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REAL ASPECT OF THE MODULI SPACE OF STABLE
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Seongchun Kwon

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REAL ASPECTS OF THE MODULI SPACE OF STABLE MAPS OF GENUS ZERO CURVES

By

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ABSTRACT

REAL ASPECTS OF THE MODULI SPACE OF STABLE MAPS OF GENUS ZERO CURVES

By

Seongchun Kwon

We show that the moduli space of stable maps from a genus 0 curve into a nonsingular real convex projective variety having a real structure compatible with a complex conjugate involution on $\mathbb{CP}^k$ has a real structure. The real part of this moduli space consists of real maps having marked points on the real part of domain curves. This real part analysis enables us to relate the studies of real intersection cycles with real enumerative problems.

To my parents

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1 Introduction

A Gromov-Witten invariant and its applications to enumerative problems in the *complex* world has been studied by many people. That invariant is defined on the moduli space of stable maps (Definition in section 3). In this thesis, we investigate the real aspect of the moduli space $\overline{M}_n(X,\beta)$ of stable maps from genus 0 curves when the target space is a convex (i.e. $H^1(\mathbb{CP}^1, \mu^*(TX)) = 0$, for every $\mu : \mathbb{CP}^1 \to X$, where TX is a tangent bundle) nonsingular projective real variety whose real structure corresponds to the complex conjugation map on $\mathbb{CP}^k$. Here, a projective real variety is a projective variety having an anti-holomorphic involution. To search for the ways to use the above moduli space of stable maps $\overline{M}_n(X,\beta)$ in studying real enumerative problems, we have to see whether we can understand the moduli space $\overline{M}_n(X,\beta)$ as a *real* projective variety or not. If the answer is positive, then we need to understand the nature of the real part (Definition in sec 2), for example, whether each point in the real part of $\overline{M}_n(X,\beta)$ represents real maps or not. We will show the following:

- (Section 3, 4.2) The moduli space of stable maps of genus zero curves $\overline{M}_n(X,\beta)$, where X satisfies the above conditions, is a real projective variety.
- (Section 4.1) The real part of $\overline{M}_n(X,\beta)$ consists of real maps having marked points on the real part of the domain curves.

The real model (Definition in sec 2) of $\overline{M}_n(X,\beta)$ has a $\mathbb{Z}$-module Chow group fundamental cycle. And the real part of $\overline{M}_n(X,\beta)$ has $\mathbb{Z}/2\mathbb{Z}$-module ordinary homology fundamental cycle. So, it is natural to consider whether we can define real enumerative invariants on the real model or the real part of $\overline{M}_n(X,\beta)$ which count the number of real curves on the real model or the real part of $\overline{M}_n(X,\beta)$. Unfortunately, we cannot define nice enumerative invariants using fundamental cycles. The reason is explained in section 5. The possible way to use the real aspect of $\overline{M}_n(X,\beta)$ will be developing an efficient method to construct real cycles meeting transversally, maximizing the number of intersection points of cycles in the real part of $\overline{M}_n(X, \beta)$. Its enumerative implication will concern how many real solutions we can have for the given enumerative problem, improving the minimum bound of the real solutions. But the technique to construct such real cycles is open.

A Gromov-Witten invariant in the *real* world with Quantum Schubert calculus has been widely studied by F. Sottile. See [Sot1], [Sot2], [Sot4], [Sot5].

2 Preliminaries

We begin with reviews of some standard notions and facts in real algebraic geometry. A more detailed exposition can be found in [Sil, I. sec.1,4].

Definition. Let X be a scheme over $\mathbb{C}$. We will say that (X, s), or simply s, is a *real structure on* X if s is an involution on X such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{s} & X \\ \downarrow & & \downarrow \\ Spec(\mathbb{C}) & \xrightarrow{j^*} & Spec(\mathbb{C}) \end{array}$$

commutes, where $j : \mathbb{C} \to \mathbb{C}$ is the complex conjugation.

We then call such a scheme X a *real scheme with a real structure* s.

Remark 2.1 If X is a projective variety over $\mathbb{C}$, then having a real structure is equivalent to having an anti-holomorphic involution on the set of complex points $X(\mathbb{C})$. See [Sil, p4, (1.4) Proposition].

Definition. Let X be a scheme over $\mathbb{C}$. We will say that X *has a real model* if there exists a scheme $X^{\mathbb{R}}$ over $\mathbb{R}$ such that $X \cong X^{\mathbb{R}} \times_{\mathbb{R}} \mathbb{C}$, where $X^{\mathbb{R}} \times_{\mathbb{R}} \mathbb{C}$ is the fibre product of $X^{\mathbb{R}}$ and Spec($\mathbb{C}$) over Spec($\mathbb{R}$). We will call $X^{\mathbb{R}}$ a *real model of* X, and X a *complexification of* $X^{\mathbb{R}}$.

Proposition 2.1 *The category of quasi-projective or projective schemes over* $\mathbb{R}$ *and that of quasi-projective or projective schemes over* $\mathbb{C}$*, endowed with real structures are equivalent categories.*

More precisely, there exists a real structure (X, s) *on a projective or quasi-projective scheme* X *over* $\mathbb{C}$ *if and only if there exists a real model* $X^{\mathbb{R}}$ *for* X *and an isomorphism* $\varphi : X \to X^{\mathbb{R}} \times_{\mathbb{R}} \mathbb{C}$ *such that* $s = \varphi^{-1} \circ \sigma \circ \varphi$*, where* σ *is induced by complex conjugation in* $X^{\mathbb{R}} \times_{\mathbb{R}} \mathbb{C}$*. For a fixed* (X, s)*,* φ *and* $X^{\mathbb{R}}$ *are unique up to real isomorphism.*

Proof. See [Sil, p5] □

Definition. Let (X, s) be a real structure on a projective or quasi-projective scheme X over $\mathbb{C}$. We will call the fixed points by the action s in the complex points $X(\mathbb{C})$ the *real part of* X and denote it by X^{re}.

If a projective scheme Y is defined over $\mathbb{R}$, then it consists of real and non-real closed points because the real number field $\mathbb{R}$ is not algebraically closed.

A real model of the real projective scheme X is an algebraic geometric notion including non-real points also. But the real part of $X(\mathbb{C})$ and the set of real closed points in a scheme Y defined over $\mathbb{R}$ are differential geometric notions. If $X(\mathbb{C})$ is real isomorphic to $Y \times_{\mathbb{R}} \mathbb{C}$, then each point in the real part of $X(\mathbb{C})$ uniquely corresponds to real points in Y, and vice versa.

We will use notations $\mathbb{CP}^k$, $\mathbb{RP}^k$ to represent Proj$\mathbb{C}[x_0, \ldots, x_k]$, Proj$\mathbb{R}[x_0, \ldots, x_k]$, that is, projective k-spaces over $\mathbb{C}$, $\mathbb{R}$ in algebraic geometry. Note that $\mathbb{CP}^k$ can be considered as a k-dimensional complex projective space in the differential geometric sense. But $\mathbb{RP}^k$, an algebraic variety containing non-real points, cannot be identified with a differential geometer's real projective space. The set of real points in $\mathbb{RP}^k$ is identifiable to a differential geometer's k-dimensional real projective space which we will denote it by $R\mathbb{P}^k$.

Definition. Let (X, s), (Y, t) be real schemes.
We will say the morphism $f : X \to Y$ is a *real map* if the morphism f commutes with real structures, i.e., $f \circ s = t \circ f$.

Such a morphism f obviously preserves the real parts, i.e., $f(X^{re}) \subset Y^{re}$.

If $X^{\mathbb{R}}$, $Y^{\mathbb{R}}$ are separated schemes of finite type over $\mathbb{R}$, then giving a morphism $f^{\mathbb{R}} : X^{\mathbb{R}} \to Y^{\mathbb{R}}$ is equivalent to giving a morphism $f : X \to Y$ which commutes with the real structures. See [Har, p107, 4.7. (c)]. We will call $f^{\mathbb{R}}$ a *real model map* of a real map f, f a *complexified map* of $f^{\mathbb{R}}$, the restriction map $f^{re} : X^{re} \to Y^{re}$ of f to the set of real points a *real part map* of f.

Example 2.1 1. $\mathbb{CP}^k$ is a real scheme having an anti-holomorphic involution given by a standard complex conjugation map. Then, $\mathbb{CP}^k$ is isomorphic to $\mathbb{RP}^k \times_{\mathbb{R}} \mathbb{C}$. We illustrate a non-real point in $\mathbb{RP}^1$. Let $O^{\mathbb{R}}$ be the standard open set $\{x_0 \neq 0\}$. Then, $O^{\mathbb{R}}$ is isomorphic to $\mathrm{Spec}\mathbb{R}[y]$, where $y = x_1/x_0$. Note that $y^2 + 1$ is an irreducible polynomial in $\mathbb{R}[y]$. Therefore, it generates a prime ideal and obviously corresponds to a non-real closed point. This non-real point splits into two complex points $[1 : i]$, $[1 : -i]$ corresponding to $(y + i)$, $(y - i)$ in a standard open set $O^{\mathbb{R}} \times_{\mathbb{R}} \mathbb{C}$ isomorphic to $\mathrm{Spec}\mathbb{C}[y] = \mathrm{Spec}(\mathbb{R}[y] \otimes_{\mathbb{R}} \mathbb{C})$ in $\mathbb{CP}^1$. The set of these two points is preserved by the involution. In this special case of dimension 1, there is a set theoretic one-to-one and onto correspondence between $\mathbb{CP}^1/\sim$ and the scheme $\mathbb{RP}^1$, where $\sim$ is the equivalence relation by the conjugation action, because every irreducible polynomial having degree higher than one has degree 2. Note that $\mathbb{CP}^1/\sim$ is diffeomorphic to a closed disk. More generally, real points in $\mathbb{RP}^k$ correspond to the points in the real part of $\mathbb{CP}^k$ which are fixed by the involution. Each non-real point in $\mathbb{RP}^k$ splits into an even number of non-real complex points in $\mathbb{CP}^k$ preserved by the complex conjugation action.

2. We may have more than one real structure on the same scheme. Not every real structure induces a real part. For example, $\mathbb{CP}^1$ has two non-isomorphic real structures. One is explained in 1, having a real part diffeomorphic to $R\mathbb{P}^1$, with real model isomorphic to $\mathbb{RP}^1$. The other is an anti-holomorphic map $s([z:w]) = [-\bar{w}:\bar{z}]$, having no fixed points, i.e. no real part, with real model isomorphic to the conic in $\mathbb{RP}^2$ given by the homogeneous equation $x_0^2 + x_1^2 + x_2^2 = 0$. See [Har, p107, 4.7,(e)].

In general, if a smooth real scheme X has a real part, then the dimension of the real part is half of that of the original scheme. i.e., $dim_{\mathbb{C}} X(\mathbb{C}) = dim_{\mathbb{R}} X^{re}$. See [Sil, p8].

3. The Deligne-Mumford moduli space $\overline{M}_n$ with n marked points, is a real moduli space whose anti-holomorphic involution is induced by the involution in example 1 as described in [G-M, 2.3], [F-Oh, sec.10], [Cey, sec.4.1]. More precisely,

i. For non-singular curve; $(\mathbb{CP}^1, a_1, \ldots, a_n) \mapsto (\mathbb{CP}^1, \bar{a}_1, \ldots, \bar{a}_n)$

ii. For singular curve with two irreducible components;

$(\mathbb{CP}^1_\alpha, \delta; a_1, \ldots, a_k) \cup (\mathbb{CP}^1_\beta, \delta'; b_1, \ldots, b_l) \mapsto (\mathbb{CP}^1_\alpha, \bar{\delta}; \bar{a}_1, \ldots, \bar{a}_k) \cup (\mathbb{CP}^1_\beta, \bar{\delta}'; \bar{b}_1, \ldots, \bar{b}_l)$,

where $\mathbb{CP}^1_\alpha$, $\mathbb{CP}^1_\beta$ are irreducible components after the normalization, δ, δ' are gluing points, a_i, b_j are marked points, $k + l = n$.

iii. General cases are obvious from i, ii.

We will prove this map defines an anti-holomorphic involution in section 3. In fact, this is an involution we get when we consider the $\mathbb{C}$-scheme Deligne-Mumford moduli space as a complexification of the $\mathbb{R}$-scheme Deligne-Mumford moduli space. That is, the real model of $\overline{M}_n$ is an $\mathbb{R}$-scheme Deligne-Mumford moduli space $\overline{M}_n^{\mathbb{R}}$ with n marked points.

Not every universal family of curves $\overline{U}_n$ on $\overline{M}_n$ is real. For example, any rational curve with 3 marked points can become a universal family over $\overline{M}_3$. But that can be considered as a real universal family of curves only when all 3 marked points are on the real part of $\mathbb{CP}^1$. To analyze the real part of $\overline{M}_n$, we need a real universal family of curves. The real universal family of curves can be constructed by complexification of the $\mathbb{R}$-scheme universal family of curves over the $\mathbb{R}$-scheme Deligne-Mumford moduli space $\overline{M}_n^{\mathbb{R}}$. Obviously, the one point moduli space $\overline{M}_3$ can be represented by a rational curve $(\mathbb{CP}^1, a_1, a_2, a_3)$ with 3 marked points a_i in the real part. Each general point $x(\neq a_i)$ in $\overline{U}_3 := (\mathbb{CP}^1, a_1, a_2, a_3)$ can be understood as a general point in $\overline{M}_4$ representing a rational curve with marked points at a_1, a_2, a_3, x. Thus, non-singular curves represented by points in the real parts of $\overline{M}_4$ are real curves with 4 real marked points. When these curves degenerate, they make singular curves having real irreducible components with real marked points and real gluing points. If two points on the real part of the rational curve collide, then the colliding place becomes a gluing point with the other new real irreducible component. And then, the collided points split into two points in the real part of the new irreducible component. Since the construction of Deligne-Mumford moduli space is inductive using real isomorphisms $\overline{U}_{n-1} \cong \overline{M}_n$, we see that points in the real part of $\overline{M}_n$, $n \geq 3$, are represented by rational curves with real marked points or singular curves having real irreducible rational components with real marked and real gluing points. See [G-M, sec2.3].

Recall there is a 1-1 correspondence between isomorphism classes of locally free sheaves of rank n on the scheme X, and isomorphism classes of vector bundles of rank n over X. We won't distinguish the words between a 'locally free sheaf' and a 'vector bundle'. See [Har, p129, 5.18(d)].

Note that a real structure s on X induces a canonical morphism on the structure sheaf $\mathcal{O}_X$ by

$$\begin{array}{rcl} \Gamma(U, \mathcal{O}_X) & \to & \Gamma(s(U), \mathcal{O}_X) \\ f & \mapsto & j \circ f \circ s := f^s, \end{array}$$

which is an isomorphism of rings, where U is any open set in X.

Let U be an affine open set in X and $\mathcal{L}$ a locally free sheaf. Then, $\mathcal{L}(s(U))$ is an $\mathcal{O}_X(s(U))$-module. We make $\mathcal{L}(s(U))$ an $\mathcal{O}_X(U)$-module by changing exterior multiplication,

$$\begin{array}{ccc} \mathcal{O}_X(U) \times \mathcal{L}(s(U)) & \rightarrow & \mathcal{L}(s(U)) \\ (f, v) & \mapsto & (j \circ f \circ s)v \end{array}$$

leaving the underlying additive group structure as it was. We call the locally free sheaf $\mathcal{L}^s$ of $\mathcal{O}_X$-modules defined in this way the *conjugate vector bundle with respect to the real structure* s on (X, s).

For example, if $\mathcal{L}$ is a sheaf of functions with values in $\mathbb{C}^r$, we may describe $\mathcal{L}^s$ by

$$\mathcal{L}^s(U) = \{j \circ h \circ s | h \in \mathcal{L}(s(U))\}$$

We call the vector bundle V over the real scheme (X, s) a *s-real bundle* if its conjugate bundle V^s is identical to the bundle V. Here, 'identical' means exactly the same, not meaning isomorphic. The line bundle from the structure sheaf $\mathcal{O}_X$ on the real scheme (X, s) is a trivial example of a real vector bundle.

Remark 2.2 Let $D = \Sigma n_i D_i$ be a Weil divisor on a real scheme (X, s). Let D^s be a conjugate Weil divisor $\Sigma n_i s(D_i)$. If we consider a Cartier divisor $\{(U_i, f_i)\}$ associated to the Weil divisor D, then its conjugate Cartier divisor, the Cartier divisor associated to the conjugate Weil divisor D^s, can be written as $\{(U_i, f_i^s)\}$. Hence, if $\mathcal{O}(D)$ is the invertible sheaf associated to D, then its conjugate line bundle comes from its conjugate Weil divisor, i.e. $(\mathcal{O}(D))^s = \mathcal{O}(D^s)$. Conversely, if $\mathcal{L}$ is an invertible sheaf on X and $D(\mathcal{L})$ is the associated Weil divisor, then the associated Weil divisor for the conjugate line bundle $\mathcal{L}^s$ is the conjugate Weil divisor of $D(\mathcal{L})$, i.e., $D(\mathcal{L}^s) = (D(\mathcal{L}))^s$. Consequently, the line bundle is real if and only if its associated Weil divisor is fixed by an involution s. More generally, the vector bundle V on X is real only when there exists a locally free sheaf $V^{\mathbb{R}}$ on $X^{\mathbb{R}}$ whose complexification becomes V. See [Sil, p6, (1.8) Lemma].

Example 2.2 1. Line bundles on $\mathbb{CP}^k$, $\mathbb{RP}^k$ are classified by their degree. That is, any invertible sheaf on $\mathbb{CP}^k$, $\mathbb{RP}^k$ is isomorphic to $\mathcal{O}(l)$ for some $l \in \mathbb{Z}$. See [Har, p145]. But the restrictions of same degree line bundles on $\mathbb{RP}^k$ to the real points, so line bundles on the differential geometer's real projective space $R\mathbb{P}^k$, are not necessarily isomorphic. Let s be a real structure from the complex conjugation map on $\mathbb{CP}^1$. Then, the Weil divisors $[i:1]+[-i:1]$ and $[1:1]+[-1:1]$ define degree 2 s-real line bundles, say L_1, L_2 respectively. The natural holomorphic section s_1 to L_1 induced from the associated Cartier divisors(see [Grif-H, p135]) vanishes at $[i:1]$ and $[-i:1]$ which are not in the real part of $\mathbb{CP}^1$. Thus, it induces a trivial line bundle on $R\mathbb{P}^1$. But the s-real line bundle L_2 induces a nontrivial line bundle on $R\mathbb{P}^1$. If we restrict the s-real line bundles to the upper-hemisphere so that the fibers along the boundary come from the real parts of L_1 and L_2, then these give an example of line bundles whose Chern classes are the same after the complex double, i.e. in this case, line bundles L_1, L_2 on $\mathbb{CP}^1$, but the real line bundles along the boundary are not isomorphic. The invariant for line bundles on the upper-hemisphere is called a relative Chern class.

2. Not every degree's line bundle on $\mathbb{CP}^k$ allows a real line bundle. Let s be a real structure on $\mathbb{CP}^1$ from the antiholomorphic involution $[z:w] \mapsto [-\bar{w}:\bar{z}]$, which doesn't have any fixed point. Then, this real structure doesn't have any odd degree real line bundles $\mathcal{O}(2r+1)$ because there is a 1-1 correspondence between Weil divisors, and invertible sheaves (see [Har, p144]) and none of the odd degree's Weil divisors can be fixed. Remark 2.2 leads us to the conclusion.

3 The moduli space of stable maps is a real moduli space

The moduli space $\overline{M}_n(X;\beta)$ of stable maps $(f, \mathcal{C}, x_1, \ldots, x_n)$ from a genus zero curve with n-marked points consists of the equivalent classes of stable maps $(f, \mathcal{C}, x_1, \ldots, x_n)$ satisfying the following conditions by its definition;

(1)$f_*([\mathcal{C}])$ represents the homology class β in $H_2(X;\mathbb{Z})$;

(2) The arithmetic genus of domain curves having n-marked points is zero ;

(3) (stability condition) If the domain curve $\mathcal{C}$ has some irreducible components $\mathcal{C}_i^0$ such that $f_*([\mathcal{C}_i^0]) = 0$, then each of these components, $\mathcal{C}_i^0$, contain at least 3 special points(marked or gluing points);

(4) Two stable curves $(f, \mathcal{C}, x_1, \ldots, x_n)$, $(f', \mathcal{C}', x_1', \ldots, x_n')$ are equivalent if there exists an isomorphism $\sigma; \mathcal{C} \to \mathcal{C}'$ such that $f' \circ \sigma = f$ and $\sigma(x_i) = x_i'$, $i = 1, \ldots, n$.

Let $(\mathbb{CP}^1, s)$, (X, t) be real structures. Then, it is natural to be concerned whether the set theoretic involution $(f, \mathbb{CP}^1, x_1, \ldots, x_n) \mapsto (t \circ f \circ s, \mathbb{CP}^1, s(x_1), \ldots, s(x_n))$ defines an anti-holomorphic involution on $\overline{M}_n(X;\beta)$. We will consider the real structure coming from the complex conjugation map on $\mathbb{CP}^k$ and real projective varieties X related to this real structure and show the above involution is an anti-holomorphic involution on $\overline{M}_n(X;\beta)$. At the end, we will see this result doesn't always hold for any real structures on a domain and a target space.

We will follow Fulton-Pandharipande's construction in [F-P]. The moduli space of stable maps of genus zero curves was constructed by gluing the quotient of projective varieties which are the universal space of an $\overline{h}$-rigid stable family of degree d maps (See a section 3.1 for the definition). The strategy for showing the moduli space of stable maps is a real moduli space is showing each of the ingredients they used are real. The universal space for an $\overline{h}$-rigid stable family was constructed by using a certain locus of the Deligne-Mumford moduli space and a universal curve on it. Their construction is not dependent on the chosen universal curve model. However, we need a *real* universal curve model for our proof. The existence of a real universal curve was explained in Example 2.1,3.

Lemma 3.1 *The Deligne-Mumford moduli space $\overline{M}_m$ is a real moduli space with a real structure induced by a complex conjugation map on $\mathbb{CP}^1$.*

Proof. method 1: (simplest) The Deligne-Mumford moduli space is originally defined over $\mathbb{Z}$. So, it is defined over any field. The $\mathbb{C}$-scheme Deligne-Mumford moduli space can be obtained by a scalar extension from the $\mathbb{R}$-scheme Deligne-Mumford moduli space. That is, the $\mathbb{C}$-scheme Deligne-Mumford moduli space is a complexification of the $\mathbb{R}$-scheme Deligne-Mumford moduli space $\overline{M}_m^{\mathbb{R}}$. So, Lemma 3.1 is proved by Proposition 2.1.
method 2: (geometric) We consider the involution defined in Example 2.1 3. The map we defined is an antiholomorphic involution because the image curve's marked and gluing points are induced by the complex conjugation map on that curve and the splitting of a tangent space at $(C, a_1, \ldots, a_m)$ is;
$T_{(C, a_1, \ldots, a_k)}\overline{M}_m \cong H^1(C, \mathcal{T}_C(-a_1 \ldots - a_m)) \bigoplus \oplus_{s \in sing(C)} T_s' \otimes T_s''$
$\cong \oplus_{C_\alpha : irreducible} H^1(C_\alpha, \mathcal{T}_{C_\alpha}(-a_1 \ldots - a_\alpha)) \bigoplus \oplus_{s \in sing(C)} T_s' \otimes T_s''$
$\cong \oplus_{C_\alpha : irreducible} H^0(C_\alpha, \mathcal{T}^*_{C_\alpha}(a_1 + \ldots + a_\alpha) \otimes \omega_{C_\alpha})^* \bigoplus \oplus_{s \in sing(C)} T_s' \otimes T_s''$, by Serre's duality. □

Remark 3.1 Araujo - Kollar constructed the moduli space of stable maps on any Noetherian scheme in [A-K, sec.10]. However, the relation between an $\mathbb{R}$-scheme version's moduli space of stable maps and a $\mathbb{C}$-scheme version's moduli space of stable maps is different from that of $\mathbb{R}$-, $\mathbb{C}$-scheme Deligne-Mumford moduli space. That is, a $\mathbb{C}$-scheme version's moduli space of stable maps is not the complexification of an $\mathbb{R}$-scheme version's moduli space of stable maps. A counterexample showing that the real model of the moduli space of stable maps and an $\mathbb{R}$-scheme version's moduli space of stable maps are different is given by, $z \mapsto z^2$ and $z \mapsto -z^2$. These maps are different in $\mathbb{R}$-scheme version's moduli space of stable maps but they are equivalent in the real model of the moduli space of stable maps $\overline{M}_0(\mathbb{CP}^1, 2)$ by an isomorphism between the domain curves defined by multiplication by i.

3.1 Fulton-Pandharipande's construction of the moduli space of stable maps $\overline{M}_n(\mathbb{CP}^k, d)$

Fulton-Pandharipande constructed the moduli space $\overline{M}_n(\mathbb{CP}^k, d)$ for $k > 0$, $d > 0$ and $(n, k, d) \neq (0, 1, 1)$. Other cases, $\overline{M}_n(\mathbb{CP}^0, 0)$, $\overline{M}_n(\mathbb{CP}^k, 0)$, $\overline{M}_0(\mathbb{CP}^1, 1)$ are isomorphic to $\overline{M}_n$, $\overline{M}_n \times \mathbb{CP}^k$, $Spec(\mathbb{C})$ respectively.

[I] Construction of the universal space $\overline{M}_n(\mathbb{CP}^k, d, \overline{h})$ for the $\overline{h}$-rigid stable family of curves

We call the correspondence between irreducible components of the curve $\mathcal{C}$ and the degree of the restriction of the line bundle $\mathcal{L}$ to each component of $\mathcal{C}$ as the *multidegree* of $\mathcal{L}$ on the curve $\mathcal{C}$. We will say bundles $\mathcal{L}$, $\mathcal{V}$ on $\mathcal{C}$ *satisfy equal mutidegree condition* if their degrees on each component are the same.

Definition. [F-P, 3.2] Let $\mathbb{CP}^k = \mathbb{P}(H)$, where $H^* = H^0(\mathbb{CP}^k, \mathcal{O}_{\mathbb{CP}^k}(1))$. Let $\overline{h} = (h_0, \ldots, h_k)$ be an ordered hyperplane basis of H^*. *A $\overline{h}$-rigid stable family of degree d maps* from n-pointed, genus 0 curves to $\mathbb{CP}^k$ consists of the data

$$(\pi : \mathcal{C} \to S, \{p_i\}_{1 \le i \le n}, \{q_{i,j}\}_{0 \le i \le k, 1 \le j \le d}, \mu),$$

where

(i) $(\pi : \mathcal{C} \to S, \{p_i\}, \mu)$ is a stable family of degree d maps from n-pointed, genus 0 curves to $\mathbb{CP}^k$, where $\mu; \mathcal{C} \to \mathbb{CP}^k$;

(ii) $(\pi : \mathcal{C} \to S, \{p_i\}_{1 \le i \le n}, \{q_{i,j}\}_{0 \le i \le k, 1 \le j \le d})$ is a flat, projective family of $n + d(k+1)$-pointed, genus 0, Deligne-Mumford stable curves with sections $\{p_i\}$ and $\{q_{i,j}\}$;

(iii)(Transversality condition) For $0 \le i \le k$, there is an equality of Weil divisors

$$\mu^*(h_i) = q_{i,1} + q_{i,2} + \ldots + q_{i,d}.$$

Remark 3.2 1. An $\overline{h}$-rigid stable family is a special kind of flat family of degree d maps from n-pointed genus 0 to $\mathbb{CP}^k$ such that the image of each fibre curve intersects each chosen hyperplane basis $(h_0, \ldots, h_k)$ of $\mathbb{P}(V)$ transversally at unmarked, nonsingular points.

2. The condition (iii) implies the last $d(k+1)$-marked points $\{q_{ij}\}$ are from the hyperplane intersection divisors. Fulton-Pandharipande added those ordered hyperplane intersection marked points to relate the geometry of the moduli space of stable maps of genus zero with that of Deligne-Mumford moduli space.

3. Note that the condition (iii) combined with (i) implies that the number of marked points from each set of $\{q_{i,j}\}$, $i = 0, \ldots, k$, on each irreducible component in each fibre is exactly the same as the degree of the map on each component. That implies $k+1$ line bundles on $\mathcal{C}$ constructed by using Weil divisors $q_{i,1} + \ldots + q_{i,d}$ from the last $d(k+1)$ marked points satisfy the equal multi-degree condition.

There is a universal locus B in Deligne-Mumford moduli space $\overline{M}_m$, $m = n+d(k+1)$ that every $\overline{h}$-rigid stable family in (ii) factors through. But the map's information we can get from the points in B is limited to the hyperplane intersection points. To distinguish the $\overline{h}$-stable maps sharing the same hyperplane intersection points, Fulton-Pandharipande constructed a k-dimensional $\mathbb{C}^*$-fibration on B by using the $k+1$ Weil divisors $q_{i,1} + \ldots + q_{i,d}$. The following notion of *$\mathcal{H}$-balanced* is satisfied by the sublocus B and enables them to construct the desired fibre bundle, which is a universal space for the $\overline{h}$-rigid stable family of maps.

Notation. We will denote the line bundle $\mathcal{O}_{\overline{U}_m}(q_{i,1} + q_{i,2} + \ldots + q_{i,d})$ on $\overline{U}_m$ by $\mathcal{H}_i$ $i = 0, \ldots, k$.

Definition. [F-P, 3.3] Let $\overline{M}_m$ be the Deligne-Mumford moduli space of genus

0, m-pointed curves. Let $\pi : \overline{U}_m \to \overline{M}_m$ be the universal curve with m-sections $\{p_i\}_{1\leq i\leq n}$ and $\{q_{i,j}\}_{0\leq i\leq k, 1\leq j\leq d}$. For any morphism $\gamma : X \to \overline{M}_m$, consider the fiber product:

$$\begin{array}{ccc} X \times_{\overline{M}_m} \overline{U}_m & \xrightarrow{\bar{\gamma}} & \overline{U}_m \\ \downarrow \pi_X & & \downarrow \pi \\ X & \xrightarrow{\gamma} & \overline{M}_m \end{array}$$

The morphism $\gamma : X \to \overline{M}_m$ is $\mathcal{H} -$ *balanced* if

(i) for $1 \leq i \leq k$, $\pi_{X*}\bar{\gamma}^*(\mathcal{H}_i \otimes \mathcal{H}_0^{-1})$ is locally free;

(ii) for $1 \leq i \leq k$, the canonical map $\pi_X^*\pi_{X*}\bar{\gamma}^*(\mathcal{H}_i \otimes \mathcal{H}_0^{-1}) \to \bar{\gamma}^*(\mathcal{H}_i \otimes \mathcal{H}_0^{-1})$ is an isomorphism.

The condition (ii) implies that X goes to the locus in $\overline{M}_m$ satisfying the equal mutidegree condition for any pair of line bundles $(\mathcal{H}_i, \mathcal{H}_0)$, $i = 1, \ldots, k$ on each fibre of the universal curve $\overline{U}_m \mid_{\gamma(X)}$. The reason is direct image sheaves may change the rank of the sheaves. If that happens, then the pull back of the bundle $\pi_X^*\pi_{X*}\bar{\gamma}^*(\mathcal{H}_i \otimes \mathcal{H}_0^{-1})$ in (ii) has different rank, preventing it from becoming isomorphic to $\bar{\gamma}^*(\mathcal{H}_i \otimes \mathcal{H}_0^{-1})$. Examples showing the rank changes of direct image sheaves are the following :

Let $\pi : \mathbb{CP}^1 \to Spec\mathbb{C}$. Then,

$\pi_*(\mathcal{O}_{\mathbb{CP}^1}(1)) = H^0(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1}(1)) \cong \mathbb{C} \oplus \mathbb{C}$

$\pi_*(\mathcal{O}_{\mathbb{CP}^1}) = H^0(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1}) \cong \mathbb{C}$

$\pi_*(\mathcal{O}_{\mathbb{CP}^1}(-1)) = H^0(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1}(-1)) \cong 0$

and $\pi^*\pi_*(\mathcal{O}_{\mathbb{CP}^1}(1))$, $\pi^*\pi_*(\mathcal{O}_{\mathbb{CP}^1})$, $\pi^*\pi_*(\mathcal{O}_{\mathbb{CP}^1}(-1))$ are trivial bundles of rank 2, 1, 0 on $\mathbb{CP}^1$ respectively. We can calculate direct image sheaves for the reducible curve cases by using a short exact sequence of locally free sheaves related to a normalization, by noticing that a genus zero curve is a tree, which implies the number of gluing points is one less than the number of connected components, and taking a long exact sheaf cohomology sequence induced from that. What we can see is the rank of the line bundle is preserved by $\pi^*\pi_*$ only when the line bundle is trivial on the fibre. Hence, the image of X by a morphism γ sits inside of a certain locus in $\overline{M}_m$ on which the $\mathcal{H}_i \otimes \mathcal{H}_0^{-1}$ are trivial line bundles on each geometric fibre, equivalently, the locus satisfying equal multi degree conditions for any pair of $(\mathcal{H}_i, \mathcal{H}_0)$, $i = 1, \dots, k$. The universal sublocus B in $\overline{M}_m$ for the flat families in (ii) of the definition of the $\overline{h}$-rigid stable family is the largest sublocus satisfying equal multi degree conditions for any pair of $(\mathcal{H}_i, \mathcal{H}_0)$. Then, B is closed by an upper semicontinuity property [Har, p288]. By vanishing of higher direct image sheaf $\mathcal{R}^i\pi_*$ for $i \geq 1$ and the Cohomology and Base Change Theorem [Har, p290], the direct image sheaf of the line bundle $\mathcal{H}_i \otimes \mathcal{H}_0^{-1}$ on the universal curve $\overline{U}_m$ becomes a well-defined line bundle on the locus B in $\overline{M}_m$. The subscheme B itself is $\mathcal{H}$-balanced by an inclusion map to $\overline{M}_m$ because the natural morphism $\pi_B^*(\pi_{B*}(\mathcal{H}_i \otimes \mathcal{H}_0^{-1}))_x \otimes_{\mathcal{O}_x} \mathbb{C} \to (\mathcal{H}_i \otimes \mathcal{H}_0^{-1})_x \otimes_{\mathcal{O}_x} \mathbb{C}$ is surjective for all $x \in \pi^{-1}(B)$ by noting $\pi_B^*(\pi_{B*}(\mathcal{H}_i \otimes \mathcal{H}_0^{-1}))_x \otimes_{\mathcal{O}_x} \mathbb{C}$ is isomorphic to the global section sheaf of $\mathcal{H}_i \otimes \mathcal{H}_0^{-1}$ on the fibre of $\pi_B^{-1}(\pi_B(x))$.

Lemma 3.2 *The universal closed sublocus B in $\overline{M}_m$ is a real projective variety.*

proof. Since B satisfies the equal multidegree condition, for any i and a chosen irreducible component in the fibre over $b \in B$, the number of marked points from $\{q_{i,j}\}_{1 \leq j \leq d}$ is the same. The closed sublocus B in the projective variety $\overline{M}_m$ is invariant under the antiholomorphic involution described in 3 in Example 2.1 because the involution preserves the number of marked points from $\{q_{i,j}\}_{1 \leq j \leq d}$ on each irreducible component in any pair of conjugate curves. The Lemma follows from Lemma 3.1. □

Before we do a fibration over B, we see how the fibration can compensate the missing data with an example. Recall the following standard facts.

Lemma 3.3 *[Har, p150] Let C be a scheme over $\mathbb{C}$.*

If $\mathcal{L}$ is an invertible sheaf on C, and if $s_0, \ldots, s_k \in H^0(C, \mathcal{L})$ are global sections which generate $\mathcal{L}$, then there exists a unique morphism $\varphi : C \to \mathbb{CP}^k$ such that $\mathcal{L} \cong \varphi^*(\mathcal{O}_{\mathbb{CP}^k}(1))$ and $s_i = \varphi^*(w_i)$ under this isomorphism.

Lemma 3.4 *[Har, p157] Let C be a nonsingular projective variety over $\mathbb{C}$. Let D_0 be a divisor on C and let $\mathcal{L} \cong \mathcal{L}(D_0)$ be the corresponding invertible sheaf. Then,*
(a) Every effective divisor linearly equivalent to D_0 is $(s)_0$ for some $s \in H^0(C, \mathcal{L})$, where $(s)_0$ denotes the divisor of zeros of s.
(b) Two sections $s, s' \in H^0(C, \mathcal{L})$ have the same divisor of zeros if and only if there is a $\lambda \in \mathbb{C}^*$ such that $s' = \lambda s$.

Example 3.1 Let's consider the geometric fibre on a geometric point in B, i.e., π_B^b : $(\mathbb{CP}^1; \{p_i\}, \{q_{i,j}\}) \to Spec\mathbb{C} \cong b \in B$. We will use each set $\{q_{i,j}\}_{1 \le j \le d}$, $0 \le i \le k$ from the last $d(k+1)$-marked points as a Weil divisor, so, effective Cartier, $q_{i,1} + \ldots + q_{i,d}$. To use Lemma 3.3, we have to use one line bundle and select $k+1$ global sections s_i, telling the actual morphism to $\mathbb{CP}^k$, satisfying the condition (iii) in the definition of a $\overline{h}$-rigid stable family of curves, i.e., vanishing at $\{q_{i,j}\}_{1 \le j \le d}$, for each $i = 0, 1, \ldots, k$. Let's consider a line bundle $\mathcal{O}_{\mathbb{CP}^1}(q_{0,1} + \ldots + q_{0,d})$ although we may consider any other line bundle $\mathcal{O}_{\mathbb{CP}^1}(q_{i,1} + \ldots + q_{id})$. Note that all same degree effective divisors are linearly equivalent since line bundles over $\mathbb{CP}^k$ are classified by their degrees. Hence, we can choose $k+1$ global sections $s_i \in H^0(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1}(q_{0,1} + \cdots + q_{0,d}))$ satisfying the requirement by Lemma 3.4 (a). On the other hand, the linear system on $\mathbb{CP}^1$ generated by s_i, $i = 0, \ldots, k$, has no base point because the $q_{i,j}$ are distinct points in $\mathbb{CP}^1$. Thus, we can use Lemma 3.3.

Now, we describe the morphism to $\mathbb{CP}^k$ decided by the chosen global sections s_i, $i = 0, \ldots, k$. Let's denote $S_0 = \{p \in \mathbb{CP}^1 | s_0(p)$ doesn't vanish $\}$ and $U = \{w_0 \neq 0\} \subset \mathbb{CP}^k$. Then, the actual morphism restricted to $S_0 \to U$ comes from the ring homomorphism $\mathbb{C}[z_1, \ldots, z_k] \to \Gamma(S_0, \mathcal{O}_{S_0})$ by sending

$$z_i \mapsto s_i/s_0, \tag{1}$$

and making it $\mathbb{C}$-linear, where $z_i = w_i/w_0$, $i = 1, \ldots, k$. Since the set S_0 is $\mathbb{CP}^1\setminus$ finite points, the above restriction map is uniquely extended to the whole space $\mathbb{CP}^1$. Observe that the morphism in Lemma 3.3, i.e., the morphism (1), is dependent on the actual choice of $s_i \in H^0(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1}(q_{0,1} + \cdots + q_{0,d}))$, but up to ratio of s_i/s_0, $i = 1, \ldots, k$. More precisely, $\{\lambda_0 s_0, \ldots, \lambda_k s_k\}$, $\{\lambda'_0 s_0, \ldots, \lambda'_k s_k\}$, $\lambda_i, \lambda'_i \in \mathbb{C}^*$ induce the same morphism if and only if $\lambda_i/\lambda_0 = \lambda'_i/\lambda'_0$, $i = 1, \ldots, k$.

Constructing a space recording all possible ratios $\lambda_i s_i/\lambda_0 s_0$, $\lambda_i, \lambda_0 \in \mathbb{C}^*$ is our goal. Then, there will be a one-to-one correspondence between points in the constructed universal space over $Spec\mathbb{C} \cong b$ and maps whose hyperplane intersection points are $\{q_{i,j}\}_{1\leq j\leq d}$, $i = 0, \ldots, k$.

Let s_i, $i = 0, \ldots, k$ be the chosen global sections. Observe that invertible sheaves $\mathcal{H}_i \otimes \mathcal{H}_0^{-1}$ are generated by any $r_i(s_0/s_i)$, $r_i \in \mathbb{C}^*$ and the coefficients r_i, which are degree 0 polynomials, can be considered as elements in the $H^0(\mathbb{CP}^1, \mathcal{H}_i \otimes \mathcal{H}_0^{-1})$. Therefore, all possible ratios $\lambda_i s_i/\lambda_0 s_0$, $\lambda_i, \lambda_0 \in \mathbb{C}^*$ can be recorded by $[H^0(\mathbb{CP}^1, \mathcal{H}_1 \otimes \mathcal{H}_0^{-1}) \setminus 0] \times \ldots \times [H^0(\mathbb{CP}^1, \mathcal{H}_k \otimes \mathcal{H}_0^{-1}) \setminus 0] \cong \mathbb{C}^* \times \ldots \times \mathbb{C}^*$, where $\mathcal{H}_i = \mathcal{O}_{\mathbb{CP}^1}(q_{i,1} + \ldots + q_{i,d})$.

Let's summarize the geometric procedure of the above construction.
Let $\pi : (\mathbb{CP}^1, \{p_i\}, \{q_{i,j}\}) \to Spec\mathbb{C}$ be a geometric fibre on the geometric point b in B and $s_0, \ldots, s_k$ be global sections in $H^0(\mathbb{CP}^1, \mathcal{H}_0)$, whose zeros generate effective divisors $q_{i,1}+\ldots+q_{i,d}$, $i = 0, \ldots, k$. We constructed bundles $\mathcal{H}_i = \mathcal{O}_{\mathbb{CP}^1}(q_{i,1}+\ldots+q_{i,d})$ using the last $d(k+1)$ marked points and considered the tensor bundles $\mathcal{H}_i \otimes \mathcal{H}_0^{-1}$ on $\mathbb{CP}^1$. Then, we considered the direct image sheaves $\pi_{B*}(\mathcal{H}_i \otimes \mathcal{H}_0^{-1}) \cong \mathbb{C}$, deleted a zero element from each $i = 1, \ldots, k$ because of Lemma 3.4 (b), denoting them by

$Y_i'^b$, and constructed a k-dimensional $\mathbb{C}^*$-bundle $Y'^b := Y_1'^b \times \ldots \times Y_k'^b$ on $Spec\mathbb{C} \cong b$.

$$\begin{array}{ccccccc} \mathbb{CP}^k & \stackrel{\mu^b}{\leftarrow} & Y'^b \times_{\mathbb{C}} \mathbb{CP}^1 & \stackrel{\overline{\gamma}^b}{\rightarrow} & \mathbb{CP}^1 & \hookrightarrow & B\overline{U}_{0,m} \\ & & \pi_B'^b \downarrow & & \downarrow \pi_B^b & & \downarrow \pi_B \\ & & Y'^b & \stackrel{\gamma^b}{\rightarrow} & Spec\mathbb{C} \cong b & \hookrightarrow & B \quad \subset \quad \overline{M}_m \end{array}$$

where $B\overline{U}_{0,m}$ is the restriction of the universal curve $\overline{U}_m$ over $\overline{M}_m$ to B.

The fibres of any elements in the k-dimensional $\mathbb{C}^*$-bundle Y'^b are naturally equipped with $k+1$ sections in $H^0(Y'^b \times_{\mathbb{C}} \mathbb{CP}^1, \overline{\gamma}^{b*}(\mathcal{H}_0))$, representing pull-back divisors $\overline{\gamma}^{b*}(q_{i,1} + \ldots + q_{i,d})$. By Lemma 3.3, there is a morphism μ to $\mathbb{CP}^k$ whose restriction to each fibre over Y'^b is similar to the morphism described in (1). In fact, that morphism is given by $z_i \mapsto r_i(y) \cdot \overline{\gamma}^{b*}(s_i)/\overline{\gamma}^{b*}(s_0)$, where $y \in Y'^b$, r_i is a $\mathbb{C}^*$-valued function on Y'^b which may be understood as an i-th projection map from Y'^b to $Y_i'^b = H^0(\mathbb{CP}^1, \mathcal{H}_i \otimes \mathcal{H}_0^{-1}) \setminus 0 \cong \mathbb{C}^*$.

What we have seen is the construction of the universal space $\overline{M}_n(\mathbb{CP}^k, d, \overline{h})$ for the $\overline{h}$-rigid stable family of maps over a geometric point in B. The way we construct the k-dimensional $\mathbb{C}^*$-bundle $\overline{M}_n(\mathbb{CP}^k, d, \overline{h})$, which will be also denoted as Y, on B is the same. Conditions in $\mathcal{H}$-balanced allow us to globalize the above construction. The first condition in $\mathcal{H}$-balanced guarantees direct image sheaves $\pi_{B*}(\mathcal{H}_i \otimes \mathcal{H}_0^{-1})$ of $\mathcal{H}_i \otimes \mathcal{H}_0^{-1}$ define line bundles on B and induce a nice geometric object, a k-dimensional $\mathbb{C}^*$-fibration $Y \equiv Y_1 \times_B \ldots \times_B Y_k$ on B, where $Y_i = \pi_{B*}(\mathcal{H}_i \otimes \mathcal{H}_0^{-1}) \setminus 0$, $i = 1, \ldots, k$. The second condition in $\mathcal{H}$-balanced is used to get canonical sections in $H^0(Y \times_B B\overline{U}_m, \overline{\gamma}^*(\mathcal{H}_i \otimes \mathcal{H}_0^{-1}))$. Those global sections give global sections in $H^0(Y \times_B B\overline{U}_m, \overline{\gamma}^*(\mathcal{H}_0))$ representing Weil divisors $\overline{\gamma}^*(q_{i,1} + \ldots + q_{i,d})$, $i = 1, \ldots, k$, reflecting the meaning of each point in Y which was explained at the end of Example 3.1. By Lemma 3.3, we can define a morphism μ from $Y \times_B B\overline{U}_{0,m}$ to $\mathbb{CP}^k$. Let's describe that more precisely.

$$\begin{tikzcd}
\mathbb{CP}^k & Y \times_B B\overline{U}_m \arrow[l, "\mu"'] \arrow[r, "\bar{\gamma}"] \arrow[d, "\pi'_B"'] & B\overline{U}_m \arrow[d, "\pi_B"] \\
& Y = Y_1 \times_B \ldots \times_B Y_k \arrow[r, "\gamma"] \arrow[dr, "\pi'_{B_i}"'] & B \\
& & Y_i \arrow[u, "\gamma_i"']
\end{tikzcd}$$

We observe the following:

$$1.\bar{\gamma}^*(\mathcal{H}_i \otimes \mathcal{H}_0^{-1}) \cong \bar{\gamma}^*\pi_B^*\pi_{B*}(\mathcal{H}_i \otimes \mathcal{H}_0^{-1}) \quad \text{by the second condition in } \mathcal{H}\text{-balanced}$$
$$\cong \pi_B'^*\pi_{B_i}'^*\gamma_i^*\pi_{B*}(\mathcal{H}_i \otimes \mathcal{H}_0^{-1})$$

2. $\gamma_i^*\pi_{B*}(\mathcal{H}_i \otimes \mathcal{H}_0^{-1})$ has a tautological section because of the definition of Y_i.
3. The pull-back of the tautological section to $\pi_B'^*\pi_{B_i}'^*\gamma_i^*\pi_{B*}(\mathcal{H}_i \otimes \mathcal{H}_0^{-1})$ gives a globally non-vanishing section.
2 and 3 imply $\bar{\gamma}^*(\mathcal{H}_i \otimes \mathcal{H}_0^{-1})$ is a trivial line bundle with tautological non-vanishing sections r_i, $i = 1, \ldots, k$ which are constant along the fibres of Y. We can treat those sections r_i as functions from Y to $\mathbb{C}^*$. Now, we got the desired canonical induced sections $r_i\bar{\gamma}^*(s_i) \in H^0(Y \times_B B\overline{U}_m, \bar{\gamma}^*(\mathcal{H}_0))$, $i = 0, \ldots, k$, where $r_0 = 1$, and $r_i : Y \to \mathbb{C}^*$. Lemma 3.3 gives a morphism from $Y \times_B B\overline{U}_m$ to $\mathbb{CP}^k$ such that $\mu^*(h_i) = r_i\bar{\gamma}^*(s_i)$, $i = 0, \ldots, k$, and $\mu^*(\mathcal{O}_{\mathbb{CP}^k}(1)) \cong \bar{\gamma}^*(\mathcal{H}_0)$.

All we have explained is the following Proposition.

Proposition 3.1 *[F-P] The moduli space of $\overline{h}$-rigid stable family of degree d maps from n-pointed, genus 0 curves to $\mathbb{CP}^k$ is a fine moduli space $\overline{M}_n(\mathbb{CP}^k, d, \overline{h})$ which is a nonsingular projective variety.*

[II] Quotients and gluing;

The moduli space of stable maps of genus zero $\overline{M}_n(\mathbb{CP}^k, d)$ was constructed by gluing quotients of the moduli spaces of $\overline{h}$-rigid stable family of maps $\overline{M}_n(\mathbb{CP}^k, d, \overline{h})$, where $\overline{h}$ is any basis of $H^* = H^0(\mathbb{CP}^k, \mathcal{O}_{\mathbb{CP}^k}(1))$.

We need to consider the followings:

1. Is there any ordered basis $\overline{h} = (h_0, \ldots, h_k)$ in H^* for a given n-marked, genus 0, degree d curve to $\mathbb{CP}^k$, such that the curve intersects with any chosen hyperplane basis h_i transversally at unmarked, nonsingular points? That is, can we get enough gluing pieces from the $\overline{M}_n(\mathbb{CP}^k, d, \overline{h})$?
2. The last $d(k+1)$-marked points $\{q_{i,j}\}_{1\leq j\leq d}$, $i = 0, \ldots, k$ played a role as hyperplane intersection divisors $q_{i,1} + \ldots + q_{i,d}$, $i = 0, \ldots, k$. How can we forget orders of points in each set $\{q_{i,j}\}_{1\leq j\leq d}$, $i = 0, \ldots, k$?
3. How can we glue quotients of moduli spaces $\overline{M}_n(\mathbb{CP}^k, d, \overline{h})$ for various choices of basis $\overline{h}$ of H^*?

The answers are the following:

1. Bertini's theorem tells us that most hyperplanes in H^* intersect with the given curve transversally at nonsingular points. So, we can always find the ordered basis $\overline{h}$ satisfying the conditions.
2. We make the product of the symmetric group $G = G_d^0 \times \ldots \times G_d^k$ act on the moduli space $(\pi : \mathcal{U} \to \overline{M}_n(\mathbb{CP}^k, d, \overline{h}), \{p_i\}, \{q_{i,j}\}_{0\leq i\leq k, 1\leq j\leq d}, \mu)$, where G_d^i acts on the set $\{q_{i,j}\}_{1\leq j\leq d}$ by permuting the orders.
Since the finite group G acts on the projective variety $\overline{M}_n(\mathbb{CP}^k, d, \overline{h})$, its quotient $\overline{M}_n(\mathbb{CP}^k, d, \overline{h})/G$ is also a projective variety.
3. Let $\bar{h}$, $\bar{h}'$ be different choices of basis of H^*. There are G-invariant open subloci $\overline{M}(\bar{h}, \bar{h}')$, $\overline{M}(\bar{h}', \bar{h})$ in $\overline{M}_n(\mathbb{CP}^k, d, \overline{h})$, $\overline{M}_n(\mathbb{CP}^k, d, \overline{h'})$ respectively, consisting of curves intersecting with all hyperplane basis $h_0, \ldots, h_k, h'_0, \ldots, h'_k$ transversally at nonsingular unmarked points. Clearly, $\overline{M}(\bar{h}, \bar{h}')$ and $\overline{M}(\bar{h}', \bar{h})$ are isomorphic. And Fulton-

Pandharipande showed $\overline{M}(\bar{h}, \bar{h}')/G$ and $\overline{M}(\bar{h}', \bar{h})/G$ are also isomorphic in [F-P] Proposition 4.

3.2 Proof: The moduli space of stable maps is a real moduli space

First, we show that the ingredients used in Section 3.1 [I] are real with respect to the antiholomorphic involution induced by a complex conjugation map on $\mathbb{CP}^1$, $\mathbb{CP}^k$.

Lemma 3.5 *The fine moduli space $\overline{M}_n(\mathbb{CP}^k, d, \overline{h})$ is a real projective variety whose real structure is induced by complex conjugation maps on $\mathbb{CP}^1$, $\mathbb{CP}^k$, where $\overline{h} = (h_0, \ldots, h_k)$ is a real ordered hyperplane basis of $H^0(\mathbb{CP}^k, \mathcal{O}_{\mathbb{CP}^k}(1))$.*

proof. By Lemma 3.2, B is a real projective variety. Let $\pi_B : B\overline{U}_m \to B$ be the real universal curve with m real sections $\{p_i\}_{1\leq i\leq n}$ and $\{q_{i,j}\}_{0\leq i\leq k, 1\leq j\leq d}$ from B to $B\overline{U}_m$. Then, the Weil divisors, $q_{i,1}+\ldots+q_{i,d}$, $i = 0, \ldots, k$ and $q_{i,1}+\ldots+q_{i,d}-q_{0,1}-\ldots-q_{0,d}$, $i = 1, \ldots, k$ are all invariant under the anti-holomorphic involution on $B\overline{U}_m$. That implies the associated line bundles $\mathcal{H}_i$, $\mathcal{H}_i \otimes \mathcal{H}_0^{-1}$ are all real line bundles by Remark 2.2. Equivalently, there is an anti-holomorphic bundle involution on each bundle. Let $\tau, \tilde{\tau}, \tilde{\tau}_i'$ denote anti-holomorphic involutions on $B, B\overline{U}_m, \mathcal{H}_i \otimes \mathcal{H}_0^{-1}$ respectively and π_B' be a natural projection map from the line bundles $\mathcal{H}_i \otimes \mathcal{H}_0^{-1}$ on $B\overline{U}_m$ to $B\overline{U}_m$.

$$\begin{array}{ccc}
\mathcal{H}_i \otimes \mathcal{H}_0^{-1} & \stackrel{\tilde{\tau}_i'}{\rightarrow} & \mathcal{H}_i \otimes \mathcal{H}_0^{-1} \\
\pi_B' \downarrow & & \pi_B' \downarrow \\
B\overline{U}_m & \stackrel{\tilde{\tau}}{\rightarrow} & B\overline{U}_m \\
\pi_B \downarrow & & \downarrow \pi_B \\
B & \stackrel{\tau}{\rightarrow} & B
\end{array}$$

$$\begin{array}{ccc}
\mathcal{H}_i \otimes \mathcal{H}_0^{-1}|_{\pi_B'^{-1}(\mathcal{C}_b)} & \stackrel{\tilde{\tau}_i'|_{\pi_B'^{-1}(\mathcal{C}_b)}}{\rightarrow} & \mathcal{H}_i \otimes \mathcal{H}_0^{-1}|_{\pi_B'^{-1}(\mathcal{C}_{\tau(b)})} \\
\pi_B' \downarrow & & \downarrow \pi_B' \\
\mathbb{CP}^1 \cong \mathcal{C}_b & \stackrel{\tilde{\tau}|_{\pi_B^{-1}(b)}}{\rightarrow} & \mathcal{C}_{\tau(b)} \cong \mathbb{CP}^1
\end{array}$$

We'll show the line bundles $\pi_{B*}(\mathcal{H}_i \otimes \mathcal{H}_0^{-1})$ have natural anti-holomorphic bundle involutions induced from the anti-holomorphic bundle involutions on the $\mathcal{H}_i \otimes \mathcal{H}_0^{-1}$. Since $\pi_{B*}(\mathcal{H}_i \otimes \mathcal{H}_0^{-1})$ is a bundle over the real scheme B, it is enough to show that there is a natural anti-holomorphic involution between fibres over b and $\tau(b)$. Let's see the bundle map $\tilde{\tau}_i'|_{\pi_B'^{-1}(\mathcal{C}_b)}$ restricted to the pointed curve $\mathcal{C}_b \equiv \pi_B^{-1}(b)$ and its pointed conjugate curve $\mathcal{C}_{\tau(b)} \equiv \pi_B^{-1}(\tau(b))$, where $b \in B$ represents a nonsingular pointed curve isomorphic to $\mathbb{CP}^1$. Then, $\mathcal{C}_{\tau(b)}$ is isomorphic to $\mathbb{CP}^1$ with conjugate marked points. For notational convenience, we denote both $\mathcal{C}_b, \mathcal{C}_{\tau(b)}$ as $\mathbb{CP}^1$. Since the divisor $D_{i,b} \equiv q_{i,1}(b) + \ldots + q_{i,d}(b) - q_{0,1}(b) - \ldots - q_{0,d}(b)$ has degree zero on $\mathbb{CP}^1$, it is a principal divisor. Let $D_{i,b}$ be defined by $f_i \cdot f_0^{-1} \in H^0(\mathbb{CP}^1, \mathcal{K}^*)$, where $\mathcal{K}^*$ consists of invertible elements in the sheaf of total quotient rings of $\mathcal{O}_{\mathbb{CP}^1}$. Then, $\mathcal{O}_{\mathbb{CP}^1}(D_{i,b})$ is globally generated by $f_0 \cdot f_i^{-1}$. The divisor $D_{i,\tau(b)}$ on the conjugate curve is defined by $\bar{f}_0 \cdot \bar{f}_i^{-1} \in H^0(\mathbb{CP}^1, \mathcal{K}^*)$ because $D_{i,\tau(b)}$ is $\bar{q}_{i,1}(b) + \ldots + \bar{q}_{i,d}(b) - \bar{q}_{0,1}(b) - \ldots - \bar{q}_{0,d}(b)$, where $\bar{f}_j$ is a conjugate polynomial whose coefficients are complex conjugates of the f_j. Then, $\mathcal{O}_{\mathbb{CP}^1}(D_{i,\tau(b)})$ is globally generated by $\bar{f}_0 \cdot \bar{f}_i^{-1}$. So, the restriction of the globally defined anti-holomorphic bundle involution $\tilde{\tau}_i'$ on $\mathcal{H}_i \otimes \mathcal{H}_0^{-1}$ to the map between $\mathcal{O}_{\mathbb{CP}^1}(D_{i,b})$ and $\mathcal{O}_{\mathbb{CP}^1}(D_{i,\tau(b)})$ is the map sending $f_0 \cdot f_i^{-1}$ to $\bar{f}_0 \cdot \bar{f}_i^{-1}$. We can describe the similar situation when $b \in B$ represents a pointed singular curve with a little more work by using the sheaf exact sequence of a normalization.

The canonical anti-holomorphic bundle involution τ_i' on $\pi_{B*}(\mathcal{H}_i \otimes \mathcal{H}_0^{-1})$ is induced from $\tilde{\tau}_i'$. Observe the restriction map $\tau_i'|_b : \pi_{B*}(\mathcal{H}_i \otimes \mathcal{H}_0^{-1})|_b \to \pi_{B*}(\mathcal{H}_i \otimes \mathcal{H}_0^{-1})|_{\tau(b)}$ can be considered as a complex conjugation map on the induced local charts because $\pi_{B*}(\mathcal{H}_i \otimes \mathcal{H}_0^{-1})|_b \cong H^0(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1}(D_{i,b})) \cong \mathbb{C}$ and the bundle map $\tilde{\tau}_i'|_{\pi'^{-1}(\mathcal{C}_b)}$ goes $\alpha \cdot f_0 \cdot f_i^{-1} \mapsto \bar{\alpha} \cdot \bar{f}_0 \cdot \bar{f}_i^{-1}$ for any $\alpha \in \mathbb{C}$, where $\bar{\alpha}$ denotes a complex conjugate of $\alpha \in \mathbb{C}$. This shows τ_i' is an anti-holomorphic bundle involution on $\pi_{B*}(\mathcal{H}_i \otimes \mathcal{H}_0^{-1})$.

Let Y_i, $i = 1, \ldots, k$ be the $\mathbb{C}^*$-bundle coming from $\pi_{B*}(\mathcal{H}_i \otimes \mathcal{H}_0^{-1})$ by removing a zero section. The restriction maps of τ_i' to Y_i, $i = 1, \ldots, k$ are anti-holomorphic $\mathbb{C}^*$-bundle involutions. The k-dimensional $\mathbb{C}^*$-bundle $Y \equiv Y_1 \times_B \ldots \times_B Y_k$ has an induced anti-holomorphic bundle involution. That means $\overline{M}_n(\mathbb{CP}^k, d, \overline{h}) \equiv Y$ is a quasi-projective real variety.

Since the moduli space of $\overline{h}$-stable degree d, n-pointed curves $\overline{M}_n(\mathbb{CP}^k, d, \overline{h})$ is a fine moduli space, we want to show $\overline{M}_n(\mathbb{CP}^k, d, \overline{h})$ is equipped with a real universal curve and a real projective morphism from the universal curve over $\overline{M}_n(\mathbb{CP}^k, d, \overline{h})$ to $\mathbb{CP}^k$.

Note that Y has a universal family $Y \times_B B\overline{U}_m$ induced from the real universal family $B\overline{U}_m \to B \subset \overline{M}_m$ and $Y \times_B B\overline{U}_m$ has a natural morphism to $\mathbb{CP}^k$ as explained in section 3.1 [I]. We will show $Y \times_B B\overline{U}_m$ is real and μ is a real morphism to $\mathbb{CP}^k$. It is easy to see $Y \times_B B\overline{U}_m$ is real. Note that Y, B, $B\overline{U}_m$ are real varieties. Since a question is local, we may consider Y, B, $B\overline{U}_m$ as $Spec(\mathcal{Y}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C})$, $Spec(\mathcal{B}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C})$, $Spec(\mathcal{B}\overline{U}_{\mathbb{R}m} \otimes_{\mathbb{R}} \mathbb{C})$ respectively. Let's denote $\mathcal{Y}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} := \mathcal{Y}$, $\mathcal{B}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} := \mathcal{B}$, $\mathcal{B}\overline{U}_{\mathbb{R}m} \otimes_{\mathbb{R}} \mathbb{C} := \mathcal{B}\overline{U}_m$, real models $Spec(\mathcal{Y}_{\mathbb{R}}) := Y^{\mathbb{R}}$, $Spec(\mathcal{B}_{\mathbb{R}}) := B^{\mathbb{R}}$, $Spec(\mathcal{B}\overline{U}_{\mathbb{R}m}) := B\overline{U}_m^{\mathbb{R}}$.

$$\begin{aligned} \mathcal{Y} \otimes_{\mathcal{B}} \mathcal{B}\overline{U}_m &\cong [\mathcal{Y}_{\mathbb{R}} \otimes_{\mathcal{B}_{\mathbb{R}}} (\mathcal{B}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C})] \otimes_{\mathcal{B}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}} (\mathcal{B}\overline{U}_{\mathbb{R}m} \otimes_{\mathbb{R}} \mathbb{C}) \\ &\cong (\mathcal{Y}_{\mathbb{R}} \otimes_{\mathcal{B}_{\mathbb{R}}} \mathcal{B}\overline{U}_{\mathbb{R}m}) \otimes_{\mathbb{R}} \mathbb{C} \end{aligned}.$$

This means $Y \times_B B\overline{U}_m \cong (Y^{\mathbb{R}} \times_{B^{\mathbb{R}}} B\overline{U}_m^{\mathbb{R}}) \times_{\mathbb{R}} \mathbb{C}$.

Finally, we see that there is a canonical morphism $\mu^{\mathbb{R}}$ from $Y^{\mathbb{R}} \times_{B^{\mathbb{R}}} B\overline{U}_m^{\mathbb{R}}$ to $\mathbb{RP}^k$ by the similar construction we saw in the section 3.1 [I]. Therefore, we may consider the morphism μ as a complexification of $\mu^{\mathbb{R}}$. So, the Lemma follows. □

We are ready to prove the main Theorem in this section. We will consider the similar questions written in section 3.1 [II] in a real setting.

Theorem 3.1 *The moduli space of stable maps of genus zero $\overline{M}_n(\mathbb{CP}^k, d)$ is a real moduli space whose real structure is induced from anti-holomorphic involutions by complex conjugations on $\mathbb{CP}^1$, $\mathbb{CP}^k$.*

proof. Recall that Fulton-Pandharipande's construction was about the moduli space $\overline{M}_n(\mathbb{CP}^k, d)$ for $k > 0$, $d > 0$ and $(n, k, d) \neq (0, 1, 1)$. Other cases, $\overline{M}_n(\mathbb{CP}^0, 0)$, $\overline{M}_n(\mathbb{CP}^k, 0)$, $\overline{M}_0(\mathbb{CP}^1, 1)$ are isomorphic to $\overline{M}_n$, $\overline{M}_n \times \mathbb{CP}^k$, $Spec(\mathbb{C})$ respectively and so, they are obviously real moduli spaces.

We showed $\overline{M}_n(\mathbb{CP}^k, d, \overline{h})$ is a *real* fine moduli space, where $\overline{h}$ is a real ordered basis of $H^* = H^0(\mathbb{CP}^k, \mathcal{O}_{\mathbb{CP}^k}(1))$. We have to consider the following questions :

1. Is there any *real* ordered basis $\overline{h} = (h_0, \ldots, h_k)$ in H^* for a given n-marked, genus 0, degree d curve to $\mathbb{CP}^k$ such that the given curve intersects with any chosen hyperplane basis h_i transversally at unmarked, nonsingular points?

2. Does the product of the symmetric group G action on the moduli space $\overline{M}_n(\mathbb{CP}^k, d, \overline{h})$ commute with the anti-holomorphic involution on $\overline{M}_n(\mathbb{CP}^k, d, \overline{h})$ so that the quotient space $\overline{M}_n(\mathbb{CP}^k, d, \overline{h})/G$ has an anti-holomorphic involution, (i.e. becomes a real variety)?

3. Does the gluing commute with an anti-holomorphic involution so that the anti-holomorphic involution on each $\overline{M}_n(\mathbb{CP}^k, d, \overline{h})/G$ extends to the whole moduli space $\overline{M}_n(\mathbb{CP}^k, d)$?

The answers are the following:

1. As F. Sottile [Sot3] pointed out, real points are Zariski dense in $\mathbb{CP}^k$. That implies we can always find a *real* ordered basis satisfying the transversality condition.

2. The symmetric group G action on $\overline{M}_n(\mathbb{CP}^k, d, \overline{h})$ to forget the orders of the marked points $\{q_{i,j}\}$ in each of the last $k+1$ sets commutes with the anti-holomorphic involution.

i.e.
$$\begin{array}{ccc}
(\mathcal{C}_b, \{p_i\}, \{q_{i,j}\}, \mu_b) & \overset{\sigma}{\rightarrow} & (\mathcal{C}_b, \{p_i\}, \{q_{i,\sigma(j)}\}, \mu_b) \\
\downarrow \tau' & & \downarrow \tau' \\
(\mathcal{C}_{\tau(b)}, \{\overline{p}_i\}, \{\overline{q}_{i,j}\}, \overline{\mu}_{\tau(b)}) & \overset{\sigma}{\rightarrow} & (\mathcal{C}_{\tau(b)}, \{\overline{p}_i\}, \{\overline{q}_{i,\sigma(j)}\}, \overline{\mu}_{\tau(b)})
\end{array}$$

,where τ', τ denote the involution on $\overline{M}_n(\mathbb{CP}^k, d, \overline{h})$ described in the proof of Lemma 3.5, on B respectively, and σ is an element in G.

Note that the description in the above diagram is about up to isomorphism according to the equivalence relation in the moduli space $\overline{M}_n(\mathbb{CP}^k, d, \overline{h})$ rather than about the actual model $(\mathcal{C}_b, \{p_i\}, \{q_{i,\sigma(j)}\}, \mu_b)$. But there is no problem.

For example, if $(\mathcal{C}_b, \{p_i\}, \{q_{i,\sigma(j)}\}, \mu_b)$ is isomorphic to $(\mathcal{C}_{b'}, \{p'_i\}, \{q'_{i,j}\}, \mu_{b'})$, then $(\mathcal{C}_{\tau(b)}, \{\bar{p}_i\}, \{\bar{q}_{i,\sigma(j)}\}, \bar{\mu}_{\tau(b)})$ is isomorphic to $(\mathcal{C}_{\tau(b')}, \{\bar{p}'_i\}, \{\bar{q}'_{i,j}\}, \bar{\mu}_{\tau(b')})$. More precisely, if the linear fractional transformation $(az+b)/(cz+d)$, $ad-bc \neq 0$, gives an equivalence relation between $(\mathcal{C}_b, \{p_i\}, \{q_{i,\sigma(j)}\}, \mu_b)$ and $(\mathcal{C}_{b'}, \{p'_i\}, \{q'_{i,j}\}, \mu_{b'})$, then the linear fractional transformation $(\bar{a}z+\bar{b})/(\bar{c}z+\bar{d})$ gives an equivalence relation between $(\mathcal{C}_{\tau(b)}, \{\bar{p}_i\}, \{\bar{q}_{i,\sigma(j)}\}, \bar{\mu}_{\tau(b)})$ and $(\mathcal{C}_{\tau(b')}, \{\bar{p}'_i\}, \{\bar{q}'_{i,j}\}, \bar{\mu}_{\tau(b')})$, where $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ are the complex conjugates of a, b, c, d if $\mathcal{C}_b$ is nonsingular.

3. It is easy to see that the isomorphism which is a gluing map between $\overline{M}(\overline{h}, \overline{h}')/G$ and $\overline{M}(\overline{h}', \overline{h})/G$ commutes with the anti-holomorphic involutions on $\overline{M}_n(\mathbb{CP}^k, d, \overline{h})/G$ and on $\overline{M}_n(\mathbb{CP}^k, d, \overline{h'})/G$ from the proof of [F-P] Proposition 4. Here, $\overline{M}(\overline{h}, \overline{h}')$ denotes a Zariski open sublocus in $\overline{M}_n(\mathbb{CP}^k, d, \overline{h})$ consisting of maps intersecting transversally with each of the hyperplanes in the basis $\overline{h}'$ of $H^0(\mathbb{CP}^k, \mathcal{O}_{\mathbb{CP}^k}(1))$. Since the gluing maps commute with the anti-holomorphic involutions on each of the quotients of the projective varieties $\overline{M}_n(\mathbb{CP}^k, d, \overline{h})$, the moduli space of stable maps of genus zero $\overline{M}_n(\mathbb{CP}^k, d)$ has a globally well-defined anti-holomorphic involution. We are done. □

Corollary 3.1 *Let X be a real projective variety having a real structure corresponding to the complex conjugate involution on $\mathbb{CP}^k$. Then, $\overline{M}_n(X;\beta)$ is a real projective variety.*

proof. It is natural from the construction. See section 5 in [F-P]. □

Remark 3.3 1. Corollary 3.1 cannot be extended to any real structures $(\mathbb{CP}^1, s)$, (X, t). Sometimes, the natural set theoretic correspondence $f \mapsto t \circ f \circ s$ doesn't define an anti-holomorphic involution on $\overline{M}_n(X, \beta)$. Let's consider the case on $\overline{M}_n(\mathbb{CP}^k, d)$. An anti-holomorphic involution on $\overline{M}_n(\mathbb{CP}^k, d)$ comes from an anti-holomorphic involution on the projective variety $\overline{M}_n(\mathbb{CP}^k, d, \overline{h})$ where the $\overline{h}$ form a real hyperplane basis. But not every involution on $\mathbb{CP}^k$ allows a real hyperplane basis $\overline{h}$. For example, an involution $z \mapsto -1/\bar{z}$ on $\mathbb{CP}^1$ does not allow such a basis $\overline{h}$. Then, there is no way to make real gluing pieces.

2. The implication of this section is that the moduli space of stable maps of genus

zero $\overline{M}_n(\mathbb{CP}^k, d)$ is isomorphic to $\overline{M}_n(\mathbb{CP}^k, d)^{\mathbb{R}} \times_{\mathbb{R}} \mathbb{C}$, where $\overline{M}_n(\mathbb{CP}^k, d)^{\mathbb{R}}$ is a real model. Hence, there is a natural Chow ring homomorphism from $A^d(\overline{M}_n(\mathbb{CP}^k, d)^{\mathbb{R}})$ to $A^d(\overline{M}_n(\mathbb{CP}^k, d))$, induced by complexification of cycles.

3. We introduce an adequate notion of a real group action which gives a natural correspondence in equivariant Chow cycles similar to Remark 3.3 2. We introduce the concept of a real group action on a real scheme X.

Definition. Let G be a real Lie group, i.e. Lie group having an antiholomorphic involution, and X be a real scheme. We call a group action G *a real group action on* X if a morphism μ defining a Lie group action

$$\mu : G \times X \longrightarrow X$$

commutes with real structures, i.e. μ is a real morphism.

With this notion, we have a natural equivariant Chow ring version's morphism from $A^*(X^{\mathbb{R}} \times_{G^{\mathbb{R}}} EG^{\mathbb{R}})$ to $A^*(X \times_G EG)$ by complexification of cycles.

We see examples of real group actions.

Example 3.2 1. Let $T = (\mathbb{C}^*)^{k+1}$ act on $\mathbb{CP}^k$ in the following way;

$$\begin{array}{ccc} T \times \mathbb{CP}^k & \longrightarrow & \mathbb{CP}^k \\ (t_0, \ldots, t_k) \cdot [z_0; \ldots; z_k] & \longmapsto & [t^{\lambda_0} \cdot z_0; \ldots; t^{\lambda_k} \cdot z_k] \end{array}$$

Then, it naturally induces a T-action on $\overline{M}_n(\mathbb{CP}^k, d)$. We can check it is a real group action.

2. Group actions inducing $\mathbb{CP}^k$ and $\mathcal{O}_{\mathbb{CP}^k}(m)$ are real group actions. i.e.,

$$\begin{array}{ccc} \mathbb{C}^* \times \mathbb{C}^{k+1} & \overset{\phi}{\to} & \mathbb{C}^{k+1} \\ (c; c_0, \ldots, c_k) & \overset{\phi_{\mathbb{C}}}{\mapsto} & (c \cdot c_0, \ldots, c \cdot c_k) \end{array}$$

and

$$\begin{array}{ccc} \mathbb{C}^* \times \mathbb{C}^{k+1} \times \mathbb{C} & \stackrel{\psi}{\to} & \mathbb{C}^{k+1} \times \mathbb{C} \\ (t; z_0, \dots, z_k; z) & \stackrel{\psi_t}{\mapsto} & (tz_0, \dots, tz_k, t^m z) \end{array}$$

are real group actions.

4 Real part of the moduli space of stable maps and Projectivity

4.1 Real part of the moduli space of stable maps

We describe the last section's construction more concretely. All homogeneous coordinate forms on a domain curve $\mathbb{CP}^1$ in discussion will be standard homogeneous coordinate forms. We will denote any irreducible component as $\mathbb{CP}^1$ for easier looking without mentioning a normalization. We may interpret choosing a real ordered hyperplane basis $\overline{h} = (h_0, \dots, h_k)$ of $H^0(\mathbb{CP}^k, \mathcal{O}_{\mathbb{CP}^k}(1))$ as choosing a homogeneous coordinate system for $\mathbb{CP}^k$. Then, the last $d(k+1)$ marked points in the definition of $\overline{h}$-rigid stable family gives us some information about the morphism's nature with polynomials' splitting forms. For example, if we express the hyperplane intersection points $\{q_{i,j}\}_{0 \le i \le k, 1 \le j \le d}$ by homogeneous coordinates $\{[q_{i,j}^{(1)} : q_{i,j}^{(2)}]\}_{0 \le i \le k, 1 \le j \le d}$, then a degree d morphism f can be expressed with a homogeneous polynomial form $[a_0 \cdot \prod_{j=1}^d (q_{0,j}^{(2)} z - q_{0,j}^{(1)} w) : \dots : a_k \cdot \prod_{j=1}^d (q_{k,j}^{(2)} z - q_{k,j}^{(1)} w)]$, $a_i \neq 0$, where a domain curve is irreducible. We know there is a universal closed locus B in $\overline{M}_{n+d(k+1)}$ through which every morphism from the base scheme S of a $\overline{h}$-stable family factors. But marked points information doesn't contain enough data to recover an actual morphism. To recover an exact morphism f, we need to record the ratios a_i/a_0, $i = 1, \dots, k$. That could be done by constructing a k-dimensional $\mathbb{C}^*$-bundle on a universal closed locus B. Roughly, an associated morphism f with a point $(\alpha_1, \dots, \alpha_k)$ in a fibre

$[H^0(\mathbb{CP}^1, \mathcal{H}_1 \otimes \mathcal{H}_0^{-1}) \setminus 0] \times \ldots \times [H^0(\mathbb{CP}^1, \mathcal{H}_k \otimes \mathcal{H}_0^{-1}) \setminus 0]$ on a geometric point representing an irreducible curve $(\mathbb{CP}^1, \{p_i\}_{1\le i\le n}, \{[q_{i,j}^{(1)}; q_{i,j}^{(2)}]\}_{0\le i\le k, 1\le j\le d})$ can be thought of as

$$[\prod_{j=1}^{d}(z - \frac{q_{0,j}^{(1)}}{q_{0,j}^{(2)}}w); \alpha_1 \cdot \prod_{j=1}^{d}(z - \frac{q_{1,j}^{(1)}}{q_{1,j}^{(2)}}w); \ldots ; \alpha_k \cdot \prod_{j=1}^{d}(z - \frac{q_{k,j}^{(1)}}{q_{k,j}^{(2)}}w)] \tag{2}$$

, where $z - (q_{i,j}^{(1)}/q_{i,j}^{(2)})w = w$ if $q_{i,j}^{(1)}/q_{i,j}^{(2)} = q_{i,j}^{(1)}/0$.

Giving an anti-holomorphic involution on the quotient of the moduli space $\overline{M}_n(\mathbb{CP}^k, d, \overline{h})/G$ is just sending $(\mathbb{CP}^1, \{\alpha_i\}_{i=1,\ldots,n}, [p_0(z;w); \ldots ; p_k(z;w)])$ to $(\mathbb{CP}^1, \{\overline{\alpha}_i\}_{i=1,\ldots,n}, [\overline{p}_0(z;w); \ldots ; \overline{p}_k(z;w)])$, where G is a product of symmetric groups (see the proof of Theorem 3.1), $\overline{\alpha}_i$ denotes a complex conjugate point of α_i, and $\overline{p}_i(z;w)$ denotes a homogeneous polynomial whose coefficients are the complex conjugates of those of $p_i(z;w)$. Note that although the polynomial expression depends on a chosen hyperplane basis $\overline{h}$, the anti-holomorphic involution defined as above isn't dependent on the choice of a real ordered basis $\overline{h}$ because they are related by the $PGL(\mathbb{R}, k+1)$ action which commutes with the anti-holomorphic involution on $\mathbb{CP}^k$. Thus, we may think of the homogeneous polynomials' image by an anti-holomorphic involution as their conjugate polynomials regardless of which chosen ordered real hyperplane basis makes that polynomial expression. The same way of thinking works when we consider reducible curve cases by gluing operation and restricting our polynomial expressions to each irreducible component. Since the quotients of moduli spaces of various $\overline{h}$-rigid stable families with real hyperplane basis $\overline{h}$ cover the moduli space $\overline{M}_n(\mathbb{CP}^k, d)$, we may think of a global anti-holomorphic involution on $\overline{M}_n(\mathbb{CP}^k, d)$ in the same way.

Definition. The i-th evaluation map ev_i is a morphism from $\overline{M}_n(\mathbb{CP}^k, d)$ to $\mathbb{CP}^k$, sending $(\mathcal{C}_b, p_1, \ldots, p_n, f)$ to $f(p_i)$.

It is easy to see that an evaluation map commutes with an anti-holomorphic involution. The definitions and properties of forgetting maps in Corollary 4.1 can be found in [C-K][7.1.1, 10.1.1].

Corollary 4.1 *(i) The evaluation map is a real morphism.*

(ii) The forgetting morphism from $\overline{M}_n(\mathbb{CP}^k, d)$ to $\overline{M}_n$ is a real morphism.

(iii) The forgetting morphism from $\overline{M}_n(\mathbb{CP}^k, d)$ to $\overline{M}_{n-1}(\mathbb{CP}^k, d)$ is a real morphism.

proof The proof is immediate from the above explanation and the way we gave the anti-holomorphic involution on $\overline{M}_n(\mathbb{CP}^k, d, \overline{h})$ in Lemma3.5 because that involution commutes with the product of the symmetric group actions and gluing maps, i.e. we can globalize the anti-holomorphic involution to the whole moduli space. □

Corollary 4.1 implies that we have corresponding real model maps for the evaluation map from $\overline{M}_n(\mathbb{CP}^k, d)^{\mathbb{R}}$ to $\mathbb{RP}^k$ and forgetting maps from $\overline{M}_n(\mathbb{CP}^k, d)^{\mathbb{R}}$ to $\overline{M}_n^{\mathbb{R}}$, from $\overline{M}_n(\mathbb{CP}^k, d)^{\mathbb{R}}$ to $\overline{M}_{n-1}(\mathbb{CP}^k, d)^{\mathbb{R}}$. Note that real points $\overline{M}_n(\mathbb{CP}^k, d)^{re}$ go to real points $\overline{M}_n^{re}$.

Lemma 4.1 *Every point in the real part of $M_n(\mathbb{CP}^k, d)$, before a compactification, represents a real degree d map with real marked points if $n \geq 3$.*

proof Let $(\mathbb{CP}^1, p_1, \ldots, p_n, f)$ be a real point. Then, there is a linear fractional transformation T such that $T(p_i) = \overline{p}_i$ and $\overline{f} \circ T = f$, where $\overline{f} : (\mathbb{CP}^1, \overline{p}_1, \ldots, \overline{p}_n) \rightarrow \mathbb{CP}^k$ is a conjugate map of f. As we have seen in Example 2.1, 3, real points of the Deligne-Mumford moduli space are represented by real pointed curves. By Corollary 4.1 (ii), we see the domain curve $(\mathbb{CP}^1, p_1, \ldots, p_n)$ of the map f representing a real point is equivalent to a real pointed curve $(\mathbb{CP}^1, r_1, \ldots, r_n)$ by a linear fractional transformation R such that $R(r_i) = p_i$. Note that the conjugation map of a composition map $f \circ R$ is that map itself because of the number of marked points($n \geq 3$) in a domain curve. That implies f is a real map. □

Lemma 4.1 can be generalized to any n.

Proposition 4.1 *Every point in the real part of $M_n(\mathbb{CP}^k, d)$ represents a real degree d map with real marked points on the domain curve for any n.*

proof The domain curve's marked points condition comes from Corollary 4.1 (iii). The map's condition follows from Lemma 4.1 because a forgetting morphism from $\overline{M}_n(\mathbb{CP}^k, d)$ to $\overline{M}_{n-1}(\mathbb{CP}^k, d)$ is a submersion. □

Corollary 4.2 *If X is a real projective variety, having a compatible real structure to the complex conjugation map on $\mathbb{CP}^k$, then every point in the real part $M_n(X, \beta)^{re}$ represents a real map with real marked points on the domain curve for any n.*

proof It is obvious. □

In contrast to the stable map with a nonsingular domain curve case, the image of the stable map with a singular domain curve by a forgetting map to $\overline{M}_n$ is not necessarily equivalent to a domain curve of the map because of contractions due to a stability condition.

Typical type's degenerations of domain curves on the real points are:

1. Singular curve with real marked, real gluing points

2. Singular curve with or without components described in 1 and added conjugate pairs of irreducible components without marked points

3. For $n = 0$: Singular curve with two irreducible components having a real gluing point such that the gluing point is the unique point in the real part

i.e., Singular curve we get by squeezing the equator of the sphere

Note that a forgetting map to $\overline{M}_n$ sends all domain curves of type 1 or 2 to the real points of $\overline{M}_n$ by a contraction.

A stable map in the real part $\overline{M}_n(\mathbb{CP}^1, 2)^{re}$ with a singular domain curve having 3 irreducible components $\mathbb{CP}^1_A$, $\mathbb{CP}^1_{A'}$, $\mathbb{CP}^1_B$, P.Deligne [Del] constructed is:

A component $\mathbb{CP}^1_B$ has marked points at 0, 1, ∞

A point 0 in $\mathbb{CP}^1_A$ is glued to i in $\mathbb{CP}^1_B$

A point 0 in $\mathbb{CP}^1_{A'}$ is glued to $-i$ in $\mathbb{CP}^1_B$

An anti-holomorphic involution on a domain curve is given by:

$\mathbb{CP}^1_B \to \mathbb{CP}^1_B,\ z \mapsto \overline{z}$

$\mathbb{CP}^1_A \to \mathbb{CP}^1_{A'},\ z \mapsto \overline{z}$

$\mathbb{CP}^1_{A'} \to \mathbb{CP}^1_A,\ z \mapsto \overline{z}$

A stable map is defined by:

Identity maps on $\mathbb{CP}^1_A$, $\mathbb{CP}^1_{A'}$

A zero map on $\mathbb{CP}^1_B$

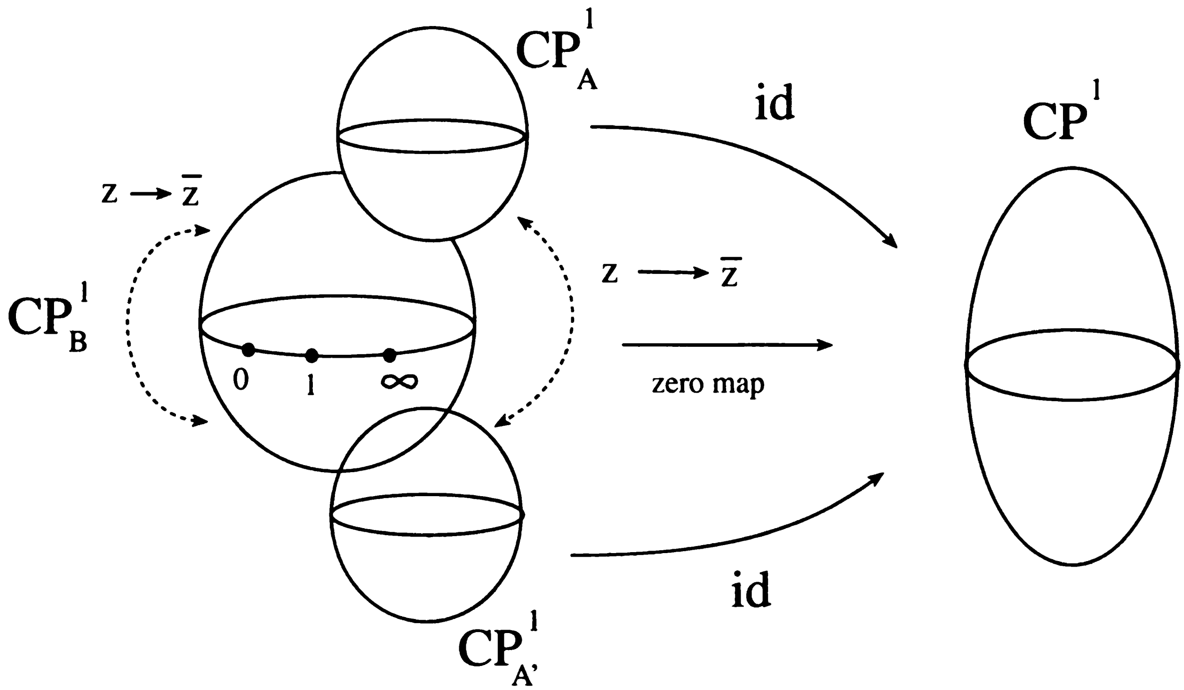

Figure 1:

A *real* $\overline{h}$-stable family of maps is a $\overline{h}$-stable family of maps which comes from a complexification of an $\mathbb{R}$-scheme stable family of maps, i.e. $\omega_{\mathcal{C}^{\mathbb{R}}}(p_1^{\mathbb{R}} + \ldots + p_n^{\mathbb{R}}) \otimes \mu^{\mathbb{R}*}(\mathcal{O}_{\mathbb{RP}^k}(3))$ is ample on $\mathcal{C}^{\mathbb{R}}$, satisfying similar conditions in the $\overline{h}$-stable family of maps. The real part of this family of maps consists of stable degree d real maps having marked points on the real parts of the domain curves.

Definition.

Let $\mathbb{CP}^k = \mathbb{P}(V)$, where $V^* = H^0(\mathbb{CP}^k, \mathcal{O}_{\mathbb{CP}^k}(1))$.

Let $\overline{h} = (h_0, ..., h_k)$ be an *ordered real basis* of V^*.

A real $\overline{h}$-stable family of degree d maps from n-pointed, genus 0 curves to $\mathbb{CP}^k$ consists of the data:

$(\pi : \mathcal{C}^{\mathbb{R}} \times_{\mathbb{R}} \mathbb{C} \to \mathcal{S}^{\mathbb{R}} \times_{\mathbb{R}} \mathbb{C}, \{p_i\}_{i=1,...,n}, \{q_{i,j}\}_{i=0,..,k,j=1,...d}, \mu)$, where

(i) $(\pi^{\mathbb{R}} : \mathcal{C}^{\mathbb{R}} \to \mathcal{S}^{\mathbb{R}}, \{p_i^{\mathbb{R}}\}, \mu^{\mathbb{R}})$ is an $\mathbb{R}$-scheme stable family of degree d maps from n-pointed, $\mathbb{R}$-scheme genus 0 curves to $\mathbb{RP}^k$ and π comes from a complexification of an $\mathbb{R}$-scheme map $\pi^{\mathbb{R}}$.

(ii) $(\pi : \mathcal{C}^{\mathbb{R}} \times_{\mathbb{R}} \mathbb{C} \to \mathcal{S}^{\mathbb{R}} \times_{\mathbb{R}} \mathbb{C}, \{p_i\}_{i=1,...,n}, \{q_{i,j}\}_{i=0,..,k,j=1,...d})$ is a flat, projective family of $n + d(k + 1)$-pointed, genus 0, Deligne-Mumford stable curves with sections $\{p_i\}$ and $\{q_{i,j}\}$

(iii) For $0 \leq i \leq k$, there is an equality of Cartier divisors, $\mu^*(t_i) = q_{i,1} + q_{i,2} + ... + q_{i,d}$

Remark 4.1 (a) By base changes, $(\pi : \mathcal{C}^{\mathbb{R}} \times_{\mathbb{R}} \mathbb{C} \to \mathcal{S}^{\mathbb{R}} \times_{\mathbb{R}} \mathbb{C}, \{p_i\}_{i=1,...,n}, \mu)$ is a stable family of degree d maps to $\mathbb{CP}^k$.

(b) Along real points in $\mathcal{S}^{\mathbb{R}} \times_{\mathbb{R}} \mathbb{C}$, $\{q_{i,j}\}$ consists of reals and complex conjugate pairs because each fibre along this locus comes from complexifications of $\mathbb{R}$-scheme maps. More precisely, let $f : \mathbb{CP}^1 \to \mathbb{CP}^k$ be a real degree d map. Then, f can be represented by real degree d homogeneous polynomials with standard hyperplane basis of $\mathbb{CP}^1$, $\mathbb{CP}^k$. Since a chosen basis in the definition of a real $\overline{h}$-stable family of maps is real, that basis is related to a standard basis by the $PGL(\mathbb{R}, k + 1)$ action. And we get another real polynomial representation which splits into linear factors. It is obvious that solutions of each homogeneous polynomial consist of reals and complex conjugate pairs.

(c) Note that the restriction of an anti-holomorphic involution on $\mathcal{C}^{\mathbb{R}} \times_{\mathbb{R}} \mathbb{C}$ along real points $\mathcal{S}^{re}$ is complex conjugation maps on each fibre fixing the first n marked points. (b) in this remark says the complex line bundles $\mathcal{H}_i$ defined by Cartier divisors $\mu^*(h_i)$, $i = 0, \ldots, k$ are real line bundles on each geometric fibre along real points $\mathcal{S}^{re}$ of the base scheme because Cartier divisors $\mu^*(h_i)$, $i = 0, \ldots, k$ are fixed by an anti-holomorphic involution. That means those line bundles come from the complexifications of the line bundles $\mu^{\mathbb{R}*}(h_i^{\mathbb{R}})$, $i = 0, \ldots, k$ on each fibre. See remark 2.2.

(d) Along the real points $\mathcal{S}^{re}$, an associated morphism $\mu^{\mathbb{R}}$ with the last $d(k+1)$ marked points $\{[q_{i,j}^{(1)}; q_{i,j}^{(2)}]\}_{0 \le i \le k, 1 \le j \le d}$ can be related by real homogeneous polynomials

$$[\prod_{j=1}^{d}(z - \frac{q_{0,j}^{(1)}}{q_{0,j}^{(2)}}w); \alpha_1 \cdot \prod_{j=1}^{d}(z - \frac{q_{1,j}^{(1)}}{q_{1,j}^{(2)}}w); \ldots ; \alpha_k \cdot \prod_{j=1}^{d}(z - \frac{q_{k,j}^{(1)}}{q_{k,j}^{(2)}}w)]$$

, where $z-(q_{i,j}^{(1)}/q_{i,j}^{(2)})w = w$ if $q_{i,j}^{(1)}/q_{i,j}^{(2)} = q_{i,j}^{(1)}/0$. The data $(\alpha_1, \ldots, \alpha_k)$ can be recorded by constructing a k-dimensional $\mathbb{R}^*$-bundle induced from $\mu^{\mathbb{R}*}(h_i^{\mathbb{R}})$, $i = 0, \ldots, k$.

(e) As we have seen, the topology of $\overline{M}_n(\mathbb{CP}^k, d)$ is related to the closed sublocus B in $\overline{M}_{n+d(k+1)}$. But the topology of real points $\overline{M}_n(\mathbb{CP}^k, d)^{re}$ of $\overline{M}_n(\mathbb{CP}^k, d)$ doesn't come from real points of B. It comes from an extended sense's real locus in B on which we can construct real line bundles to record the additional data $(\alpha_1, \ldots, \alpha_k)$ in (d). The reason is explained in (b), (c), (d) in this remark.

(f) Note that the first n-marked points are on the real points $\mathcal{C}^{re}$ of the domain curves along the geometric fibres of real points $\mathcal{S}^{re}$. But the last $d(k+1)$ marked points are not necessarily on $\mathcal{C}^{re}$.

Definition. *The derived real $\overline{h}$-stable family of degree d maps for a real $\overline{h}$-stable family of degree d maps* from n-pointed, genus 0 curves to $\mathbb{CP}^k$
$(\pi : \mathcal{C}^{\mathbb{R}} \times_{\mathbb{R}} \mathbb{C} \to \mathcal{S}^{\mathbb{R}} \times_{\mathbb{R}} \mathbb{C}, \{p_i\}_{i=1,\ldots,n}, \{q_{i,j}\}_{i=0,\ldots,k,j=1,\ldots,d}, \mu)$ is
$(\pi_{\mathbb{R}} : \mathcal{C}_{\mathbb{R}} \to \mathcal{S}^{re}, \{p_i\}_{i=1,\ldots,n}, \{q_{i,j}\}_{i=0,\ldots,k,j=1,\ldots,d}, \mu_{\mathbb{R}})$, where $\mathcal{C}_{\mathbb{R}} = \pi^{-1}(\mathcal{S}^{re})$, $\mu_{\mathbb{R}} : \mathcal{C}_{\mathbb{R}} \to \mathbb{CP}^k$.

Remark 4.2 The construction of real points tends to be geometric because we have to use an extended sense's real locus B^{er} in B. The following picture helps to understand the construction in proposition 4.2.

$$
\begin{array}{ccccccc}
\mathcal{H}_i \otimes \mathcal{H}_0^{-1} & & & & & & \mathcal{H}_i^{\mathbb{R}} \otimes \mathcal{H}_0^{\mathbb{R}-1} \\
& \searrow & & & & \swarrow & \\
& & \mathbb{CP}^k & \to & \mathbb{RP}^k & & \\
\pi_*(\mathcal{H}_i \otimes \mathcal{H}_0^{-1}) & \searrow & \downarrow \pi & & \downarrow \pi' & \swarrow & \pi'_*(\mathcal{H}_i^{\mathbb{R}} \otimes \mathcal{H}_0^{\mathbb{R}-1}) \\
& & spec\mathbb{C} & \to & spec\mathbb{R} & &
\end{array}
$$

Note that $\pi_*(\mathcal{H}_i \otimes \mathcal{H}_0^{-1}) \cong \pi'_*(\mathcal{H}_i^{\mathbb{R}} \otimes \mathcal{H}_0^{\mathbb{R}-1}) \otimes_{\mathbb{R}} \mathbb{C}$.

Proposition 4.2 *There is a universal real sublocus* $\overline{M}_n(\mathbb{CP}^k, d, \overline{h})^{er}$ *for the derived real* $\overline{h}$*-stable family of degree* d *maps in* $\overline{M}_n(\mathbb{CP}^k, d, \overline{h})$. *The real part* $\overline{M}_n(\mathbb{CP}^k, d)^{re}$ *of the moduli space of stable maps* $\overline{M}_n(\mathbb{CP}^k, d)$ *is obtained by gluings of the quotient by the product of a symmetric group action* G. *See section 3.2 [II] for the definition of* G.

proof Let $\overline{h}$ be given. The universal closed sublocus B is a real projective variety. The quotient variety B/G has an antiholomorphic involution induced from that of B. Then, the points in the real part $(B/G)^{re}$ represent pointed curves with n first real marked points and $d(k+1)$ last reals and complex conjugate marked points and singular curves described right after Proposition 4.1.

$$
\begin{array}{lcl}
B\overline{U}_{n+d(k+1)} & & \\
\downarrow \pi_B & & \\
B & \xrightarrow{pr} & B/G
\end{array}
$$

Let's denote $\pi_B^{-1} \circ pr^{-1}((B/G)^{re})$ by $B\overline{U}_{\mathbb{R}n+d(k+1)}$. When we restrict our attention to the locus $B\overline{U}_{\mathbb{R}n+d(k+1)}$, $\mathcal{H}_i$ has a natural fibrewise antiholomorphic automorphism. So does $\mathcal{H}_i \otimes \mathcal{H}_0^{-1}$. This fibrewise antiholomorphic automorphism induces a fibrewise antiholomorphic automorphism when we consider the direct image sheaf $\pi_{B*}(\mathcal{H}_i \otimes \mathcal{H}_0^{-1})$. See remark 4.2. This allows us to construct real line bundles and then, the desired k-dimensional $\mathbb{R}^*$-bundle $\overline{M}_n(\mathbb{CP}^k, d, \overline{h})^{er}$ over B^{er}.

To see the constructed space $\overline{M}_n(\mathbb{CP}^k, d, \overline{h})^{er}$ is a universal locus for the derived real $\overline{h}$-stable family of maps, we observe that the natural morphism from $\mathcal{S}^{\mathbb{R}} \times_{\mathbb{R}} \mathbb{C}$ to B sends the real points of a real $\overline{h}$-stable family to B^{er} due to the types of the last $d(k+1)$ marked points. Then, the additional information canonically corresponds to a point in the k-dimensional $\mathbb{R}^*$-bundle as described in remark 4.1. This correspondence is consistent with what Fulton-Pandharipande did in [F-P]. We will see a concrete example right after this proof. This locus is preserved by a symmetric group G action. And the moduli space we get by gluing the quotient spaces $\overline{M}_n(\mathbb{CP}^k, d, \overline{h})^{er}/G$ for various $\overline{h}$ is the real points $\overline{M}_n(\mathbb{CP}^k, d)^{re}$ of the moduli space of stable maps of genus zero. $\square$

Example 4.1 We see the concrete case.

$$\begin{array}{ccccccccc}
\mathbb{CP}^2 & \overset{f}{\leftarrow} & (\mathcal{C}_{\mathbb{R}} & \supset) & (\mathbb{CP}^1_a, \{q'_{i,j}\}) & \overset{\varphi}{\rightarrow} & (\mathbb{CP}^1_b, \{q_{i,j}\}) & \subset & B\overline{U}_{\mathbb{R}n+d(k+1)} \\
 & & \downarrow & & \pi' \downarrow & & \pi \downarrow & & \downarrow \\
 & & \mathcal{S}^{re} & \ni & s & \rightarrow & c & \in & B^{er}
\end{array}$$

Let $(\mathbb{CP}^1_a, f)$ be a *real* degree 2 map to $\mathbb{CP}^2$, where

$f([z;w]) = [f_0([z;w]); \alpha_1 f_1([z;w]); \alpha_2 f_2([z;w])] = [z^2 - w^2; \alpha_1(z^2 + w^2); \alpha_2(2z^2 + w^2)]$, where $\alpha_i \in \mathbb{R} \setminus 0$. Assume marked points from the hyperplane intersection points $\{q'_{i,j}\}$ in $\mathbb{CP}^1_a$, i.e. zeros of f_i, are given by

$\{\{[1;1],[1;-1]\}, \{[i;-1],[i;1]\}, \{[i;-\sqrt{2}],[i;\sqrt{2}]\}\}$. And assume $\mathbb{CP}^1_b$ has marked points $\{\{[i;1],[i;-1]\}, \{[-1;-1],[-1;1]\}, \{[-1;-\sqrt{2}],[-1;\sqrt{2}]\}\}$.

(i) Then, we see $(\mathbb{CP}^1_b, \{q_{i,j}\})$ and $(\mathbb{CP}^1_a, \{q'_{i,j}\})$ are equivalent by $\varphi([z;w]) = [iz;w]$, where $\varphi; \mathbb{CP}^1_a \rightarrow \mathbb{CP}^1_b$.

(ii) Note that there is a bundle isomorphism between $\mathcal{H}'_i \otimes \mathcal{H}_0'^{-1}$ and $\mathcal{H}_i \otimes \mathcal{H}_0^{-1}$, sending a generator g_0/g_i to a generator $g_0 \circ \varphi / g_i \circ \varphi = f'_0/f'_i$, where $\mathcal{H}'_i$, $\mathcal{H}_i$ come from effective Weil divisors $q'_{i,1} + q'_{i,2}$, $q_{i,1} + q_{i,2}$ and $[g_0([z;w]); g_1([z;w]); g_2([z;w])]$
$= [-z^2 - w^2; -z^2 + w^2; -2z^2 + w^2]$.

That induces an isomorphism

between $\pi'_*(\mathcal{H}'_i \otimes \mathcal{H}_0'^{-1}) \setminus 0 = H^0(\mathbb{CP}^1_a, \mathcal{H}'_i \otimes \mathcal{H}_0'^{-1}) \setminus 0$ and $\pi_*(\mathcal{H}_i \otimes \mathcal{H}_0^{-1}) \setminus 0 = H^0(\mathbb{CP}^1_b, \mathcal{H}_i \otimes \mathcal{H}_0^{-1}) \setminus 0$. Both are isomorphic to $\mathbb{C}^*$ and we may consider the numbers α_i (of course, degree 0 polynomial) as elements of $\pi'_*(\mathcal{H}'_i \otimes \mathcal{H}_0'^{-1}) \setminus 0$ and $\pi_*(\mathcal{H}_i \otimes \mathcal{H}_0^{-1}) \setminus 0$, which canonically correspond to each other by an induced isomorphism. The reason is $\alpha_i \cdot g_i \circ \varphi = \alpha_i \cdot f'_i$, where $\alpha_0 = 1$. It is easy to see that $\alpha_i \in \mathbb{R} \setminus 0$.

The way we extend what we observed with geometric points to the morphism from $\mathcal{S}^{re}$ to $\overline{M}_n(\mathbb{CP}^k, d, \overline{h})^{er}$ is similar to what we saw in section 3.2 [II].

Remark 4.3 1. The general construction for the real part $\overline{M}_n(X, \beta)^{re}$ comes from the modification of sec.5 in [F-P].
2. The general procedure to decide the number of connected components of the real points $\overline{M}_n(\mathbb{CP}^k, d)^{re}$ is not yet well understood. Note that the number of connected components in the Deligne-Mumford moduli space $\overline{M}_n$ is $\frac{(n-1)!}{2}$ which is from half of the possible cyclic orderings of marked points, before a compactification. But it is connected after a compactification.
3. Orientability of $\overline{M}_n(\mathbb{CP}^k, d)^{re}$ is also not yet clear. But we may expect most of the cases are non-orientible even for the degree 0 cases because the Deligne-Mumford moduli space $\overline{M}_n$, $n \geq 5$ is non-orientible. Note that $\overline{M}_5$ is from blowing up four points of $\mathbb{CP}^2$ whose corresponding real model is non-orientible.

4.2 Projectivity of the real model $\overline{M}_n(X, \beta)^{\mathbb{R}}$

Fulton-Pandharipande have shown the moduli space $\overline{M}_n(\mathbb{CP}^k, d)$ is projective in [F-P]. They used Kollar's semipositivity approach in [Kol] to apply the Nakai-Moishezon criterion to a certain power of a determinant line bundle $Det(Q)$. Ampleness of $Det(Q)^p$ implies the projectivity of $\overline{M}_n(\mathbb{CP}^k, d)$. We summarize the definition of a vector bundle Q on $\overline{M}_n(\mathbb{CP}^k, d)$ in [F-P]. The moduli space $\overline{M}_n(\mathbb{CP}^k, d, \overline{h})$ is a fine moduli space equipped with a universal family $(\pi; \mathcal{U} \to \overline{M}_n(\mathbb{CP}^k, d, \overline{h}), \{p_i\}, \mu)$. Let $\mathbb{P}(L_l^*)$ be a projective bundle coming from the projectivization of fibres of L_l^*, where $L_l^* = \pi_*(\omega_\pi^l(\sum_{i=1}^n p_i) \otimes \mu^*(\mathcal{O}(3l)))$. We can decide the power l which allows a $\overline{M}_n(\mathbb{CP}^k, d, \overline{h})$-canonical embedding $e : \mathcal{U} \to \mathbb{P}(L_l^*)$ by using Riemann-Roch Theorem and Lemma3.3. The morphism μ induces a $\overline{M}_n(\mathbb{CP}^k, d, \overline{h})$-canonical embedding $\varepsilon : \mathcal{U} \to \mathbb{P}(L_l^*) \times \mathbb{CP}^k$ and the n sections $\{p_i\}$ define n sections $\{(e \circ p_i, \mu \circ p_i)\}$ of $\mathbb{P}(L_l^*) \times \mathbb{CP}^k$ over $\overline{M}_n(\mathbb{CP}^k, d, \overline{h})$. Let π' denote a natural projection map from $\mathbb{P}(L_l^*) \times \mathbb{CP}^k$ to $\overline{M}_n(\mathbb{CP}^k, d, \overline{h})$ and P_i a subscheme defined by the i-th section, $\mathcal{U}'$ an embedded $\mathcal{U}$ by ε. The sum of direct image sheaves $\pi'_*(\mathcal{L}^m \otimes \mathcal{O}_{\mathcal{U}'}) \oplus \oplus_{i=1}^n \pi'_*(\mathcal{L}^m \otimes \mathcal{O}_{P_i})$ of the line bundles $\mathcal{L}^m$ along $\mathcal{U}'$ and P_i becomes a vector bundle for sufficiently large m by vanishing of higher direct images. That is the definition of the vector bundle Q. We consider the determinant line bundle $det(Q)$ on $\overline{M}_n(\mathbb{CP}^k, d, \overline{h})$. $\overline{M}_n(\mathbb{CP}^k, d)$ is locally a quotient of $\overline{M}_n(\mathbb{CP}^k, d, \overline{h})$. The induced line bundle $Det(Q)$ is a well-defined line bundle except at the singular points. But we get a well-defined line bundle $Det(Q)^p$ by raising the power of p for p large enough.

In Lemma 3.5, we showed that the moduli space $\overline{M}_n(\mathbb{CP}^k, d, \overline{h})$ is a real fine moduli space equipped with a real universal family $(\pi : \mathcal{U} \to \overline{M}_n(\mathbb{CP}^k, d, \overline{h}), \{p_i\}, \mu)$. As we explained in section 2, that implies that there exists a corresponding real model map $(\pi^{\mathbb{R}} : \mathcal{U}^{\mathbb{R}} \to \overline{M}_n(\mathbb{CP}^k, d, \overline{h})^{\mathbb{R}}, \{p_i^{\mathbb{R}}\}, \mu^{\mathbb{R}})$. Fulton-Pandharipande's construction to show projectivity of $\overline{M}_n(\mathbb{CP}^k, d)$ works in this setting and it shows the following Proposition.

Proposition 4.3 *The real model $\overline{M}_n(\mathbb{CP}^k, d)^{\mathbb{R}}$ of $\overline{M}_n(\mathbb{CP}^k, d)$ is real projective.*

Corollary 4.3 *The real model $\overline{M}_n(X, \beta)^{\mathbb{R}}$ of $\overline{M}_n(X, \beta)$ is real projective, where X is a nonsingular real projective variety having real structure compatible with the complex conjugation map on $\mathbb{CP}^k$.*

proof. It is obvious because X is a real projective variety. $\square$

Remark 4.4 1. The real dimension of the real model $\overline{M}_n(X,\beta)^{\mathbb{R}}$ has a pure dimension equal to the complex dimension of $\overline{M}_n(X,\beta)$. 2 in Remark 3.3 implies that $\overline{M}_n(X,\beta)^{\mathbb{R}}$ carries a fundamental cycle in Chow group.

2. When we pick a triangulation on the real part $\overline{M}_n(X,\beta)^{re}$, the sum of n-dimensional simplexes is a cycle modulo 2 of $\overline{M}_n(X,\beta)$. Therefore, the real part $\overline{M}_n(X,\beta)^{re}$ carries a fundamental cycle with $\mathbb{Z}/2\mathbb{Z}$ - module version's ordinary homology.

3. Fukaya-Oh-Ohta-Ono's moduli space FM_n of pointed disks consists of all isomorphism classes of pointed stable disks with n marked points on the boundary, where $(\Sigma, z_1, \ldots, z_n)$ and $(\Sigma', z_1', \ldots, z_n')$ are isomorphic if there exists an orientation preserving diffeomorphism $\tau : \Sigma \to \Sigma'$ such that $\tau(z_i) = z_i'$. 'Stable' means each irreducible component has at least three special points, i.e., marked or gluing points. Geometrically, FM_n has $(n-1)!$ contractible diffeomorphic orientable components. After the compactification, its number of connected components doesn't change and each component is diffeomorphic to an $n-3$ dimensional disk with boundary. Thus, $\overline{FM_n}$ doesn't have a $\mathbb{Z}$-module version's fundamental cycle. It is interesting to see the differences of geometric properties of a real part of a Deligne-Mumford moduli space before and after the compactification. Before the compactification, a real part of Deligne-Mumford moduli space M_n consists of $\frac{(n-1)!}{2}$ contractible diffeomorphic orientable components and FM_n is a double cover of M_n. But after the compactification, a real part of Deligne-Mumford moduli space $\overline{M}_n$, $n > 4$, (resp. $\overline{M}_3$, $\overline{M}_4$) becomes a non-orientable(resp. orientable) smooth connected manifold, having a $\mathbb{Z}/2\mathbb{Z}$-module version's fundamental cycle. This big difference in geometric properties after a compactification comes from differences in equivalence relations, i.e. whether it preserves orientations or not. We see more equivalence relation tends to make more convergence property and so make the moduli space have a fundamental cycle. An intuitive example for this is when the number of marked points is 4. We may think of the

real part of M_4 as a circle with 3 points removed, FM_4 as two circles with 3 points removed from each. But after the compactification $\overline{M}_4$ becomes diffeomorphic to a circle and $\overline{FM}_4$ to 6 disjoint closed intervals. We observe that $\overline{FM}_4$ is a generically double cover of $\overline{M}_4$, but at the singular divisor, it becomes a 4-uple cover. Generally, the number of inverse images at the compactification divisors are dependent on the number of connected components and the number of marked, gluing points on each component. See [F-Oh, sec.10], [F-Oh-Ohta-Ono] for more detailed descriptions about Fukaya-Oh-Ohta-Ono's moduli space.

5 The Gromov-Witten invariant and real enumerative problems

As we have shown in the previous sections, the moduli space of stable maps $\overline{M}_n(X,\beta)$ is a real moduli space if X is a convex real projective variety, having a real structure corresponding to the complex conjugate involution on $\mathbb{CP}^m$. The analysis of the real part of $\overline{M}_n(X,\beta)$ and the existence of the fundamental cycle as described in Remark 4.4,1,2 allows us to consider whether there is any way to define a real enumerative invariant on the real part $\overline{M}_n(X,\beta)^{re}$ or on the corresponding real model $\overline{M}_n(X,\beta)^{\mathbb{R}}$ by using homology and cohomology, or Chow group and Chow ring bilinear pairing. In this section, we assume that the variety X is a homogeneous variety. More detailed properties about the homogeneous variety can be found in [F-P, sec.0.2, sec.7]. Since in most cases, the real part $\overline{M}_n(X,\beta)^{re}$ is non-orientable, it is natural to consider working with a $\mathbb{Z}/2\mathbb{Z}$-module ordinary (co)homology. The invariant on $\overline{M}_n(X,\beta)^{re}$ can be defined by using the real part maps ev_i^{re} of evaluation maps,
i.e., $<[\overline{M}_n(X,\beta)^{re}], ev_1^{re*}(\xi_1)\cup\ldots\cup ev_n^{re*}(\xi_n)>$, where $<,>$ is the bilinear pairing

$$<,>: H_l(\overline{M}_n(X,\beta)^{re},\mathbb{Z}/2\mathbb{Z})\times H^l(\overline{M}_n(X,\beta)^{re},\mathbb{Z}/2\mathbb{Z})\longrightarrow\mathbb{Z}/2\mathbb{Z},$$

the dimension of $\overline{M}_n(X,\beta)^{re}$ is l and the real part map of the evaluation map is $ev_i^{re} : \overline{M}_n(X,\beta)^{re} \rightarrow X^{re}$. Note that the Poincarè duality doesn't hold in this case because $\overline{M}_n(X,\beta)^{re}$ is an orbifold.

We can define an invariant on the real model $\overline{M}_n(X,\beta)^{\mathbb{R}}$ by using the real model map $ev_i^{\mathbb{R}}$ of the evaluation map. We use $\mathbb{Z}$-module Chow group and Chow ring's bilinear pairing $< [\overline{M}_n(X,\beta)^{\mathbb{R}}], ev_1^{\mathbb{R}*}(\xi_1^{\mathbb{R}}) \dots ev_n^{\mathbb{R}*}(\xi_n^{\mathbb{R}}) >$, where

$$<,>: A_l(\overline{M}_n(X,\beta)^{\mathbb{R}}) \times A^l(\overline{M}_n(X,\beta)^{\mathbb{R}}) \longrightarrow \mathbb{Z},$$

l is the dimension of $\overline{M}_n(X,\beta)^{\mathbb{R}}$ and the real model map of the evaluation map is $ev_i^{\mathbb{R}} : \overline{M}_n(X,\beta)^{\mathbb{R}} \rightarrow X^{\mathbb{R}}$. This time, the invariant is equal to the usual Gromov-Witten invariant $< [\overline{M}_n(X,\beta)], ev_1^*(\xi_1) \dots ev_n^*(\xi_n) >$ on $\overline{M}_n(X,\beta)$ coming from the bilinear pairing of the complexifications of cycles in $\overline{M}_n(X,\beta)^{\mathbb{R}}$. So, the invariant defined in this way cannot have a significance as a real enumerative invariant. To relate the previous sections' results with the real enumerative problem, we will start with the explanation about why the Gromov-Witten invariant has an implication in enumerative problems in $\mathbb{C}$-scheme case. Readers can see more explicit details in [F-P, sec.7], [C-K, Chapter 7].

Let $\xi_1, \dots, \xi_n$ be given classes in a Chow ring $A^*(X)$ corresponding to subvarieties $\Gamma_1, \dots, \Gamma_n$ in general position in X. The Gromov-Witten invariant

$I_\beta(\xi_1, \dots, \xi_n) = \int_{\overline{M}_n(X,\beta)} ev_1^*(\xi_1) \cdot \dots \cdot ev_n^*(\xi_n) =< [\overline{M}_n(X,\beta)], ev_1^*(\xi_1) \dots ev_n^*(\xi_n) >$, where ev_i is an i-th evaluation map, can be well-defined only when $\sum \mathrm{Codim}\Gamma_i$ is the same as the dimension of the moduli space.

Roughly speaking, when it has an enumerative meaning, this invariant counts the number of pointed maps $(C, p_1, \dots, p_n; f)$ such that $f_*([C]) = \beta$ and $f(p_i) \in \Gamma_i$. That is, it counts the number of points in $ev_1^{-1}(\Gamma_1) \cap \dots \cap ev_n^{-1}(\Gamma_n)$.

Now, suppose Γ_i, $i = 1, \dots, n$, is a real subscheme in X, i.e. Γ_i comes from the complexification of $\Gamma^{\mathbb{R}}$ in $X^{\mathbb{R}}$. Then, $ev_i^{-1}(\Gamma_i)$ is a real subscheme in $\overline{M}_n(X,\beta)$. So, the real subscheme $ev_1^{-1}(\Gamma_1) \cap \dots \cap ev_n^{-1}(\Gamma_n)$ consists of points preserved by the anti-holomorphic involution on $\overline{M}_n(X, \beta)$. Counting real curves for the given enumerative problem will be related to the number of points in $ev_1^{-1}(\Gamma_1) \cap \ldots \cap ev_n^{-1}(\Gamma_n)$ in the real part of $\overline{M}_n(X, \beta)$ when cycles meet transversally. But the number of points in the real part is dependent on the choice of the actual cycle representatives because the real number field $\mathbb{R}$ is not algebraically closed. Note that those numbers are $\mathbb{Z}/2\mathbb{Z}$-module invariant because the complex number field $\mathbb{C}$ has a field extension degree 2 over $\mathbb{R}$.

Therefore, to relate the previous sections' results with real enumerative problems, studying the existence of real cycles rationally equivalent to the pull-back of real cycles, meeting transversally at real points $\overline{M}_n(X, \beta)^{re}$ becomes important. Equality in the Gromov-Witten invariant and the actual numbers of intersection points means curves whose i-th marked points go to the real part of Γ_i are all real curves, i.e. the given enumerative problem is fully real. Developing methods to construct real cycles to improve the expected number of real solutions for the enumerative problems should be the main goal of further study.

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