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**DECAY OF CORRELATIONS
FOR PIECEWISE SMOOTH MAPPINGS
WITH UNBOUNDED DISTORTION**

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Charles Howard Morgan, Jr.

A DISSERTATION

Submitted to
Michigan State University
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ABSTRACT

DECAY OF CORRELATIONS FOR PIECEWISE SMOOTH MAPPINGS WITH UNBOUNDED DISTORTION

By

Charles Howard Morgan, Jr.

In [You98], Lai-Sang Young introduces a method for showing that a large class of dynamical systems have exponential decay of correlations for Hölder continuous real-valued functions (observables). She assumes that cumulative distortion along orbits is uniformly bounded. We extend this result to include dynamical systems which have unbounded cumulative distortion along orbits.

Furthermore, Young shows that piecewise hyperbolic systems with finitely many domains of invertibility fit into her theorem. We show that Young's proof can be extended to include systems with countably many domains of invertibility, provided that the domains of invertibility decay at least exponentially quickly in measure.

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Physically observable measures and correlation functions

1.1 SRB measures

In his 1976 paper [Rue76], David Ruelle extended the work of Sinai [Sin68] by introducing a new type of measure for Axiom A attractors. This measure is invariant¹ with respect to the given map, and it maximizes a quantity related to Pesin's Formula,² and it governs the behavior of a set of orbits of positive Lebesgue measure. This measure μ is supported on the attractor Σ , and for some positive Lebesgue measure set of points in some neighborhood of Σ , we have $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i x) \rightarrow \int \varphi d\mu$ for all continuous real- or complex-valued functions φ . We say that such a measure is **physically observable** and call it a **physical measure**, and we call the functions such as φ **observables**.

¹A Borel measure μ is said to be *g-invariant* if, for every Borel set E , $\mu(g^{-1}E) = \mu(E)$.

²We discuss Pesin's Formula in the next two pages.

Throughout this thesis, we shall use M to denote a smooth compact finite-dimensional Riemannian manifold (possibly with boundary). Suppose that $f : M \rightarrow M$ is C^2 . Let $\lambda_1 > \lambda_2 > \cdots > \lambda_r$ be the distinct Lyapunov exponents of (f, μ) , and let $E_1, E_2, \dots, E_r$ be the corresponding eigenspaces, respectively. In the same paper, Ruelle also showed that for Axiom A attractors μ is the unique f -invariant measure which maximizes the quantity $h_\mu(f) - \int \sum_{i=1}^r \lambda_i \dim E_i d\mu$, where this sum is taken over all Lyapunov exponents greater than 1 and where $h_\mu(f)$ is the metric entropy of f with respect to μ and that this maximum is zero. In other words, this measure μ satisfies Pesin's Formula

$$(1.1) \quad h_\mu(f) = \int \sum_{i=1}^r \lambda_i \dim E_i d\mu.$$

In 1968, Sinai [Sin68] showed that measures satisfying (1.1) for Anosov systems had the interesting property that their conditional measures on unstable manifolds are equivalent to Riemannian measure on those manifolds. In [Rue76], Ruelle proved the same for Axiom A attractors. Later, Ledrappier, Strelcyn, and Young (both independently and collectively in [Led84], [LS82], and [LY85]) showed that, assuming (f, μ) has a positive Lyapunov exponent μ -a.e., Pesin's Formula holds if and only if the conditional measures of μ on unstable manifolds are absolutely continuous with respect to Lebesgue measure. This leads us then to a definition of SRB measure which is as follows.

Definition 1.1 *Let f , M , $\mathcal{B}$, and μ be as above. An f -invariant Borel probability measure μ is called an **SRB measure** for f if it is ergodic, if f has a positive Lyapunov exponent μ -a.e., and if the conditional measures of μ on unstable manifolds*

are absolutely continuous with respect to Lebesgue measure.

We should perhaps mention that the uniqueness of the SRB measure for Axiom A systems as discussed by Ruelle has nothing to do with Pesin's Formula. Instead, uniqueness is a consequence of the fact that Axiom A systems are mixing³ with respect to their SRB measures. It is possible for some dynamical systems to have many SRB measures, and we discuss such a system in the following paragraph.

It is in general not an easy task to decide whether a dynamical system has an SRB measure or whether it is unique when it does have one. In [Ryc83], Rychlik constructs a piecewise expanding interval map $f : [0, 1] \rightarrow [0, 1]$ with countably infinitely many domains of invertibility which has finitely many SRB measures; however, Hu and Young in [HY95] construct examples of dynamical systems which are hyperbolic everywhere except at a single point and which do not have SRB measures at all.

In the examples we consider here, all SRB measures for piecewise expanding maps are absolutely continuous;⁴ however, Rychlik also considers a map $f : [0, 1] \rightarrow [0, 1]$ for which the Dirac measure δ_0 supported at zero is an asymptotic measure. This map is defined as follows:

$$(*) \quad f(0) = 0 \quad \text{and} \quad f(x) = 2x - 2^{-j+1} \quad \text{if } x \in (2^{-j}, 2^{-j+1}] \text{ for all } j \in \mathbb{N}.$$

Rychlik's example has two fundamental difficulties which our examples do not have, and so we avoid the possibility of having only singular asymptotic measures (for one-

³A dynamical system (f, μ) is said to be *mixing* if, for every borel sets A and B we have $\lim_{n \rightarrow \infty} \mu(f^{-n}A \cap B) = \mu(A)\mu(B)$.

⁴Our dynamical systems live on Riemannian manifolds, so when we say that a measure is absolutely continuous, we mean of course that it is absolutely continuous with respect to Lebesgue measure.

dimensional maps at least). The first of these problems is that f has countably many invariant measures which are indistinguishable from δ_0 by some regularity properties (absolute continuity, for example); in fact, $\delta_{2^{-k}}$ is an f -invariant measure for every $k \in \mathbb{N} \cup \{0\}$; and the second of these problems is that f is not topologically mixing. We say that the system $f : M \rightarrow M$ is **topologically mixing** if, for any two nonempty open sets U and V in M , there is a positive integer $N = N(U, V)$ such that, for every $n > N$, $f^n(U) \cap V \neq \emptyset$. For Rychlik's map $(*)$, the point 1 is a global repeller of the system, and 0 is a global attractor; thus, we can clearly find U and V in the unit interval so that $f^n U$ and V never intersect.

1.2 Correlation functions

An important statistical property related to the SRB measure is the rate at which two random variables become increasingly more independent. Let $f : M \rightarrow M$ be a map which has a unique SRB measure μ , and let $\varphi : M \rightarrow \mathbb{C}$ and $\psi : M \rightarrow \mathbb{C}$ be random variables (observables). We say that φ and ψ are **independent** if

$$\int_M \varphi \psi d\mu = \int_M \varphi d\mu \int_M \psi d\mu.$$

We define the **auto-correlation function** of φ as

$$C_\varphi(n) = \int_M (\varphi \circ f^n) \varphi d\mu - \left(\int_M \varphi d\mu \right)^2$$

and the **correlation function** of φ and ψ as

$$C_{\varphi, \psi}(n) = \int_M (\varphi \circ f^n) \psi d\mu - \int_M \varphi d\mu \int_M \psi d\mu.$$

Thus, the correlation function $C_{\varphi,\psi}(n)$ is a measure of the independence of $\varphi \circ f^n$ and ψ ; i.e., $C_{\varphi,\psi}(n)$ measures how much influence values of ψ have on values of $\varphi \circ f^n$. Clearly, $\varphi \circ f^n$ and φ are not independent random variables, but if $C_{\varphi,\psi}(n)$ tends to zero as n tends to infinity, that tells us that values of $\varphi \circ f^n$ depend less and less on the initial values of φ .

Definition 1.2 *Let ν be an f -invariant Borel probability measure on M , and let $\mathcal{F}$ be some Banach space of observables (real-valued functions) on M . We say that (f, ν) has **exponential decay of correlations** for functions in $\mathcal{F}$ if there exists $\tau \in (0, 1)$ such that, for all $\varphi, \psi \in \mathcal{F}$, there exists $C(\varphi, \psi) > 0$ such that*

$$\left| \int (\varphi \circ f^n) \psi \, d\nu - \int \varphi \, d\nu \int \psi \, d\nu \right| < C(\varphi, \psi) \tau^n$$

for all $n \geq 0$.

Similarly, we say that (f, ν) has **polynomial decay of correlations** if there is $p > 1$ such that

$$\left| \int (\varphi \circ f^n) \psi \, d\nu - \int \varphi \, d\nu \int \psi \, d\nu \right| < C(\varphi, \psi) n^{-p}$$

for all $n \geq 0$.

In particular, we are interested in knowing the rate at which the correlation functions $C_{\varphi,\psi}(n)$ tend to zero as $n \rightarrow \infty$; i.e., when φ and ψ become increasingly more independent. One goal of our work is to find some rapid rate of decay of these correlation functions. One reason for wishing to know how quickly they decay to zero is that this rate of decay is an important factor in determining whether the Central Limit Theorem holds for some class of random variables. One rule of thumb is that, if the

sum $\sum_{n \in \mathbb{N}} C_{\varphi, \psi}(n)$ is finite for all random variables in some class, then the Central Limit Theorem holds for the subclass of those random variables which are in $L^2(\mu)$. (See [Bal01].)

It is perhaps easier to think of the rate of decay of the correlation functions as the speed at which mixing occurs. Recall that a dynamical system (f, μ) is said to be **mixing** if $\lim_{n \rightarrow \infty} \mu(f^{-n}A \cap B) = \mu(A)\mu(B)$. If the space of observables in question contains the characteristic functions of the Borel sets, then we have that

$$C_{1_A, 1_B}(n) = \mu(f^{-n}A \cap B) - \mu(A)\mu(B).$$

Therefore, it is sometimes said that a system (f, ν) is mixing if $C_{\varphi, \psi}(n) \rightarrow 0$ as $n \rightarrow \infty$ for all $\varphi, \psi \in L^2(\nu)$.

While different speeds of mixing have been considered in dynamical systems, dynamical systems with exponential decay of correlations have received the most attention, in part because many elegant systems exhibit this speed of mixing; however, the real reason they have gotten the most attention is that the two most well-understood methods for determining the rate of decay of correlations (contraction in a Hilbert metric on a lattice and spectral gap of a transfer operator) are designed precisely to determine whether or not speed of mixing is exponential. These methods do not detect slower rates of decay of correlations.

Systems with true polynomial decay of correlations (i.e., polynomial upper and lower bounds on correlation functions) are relatively new. (See, for example, [Hu].) Not much is known in general about these systems. To this author's knowledge, all dynamical systems with true polynomial decay of correlations have one of a few very

specific limited forms.

We consider here a class of dynamical systems which have exponential decay of correlations for functions which are piecewise Hölder continuous, and we do this by looking at recurrence times of a subset of the phase space. Young has recently shown in [You99] that the rate at which certain subsets of the phase space return to another given subset determines the rate of decay of correlations; therefore, a slow rate of return of these subsets causes a slow rate of mixing. It is hoped that the extension we have been able to afford here to the work in [You98] will also work essentially unchanged in [You99] so that we can add to Young's library of dynamical systems which have polynomial decay of correlations.

1.3 Invariant measures and the transfer operator

The primary tool for proving exponential decay of correlations is the Perron-Frobenius operator; in fact, its spectrum is the key point of our discussion. Several of its properties make this operator a natural object for study. Let μ be a finite Borel measure on $(M, \mathcal{B})$, where $\mathcal{B}$ is the Borel σ -algebra and M is as already described. Let $g : M \rightarrow M$ be some smooth measurable function, and let $\varphi : M \rightarrow \mathbb{R}$ (or $\mathbb{C}$) be some real- or complex-valued observable. We define the Perron-Frobenius operator $P : L^1(\mu) \rightarrow L^1(\mu)$ by

$$(1.2) \quad P\varphi(y) = \sum_{x: g(x)=y} \frac{\varphi(x)}{|\det Dg(x)|}.$$

A few basic properties of P are as follows:

$$(i) \quad \int_M (P\varphi) \psi d\mu = \int_M \varphi (\psi \circ g) d\mu \text{ for all } \varphi, \psi \in L^1(\mu);$$

$$(ii) \int_M P\varphi d\mu = \int_M \varphi d\mu;$$

(iii) if $\nu = \phi_0 \mu$ is an absolutely continuous g -invariant measure, then $P\phi_0 = \phi_0$; and

(iv) if $P\phi_0 = \phi_0$, then $\nu = \phi_0 \mu$ is an absolutely continuous g -invariant measure.

All of these properties are immediate from the definition of P . We refer the reader to [Bro96] for excellent, concise proofs of all four properties. We note also that (i) alone may be taken as the definition of P , and then the remaining four properties and the explicit formula for P follow immediately.

1.4 An introduction to the tower method

In this paper, we use a construction which has only recently become fashionable in proving exponential decay of correlations. Instead of working directly with our dynamical system $f : M \rightarrow M$, where M is some finite-dimensional compact Riemannian manifold and f is some piecewise smooth measurable map, we construct a tower sitting over some prudently chosen subset Λ of M , and we use information about when points from Λ return to Λ under iterates of f . We discuss here how to do this in a more general setting.

Let $(X, \mathcal{B}, \mu)$ be some probability space, and suppose that $\Lambda \subset X$ is a measurable set such that $\mu(\Lambda) > 0$. We do not exclude the possibility that $\Lambda = X$, as one could do when considering Anosov diffeomorphisms. Suppose also that $f : X \rightarrow X$ preserves μ ; i.e., $\mu(f^{-1}E) = \mu(E)$ for all $E \in \mathcal{B}$. For each $x \in \Lambda$, let $R(x) \in \mathbb{N}$ be the smallest positive integer such that $f^{R(x)}(x) \in \Lambda$. We call $R(x)$ the **return time**

of x . In [Shi96] one finds the following theorem (and proof) about returns of points in Λ to Λ .

Theorem 1.3 *Let f , $(X, \mathcal{B}, \mu)$, and Λ be as above. Then $R(x) < \infty$ for μ -a.e. $x \in \Lambda$.*

Proof: For each $n \in \mathbb{N}$, let us define $\Lambda_n = \{x \in \Lambda : R(x) = n\}$; i.e., Λ_n is the set of all points which return to Λ at time n . Let $\Lambda_0 = \{x \in \Lambda : f^n(x) \notin \Lambda \text{ for all } n \in \mathbb{N}\}$ denote the set of points which never return to Λ . It is clear that all of the Λ_n s are pairwise disjoint, and it is clear that

$$\Lambda = \bigcup_{n \geq 0} \Lambda_n.$$

Since f is measurable, all the Λ_n s are measurable since

$$\Lambda_1 = \Lambda \cap f^{-1}\Lambda$$

$$\Lambda_n = \Lambda \cap f^{-n}\Lambda \cap \bigcap_{i=1}^{n-1} f^{-i}(M \setminus \Lambda) \text{ for } n \geq 2, \text{ and}$$

$$\Lambda_0 = \Lambda \setminus \bigcup_{n \in \mathbb{N}} \Lambda_n.$$

Because $\Lambda_0 \subset \Lambda$, no points in Λ_0 can return to Λ_0 ; in other words, if $x \in \Lambda_0$, then $f^i(x) \notin \Lambda_0$ for all $i \in \mathbb{N}$, otherwise we would have a return of x to Λ . This implies that $f^{-i}\{x\} \cap \Lambda_0 = \emptyset$ for all $i \in \mathbb{N}$ whenever $x \in \Lambda_0$ as well. Thus, the collection of sets $\{f^{-i}\Lambda_0 : i \in \mathbb{N}\}$ is pairwise disjoint. Thus, $\mu\left(\bigcup_{i \in \mathbb{N}} f^{-i}\Lambda_0\right) = \sum_{i \in \mathbb{N}} \mu(f^{-i}\Lambda_0)$. Since this sum must converge to something less than 1 and since f preserves the measure μ , it is immediate that $\mu(\Lambda_0) = 0$, which is what we wished to prove. ■

This theorem permits us to partition Λ , modulo a set of μ -measure zero, into the

collection of Λ_n s. For $i \neq j$ it is clear that all the images of Λ_i are disjoint from all the images of Λ_j , except perhaps when they return to Λ .

For those who are familiar with the use of towers in recent results on decay of correlations, we should note that in the current context we are not making any assumptions about how the Λ_n s return to Λ ; in particular, we are not assuming that they map onto Λ when they return.

Imagine now that we make a tower by constructing a column above each Λ_n by placing above it a copy of $f\Lambda_n$ and above that a copy of $f^2\Lambda_n$ and so on until we reach $f^{n-1}\Lambda_n$, which is the top level of the column since $f^n\Lambda_n$ goes back to Λ , the bottom of the tower. Let us denote this tower by Δ . We coordinatize the tower by identifying each point with its preimage in Λ and the image of Λ_i in which it lies; i.e., if $y \in f^k\Lambda_i$ for $0 \leq k < i$, then let $x = f^{-k}(y)$, and identify y with the point (x, k) . We say that (x, k) is on the k th level of Δ . We may define the tower explicitly as

$$\Delta = \{(x, i) : x \in \Lambda \text{ and } 0 \leq i < R(x)\}.$$

This tower Δ and the map $f : M \rightarrow M$ induce a map $\hat{f} : \Delta \rightarrow \Delta$ which moves points up the tower, except for points which are already at the top, and $\hat{f}$ moves these points to the bottom of the tower. The definition of $\hat{f}$ is as follows:

$$\hat{f}(x, i) = \begin{cases} (x, i+1) & \text{if } R(x) > i+1 \\ (f^{R(x)}(x), 0) & \text{if } R(x) = i+1 \end{cases}.$$

We refer to $\hat{f}$ as the **tower map**, and we refer to the map $f^R : \Lambda \rightarrow \Lambda$, defined by $f^R(x) = f^{R(x)}(x)$ as the **return map** or (in the language of Kakutani [Kak43]) as the **induced map**.

The measure μ on M also induces a probability measure $\hat{\mu}$ on Δ . Ergodic properties of the induced system $(f^R, \hat{\mu})$ and (f, μ) are proved in [Kak43] and in [Kac47]. Some of these results are summarized very nicely in [Shi96, pp. 23ff].

The above results are purely measure-theoretic, but because we are dealing in this paper with differentiable dynamical systems and because our primary goal involves mixing, we must add some additional conditions to the definition of return time to guarantee that when the Λ_n s return to Λ they satisfy some geometric conditions which are essential in our determination of the speed of mixing. We will have the additional problem that two points which return at the same time to Λ may not be comparable to each other because they land on opposite sides of some discontinuity set before they return to Λ , and so they will have to be placed in different Λ_i s which return at the same time. Hence, the tower we consider in Chapter 2 will be more complicated than the above tower. It may have countably many columns of the same height, and this does present some complications which are not present in the above tower and which are not present in [You98]. We will discuss in Chapter 2 the precise definition of return time we require for our purposes.

Since the publication of Lai-Sang Young's paper [You98], the tower method approach to proving exponential decay of correlations has become a relatively widely-used method. We claim that this justifies our attempt to extend her result so that it applies to a wider class of dynamical systems. In her paper, Young showed how her method applies to piecewise hyperbolic (with finitely many pieces) systems in two dimensions. Later Chernov in [Che99] showed how her method can also be applied to piecewise hyperbolic systems in higher dimensions. Alvez, Luzzatto, and Pinheiro in

[ALP] use the tower method to show that the rate of growth of Lyapunov exponents determines the rate of decay of the return times. These are just a few examples of uses of the tower method in this fashion; a multitude of authors are using the tower method in some way to determine rate of decay of correlations for various dynamical systems.

Some authors such as Young in [You99] and Buzzi and Maume-Deschamps in [BMD] consider only tower systems instead of beginning with a dynamical system and constructing its tower. This approach usually assumes *a priori* the existence of a bounded SRB measure for the tower system and the result of Lemma 2.4 (iii) on p. 29; however, we shall prove Lemma 2.4 from a few assumptions on our dynamical system; we state these assumptions in the next chapter.

The Main Theorem

In this chapter, we prove two primary things:

1. dynamical systems satisfying certain conditions have SRB measures, and
2. dynamical systems with SRB measures which satisfy certain other properties have unique SRB measures and exponential decay of correlations for functions which are piecewise Hölder continuous.

We follow Young's argument very closely; however, we make two changes which allow us to consider piecewise hyperbolic maps with countably many domains of invertibility instead of only finitely many and maps for which certain types of distortion estimates grow quite quickly instead of remaining uniformly bounded.

2.1 Setting and assumptions

We assume that M is a smooth compact finite-dimensional Riemannian manifold, and that $f : M \rightarrow M$ is a map. In all that follows, we let m denote Riemannian measure on M . If $\gamma \subset M$ is a submanifold, then we use m_γ to denote the Riemannian measure on γ induced by the restriction of the Riemannian structure to γ .

Definition 2.1 *Let $H \subset M$. We say that H has a **hyperbolic product structure** if there are a continuous family of stable disks $\Gamma^s = \{\gamma^s\}$ and a continuous family of unstable disks $\Gamma^u = \{\gamma^u\}$ such that the following are true:*

- (a) *for each $\gamma^s \in \Gamma^s$ and each $\gamma^u \in \Gamma^u$, $\dim \gamma^s + \dim \gamma^u = \dim M$;*
- (b) *each γ^s disk is transverse to each γ^u disk, and the angles between them are uniformly bounded away from zero;*
- (c) *each γ^s disk intersects each γ^u disk in one point; and*
- (d)
$$H = \left(\bigcup_{\gamma^u \in \Gamma^u} \gamma^u \right) \cap \left(\bigcup_{\gamma^s \in \Gamma^s} \gamma^s \right).$$

In other words, H is like a rectangle which is coordinatized by two transverse foliations. We call Γ^s and Γ^u the defining sets of H . We shall call $H' \subset H$ an **s-subset** if it also has a hyperbolic product structure and if its defining sets are Γ^u and some proper subset of Γ^s . We define **u-subsets** similarly.

We assume that we have some set $\Lambda \subset M$ with a hyperbolic product structure, and we do not exclude the possibility that $\Lambda = M$. Furthermore, we require that $m_\gamma(\gamma \cap \Lambda) > 0$ for all $\gamma \in \Gamma^u$.

We assume also that Λ is decomposed into countably many pairwise disjoint s -subsets $\Lambda_1, \Lambda_2, \dots$ such that the following are true:

- (a) for each $\gamma^u \in \Gamma^u$, $m_{\gamma^u}((\Lambda \setminus \bigcup_{i \in \mathbb{N}} \Lambda_i) \cap \gamma^u) = 0$;
- (b) for each $i \in \mathbb{N}$, there is some minimal $R_i \in \mathbb{N}$ such that $f^{R_i} \Lambda_i$ is a u -subset of Λ such that, for all $x \in \Lambda_i$, $f^{R_i}(\gamma^s(x)) \subset \gamma^s(f^{R_i}(x))$ and $f^{R_i}(\gamma^u(x)) \supset \gamma^u(f^{R_i}(x))$;
- (c) there is some $R_0 > 1$ such that $R_i \geq R_0$ for all $i \in \mathbb{N}$; and
- (d) for each $i \in \mathbb{N}$ and for each $\gamma \in \Gamma^u$, $m_\gamma(\gamma \cap \Lambda_i) > 0$; and
- (e) for each $i \in \mathbb{N}$, $f^{R_i}|_{\Lambda_i}$ is injective.

We should note that in the general situation, given a collection of Λ_i s, there is a first return time for each Λ_i , and it is possible that the first return time could be 1, as it is in the examples we consider in later chapters. To make the first return time larger than some arbitrary R_0 , we simply run the system forward R_0 iterates, and then pull back Λ by each $f|_{\Lambda_i}$ for R_0 times so that we divide each Λ_i more finely.

The tower map and Markov partition

We obtain our results on decay of correlations by working with a new map for which our current system is a factor system. Let $\bigcup_{n \in \mathbb{N}} f^n \Lambda$ denote the union of all the images of Λ . We shall construct a tower Δ and the induced map $F : \Delta \rightarrow \Delta$ as we did in Section 1.4 such that there is a projection $\pi : \Delta \rightarrow \bigcup_{n \in \mathbb{N}} f^n \Lambda$ with the property that $f \circ \pi = \pi \circ F$. For simplicity, we shall denote by $R : \Lambda \rightarrow \mathbb{N}$ the return time

function $R|_{\Lambda_i} \equiv R_i$ and by $f^R : \Lambda \rightarrow \Lambda$ the return map defined by $f^R(x) = f^{R_i}(x)$ if $x \in \Lambda_i$.

Let us define the set Δ as follows:

$$\Delta = \{(x, l) : x \in \Lambda \text{ and } l = 0, 1, \dots, R(x) - 1\}.$$

We shall define $F : \Delta \rightarrow \Delta$ by

$$F(x, l) = \begin{cases} (x, l+1) & \text{if } l < R(x) - 1 \\ (f^R(x), 0) & \text{if } l = R(x) - 1 \end{cases}.$$

Then it is clear that the projection $\pi : \Delta \rightarrow \bigcup_{n \in \mathbb{N}} f^n \Lambda$ should be $\pi(x, l) = f^l(x)$.

We denote by Δ_l the set of all points in Δ whose second coordinate is l . Then $\Delta_0 = \Lambda \times \{0\}$. In fact, for each $i \in \mathbb{N}$, $\Delta_l = \{x \in \Lambda : R(x) > l\} \times \{l\}$. Then each Δ_l is a canonical copy of the union of Λ_i s which return after time l , and we imagine this copy of Λ_i as sitting over the copy of Λ_i in Δ_0 . The map F simply either carries the copy of Λ_i on Δ_l bijectively up to the copy of Λ_i on Δ_{l+1} if $R_i > l + 1$ or injectively onto a u-subset of Δ_0 if $R_i = l + 1$.

The Λ_i s induce a countable partition $\mathcal{M}_l$ on each Δ_l and, hence, a countable partition $\mathcal{M}$ on Δ for which the map $F : \Delta \rightarrow \Delta$ is fully Markov. Throughout this paper, we shall use $\Delta_{l,i}$ to denote the copy of Λ_i on the l th level of the tower Δ , provided that $R_i > l$; i.e.,

$$\Delta_{l,i} = \Delta_l \cap F^l(\Lambda_i \times \{0\}), \text{ if } R_i > l.$$

A useful tool for defining a metric on the tower Δ is a function called the **separation time** function $s : \Delta \times \Delta \rightarrow \mathbb{N}$, which we shall define now. Let $x, y \in \Delta$, and

let $\mathcal{M}(x)$ and $\mathcal{M}(y)$ denote the element of $\mathcal{M}$ containing x and y , respectively. We may now define s as follows:

$$s(x, y) = \sup \{n \in \mathbb{N} : \mathcal{M}(F^j x) = \mathcal{M}(F^j y) \text{ for all } 0 \leq j \leq n\}.$$

In other words, the separation time of x and y is the last time that they are together is they travel through the elements of the partition $\mathcal{M}$, or it is equal to ∞ . It is not necessary to say what $s(x, y)$ should be if $\mathcal{M}(x) \neq \mathcal{M}(y)$ since all of our conditions are concerned only with two points which start in the same element of $\mathcal{M}$ anyway; however, one could for completeness simply set $s(x, y) = 0$ if $\mathcal{M}(x) \neq \mathcal{M}(y)$. The separation time function $s : \Delta \times \Delta \rightarrow \mathbb{N}$ induces a separation time function on $\Lambda \times \Lambda$, and we will deliberately be sloppy and call them both s .

Now that we have established a separation time function, we may state the remainder of our major assumptions. In the following, let f^u denote the restriction of f to γ^u disks, and let Df^u denote its derivative, called the **unstable derivative**; we call $\det Df^u$ the **unstable jacobian**. We suppose that there exists $\alpha \in (0, 1)$ and that, for each $i \in \mathbb{N}$, there exists some $C(i) > 0$ such that:

(A) for all $x, y \in \Lambda_i$ such that $y \in \gamma^s(x)$, we have $d(f^n x, f^n y) \leq C(i)\alpha^n$ for all $n \geq 0$;

(B) for all $x, y \in \Lambda_i$ such that $y \in \gamma^u(x)$ and $0 \leq k \leq n < s(x, y)$, we have

$$(a) \quad d(f^n x, f^n y) \leq C(i)\alpha^{s(x, y)-n}, \text{ and}$$

$$(b) \quad \log \prod_{j=k}^n \frac{\det Df^u(f^j x)}{\det Df^u(f^j y)} \leq C(i)\alpha^{s(x, y)-n};$$

(C) for all $x, y \in \Lambda_i$,

- (a) for $y \in \gamma^s(x)$, we have $\log \prod_{j=n}^{\infty} \frac{\det Df^u(f^j x)}{\det Df^u(f^j y)} \leq C(i)\alpha^n$ for all $n \geq 0$, and
- (b) for $\gamma_1, \gamma_2 \in \Gamma^u$ and $\theta_{\gamma_1, \gamma_2} : \gamma_1 \cap \Lambda_i \rightarrow \gamma_2 \cap \Lambda_i$, the stable holonomy map, we require that $\theta_{\gamma_1, \gamma_2}$ is absolutely continuous and that $\frac{d(\theta_{\star}^{-1} m_{\gamma_2})}{dm_{\gamma_1}} = \prod_{j=0}^{\infty} \frac{\det Df^u(f^j x)}{\det Df^u(f^j(\theta x))}$;
- (D) there is a function $\varrho : \mathbb{N} \rightarrow \mathbb{R}$ such that $\varrho(n) \rightarrow 0$ as $n \rightarrow \infty$ and, for all $n \in \mathbb{N}$ and for all $\gamma \in \Gamma^u$,
- $$m_{\gamma} \left(\gamma \cap \bigcup_{i>n} \Lambda_i \right) \leq \varrho(n);$$
- (E) the sum $\sum_{i \in \mathbb{N}} C(i) e^{C(i)} \varrho(i)$ converges.

2.2 Statements of main results

Theorem 2.2 (Existence of SRB measures) *Suppose that $f : M \rightarrow M$ with $\Lambda \subset M$ satisfies the construction given in the previous parts of this chapter and that the following additional criteria are satisfied:*

- (i) $\sum_{i \in \mathbb{N}} C(i) \alpha^{R_i}$ converges;
- (ii) $\sum_{i \in \mathbb{N}} R_i e^{C(i)} \varrho(i) < \infty$ for some $\gamma \in \Gamma^u$.

Then f has a finite SRB measure, which we call ν .

Let $0 < \eta < 1$, and let $\mathfrak{H}_{\eta}$ be the space of observables defined by

$$\mathfrak{H}_{\eta} = \left\{ \varphi : M \rightarrow \mathbb{C} : \exists \mathfrak{C}(\varphi) > 0 \text{ such that } |\varphi(x) - \varphi(y)| \leq \mathfrak{C}(\varphi) d(x, y)^{\eta} \right\}.$$

Theorem 2.3 (Decay of correlations) *Suppose that $f : M \rightarrow M$ with $\Lambda \subset M$ satisfies the construction given in the previous parts of this chapter and that there are some $C_0 > 0$ and $\theta_0 < 1$ such that $m_\gamma \{x \in \gamma \cap \Lambda : R(x) > l\} < C_0 \theta_0^l$, and*

- (i) $\gcd\{R_i\} = 1$, or
- (ii) (f, ν) is totally ergodic (i.e., (f^n, ν) is ergodic for all $n \in \mathbb{N}$).

Then (f, ν) has exponential decay of correlations for functions in $\mathfrak{H}_\eta$ for all $0 < \eta < 1$.

2.3 Differences between Young's result and ours

It is important here to list some of the ways in which our approach here is fundamentally different from that of Young in [You98].

Countable Markov partition

The proof of the quasicompactity of the Perron-Frobenius operator in [You98] requires that the Markov partition $\mathcal{M}_l$ on Δ_l be finite for each $l \in \mathbb{N}$. To construct this partition, one requires a separation time function s_0 , as we discuss below, which is different from the one we define above. This function s_0 is used to define the Markov partition and to define a new separation time function s like the one we constructed above.

By including the rate of decay of the Λ_i s in our estimates of the spectrum of the Perron-Frobenius operator, we are able to allow for a countable partition $\mathcal{M}_l$ on Δ_l for each $l \in \mathbb{N}$. We are also able to take the natural partition of Δ into copies of

the Λ_i s so that the partition is given by the behavior of the map f on Λ . An added advantage comes in the way we are able to define a universal separation time function, as we discuss next.

Universal separation time

The construction of a partition similar to our partition $\mathcal{M}$ as given in [You98] requires the *a priori* existence of some separation time function $s_0 : \Lambda \times \Lambda \rightarrow \mathbb{N}$ satisfying the following three conditions:

- (a) $s_0(x, y) \geq 0$ depends only on the γ^s -disks containing x and y ;
- (b) for $x, y \in \Lambda_i$, $s_0(x, y) \geq R_i + s_0(f^{R_i}x, f^{R_i}y)$;
- (c) for $x \in \Lambda_i$ and $y \in \Lambda_j$ with $R_i = R_j$ and $i \neq j$, $s_0(x, y) < R_i - 1$.

This function is not necessarily unique for any given dynamical system, but it is in some way intrinsically constructed for a particular dynamical system. Note that a fourth condition in [You98] that only finitely many Λ_i s are separated by time n for all $n \in \mathbb{N}$ is not satisfied by our set of assumptions, but we show in this thesis that this requirement is no longer needed.

Instead of requiring the *a priori* existence of some function $s : \Lambda \times \Lambda \rightarrow \mathbb{N}$ satisfying the four conditions given above and then defining the partition $\mathcal{M}$ so that s may be extended to s_0 to be compatible with it in terms of those four conditions, we define our partition $\mathcal{M}$ by the natural structure of Λ , and then we use $\mathcal{M}$ to define a return time function s which trivially satisfies the first three properties for any dynamical

system satisfying Condition (B). The fourth property of s_0 cannot be satisfied since we may have countably many Λ_i s returning at any given time, but we avoid this problem as we mentioned above.

The disadvantage of our approach is that our Conditions (A)-(C) (See p. 17.) cannot be stated without first constructing the tower Δ ; however, this is only an æsthetic problem, not a fundamental one since the tower is a canonical structure given by the hyperbolic product structure of Λ and the fact that f^R is fully Markov.

Non-uniform contraction

Recall that C and α are fixed in [You98] and that, for each $i \in \mathbb{N}$, $C(i)$ is fixed in our result. By Condition (P3) in [You98] for the case $n = 1$, we have

$$(2.1) \quad d(fx, fy) \leq C\alpha;$$

however, by Condition (A) in our result we have that, for each $i \in \mathbb{N}$,

$$(2.2) \quad d(f^m x, f^m y) \leq C(i)\alpha^m$$

for all $m \in \mathbb{N}$. In other words, in [You98] all stable disks grow to a fixed size before contracting. In our result, for every $m \in \mathbb{N}$, there can be $i \in \mathbb{N}$ such that $C(i)\alpha^m > 1$ so that only finitely many stable disks need to contract by the m th iterate, but countably infinitely many may still be growing before their contraction begins.

Unbounded distortion

By assumption (P4)(b) in [You98], the distortion along unstable leaves is uniformly bounded; i.e., for $x, y \in \Lambda_i$ with $y \in \gamma^u(x)$ and $S = s(x, y)$. Then

$$(2.3) \quad \frac{\det Df^u(f^S x)}{\det Df^u(f^S y)} \leq e^C;$$

however, in our paper we allow that

$$(2.4) \quad \frac{\det Df^u(f^S x)}{\det Df^u(f^S y)} \leq e^{C(i)}$$

for all $y \in \gamma^u(x) \cap \Lambda_i$. The only restrictions on the speed of growth of $C(i)$ is that the sums $\sum_{i \in \mathbb{N}} R_i e^{C(i)} \varrho(i)$, $\sum_{i \in \mathbb{N}} e^{C(i)} \alpha^{R_i}$, and $\sum_{i \in \mathbb{N}} C(i) e^{C(i)} \varrho(i)$ must converge; thus, as long as $\varrho(i)$ decays sufficiently quickly, $C(i)$ can grow quite quickly as well. Hence, the distortion between two points is not necessarily uniformly bounded.

Worsening of absolute continuity of stable foliation

Let $x, y \in \Lambda_i$. Assumption (P5)(b) in [You98] requires that

$$(2.5) \quad \frac{d(\theta_*^{-1} m_{\gamma'})}{dm_\gamma} \leq e^C,$$

but in our paper we allow the Radon-Nikodym derivatives to grow but require

$$(2.6) \quad \frac{d(\theta_*^{-1} m_{\gamma'})}{dm_\gamma} \leq e^{C(i)}.$$

In particular, the Radon-Nikodym derivatives may grow without bound as i tends to infinity.

2.4 Proof of Theorem 2.2

We follow very closely the proof in [You98]; however, there are some significant differences between our proof and that one because we allow countably many Λ_i s to return at any given time, so we include the entire proof, but the biggest complications come because we allow $C(i)$ to vary with $i \in \mathbb{N}$. Let γ_0 be some arbitrary γ^u -disk which is full-width in some Λ_{i_0} , and let $\mu_0 := m_{\gamma_0}|_{\gamma_0 \cap \Lambda_{i_0}}$. We have already assumed that $\mu_0(\gamma_0) > 0$. For each $j \in \mathbb{N}$ and for each $\gamma \in \Gamma^u$, let $\rho_j^\gamma = \frac{1}{(f^R)_*^j \mu_0(\gamma)} \cdot \frac{d(f^R)_*^j \mu_0}{dm_\gamma}$ whenever $(f^R)_*^j \mu_0(\gamma) > 0$; otherwise, let $\rho_j^\gamma \equiv 0$.

Suppose that $\gamma \in \Gamma^u$ is such that $(f^R)_*^j \mu_0(\gamma) > 0$, and let $x, y \in \gamma \cap \Lambda_i$. Let $x_0, y_0 \in \gamma_0 \cap \Lambda_{i_0}$ be such that $(f^R)^j x_0 = x$ and $(f^R)^j y_0 = y$. Since x and y are both in Λ_i , we know that $s(x_0, y_0) \geq R_{i_0} j$. Suppose that $x_0, y_0 \in \gamma_0 \cap \Lambda_{i_0}$. Then, by Condition (B)(b) for all $0 \leq k \leq R_{i_0} j - 1$ we have

$$(2.7) \quad \prod_{m=k}^{R_{i_0} j - 1} \frac{\det Df^u(f^m x_0)}{\det Df^u(f^m y_0)} \leq C(i_0) \alpha^{s(x_0, y_0) - R_{i_0} j + 1}.$$

By the Change of Variables Theorem and the Chain Rule we have,

$$\begin{aligned} \frac{\rho_j^\gamma(x)}{\rho_j^\gamma(y)} &= \frac{\det Df^u(f^{R_{i_0} j} x_0)}{\det Df^u(f^{R_{i_0} j} y_0)} \prod_{m=0}^{R_{i_0} j - 1} \frac{\det Df^u(f^m x_0)}{\det Df^u(f^m y_0)} \\ &= \frac{\det Df^u(x)}{\det Df^u(y)} \prod_{m=0}^{R_{i_0} j - 1} \frac{\det Df^u(f^m x_0)}{\det Df^u(f^m y_0)} \\ &\leq C(i) \prod_{m=0}^{R_{i_0} j - 1} \frac{\det Df^u(f^m x_0)}{\det Df^u(f^m y_0)} \\ &\leq C(i) C(i_0) \alpha^{s(x_0, y_0) - R_{i_0} j + 1} \\ &< C(i) C(i_0) \alpha^{s(x, y)} \end{aligned}$$

because $s(x_0, y_0) \geq R_{i_0}j + s(x, y)$. Recall that our separation time function s satisfies the condition $s(x, y) \geq R_i$ for all $x, y \in \Lambda_i$. We may then define

$$L := \sup_{\substack{i \in \mathbb{N} \\ x, y \in \gamma \cap \Lambda_i}} e^{C(i)\alpha^{s(x, y)}} \leq \sup_{i \in \mathbb{N}} e^{C(i)\alpha^{R_i}},$$

which is bounded since we assume that $\sum_{i \in \mathbb{N}} C(i)\alpha^{R_i}$ converges. We have then that

$$(2.8) \quad \frac{\rho_j^\gamma(x)}{\rho_j^\gamma(y)} \leq C(i_0) \log L \quad \text{for all } x, y \in \gamma \cap \Lambda_i,$$

and this is independent of j and γ . This implies that $\frac{1}{M_0(i)} \leq \rho_j^\gamma(x) \leq M_0(i)$ for all $x \in \gamma \cap \Lambda_i$ for some $M_0(i) > 1$. In fact, $M_0(i) = C(i_0) \log L \sup_{y \in \gamma \cap \Lambda_i} \rho_j^\gamma(y)$. We must show that M_0 is also independent of i . Let $x, y \in \gamma \cap \Lambda$ with x and y in different Λ_i s, and let $c : [0, 1] \rightarrow \Lambda$ be a smooth curve connecting x and y such that, for each $i \in \mathbb{N}$, we have $c([0, 1]) \cap \text{cl } E_i$, where cl denotes the closure, contains at most one connected component. For convenience, let us say that x is on the “left” of the curve c and that y is on the “right.” Let $x_i = \partial_{\text{left}}(\gamma \cap \Lambda_i)$. Then

$$\begin{aligned} \frac{\rho_j^\gamma(x)}{\rho_j^\gamma(y)} &\leq \prod_{i \in \mathbb{N}} \frac{\rho_j^\gamma(x_i)}{\rho_j^\gamma(x_{i+1})} \\ &\leq \prod_{i \in \mathbb{N}} e^{C(i)\alpha^{R_i}} \\ &= e^{\sum_{i \in \mathbb{N}} C(i)\alpha^{R_i}}, \end{aligned}$$

which is also finite. Therefore, there exists $M_0 > 0$ independent of j , γ , and i such that

$$(2.9) \quad \frac{1}{M_0} \leq \rho_j^\gamma \leq M_0 \quad \text{on } \gamma \cap \Lambda.$$

Let $\nu_j^\gamma = \rho_j^\gamma m_\gamma$. Then ν_j^γ is a finite measure on γ . Let ν_0 be an accumulation point of

$\left\{ \frac{1}{n} \sum_{j=0}^{n-1} (f^R)_*^j \mu_0 \right\}_{n \in \mathbb{N}}$ in the weak* topology, and let ν_0^γ be the conditional measure of ν_0 on the u -disk γ .

We claim that $\nu_0^\gamma \ll m_\gamma$ for almost every γ with bounds $\frac{1}{M_1(i)} \leq \frac{d\nu_0^\gamma}{dm_\gamma} \leq M_1(i)$ for some $M_1(i) > 1$. The remainder of our proof is identical to Young's proof; however, we include it here for completeness.

Let $\omega \subset \gamma \cap \Lambda_i$ be open in $\gamma \cap \Lambda_i$ with $m_\gamma(\partial\omega) = 0$, and let S_ω denote the s -subset of Λ whose section in γ is ω . Let U be a u -subset which is also a compact neighborhood of γ . Then (2.9) together with (C) imply that for all j we have

$$(2.10) \quad \frac{1}{M_1(i)} \cdot \frac{m_\gamma(S_\omega)}{m_\gamma(\Lambda)} \leq \frac{(f^R)_*^j \mu_0(U \cap S_\omega)}{(f^R)_*^j \mu_0(U)} \leq M_1(i) \frac{m_\gamma(S_\omega)}{m_\gamma(\Lambda)}$$

for some $M_1(i)$. The bounds in (2.10) also apply to ν_0 . By taking U arbitrarily small, the Martingale Convergence Theorem gives us that

$$\frac{1}{M_1(i)} \cdot \frac{m_\gamma(S_\omega)}{m_\gamma(\Lambda)} \leq \nu_0^\gamma(S_\omega) \leq M_1(i) \cdot \frac{m_\gamma(S_\omega)}{m_\gamma(\Lambda)}$$

for almost every γ . Since ω is arbitrary, the density statement for ν_0^γ follows.

We have so far constructed an f^R -invariant finite Borel measure ν_0 on Λ with absolutely continuous conditional measures on γ^u -leaves. We may clearly identify ν_0 with an F^R -invariant measure $\tilde{\nu}_0$ on Δ_0 . We define a measure $\tilde{\nu}$ on Δ by

$$\tilde{\nu} := \sum_{j \in \mathbb{N}} F_*^j \tilde{\nu}_0|_{\{R > j\}}.$$

The fact that $\nu_0^\gamma \ll \mu_0$ along with our assumption that $m_\gamma(\gamma \cap \Lambda) \sum_{i \in \mathbb{N}} R_i e^{C(i)} \varrho(i) < \infty$ for some $\gamma \in \Gamma^u$ implies that $\tilde{\nu}(\Delta)$ is finite. We now define the measure ν to be the push-forward measure induced on Λ by the canonical projection from Δ to Λ ; i.e., $\nu := \frac{\pi_* \tilde{\nu}}{\tilde{\nu}(\Delta)}$, where $\pi : \Delta \rightarrow \bigcup_{n \in \mathbb{N}} f^n \Lambda$ is the projection $\pi(x, l) = f^l(x)$. Clearly, ν is

f -invariant and the SRB property is clear since $f_*^j \nu$ clearly has absolutely continuous conditional measures on $\{f^j \gamma^u\}$ for every $j \in \mathbb{N}$, and these are unstable manifolds. This proves Theorem 2.2.

2.5 Proof of Theorem 2.3

A related expanding system

It is not difficult to see that the Perron-Frobenius operator (1.2) improves the Hölder continuity of observables when the map $g : M \rightarrow M$ is an expanding map and that it worsens the Hölder continuity of observables when the map g is contracting. Because our dynamical system $f : \Lambda \rightarrow \Lambda$ has both expansion and contraction coexisting, we must somehow “eliminate” the contracting direction. Expanding systems are mixing with respect to a measure equivalent to Lebesgue measure, but contracting systems clearly are not. Furthermore, expansion causes the Perron-Frobenius operator to improve certain properties of observables (e.g., Hölder continuity or essential variation), but contraction causes the operator to worsen these properties. There are two common methods for “factoring out” the contracting direction: (1) identifying points in the same stable leaf and considering the resulting quotient space, and (2) taking averages of observables along stable leaves with respect to some cone (in the lattice theory sense) of densities and considering the action of the Perron-Frobenius operator on these averages. The latter method requires a good deal of knowledge about the smoothness of the stable bundle (the collection of stable leaves); however,

the former method requires only absolute continuity of the stable foliation. We use the former method in our work, while Viana uses the latter method. (In particular, see pp. 79-122 of [Via].)

We have a tower Δ and a map $F : \Delta \rightarrow \Delta$ induced by the map $f : \Lambda \rightarrow \Lambda$. We want to define a related dynamical system which is expanding so that the related Perron-Frobenius operator will reduce the norm of observables in the sense that $\|P^n \varphi\| \leq \lambda \|\varphi\| + K|\varphi|_1$ for some $n \in \mathbb{N}$, some $\lambda < 1$, some $K > 0$ and for some appropriately chosen norm $\|\cdot\|$ on some space of real-valued functions on M . If x and y are in Λ , let $x \sim y$ if and only if $y \in \gamma^s(x)$. Then $\sim$ defines an equivalence relation on Δ , so $\overline{\Delta} = \Delta / \sim$ is a quotient space. Let $\overline{F} : \overline{\Delta} \rightarrow \overline{\Delta}$ be the map defined by $\overline{F}(\overline{x}) = \overline{F(x)}$, where $\overline{x}$ is the equivalence class of the points in $\gamma^s(x)$. Because F maps γ^s disks to γ^s disks, it is clear that, if $\overline{x} = \overline{y}$, $\overline{F}(\overline{x}) = \overline{F}(\overline{y})$; therefore, $\overline{F}$ is well-defined. In all that follows, we shall let $\overline{\Lambda}$, $\overline{\Delta}_l$, $\overline{\Delta}_{l,i}$, and $\overline{f}^R$ have the obvious meanings as given by the equivalence relation $\sim$.

We show here that for many dynamical systems there is a measure μ on Λ whose conditional measures on γ^u disks are preserved by the stable holonomy map; i.e., if $\theta_{\gamma, \gamma'}$ is the stable holonomy map taking points from γ to γ' by sliding them along the stable foliation, then $\theta_* \mu_\gamma = \mu_{\gamma'}$. This fact allows us to “collapse” Δ to $\overline{\Delta}$ as described above in a way that ensures that the essential properties of the dynamical system $F : \Delta \rightarrow \Delta$ are carried to $\overline{F} : \overline{\Delta} \rightarrow \overline{\Delta}$.

We shall define a reference measure $\overline{\mu}$ on $\overline{\Lambda}$ and then extend it to all of $\overline{\Delta}$ by letting $\overline{\mu}|_{\overline{\Delta}_l} = \overline{F}_* \overline{\mu}|_{\overline{\Delta}_{l-1}}$. We shall also use the name $\overline{\mu}$ when talking about this measure on $\overline{\Delta}$. This measure $\overline{\mu}$ will be such that $J\overline{F} \equiv 1$ on the points in $\overline{\Delta}$ which are not on

the top level of the tower, but $J\bar{F} = \bar{f}^R \circ \bar{F}^{-(R-1)}$ at the points which are on the top level of the tower.

Let $\hat{\gamma} \in \Gamma^u$ be some fixed γ^u disk. For each $x \in \Lambda$, let $\hat{x}$ denote the single point of Λ in $\gamma^s(x) \cap \hat{\gamma}$. For each $n \in \mathbb{N}$, let

$$u_n(x) = \log \prod_{i=0}^{n-1} \left| \frac{\det Df^u(f^i x)}{\det Df^u(f^i \hat{x})} \right|.$$

By our assumption (C)(a) (See p. 17.) on each Λ_i the sequence of u_n s converges uniformly to some function $u|_{\Lambda_i}$. On each $\gamma \in \Gamma^u$, we define μ_γ to be the measure $\mu_\gamma = e^u \cdot 1_{\gamma \cap \Lambda} m_\gamma$. By Conditions (D) and (E), the measure μ_γ is a finite measure for each $\gamma \in \Gamma^u$.

We note that $f^{R_i}|_{\gamma \cap \Lambda_i}$ is nonsingular with respect to the measures μ_γ . If we have $f^{R_i}(\gamma \cap \Lambda_i) \subset \gamma'$, then we shall write $J(f^R)(x)$ for $J_{\mu_\gamma, \mu_{\gamma'}}(f^{R_i}|_{\gamma \cap \Lambda_i})(x)$, where $J_{m_1, m_2}(T) := \frac{d(T_*^{-1}m_2)}{dm_1}$, since it will be clear in our estimates in which Λ_i a particular x lies.

Lastly, we note that

$$\begin{aligned} \bar{\mu}(\bar{\Delta}) &= \sum_{i \in \mathbb{N}} R_i \bar{\mu}(\bar{\Delta}_{0,i}) \\ &\leq \sum_{i \in \mathbb{N}} R_i e^{C(i)} m_\gamma(\gamma \cap \Lambda_i) \\ &\leq m_\gamma(\gamma \cap \Lambda) \sum_{i \in \mathbb{N}} R_i e^{C(i)} \varrho(i), \end{aligned}$$

and this sum is finite by assumption (ii) of Theorem 2.2; therefore, $\bar{\mu}$ is a finite measure on $\bar{\Delta}$.

We prove next a lemma which shows that this measure $\bar{\mu}$ allows us to collapse Δ along the stable foliation Γ^s in a way that preserves $J(f^R)$.

Lemma 2.4 *Let γ and γ' be two unstable disks in Γ^u , and let $\theta = \theta_{\gamma, \gamma'} : \gamma \cap \Lambda \rightarrow \gamma' \cap \Lambda$ be the stable holonomy map discussed earlier. Then the following are true:*

- (i) $\theta_* \mu_\gamma = \mu_{\gamma'}$;
- (ii) $J(f^R)(x) = J(f^R)(y)$ for all $y \in \gamma^s(x)$; and
- (iii) for each $i \in \mathbb{N}$, there exists $C_1(i)$ such that, for all x and y in $\gamma \cap \Lambda_i$,

$$\left| \frac{J(f^R)(x)}{J(f^R)(y)} - 1 \right| \leq C_1(i) \alpha^{s(f^R x, f^R y)/2}.$$

The proofs of (i) and (ii) are exactly the same as in [You98]. For (iii) we follow this proof but take $C_1(i) = 5C(i)$. The proofs of all three parts are very short, and we refer the reader to [You98] for them.

A space of observables for the factor system

We shall define a space of real-valued functions $\bar{\varphi} : \bar{\Delta} \rightarrow \mathbb{R}$. We use the bar notation for the function names as well since we will introduce later a related class of functions $\varphi : M \rightarrow \mathbb{R}$. It is this latter class of functions in which we are interested; however, we will demonstrate decay of correlations for the factored expanding system with respect to the former class of observables and then show how the correlation functions of the original system are related to the correlation functions of the factored expanding system.

Throughout the rest of this paper, we shall let $|\bar{\varphi}|_1$ and $|\bar{\varphi}|_\infty$ denote the L^1 and L^∞ norms of $\bar{\varphi}$, respectively, with respect to the reference measure $\bar{\mu}$. By $\bar{\varphi}_{l,i}$ we shall mean $\bar{\varphi}|_{\bar{\Delta}_{l,i}}$, where $\bar{\Delta}_{l,i}$ has the same meaning as earlier. We choose $\epsilon > 0$ such that

$e^\epsilon \theta_0 < 1$. (See Theorem 2.3, p. 19, for the definition of θ_0 .) For future reference, we note here also that there is some $K > 0$ such that

$$(2.11) \quad \frac{1}{\bar{\mu}(\bar{\Delta}_0)} \sum_{l,i} \bar{\mu}(\bar{\Delta}_{l,i}^*) e^{l\epsilon} \leq K$$

where K depends upon our choice of ϵ and where the $\bar{\Delta}_{l,i}^*$ s are the $\bar{\Delta}_{l,i}$ s which get mapped by F onto $\bar{\Delta}_0$, the base of the tower $\bar{\Delta}$. (In other words, the $\bar{\Delta}_{l,i}^*$ s are the $\bar{\Delta}_{l,i}$ s at the top of the tower.) This is true because of our assumptions that $\bar{\mu}(\bar{\Delta}_{l,i}^*) < \bar{\mu}(\bar{\Delta}_{l,i}) < C_0 \theta_0^l$ and that $\theta_0 e^\epsilon < 1$. In fact, we may choose ϵ so that $K = 2$, if we wish; however, it is not necessary. For each $i \in \mathbb{N}$, let $0 < l(i) \leq m_\gamma(\gamma \cap \Lambda_i)$ for each $\gamma \in \Gamma^u$. Also, for each $i \in \mathbb{N}$, we may choose $d(i) \in (0, 1)$ such that $d(i) \leq l(i)\varrho(i)$.

We choose $N \in \mathbb{N}$ large enough so that

$$(2.12) \quad (1 + \beta^N) e^{-N\epsilon} + 3K\beta^N \left(\sum_{i \in \mathbb{N}} C_1(i) e^{C_1(i)} \varrho(i) \right) < 1,$$

where $\beta = \alpha^{1/2}$. We shall then require that, for all $i \in \mathbb{N}$, $R_i \geq N$. As we show in the examples we consider in subsequent chapters, we may choose Λ differently so that we may make the first return time as large as we require.

Let us define $\|\bar{\varphi}_{l,i}\|_h$ by

$$\|\bar{\varphi}_{l,i}\|_h = \left(\text{ess sup}_{\bar{x}, \bar{y} \in \bar{\Delta}_{l,i}} \frac{|\bar{\varphi}(\bar{x}) - \bar{\varphi}(\bar{y})|}{\beta^{s(\bar{x}, \bar{y})}} \right) e^{-l\epsilon} d(i)$$

and $\|\bar{\varphi}_{l,i}\|_\infty$ by

$$\|\bar{\varphi}_{l,i}\|_\infty = |\bar{\varphi}_{l,i}|_\infty e^{-l\epsilon} d(i).$$

Then we define

$$\|\bar{\varphi}_l\|_h = \sum_{i \in \mathbb{N}} \|\bar{\varphi}_{l,i}\|_h$$

and

$$\|\bar{\varphi}_l\|_\infty = \sum_{i \in \mathbb{N}} \|\bar{\varphi}_{l,i}\|_\infty.$$

Lastly, we take

$$\|\bar{\varphi}\|_h = \sup_{l \in \mathbb{N}} \|\bar{\varphi}_l\|_h \quad \text{and} \quad \|\bar{\varphi}\|_\infty = \sup_{l \in \mathbb{N}} \|\bar{\varphi}_l\|_\infty$$

and then define

$$\|\bar{\varphi}\| = \|\bar{\varphi}\|_h + \|\bar{\varphi}\|_\infty + |\bar{\varphi}|_1.$$

For convenience in later parts, we shall let

$$\|\bar{\varphi}\|' = \|\bar{\varphi}\|_h + \|\bar{\varphi}\|_\infty$$

and write

$$\|\bar{\varphi}\| = \|\bar{\varphi}\|' + |\bar{\varphi}|_1.$$

Now we define our space $\mathcal{F}$ of observables by

$$\mathcal{F} = \{\bar{\varphi} : \bar{\Delta} \rightarrow \mathbb{R} : \|\bar{\varphi}\| < \infty\}.$$

While the norm $\|\cdot\|$ looks quite complicated, we shall see near the end of the proof that a very natural class of piecewise Hölder continuous observables is relate to it.

Our first step now is to show that there is some absolutely continuous $\bar{F}$ -invariant measure whose density belongs to $\mathcal{F}$.

Lemma 2.5 *Assume the setting of Theorem 2.3, and let β be as above. Then $\bar{F} : \bar{\Delta} \rightarrow \bar{\Delta}$ has an invariant absolutely continuous probability measure $\bar{\nu} = \bar{\rho} \bar{\mu}$, where $\bar{\rho}$ satisfies $c_0^{-1} \leq \bar{\rho} \leq c_0$ on $\bar{\Delta}_{l,i}$ for some $c_0 > 1$. Furthermore, $|\bar{\rho}(\bar{x}) - \bar{\rho}(\bar{y})| \leq C'(i) \beta^{s(\bar{x}, \bar{y})}$ for some $C'(i)$.*

Proof: We take Young's proof as it is, but we make the obvious changes required by our use of $C(i)$. For each $n \in \mathbb{N}$, let $\bar{\nu}_n = \frac{1}{n} \sum_{j=0}^{n-1} \bar{F}_*^j \left(\bar{\mu}|_{\bar{\Delta}_0} \right)$. Consider $\bar{\nu}_n|_{\bar{\Delta}_0}$, and let $\bar{\rho}_n = \frac{d\bar{\nu}_n}{d\bar{\mu}}$. Then $\bar{\rho}_n|_{\bar{\Delta}_0} = \frac{1}{n} \sum_{j \in \mathbb{N}} \bar{\rho}_n^j$, where $\bar{\rho}_n^j$ is the density of $\bar{F}_*^k \left(\bar{\mu}|_{\sigma^j} \right)$, and the σ^j s are the components of $F^{-i}\Delta_0 \cap \Delta_0$ for $i \leq n$. Let $\bar{x}, \bar{y} \in \Delta_{0,i}$, the i th rectangle on Δ_0 , and let $\bar{x}', \bar{y}' \in \sigma^j$ be such that $\bar{F}^i \bar{x}' = \bar{x}$ and $\bar{F}^i \bar{y}' = \bar{y}$. Then

$$\begin{aligned} \frac{\bar{\rho}_n^j(\bar{y})}{\bar{\rho}_n^j(\bar{x})} &= \frac{J\bar{F}^i(\bar{x}')}{J\bar{F}^i(\bar{y}')} \\ &= \prod_{k=1}^q \frac{J\bar{F}(\bar{F}^{i_k} - 1_{\bar{x}'})}{J\bar{F}(\bar{F}^{i_k} - 1_{\bar{y}'})}, \end{aligned}$$

where $i_1 < i_2 < \dots < i_q = i$ are the times when $\bar{F}^p \sigma^j \subset \bar{\Delta}_0$. Furthermore, we have that

$$\frac{J\bar{F}(\bar{F}^{i_k-1}\bar{x}')}{J\bar{F}(\bar{F}^{i_k-1}\bar{y}')} \leq \exp \left(C_1(i) \beta^{s(\bar{F}^{i_k}\bar{x}', \bar{F}^{i_k}\bar{y}')} \right) \leq \exp(C_1(i) \beta^{s(\bar{x}, \bar{y})}).$$

Thus, we have that $\bar{\rho}_n^j(\bar{y}) \leq \bar{\rho}_n^j(\bar{x}) \exp(C_1(i) \beta^{s(\bar{x}, \bar{y})})$; hence, it follows that $\bar{\rho}_n(\bar{y}) \leq \bar{\rho}_n(\bar{x}) \exp(C_1(i) \beta^{s(\bar{x}, \bar{y})})$. Letting $n \rightarrow \infty$, we have $\bar{\rho}(\bar{y}) \leq \bar{\rho}(\bar{x}) \exp(C_1(i) \beta^{s(\bar{x}, \bar{y})})$ for all $\bar{x}, \bar{y} \in \Delta_{0,i}$.

Following the proof of Theorem 2.3, we know that the sequence $\{\bar{\nu}_n\}_{n \in \mathbb{N}}$ has an accumulation point, which we call $\bar{\nu}$ on $\bar{\Delta}$ with $0 < \bar{\nu}(\bar{\Delta}) < \infty$ because of our assumption that $\sum_{i \in \mathbb{N}} R_i e^{C(i)} \varrho(i)$ is finite. Also by the proof of Theorem 2.3, we have some $c_0 > 0$ such that $c_0^{-1} \leq \bar{\rho} \leq c_0$ on $\bar{\Delta}$. Thus, we have that

$$|\bar{\rho}(\bar{x}) - \bar{\rho}(\bar{y})| \leq \left| \bar{\rho}|_{\bar{\Delta}_{l,i}} \right|_{\infty} \cdot \left| \frac{\bar{\rho}(\bar{x})}{\bar{\rho}(\bar{y})} - 1 \right| \leq c_0 C_1(i) \beta^{s(\bar{x}, \bar{y})}$$

whenever $\bar{x}, \bar{y} \in \bar{\Delta}_{l,i}$. ■

Decay of correlations for the factor system

The Perron-Frobenius operator P associated with the dynamical system $\overline{F} : \overline{\Delta} \rightarrow \overline{\Delta}$ with the reference measure $\overline{\mu}$ is defined as usual by

$$P(\overline{\varphi})(\overline{x}) = \sum_{\overline{y} : \overline{F}(\overline{y}) = \overline{x}} \frac{\overline{\varphi}(\overline{y})}{J\overline{F}(\overline{y})}.$$

In order to prove that the factor system $(\overline{F}, \overline{\nu})$ has exponential decay of correlations for functions in $\mathcal{F}$, we have to demonstrate three properties of the Perron-Frobenius operator, to which we shall simply refer hereafter as the **transfer operator**. We must show that

- it is a bounded operator with its spectrum is contained entirely within the unit disk;
- it can be approximated by a compact operator from $\mathcal{F}$ to $\mathcal{F}$; i.e., there is some compact operator $Q : \mathcal{F} \rightarrow \mathcal{F}$ such that $\|P^N - Q\| < \lambda^N$ for some $N \in \mathbb{N}$ and some $\lambda < 1$; and
- its only spectral point of modulus 1 is 1 and the corresponding eigenspace is one-dimensional.

The first two properties are proved by showing that P satisfies what is commonly referred to as a **Lasota-Yorke inequality**:

$$(2.13) \quad \|P^N \overline{\varphi}\|' \leq \lambda^N \|\overline{\varphi}\|' + K \|\overline{\varphi}\|_1,$$

for some $N \in \mathbb{N}$, some $\lambda < 1$, and some constant $K > 0$, all of which are independent of $\overline{\varphi}$. This is precisely the inequality found in [IM50] and used there to prove the first

two properties as well. The third property is a consequence of our assumption that (f, ν) is totally ergodic or $\gcd\{R(x) : x \in \Lambda\} = 1$. In other words, the third property will be true if the system (f, ν) is mixing.

Having derived (2.13), one could conceivably use the classical result of Ionescu Tulcea and Marinescu [IM50] to prove the first two desired properties; however, this author has found it a more difficult approach than the one in which one approximates P by a compact operator.

Recall that the function $\bar{F} : \bar{\Delta} \rightarrow \bar{\Delta}$ is one-to-one on the parts of $\bar{\Delta}$ which move up the tower under $\bar{F}$, but it certainly is not one-to-one on the parts of $\bar{\Delta}$ which get mapped onto $\bar{\Delta}_0$. Also, we have assumed that there is some $N \in \mathbb{N}$ such that $R_i \geq N$ for all $i \in \mathbb{N}$. This means that the first N levels of the tower $\bar{\Delta}$ are complete copies of Λ ; i.e., they all contain all copies of the Λ_i s; therefore, as we try to prove (2.13), we will have to consider how P behaves on the first N levels of $\bar{\Delta}$ separately from how it behaves on the levels above N . To prove (2.13), we shall require four estimates; two estimates for each of the norms $\|\cdot\|_h$ and $\|\cdot\|_\infty$ on each of these two pieces of $\bar{\Delta}$.

Estimate 1 For all $l \geq N$ and for every $\bar{\varphi} \in \mathcal{F}$, $\|(P^N \bar{\varphi})_{l,i}\|_\infty = e^{-\epsilon N} \|\bar{\varphi}_{l-N,i}\|_\infty$.

Proof: Fix l and i . Because $l \geq N$, $\bar{F}^N$ is one-to-one on $\bar{F}^{-N} \bar{\Delta}_{l,i}$, so we have

$$\begin{aligned} \|(P^N \bar{\varphi})_{l,i}\|_\infty &= |(P^N \bar{\varphi})_{l,i}|_\infty e^{-l\epsilon} d(i) \\ &= \left(\operatorname{ess\,sup}_{\bar{y} \in \bar{F}^{-N} \bar{\Delta}_{l,i}} |\bar{\varphi}(\bar{y})| e^{-(l-N)\epsilon} d(i) \right) e^{-N\epsilon} \\ &= \|\bar{\varphi}_{l-N,i}\|_\infty e^{-N\epsilon}, \end{aligned}$$

which is what we wanted to show. ■

Estimate 2 For all l with $0 \leq l < N$ and for all $\bar{\varphi} \in \mathcal{F}$, we have

$$\begin{aligned} \left\| (P^N \bar{\varphi})_{l,i} \right\|_{\infty} &\leq \sum_{br} \frac{e^{C_1(i)}}{\bar{\mu}(\bar{\Delta}_{l,i})} e^{-l\epsilon} d(i) \left| \bar{\varphi} \cdot 1_{\bar{F}^{-N} \bar{\Delta}_{l,i}} \right|_1 + \\ &\quad K e^{C_1(i)} \beta^N e^{-l\epsilon} \varrho(i) \left\| \bar{\varphi} \cdot 1_{\bar{F}^{-N} \bar{\Delta}_{l,i}} \right\|_h. \end{aligned}$$

Proof: Fix l and i . Then we have

$$(2.14) \quad \left\| (P^N \bar{\varphi})_{l,i} \right\|_{\infty} \leq \sum_{br} \left| \frac{1}{J \bar{F}^N} \cdot 1_{\bar{F}^{-N} \bar{\Delta}_{l,i}} \right|_{\infty} \cdot \left| \bar{\varphi} \cdot 1_{\bar{F}^{-N} \bar{\Delta}_{l,i}} \right|_{\infty} e^{-l\epsilon} d(i),$$

where $\sum_{br}$ means that we sum over all the inverse branches of $\bar{F}^{-N}$. We note now that

$$(2.15) \quad \left| \bar{\varphi} \cdot 1_{\bar{F}^{-N} \bar{\Delta}_{l,i}} \right|_{\infty} \leq \left| \frac{1}{\bar{\mu}(\bar{F}^{-N} \bar{\Delta}_{l,i})} \int_{\bar{F}^{-N} \bar{\Delta}_{l,i}} \bar{\varphi} d\bar{\mu} \right| + \operatorname{ess\,sup}_{\bar{y}_1, \bar{y}_2 \in \bar{F}^{-N} \bar{\Delta}_{l,i}} |\bar{\varphi}(\bar{y}_1) - \bar{\varphi}(\bar{y}_2)|.$$

By (2.15) we decompose (2.14) into the sum of two parts:

$$\begin{aligned} \left\| (P^N \bar{\varphi})_{l,i} \right\|_{\infty} &\leq \sum_{br} \left| \frac{1}{J \bar{F}^N} \cdot 1_{\bar{F}^{-N} \bar{\Delta}_{l,i}} \right|_{\infty} \cdot \left| \frac{1}{\bar{\mu}(\bar{F}^{-N} \bar{\Delta}_{l,i})} \int_{\bar{F}^{-N} \bar{\Delta}_{l,i}} \bar{\varphi} d\bar{\mu} \right| e^{-l\epsilon} d(i) \\ &\quad + \sum_{br} \left| \frac{1}{J \bar{F}^N} \cdot 1_{\bar{F}^{-N} \bar{\Delta}_{l,i}} \right|_{\infty} \left(\operatorname{ess\,sup}_{\bar{y}_1, \bar{y}_2 \in \bar{F}^{-N} \bar{\Delta}_{l,i}} |\bar{\varphi}(\bar{y}_1) - \bar{\varphi}(\bar{y}_2)| \right) e^{-l\epsilon} d(i). \end{aligned}$$

For convenience, let us call the former sum (Sum1) and the latter sum (Sum2). Then the distortion estimate

$$(2.16) \quad \left| \frac{1}{J \bar{F}^N} \cdot 1_{\bar{F}^{-N} \bar{\Delta}_{l,i}} \right|_{\infty} \leq e^{C_1(i)} \frac{\bar{\mu}(\bar{F}^{-N} \bar{\Delta}_{l,i})}{\bar{\mu}(\bar{\Delta}_{l,i})}$$

yields

$$(\text{Sum1}) \leq \sum_{br} \frac{e^{C_1(i)}}{\bar{\mu}(\bar{\Delta}_{l,i})} \left| \bar{\varphi} \cdot 1_{\bar{F}^{-N} \bar{\Delta}_{l,i}} \right|_1 e^{-l\epsilon} d(i)$$

$$\begin{aligned}
&\leq \sum_{br} e^{C_1(i)} \left| \bar{\varphi} \cdot 1_{\bar{F}^{-N} \bar{\Delta}_{l,i}} \right|_1 e^{-l\epsilon} \frac{1}{l(i)} d(i) \\
&\leq \sum_{br} e^{C_1(i)} \left| \bar{\varphi} \cdot 1_{\bar{F}^{-N} \bar{\Delta}_{l,i}} \right|_1 e^{-l\epsilon} \varrho(i).
\end{aligned}$$

To estimate (Sum2), let us use l_{br} to denote the level of the tower in which $\bar{F}^{-N} \bar{\Delta}_{l,i}$ in the branch in question, and let $\bar{\Delta}_{br}^* = \bar{\Delta}_{l',i'}^*$, where $\bar{\Delta}_{l',i'}^*$ is the element of the top level of the tower through which the branch in question passes. Then the distortion estimate

$$\left| \frac{1}{J\bar{F}^N} \cdot 1_{\bar{F}^{-N} \bar{\Delta}_{l,j}} \right|_\infty \leq e^{C_1(i)} \frac{\bar{\mu}(\bar{\Delta}_{br}^*)}{\bar{\mu}(\bar{\Delta}_{l,i})}$$

yields

$$\begin{aligned}
&\text{(Sum2)} \\
&\leq \left[\sum_{br} \left| \frac{1}{J\bar{F}^N} \cdot 1_{\bar{F}^{-N} \bar{\Delta}_{l,i}} \right|_\infty \left(\text{ess sup}_{\bar{y}_1, \bar{y}_2 \in \bar{F}^{-N} \bar{\Delta}_{l,i}} \frac{|\bar{\varphi}(\bar{y}_1) - \bar{\varphi}(\bar{y}_2)|}{\beta^{s(\bar{y}_1, \bar{y}_2)}} e^{-l_{br}\epsilon} \right) e^{l_{br}\epsilon} \right] \beta^N e^{-l\epsilon} d(i) \\
&\leq e^{C_1(i)} \beta^N \left\| \bar{\varphi} \cdot 1_{\bar{F}^{-N} \bar{\Delta}_{l,i}} \right\|_h \left(\frac{1}{\bar{\mu}(\bar{\Delta}_{l,i})} \sum_{br} \bar{\mu}(\bar{\Delta}_{br}^*) e^{l_{br}\epsilon} \right) e^{-l\epsilon} d(i) \\
&\leq K e^{C_1(i)} \beta^N \left\| \bar{\varphi} \cdot 1_{\bar{F}^{-N} \bar{\Delta}_{l,i}} \right\|_h e^{-l\epsilon} \varrho(i).
\end{aligned}$$

Combining (Sum1) and (Sum2), we have immediately the inequality which we sought to prove. ■

Estimate 3 For all $l \geq N$ and for all $\bar{\varphi} \in \mathcal{F}$, we have $\left\| (P^N \bar{\varphi})_{l,i} \right\|_h \leq \beta^N e^{-N\epsilon} \left\| \bar{\varphi}_{l,i} \right\|_h$.

Proof: As in Estimate 1, this estimate is very quick to prove since $\bar{F}$ is one-to-one on the first N levels of the tower. From the definitions of $\|\cdot\|_h$ and P , we have

$$\left\| (P^N \bar{\varphi})_{l,i} \right\|_h = \left(\text{ess sup}_{\bar{x}_1, \bar{x}_2 \in \bar{\Delta}_{l,i}} \frac{|(P^N \bar{\varphi})(\bar{x}_1) - (P^N \bar{\varphi})(\bar{x}_2)|}{\beta^s(\bar{x}_1, \bar{x}_2)} \right) e^{-l\epsilon} d(i)$$

$$\begin{aligned}
&= \left(\operatorname{ess\,sup}_{\bar{y}_1, \bar{y}_2 \in \bar{F}^{-N} \bar{\Delta}_{l,i}} \frac{|\bar{\varphi}(\bar{y}_1) - \bar{\varphi}(\bar{y}_2)|}{\beta^s(\bar{y}_1, \bar{y}_2)} e^{-(l-N)\epsilon} \right) \beta^N e^{-N\epsilon} d(i) \\
&= \left(\operatorname{ess\,sup}_{\bar{y}_1, \bar{y}_2 \in \bar{\Delta}_{l-N,i}} \frac{|\bar{\varphi}(\bar{y}_1) - \bar{\varphi}(\bar{y}_2)|}{\beta^s(\bar{y}_1, \bar{y}_2)} e^{-(l-N)\epsilon} \right) \beta^N e^{-N\epsilon} d(i) \\
&\leq \|\bar{\varphi}_{l-N,i}\|_h \beta^N e^{-N\epsilon},
\end{aligned}$$

which is what we wished to prove. ■

Estimate 4 For all l with $0 \leq l < N$ and for all $\bar{\varphi} \in \mathcal{F}$, we have

$$\begin{aligned}
\|(P^N \bar{\varphi})_{l,i}\|_h &\leq \sum_{br} \frac{C_1(i) e^{C_1(i)}}{\bar{\mu}(\bar{\Delta}_{l,i})} e^{-l\epsilon} d(i) \left| \bar{\varphi} \cdot 1_{\bar{F}^{-N} \bar{\Delta}_{l,i}} \right|_1 \\
&\quad + K C_1(i) e^{C_1(i)} \beta^N e^{-l\epsilon} \varrho(i) \left\| \bar{\varphi} \cdot 1_{\bar{F}^{-N} \bar{\Delta}_{l,i}} \right\|_h \\
&\quad + K e^{C_1(i)} \beta^N \varrho(i) \left\| \bar{\varphi} \cdot 1_{\bar{F}^{-N} \bar{\Delta}_{l,i}} \right\|_h,
\end{aligned}$$

Proof: Consider $\bar{x}_1, \bar{x}_2 \in \bar{\Delta}_{l,i}$. We write $\bar{y}_j = \bar{F}^{-N} \bar{x}_j$ for $j = 1, 2$ for the preimages of $\bar{x}_1$ and $\bar{x}_2$ in a given branch of $\bar{F}^{-N}$. In our estimates here, we shall restrict our attention to one branch of $\bar{F}^{-N}$, show that the estimates are the same for all branches, and then sum over all the branches to obtain the final estimate.

We have from the definition of $\|\cdot\|_h$ that

$$\begin{aligned}
\|(P^N \bar{\varphi})_{l,i}\|_h &= \operatorname{ess\,sup}_{\bar{x}_1, \bar{x}_2 \in \bar{\Delta}_{l,i}} \left(\left| \sum_{br} \left(\frac{\bar{\varphi}(\bar{F}^{-N} \bar{x}_1)}{J \bar{F}^N(\bar{F}^{-N} \bar{x}_1)} - \frac{\bar{\varphi}(\bar{F}^{-N} \bar{x}_2)}{J \bar{F}^N(\bar{F}^{-N} \bar{x}_2)} \right) \right| \right. \\
(2.17) \quad &\quad \cdot \beta^{-s(\bar{F}^{-N} \bar{x}_1, \bar{F}^{-N} \bar{x}_2)} \Big) e^{-l\epsilon} d(i) \\
&\leq \sum_{br} \operatorname{ess\,sup}_{\bar{y}_1, \bar{y}_2 \in \bar{F}^{-N} \bar{\Delta}_{l,i}} \left(\left| \frac{\bar{\varphi}(\bar{y}_1)}{J \bar{F}^N(\bar{y}_1)} - \frac{\bar{\varphi}(\bar{y}_2)}{J \bar{F}^N(\bar{y}_2)} \right| \beta^{-s(\bar{y}_1, \bar{y}_2)} \right) \beta^N d(i).
\end{aligned}$$

For each of the inverse branches we have that

$$\left| \frac{\bar{\varphi}(\bar{y}_1)}{J \bar{F}^N(\bar{y}_1)} - \frac{\bar{\varphi}(\bar{y}_2)}{J \bar{F}^N(\bar{y}_2)} \right| \leq \frac{|\bar{\varphi}(\bar{y}_1) - \bar{\varphi}(\bar{y}_2)|}{J \bar{F}^N(\bar{y}_1)} + |\bar{\varphi}(\bar{y}_2)| \cdot \left| \frac{1}{J \bar{F}^N(\bar{y}_1)} - \frac{1}{J \bar{F}^N(\bar{y}_2)} \right|$$

$$\leq \frac{|\bar{\varphi}(\bar{y}_1) - \bar{\varphi}(\bar{y}_2)|}{J\bar{F}^N(\bar{y}_1)} + \frac{|\bar{\varphi}(\bar{y}_2)|}{J\bar{F}^N(\bar{y}_2)} C_1(i) \beta^s \left(\bar{F}^{-(l_{br}-N)}(\bar{y}_1), \bar{F}^{-(l_{br}-N)}(\bar{y}_2) \right),$$

where l is as before the level on which y_j lies in its branch. Combining this with (2.17) we have the following:

$$\begin{aligned} \left\| (P^N \bar{\varphi})_{l,i} \right\|_h &\leq \sum_{br} \operatorname{ess\,sup}_{\bar{y}_1, \bar{y}_2 \in \bar{F}^{-N} \bar{\Delta}_{l,i}} \left(\left(\frac{|\bar{\varphi}(\bar{y}_1) - \bar{\varphi}(\bar{y}_2)|}{J\bar{F}^N(\bar{y}_1)} + \frac{|\bar{\varphi}(\bar{y}_2)|}{J\bar{F}^N(\bar{y}_2)} C_1(i) \beta^s \left(\bar{F}^{-(l_{br}-N)}(\bar{y}_1), \bar{F}^{-(l_{br}-N)}(\bar{y}_2) \right) \right) \beta^{-s(\bar{y}_1, \bar{y}_2)} \right) \beta^N d(i) \\ &\leq \sum_{br} \left| \frac{1}{J\bar{F}^N} \cdot 1_{\bar{F}^{-N} \bar{\Delta}_{l,i}} \right|_\infty \cdot \left(\operatorname{ess\,sup}_{\bar{y}_1, \bar{y}_2 \in \bar{F}^{-N} \bar{\Delta}_{l,i}} \frac{|\bar{\varphi}(\bar{y}_1) - \bar{\varphi}(\bar{y}_2)|}{\beta^{s(\bar{y}_1, \bar{y}_2)}} \right) \beta^N d(i) \\ &\quad + \sum_{br} C_1(i) \sum_{br} \left| \frac{1}{J\bar{F}^N} \cdot 1_{\bar{F}^{-N} \bar{\Delta}_{l,i}} \right|_\infty \cdot \left| \bar{\varphi} \cdot 1_{\bar{F}^{-N} \bar{\Delta}_{l,i}} \right|_\infty \beta^N d(i). \end{aligned}$$

Note that we used the fact that

$$\beta^s \left(\bar{F}^{-(l_{br}-N)} \bar{y}_1, \bar{F}^{-(l_{br}-N)} \bar{y}_2 \right) \leq \beta^{s(\bar{y}_1, \bar{y}_2)}$$

because $s \left(\bar{F}^{-(l_{br}-N)} \bar{y}_1, \bar{F}^{-(l_{br}-N)} \bar{y}_2 \right) \geq s(\bar{y}_1, \bar{y}_2)$. We have already seen both of these sums in Estimate 2, and so we simply now follow our work there from which we get

$$\begin{aligned} \left\| (P^N \bar{\varphi})_{l,i} \right\|_h &\leq \sum_{br} \frac{C_1(i) e^{C_1(i)}}{\bar{\mu}(\bar{\Delta}_{l,i})} e^{-l_\epsilon} d(i) \left| \bar{\varphi} \cdot 1_{\bar{F}^{-N} \bar{\Delta}_{l,i}} \right|_1 \\ &\quad + K C_1(i) e^{C_1(i)} \beta^N e^{-l_\epsilon} \varrho(i) \left\| \bar{\varphi} \cdot 1_{\bar{F}^{-N} \bar{\Delta}_{l,i}} \right\|_h \\ &\quad + K e^{C_1(i)} \beta^N \varrho(i) \left\| \bar{\varphi} \cdot 1_{\bar{F}^{-N} \bar{\Delta}_{l,i}} \right\|_h, \end{aligned}$$

and this is the inequality we sought. ■

Combination of Estimates 1 through 4. We now derive our Lasota-Yorke inequality (2.13). Combining the above estimates and using the definitions of $\|\bar{\varphi}_{l,i}\|$ we

have

$$\begin{aligned}
\|P^N \bar{\varphi}\|' &\leq e^{-N\epsilon} \|\bar{\varphi}\|_\infty + K\beta^N \left(\sum_{i \in \mathbb{N}} e^{C_1(i)} \varrho(i) \right) \|\bar{\varphi}\|_h \\
&\quad + \beta^N e^{-N\epsilon} \|\bar{\varphi}\|_h + 2K\beta^N \left(\sum_{i \in \mathbb{N}} C_1(i) e^{C_1(i)} \varrho(i) \right) \|\bar{\varphi}\|_h \\
&\quad + \left(\sum_{i \in \mathbb{N}} \frac{e^{C_1(i)} d(i)}{\bar{\mu}(\bar{\Delta}_{l,i})} \right) |\bar{\varphi}|_1 + \left(\sum_{i \in \mathbb{N}} \frac{C_1(i) e^{C_1(i)} d(i)}{\bar{\mu}(\bar{\Delta}_{l,i})} \right) |\bar{\varphi}|_1 \\
&\leq \left((1 + \beta^N) e^{-N\epsilon} + 3K\beta^N \left(\sum_{i \in \mathbb{N}} C_1(i) e^{C_1(i)} \varrho(i) \right) \right) \|\bar{\varphi}\|' \\
&\quad + 2 \left(\sum_{i \in \mathbb{N}} \frac{C_1(i) e^{C_1(i)} d(i)}{\bar{\mu}(\bar{\Delta}_{l,i})} \right) |\bar{\varphi}|_1 \\
&\leq \left((1 + \beta^N) e^{-N\epsilon} + 3K\beta^N \left(\sum_{i \in \mathbb{N}} C_1(i) e^{C_1(i)} \varrho(i) \right) \right) \|\bar{\varphi}\|' \\
&\quad + 2 \left(\sum_{i \in \mathbb{N}} C_1(i) e^{C_1(i)} \varrho(i) \right) |\bar{\varphi}|_1.
\end{aligned}$$

Choosing N large enough so that it satisfies (2.12), we have some $\lambda < 1$ so that

$$(2.13) \quad \|P^N \bar{\varphi}\|' \leq \lambda^N \|\bar{\varphi}\|' + K' |\bar{\varphi}|_1$$

for some $K' > 0$ since the sums in the last two lines are finite by Condition (E) on p. 18. This is the Lasota-Yorke inequality (2.13) which we sought to verify.

Spectral radius of P . In order to make use of the approximation of P by a compact operator, a fact which we prove in the next section, we require that the spectrum of P is contained in the unit disk. By (2.13) we know that, for all $k \in \mathbb{N}$,

$$\|P^{kN} \bar{\varphi}\|' = \|P^N (P^{(k-1)N} \bar{\varphi})\|' \leq \lambda^N \|P^{(k-1)N} \bar{\varphi}\|' + K' |P^{(k-1)N} \bar{\varphi}|_1.$$

It is a basic property of the Perron-Frobenius operator that $|P\bar{\varphi}|_1 = |\bar{\varphi}|_1$ for any real-valued function $\bar{\varphi}$. (See [Bro96] for an excellent discussion of important properties of

P .) By this property and by induction on the above inequality we have

$$\|P^{kN}\bar{\varphi}\|' \leq \lambda^{kN} \|\bar{\varphi}\|' + \left(K' \sum_{j=0}^{k-1} \lambda^{jN} \right) |\bar{\varphi}|_1$$

for all $k \in \mathbb{N}$; therefore, we obtain

$$\|P^{kN}\bar{\varphi}\| = \|P^{kN}\bar{\varphi}\|' + |P^{kN}\bar{\varphi}|_1 \leq \left(1 + \lambda^{kN} + K' C_0 \sum_{j \in \mathbb{N}} \lambda^{jN} \right) \|\bar{\varphi}\|.$$

Thus, $\|P^{kN}\bar{\varphi}\| \leq \mathcal{K} \|\bar{\varphi}\|$ for all $k \in \mathbb{N}$, for some $\mathcal{K} > 0$, and for all $\bar{\varphi} \in \mathcal{F}$. Let $n \in \mathbb{N}$.

Then there exists $i \in \mathbb{N}$ such that $0 \leq i < N$ and such that $n = kN + i$; therefore,

$$\|P^n \bar{\varphi}\| \leq \mathcal{K} \|P^i \bar{\varphi}\| \leq \mathcal{K} \left(\sup_{0 \leq i < N} \|P^i\| \right) \|\bar{\varphi}\|.$$

Since this is true for all $n \in \mathbb{N}$ and for all $\bar{\varphi} \in \mathcal{F}$, the spectrum of P lies entirely within the closed unit disk.

Approximation of P by a compact operator

In this section, we shall construct a finite-rank operator and show that this operator

is close to P in the sense used in [DS58]; i.e., that there is $\tau_0 < 1$ and $m \in \mathbb{N}$ such that

$$\|P^m - Q\| < \tau^m. \text{ Let } \mathcal{M} \text{ denote our original partition of } \bar{\Delta} \text{ into the } \bar{\Delta}_{l,i} \text{ components.}$$

Recall that each $\bar{\Delta}_l$ is partitioned into countably many pieces. For $k \in \mathbb{N}$, let $\mathcal{P}_k$ be

some finite collection of $\bar{\Delta}_{l,i}$ s for $l \leq k$ such that $\sum_{\bar{\Delta}_{l,i} \notin \mathcal{P}_k} \bar{\mu}(\bar{\Delta}_{l,i}) e^{l\epsilon} < \epsilon_k$, where $\epsilon_k \rightarrow 0$

as $k \rightarrow \infty$. We can do this for the following reasons. The partition on each level $\bar{\Delta}_l$

of the tower $\bar{\Delta}$ is countable, and $\bar{\mu}(\bar{\Delta}_l)$ is finite. On the first k levels of the tower,

we include in $\mathcal{P}_k$ sufficiently many of the $\bar{\Delta}_{l,i}$ s so that what remains has measure as

small as we wish. For the levels above k , we know that their total measure is not

more than $C_0 \theta_0^k$.

For convenience of notation, we will use $\mathcal{P}_k$ for both the finite collection and the

union of its elements. Let $\bar{\varphi}^{\leq k} = \bar{\varphi} \cdot 1_{\mathcal{P}_k}$ and $\bar{\varphi}^{> k} = \bar{\varphi} - \bar{\varphi}^{\leq k}$. For $\bar{\varphi} : \bar{\Delta} \rightarrow \mathbb{R}$, we shall

let $E_N(\bar{\varphi})$ denote the conditional expectation of $\bar{\varphi}$ with respect to the partition $\mathcal{M}$ and the reference measure $\bar{\mu}$. Let $Q_k : \mathcal{F} \rightarrow \mathcal{F}$ be defined by $Q_k(\bar{\varphi}) = P^N(E_N(\bar{\varphi}^{\leq k}))$.

Because the number of $\bar{\Delta}_{l,i}$ s in $\mathcal{P}_k$ is finite, Q_k is clearly a finite-rank operator.

Let $\bar{\psi} = (\bar{\varphi} - E_N(\bar{\varphi}))^{\leq k}$. Then

$$E_N(\bar{\psi}) = E_N(\bar{\varphi} \cdot 1_{\mathcal{P}_k}) - E_N(E_N(\bar{\varphi} \cdot 1_{\mathcal{P}_k})) = 0.$$

Note that

$$\begin{aligned} (P^N - Q_k)(\bar{\varphi}) &= P^N(\bar{\varphi}^{\leq k} + \bar{\varphi}^{>k}) - P^N(E_N(\bar{\varphi}^{\leq k})) \\ &= P^N(\bar{\varphi}^{\leq k} - E_N(\bar{\varphi}^{\leq k})) + P^N(\bar{\varphi}^{>k}) \\ &= P^N(\bar{\psi}) + P^N(\bar{\varphi}^{>k}). \end{aligned}$$

In order to estimate $\|(P^N - Q_k)(\bar{\varphi})\|$, we must break this down into the four pieces corresponding to those in Estimates 1 through 4. In fact, we will encounter here nearly those very same estimates; however, these estimates will be easier since certain terms which appear in Estimates 1 through 4 will not be present here because $E_N(\bar{\psi}) = 0$.

Estimate 5 For $l \geq N$, we have $\|(P^N \bar{\psi})_{l,i}\|_{\infty} \leq \beta^N e^{-N\epsilon} \|\bar{\varphi}_{l-N,i}\|_h$.

Proof: Since $l \geq N$, $\bar{F}^N$ is one-to-one on $\bar{F}^{-N} \bar{\Delta}_{l,i}$, from the definition of $\|\cdot\|_{\infty}$

we get

$$\begin{aligned} &\|(P^N \bar{\psi})_{l,i}\|_{\infty} \\ &= \left| \bar{\psi} \cdot 1_{\bar{F}^{-N} \bar{\Delta}_{l,i}} \right|_{\infty} e^{-l\epsilon} d(i) \\ &\leq \left(\left| \frac{1}{\bar{\mu}(\bar{F}^{-N} \bar{\Delta}_{l,i})} \int_{\bar{F}^{-N} \bar{\Delta}_{l,i}} \bar{\psi} d\bar{\mu} \right| + \operatorname{ess\,sup}_{\bar{y}_1, \bar{y}_2 \in \bar{F}^{-N} \bar{\Delta}_{l,i}} |\bar{\psi}(\bar{y}_1) - \bar{\psi}(\bar{y}_2)| \right) e^{-l\epsilon} d(i). \end{aligned}$$

By the definition of $\bar{\psi}$, the integral above is zero. Furthermore, we also note that

$$\begin{aligned} |\bar{\psi}(\bar{y}_1) - \bar{\psi}(\bar{y}_2)| &= |\bar{\varphi}(\bar{y}_1) - E_N(\bar{\varphi})(\bar{y}_1) - \bar{\varphi}(\bar{y}_2) + E_N(\bar{\varphi})(\bar{y}_2)| \\ &= |\bar{\varphi}(\bar{y}_1) - \bar{\varphi}(\bar{y}_2)| \end{aligned}$$

because $E_N(\bar{\varphi})(\bar{y}_1) = E_N(\bar{\varphi})(\bar{y}_2)$ since both $\bar{y}_1$ and $\bar{y}_2$ are in the same branch of $\bar{F}^{-N}$ and, therefore, in the same element of the partition $\mathcal{P}_k$. Thus, we have

$$\begin{aligned} \left\| (P^N \bar{\psi})_{l,i} \right\|_{\infty} &\leq \operatorname{ess\,sup}_{\bar{y}_1, \bar{y}_2 \in \bar{F}^{-N} \bar{\Delta}_{l,i}} |\bar{\varphi}(\bar{y}_1) - \bar{\varphi}(\bar{y}_2)| e^{-l\epsilon} d(i) \\ &\leq \operatorname{ess\,sup}_{\bar{y}_1, \bar{y}_2 \in \bar{\Delta}_{l-N,i}} \frac{|\bar{\varphi}(\bar{y}_1) - \bar{\varphi}(\bar{y}_2)|}{\beta^{s(\bar{y}_1, \bar{y}_2)}} e^{-(l-N)\epsilon} e^{-N\epsilon} \beta^N d(i) \\ &\leq \left\| \bar{\varphi}_{l-N,i} \right\|_h e^{-N\epsilon} \beta^N, \end{aligned}$$

and this completes this estimate. $\blacksquare$

Estimate 6 For $l \geq N$, we have $\left\| (P^N \bar{\psi})_{l,i} \right\|_h \leq \left\| \bar{\varphi}_{l-N,i} \right\|_h \beta^N e^{-N\epsilon}$.

Proof: From the definition of $\|\cdot\|_h$ we get

$$\begin{aligned} \left\| (P^N \bar{\psi})_{l,i} \right\|_h &= \left(\operatorname{ess\,sup}_{\bar{x}_1, \bar{x}_2 \in \bar{\Delta}_{l,i}} \frac{|(P^N \bar{\psi})(\bar{x}_1) - (P^N \bar{\psi})(\bar{x}_2)|}{\beta^{s(\bar{x}_1, \bar{x}_2)}} \right) e^{-l\epsilon} d(i) \\ &= \left(\operatorname{ess\,sup}_{\bar{y}_1, \bar{y}_2 \in \bar{F}^{-N} \bar{\Delta}_{l,i}} \frac{|\bar{\psi}(\bar{y}_1) - \bar{\psi}(\bar{y}_2)|}{\beta^{s(\bar{y}_1, \bar{y}_2)}} e^{-(l-N)\epsilon} \right) e^{-N\epsilon} \beta^N d(i) \\ &\leq \left\| \bar{\varphi}_{l-N,i} \right\|_h \beta^N e^{-N\epsilon}, \end{aligned}$$

which is what we wished to prove. $\blacksquare$

Estimate 7 For $l < N$, we have $\left\| (P^N \bar{\psi})_l \right\|_{\infty} \leq K \beta^N e^{C_1(i)} \varrho(i) \left\| \bar{\varphi} \cdot 1_{\bar{F}^{-N} \bar{\Delta}_{l,i}} \right\|_h$.

Proof: All of the hard work for this estimate was done in Estimate 2, but here we have only the latter term from Estimate 2 since $\left| \bar{\psi} \cdot 1_{\bar{F}^{-N} \bar{\Delta}_{l,i}} \right|_1 = 0$. Therefore, the estimate we seek follows immediately. ■

Estimate 8 For $l < N$, we have

$$\begin{aligned} \left\| (P^N \bar{\psi})_{l,i} \right\|_h &\leq K C_1(i) e^{C_1(i)} \beta^N e^{-l\epsilon} \varrho(i) \left\| \bar{\varphi} \cdot 1_{\bar{F}^{-N} \bar{\Delta}_{l,i}} \right\|_h + \\ &\quad K e^{C_1(i)} \beta^N \varrho(i) \left\| \bar{\varphi} \cdot 1_{\bar{F}^{-N} \bar{\Delta}_{l,j}} \right\|_h. \end{aligned}$$

Proof: Analogous to Estimate 7, this argument mimics the proof of Estimate 4 except that again $\left| \bar{\psi} \cdot 1_{\bar{F}^{-N} \bar{\Delta}_{l,i}} \right|_1 = 0$. Thus, our estimate follows immediately. ■

Combination of Estimates 5 through 8 and estimate of $\|P^N(\bar{\varphi}^{>k})\|$. Combining Estimates 5 through 8 and using the definitions of $\|\cdot\|_\infty$ and $\|\cdot\|_h$ we have

$$\begin{aligned} \|P^N \bar{\psi}\| &= \|P^N \bar{\psi}\|' + |P^N \bar{\psi}|_1 = \|P^N \bar{\psi}\|' \\ &\leq 2\beta^N e^{-N\epsilon} \|\bar{\varphi}\|_h + K\beta^N \|\bar{\varphi}\|_h \sum_{i \in \mathbb{N}} e^{C_1(i)} \varrho(i) \\ &\quad + K\beta^N \|\bar{\varphi}\|_h \sum_{i \in \mathbb{N}} C_1(i) e^{C_1(i)} \varrho(i) + K\beta^N \|\bar{\varphi}\|_h \sum_{i \in \mathbb{N}} e^{C_1(i)} \varrho(i) \\ &\leq \beta^N \left(2 + 3K \sum_{i \in \mathbb{N}} C_1(i) e^{C_1(i)} \varrho(i) \right) \|\bar{\varphi}\|. \end{aligned}$$

Lastly, we show the most difficult calculation for $\|P^N(\bar{\varphi}^{>k})\|$, and the others are nearly identical. We have the following:

$$\begin{aligned} \left\| (P^N(\bar{\varphi}^{>k}))_{l,i} \right\|_\infty &\leq \sum_{\substack{br \\ > k}} \left| \frac{1}{J\bar{F}^N} \cdot 1_{\bar{F}^{-N} \bar{\Delta}_{l,i}} \right|_\infty \left| \bar{\varphi} \cdot 1_{\bar{F}^{-N} \bar{\Delta}_{l,i}} \right|_\infty d(i) \|\bar{\varphi}\|_\infty \\ &\quad + \sum_{\bar{\Delta}_{l,i} \notin \mathcal{P}_k : l \leq k} \left| \frac{1}{J\bar{F}^N} \cdot 1_{\bar{F}^{-N} \bar{\Delta}_{l,i}} \right|_\infty \left| \bar{\varphi} \cdot 1_{\bar{F}^{-N} \bar{\Delta}_{l,i}} \right|_\infty d(i) \|\bar{\varphi}\|_\infty \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{\substack{br \\ > k}} e^{C_1(i)} d(i) \frac{\bar{\mu}(\bar{F}^{-N} \bar{\Delta}_{l,i})}{\bar{\mu}(\bar{\Delta}_{l,i})} e^{l_{br} \epsilon} \|\bar{\varphi}\|_{\infty} \\
&\quad + \sum_{\bar{\Delta}_{l,i} \notin \mathcal{P}_k : l \leq k} \left| \frac{1}{J \bar{F}^N} \cdot 1_{\bar{F}^{-N} \bar{\Delta}_{l,i}} \right|_{\infty} \left| \bar{\varphi} \cdot 1_{\bar{F}^{-N} \bar{\Delta}_{l,i}} \right|_{\infty} d(i) \|\bar{\varphi}\|_{\infty} \\
&\leq e^{C_1(i)} d(i) \frac{\varrho(i)}{s(i)} \|\bar{\varphi}\|_{\infty} \sum_{l > k} \bar{\mu}(\bar{\Delta}_l) e^{l \epsilon} \\
&\quad + e^{C_1(i)} d(i) \frac{\varrho(i)}{l(i)} \|\bar{\varphi}\|_{\infty} \sum_{\bar{\Delta}_{l,i} \notin \mathcal{P}_k : l \leq k} \bar{\mu}(\bar{\Delta}_{l,i}) e^{l \epsilon} \\
&\leq \left(e^{C_1(i)} d(i) \frac{\varrho(i)}{l(i)} \right) \|\bar{\varphi}\|' (\epsilon'_k + \epsilon_k),
\end{aligned}$$

for some $\epsilon'_k \rightarrow 0$ as $k \rightarrow \infty$. Also

$$|P^N(\bar{\varphi}^{>k})|_1 = |\bar{\varphi}^{>k}|_1 = \sum_{\bar{\Delta}_{l,i} \notin \mathcal{P}_k} |\bar{\varphi}_{l,i}^{>k}|_1 < \epsilon''_k |\bar{\varphi}|_1$$

for some $\epsilon''_k \rightarrow 0$ as $k \rightarrow \infty$.

We may then choose $k \in \mathbb{N}$ and $N \in \mathbb{N}$ sufficiently large so that

$$\mathcal{Z} := (\epsilon_k + \epsilon'_k + \epsilon''_k) \sum_{i \in \mathbb{N}} e^{C_1(i)} \varrho^2(i) + \beta^N \left(2 + 3K \sum_{i \in \mathbb{N}} C_1(i) e^{C_1(i)} d(i) \right) < 1.$$

Let τ_0 be such that $\mathcal{Z} < \tau_0^N < 1$. Then $\|(P^N - Q_k) \bar{\varphi}\| \leq \tau_0^N \|\bar{\varphi}\|$. We now apply the following proposition from Dunford and Schwartz. (See pp. 709-711 [DS58] for the proof.)

Proposition 2.6 *If P is a bounded linear operator, if there is some compact operator Q , if there is some $\tau_0 < 1$, and if there is some $N \in \mathbb{N}$ such that $\|P^N - Q\| < \tau_0^N$, then any spectral point λ such that $|\lambda|^N > \|P^N - Q\|$ is isolated and its eigenspace is finite-dimensional.*

Thus, P is quasi-compact, and our only task now is to isolate 1 as the only spectral point on the unit circle.

Showing that $(\overline{F}, \overline{\nu})$ is mixing

We have established that the Perron-Frobenius operator $P : \mathcal{F} \rightarrow \mathcal{F}$ is quasi-compact; i.e., that all spectral points lie in the closed unit disk and that there is some disk of radius $\mathfrak{r} < 1$ such that all spectral points outside this small disk are isolated. When this happens, it is often said that P has a gap in its spectrum. In order to prove that $(\overline{F}, \overline{\nu})$ is mixing, we must show that 1 is the only spectral point of modulus 1. Young proves in [You98] that $(\overline{F}, \overline{\nu})$ is exact, and her proof works here unchanged since this part of the proof does not rely upon the behavior of $C(i)$; thus, we refer the reader there for the complete very brief proof. It is a basic fact that exact systems are mixing.

Decay of correlations for $(\overline{F}, \overline{\nu})$

It is worth noting that the mixing property of $(\overline{F}, \overline{\nu})$ along with the quasicompacity of P implies directly that 1 is the only spectral point of P on the unit circle (and it is of course an eigenvalue since it is isolated) and its eigenspace is one-dimensional.

To see this, let $\phi \in L^1(\overline{\mu})$ and $\psi \in L^\infty(\overline{\mu})$, and suppose that $P\phi = \phi$. Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \int \psi(P^n \phi) d\overline{\mu} &= \lim_{n \rightarrow \infty} \int (\psi \circ \overline{F}^n) \phi d\overline{\mu} \\ &= \lim_{n \rightarrow \infty} \int (\psi \circ \overline{F}^n) \frac{\phi}{(\overline{\rho})} d\overline{\nu} \\ &= \int \psi d\overline{\nu} \int \frac{\phi}{(\overline{\rho})} d\overline{\nu} \\ &= \int \psi \left(\overline{\rho} \int \phi d\overline{\mu} \right) d\overline{\mu}. \end{aligned}$$

Thus, $\sigma^n \phi = P^n \phi$ converges pointwise to $\bar{\rho} \int \phi d\bar{\mu}$, which implies that $\sigma = 1$ and that the eigenspace of σ is one-dimensional.

We will show explicitly at the end of this chapter how this puts an exponentially decaying bound on the correlation functions of $(\bar{F}, \bar{\nu})$ as well as for (F, ν) , which is the goal of our proof.

Decay of correlations for the original system

In this section we show that we may compare the correlation functions, which we derived in the previous sections for the factor system, to the correlation functions for the original dynamical system (f, ν) . We have so far a Markov system $F : (\Delta, \tilde{\nu}) \rightarrow (\Delta, \tilde{\nu})$ over the dynamical system $f : (M, \nu) \rightarrow (M, \nu)$ where ν is an SRB measure. We note also that we have a projection $\pi : \Delta \rightarrow M$ such that $\nu = \frac{\pi_* \tilde{\nu}}{\tilde{\nu}(\Delta)}$. We also have the factor system $\bar{F} : (\bar{\Delta}, \bar{\nu}) \rightarrow (\bar{\Delta}, \bar{\nu})$ and the projection $\bar{\pi} : \Delta \rightarrow \bar{\Delta}$ such that $\bar{\pi}_* \tilde{\nu} = \bar{\nu}$. For $\varphi : M \rightarrow \mathbb{C}$ we let $\tilde{\varphi} := \varphi \circ \pi : \Delta \rightarrow \mathbb{C}$ denote the lift of φ to the tower Δ . Recall that π is simply an identification between the l th level of the tower and $f^l \Lambda$. Let $\eta > 0$ be as we previously defined it. We defined our space of observables $\mathfrak{H}_\eta$ earlier.

For convenience of notation, let us use $D_n(\varphi, \psi; \nu)$ to denote the n th correlation function with respect to the measure ν :

$$D_n(\varphi, \psi; \nu) = \int \varphi(\psi \circ f^n) d\nu - \int \varphi d\nu \int \psi d\nu$$

and the analogous definition of $D_n(\tilde{\varphi}, \tilde{\psi}; \tilde{\nu})$, the n th correlation function with respect to the measure $\tilde{\nu}$. Note that $D_n(\varphi, \psi; \nu)$ is with respect to the map $f : M \rightarrow M$, but

that $D_n(\tilde{\varphi}, \tilde{\psi}; \tilde{\nu})$ is with respect to the map $F : \Delta \rightarrow \Delta$. It is obvious by the Change of Variables Theorem that

$$(2.18) \quad D_n(\varphi, \psi; \nu) = D_n(\tilde{\varphi}, \tilde{\psi}; \tilde{\nu}).$$

It is our goal in the remainder of this chapter to show that $D_n(\tilde{\varphi}, \tilde{\psi}; \tilde{\nu})$ can be approximated arbitrarily closely by quantities involving things related to the system $(\overline{F}, \overline{\Delta}, \overline{\nu})$ which we know already has exponential decay of correlations for functions in $\mathcal{F}$. Because of (2.18), we shall show that $D_n(\tilde{\varphi}, \tilde{\psi}; \tilde{\nu})$ can be approximated well by the correlation functions of the system $(\overline{F}, \overline{\Delta}, \overline{\nu})$. We show in the last two pages how the observables $\varphi : M \rightarrow \mathbb{R}$ in $\mathfrak{H}_\eta$ are related to observables in $\mathcal{F}$ for the factor system $(\overline{F}, \overline{\Delta}, \overline{\nu})$.

We state now a lemma which will be of much use later. We shall fix k later. In what follows, let $\Delta(i)$ denote the part of Δ sitting over Λ_i .

Lemma 2.7 *Let $x \in \Delta(i)$. Then $\text{diam}(\pi F^k \mathcal{M}_{2k}(x)) \leq 2C(i)\alpha^k$.*

Proof: Let $y_1, y_2 \in \mathcal{M}_{2k}(x) \cap \Delta(i)$. Then there exists $\hat{y} \in \gamma^u(y_1) \cap \gamma^s(y_2)$. Suppose without loss of generality that $\mathcal{M}_{2k}(x) \cap \Delta(i) \subset \Delta_l$. Then $\pi F^{-l} \hat{y}, \pi F^{-l} y_2 \in \Lambda_i$, and they both lie in the same γ^s -leaf. By (P3) we have

$$d(\pi F^k \hat{y}, \pi F^k y_2) \leq C(i)\alpha^{l+k} \leq C(i)\alpha^k.$$

Similarly, $\pi F^{-l} \hat{y}, \pi F^{-l} y_1 \in \Lambda_i$, and they both lie in the same γ^u -leaf. By (P4)(a) we have

$$d(\pi F^k \hat{y}, \pi F^k y_1) \leq C(i)\alpha^{s(F^k \hat{y}, F^k y_1) - (l+k)} \leq C(i)\alpha^k.$$

Thus, we have that $d(\pi F^k y_1, \pi F^k y_2) \leq 2C(i)\alpha^k$. ■

Estimate 9 Define $\bar{\psi}_k$ on Δ by $\bar{\psi}_k|_A \equiv \inf\{\tilde{\psi}(x) : x \in F^k A\}$ for every $A \in \mathcal{M}_{2k}$.

Then

$$\left| D_{n-k}(\tilde{\varphi}, \tilde{\psi} \circ F^k; \tilde{\nu}) - D_{n-k}(\tilde{\varphi}, \bar{\psi}_k; \tilde{\nu}) \right| \leq C'(\varphi, \psi) \alpha^{k\eta}$$

for some $C'(\varphi, \psi) > 0$.

Proof: Note that

$$\begin{aligned} & \left| D_{n-k}(\tilde{\varphi}, \tilde{\psi} \circ F^k; \tilde{\nu}) - D_{n-k}(\tilde{\varphi}, \bar{\psi}_k; \tilde{\nu}) \right| \\ &= \left| \int \tilde{\varphi}(\tilde{\psi} \circ F^k \circ F^{n-k}) d\tilde{\nu} - \int \tilde{\varphi} d\tilde{\nu} \int \tilde{\psi} \circ F^k d\tilde{\nu} \right. \\ & \quad \left. - \int \tilde{\varphi}(\bar{\psi}_k \circ F^{n-k}) d\tilde{\nu} + \int \tilde{\varphi} d\tilde{\nu} \int \bar{\psi}_k d\tilde{\nu} \right| \\ &\leq \left| \int (\tilde{\psi} \circ F^k - \bar{\psi}_k) \circ F^{n-k} \tilde{\varphi} d\tilde{\nu} \right| + \left| \int (\tilde{\psi} \circ F^k - \bar{\psi}_k) d\tilde{\nu} \cdot \int \tilde{\varphi} d\tilde{\nu} \right| \end{aligned}$$

It follows from Lemma 2.7 that

$$\begin{aligned} & \left| \int (\tilde{\psi} \circ F^k - \bar{\psi}_k) \circ F^{n-k} \tilde{\varphi} d\tilde{\nu} \right| + \left| \int (\tilde{\psi} \circ F^k - \bar{\psi}_k) d\tilde{\nu} \cdot \int \tilde{\varphi} d\tilde{\nu} \right| \\ &= \left| \sum_{i \in \mathbb{N}} \int_{\Lambda_i} (\tilde{\psi} \circ F^k - \bar{\psi}_k) \circ F^{n-k} \tilde{\varphi} d\tilde{\nu} \right| + \left| \sum_{i \in \mathbb{N}} \int_{\Lambda_i} (\tilde{\psi} \circ F^k - \bar{\psi}_k) d\tilde{\nu} \cdot \int_{\Lambda_i} \tilde{\varphi} d\tilde{\nu} \right| \\ &\leq \sum_{i \in \mathbb{N}} \left| \int_{\Lambda_i} (\tilde{\psi} \circ F^k - \bar{\psi}_k) \circ F^{n-k} \tilde{\varphi} d\tilde{\nu} \right| + \sum_{i \in \mathbb{N}} \left| \int_{\Lambda_i} (\tilde{\psi} \circ F^k - \bar{\psi}_k) d\tilde{\nu} \cdot \int_{\Lambda_i} \tilde{\varphi} d\tilde{\nu} \right| \\ &\leq 2 \sum_{i \in \mathbb{N}} \mathfrak{C}(\psi) [\text{diam}(\pi F^k(\mathcal{M}_{2k}(x)))]^\eta \cdot \int_{\Lambda_i} |\tilde{\varphi}| d\tilde{\nu} \\ &\leq 2 \mathfrak{C}(\psi) \sum_{i \in \mathbb{N}} C(i) [\alpha^k]^\eta |\varphi|_\infty \tilde{\nu}(\Lambda_i) \\ &\leq 2 \mathfrak{C}(\psi) \sum_{i \in \mathbb{N}} C(i) [\alpha^k]^\eta |\varphi|_\infty B_\varrho(\Lambda_i) \\ &\leq C'(\varphi, \psi) \alpha^{k\eta}, \end{aligned}$$

for some $B > 0$ where $C'(\varphi, \psi) = 2\mathfrak{C}(\psi) \sum_{i \in \mathbb{N}} C(i) |\varphi|_\infty B \varrho(\Lambda_i)$, and this sum is finite by our assumptions on $\mathfrak{H}_\eta$. ■

Note that each set $A \in \mathcal{M}_{2k}$ is a union of stable leaves; thus, $\bar{\psi}_k$ is constant on stable leaves, and so $\bar{\psi}_k$ induces a well-defined map on $\bar{\Delta}$, and we shall also call this map $\bar{\psi}_k$ without confusion since it will be clear from the context the particular map to which we are referring.

Estimate 10 *Let $\bar{\psi}_k$ be as defined above, and let $\bar{\varphi}_k$ be defined analogously. Let $\bar{\varphi}_k \tilde{\nu}$ denote the signed measure whose density with respect to $\tilde{\nu}$ is $\bar{\varphi}_k$, and let $\tilde{\varphi}_k = \frac{d(F_*^k(\bar{\varphi}_k \tilde{\nu}))}{d\tilde{\nu}}$. Then*

$$|D_{n-k}(\tilde{\varphi}, \bar{\psi}_k; \tilde{\nu}) - D_{n-k}(\tilde{\varphi}_k, \bar{\psi}_k; \tilde{\nu})| \leq C''(\varphi, \psi) \alpha^{k\eta}$$

for some $C'''(\varphi, \psi) > 0$.

Proof: Just as in the proof of Estimate 9, Lemma 2.7 gives us

$$\begin{aligned} & \left| \sum_{i \in \mathbb{N}} \int_{\Lambda_i} (\bar{\psi}_k \circ F^{n-k}) (\tilde{\varphi} - \tilde{\varphi}_k) d\tilde{\nu} \right| + \left| \sum_{i \in \mathbb{N}} \int_{\Lambda_i} \bar{\psi}_k d\tilde{\nu} \cdot \int_{\Lambda_i} (\tilde{\varphi} - \tilde{\varphi}_k) d\tilde{\nu} \right| \\ & \leq \sum_{i \in \mathbb{N}} \int_{\Lambda_i} |(\bar{\psi}_k \circ F^{n-k}) (\tilde{\varphi} - \tilde{\varphi}_k)| d\tilde{\nu} + \sum_{i \in \mathbb{N}} \int_{\Lambda_i} |\bar{\psi}_k| d\tilde{\nu} \cdot \int_{\Lambda_i} |\tilde{\varphi} - \tilde{\varphi}_k| d\tilde{\nu} \\ & \leq 2B \max |\psi| \cdot \sum_{i \in \mathbb{N}} \varrho(i) \int_{\Lambda_i} |\tilde{\varphi} - \tilde{\varphi}_k| d\tilde{\nu} \\ & \leq 2B \alpha^{k\eta} \max |\psi| \cdot \sum_{i \in \mathbb{N}} \mathfrak{C}(\varphi) C(i) \varrho(i) \\ & \leq C_2(\varphi, \psi) \alpha^{k\eta}, \end{aligned}$$

where η is chosen as before and $C_2(\varphi, \psi) = 2B \alpha^{k\eta} \max |\psi| \mathfrak{C}(\varphi) \sum_{i \in \mathbb{N}} C(i) \varrho(i)$, and this sum is also obviously finite.

Note that in Estimates 9 and 10, we did not use the fact that φ and ψ were Hölder on M with a uniform Hölder coefficient. Rather, we could have simply required that both φ and ψ were Hölder on each Λ_i with some Hölder coefficient $\mathfrak{C}(i, \varphi)$, for example, such that the sum $\sum_{i \in \mathbb{N}} \mathfrak{C}(i, \varphi) C(i) \varrho(i)$ is finite. In particular, this would allow $\mathfrak{C}(i, \varphi)$ to grow without bound as $i \rightarrow \infty$.

We now observe that $D_{n-k}(\tilde{\varphi}_k, \bar{\psi}_k; \tilde{\nu})$ can be expressed in terms of objects only from $\bar{F} : (\bar{\Delta}, \bar{\nu}) \rightarrow (\bar{\Delta}, \bar{\nu})$. First,

$$\begin{aligned}
\int (\bar{\psi}_k \circ F^{n-k}) \tilde{\varphi}_k d\tilde{\nu} &= \int \bar{\psi}_k d(F_*^{n-k}(\tilde{\varphi}_k \tilde{\nu})) \\
&= \int \bar{\psi}_k d(F_*^n(\bar{\varphi}_k \tilde{\nu})) \\
&= \int \bar{\psi}_k d(\bar{\pi}_* F_*^n(\bar{\varphi}_k \tilde{\nu})) \\
&= \int \bar{\psi}_k d(\bar{F}_*^n(\bar{\varphi}_k \bar{\nu})) \\
&= \int \bar{\psi}_k d(\bar{F}_*^n(\bar{\varphi}_k \bar{\rho} \bar{\mu})) \\
&= \int (\bar{\psi}_k \circ \bar{F}^n)(\bar{\varphi}_k \bar{\rho}) d\bar{\mu} \\
&= \int \bar{\psi}_k P^n(\bar{\varphi}_k \bar{\rho}) d\bar{\mu}.
\end{aligned}$$

Also, we observe that

$$\int \tilde{\varphi}_k d\tilde{\nu} \cdot \int \bar{\psi}_k d\tilde{\nu} = \int d(F_*^k(\bar{\varphi}_k \tilde{\nu})) \cdot \int \bar{\psi}_k d\tilde{\nu} = \int \bar{\varphi}_k d\tilde{\nu} \cdot \int \bar{\psi}_k d\tilde{\nu}.$$

Estimation of the correlation functions and end of the proof

We note that we have

$$D_{n-k}(\tilde{\varphi}_k, \bar{\psi}_k; \tilde{\nu}) = \left| \int (\bar{\psi}_k \circ F^{n-k}) \tilde{\varphi}_k d\tilde{\nu} - \int \tilde{\varphi}_k d\tilde{\nu} \cdot \int \bar{\psi}_k d\tilde{\nu} \right|$$

$$\begin{aligned}
&= \left| \int \bar{\psi}_k P^n(\bar{\varphi}_k \bar{\rho}) d\bar{\mu} - \int \bar{\varphi}_k \bar{\rho} d\bar{\mu} \cdot \int \bar{\psi}_k \bar{\rho} d\bar{\mu} \right| \\
&= \left| \int \bar{\psi}_k \left(P^n(\bar{\varphi}_k \bar{\rho}) - \left(\int \bar{\varphi}_k \bar{\rho} d\bar{\mu} \right) \bar{\rho} \right) d\bar{\mu} \right| \\
&\leq |\psi|_{L^\infty(\bar{\mu})} \cdot \left\| P^n \left(\bar{\varphi}_k \bar{\rho} - \left(\int \bar{\varphi}_k \bar{\rho} d\bar{\mu} \right) \bar{\rho} \right) \right\|_{L^1(\bar{\mu})} \\
&\leq |\psi|_{L^\infty(\bar{\mu})} \cdot C_0'' \left\| P^n \left(\bar{\varphi}_k \bar{\rho} - \left(\int \bar{\varphi}_k \bar{\rho} d\bar{\mu} \right) \bar{\rho} \right) \right\|_\infty \\
&\leq |\psi|_{L^\infty(\bar{\mu})} \cdot C_0'' \left\| P^n \left(\bar{\varphi}_k \bar{\rho} - \left(\int \bar{\varphi}_k \bar{\rho} d\bar{\mu} \right) \bar{\rho} \right) \right\|,
\end{aligned}$$

where $\|\cdot\|$ is the norm on $\mathcal{F}$ we introduced earlier.

Next we observe that $D_n(\varphi, \psi; \nu) = D_n(\tilde{\varphi}, \tilde{\psi}; \tilde{\nu}) = D_{n-k}(\tilde{\varphi}, \tilde{\psi} \circ F^k; \tilde{\nu})$. We have proved earlier that P is quasi-compact, so we may define

$$\tau = \sup\{|\zeta| : \zeta \in \sigma(P) \text{ such that } \zeta \neq 1\}.$$

By Estimates 9 and 10 we have

$$\begin{aligned}
D_n(\varphi, \psi; \nu) &\leq |D_n(\varphi, \psi; \nu) - D_{n-k}(\tilde{\varphi}_k, \bar{\psi}_k; \tilde{\nu})| + D_{n-k}(\tilde{\varphi}_k, \bar{\psi}_k; \tilde{\nu}) \\
&\leq (C'(\varphi, \psi) + C''(\varphi, \psi)) \alpha^{k\eta} + |\psi|_{L^\infty(\bar{\mu})} C_0'' \tau^n \left\| \bar{\varphi}_k \bar{\rho} - \left(\int \bar{\varphi}_k \bar{\rho} d\bar{\mu} \right) \bar{\rho} \right\|.
\end{aligned}$$

Now choose $k \approx \frac{n}{2}$ and $\tau_1 = \max\{\alpha^{k\eta}, \tau\}$. Then τ is the rate of decay of the correlation functions, and we define

$$C(\varphi, \psi) = C'(\varphi, \psi) + C''(\varphi, \psi) + |\psi|_{L^\infty(\bar{\mu})} C_0'' \left\| \bar{\varphi}_k \bar{\rho} - \left(\int \bar{\varphi}_k \bar{\rho} d\bar{\mu} \right) \bar{\rho} \right\|,$$

and the last factor is finite since it is clearly bounded above by $\|\bar{\varphi}\bar{\rho}\| + \left\| \left(\int \bar{\varphi} \bar{\rho} d\bar{\mu} \right) \bar{\rho} \right\|$, and this completes the proof of Theorem 2.3.

One-dimensional examples

In this chapter, we construct some one-dimensional dynamical systems which do not fit into Young's original construction but which fit into the setting of our extension of Young's theorem. We begin with a map with two domains of invertibility and show how our technique extends trivially to a map with finitely many domains of invertibility as long as we choose Λ prudently. Finally, we show that under an additional assumption about the speed of decay of the measures of the domains of invertibility, we may extend our result to include maps with countably many domains of invertibility. Some of the properties of our examples are not really essential for the dynamics we consider; however, our aim is to construct simple examples of dynamical systems which are not (as far as this author knows) covered by previously known theorems. We claim that it is clear which properties of our examples are essential and which are merely æsthetic.

3.1 Two domains of invertibility: a motivational example

Let $a \in (0, 1)$. We partition the unit interval $[0, 1]$ into two pieces $[0, a)$ and $[a, 1]$. Furthermore, we partition the interval $[a, 1] \pmod{0}$ into subintervals by a countable collection $\{\tau_{-i} : i \in \mathbb{N}\}$ such that $a < \dots < \tau_{-i} < \tau_{-i+1} < \dots < \tau_{-1} = 1$ and such that $[a, 1] = \bigcup_{i \in \mathbb{N}} [\tau_{-i-1}, \tau_{-i}]$. To use the notation of Chapter 2, we let $\Lambda_i = (\tau_{-i-1}, \tau_{-i})$.

For the moment, we wish to define our function $f : [0, 1] \rightarrow [0, 1]$. We let

$$(i) \quad f|_{[0, a)}(x) = \frac{x}{a}.$$

Then for each $i \in \mathbb{N}$ we suppose that $f|_{\Lambda_i}$ is C^2 , and f satisfies the following conditions:

$$(ii) \quad \text{there is } \alpha < 1 \text{ such that, for each } i \in \mathbb{N}, \left| f'|_{\Lambda_i} \right| \geq \frac{1}{\alpha} > 1,$$

$$(iii) \quad \text{for each } i \in \mathbb{N}, f\Lambda_i = (a^i, a^{i-1}),^1$$

$$(iv) \quad \text{there is } K > 0 \text{ such that for each } i \in \mathbb{N} \text{ we have } \sup_{x, y \in \Lambda_i} \left| \frac{f'(x)}{f'(y)} \right| \leq K,$$

$$(v) \quad \text{for each } i \in \mathbb{N} \text{ there is } D_i > 0 \text{ such that } \left| f''|_{\Lambda_i} \right| \equiv D_i, \text{ and}$$

$$(vi) \quad \text{for each } i \in \mathbb{N}, \left| \frac{f''(x)}{f'(x)} \right| \leq K.$$

We shall also assume that there is $b < e^{-1}$ such that $m\left(\bigcup_{i>l} \Lambda_i\right) \leq b^{\alpha^{-l}}$. We shall see later that this will ensure that Condition (E) is satisfied.

¹This property is not essential, but we use it here to make all the estimates easy.

To understand how this map fits into our scheme from Chapter 2, let us examine how it behaves on each Λ_i . First, we let $\Lambda = (a, 1)$. The subinterval Λ_1 is mapped onto Λ in the first iterate, so its return time is $R_1 = 1$. Then f maps the subinterval Λ_i onto (a^i, a^{i-1}) and then onto (a^{i-1}, a^{i-2}) and so on until it is finally mapped onto Λ . The return time of Λ_i is $R_i = i$, so we have the simplest possible example with one Λ_i returning at each time $n \in \mathbb{N}$.

Let $x, y \in \Lambda_i$, and let $C(i) = K'\alpha^{-i+1}$, where $K' = K \sum_{j \in \mathbb{N}} \alpha^{-j}$. We claim that

$$\log \prod_{j=0}^n \left| \frac{f'(f^j x)}{f'(f^j y)} \right| \leq iK \sum_{j \in \mathbb{N}} \alpha^{-j} \leq K'\alpha^{-i} = C(i)\alpha^{-1}.$$

First, let us note that, if x and y are far enough apart in Λ_i , they will separate when they return to Λ , and so we need only look at the quantity $\log \prod_{j=0}^n \left| \frac{f'(f^j x)}{f'(f^j y)} \right|$ in Condition (B)(b) for $0 \leq n < s(x, y)$. (See p. 17.) We should note that Conditions (A) and (C) are vacuous here since we do not have a contracting direction. For these x and y , we have

$$\log \prod_{j=0}^n \left| \frac{f'(f^j x)}{f'(f^j y)} \right| = \sum_{j=0}^n \log \left| \frac{f'(f^j x)}{f'(f^j y)} \right| = \log \left| \frac{f'(x)}{f'(y)} \right| \leq K \leq \alpha^{-i} = C(i)\alpha^{-1},$$

which is what we require.

On the other hand, if x and y are quite close together in Λ_i , they could return together to another Λ_{j_1} in Λ . If they land close enough together in Λ_{j_1} , they could return together to another Λ_{j_2} , and so on; however, because of the minimum expansion by α^{-1} , they must separate in finite time. Let us consider the case when $|x - y| = \alpha m(\Lambda_i)$, where m denotes ordinary Lebesgue measure. Because $f|_{[0, a]}$ is linear, distortion is introduced only when $f^j x$ and $f^j y$ are in Λ . If they land back in the

same Λ_i in which they started, then they must now be far enough apart to separate when they return again, and so we would have

$$\log \prod_{j=0}^n \left| \frac{f'(f^j x)}{f'(f^j y)} \right| \leq (1+a)K \leq C(i)\alpha^{-1}.$$

In fact, the only way that x and y can avoid separating when they return to Λ is if they land together in some Λ_j with $j < i$. If they land near the left endpoint of Λ_j , then they will certainly separate on their next return, and so they will pick up no more distortion. It is not hard to see that the most distortion which can be introduced is given by

$$\log \prod_{j=0}^n \left| \frac{f'(f^j x)}{f'(f^j y)} \right| \leq iK \leq K'\alpha^{-i} = C(i)\alpha^{-1}.$$

Continuing this line of reasoning for any two points x and y in Λ_i , we see that

$$\log \prod_{j=0}^n \left| \frac{f'(f^j x)}{f'(f^j y)} \right| \leq iK \sum_{j \in \mathbb{N}} \alpha^{-j} \leq K'\alpha^{-i} = C(i)\alpha^{-1}.$$

We should point out that because of assumption (iv) above, these systems do not fit into the original theory in [You98]. This is because Young's original theorem requires that there is some $\delta < 1$ such that, for all $x, y \in \Lambda_i$, $\left| \frac{f'(x)}{f'(y)} \right| < T\delta^i$ for some $T > 0$.

Finally, we note that Condition (D) (See p. 18.) is clearly true, and Condition (E) is true by our choice of b since $e^{C(i)}\varrho(i) \leq e^{\alpha^{-i}}b^{\alpha^{-i}} = (eb)^{\alpha^{-i}}$, and this decays faster than exponentially since $eb < 1$ and $\alpha < 1$.

Note that we have only verified the conditions for Theorem 2.3. With the amount of distortion present on each Λ_i , we cannot verify the assumptions of Theorem 2.2. In particular, the sum $\sum_{i \in \mathbb{N}} e^{C(i)}\alpha^i$ does not converge, but this is not important since

the existence of SRB measures for systems satisfying assumptions (ii) and (vi) above has been proved by Broise in [Bro96].

3.2 Finitely many domains of invertibility

It is clear from the construction in the previous section that we may easily consider a function $f : [0, 1] \rightarrow [0, 1]$ with finitely many domains of invertibility. Suppose now that we have some finite collection of a_i s such that $0 < a_1 < a_2 < \dots < a_l < 1$. We suppose that

$$(i) \quad f|_{[0, a_1)}(x) = \frac{x}{a_1},$$

and we suppose that each subinterval (a_j, a_{j+1}) is partitioned (mod 0) into countably many subintervals $\Lambda_{j,i}$ such that

$$(ii) \quad \text{for each } j \text{ and for each } i \in \mathbb{N}, f|_{\Lambda_{j,i}} \text{ satisfies all the assumptions for } f|_{\Lambda_i} \text{ in the previous section } \textit{mutatis mutandis} \text{ with } a \text{ replaced here by } a_1.$$

In particular, for every j we assume that $f\Lambda_{j,i}$ maps onto (a_1^i, a_1^{i-1}) . Also, we replace the assumption $m(\bigcup_{i>l} \Lambda_i) < b^{a^{-l}}$ for some $b < e^{-1}$ by the corresponding assumption $m(\bigcup_{R_i>l} \Lambda_i) = m(\bigcup_{i>l} \Lambda_{j,i}) < b^{a^{-l}}$. We define $\Lambda = (a_1, 1)$. In each subinterval (a_i, a_{i+1}) , there is precisely one $\Lambda_{j,i}$ which returns at time $R_i = i$, and since there are finitely many such subintervals, it is clear that the sums we consider in the previous example are all still finite here.

3.3 Countably many domains of invertibility

In this section, we show how we can extend our work in the previous section to include maps with countably many domains of invertibility. As one imagines, the extension is not trivial. In fact, we require further knowledge of the rate of decay of the domains of invertibility.

We assume that we have some countable collection of a_i s such that $0 < a_1 < \dots < a_i < a_{i+1} < 1$. All of the assumptions in the previous section, including the assumption that $m\left(\bigcup_{i>l} \Lambda_{j,i}\right) < b^{a^{-l}}$ for some $b < e^{-l}$, are taken here. Even in this case, [Bro96] proves the existence of SRB measures under assumptions (ii) and (vi), which we established at the beginning of this chapter; therefore, we only have to show that the sum $\sum_{i \in \mathbb{N}} e^{C(i)} \varrho(i)$ converges, where $\varrho(i) = b^{a^{-i}}$. First, note that

$$(3.1) \quad \sum_{i \in \mathbb{N}} e^{5C(i)} \varrho(i) = \sum_{n \in \mathbb{N}} \sum_{\substack{i \in \mathbb{N} \\ R_i = n}} e^{5C(i)} \varrho(i).$$

Let us use E_j to denote the subinterval (a_j, a_{j+1}) . In our construction, there is precisely one Λ_i in each subinterval E_j which returns at time n , and $C(i)$ is the same for all of these Λ_i s with $R_i = n$. We see that we require sufficiently fast decay of the lengths of the intervals E_j in order to guarantee convergence of the sum (3.1). In particular, exponential decay will suffice. We assume that there is $\rho < 1$ such that $|E_j| \leq \rho^j |E_1|$, where $|E_j|$ denotes the length of the interval E_j . There is some constant $K_2 > 0$ such that $|\Lambda_{j,n}| \leq K_2 \rho^j |\Lambda_{1,n}|$ for all $j \in \mathbb{N}$ and for all $n \in \mathbb{N}$. Note that $C(i)$ is constant on $\bigcup_{R_i=n} \Lambda_i$, so without ambiguity we shall write $C(n)$ for this

constant. Then (3.1) is bounded above term-by-term by

$$\sum_{n \in \mathbb{N}} e^{5C(n)} |\Lambda_{1,n}| K_2 \sum_{j \in \mathbb{N}} \rho^j,$$

and this is clearly finite since this is the same sum as we have in the system with only two domains of invertibility, except with a constant multiple.

Newhouse-Jakobson maps

We finally address the maps which motivate all our work. In [JN00], Jakobson and Newhouse prove that certain types of piecewise smooth hyperbolic maps on the unit square in $\mathbb{R}^2$ have finite SRB measures. Furthermore, they prove that the natural extensions of the systems they consider are K-automorphisms; therefore, they are mixing, but the issue of the speed of mixing is left open. We make some meager progress in addressing this issue through the use of our theorem, but unfortunately we are not able to include the family of Newhouse-Jakobson maps in their full generality under the umbrella of our theorem. For the convenience of the reader, we give here a brief summary of some Newhouse-Jakobson maps which do fit into our theory, and we refer the reader to [JN00] for a discussion of them in their more general context.

We assume that the unit square I^2 is partitioned (modulo a set of measure zero) into countably many full-height curvilinear rectangles $\{E_i : i \in \mathbb{N}\}$. For convenience we shall use the notation and vocabulary of [JN00] and call E_i the i th *post*. The upper

and lower boundaries of each E_i are subintervals of the upper and lower boundaries of I^2 ; the left and right boundaries of each E_i are the graphs of smooth functions $x_i(y)$ such that $\left| \frac{dx_i}{dy} \right| \leq \alpha$ for some $\alpha \in (0, 1)$ which is independent of the function x_i . We assume that $F|_{E_i}$ extends to a C^2 map on some neighborhood $\mathcal{E}_i$ of E_i and that $S_i := FE_i \subset I^2$ is a full-width *strip*. The left and right boundaries of each S_i are subintervals of the left and right boundaries of I^2 ; the upper and lower boundaries of each S_i are the graphs of smooth functions $y_i(x)$ such that $\left| \frac{dy_i}{dx} \right| \leq \alpha$. For each $i \in \mathbb{N}$, we do not permit the upper and lower boundaries of S_i to meet, nor do we permit the left and right boundaries of E_i to meet. For each $i \in \mathbb{N}$, we let $f_i := F|_{E_i}$ denote the restriction of F to the i th post. There are some technical requirements on the $\mathcal{E}_i$ s which are essential in the proof of the existence of SRB measures; these requirements are discussed in [JN00], but they are not important to our work here since they do not enter into the discussion about the correlation functions.

For each $z \in I^2$, let l_z denote the horizontal line containing z . We define the following:

$$\delta_z(E_i) = \text{diam}(l_z \cap E_i)$$

$$\delta_{i,\max} = \max_{z \in Q} \delta_z(E_i)$$

$$\delta_{i,\min} = \min_{z \in Q} \delta_z(E_i),$$

and we assume the following conditions on the geometry of the E_i s:

$$(H1) \quad \text{int } E_i \cap \text{int } E_j = \emptyset \text{ if } i \neq j,$$

$$(H2) \quad m(I^2 \setminus \bigcup_{i \in \mathbb{N}} \text{int } E_i) = 0,$$

$$(H3) \quad -\sum_{i \in \mathbb{N}} \delta_{i,\max} \log \delta_{i,\min} < \infty.$$

We assume also that $\tilde{Q} := I^2 \setminus \bigcup_{i \in \mathbb{N}} \text{int } E_i$ is hyperbolic for F ; i.e., for each $x \in \tilde{Q}$ there is a splitting $T_x M = E_x^s \oplus E_x^u$ which varies continuously with $x \in \tilde{Q}$, a constant $K_0 > 1$, and a Riemannian norm $|\cdot|$ such that

$$1. \quad D_x f(E_x^s) = E_{f(x)}^s \text{ and } D_x f(E_x^u) = E_{f(x)}^u,$$

$$2. \quad |D_x f(v)| \leq \frac{1}{K_0} |v| \text{ for all } v \in E_x^s, \text{ and } |D_x f(v)| \geq K_0 |v| \text{ for all } v \in E_x^u.$$

For each $i \in \mathbb{N}$ and for all $(x, y) \in E_i$, we write $f_i(x, y) = (f_{i1}(x, y), f_{i2}(x, y))$, where we are using the canonical coordinates on the unit square, and we use f_{i1xx} , f_{i1xy} , f_{i2yy} , etc. to denote the partial derivatives of f_{i1} and f_{i2} for each $i \in \mathbb{N}$. For each $i \in \mathbb{N}$, let

$$|D^2 f_i(x_0, y_0)| = \max_{\substack{j=1,2 \\ (k,l)=(x,x),(x,y),(y,y)}} |f_{ijkl}(x_0, y_0)|$$

denote the maximum second derivative at (x_0, y_0) . We may now state the final condition found in [JN00]:

$$(D1) \quad \text{There is some } C_0 > 0 \text{ such that } \sup_{\substack{i \in \mathbb{N} \\ z \in E_i}} \frac{|D^2 f_i(z)|}{|f_{i1x}(z)|} \delta_z(E_i) < C_0.$$

The reason we are not able to include all Newhouse-Jakobson maps in our theorem is that Newhouse and Jakobson make very weak assumptions about the stable foliations of their systems. Aside from the requirement that the stable foliation Γ^s is preserved by F , the only condition relating to the absolute continuity of Γ^s is Condition (H3).

Let $\gamma \in \Gamma^u$. From (H3) Newhouse and Jakobson show that for every $\epsilon > 0$, there is some compact set $A \subset \gamma$ and some constant $K_4 = K_4(A) > 0$ such that

$m_\gamma(\gamma \setminus A) < \epsilon$ and such that, for all $z \in A$ and all $n \geq 0$,

$$(4.1) \quad \prod_{j=0}^n \frac{\det Df^u(f^j z)}{\det Df^u(f^j(\theta z))} < K_4,$$

where θ is the stable holonomy map we introduced in Chapter 2. This is very nearly Condition (C) for $C(i) = \text{const}$ independent of i , but of course we have no idea how the ratio on the left-hand side of (4.1) behaves off the set A .

Thinking of K_4 as a function of ϵ , we then have to make some comparatively strong assumption on the regularity of the stable foliation Γ^s so that we can guarantee how K_4 will behave as $m_\gamma(\gamma \setminus A) \rightarrow 0$. One way to do this is to assume that $\det Df^u$ is Lipschitz; i.e.,

$$(4.2) \quad |\det Df^u(x_0) - \det Df^u(y_0)| \leq K_5 |x_0 - y_0|$$

for some $K_5 > 0$ whenever $y_0 \in \gamma^s(x_0)$. Note that

$$\frac{\det Df^u(x)}{\det Df^u(y)} = 1 + \frac{\det Df^u(x) - \det Df^u(y)}{\det Df^u(y)} \leq \exp \left(\frac{|\det Df^u(x) - \det Df^u(y)|}{\det Df^u(y)} \right).$$

Thus, if $y \in \gamma^s(x)$, we have

$$\begin{aligned} \log \prod_{j=0}^{\infty} \left| \frac{\det Df^u(f^j x)}{\det Df^u(f^j y)} \right| &= \sum_{j=0}^{\infty} \log \left| \frac{\det Df^u(f^j x)}{\det Df^u(f^j y)} \right| \\ &\leq \sum_{j=0}^{\infty} \frac{1}{|\det Df^u(f^j y)|} \cdot |\det Df^u(f^j x) - \det Df^u(f^j y)| \\ &\leq \frac{1}{K_0} \sum_{j=0}^{\infty} |\det Df^u(f^j x) - \det Df^u(f^j y)| \\ &\leq \frac{K_5}{K_0} \sum_{j=0}^{\infty} |f^j x - f^j y| \\ &\leq \frac{K_5}{K_0} |x - y| \sum_{j=0}^{\infty} \left(\frac{1}{K_0} \right)^j \end{aligned}$$

$$=: K_6,$$

since x and y are in the same stable leaf. Similarly, for each $n \in \mathbb{N}$ we have

$$\begin{aligned} \log \prod_{j=n}^{\infty} \left| \frac{\det Df^u(f^j x)}{\det Df^u(f^j y)} \right| &= \sum_{j=n}^{\infty} \log \left| \frac{\det Df^u(f^j x)}{\det Df^u(f^j y)} \right| \\ &\leq \frac{K_5}{K_0} |x - y| \sum_{j=n}^{\infty} \left(\frac{1}{K_0} \right)^j \\ &= \frac{K_5}{K_0} |x - y| \left(\frac{1}{K_0} \right)^n \sum_{j=0}^{\infty} \left(\frac{1}{K_0} \right)^j \\ &= K_6 \left(\frac{1}{K_0} \right)^n. \end{aligned}$$

Thus, the system satisfies Condition (C) for $C(i) = K_6$ for all $i \in \mathbb{N}$ and for $\alpha = K_0^{-1}$.

As we did in the previous two chapters, let us take $\Lambda = \bigcup_{i \geq 2} E_i$. As with the previous examples, this will allow us to define the Λ_i s so that each E_j has precisely one Λ_i which returns at time n for each $n \in \mathbb{N}$. Furthermore, because F maps each full-height post E_j onto a full-width strip S_j , it will be very easy for us to define the Λ_i s explicitly. In more general hyperbolic systems in which full-height sets might be mapped across some proper subset of the posts, constructing the Λ_i s is significantly more difficult. Let us fix a post E_i . We shall define $\Lambda_{i,1}$ to be that part of E_i which is mapped across Λ by F ; i.e., $\Lambda_{i,1} = f_i^{-1}\Lambda$. We define $\Lambda_{i,2}$ to be that part of E_i which is first mapped to E_1 by F and then across Λ ; i.e., $\Lambda_{i,2} = f_i^{-1}f_1^{-1}\Lambda$. We continue inductively so that we may define $\Lambda_{i,j} = f_i^{-1}f_1^{-j+1}\Lambda$. We do this for each $i \in \mathbb{N}$. Then $\Lambda_{i,j}$ is that part of E_i which returns to Λ at time j .

The final condition we must verify is that there is $C > 0$ and $\theta_0 < 1$ such that $m_\gamma \{x \in \gamma \cap \Lambda : R(x) > l\} \leq C\theta_0^l$ for all $\gamma \in \Gamma^u$ and for all $l \in \mathbb{N}$. For general

hyperbolic systems, this requires a bit of work to prove; however, for the system we are considering here, this is easily seen to be true since each full-height post E_i is mapped onto a full-width strip S_i . This is clear from our construction of the $\Lambda_{i,j}$ s above. Let $\gamma \in \Gamma^u \cap \Lambda$ be a full-width u-disk in Λ , and let $\gamma' \in \Gamma^u \cap \Lambda_{i,j}$ be its preimage in $\Lambda_{i,j}$; i.e., $F^j \gamma' = \gamma$. Then by the hyperbolicity of F , the length of γ' is no greater than $K_0^{-j} \cdot \delta_{i,\max}$. As we did in Section 3.3, since we have countably many domains of invertibility, we assume that there is some $\rho < 1$ such that $\delta_{i,\max} < \rho^i$. Then $m_\gamma \{x \in \gamma \cap \Lambda : R(x) \geq j\} \leq K_0^{-j} \sum_{i \in \mathbb{N}} \delta_{i,\max}$, which is what we wanted to verify.

Notice that we did not worry about verifying the conditions of Theorem 2.2 to determine existence of SRB measures; however, this is not important here since Newhouse and Jakobson prove the existence of SRB measures whose conditional measures on unstable leaves are equivalent to Lebesgue measure, and this is precisely what we require in order to apply Theorem 2.3. Furthermore, they prove that the natural extension is a K-automorphism; therefore, the dynamical system we consider here is exact and, thus, mixing.

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