



LIBRARY
Michigan State
University

This is to certify that the
dissertation entitled

INTERMEDIATE DOMAINS BETWEEN A LOCAL RING AND
ITS COMPLETION: CONDITIONS FOR NORMALITY AND
FACTORIALITY

presented by

SARAH ELIZABETH SWORD

has been accepted towards fulfillment
of the requirements for the

PHD

degree in

MATHEMATICS

Christel Rowland

Major Professor's Signature

8/22/03

Date

PLACE IN RETURN BOX to remove this checkout from your record.
TO AVOID FINES return on or before date due.
MAY BE RECALLED with earlier due date if requested.

DATE DUE	DATE DUE	DATE DUE

**INTERMEDIATE DOMAINS BETWEEN A
LOCAL RING AND ITS COMPLETION:
CONDITIONS FOR NORMALITY AND
FACTORIALITY**

By

Sarah Elizabeth Sword

A DISSERTATION

Submitted to

Michigan State University

in partial fulfillment of the requirements

for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

2003

ABSTRACT

INTERMEDIATE DOMAINS BETWEEN A LOCAL RING AND ITS COMPLETION: CONDITIONS FOR NORMALITY AND FACTORIALITY

By

Sarah Elizabeth Sword

The study of intersection rings between an excellent local noetherian ring (R, m) and its m -adic completion $\widehat{R}$ has provided a rich ground for constructing examples of “bad” noetherian local domains. The general strategy is to look at a noetherian local domain R that is essentially finitely generated over a field k . We let J be an ideal in $\widehat{R}$ so that the associated primes of J are in the generic formal fiber of R . We can then embed $R \hookrightarrow \widehat{R}/J$ so that regular elements of R map to regular elements of $\widehat{R}/J$. The intermediate ring of interest is $Q(R) \cap (\widehat{R}/J)$, where $Q(R)$ is the field of fractions of R . We look at two such constructions and determine conditions for normality and factoriality of the constructed intermediate rings.

To the women who inspire me, especially:

Christel Rotthaus

Glenda Lappan,

Diane Wakoski,

and my mother, Louise Drucker.

ACKNOWLEDGMENTS

My father's Masters' thesis began with a quotation from Dr. Seuss' "I Had Trouble in Getting to Solla Sollew". I don't know if I've gotten to Solla Sollew, but I have gotten a lot of help getting to the end of the PhD. Michigan State has been a wonderful place to be a student. How can I ever thank Christopher Danielson, JD Ferry, Laura Ghezzi, Payman Kassaei, Jim Larson, Carrie Preston, Kelly Rivette and Jason Troye for their friendship?

Working at the Connected Mathematics Project in the "Penthouse of Wells Hall" transformed my graduate school experience. I'd be a very poor woman if Betty Phillips actually charged me 5 cents every time I came to her with my troubles. Yvonne Grant's sense of humor carried me through a lot of days when I was sure I would never finish. Judi Miller's resourcefulness and kindness were a tremendous help. Lisa Keller's patience is inspiring. I flat out loved working with the other students, both graduate and undergraduate: Evo Pedawi, Bulent Buyukbozkirli, Billie Keusch, Marie Turini, Kuo-Liang Chang, Carl Oliver, Alisha Harold, Sri Dharmavaram, Lauren Rebrovich, Brian Oliver, Emily Bouck and Deb Johanning.

I owe a huge debt to Diane Wakoski, MSU's Poet-in-Residence, and East Lansing's Lady of Light, as well as her Sapphos and Alchemists. I don't know what I will do without them.

I am very lucky for my family, who somehow managed never to doubt that I would finish but be ecstatically surprised when I did.

Thank you, thank you thank you to my committee: Dr. Jeanne Wald, who has been smiling at me and encouraging me since I arrived; Dr. Susan Schuur, who has been very supportive; Dr. Ron Fintushel, the only mathematics professor I've ever had who regularly begins class quoting Andrew Marvell; and Dr. Glenda Lappan whose grace, wisdom, humor and hard work will always inspire me.

And most importantly, thanks to my advisor, Dr. Christel Rotthaus, without whose guidance, tremendous knowledge, and incredible patience, I would never have been

able to do this project. I feel honored to be her student.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION	1
--------------------	---

CHAPTER 2

THE NISHIMURA CONSTRUCTION	6
----------------------------------	---

Setting and Notation	6
----------------------------	---

Heitmann's Numbering Theorem	8
------------------------------------	---

Construction of the Ring B	14
------------------------------------	----

CHAPTER 3

CONDITIONS FOR NORMALITY AND FACTORIALITY OF B	20
--	----

Conditions for Normality	20
--------------------------------	----

Illustrating the Normality of B : Examples	25
--	----

The Factoriality of B	33
-------------------------------	----

Constructing Examples in which B is Factorial	35
---	----

CHAPTER 4

THE HEINZER-ROTTAUS-WIEGAND CONSTRUCTION	38
--	----

A Reference Ring	39
------------------------	----

Construction of the Intermediate Ring B	41
---	----

Conditions for Normality and Factoriality	45
A More Specific Setting	49
REFERENCES	52

1 Introduction

The study of intermediate rings between an excellent, local noetherian ring (R, m) and its m -adic completion $\widehat{R}$ has provided a rich ground for constructing examples of “bad” noetherian local domains. The general principle is to look at a noetherian local domain R that is essentially finitely generated over a field k . We let J be an ideal in $\widehat{R}$ so that the associated primes of J are in the generic formal fiber of R ; that is, so that $Q \cap R = (0)$ for every associated prime ideal of $\widehat{R}/J$. Then we have an embedding $R \hookrightarrow \widehat{R}/J$ in which non-zerodivisors of R map to non-zerodivisors of $\widehat{R}/J$. The intermediate ring of interest is:

$$B := Q(R) \cap (\widehat{R}/J)$$

where $Q(R)$ is the field of fractions of the domain R .

B is a local ring birationally dominating R . In many circumstances, B is noetherian with completion equal to $\widehat{R}/J$. Furthermore, B can often be realized as a direct limit of essentially finitely generated extensions of R , greatly assisting the study of the structure of B .

The construction of these “insider” rings was pioneered by Akizuki in the 1930s [1]. He used insider rings to construct a local noetherian domain C whose normalization is not a finite C -module. In the 1950s, Nagata used similar techniques to construct noetherian rings with “bad” completions. For example, he constructed a local noetherian normal domain (R, m) whose completion is not reduced [14], [15]. The examples Rotthaus constructed using this method include a normal local domain which is Nagata but not excellent [23]. Ogoma modified the construction to find an example of a normal non-catenary domain [18]. These techniques have also been used by Brodman-Rotthaus [2], [3], Heitmann [11], Weston [25] and others. Heinzer-Rotthaus-Weigand continue to create variations of this construction to create examples of noetherian and non-noetherian as well as excellent and non-excellent rings (see, for example, [7], [8], [9], [10]).

A local noetherian ring is called *excellent* if it is universally catenary and has geometrically regular formal fibers [5]. The class of excellent rings is closed under localization, passing to quotients, and finitely generated extensions. Rotthaus showed that the class of excellent local rings is also closed under ideal-adic completions [21]. Matsumura writes that “practically all noetherian rings in applications” are excellent [13], p. 260. In general, the intermediate rings we consider are *not* excellent, which is one reason they are such a good source of examples: the formal fibers of our constructed rings are not geometrically regular. In general, these rings are neither analytically normal nor analytically unramified.

In his preprint, *A Few Examples of Local Rings, I* [16], Jun-Ichi Nishimura compiles some of the work done by the aforementioned authors. He proves the following:

Theorem ([16], Theorem 1.4) *Let K be a purely transcendental extension field of countable degree over a countable field K_0 . Take polynomials in m variables over K_0 without constant term:*

$$F_1(Z), \dots, F_r(Z) \in K_0[Z_1, \dots, Z_m].$$

Then, for any $n > m$, there exists a local domain B which satisfies the following:

- $K[[\zeta_1, \dots, \zeta_n]]/(F_1(\zeta), \dots, F_r(\zeta)) \cong \widehat{R}/(f_1, \dots, f_r) \cong \widehat{B}$.
- $\hat{p} = (\zeta_1, \dots, \zeta_m)\widehat{B}$ is a prime ideal of $\widehat{B}$ and $\hat{p} \cap B = (0)$.
- B/p is essentially of finite type over K for any non-zero prime ideal $p \in \text{Spec}(B)$.

The first part of our paper is a detailed examination of the construction of this domain B , which we call “The Nishimura Construction”. We explore in particular Nishimura’s use of a theorem of Heitmann [11]. In this construction, we choose R to be a localized polynomial ring over a field, and let $I \subset R$ be an ideal of R . We will

need to put a certain condition on the associated primes of the extended ideal $I\hat{R}$. One important feature of Nishimura's work is the construction of an automorphism φ of $\hat{R}$, which maps the ideal $I\hat{R}$ to an ideal $J \subset \hat{R}$. Our condition on $I\hat{R}$ will yield the condition that if $Q \in \text{Ass}(\hat{R}/J)$, then $Q \cap R = (0)$. Thus we have an embedding $R \hookrightarrow \hat{R}/J$ in which non-zero-divisors of R map to non-zero-divisors of $\hat{R}/J$.

In this construction, $B = Q(R) \cap \hat{R}/J$. The constructed ring B is a local noetherian domain birationally dominating R with completion equal to $\hat{R}/J$. Furthermore, B is noetherian and realizable as a nested union of essentially finitely generated algebraic extensions of R . The choice of our automorphism φ of $\hat{R}$ allows us to identify a prime ideal $\hat{P}$ in the generic formal fiber of the constructed ring B . We use the prime ideal $\hat{P}$, corresponding to $\hat{p}$ in Nishimura's theorem, to facilitate our goal: to study conditions for normality and factoriality of B . With certain restrictions, we find necessary and sufficient conditions for B to be normal. We also find sufficient conditions for B to be factorial.

The very automorphism of $\hat{R}$ that makes this construction so powerful also makes finding necessary conditions for the factoriality of B extremely difficult. Ogoma [18] and Weston [25] made substantial modifications to the Nishimura construction in order to construct local factorial domains with non-factorial completions. In this paper, we are able to construct such domains using the Nishimura construction, but our constructed domains must be complete intersections.

Our results culminate in straightforward "recipes" for constructing normal rings with non-normal completions and factorial rings with non-factorial completions. Using our machinery to construct examples of bad local domains is very simple, as we see in the following two motivating examples:

Example: *A noetherian normal local domain whose completion is not normal.*

We begin with a localized polynomial ring in three variables over a field K ; i.e., $R = K[z_1, z_2, z_3]_{(z_1, z_2, z_3)}$. Let $I = (z_1 z_2)$. Neither the ring R/I nor its completion is normal. We can use the Nishimura construction together with our recipe to see

that the constructed ring B is a noetherian, local, normal domain with completion isomorphic to the non-normal ring $\widehat{R}/I\widehat{R}$.

Example: *A local factorial complete intersection whose completion is not factorial.*

We choose R to be a localized polynomial ring in five variables over a field K ; i.e., $R = K[z_1, z_2, z_3, z_4, z_5]_{(z_1, z_2, z_3, z_4, z_5)}$. Let $I = (z_1 z_3 - z_2 z_4)$. Neither the ring R/I nor its completion is factorial, however, we can again use the Nishimura construction together with our techniques to see that the constructed ring B yields the desired example.

In the second part of the paper, we look at a similar construction, which we will call the “HRW Construction”. Heinzer, Rotthaus and Wiegand have generalized a construction due to Akizuki in a series of papers on intermediate rings, including [7], [8], [9], [10]. In [10], the authors begin with a local noetherian domain R that is essentially finitely generated over a field k and with fraction field L . In this paper, they are interested in identifying ideals I for which intersection domains of the form $L \cap (\widehat{R}/I)$ are noetherian. To do this, they restrict to the completion R^* of R with respect to a non-zero non-unit x of R . They assume that I is an ideal of R^* such that $p \cap R = (0)$ for all associated primes p of R^*/I . Hence we have again an injection $R \hookrightarrow R^*/I$ so that non-zerodivisors of R map to non-zerodivisors of R^*/I . They show:

Theorem ([10], Theorem 3.2) *Let R be a noetherian integral domain with fraction field L . Let x be a non-zero non-unit of R and let R^* denote the (x) -adic completion of R . Suppose I is an ideal of R^* with the property that $p \cap R = (0)$ for each $p \in \text{Ass}(R^*/I)$. Then $R \longrightarrow (R^*/I)_x$ is flat if and only if $A := L \cap (R^*/I)$ is noetherian and A_x is realizable as a subring of $R_x = R[1/x]$.*

In this construction, A is approached with a sequence of finitely generated birational extensions of R , all of which are contained in the fraction field of R .

We again determine conditions for the normality and factoriality of the intermediate

ring A . We make use of the theorem quoted above to assume that our intersection rings are noetherian. We then use our conditions to generate more examples of bad local noetherian rings.

In the work that follows, all rings are assumed to be commutative with identity.

2 The Nishimura Construction

We begin with a detailed examination of the construction found in Nishimura's paper [14]. For any unexplained terminology, the reader can refer to [13].

Let K be a field, and R a localized polynomial ring over the field in n variables, i.e., $R = K[z_1, \dots, z_n]_{(z_1, \dots, z_n)}$. We denote by $\widehat{R}$ the completion of R with respect to the maximal ideal $(z_1, \dots, z_n)R$; so then $\widehat{R} = K[[z_1, \dots, z_n]]$.

If I is an ideal of R , we want to construct a local noetherian domain B birationally dominating R so that

$$\frac{\widehat{R}}{I\widehat{R}} \cong \widehat{B}$$

where $\widehat{B}$ is the completion of B with respect to the maximal ideal of B .

We construct an automorphism φ of $\widehat{R}$ taking the ideal $I\widehat{R}$ to an ideal $J = \varphi(I\widehat{R})$ satisfying the condition that if $Q \in \text{Ass}(\widehat{R}/J)$, then $Q \cap R = 0$. Our ring of interest is the ring:

$$A = Q(R) \cap \frac{\widehat{R}}{J}$$

In order to do this, we put a condition on the field K , namely, that K is a purely transcendental, countable extension of a countably infinite field. To achieve the condition on the associated primes of J , we put a condition on the ideal I involving the associated primes of $I\widehat{R}$. To facilitate our study of this intersection ring, we look at a set of finitely generated R -algebras inside A and show that the nested union of these R -algebras is actually equal to A .

2.1 Setting and Notation

To begin, let K_0 be a countable field. Let $\{a_{ij} \mid i = 1, \dots, \infty; j = 1, \dots, m\}$ be a set of elements which are algebraically independent over K_0 . In order to use Heitmann's lemma, we will need $m \geq 2$. Define a collection of purely transcendental extension

fields over K_0 as follows:

$$\begin{aligned}
K_1 &:= K_0(a_{11}, \dots, a_{1m}) \\
K_2 &:= K_1(a_{21}, \dots, a_{2m}) \\
&\vdots \\
K_k &:= K_{k-1}(a_{k1}, \dots, a_{km}) \\
&\vdots
\end{aligned}$$

Let K be the union of the fields K_k :

$$K = \bigcup_k K_k$$

Let $n > m$ and let $z_1, \dots, z_n$ be variables over K , and define:

$$\begin{aligned}
S_0 &= K_0[z_1, \dots, z_n] & \text{with prime ideal} & \mathfrak{N}_0 = (z_1, \dots, z_n)S_0 \\
S_k &= S_{k-1}[a_{k1}, \dots, a_{km}] & \text{with prime ideal} & \mathfrak{N}_k = (z_1, \dots, z_n)S_k \\
S &= \bigcup_k S_k = K_0[a_{ik}][z_1, \dots, z_n] & \text{with prime ideal} & \mathfrak{N} = (z_1, \dots, z_n)S.
\end{aligned}$$

Localizing at the prime ideals $\mathfrak{N}_k$ and $\mathfrak{N}$, we get corresponding rings:

$$\begin{aligned}
R_0 &= (S_0)_{\mathfrak{N}_0} = K_0[z_1, \dots, z_n]_{(z_1, \dots, z_n)} & \text{with} & \mathfrak{n}_0 = (z_1, \dots, z_n)R_0 \\
R_k &= (S_k)_{\mathfrak{N}_k} = K_k[z_1, \dots, z_n]_{(z_1, \dots, z_n)} & \text{with} & \mathfrak{n}_k = (z_1, \dots, z_n)R_k \\
R &= (S)_{\mathfrak{N}} = K[z_1, \dots, z_n]_{(z_1, \dots, z_n)} & \text{with} & \mathfrak{n} = (z_1, \dots, z_n)R.
\end{aligned}$$

Since $R_k = R_{k-1}[a_{1k}, \dots, a_{mk}]_{(z_1, \dots, z_n)}$, the ring R can be written as a nested union of localized polynomial rings. Furthermore, R is a localized polynomial ring in n variables $z_1, \dots, z_n$ over the field K , and hence a regular local ring. Also notice that since K is countable, R is countable.

We choose a sequence of elements $p_1, \dots, p_k, \dots$ contained in the ideal $\mathfrak{N} = (z_1, \dots, z_n)S$ such that:

$$1. \quad p_1 = z_1 + \dots + z_n$$

2. if $0 \neq P \in \text{Spec}(R)$, then there exists $k \in \mathbb{N}$ so that $p_k \in P$
3. For all $k \in \mathbb{N}$, $S_k \cap \{p_i \mid i = 1, \dots, \infty\}$ is infinite
4. $p_k \in S_k - 2$ for every $k \geq 2$

We make the following definitions:

$$\begin{aligned}
z_{i0} &= z_i \quad \text{for } 1 \leq i \leq m \\
q_k &= p_1 \cdots p_k \quad \text{for } k \geq 1 \\
z_{ik} &= z_i + a_{i1}q_1 + \cdots + a_{ik}q_k^k \quad \text{for } k \geq 1 \quad \text{and } 1 \leq i \leq m \\
\mathfrak{P}_k &= (z_{1k}, \dots, z_{mk})R
\end{aligned}$$

The elements $z_1, \dots, z_m$ form part of a system of parameters of R . Further, since $z_{ik} \equiv z_i + a_{i1}(z_1 + \dots + z_n) \pmod{\mathfrak{n}^2 R}$ for every $i = 1, \dots, m$ and for all $k \geq 0$, the elements $z_{1k}, \dots, z_{mk}$ form part of a regular system of parameters of R . Thus for all $k \geq 0$ the ideal $\mathfrak{P}_k = (z_{1k}, \dots, z_{mk})R$ is a prime ideal of height m . We now introduce a theorem of Heitmann that will continue to be useful throughout the construction.

2.2 Heitmann's Numbering Theorem

Theorem 2.2.1. (Heitmann's Numbering Theorem) [11], *Proposition 1.* Suppose that $m < n$. Then for all integers $k \geq 0$ and all positive integers $h \leq k+1$, $p_h \notin \mathfrak{P}_k$.

Proof. If we can show that for all k , $(z_{1k}, \dots, z_{mk})S_k$ is a prime ideal of S_k , then

$$\begin{aligned}
\mathfrak{P}_k \cap S_{k-1} &= (z_{1k}, \dots, z_{mk})R \cap S_{k-1} \\
&= (z_{1k}, \dots, z_{mk})R \cap R_k \cap S_{k-1} \\
&= (z_{1k}, \dots, z_{mk})R_k \cap S_{k-1} \\
&= (z_{1k}, \dots, z_{mk})S_k \cap S_{k-1} \\
&= 0
\end{aligned}$$

where the last intersection is 0 because the elements $a_{1k}, \dots, a_{mk}$ are algebraically independent over S_{k-1} . Since $0 \neq p_h \in S_{k-1}$ for $h \leq k+1$, we see that $p_h \notin \mathfrak{P}_k$. The difficulty, then, is to show that for all k , $(z_{1k}, \dots, z_{mk})S_k$ is a prime ideal of S_k . To do this, we first need a lemma.

Lemma 2.2.2. *Let A be a noetherian domain, $t_1, \dots, t_m$ variables over A , and $w_1, \dots, w_m, q$ a regular sequence in A . Then $(qt_1 - w_1, \dots, qt_m - w_m)$ is a prime ideal in $A[t_1, \dots, t_m]$.*

Proof. We will prove by induction on m :

- (1) $qt_1 - w_1, \dots, qt_m - w_m, q$ is a regular sequence in $A[t_1, \dots, t_m]$
- (2) $(qt_1 - w_1, \dots, qt_m - w_m)$ is a prime ideal in $A[t_1, \dots, t_m]$.

Suppose $m = 1$. For convenience, write $t_1 = t$ and $w_1 = w$. Then w, q form a regular sequence in A . We need to show that the ideal $(qt - w)$ is prime in the ring $A[t]$. Note that $(qt - w)$ is prime in the ring $A_q[t]$. Since q is a regular element of A , we need only show that if $f(t) \in A[t]$ with $q^r f(t) \in (qt - w)$ then $f(t) \in (qt - w)$. Write:

$$f(t) = \sum_{i=0}^s a_i t^i.$$

where $q^r f(t) \in (qt - w)$. Hence $q^r a_0 \in wA$. Since q, w is a regular sequence of A , $a_0 = wb_0$ for some $b_0 \in A$. Define $b_1 = a_1 + b_0 q$ and $b_i = a_i$ for $i > 1$. Write:

$$g(t) = \sum_{i=1}^s b_i t^i.$$

Note that since $g(t) - f(t) = b_0 qt - a_0 = b_0 qt - b_0 w = b_0 (qt - w)$, $f(t) \in (qt - w)$ if and only if $g(t) \in (qt - w)$. Hence we need only show that $g(t) \in (qt - w)$. An easy argument shows that $qt - w, t$ is a regular sequence in the domain $A[t]$. Thus, with

$$h(t) = \sum_{i=0}^{s-1} b_{i+1} t^i$$

the condition

$$q^r g(t) = tq^r h(t) \in (qt - w)$$

implies that

$$q^r h(t) \in (qt - w)$$

and we may proceed with the polynomial $h(t)$. Since $\deg h(t) < \deg f(t)$, it follows by induction on the degree of $f(t)$ that $f(t) \in (qt - w)$.

Now we assume that $(qt_1 - w_1, \dots, qt_m - 1 - w_m - 1)$ is prime in $A[t_1, \dots, t_m - 1]$ and that $qt_1 - w_1, \dots, qt_m - 1 - w_m - 1, q$ is a regular sequence in $A[t_1, \dots, t_m - 1]$. Note that:

$$\begin{aligned} (qt_1 - w_1, \dots, qt_m - 1 - w_m - 1, q)A[t_1, \dots, t_m - 1] = \\ (w_1, \dots, w_m - 1, q)A[t_1, \dots, t_m - 1] \end{aligned}$$

and that $A[t_1, \dots, t_m - 1]$ is flat over A . The fact that $w_1, \dots, w_m, q$ is a regular sequence in A yields that $w_1, \dots, w_m, q$ is a regular sequence in $A[t_1, \dots, t_m - 1]$. Moreover, w_m is a regular element of

$$\begin{aligned} \frac{A[t_1, \dots, t_m - 1]}{(q, w_1, \dots, w_m - 1)A[t_1, \dots, t_m - 1]} = \\ = \frac{A[t_1, \dots, t_m - 1]}{(qt_1 - w_1, \dots, qt_m - 1 - w_m - 1, q)A[t_1, \dots, t_m - 1]}. \end{aligned}$$

Thus $qt_1 - w_1, \dots, qt_m - 1 - w_m - 1, q, w_m$ is a regular sequence in $A[t_1, \dots, t_m - 1]$. By our induction hypothesis, the ring

$$B = \frac{A[t_1, \dots, t_m - 1]}{(qt_1 - w_1, \dots, qt_m - 1 - w_m - 1)}$$

is a noetherian domain, t_m is a variable over B , and q, w_m is a regular sequence in B . It is a well known fact (see, for example, the discussion on page 5 of [4]) that if q, w_m is a regular sequence in the domain B , so is w_m, q . Hence by our work in case $m = 1$, the ideal $(qt_m - w_m)$ is prime in $B[t_m]$. In particular, $(qt_1 - w_1, \dots, qt_m - 1 - w_m - 1, qt_m - w_m)$ is a prime ideal of $A[t_1, \dots, t_m]$, and the sequence $qt_1 - w_1, \dots, qt_m - w_m, q$ is regular in $A[t_1, \dots, t_m]$. $\square$

To prove Heitmann's theorem, we will show by induction on k :

- (1) $z_{1k}, \dots, z_{mk}, q_{k+1}^{k+1}$ is a regular sequence in S_k .
- (2) For all $1 \leq s \leq m$ the ideal $(z_{1k}, \dots, z_{sk})$ is prime in S_k .
- (3) $p_h \notin \mathfrak{P}_k$ for $h \leq k+1$.

The case $k = 0$ is clear:

- (1) The sequence $z_1, \dots, z_m, z_1 + \dots + z_n$ is regular in S_0 .
- (2) For all $1 \leq s \leq m$, the ideal $(z_1, \dots, z_s)$ is prime in S_0 .
- (3) $z_1 + \dots + z_n \notin (z_1, \dots, z_m)$.

Assume the statement is true for $k-1$. This assumption combined with the work above the statement of the lemma yields that $q_k = p_1 p_2 \dots p_k \notin \mathfrak{P}_{k-1}$ and so $q_k \notin (z_{1(k-1)}, \dots, z_{m(k-1)})S_{k-1}$. Thus

$$z_{1(k-1)}, \dots, z_{m(k-1)}, q_k$$

is a regular sequence in S_{k-1} . Furthermore, for all $1 \leq s \leq m$, the ideal

$(z_{1(k-1)}, \dots, z_{s(k-1)})S_{k-1}$ is prime in S_{k-1} . Hence, $z_{1(k-1)}, \dots, z_{m(k-1)}, q_k$ and

$z_{1(k-1)}, \dots, z_{m(k-1)}, q_k^k$ are regular sequences in S_{k-1} .

The ring S_{k-1} is a noetherian domain, $a_{1k}, \dots, a_{mk}$ are variables over S_{k-1} , and

$z_{1(k-1)}, \dots, z_{m(k-1)}, q_k^k$ is a regular sequence in S_{k-1} . Writing:

$$z_{ik} = q_k^k a_{ik} + z_{i(k-1)}$$

Lemma 2.2.2 yields that the ideal $(z_{1k}, \dots, z_{sk})$ is prime in S_k . (A similar argument shows that $z_{1k}, \dots, z_{mk}, q_{k+1}^{k+1}$, is a regular sequence in S_k .) Hence $p_h \notin \mathfrak{P}_k$ for all $h \leq k+1$.

□

Recall the general assumption that $m < n$. Consider the following elements of $\hat{R} = K[[z_1, \dots, z_n]]$:

$$\zeta_i := \begin{cases} z_i + \sum_{l=1}^{\infty} a_{il} q_l^l & \text{if } 1 \leq i \leq m \\ z_i & \text{if } m < i \leq n \end{cases}$$

We define an automorphism $\varphi : \hat{R} \rightarrow \hat{R}$ as follows:

$$\varphi(z_i) = \zeta_i \quad \text{for } i = 1, \dots, n$$

For $1 \leq i \leq m$, we have:

$$\begin{aligned} \zeta_i &\equiv z_i + a_{i1} q_1 \pmod{(z_1, \dots, z_n)^2 \hat{R}} \\ &\equiv z_i + a_{i1} (z_1 + \dots + z_n) \pmod{(z_1, \dots, z_n)^2 \hat{R}} \end{aligned}$$

so the elements $\zeta_1, \dots, \zeta_m, z_{m+1}, \dots, z_n$ form a regular system of parameters of $\hat{R}$. Hence, φ is an automorphism of $\hat{R}$. Moreover, the elements $\zeta_1, \dots, \zeta_m$ generate a prime ideal of height m . We will now use Heitmann's Numbering Theorem to show that $\hat{P} := (\zeta_1, \dots, \zeta_m) \cap R = 0$.

Lemma 2.2.3. *Let $\hat{P} := (\zeta_1, \dots, \zeta_m)$. Then $\hat{P} \cap R = 0$.*

Proof. Suppose $\hat{P} \cap R \neq 0$. Then by our choice of the sequence $\{p_i\}$, there exists $h \in \mathbb{N}$ so that $p_h \in \hat{P} \cap R$. Recall that for $1 \leq i \leq m$:

$$\zeta_i = z_i + \sum_{l=1}^{\infty} a_{il} q_l^l$$

Thus, $p_h \in \hat{P} \cap R$ implies that $z_{i(h-1)} \in \hat{P} \cap R$ for $1 \leq i \leq m$; i.e., that

$$\mathfrak{P}_{h-1} = (z_{1(h-1)}, \dots, z_{m(h-1)}) \subseteq \hat{P} \cap R.$$

A quick examination of the heights:

$$m = ht(\mathfrak{P}_{h-1}) \leq ht(\hat{P} \cap R) \leq m$$

yields that $\mathfrak{P}_{h-1} = \hat{P} \cap R$. By Heitmann's lemma, $p_h \notin \mathfrak{P}_{h-1}$. Hence, $p_h \notin \hat{P} \cap R$, a contradiction. $\square$

Let I be an ideal of R . Recall that our goal for the construction is to find a local noetherian domain B birationally dominating R so that

$$\widehat{B} \cong \widehat{R}/I\widehat{R}.$$

To do this, we need a condition on the ideal I . In particular, let

$$I := (F_1(z_1, \dots, z_n), \dots, F_r(z_1, \dots, z_n)) \subset R$$

be an ideal satisfying the following condition:

For every associated prime ideal Q in $\widehat{R}/I\widehat{R}$ we have $Q \subseteq (z_1, \dots, z_m)\widehat{R}$.

Under the automorphism $\varphi : \widehat{R} \longrightarrow \widehat{R}$, we map the ideal $I\widehat{R}$ into an ideal $J \subseteq \widehat{R}$:

$$J := \varphi(I\widehat{R}) = (\varphi(F_1), \dots, \varphi(F_r)).$$

Furthermore,

$$\varphi((z_1, \dots, z_m)\widehat{R}) = (\zeta_1, \dots, \zeta_m)\widehat{R} = \widehat{P}.$$

The condition on the associated primes of I yields the following condition on the ideal J :

For every associated prime ideal $\widetilde{Q}$ of $\widehat{R}/J$ we have $\widetilde{Q} \subseteq \widehat{P} := (\zeta_1, \dots, \zeta_m)\widehat{R}$.

From Lemma 2.2.3 we know that $\widehat{P} \cap R = 0$. Thus the condition gives us that the associated primes of J are in the generic formal fiber of R ; i.e.,

For every associated prime ideal $\widetilde{Q}$ of $\widehat{R}/J$ we have $\widetilde{Q} \cap R = 0$.

Thus the composition

$$\delta : R \longrightarrow \widehat{R} \longrightarrow \widehat{R}/J$$

is an embedding. Moreover, if $0 \neq r \in R$, $\delta(r)$ is a nonzero divisor of $\widehat{R}/J$.

We summarize this in a corollary:

Corollary 2.2.4. *If $\widetilde{Q} \in \text{Ass}(\widehat{R}/J)$, then $\widetilde{Q} \cap R = 0$. In particular, $J \cap R = 0$.*

In other words, all the associated primes of $\widehat{R}/J$ are in the generic formal fiber of R , and we have an embedding $\delta : R \hookrightarrow \widehat{R}/J$. Moreover, if $0 \neq r \in R$, $\delta(r)$ is a nonzero

divisor of $\widehat{R}/J$. We can then define an intermediate ring A between R and $\widehat{R}/J$:

$$A = Q(R) \cap \widehat{R}/J$$

We will show that A is indeed a local noetherian ring with completion $\widehat{R}/J$. In order to do this, we construct a subring B of A as a nested union of local K -algebras with $Q(A) = Q(B)$. The main objective will be to show that B is a local noetherian ring, which will give us that $A = B$.

2.3 Construction of the ring B

Recall the following definitions:

$$z_{ik} = z_i + a_{i1}q_1 + \cdots + a_{ik}q_k^k \quad \text{for } k \geq 1$$

$$\zeta_i := \begin{cases} z_i + \sum_{l=1}^{\infty} a_{il}q_l^l & \text{if } 1 \leq i \leq m \\ z_i & \text{if } m < i \leq n \end{cases}$$

$$I := (F_1(z_1, \dots, z_n), \dots, F_r(z_1, \dots, z_n))$$

and

$$\varphi : \widehat{R} \rightarrow \widehat{R}$$

with:

$$\varphi(z_i) = \zeta_i \quad \text{for } i = 1, \dots, n$$

Now define:

$$f_i = \varphi(F_i) = F_i(\zeta_1, \dots, \zeta_n)$$

Then $J = (f_1, \dots, f_r) \subseteq K[[z_1, \dots, z_n]] = \widehat{R}$.

We define for $1 \leq j \leq r$ and $k \geq 1$:

$$\alpha_{jk} = \frac{1}{q_k^k} F_j(z_{1k}, \dots, z_{mk}, z_{m+1}, \dots, z_n) \in Q(R)$$

Lemma 2.3.1. *For $1 \leq j \leq r$ and $k \geq 1$, $\alpha_{jk} = r_{1k}\alpha_{j(k+1)} + r_{2k}$, where r_{1k}, r_{2k} are in R .*

Proof. By definition,

$$\alpha_{jk} = \frac{1}{q_k^k} F_j(z_{1k}, \dots, z_{mk}, z_{m+1}, \dots, z_n)$$

We write:

$$\begin{aligned} \alpha_{j(k+1)} &= \frac{1}{q_{k+1}^{k+1}} F_j(z_{1(k+1)}, \dots, z_{m(k+1)}, z_{m+1}, \dots, z_n) \\ &= \frac{1}{q_{k+1}^{k+1}} F_j(z_{1k} + a_{1(k+1)}q_{k+1}^{k+1}, \dots, z_{mk} + a_{m(k+1)}q_{k+1}^{k+1}, \dots, z_n) \end{aligned}$$

Hence, using Taylor's formula:

$$\alpha_{jk} = \frac{q_{k+1}^{k+1}}{q_k^k} [\alpha_{j(k+1)} + s_{jk}]$$

where $s_{jk}, \frac{q_{k+1}^{k+1}}{q_k^k} \in R$. □

Lemma 2.3.2. *For every $j \in \{1, \dots, r\}$ and $k \geq 1$, the elements α_{jk} are in $A = (\widehat{R}/J) \cap Q(R)$.*

Proof. Since every α_{jk} is an element of the quotient field of R , we need only show that $\alpha_{jk} \in \widehat{R}/J$. As

$$\alpha_{jk} = \frac{1}{q_k^k} F_j(z_{1k}, \dots, z_{nk}),$$

we have that

$$F_j(z_{1k}, \dots, z_{nk}) = \alpha_{jk} q_k^k.$$

Then by Taylor's formula:

$$\begin{aligned} F_i(\zeta_1, \dots, \zeta_n) - \alpha_{jk} q_k^k &= F_i(\zeta_1, \dots, \zeta_n) - F_i(z_{1k}, \dots, z_{nk}) \\ &= q_k^{k+1} \kappa_{jk} \end{aligned}$$

for some $\kappa_{jk} \in \widehat{R}$. Since $F_i(\zeta_1, \dots, \zeta_n) \in J$,

$$-\alpha_{jk} q_k^k \equiv q_k^{k+1} \kappa_{jk} \pmod{J}$$

That is, $\alpha_{jk} \in \widehat{R}/J$. $\square$

We use the elements α_{ij} to define a sequence of subrings of A . Recall that $\frac{\widehat{R}}{J}$ is R -torsion free; that is, for all $Q \in \text{Ass}(\frac{\widehat{R}}{J})$, the contraction $Q \cap R = 0$. Using this, we can extend the canonical injection:

$$R \hookrightarrow \frac{\widehat{R}}{J}$$

to a commutative diagram:

$$\begin{array}{ccc} R & \hookrightarrow & \widehat{R}/J \\ \downarrow & & \downarrow \tau \\ Q(R) & \xrightarrow{\lambda} & (\widehat{R}/J) \otimes_R Q(R) \end{array}$$

where τ and λ are injective. We restrict λ to get injective morphisms:

$$\lambda_k : R[\alpha_{1k}, \dots, \alpha_{rk}] \hookrightarrow \widehat{R}/J$$

for all $k \in \mathbb{N}$.

For all $k \in \mathbb{N}$, define

$$\mathfrak{w}_k := \lambda_k^{-1}(\mathfrak{m})$$

where $\mathfrak{m} \subseteq \widehat{R}/J$ is the maximal ideal, and

$$B_k := R[\alpha_{1k}, \dots, \alpha_{rk}] \mathfrak{w}_k.$$

Because of the recursion formulas

$$\alpha_{jk} = \frac{q_k^{k+1}}{q_k^k} [\alpha_{j(k+1)} + s_{jk}]$$

we get local inclusions $B_k \subseteq B_{k+1}$ for all $k \in \mathbb{N}$. We define the nested union ring:

$$B = \bigcup_{k \in \mathbb{N}} B_k = \varinjlim B_k$$

Proposition 2.3.3. *B is a local noetherian ring. Furthermore, for any prime ideal $0 \neq q \in \text{Spec}(B)$, the ring B/q is a K -algebra of essentially finite type.*

Proof. That B is quasi-local follows immediately from the definition of B . To show that B is noetherian, by a theorem of Cohen (Matsumura 3.4), it is enough to show that every prime ideal of B is finitely generated. Let $0 \neq q \in \text{Spec}(B)$. Then $q \cap R \neq 0$, and there exists an $l \in \mathbb{N}$ so that $p_l \in q$.

We claim that there exists a surjection $R \longrightarrow B/(p_l B)$.

To prove the claim, let $x \in B = \bigcup B_k$. Then there exists $k \in \mathbb{N}$ so that $x \in B_k = R[\alpha_{1k}, \dots, \alpha_{rk}] \mathfrak{w}_k$. We may assume that $k \geq l$. Write:

$$x = \frac{h(\alpha_{1k}, \dots, \alpha_{rk})}{g(\alpha_{1k}, \dots, \alpha_{rk})}$$

where $h, g \in R[X_1, \dots, X_r]$ are polynomials in r variables over R and $g(\alpha_{1k}, \dots, \alpha_{rk})$ is a unit in $\widehat{R}/J$. Note that this implies that the constant term g_0 of g is a unit in $\widehat{R}/J$ and hence in R . Denote by h_0 the constant term of h and by g_0 the constant term of g .

Recall that:

$$\alpha_{jk} = \frac{q_k^{\epsilon_k + 1}}{q_k^{\epsilon_k}} [\alpha_{j(k+1)} + s_{jk}].$$

Using this formula, we can write

$$x = \frac{h_0 + p_l \rho}{g_0 + p_l \tau}$$

where ρ and τ are elements of B_{k+1} . Thus:

$$x \equiv \frac{h_0}{g_0} \pmod{p_l B}$$

where $h_0, g_0 \in R$ and g_0 is a unit in R . Thus $\frac{B}{p_l B}$ is a homomorphic image of R , so q is finitely generated. In particular, B/q is a homomorphic image of an algebra that is essentially of finite type over K . $\square$

Proposition 2.3.4. $\hat{B} = \hat{R}/J$.

Proof. Consider the sequence of injections:

$$R \hookrightarrow B \hookrightarrow \frac{\hat{R}}{J}$$

These induce morphisms on the completions:

$$\hat{R} \xrightarrow{\iota} \hat{B} \longrightarrow \frac{\hat{R}}{J}$$

We know that $R \longrightarrow \frac{B}{p_s B}$ is surjective for any $s \in \mathbb{N}$, so $\hat{R} \longrightarrow \hat{B}$ is surjective. We need to show that:

$$\ker(\iota) = J$$

Clearly $\ker(\iota) \subset J$. We now show $J \subset \ker(\iota)$. By definition of α_{jk} :

$$F_j(z_{1k}, \dots, z_{nk}) = \alpha_{jk} q_k^k$$

Hence:

$$\begin{aligned} F_j(\zeta_1, \dots, \zeta_n) - q_k^k \alpha_{jk} &= F_j(\zeta_1, \dots, \zeta_n) - F_j(z_{1k}, \dots, z_{nk}) \\ &= q_{k+1}^{k+1} \kappa_{jk} \end{aligned}$$

for some $\kappa_{jk} \in \hat{R}$. Hence $\iota(F_j) \in q_k^k \hat{B}$ for all $k \in \mathbb{N}$; that is, the generators of the ideal J are contained in the kernel of the map ι . Thus $J \subseteq \ker(\iota)$, so $J = \ker(\iota)$. $\square$

Proposition 2.3.5. The ideal $\hat{P} = (\zeta_1, \dots, \zeta_m) \subset \hat{B} = \hat{R}/J$ is in the generic formal fiber of B ; i.e.

$$\hat{P} \cap B = 0.$$

Proof. Since $Q(R) = Q(B)$, if $\hat{P} \cap B \neq 0$, then $\hat{P} \cap R \neq 0$, a contradiction to Lemma 3. $\square$

Finally, we note that $A = B$. This is clear because $B \subseteq A$ is a noetherian domain with $Q(B) = Q(A)$. The flatness of $B \longrightarrow \widehat{B}$ yields $B = A$.

In summary, we have the following Theorem:

Theorem 2.3.6. *Let K be a field of countably infinite transcendence degree over a countably infinite field k . Let $R = K[z_1, \dots, z_n]_{(z_1, \dots, z_n)}$ be a localized polynomial ring in n variables over K . Let m be a positive integer so that $m < n$. If I is an ideal of R so that for every $\mathcal{Q} \in \text{Ass}(\frac{\widehat{R}}{I\widehat{R}})$, $\mathcal{Q} \subseteq (z_1, \dots, z_m)\widehat{R}$, we can find a local noetherian domain B which birationally dominates R and satisfies:*

- $\widehat{B} \cong \frac{\widehat{R}}{I\widehat{R}}$
- For every $0 \neq q \in \text{Spec}(B)$, there is a natural surjection $R \longrightarrow \frac{B}{q}$
- The prime ideal $\widehat{P} = (\zeta_1, \dots, \zeta_m)$ is in the generic formal fiber of B .

3 Conditions for the Normality and Factoriality of B

In this section, look at conditions under which the constructed ring B is normal and factorial. We then find "recipes" for constructing normal domains with non-normal completions and factorial rings with non-factorial completions.

3.1 Conditions for Normality

This section contains a result giving a necessary and sufficient condition for B to be a normal domain. The ring R is excellent, so R/I is. If R/I is normal, then $\widehat{R/I} \cong \widehat{B}$ is normal. Then B is normal. On the other hand, if R/I is not normal, we consider the nonnormal locus of $\widehat{R}/J$ and use the defining ideals of this non-normal locus to describe a criterion for the normality of B . We start with some well known results from Matsumura.

Theorem 3.1.1. *Let (A, m, k) and (B, n, k') be local Noetherian rings and*

$$\lambda : A \rightarrow B$$

a faithfully flat map. Then:

- *If B is a regular local ring, so is A*
- *If A and $\frac{B}{mB}$ are regular local rings, then B is a regular local ring.*

Next we recall the *Serre Conditions* on a Noetherian ring A . These are used as in the subsequent theorem to give equivalent conditions for a noetherian ring to be reduced or normal.

(R _{i}) A_P is regular for all $P \in \text{Spec}A$ with $ht(P) \leq i$;

(S _{i}) $\text{depth } A_P \geq \min(htP, i)$ for all $P \in \text{Spec}A$

Theorem 3.1.2. (Serre) *Let A be a noetherian ring. A is reduced if and only if the Serre conditions (R_0) and (S_1) are satisfied. A is normal if and only if the Serre conditions (R_1) and (S_2) are satisfied.*

One of the most important features of the ring R/I is that it is an excellent ring. We recall the definition and some well-known facts about excellent rings:

Definition 3.1.3. *A noetherian local ring A is said to be excellent if A satisfies the following two conditions:*

- (i) *A is universally catenary*
- (ii) *The formal fibers of A are geometrically regular.*

Note that if a noetherian local ring A is excellent, then a localization, finitely generated extension, or quotient ring of A is also excellent. In particular, our ring R/I is an excellent ring, since R is essentially finite over a field.

Theorem 3.1.4. *If (A, m, k) is a local noetherian ring and $\hat{A}$ is the completion of A with respect to the maximal ideal m , then:*

- (1) *A is regular if and only if $\hat{A}$ is.*
- (2) *A is Cohen-Macaulay if and only if $\hat{A}$ is.*

If we assume in addition that A is excellent, then:

- (3) *A satisfies the Serre condition (S_i) if and only if $\hat{A}$ does.*
- (4) *A satisfies the Serre condition (R_i) if and only if $\hat{A}$ does.*

As a result of (3) and (4), if A is excellent, we also have:

- (5) *A is normal if and only if $\hat{A}$ is.*
- (6) *A is reduced if and only if $\hat{A}$ is.*

It is well known that the normal locus of an excellent, local, Noetherian ring is open. In fact, we propose the following lemma:

Lemma 3.1.5. *Let A be a reduced, excellent, Noetherian local ring satisfying the Serre condition (S_2) . The non-normal locus of A can be described as follows:*

$$\text{NNor}(A) = V(Q_1 \cap \dots \cap Q_s)$$

where the ideals $Q_1, \dots, Q_s$ are height one primes of A . We call the ideals $Q_1, \dots, Q_s$ the defining ideals of the non-normal locus of A .

Proof. The excellence of A implies that the non-normal locus of A is closed. In particular, there exists a reduced ideal $\mathfrak{J}$ of A with:

$$\text{NNor}(A) = V(\mathfrak{J}).$$

Since $\mathfrak{J}$ is reduced, write $\mathfrak{J} = Q_1 \cap \dots \cap Q_s$, where the ideals $Q_1, \dots, Q_s$ are minimal primes over $\mathfrak{J}$. Then:

$$\text{NNor}(A) = V(\mathfrak{J}) = V(Q_1 \cap \dots \cap Q_s) = V(Q_1) \cup \dots \cup V(Q_s).$$

Consider $Q_i \in \{Q_1, \dots, Q_s\}$. Clearly Q_i is in the singular locus of A . Note that A is reduced, so A satisfies the Serre condition (R_0) . Thus, Q_i has height greater than zero.

Suppose the height of Q_i is greater than 1. Then A_{Q_i} is a non-normal ring satisfying the Serre condition (S_2) . In other words, A_{Q_i} does not satisfy the Serre condition (R_1) . Thus there is a height one prime Q of A with $Q \subset Q_i$ so that A_Q is not regular. Then $Q \in \text{NNor}(A) = V(\mathfrak{J})$, contradicting the minimality of Q_i over $\mathfrak{J}$. Thus Q_i is a height one prime ideal of A for $i = 1, \dots, s$. $\square$

In order to make use of this lemma, later in this section we will assume that R/I is a reduced ring satisfying the Serre condition (S_2) . Many of the rings we encounter will also be excellent; for example, the complete local noetherian ring $\widehat{R}/J$ is excellent.

In general, our constructed ring B is not excellent, so it will be useful to work with an ideal-adic completion of B that is excellent.

For some $h \in \mathbb{N}$, consider the element p_h of R . Let R^* denote the (p_h) -adic completion of R and let B^* denote the (p_h) -adic completion of B . By results of Rotthaus [20], [21], the ring R^* is excellent. We can use this fact to prove the following lemma:

Lemma 3.1.6. *B^* is excellent.*

Proof. Recall the definition of the ideal $J \subseteq \hat{R}$:

$$J = (F_1(\zeta_1, \dots, \zeta_n), \dots, F_r(\zeta_1, \dots, \zeta_n)).$$

For $1 \leq j \leq r$, we write:

$$F_j(\zeta_1, \dots, \zeta_n) = F_j(z_1(h-1) + c_{1h}, \dots, z_m(h-1) + c_{mh}, z_{m+1}, \dots, z_n)$$

where $c_{ih} \in R^*$ for $1 \leq i \leq m$. From this, we can see that the ideal J of $\hat{R}$ is extended from the ring R^* . Echoing the proof of Proposition 2.3.4, we can see that

$$B^* \cong \frac{R^*}{J \cap R^*}.$$

Thus B^* is excellent. □

Note: In the proof of Proposition 2.3.3 we found a surjection $R \longrightarrow B/p_h B$. As a homomorphic image of R , the ring $B/p_h B$ is excellent as well.

In order to examine conditions for the normality of B , we prove one more lemma. For this lemma, we need neither assume that R/I is reduced nor that R/I satisfies the Serre condition (S_2) .

Lemma 3.1.7. *Let R , I , and B be as in the previous section. If Q is a non-zero prime ideal in the regular locus of B , and $W \subseteq \hat{B}$ is a prime ideal of $\hat{B}$ lying over Q , then $\hat{B}_W$ is regular.*

Proof. Suppose Q is a non-zero prime ideal in the regular locus of B . Then $Q \cap R \neq 0$, so there exists some element $p_h \in Q$. Let B^* denote the (p_h) -adic completion of B and R^* denote the (p_h) -adic completion of R . The extension $Q^* := QB^*$ is a prime ideal of B^* . We have a faithfully flat homomorphism:

$$B_Q \longrightarrow B_{Q^*}^*.$$

The ring B_Q is regular, and so is the field $B_{Q^*}^*/QB_{Q^*}^*$. Hence the ring $B_{Q^*}^*$ is an excellent regular ring. Let W be a prime ideal of $\hat{B}$ lying over Q . Since $p_h \in Q^*$ and since every ideal of B^* which contains p_h is extended from B , the prime ideal W also lies over $Q^* \subseteq B^*$. Recall from the proof of Lemma 3.1.6 that B^* is excellent, and thus has regular formal fibers. Hence, the homomorphism:

$$B_{Q^*}^* \longrightarrow \hat{B}_W$$

is faithfully flat and regular. Hence $\hat{B}_W$ is regular. $\square$

We are now ready to prove the main theorem of the section. Here R/I and B are as described in the previous section.

Theorem 3.1.8. *Suppose $\frac{R}{I}$ is a reduced ring satisfying the Serre condition (S_2) .*

Let $\{Q_1, \dots, Q_s\}$ be the height one defining prime ideals of the non-normal locus of $\hat{R}/J$. Then B is normal if and only if $Q_i \cap B = 0$ for all $1 \leq i \leq s$.

Proof. First we assume B is normal. Suppose that $Q = Q_i \cap B \neq 0$ for some $i \in \{1, \dots, s\}$. Since B is a domain, Q is a height one prime of B . The normality of B yields that B_Q is a regular local ring. Then by Lemma 3.1.7, Q_i must be in the regular locus of $\hat{B} = \hat{R}/J$, a contradiction.

Now we assume that $Q_i \cap B = 0$ for all $1 \leq i \leq s$. We will use the Serre criterion to show that B is normal. Since R/I is an excellent ring satisfying (S_2) , by Theorem 3.1.4, the ring $\hat{B} \cong \hat{R}/I\hat{R}$ satisfies (S_2) . Hence B does.

To see that B satisfies (R_1) , let $\mathfrak{P}$ be a height one prime ideal of B . We need to show that $B_{\mathfrak{P}}$ is a regular local ring. Since $B \rightarrow \hat{B}$ is faithfully flat, there exists a height one prime $\mathcal{P} \subseteq \hat{B}$ so that $\mathfrak{P} = \mathcal{P} \cap B$. By assumption that $Q_i \cap B = 0$ for all $1 \leq i \leq s$, it is clear that $\mathcal{P} \notin \{Q_1, \dots, Q_s\}$. Moreover, since $\mathcal{P}$ has height one, $\mathcal{P} \notin V(Q_1 \cap \dots \cap Q_s)$. Hence, $\mathcal{P}$ is in the normal locus of $\hat{B}$. In particular, $\hat{B}_{\mathcal{P}}$ is a regular local ring. Since the map:

$$B_{\mathfrak{P}} \longrightarrow \hat{B}_{\mathcal{P}}$$

is faithfully flat, $B_{\mathfrak{P}}$ is regular. Hence B is normal. $\square$

3.2 Illustrating the Normality of B : Examples

In Theorem 3.1.8, if R/I is not reduced, we can find examples in which the other conditions of the theorem are met, but the conclusion does not hold.

Example 3.2.1.

Let $R = K[z_1, z_2]_{(z_1, z_2)}$, let $I = (z_1^2)$, and let $m = 1$. Then

$$\hat{B} = \hat{R}/(\zeta_1^2).$$

The defining ideal of the non-normal locus of $\hat{B}$ is the prime ideal $(\zeta_1) = \hat{P}$. In otherwords, the defining ideal of the non-normal locus of $\hat{B}$ is in the generic formal fiber of B .

Claim. B is not normal.

Suppose B were normal. Then B would be a one-dimensional normal noetherian local domain; i.e., B would be a DVR. Hence, by Theorem 14, $\hat{B}$ would be a DVR, and so would R/I , a contradiction.

More generally, if B is a one-dimensional normal domain, then it is regular. By

T

T

C

:

E

L

T

ti

E

L

A

0.

E.

L.

Te

Theorem 14,

$$\begin{aligned}
B \text{ is regular} &\iff \hat{B} \text{ is regular} \\
&\iff \hat{R}/I\hat{R} \text{ is regular} \\
&\iff R/I \text{ is regular.}
\end{aligned}$$

Thus, in order to use the theorem to produce interesting examples, we must consider cases where the Krull dimension of R/I is at least two. We now look at some examples that use the theorem.

Example 3.2.2. *A noetherian, normal local domain with non-normal completion.*

Let $R = K[z_1, z_2, z_3]_{(z_1, z_2, z_3)}$, let $I = (z_1 z_2)$, and let $m = 2$. Then:

$$\hat{B} = K[[\zeta_1, \zeta_2, z_3]]/(\zeta_1 \zeta_2)$$

The defining ideal of the non-normal locus of $\hat{B}$ is (ζ_1, ζ_2) . Since $(\zeta_1, \zeta_2) \cap B = 0$, the ring B is normal.

Example 3.2.3. *Another noetherian normal domain with non-normal completion.*

Let $R = K[z_1, z_2, z_3]_{(z_1, z_2, z_3)}$, $m = 2$, and $I = (z_1^2 - z_2^3)$. Then:

$$\hat{B} = K[[\zeta_1, \zeta_2, z_3]]/(\zeta_1^2 - \zeta_2^3)$$

Again, the defining ideal of the non-normal locus of $\hat{B}$ is (ζ_1, ζ_2) . Since $(\zeta_1, \zeta_2) \cap B = 0$, the ring B is normal.

Example 3.2.4. *Another normal noetherian domain with non-normal completion.*

Let $R = K[z_1, z_2, z_3]_{(z_1, z_2, z_3)}$, let $I = (z_2^2 - z_1^3 - z_1^2)$ and let $m = 2$. Then:

$$\hat{B} = \frac{K[[\zeta_1, \zeta_2, z_3]]}{(\zeta_2^2 - \zeta_1^3 - \zeta_1^2)}.$$

The non-normal locus of $\hat{B}$ is defined by the ideal $\hat{P} = (\zeta_1, \zeta_2)$, so B is normal.

Remark 3.2.5. A “recipe” for constructing B so that B is normal.

Throughout the previous examples, we relied heavily on the fact that the prime ideal $\hat{P} = (\zeta_1, \dots, \zeta_m)$ is in the generic formal fiber of B ; i.e., $\hat{P} \cap B = 0$. Determining the normality of B is often a matter of checking if the defining ideals of the non-normal locus of B are contained in the ideal $\hat{P}$. In particular, we get the following “recipe” for constructing examples:

Suppose R/I is a non-normal ring satisfying the Serre condition (S_2) . Then the non-normal locus of R/I can be described as follows:

$$NNOR(R/I) = V(\tilde{Q}_1 \cap \dots \cap \tilde{Q}_r)$$

where each $\tilde{Q}_i$ is a height one prime ideal of the excellent ring R/I . The extension $\tilde{Q}_i \hat{R} / I \hat{R}$ is reduced in $\hat{R} / I \hat{R}$ and we can write:

$$\tilde{Q}_i \hat{R} = \widetilde{W_{i1}} \cap \dots \cap \widetilde{W_{il_i}}$$

where the ideals $\widetilde{W_{ij}}$ are height one prime ideals in $\hat{R} / I \hat{R}$. Let

$$\overline{\varphi}: \frac{\hat{R}}{I \hat{R}} \longrightarrow \frac{\hat{R}}{J}$$

be the isomorphism induced by the automorphism φ on $\hat{R}$.

Proposition 3.2.6. Suppose that for all i and j , the ideals $\widetilde{W_{ij}}$ are contained in the ideal $(z_1, \dots, z_m) \hat{R} / I \hat{R}$. Then:

- (a) $NNOR(\hat{R}/J) = V(W_{11} \cap \dots \cap W_{il_1} \cap \dots \cap W_{r1} \cap \dots \cap W_{rl_r})$ where $W_{ij} = \overline{\varphi}(\widetilde{W_{ij}})$.
- (b) $\overline{\varphi}(\widetilde{W_{ij}}) = W_{ij} \subseteq \overline{\varphi}((z_1, \dots, z_m) \hat{R} / I \hat{R}) = (\zeta_1, \dots, \zeta_m) \overline{\varphi}(\hat{R}/J)$. Hence $W_{ij} \cap B = (0)$ for all i and j . Thus B is normal.

Example 3.2.7. In which we use the proposition to check that B is normal.

Let $R = k[z_1, z_2, z_3]_{(z_1, z_2, z_3)}$, let:

$$I = (z_2^2 - z_1^2(z_3^2 - z_1))$$

and let $m = 2$. Then:

$$\hat{B} = \frac{k[[\zeta_1, \zeta_2, z_3]]}{(\zeta_2^2 - \zeta_1^2(z_3^2 - \zeta_1))}.$$

The defining ideal of the non-normal locus of R/I is the ideal (z_1, z_2) . This extends to the prime ideal $(z_1, z_2)\hat{R}/I\hat{R}$. By Proposition 3.2.6, B is normal.

We look at another application of the Proposition:

Example 3.2.8. *In which we use the proposition to check that B is normal.*

Let $R = k[a, b, c, d, e, f]_{(a, b, c, d, e, f)}$, where k is a field of characteristic $\neq 2, 3$. Let:

$$I = (b^2 - a^2(c^2 - a), e^2 - d^2(f^2 - d))$$

and let $P = (a, b, c, d, e)$. Then I is a prime ideal of height 2. We use the Jacobian criterion to determine the defining ideals of the non-normal locus of the excellent domain R/I . First, we find the Jacobian matrix of I :

$$\mathbf{J} = \begin{pmatrix} -2ac^2 + 3a^2 & 0 \\ 2b & 0 \\ -2a^2c & 0 \\ 0 & -2df^2 + 3d^2 \\ 0 & 2e \\ 0 & -2d^2f \end{pmatrix}$$

Let m be the ideal of 2×2 minors of $\mathbf{J}$; i.e.,

$$M = (4ac^2df^2 - 6df^2a^2 - 6ac^2d^2 + 9a^2d^2, -4ac^2e + 6ea^2, 4ac^2d^2f - 6a^2d^2f, \\ 4bdf^2 + 6bd^2, 4be, -4bd^2f, 4a^2dcf^2 - 6a^2d^2c, -4a^2ce, -4a^2cd^2f).$$

Let $N = I + M$.

Claim $\mathfrak{J} = (ad, be, ae, bd, b^2 - a^2(c^2 - a), e^2 - d^2(f^2 - d)) \subseteq \sqrt{N}$.

Clearly $be, b^2 - a^2(c^2 - a), e^2 - d^2(f^2 - d) \in \sqrt{N}$. since $be, b^2 - a^2(c^2 - a)$, and $e^2 - d^2(f^2 - d)$ are all elements of N . To see that $bd \in \sqrt{N}$:

$$\begin{aligned}
& 4bdf^2 + 6bd^2 && \in N \\
\Rightarrow & 4bd^2f^2 + 6bd^3 && \in N \\
\Rightarrow & 4bd^2f^2 + f(-4bd^2f) + 6bd^3 && \in N \\
\Rightarrow & 6bd^3 && \in N \\
\Rightarrow & bd && \in \sqrt{N}
\end{aligned}$$

To see that $ae \in \sqrt{N}$:

$$\begin{aligned}
& 4ac^2e + 6ea^2 && \in N \\
\Rightarrow & 4a^2c^2e + 6ea^3 && \in N \\
\Rightarrow & 4a^2c^2e + c(-4a^2ce) + 6ea^3 && \in N \\
\Rightarrow & 6ea^3 && \in N \\
\Rightarrow & ea && \in \sqrt{N}
\end{aligned}$$

To see that $ad \in \sqrt{N}$:

$$\begin{aligned}
& 4ac^2df^2 - 6df^2a^2 - 6ac^2d^2 + 9a^2d^2 && \in N \\
\Rightarrow & 4a^2c^2d^2f^2 - 6d^2f^2a^3 - 6a^2c^2d^3 + 9a^3d^3 && \in N \\
\Rightarrow & 4a^2c^2d^2f^2 - 4a^2c^2d^2f^2 - 6d^2f^2a^3 - 6a^2c^2d^3 + 9a^3d^3 && \in N \\
\Rightarrow & -6d^2f^2a^3 - 6a^2c^2d^3 + 9a^3d^3 && \in N \\
\Rightarrow & -6d^2f^2a^3 - fa(4ac^2d^2f - 6a^2d^2f) - 6a^2c^2d^3 + 9a^3d^3 && \in N \\
\Rightarrow & -4a^2c^2d^2f^2 - 6a^2c^2d^3 + 9a^3d^3 && \in N \\
\Rightarrow & -4a^2c^2d^2f^2 - (cf(-4a^2cd^2f) - 6a^2c^2d^3 + 9a^3d^3) && \in N \\
\Rightarrow & -6a^2c^2d^3 + 9a^3d^3 && \in N \\
\Rightarrow & -6a^2c^2d^3 - cd(4a^2dcf^2 - 6a^2d^2c) + 9a^3d^3 && \in N \\
\Rightarrow & -4a^2d^2c^2f^2 + 9a^3d^3 && \in N \\
\Rightarrow & -4a^2d^2c^2f^2 + 4a^2c^2d^2f^2 + 9a^3d^3 && \in N \\
\Rightarrow & 9a^3d^3 && \in N \\
\Rightarrow & ad && \in \sqrt{N}
\end{aligned}$$

So $(ad, be, ae, bd, b^2 - a^2(c^2 - a), e^2 - d^2(f^2 - d)) \subseteq \sqrt{N}$.

It is easy to see that $\sqrt{N} = \sqrt{J}$ and that the associated primes of $\sqrt{N} = \sqrt{J}$ are $(a, b, e^2 - d^2(f^2 - d))$ and $(d, e, b^2 - a^2(c^2 - a))$. Since these two ideals are both contained in $P = (a, b, c, d, e)$, by the previous proposition, our ring B is normal.

In general, B is not normal, even if R/I is reduced. We can use Theorem 3.1.8 to construct an example.

Example 3.2.9. *In which B is not normal.*

Let $R = Kz_1, z_2, z_3, z_4, z_5$, let $I = (z_3^2z_2^2 - z_4^3z_1^3 - z_4^2z_1^2)$, and let $m = 2$. The ring R/I is a non-normal domain satisfying the Serre condition (S_2) .

After the automorphism of $\hat{R}$, we have:

$$\hat{B} = \frac{\hat{R}}{(z_3^2\zeta_2^2 - z_4^3\zeta_1^3 - z_4^2\zeta_1^2)}.$$

The height one prime ideal (z_3, z_4) is in the non-normal locus of $\widehat{B}$, and $(z_3, z_4) \cap B \neq 0$. Thus, B is not normal.

A noetherian local ring (A, m) is Cohen-Macaulay if and only if the m -adic completion $\widehat{A}$ is Cohen-Macaulay. If A is excellent then for any $i \in \mathbb{N}$, A satisfies the Serre Condition (S_i) if and only if $\widehat{A}$ satisfies (S_i) . Moreover, if A is a quotient of a Cohen-Macaulay ring, A satisfies the Serre Condition (S_i) if and only if $\widehat{A}$ satisfies (S_i) . This is exercise 23.2 in [13], but the proof is very short and we include it here. First we show that a Cohen-Macaulay ring has Cohen-Macaulay formal fibers. Let S be a local Cohen-Macaulay ring and let $\widehat{S}$ be the completion of S . Let $p \in \text{Spec}(S)$ be a prime ideal of height j and let $\widehat{p}$ be a prime ideal in $\text{Spec}(\widehat{S})$ so that $\widehat{p} \cap S = p$. Then $S_p \longrightarrow \widehat{S}_{\widehat{p}}$ is a local flat homomorphism, so by [13], Corollary to Theorem 23.3,

$$\text{depth}(\widehat{S}_{\widehat{p}}) = \text{depth}(S_p) + \text{depth}(\widehat{S}_{\widehat{p}}/p\widehat{S}_{\widehat{p}}).$$

Furthermore,

$$\dim(\widehat{S}_{\widehat{p}}) = \dim(S_p) + \dim(\widehat{S}_{\widehat{p}}/p\widehat{S}_{\widehat{p}}).$$

Since S_p and $\widehat{S}_{\widehat{p}}$ are Cohen-Macaulay, the depth of these two rings is equal to the respective dimensions. Thus:

$$\text{depth}(\widehat{S}_{\widehat{p}}/p\widehat{S}_{\widehat{p}}) = \dim(\widehat{S}_{\widehat{p}}/p\widehat{S}_{\widehat{p}})$$

and the formal fiber is Cohen-Macaulay. Now let $T = S/I$ for some ideal I of S . Suppose T satisfies the Serre Condition (S_i) . To show that $\widehat{T}$ satisfies (S_i) , we need to show that the fiber ring $\widehat{T} \otimes_T \mathfrak{K}(p)$ over every prime ideal p in $\text{Spec}(T)$ satisfies (S_i) . Let $p \in \text{Spec}(T)$ and let $\mathfrak{p} \in \text{Spec}(S)$ so that $\mathfrak{p}/I = p$. Then:

$$\widehat{T} \otimes_T \mathfrak{K}(p) = \widehat{T} \otimes_T (T/p)_p = \widehat{S} \otimes_S (S/\mathfrak{p})_{\mathfrak{p}} = \widehat{S} \otimes_S \mathfrak{K}(\mathfrak{p})$$

Since S has Cohen-Macaulay formal fibers, $\widehat{S} \otimes_S \mathfrak{K}(\mathfrak{p})$ satisfies (S_i) . Hence the fiber ring $\widehat{T} \otimes_T \mathfrak{K}(p)$ over every prime ideal p in $\text{Spec}(T)$ satisfies (S_i) . By a similar argument, $\widehat{T}$ satisfies (S_i) .

However, in certain circumstances, we can obtain a normal ring B even in case R/I does not satisfy (S_2) . In this case, B will satisfy (S_2) but $\widehat{B}$ will not. We propose the following:

Proposition 3.2.10. *Let $R = k[z_1, \dots, z_n]_{(z_1, \dots, z_n)}$ and I be an ideal of R so that R/I is a reduced ring of depth at least 2 and dimension d . Let $P = (z_1, \dots, z_n - 1)$, and suppose the non-normal locus of $\widehat{R}/I\widehat{R}$ is equal to $V(P\widehat{R}/I\widehat{R})$. Then B is normal.*

Proof. The ring R/I has depth 2 and satisfies (R_0) and (S_1) , hence B does. In order to check the Serre condition (S_2) , we let Q be a prime ideal of B . We need to show:

$$\text{depth}(B_Q) \geq \min(\text{ht}(Q), 2).$$

We look at three cases.

Case 1. The prime ideal Q has height d .

In this case, Q is the maximal ideal of B , so

$$\text{depth}(B_Q) = \text{depth}(B) = 2.$$

Case 2. The prime ideal Q has height $d - 1$.

Let $\widehat{Q}$ be a height $d - 1$ prime ideal of $\widehat{B}$ so that $\widehat{Q} \cap B = Q$. The ideal $\widehat{Q}$ is not in the generic formal fiber of B , so $\widehat{Q} \neq \widehat{P} = (\zeta_1, \dots, z_n - 1)$. Thus $\widehat{Q}$ is in the (S_2) locus of $\widehat{B}$ and in particular:

$$\text{depth}(B_Q) = \text{depth}(\widehat{B}_{\widehat{Q}}) \geq \min(\text{ht}(\widehat{Q}), 2) = \min(\text{ht}(Q), 2).$$

Case 3. The prime ideal Q has height $h < d - 1$.

Let $\widehat{Q}$ be a prime ideal of $\widehat{B}$ so that $\widehat{Q} \cap B = Q$ and the height of $\widehat{Q}$ is the same as that of Q . Then $\widehat{P}$ is not contained in $\widehat{Q}$, so again,

$$\text{depth}(B_Q) = \text{depth}(\widehat{B}_{\widehat{Q}}) \geq \min(\text{ht}(\widehat{Q}), 2) = \min(\text{ht}(Q), 2).$$

Thus B satisfies the Serre condition (S_2) .

To see that B satisfies (R_1) , notice that the non-normal locus of $\hat{R}/I\hat{R}$ contains the non- (R_1) locus. If $\text{ht}(P\hat{R}/I\hat{R}) > 1$, then $\hat{B}$ satisfies (R_1) . Thus B satisfies (R_1) . Since R/I is reduced, we know that $\text{ht}(P\hat{R}/I\hat{R}) > 0$. If $\text{ht}(P\hat{R}/I\hat{R}) = 1$, then by the previous proposition, B is normal. $\square$

Although the conditions of the proposition may seem technical, in practice they are easy to check.

We can generalize the previous proposition without major changes to the proof.

Proposition 3.2.11. *Let $R = k[z_1, \dots, z_n]_{(z_1, \dots, z_n)}$ and I be an ideal of R so that R/I is a reduced ring of depth at least i and dimension d , where $d > i$. Let $P = (z_1, \dots, z_{n-1})$, and let the non- (S_i) locus of $\hat{R}/I\hat{R}$ be equal to $V(P\hat{R}/I\hat{R})$. Then B satisfies (S_i) .*

3.3 The Factoriality of B

We first recall some useful theorems:

Theorem 3.3.1. *(Mori's Theorem [6]. Corollary 6.12) Let (A, m) be a noetherian local ring and $\hat{A}$ its m -adic completion. Then if $\hat{A}$ is normal, then A is normal and there is a monomorphism of groups:*

$$Cl(A) \longrightarrow Cl(\hat{A}).$$

In particular, if $\hat{A}$ is factorial, then A is factorial.

Proof. See Fossum [6], 6.12. $\square$

Note that if A is factorial, $\hat{A}$ is not factorial in general, even when A is an excellent ring. See Fossum [6] Example 19.6 and Example 19.9.

Theorem 3.3.2. *[13], Theorem 20.1. Let (A, m) be a local noetherian domain. If every height one prime ideal of A is principal, then A is factorial.*

We also recall the following fact about flat extensions from Matsumura:

Proposition 3.3.3. [13], page 63. *Let A be a local noetherian ring and $\hat{A}$ its completion. If $\mathfrak{a} \subset A$ is an ideal such that $\mathfrak{a}\hat{A}$ is principal, then $\mathfrak{a}$ is principal.*

We now demonstrate a condition for the factoriality of the constructed ring B , where R and I are as in the previous sections.

Theorem 3.3.4. *If $\frac{R}{I}$ is factorial, then B is factorial.*

Proof. Let q be a height one prime ideal of B . Then $q \cap R \neq 0$, so there exists some $p_h \in q$. As before, we denote the (p_h) -adic completion of R by R^* and the (p_h) -adic completion of B by B^* .

As in the proof of Theorem 2.3.4, we know:

$$B^* = \frac{R^*}{J \cap R^*}$$

Since $p_h \in q$, the ideal qB^* is a height one prime ideal of B^* . We need only show that qB^* is a principal ideal, and then by Proposition 3.3.2, q is principal.

Consider the maps:

$$R^* \xrightarrow{\pi} \frac{R^*}{J \cap R^*} \xrightarrow{=} B^*$$

Let $Q = \pi^{-1}(qB^*) \in \text{Spec}(R^*)$. Then $p_h \in Q$ and $J \cap R^* \subset Q$.

We define an automorphism of R by:

$$\begin{aligned} \gamma: R &\rightarrow R \\ z_i &\mapsto z_{ih} \quad \text{if } 1 \leq i \leq m \\ z_i &\mapsto z_i \quad \text{if } m < i \leq n \end{aligned}$$

Note that this is an automorphism because $z_{1h}, \dots, z_{mh}, z_{m+1}, \dots, z_n$ forms a regular system of parameters of R .

Let

$$J_h := \gamma(I) = (F_1(z_{1h}, \dots, z_{mh}, z_{m+1}, \dots, z_n), \dots, F_r(z_{1h}, \dots, z_{mh}, z_{m+1}, \dots, z_n)).$$

Then:

$$\frac{R}{I} \cong \frac{R}{J_h}$$

By assumption, $\frac{R}{I}$ is factorial, so $\frac{R}{J_h}$ is also factorial. Furthermore:

$$(J_h, p_h^2 R) R^* = (J \cap R^*, p_h^2 R^*)$$

and:

$$\frac{R^*}{(J \cap R^*, p_h^2 R^*)} = \frac{R^*}{(J_h R^*, p_h^2 R^*)} = \frac{R}{(J_h, p_h^2 R)}$$

Since $p_h \in Q$, the ideal $Q \subseteq R^*$ is extended from R . Let $\beta \in Q \cap R$ be such that

$$\frac{Q \cap R}{J_h} = \frac{\beta R}{J_h}.$$

Then:

$$Q = (Q \cap R) R^* = (J_h, \beta, p_h^2) = (J \cap R^*, p_h^2, \beta).$$

If $x \in Q$,

$$x = p_h(p_h r_1) + \beta r_2 + \delta$$

where $r_1, r_2 \in R^*$ and $\delta \in J \cap R^*$. That is,

$$Q = p_h Q + (J \cap R^*, \beta R^*)$$

By Nakayama's Lemma,

$$Q = (J \cap R^*, \beta R^*)$$

Hence the prime ideal $qB^* = \frac{Q}{J \cap R^*}$ is principal, and so $q \in \text{Spec}^1(B)$ is principal.

Thus B is factorial. □

3.4 Constructing Examples in which B is Factorial

A local noetherian ring (A, m) is regular if and only if the m -adic completion $\hat{A}$ is regular. Thus our constructed ring B is regular if and only if R/I is regular. However,

we can find circumstances in which R/I has a singular locus containing more than one prime ideal but B has an isolated singularity. This is a useful environment for constructing examples in which B is factorial with a non-factorial completion, and we give conditions for creating such an environment in the next theorem.

Theorem 3.4.1. *Let $R = K[z_1, \dots, z_n]_{(z_1, \dots, z_n)}$, and let I be an ideal of R satisfying the condition:*

$$Q \in \text{Ass}(R/I), \quad \text{then} \quad Q \subseteq (z_1, \dots, z_{n-1}) := P.$$

We assume additionally that

$$\text{Sing}(R/I) = V(P);$$

that is, if $Q \in \text{Spec}(R/I)$ and $(R/I)_Q$ is not regular, then $P \subseteq Q$. In this case, the constructed ring B has an isolated singularity.

Proof. Suppose R/I has Krull dimension d . Then P has height $d - 1$. Note that:

$$\text{Sing}(\widehat{R}/I\widehat{R}) = V(P\widehat{R}/I\widehat{R})$$

and

$$\text{Sing}(\widehat{B}) = V(\widehat{P}) = (\zeta_1, \dots, \zeta_{n-1})$$

Let q be a non-maximal prime ideal of B , and let $\widehat{q}$ be a prime ideal of $\widehat{B}$ lying over q so that $\text{height}(\widehat{q}) = \text{height}(q)$.

If q has height less than $d - 1$, then $\widehat{B}_{\widehat{q}}$ is regular, so B_q is regular. Similarly, if $\text{ht}(q) = d - 1$, then $\widehat{q} \neq \widehat{P}$, because $\widehat{P}$ is in the generic formal fiber of B . Thus B has an isolated singularity. $\square$

We can use this theorem in conjunction with the the following theorem due to Grothendieck and Samuel [6], Proposition 18.15, to find examples in which R/I is not factorial, but B is. Recall that a noetherian local ring A is a *complete intersection* if the completion $\widehat{A}$ is the factor of a regular local ring by an ideal generated by a regular sequence.

Theorem 3.4.2. (Grothendieck-Samuel, [6] Proposition 18.15.) *Let A be a complete intersection. If A_p is factorial for all p with $\text{ht}(p) \leq 3$, then A is factorial.*

From this, we get an immediate corollary:

Corollary 3.4.3. *Let R , I and P be as in Theorem 3.4.1. Assume additionally that R/I is a complete intersection of dimension at least 4. Then B is factorial.*

Proof. In this case, B is a complete intersection, and by Theorem 3.4.1, B has an isolated singularity. In particular, B_q is factorial for all prime ideals of height less than or equal to 3. Thus by Grothendieck-Samuel, B is factorial. $\square$

Using this theorem, we can present a very simple example of a factorial ring with non-factorial completion.

Example 3.4.4. *A four dimensional local factorial complete intersection with non-factorial completion.*

Let $R = K[z_1, \dots, z_5]_{(z_1, \dots, z_5)}$. Let $I = (z_1 z_3 - z_2 z_4)$, and let $P = (z_1, z_2, z_3, z_4)$. A simple application of the Jacobian Criterion shows that the P is the defining ideal of the singular locus of the complete intersection R/I . Furthermore, $Q = (z_1, z_2)$ is a non-principal height one ideal of R/I , so R/I is not factorial. Thus $\widehat{R}/I\widehat{R}$ and $\widehat{B}$ are not factorial. However, B is a four dimensional complete intersection with an isolated singularity, so B is factorial.

4 The Heinzer-Rotthaus-Wiegand Construction

In this section, we examine another construction of intermediate domains between a local noetherian ring and its completion. This construction has been explored by Heinzer, Rotthaus, and Wiegand, and we will refer to it as the “HRW” construction. Let (R, m) be an excellent, noetherian, local domain with field of fractions K . As in the previous section, we denote by $\hat{R}$ the m -adic completion of R .

We will look at intermediate domains between the ring (R, m) and the completion with respect to a principal ideal. Let x be a non-zero non-unit of R and let R^* denote the (x) -adic completion of R . Choosing $\tau_1, \dots, \tau_s$ to be certain elements of xR^* , we will consider the subring

$$C := K(\tau_1, \dots, \tau_s) \cap R^*$$

of R^* . The ring C serves as a “reference” ring for the ring we want to consider. This is a good choice because often we can easily determine whether or not C is noetherian. Furthermore, in many cases, C is excellent.

For our ring of interest, we will choose an ideal I of R^* that is extended from C . We will look at the ring:

$$B := \frac{C}{I \cap C} \bigcap K.$$

The extension

$$B_x \longrightarrow \left(\frac{C}{I \cap C}\right)_x$$

is essentially of finite type and carries much information about B .

Heinzer, Rotthaus, and Wiegand have found this ring to be a rich source of examples of excellent and non-excellent, noetherian and non-noetherian rings.

Our goal is to look at conditions under which these rings are normal and factorial. We will then use our conditions to generate examples.

4.1 Setting and Notation of the HRW Construction: A “Reference Ring”

Let (R, m) be an excellent, noetherian, local domain with field of fractions K . Let x be a non-zero non-unit of R . Let R^* denote the (x) -adic completion of R . Choose $\tau_1, \dots, \tau_s \in xR^*$ to be algebraically independent over K . Write:

$$\tau_i = \sum_{j=1}^{\infty} b_{ij} x^j$$

where $b_{ij} \in R$. For each $i \in \{1, \dots, s\}$ and each $n \in \mathbb{N}$, we define the n^{th} endpiece of τ_i with respect to x :

$$\tau_{in} := \sum_{j=n+1}^{\infty} b_{ij} x^{j-n} = \sum_{j=n+1}^{\infty} \frac{b_{ij} x^j}{x^n}.$$

For each $i \in \{1, \dots, s\}$ and each $n \in \mathbb{N}$, we have the following relationship between τ_{in} and $\tau_{i(n+1)}$:

$$\tau_{in} = b_{in} x + \tau_{i(n+1)} x.$$

Furthermore, for each $i \in \{1, \dots, s\}$ and each $n \in \mathbb{N}$:

$$\tau_{in} \in K(\tau_1, \dots, \tau_s) \cap R^*.$$

We need to define two subrings of $K(\tau_1, \dots, \tau_s) \cap R^*$:

$$U_r := R[\tau_{1r}, \dots, \tau_{sr}]$$

and:

$$T_r := (1 + xU_r)^{-1} U_r.$$

Because $\tau_{in} = b_{in} x + \tau_{i(n+1)} x$, we get:

$$U_1 \subseteq U_2 \subseteq \dots \subseteq U_n \subseteq \dots$$

and:

$$T_1 \subseteq T_2 \subseteq \dots \subseteq T_n \subseteq \dots$$

We then define the nested union rings:

$$U := \bigcup_{r=1}^{\infty} U_r$$

and:

$$T = \bigcup_{r=1}^{\infty} T_r.$$

A useful proposition from [10] gives the following:

Proposition 4.1.1. *Let U^* denote the (x) -adic completion of U . Then:*

(1) *For all $k \in \mathbb{N}$, we have: $x^k U = x^k R^* \cap U$.*

(2) *$U^* = R^*$, so $R/xR = U/xU$.*

Remark. The definitions U and T are independent of the representations of $\tau_1, \dots, \tau_s$ as power series in x with coefficients in R .

Proof. (of the Remark) To see that the definition of U is independent of the representations of $\tau_1, \dots, \tau_s$ as power series, we will imitate proofs found in several papers of Heinzer, Rotthaus, and Weigand; see, for example, [10]. The corresponding statement for T follows immediately.

For each $i \in \{1, \dots, s\}$, suppose τ_i and $\rho_i = \tau_i$ have representations:

$$\tau_i = \sum_{j=1}^{\infty} b_{ij} x^j$$

and

$$\rho_i = \sum_{j=1}^{\infty} c_{ij} x^j$$

where b_{ij} and c_{ij} are elements of R . Using the n^{th} endpieces τ_{in} and ρ_{in} , we have:

$$\tau_i = \sum_{j=1}^{\infty} b_{ij} x^j = \sum_{j=1}^n b_{ij} x^j + x^n \tau_{in} = \sum_{j=1}^{\infty} (c_{ij} x^j) = \sum_{j=1}^n c_{ij} x^j + x^n \rho_{in} = \rho_i$$

Hence for each $i \in \{1, \dots, s\}$ and each $n \in \mathbb{N}$:

$$\tau_{in} - \rho_{in} = \frac{\sum_{j=1}^n (b_{ij} - c_{ij})x^j}{x^n}$$

where

$$\sum_{j=1}^n (b_{ij} - c_{ij})x^j \in x^n R^* \cap U = x^n U.$$

Thus $\tau_{in} - \rho_{in} \in U$ and U is independent of the power series representations of $\tau_1, \dots, \tau_s$. $\square$

We define another subring of R^* :

$$C := K(\tau_1, \dots, \tau_s) \cap R^*.$$

In this setting, we can use Theorem 2.12 of Heinzer-Rotthaus-Wiegand [8].

Theorem 4.1.2. *Let (R, m) be a semilocal noetherian domain. Let $\tau_1, \dots, \tau_s \in xR^*$ be as in the previous discussion. Then the extension:*

$$\psi : R[\tau_1, \dots, \tau_s] \hookrightarrow R_x^*$$

is flat if and only if T is noetherian. If this holds, we also have $C = T$.

In light of this theorem, we will assume that the extension $R[\tau_1, \dots, \tau_s] \hookrightarrow R_x^*$ is flat in order to ensure that C is a noetherian domain equal to T . We also know [8], Proposition 2.2, that the (x) -adic completion $C^* = R^*$, and the quotient field of C is $K(\tau_1, \dots, \tau_s)$.

4.2 Construction of the Intermediate Ring B

In this section we look at an insider ring that generalizes the setting of the previous section. Again, let (R, m) be an excellent, noetherian, local domain with field of fractions K . Let x be a non-zero non-unit of R . Let R^* denote the (x) -adic completion

of R . Let $\tau_1, \dots, \tau_s \in xR^*$ be algebraically independent over K as in the previous sections, and $C := K(\tau_1, \dots, \tau_s) \cap R^*$ be as before, assuming that:

$$\psi : R[\tau_1, \dots, \tau_s] \hookrightarrow R_x^*$$

is flat so that C is noetherian and computable as a nested union of localized polynomial rings of R .

We choose elements $f_1, \dots, f_r$ of $R[\tau_1, \dots, \tau_s]$, considered as polynomials in $\tau_1, \dots, \tau_s$ with coefficients in R . Suppose $f_1, \dots, f_r$ are algebraically independent over K .

Let $I := (f_1, \dots, f_r)$ be an ideal of R^* such that:

For all $P \in \text{Ass}(R^*/I)$, we have $P \cap R = 0$.

Note that $f_1, \dots, f_r \in C$, so the ideal I is extended from C . In a slight abuse of notation, we will denote $I \cap C$ and $I \cap R[\tau_1, \dots, \tau_s]$ by I as well.

We define a ring:

$$A := \frac{C}{I} \cap Q(R).$$

Again, we want conditions under which A is realizable as a nested union of localized polynomial rings over R . To that end, we will define the t^{th} *frontpieces* of the polynomials $f_1, \dots, f_r$. First, recall for $1 \leq i \leq s$ and $n \in \mathbb{N}$:

$$\tau_i = x^n \tau_{in} + \alpha_{in}$$

for some $\alpha_{in} \in R$. For each $f \in \{f_1, \dots, f_r\}$, we use Taylor series to get:

$$\begin{aligned} f(\tau_1, \dots, \tau_s) &= f(\alpha_{1t} + \tau_{1t}x^t, \dots, \alpha_{st} + x^t\tau_{st}) \\ &= \sum_{j=0}^{\infty} \left\{ \frac{1}{j!} \left(\sum_{k=1}^n x^t \tau_{kt} \frac{\delta f}{\delta \alpha_{kt}} \right)^j f(\alpha_{1t}, \dots, \alpha_{st}) \right\} \\ &= f(\alpha_{1t}, \dots, \alpha_{st}) + x^t h \end{aligned}$$

for some $h \in R^*$.

We know $f(\tau_1, \dots, \tau_s) \in IR^*$, so

$$f(\alpha_{1t}, \dots, \alpha_{st}) \equiv x^t h \pmod{(IR^*)}.$$

Thus it makes sense to define the t^{th} frontpieces of f_j as:

$$f_{jt} := \frac{f_j(\alpha_{1t}, \dots, \alpha_{st})}{x^t}.$$

For $t \in \mathbb{N}$, define:

$$B_t := R[f_{1t}, \dots, f_{rt}](m, f_{1t}, \dots, f_{rt}).$$

We have a recursion formula for the elements f_{ij} :

$$f_{ij} = xf_{i(j+1)} + d_{ij}$$

where $d_{ij} \in R$.

As a result of this recursion formula, we again have a sequence:

$$B_1 \subseteq B_2 \subseteq \dots \subseteq B_n \subseteq \dots$$

and we can define a nested union ring:

$$B := \bigcup_{t=1}^{\infty} B_t$$

Note that $B \subseteq Q(R)$.

An additional result in [10] yields conditions under which the two rings A and B are equal and noetherian. Assume that the ring T is noetherian. Note that:

$$T_x = \left(\frac{R[\tau_1, \dots, \tau_s]}{I} \right)_x.$$

If the map

$$\varphi : R \longrightarrow \left(\frac{R[\tau_1, \dots, \tau_s]}{I} \right)_x$$

is flat, then $A = B$ and B is noetherian.

It is worth noting that the map

$$R \longrightarrow \left(\frac{R^*}{IR^*} \right)_x$$

is flat if and only if B is noetherian. On the other hand, φ being flat is a sufficient but not necessary condition for B to be noetherian. The advantage of using the condition

on φ is that R and $\left(\frac{R[\tau_1, \dots, \tau_s]}{I}\right)_x$ are both essentially finitely generated over a field, so the non-flat locus of φ is closed, by [13], Theorem 24.3, and has defining ideal

$$J := \bigcap \left\{ Q \in \operatorname{Spec} \left(\frac{R[\tau_1, \dots, \tau_s]}{I} \right)_x : R \longrightarrow \left(\left(\frac{R[\tau_1, \dots, \tau_s]}{I} \right)_x \right)_Q \text{ is not flat} \right\}.$$

We denote the (x) -adic completion of B by B^* .

Remark: Note the following useful facts:

- (1) For each positive integer n , the ideals of R containing x^n are in one-to-one correspondence with the ideals of R^* containing x^n . in particular,

$$(I, x)R^* = (d_{11}, \dots, d_{r1}, x)R^*$$

and

$$(I, x)R^* \cap R = ((d_{11}, \dots, d_{r1}, x)R^* \cap R = (d_{11}, \dots, d_{r1}, x)R$$

where $d_{11}, \dots, d_{r1}$ are the constant terms of $f_1, \dots, f_r$.

- (2) Under the identification of R as a subring of R^*/I , the ideal $(d_{11}, \dots, d_{r1}, x)R$ is also equal to $x(R^*/I) \cap R$.

We quote a theorem of Heinzer-Rotthaus-Wiegand to better understand the structure of B (Proposition 2.4 [10]):

Proposition 4.2.1. *In this setting, we have:*

- (1) $x(R^*/I) \cap B = xB$.
(2) $B/x^n B = R^*/((x^n) + I)$, and $B^* = C^*/I = R^*/I$.

Furthermore, the completions of these rings at their respective maximal ideals are equal; i.e.:

$$\widehat{B} = \frac{\widehat{C}}{I} = \frac{\widehat{R}}{I}.$$

We can now look at conditions for the normality and factoriality of these intermediate rings.

4.3 Conditions for Normality and Factoriality

As in the Nishimura construction, we rely heavily on the excellence of R to identify conditions under which B is normal. Since R (and thus R^*/I [21]) is excellent, the normal locus of R^*/I is open. Using Lemma 15 of the Nishimura Construction, we write:

$$\text{NNor}(R^*/I) = V(Q_1^* \cap \dots \cap Q_t^*)$$

where each Q_i^* is a height one prime ideal of R^*/I . As in the Nishimura construction, we call $Q_1^* \dots Q_t^*$ the defining ideals of the non-normal locus of R^*/I . We can use these ideals to identify the ideals of B which need to be “checked” in order to ensure the normality of B .

Throughout this section, we will assume that R^*/I satisfies the Serre condition (S_2) .

Proposition 4.3.1. *Assume the setting of the previous section, and let $Q_1^* \dots Q_t^*$ be the defining ideals of the non-normal locus of R^*/I . Let $q_i = Q_i^* \cap B$ for $1 \leq i \leq t$. In addition, assume that R^*/I satisfies the Serre condition (S_2) . Then B is normal if and only if for all $1 \leq i \leq t$, the ring B_{q_i} is a regular local ring.*

Proof. Assume B is normal. The height of each prime ideal q_i is at most one, so the Serre criterion for normality implies that B_{q_i} is regular.

Assume that B_{q_i} is a regular local ring for all $1 \leq i \leq t$. Let $\mathfrak{P}$ be a height one prime ideal of B . We need to show that $B_{\mathfrak{P}}$ is regular.

Let Q be a height one prime of R^*/I so that $Q \cap B = \mathfrak{P}$. In case $Q \in \{Q_1^* \dots Q_t^*\}$, we have that $\mathfrak{P} = q_i$ for some i and $B_{\mathfrak{P}}$ is regular by assumption. If $Q \notin \{Q_1^* \dots Q_t^*\}$, then B_Q^* is a one-dimensional normal ring. Hence B_Q^* is regular. The canonical map:

$$B_{\mathfrak{P}} \longrightarrow B_Q^*$$

is faithfully flat. Hence $B_{\mathfrak{P}}$ is regular. Thus B satisfies the Serre condition (R_1) .

Since $B^* = R^*/I$ satisfies (S_2) , B also satisfies (S_2) . Thus B is normal. $\square$

Let φ be the flat map:

$$\varphi : R \hookrightarrow \left(\frac{R[\tau_1, \dots, \tau_s]}{I} \right)_x = \left(\frac{C}{I} \right)_x$$

Let H be the ideal describing the non-smooth locus of φ ; i.e.:

$$H := \bigcap \left\{ P \in \text{Spec} \left(\frac{R[\tau_1, \dots, \tau_s]}{I} \right)_x \mid \varphi_P : R_{\varphi^{-1}(P)} \longrightarrow \left(\left(\frac{R[\tau_1, \dots, \tau_s]}{I} \right)_x \right)_P \text{ is not smooth} \right\}$$

Lemma 4.3.2. *Assume the setting and notation of the previous proposition. Assume additionally that B is normal. With H defined as above, if $\text{ht}(H) > 1$, then C/I is normal.*

Proof. The ring C/I satisfies the Serre condition (S_2) because R^*/I does. Thus we need only verify (R_1) . To that end, choose a height one prime $q \in \text{Spec}(C/I)$. Let $Q \in \text{Spec}(C^*/I)$ be a height one prime so that $Q \cap C/I = q$.

Recall:

$$C^*/I = R^*/I = B^*$$

Let $Q_0 = Q \cap R$. We look at two cases:

Case 1. Suppose $x \in Q$.

In this case, Q is extended from B as well as from R . Hence:

$$\frac{B^*}{Q} = \frac{B}{Q \cap B} = \frac{R}{Q \cap R}$$

and:

$$\frac{B_Q^*}{QB_q^*} = \frac{B_{Q \cap B}}{(Q \cap B)B_{Q \cap B}}$$

Since B is normal, the ring $B_{Q \cap B}$ is regular. Moreover, Q is extended from B , so $(Q \cap B)B^* = QB^*$. Hence,

$$\frac{B_Q^*}{(Q \cap B)B_Q^*} = \frac{B_Q^*}{QB_Q^*}.$$

The latter ring is a field, and in particular, a regular local ring. Thus B_Q^* is a regular local ring. So $(C/I)_q$ is a regular local ring.

Case 2. Suppose $x \notin Q$.

In this case,

$$B_Q \cap B = R_{Q_0}$$

and:

$$\left(\frac{C}{I}\right)_q = \left(\frac{R[\tau_1, \dots, \tau_s]}{I}\right)_{Q \cap R[\tau_1, \dots, \tau_s]/I}.$$

Since the height of $Q \cap B$ is at most one, $B_Q \cap B$ is a regular local ring. Hence R_{Q_0} is a regular local ring of dimension at most one. Since $Q \in \text{Spec}(C^*/I)$ is a height one prime, $Q \cap R[\tau_1, \dots, \tau_s]/I$ also has height one.

Our assumption on H yields that:

$$R_{Q_0} \longrightarrow \left(\frac{R[\tau_1, \dots, \tau_s]}{I}\right)_{Q \cap R[\tau_1, \dots, \tau_s]/I}$$

is smooth. Thus:

$$\left(\frac{C}{I}\right)_q = \left(\frac{R[\tau_1, \dots, \tau_s]}{I}\right)_{Q \cap R[\tau_1, \dots, \tau_s]/I}$$

is regular. Since (R_1) and (S_2) are satisfied, C/I is normal. $\square$

The intermediate ring B is not excellent in general. However, if C is excellent and $\text{ht}(H) > 1$, then the normality of B guarantees the normality of B^* . The condition that C is excellent is not too restrictive, particularly when R is a localized polynomial ring over a field of characteristic 0. We will discuss this in more detail in the next section.

Proposition 4.3.3. *In this setting, assume $\text{ht}(H) > 1$ and C is excellent. Then B normal implies that B^* is normal.*

Proof. As a homomorphic image of an excellent ring, C/I is excellent. Since C/I is normal, $B^* = (C/I)^*$ is normal. $\square$

If we assume that R is a normal domain, we need even fewer height one prime ideals of B to determine the normality of B .

Proposition 4.3.4. *Suppose R is an excellent, noetherian normal domain. Let $\tau_1, \dots, \tau_s$, C , I , and B be as defined in section 2. Then B is normal if and only if for all height one prime ideals q in B with $x \in q$, the ring B_q is a regular local ring.*

Proof. If B is normal, then for every height one prime ideal q in B we know that B_q is a regular local ring, so “only if” is clear. To see the other direction, assume that B_q is a regular local ring for every height one prime ideal q of B so that $x \in q$. Let Ω be a height one prime of B so that $x \notin \Omega$. Then

$$B_\Omega = R_\Omega \cap R.$$

Then $R_\Omega \cap R$ is regular, because it is a one-dimensional localization of a normal ring. Thus B_Ω is regular. By the general assumption that R^*/I satisfies the Serre condition (S_2) , the ring B is normal. □

We can also find a straightforward condition under which B is factorial.

Proposition 4.3.5. *In the setting of the previous proposition, suppose R is factorial and $x \in B$ is a prime element. Then B is factorial.*

Proof. Since B is noetherian and x is a prime element of B , if B_x is factorial, then B is factorial. [13], Theorem 20.2.

Since xB is a prime ideal of B and

$$\frac{B}{xB} = \frac{R^*}{(x, I)}$$

we know that (x, I) is prime in R^* . Thus $(x, I) \cap R$ is prime in R . Since R is factorial,

$$R_x = B_x$$

is factorial. Thus B is factorial. □

We now look at a more concrete situation in which we can generate examples.

4.4 A more specific setting

Suppose k is a field of characteristic 0 and $x, y_1, \dots, y_n$ are indeterminates over k . Let $R := k[x, y_1, \dots, y_n]_{(x, y_1, \dots, y_n)}$ be the localized polynomial ring over k . Let R^* denote the (x) -adic completion of R . Let $\tau_1, \dots, \tau_s$ be elements of $xk[[x]]$ which are algebraically independent over $k(x)$. In this case:

$$C := k(x, y_1, \dots, y_n, \tau_1, \dots, \tau_s) \cap R^*.$$

and the map $R[\tau_1, \dots, \tau_s] \longrightarrow R_x^*$ is flat. C is noetherian and computable as a nested union of polynomial rings. Furthermore, in this case, C is a regular local ring. Since k is a field of characteristic 0, the ring C is excellent, by [10], Proposition 4.1.

In this setting, many of the conditions for the propositions in the last section are met, so we can use this context to generate examples.

Example 1. *An example in which B is normal, but B^* is not normal.*

Let k be a field of characteristic 0. Let $R = k[x, y, z, w]_{(x, y, z, w)}$ and $R^* = k[y, z, w]_{(y, z, w)}[[x]]$, the x -adic completion of R . Define $\tau_1, \tau_2, \tau_3 \in xk[[x]]$ to be algebraically independent over $k(x)$. Define an ideal I of R^* by

$$I = ((w + \tau_1)(y + \tau_2)^2 - (z + \tau_3)^3).$$

A calculation shows that:

$$NNor(R^*/I) = V((y + \tau_2, z + \tau_3)).$$

Claim. B is noetherian.

Proof. In order to see that B is noetherian, we need to see that:

$$\varphi : R \longrightarrow \left(\frac{R[\tau_1, \tau_2, \tau_3]}{I} \right)_x$$

is flat. First we show that the map

$$\gamma : R \longrightarrow \frac{R[\tau_1, \tau_2, \tau_3]}{I}$$

is flat. We do this via a corollary to Theorem 22.6 in Matsumura. R is noetherian, and $R[\tau_1, \tau_2, \tau_3]$ is a polynomial ring over R . The ideal of R generated by the coefficients of τ_1, τ_2 , and τ_3 in $(w + \tau_1)(y + \tau_2)^2 - (z + \tau_3)^3$ is R . Thus $\frac{R[\tau_1, \tau_2, \tau_3]}{I}$ is flat over R , and so $\left(\frac{R[\tau_1, \tau_2, \tau_3]}{I}\right)_x$ is flat over R . Thus we see that B is noetherian.

Note that:

$$(w + \tau_1, z + \tau_3) \cap R_x = (w + \tau_1, z + \tau_3) \cap B_x = 0$$

so

$$(w + \tau_1, z + \tau_3) \cap R = 0.$$

Similarly,

$$(y + \tau_2, z + \tau_3) \cap R = 0.$$

Hence by Proposition 4.3.1, B is normal. Note that B^* is not normal.

Example 4.4.1. *An example in which B is not normal.*

Let k be a field of characteristic 0. Let $R = k[x, y, z, w]_{(x, y, z, w)}$ and $R^* = k[y, z, w]_{(y, z, w)}[[x]]$, the x -adic completion of R . Define $\tau_1, \tau_2, \tau_3 \in xk[[x]]$ to be algebraically independent over $k(x)$.

In this example, our ideal I is “tweaked” from the ideal in the previous example.

Define:

$$I = (x^2(y + \tau_2)^2 - (z + \tau_3)^3).$$

Then

$$NNor(R^*/I) = V((x, z + \tau_3) \cap (y + \tau_2, z + \tau_3)).$$

In this case, to see that B is noetherian, we can use the corollary to Theorem 22.6 in Matsumura [13]. R is noetherian, and $R[\tau_1, \tau_2, \tau_3]$ is a polynomial ring over R . The ideal of R generated by the coefficients of τ_1, τ_2 , and τ_3 in $x^2(y + \tau_2)^2 - (z + \tau_3)^3$ is R . Thus $\frac{R[\tau_1, \tau_2, \tau_3]}{I}$ is flat over R , and so $\left(\frac{R[\tau_1, \tau_2, \tau_3]}{I}\right)_x$ is flat over R .

In this case, the ideal $(x, z + \tau_3)$ is extended from B , so B is not normal.

Example 4.4.2. *An example in which B is normal but not factorial.*

Let k be a field of characteristic 0. Let $R = k[x, y, z, w]_{(x, y, z, w)}$ and $R^* = k[y, z, w]_{(y, z, w)}[[x]]$, the x -adic completion of R . which are algebraically independent over K , the quotient field of R .

Let I be the principal ideal of R^* so that

$$I := (x(z + \tau_1) - y(w + \tau_2)).$$

Again, the method of the previous example will yield that B is noetherian. Also, $\widehat{R}/I\widehat{R} \cong \widehat{R}/(xz - yw)\widehat{R}$, which is normal, since $R/(xz - yw)R$ is an excellent normal ring. The natural map $R^*/I \longrightarrow \widehat{R}/I\widehat{R}$ is faithfully flat, so R^*/I is normal. Hence B is normal. However, R^*/I is not factorial because (x, y) is a height one prime of R^*/I which is extended from B and is not principal. Then $(x, y)B$ is a height one prime ideal of B which is not principal, and B is not factorial.

References

- [1] Y. Akizuki, *Eine Bemerkung über primäre Integritätsbereiche mit Teilerketten-*
satz, Proc. Jap. phys. -math. Soc. **17** (1935), 327-336.
- [2] M. Brodmann-C. Rotthaus, *Local Domains with Bad Sets of Formal Prime Di-*
visors, J. Algebra **75** (1982), 386-394.
- [3] M. Brodmann-C. Rotthaus, *A Peculiar Unmixed Domain*, Proc. Amer. Math.
Soc. **87** (1983), 596-600.
- [4] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*; Cambridge studies in advanced
mathematics 39, Cambridge University Press (1993), revised edition.
- [5] A. Grothendieck, *Elements de geometrie algebrique I*, Publ. Math. Inst. Hautes.
Etud. Sci. **4** (1960).
- [6] R.M. Fossum *The Divisor Class Group of a Krull Domain* Springer-Verlag, 1973.
- [7] W. Heinzer, C. Rotthaus, and S. Weigand *Idealwise algebraic independence*
for elements of the completion of a local domain, Illinois J. Math. **41** (1997),
272-308.
- [8] W. Heinzer, C. Rotthaus, and S. Wiegand, *Noetherian rings between a semilocal*
domain and its completion, J. Algebra **198** (1997), 627-655.
- [9] W. Heinzer, C. Rotthaus, and S. Wiegand *Intermediate rings between a local*
domain and its completion, Illinois J. Math. **43** (1999), 19-46.
- [10] W. Heinzer, C. Rotthaus, and S. Wiegand, *Noetherian domains inside a a*
homomorphic image of a copmletion, J. Algebra **215** (1999), 666-681.
- [11] R.C. Heitmann, *A Non-Catenary, Normal, Local Domain*, Rocky Mountain J.
Math. **12** (1982), 145-148.

- [12] H. Matsumura, *Commutative Algebra*, W.A. Benjamin, 1970.
- [13] H. Matsumura, *Commutative Ring Theory*, Cambridge University Press, 1986.
- [14] M. Nagata, *An example of a normal local ring which is analytically reducible*, Mem. Coll. Sci., Univ. Kyoto **31** (1958), 83-85.
- [15] M. Nagata, *Local Rings*, John Wiley, 1962.
- [16] J. Nishimura, *A Few Examples of Local Rings I*, preprint.
- [17] J. Nishimura, *A Few Examples of Local Rings II*, preprint.
- [18] T. Ogoma, *Non-Catenary Pseudo-Geometric Normal Rings*, Japan. J. Math. **6** (1980), 147-163.
- [19] T. Ogoma, *Cohen MaCaulay Factorial Domain is not Necessarily Gorenstein*, Mem. Fac. Sci. Kochi Univ. **3** (1982), 65-74.
- [20] C. Rotthaus, *Divisorial Ascent in Rings with the Approximation Property*, J. Algebra **178** (1995), 541-560.
- [21] C. Rotthaus, *Komplettierung semilokaler quasiauxgezeichneter Ringe*, Nagoya Math. J. **76** (1979), 173-180.
- [22] C. Rotthaus, *Universell japanische Ringe mit nicht offenem regulärem Ort*, Nagoya Math. J. **74** (1979), 123-135.
- [23] C. Rotthaus, *Nicht ausgenzeichnete, universell japanische Ringe*, Math. Z. **152** (1977), 107-125.
- [24] P. Samuel, *Unique factorization domains*, Tata Inst. of Fundamental Research, Bombay, 1964.
- [25] D. Weston, *On descent in dimension two and non-split Gorenstein modules*, J. Algebra **118** (1988), 263-275.

MICHIGAN STATE UNIVERSITY LIBRARIES



3 1293 02470 8624