakly to

$$(\sigma W(1) - \pi)I(|\sigma W(1)| \leq \pi) - (-\pi)I(0 \leq \pi) + (1/2)L(1,\pi). \tag{5.14}$$

The Tanaka Formula (see Perkins (1982) p437-p439)

$$\max(W(1) - \pi, 0) = \max(-\pi, 0) + \int_0^1 I(W(r) > \pi)dW(r) + (1/2)L(1,\pi)$$

now implies that the process of Equation (5.14) can be written as $\sigma \int_0^1 I(\sigma W(r) \leq \pi)dW(r)$. We can conclude that

$$n^{-1/2}\sum_{t=1}^{n}\varepsilon_t I(n^{-1/2}y_{t-1} \leq \pi) \Rightarrow \sigma \int_0^1 I(\sigma W(r) \leq \pi)dW(r).$$

□

Proof of Theorem 5.2:

From the definition of y_{t-1}, we know

$$\varepsilon_t y_{t-1} = (1/2)\{y_t^2 - y_{t-1}^2 - \varepsilon_t^2\}$$

The pointwise convergence in distribution of the statistic follows from Park and Phillips (2001). Therefore, we only need to show stochastic equicontinuity to complete the proof. To show stochastic equicontinuity note that

$$n^{-1}\sum_{t=1}^{n}\varepsilon_t y_{t-1} I(n^{-1/2}y_{t-1} \leq \pi)$$

$$= (1/2)n^{-1}\sum_{t=1}^{n}(y_t^2 - y_{t-1}^2 - \varepsilon_t^2)I(n^{-1/2}y_{t-1} \leq \pi)$$

$$= (1/2)n^{-1}\sum_{t=1}^{n}[(y_t - n^{1/2}\pi)^2 - (y_{t-1} - n^{1/2}\pi)^2 + 2n^{1/2}\pi\varepsilon_t - \varepsilon_t^2]I(n^{-1/2}y_{t-1} \leq \pi)$$

$$= (1/2)n^{-1}\sum_{t=1}^{n}(y_t - n^{1/2}\pi)^2 I(n^{-1/2}y_{t-1} \leq \pi)$$

$$-(1/2)n^{-1}\sum_{t=1}^{n}(y_{t-1} - n^{1/2}\pi)^2 I(n^{-1/2}y_{t-1} \leq \pi)$$

$$+(1/2)n^{-1}\sum_{t=1}^{n}(2n^{1/2}\pi\varepsilon_t - \varepsilon_t^2)I(n^{-1/2}y_{t-1} \leq \pi)$$

$$= (1/2)n^{-1}\sum_{t=1}^{n-1}(y_t - n^{1/2}\pi)^2 I(n^{-1/2}y_{t-1} \leq \pi)$$

$$+(1/2)n^{-1}(y_n - n^{1/2}\pi)^2 I(n^{-1/2}y_{n-1} \leq \pi)$$

$$-(1/2)n^{-1}\sum_{t=1}^{n-1}(y_t - n^{1/2}\pi)^2 I(n^{-1/2}y_t \leq \pi)$$

$$-(1/2)n^{-1}(y_0 - n^{1/2}\pi)^2 I(n^{-1/2}y_0 \leq \pi)$$

$$+(1/2)n^{-1}\sum_{t=1}^{n}(2n^{1/2}\pi\varepsilon_t - \varepsilon_t^2)I(n^{-1/2}y_{t-1} \leq \pi)$$

$$= (1/2)n^{-1}(y_n - n^{1/2}\pi)^2 I(n^{-1/2}y_{n-1} \leq \pi)$$

$$-(1/2)n^{-1}(y_0 - n^{1/2}\pi)^2 I(n^{-1/2}y_0 \leq \pi)$$

$$+(1/2)n^{-1}\sum_{t=1}^{n-1}(y_t - n^{1/2}\pi)^2[I(n^{-1/2}y_{t-1} \leq \pi) - I(n^{-1/2}y_t \leq \pi)]$$

$$+(1/2)n^{-1}\sum_{t=1}^{n}(2n^{1/2}\pi\varepsilon_t - \varepsilon_t^2)I(n^{-1/2}y_{t-1} \leq \pi)$$

Now note that

$$g_t(\pi) = (y_t - n^{1/2}\pi)^2(I(n^{-1/2}y_{t-1} \leq \pi) - I(n^{-1/2}y_t \leq \pi))$$

$$-\varepsilon_t^2 I(n^{-1/2}y_{t-1} \leq \pi)$$

is continuous in π, and also note that $g_t(\pi)$is differentiable in π and that its first derivative is

$$g_t'(\pi) = -2n^{1/2}(y_t - n^{1/2}\pi)(I(n^{-1/2}y_{t-1} \leq \pi) - I(n^{-1/2}y_t \leq \pi)).$$

Define

$$G_n(\pi) = n^{-1}\sum_{t=1}^{n} g_t(\pi)$$

and

$$G_n'(\pi) = n^{-1}\sum_{t=1}^{n} g_t'(\pi).$$

Now

$$|G_n(\pi) - G_n(\tilde{\pi})| \leq |\pi - \tilde{\pi}| \sup_{\pi\in\mathbf{R}} |G_n'(\pi)|$$

$$= |\pi - \tilde{\pi}| \sup_{\pi\in\mathbf{R}} \sum_{t=1}^{n} |n^{-1/2}y_t - \pi||I(n^{-1/2}y_{t-1} \leq \pi) - I(n^{-1/2}y_t \leq \pi)|$$

$$= |\pi - \tilde{\pi}|(1/2) \sup_{\pi\in\mathbf{R}} |L_n(1,\pi)|.$$

Therefore, it follows that $G_n(\pi)$ is stochastically equicontinuous. The proof of stochastic equicontinuity of $n^{-1/2}\sum_{t=1}^{n}\varepsilon_t y_{t-1}I(n^{-1/2}y_{t-1}\leq\pi)$ is therefore complete if we can show that

$$2\pi n^{-1/2}\sum_{t=1}^{n}\varepsilon_t I(n^{-1/2}y_{t-1}\leq\pi)$$

is stochastically equicontinuous, which follows from Theorem 1 and the continuity of $g(\pi)=\pi$. □

APPENDICES

APPENDIX A

Introduction to local time

A.1 Definition and properties

In my dissertation, we use the concept of local time. I will give a simple introduction for local time in this appendix. Local time is a continuous two-parameter stochastic process that characterizes a continuous time martingale process. When this continuous time martingale process is Brownian motion, the associated local time function is called the Brownian local time, which we well denote by $L(t, s)$. Like Brownian motion, local time is a random function that has a well-defined distribution for any given value of the argument; the finite-dimensional distributions of $L(.,.)$ a spatial density, i.e. $L(.,.)$ are not normally distributed, however. The intuitive interpretation of local time is that it is a spatial density, i.e. $L(.,.)$ provides information about how much time a Brownian motion process spends in the neighborhood of a given point s.

To get to the standard definition of local time, we first need to define the occupation time H. Let $[M]$ denote the quadratic variation process of M, where M is a continuous time semimartingale process. Then the occupation time of $M(r)$, for any

Borel measurable set A, is given by

$$H(A, t) = \int_0^t I(M(r) \in A) d[M](r).$$

For the special case $M = B$, $[B](r) = r$, and the reason for naming $H(.,.)$ "occupation time" is clear for that case. In the more general case, we can think of the amount of time spent by $M(.)$ in the set A as being measured in units of quadratic variation. The following theorem now defines the local time function $L_M(t, s)$:

Theorem A.1 *For a continuous time semimartingale process $M(.)$, there exists a continuous function $L(.,.)$ such that*

$$H((-\infty, x], t) = \int_{-\infty}^{x} L_M(t, s) d[M](s).$$

Proof of Theorem A.1:

See Chung and Williams (1990).

The above theorem implies that

$$L_M(t, s) = \lim_{\varepsilon \to 0} (2\varepsilon)^{-1} \int_0^t I(|M(r) - s| \leq \varepsilon) d[M](r).$$

Trotter (1958) was the first to show the result of Theorem A.1 for the special case of Brownian motion, i.e. $M = B$. For the case of the Brownian local time, the above theorem implies that

$$H((-\infty, x], t) = \int_{-\infty}^{x} L(t, s) ds$$

and that

$$(d/ds) \int_0^t I(B(r) \leq s) dr = L(t, s).$$

We note that in the article by Park and Phillips (1999), that seemed to have started interest of the econometrics profession in local time, the order of the arguments of the

$L(.,.)$ function see to be reversed, compared to what is convention in the statistics literature. Here, we will follow Park and Phillips' notation.

In order to get some idea of how the local time function behaves, it may be worthwhile here to realize that since $\sup_{r\in[0,t]} B(r)$ and $\inf_{r\in[0,t]} B(r)$ are well-defined random variables, $\int_0^t I(B(r) \leq s)dr = 0$ for $s < \inf_{r\in[0,t]} B(r)$. Therefore, for such s, $L(t,s) = 0$ as well. Similarly, for all $s > \sup_{r\in[0,t]} B(r)$, $\int_0^t I(B(r) \leq s)dr = 1$ and therefore, $L(t,s) = 0$ for $s > \sup_{r\in[0,t]} B(r)$. Also,

$$\int_{-\infty}^{\infty} L(t,s)ds = \left[\int_{-\infty}^{\infty} (d/ds) \int_0^t I(B(r) \leq s)dr\right]_{s=-\infty}^{\infty} = 1.$$

These facts together complete the picture that we should have in mind for $L(.,.)$: as a function of s, $L(.,.)$ is a function with bounded (yet random) support that integrates to one.

A.2 The Tanaka formula

The It formula states that, if $(d^2/dx^2)F(x) = (d/dx)f(x) = f'(x)$ and $f'(x)$ is continuous, we have

$$F(B(t)) - F(B(0)) = \int_0^t f(B(r))dB(r) + (1/2)\int_0^t f'(B(r))dr.$$

The Tanaka formula now states that a form of the It formula holds for $f(W) = I(W \leq s)$ as well. The local time $L(t,s)$ will appear in this formula, as a replacement for the It correction term. Basically, for this choice of $f(.)$, the Tanaka formula justifies that one can consider

$$-(d/ds)\int_0^t f(W)|_{W=B(r)}dr = -L(t,s)$$

instead of the undefined

$$\int_0^t (d/dW)f(W)|_{W=B(r)}dr$$

in the It formula. For a heuristic application of the It formula along these lines, to make $F(.)$ continuous and have $f(W) = I(W \leq s)$ as its derivative at any point except $W = s$, we should choose $F(W) = (W - s)I(W \leq s)$. This heuristic implication of the Tanaka formula is then

$$(B(t) - s)I(B(t) \leq s) - (-s)I(0 \leq s) = \int_0^t I(B(r) \leq s)dB(r) - (1/2)L(t, s),$$

which can be rewritten as

$$\max(s - B(t), 0) + \max(s, 0) = \int_0^t I(B(r) \leq s)dB(r) + (1/2)L(t, s). \qquad \text{(A.1)}$$

However, the most cited form of the Tanaka formula is as follows:

Theorem A.2 *Tanaka formula*

$$L(t, s) = |B(t) - s| - |s| - \int_0^t \operatorname{sgn}(B(r) - s)dB(r). \qquad \text{(A.2)}$$

Proof of Theorem A.2

See McKean (1969).

This second form of the Tanaka formula easily results from our conjecture of Equation(A.1) by noting that $\operatorname{sgn}(W - s) = 1 - 2I(W \leq s)$.

A.3 The occupation times formula

Because of the interpretation of local time as a spatial density, we may expect a relationship between integrals over a function of Brownian motion and an expression involving local time. Specifically, for a continuous function $T(.)$, we may expect that

$$\int_0^1 T(B(r))dr$$

can be approximated, for small $\varepsilon > 0$, by

$$\int_0^1 \sum_j T(j\varepsilon) I(j\varepsilon < B(r) \leq (j+1)\varepsilon) dr.$$

For small $\varepsilon > 0$, we should now have that

$$\varepsilon^{-1} \int_0^1 I(j\varepsilon < B(r) \leq (j+1)\varepsilon) dr \approx L(1, j\varepsilon),$$

suggesting that

$$\int_0^1 T(B(r)) dr \approx \int_0^1 \sum_j T(j\varepsilon) I(j\varepsilon < B(r) \leq (j+1)\varepsilon) dr$$

$$\approx \sum_j T(j\varepsilon)\varepsilon^{-1} L(1, j\varepsilon) \approx \int_{-\infty}^{\infty} T(s) L(1, s) ds.$$

This can be formalized in the following theorem:

Theorem A.3 *Let $T : \mathbb{R} \to \mathbb{R}$ be a locally integrable function. Then*

$$\int_0^1 T(B(r)) dr = \int_{-\infty}^{\infty} T(s) L(1, s) ds.$$

Proof of Theorem A.3:

See Chung and Williams (1991).

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