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INVARIANT VECTOR SUBSPACES OF L^p WITH
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INVARIANT VECTOR SUBSPACES OF L^p WITH APPLICATIONS

By

David Allen Redett

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ABSTRACT

INVARIANT VECTOR SUBSPACES OF L^p WITH APPLICATIONS

By

David Allen Redett

For a majority of this dissertation, we study invariant vector subspaces of the Hardy and Lebesgue spaces for both the circle and torus. When studying invariant vector subspaces, some kind of completeness is required of our invariant vector subspace. Classically, one only considered those invariant vector subspaces that were closed. In this dissertation, we take a different approach. Rather than studying those invariant vector subspaces that are complete in the metric induced from the norm on the larger space, we consider those invariant vector subspaces that can support any norm that makes them complete and a Hilbert space. This idea was first introduced by de Branges in his proof of the famous Bieberbach conjecture. He considered those invariant vector subspaces of $H^2(\mathbf{T})$ that could support a norm that would make them Hilbert spaces. Since then, these spaces, affectionately called de Branges spaces, have been studied by Dinesh Singh, U. N. Singh, Vern Paulsen and Sanjeev Agrawal. These gentlemen made many nice contributions to this area. We also begin making connections to Strongly Harmonizable Stable Fields. We hope in the future some of the work from this dissertation may be used to help completely understand the "prediction" of these random fields.

To my family for their continuing love, support and understanding.

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Chapter 1

Background

1.1 Preliminaries

To keep this thesis some what self contained we include some basis definitions and theorems used throughout. The information in this section can be found in any standard text on real or functional analysis. For instance, see [19].

Let **C** denote the complex plane.

Definition 1 *A* ***complex vector space*** *is a set $\mathcal{V}$, whose elements are called vectors and in which two operations, called addition and scalar multiplication, are defined, with the following algebraic properties:*

1. $x + y \in \mathcal{V}$ *whenever* $x,\ y \in \mathcal{V}$. *Further,* $x + y = y + x$.

2. $x + (y + z) = (x + y) + z$ *when* $x, y, z \in \mathcal{V}$.

3. $0 \in \mathcal{V}$ *such that* $x + 0 = x$ *for all* $x \in \mathcal{V}$.

4. *To each* $x \in \mathcal{V}$ *there corresponds a unique vector* $-x \in \mathcal{V}$ *such that* $x+(-x) = 0$.

5. $\alpha x \in \mathcal{V}$ *whenever* $\alpha \in \mathbf{C}$ *and* $x \in \mathcal{V}$. *Further,* $1x = x$.

6. $\alpha(\beta x) = (\alpha\beta)x$ *for* $\alpha, \beta \in \mathbf{C}$ *and* $x \in \mathcal{V}$.

7. $\alpha(x + y) = \alpha x + \alpha y$, $(\alpha + \beta)x = \alpha x + \beta x$ *for* $\alpha, \beta \in \mathbf{C}$ *and* $x, y \in \mathcal{V}$.

A subset $\mathcal{M}$ of a complex vector space $\mathcal{V}$ is called a **vector subspace** of $\mathcal{V}$ if $\mathcal{M}$ is itself a complex vector space, relative to the addition and scalar multiplication which are defined on $\mathcal{V}$.

Definition 2 *A complex vector space $\mathcal{X}$ is said to be a* ***normed linear space*** *if to each x in $\mathcal{X}$ there is associated a nonnegative real number $\|x\|$, called the norm of x, such that*

1. $\|x + y\| \leq \|x\| + \|y\|$ *for all x and y in $\mathcal{X}$,*

2. $\|\alpha x\| = |\alpha|\|x\|$ *if x is in $\mathcal{X}$ and $\alpha \in \mathbf{C}$,*

3. $\|x\| = 0$ *implies* $x = 0$.

If we define $d(x, y) = \|x - y\|$, then it is easy to check that d is a metric on $\mathcal{X}$. A *Banach space* is a normed linear space which is *complete* in the metric d. Completeness, then, means if $\{x_n\}_{n\geq 0}$ is a sequence in $\mathcal{X}$ such that

$$\|x_n - x_m\| \to 0$$

as n, $m \to \infty$, then there exists an x in $\mathcal{X}$ such that

$$\|x_n - x\| \to 0$$

as $n \to \infty$.

The simplest example of a Banach space is $\mathbf{C}$ with norm $|z|$ for $z \in \mathbf{C}$. The Banach spaces that are of interest in this dissertation are the L^p-spaces, which we define now.

Definition 3 *Let Ω be an arbitrary measure space with a positive measure μ. If $1 \leq p \leq \infty$ and if f is a complex measurable function on Ω, define*

$$\|f\|_p = \begin{cases} \left(\int_\Omega |f|^p \, d\mu\right)^{\frac{1}{p}} & 1 \leq p < \infty \\ \inf\left\{\alpha : \mu\left(f^{-1}((\alpha, \infty])\right) = 0\right\} & p = \infty \end{cases}$$

and let $L^p(\Omega, \mu)$ consist of all f for which

$$\|f\|_p < \infty.$$

We call $\|f\|_p$ the L^p-norm of f.

If a Banach space $\mathcal{X}$ has a norm that satisfies the *parallelogram law*

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2),$$

we call $\mathcal{X}$ a *Hilbert space.* In such a case, we may define a *sesquilinear* functional on $\mathcal{X}$ by

$$(x, y) = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2) + i\frac{1}{4}(\|x + iy\|^2 - \|x - iy\|^2).$$

This sesquilinear functional is called the *inner product* for the Hilbert space $\mathcal{X}$ and

$$\|x\| = \sqrt{(x, x)}.$$

$\mathbf{C}$ is also the simplest example of a Hilbert space with inner product given by $(z_1, z_2) = z_1\overline{z_2}$ for $z_1, z_2 \in \mathbf{C}$. We point out that of all the $L^p(\Omega, \mu)$ spaces, $L^2(\Omega, \mu)$ is the only Hilbert space with inner product given by $(f, g) = \int_\Omega f\overline{g}\, d\mu$.

Hereafter, $\mathcal{H}$ will always denote a Hilbert space.

If $(x, y) = 0$ for some $x, y \in \mathcal{H}$, we say that x is **orthogonal** to y, and sometimes write $x \perp y$. If $\mathcal{M}$ is a vector subspace of $\mathcal{H}$, let $\mathcal{M}^\perp$ be the set of all $y \in \mathcal{H}$ which are orthogonal to every $x \in \mathcal{M}$.

A set of vectors u_α in $\mathcal{H}$, where α runs through some index set $\mathbf{A}$, is called **orthonormal (o.n.)** if it satisfies the orthogonality relations $(u_\alpha, u_\beta) = 0$ for all $\alpha \neq \beta$, $\alpha \in \mathbf{A}$ and $\beta \in \mathbf{A}$, and if it is normalized so that $\|u_\alpha\| = 1$ for each $\alpha \in \mathbf{A}$.

A vector subspace $\mathcal{M}$ of a Banach space $\mathcal{X}$ is called a **subspace** of $\mathcal{X}$ if $\mathcal{M}$ is itself a Banach space, relative to the norm which is defined on $\mathcal{X}$. The following theorem gives us some very important properties of subspaces of a Hilbert space.

Theorem 1 *Let $\mathcal{M}$ be a subspace of $\mathcal{H}$.*

1. *Every $x \in \mathcal{H}$ has a unique decomposition*

$$x = Px + Qx$$

   into a sum of $Px \in \mathcal{M}$ and $Qx \in \mathcal{M}^\perp$.

2. *Px and Qx are the nearest points to x in $\mathcal{M}$ and $\mathcal{M}^\perp$, respectively.*

3. *The mapping $P : \mathcal{H} \longrightarrow \mathcal{M}$ and $Q : \mathcal{H} \longrightarrow \mathcal{M}^\perp$ are linear.*

4. $\|x\|^2 = \|Px\|^2 + \|Qx\|^2$.

Corollary 1 *If $\mathcal{M} \neq \mathcal{H}$, then there exists $y \in \mathcal{H}$, $y \neq 0$, such that $y \perp \mathcal{M}$.*

Definition 4 *A **linear transformation** of a complex vector space $\mathcal{V}$ into a complex vector space $\mathcal{W}$ is a mapping Λ of $\mathcal{V}$ into $\mathcal{W}$ such that*

$$\Lambda(\alpha x + \beta y) = \alpha\Lambda(x) + \beta\Lambda(y)$$

*for all $x, y \in \mathcal{V}$ and for all $\alpha, \beta \in \mathbf{C}$. In the special case that $\mathcal{V} = \mathcal{W}$ we call Λ a **linear operator**.*

Consider a linear transformation Λ from a normed linear space $\mathcal{X}$ into a normed linear space $\mathcal{Y}$, and define the norm of Λ by

$$\|\Lambda\| = \sup\left\{\|\Lambda x\| : x \in \mathcal{X}, \|x\| \leq 1\right\}.$$

If $\|\Lambda\| < \infty$, then Λ is called a **bounded linear transformation**. We denote by $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ the collection of all bounded linear transformations from $\mathcal{X}$ to $\mathcal{Y}$. If $\mathcal{X} = \mathcal{Y}$, we simply write $\mathcal{B}(\mathcal{X})$.

Recall that $\mathcal{H}$ denotes a Hilbert space. We call $\Lambda \in \mathcal{B}(\mathcal{H})$ an **isometry** if

$$\|\Lambda(x)\| = \|x\| \qquad \text{for all } x \in \mathcal{H}.$$

An isometry $\Lambda \in \mathcal{B}(\mathcal{H})$ is called a **shift** or **pure isometry** if

$$\bigcap_{n=0}^{\infty} \Lambda^n(\mathcal{H}) = \{0\}$$

and **unitary** if the range of Λ is $\mathcal{H}$.

Definition 5 *A vector subspace $\mathcal{M}$ of $\mathcal{H}$ is **invariant** under $\Lambda \in \mathcal{B}(\mathcal{H})$ if*

$$\Lambda(\mathcal{M}) \subseteq \mathcal{M}.$$

*We say it is **simply invariant** if the above containment is strict. (i.e., $\Lambda(\mathcal{M}) \subset \mathcal{M}$, but $\Lambda(\mathcal{M}) \neq \mathcal{M}$.)*

A subspace $\mathcal{M}$ of $\mathcal{H}$ **reduces** $\Lambda \in \mathcal{B}(\mathcal{H})$ if both $\mathcal{M}$ and $\mathcal{M}^{\perp}$ are invariant under Λ.

Definition 6 *If $\Lambda \in \mathcal{B}(\mathcal{H})$, then the **adjoint** of Λ, denoted Λ^*, is the unique operator on $\mathcal{H}$ satisfying*

$$(\Lambda(x), y) = (x, \Lambda^*(y)).$$

for all x and y in $\mathcal{H}$.

For $\Lambda_1, \Lambda_2 \in \mathcal{B}(\mathcal{H})$ we say Λ_1 and Λ_2 are **doubly commuting** if Λ_1 commutes with Λ_2 (i.e., $\Lambda_1\Lambda_2 = \Lambda_2\Lambda_1$) and Λ_1 commutes with Λ_2^* (**Note:** Λ_1 commuting with Λ_2^* is equivalent to Λ_2 commuting with Λ_1^*).

1.2 Terminology and Notation

We let $\mathbf{U}$ and $\mathbf{T}$ denote the unit disc and unit circle in the complex plane, respectively.

The Hardy space $H^p(\mathbf{U})$, $(1 \leq p < \infty)$ is the Banach space of holomorphic functions over $\mathbf{U}$ which satisfy the inequality

$$\sup_{0 \leq r < 1} \int_{\mathbf{T}} |f(r\xi)|^p \, dm(\xi) < \infty$$

where m denotes normalized Lebesgue measure on $\mathbf{T}$. The norm $||f||_p$ of a function f in $H^p(\mathbf{U})$ is defined by

$$||f||_p = \sup_{0 \leq r < 1} \left(\int_{\mathbf{T}} |f(r\xi)|^p \, dm(\xi) \right)^{1/p}.$$

When $p = 2$, we have a Hilbert space and the norm can be simplified. For f in $H^2(\mathbf{U})$ with Taylor series $\sum_{n=0}^{\infty} \hat{f}(n) z^n$ the norm simplifies to

$$||f||_2 = \left(\sum_{n=0}^{\infty} |\hat{f}(n)|^2 \right)^{1/2}.$$

The Hardy space $H^\infty(\mathbf{U})$ is the Banach space of holomorphic functions over $\mathbf{U}$ which satisfy the inequality

$$\sup_{z \in \mathbf{U}} |f(z)| < \infty.$$

The norm $||f||_\infty$ of a function f in $H^\infty(\mathbf{U})$ is defined by

$$||f||_\infty = \sup_{z \in \mathbf{U}} |f(z)|.$$

It is well known (see [19]) that every function in $H^p(\mathbf{U})$, $(1 \leq p \leq \infty)$ has a nontangential limit at $[m]$ almost every point of $\mathbf{T}$. Let f^* denote the boundary function of an f in $H^p(\mathbf{U})$; then

$$f^* \in H^p(\mathbf{T}) \equiv \overline{span}^{L^p(\mathbf{T},m)}\Big\{\xi^n : n \geq 0\Big\}.$$

It is also know (see [19]) that f can be reconstructed by the Poisson integral as well as the Cauchy integral of f^*. Further,

$$\|f\|_p = \|f^*\|_p$$

where the second norm is the $L^p(\mathbf{T}, m)$ norm. For this reason, we identify $H^p(\mathbf{U})$ and $H^p(\mathbf{T})$ and no longer distinguish between f and f^*. Therefore, these Banach spaces of holomorphic functions $H^p(\mathbf{U})$ may be viewed as a subspace of $L^p(\mathbf{T}, m)$.

For f in $L^p(\mathbf{T}) = L^p(\mathbf{T}, m)$, S will denote the operator of multiplication by the coordinate function. That is,

$$S(f)(z) = zf(z).$$

A good part of this dissertation is spent studying vector subspaces of $L^p(\mathbf{T})$ invariant under S.

We let $\mathbf{C}^2$ denote the cartesian product of two copies of $\mathbf{C}$. The unit bidisc in $\mathbf{C}^2$ is denoted by $\mathbf{U}^2$ and the distinguished boundary $\mathbf{T}^2$, where $\mathbf{U}$ and $\mathbf{T}$ are the unit disc and unit circle in the complex plane, respectively.

The Hardy space $H^p(\mathbf{U}^2)$, $(1 \leq p < \infty)$ is the Banach space of holomorphic functions over $\mathbf{U}^2$ which satisfy the inequality

$$\sup_{0 \leq r < 1} \int_{\mathbf{T}^2} |f(r\xi_1, r\xi_2)|^p \, dm_2(\xi_1, \xi_2) < \infty$$

where m_2 denotes normalized Lebesgue measure on $\mathbf{T}^2$. Note, holomorphic here means holomorphic in each variable. The norm $\|f\|_p$ of a function f in $H^p(\mathbf{U}^2)$ is defined by

$$\|f\|_p = \sup_{0\le r<1}\left(\int_{\mathbf{T}^2} |f(r\xi_1, r\xi_2)|^p \, dm_2(\xi_1,\xi_2)\right)^{1/p}.$$

When $p = 2$, we have a Hilbert space and the norm can be simplified. For f in $H^2(\mathbf{U}^2)$ with multiple Taylor series $\sum_{n,m\ge 0}\hat{f}(n,m)z_1^n z_2^m$ the norm simplifies to

$$\|f\|_2 = \left(\sum_{n,m\ge 0} |\hat{f}(n,m)|^2\right)^{1/2}.$$

The Hardy space $H^\infty(\mathbf{U}^2)$ is the Banach space of holomorphic functions over $\mathbf{U}^2$ which satisfy the inequality

$$\sup_{(z_1,z_2)\in\mathbf{U}^2} |f(z_1,z_2)| < \infty.$$

The norm $\|f\|_\infty$ of a function f in $H^\infty(\mathbf{U}^2)$ is defined by

$$\|f\|_\infty = \sup_{(z_1,z_2)\in\mathbf{U}^2} |f(z_1,z_2)|.$$

It is well known (see [18]) that every function in $H^p(\mathbf{U}^2)$, $(1 \le p \le \infty)$ has a nontangential limit at $[m_2]$ almost every point of $\mathbf{T}^2$. Let f^* denote the boundary function of an f in $H^p(\mathbf{U}^2)$, then

$$f^* \in H^p(\mathbf{T}^2) \equiv \overline{span}^{L^p(\mathbf{T}^2,m_2)}\left\{\xi_1^n\xi_2^m : n,m \ge 0\right\}.$$

It is also know (see [18]) that f can be reconstructed by the Poisson integral as well as the Cauchy integral of f^*. Further,

$$\|f\|_p = \|f^*\|_p$$

where the second norm is the $L^p(\mathbf{T}^2, m_2)$ norm. For this reason, we identify $H^p(\mathbf{U}^2)$ and $H^p(\mathbf{T}^2)$ and no longer distinguish between f and f^*. Therefore, these Banach spaces of holomorphic functions $H^p(\mathbf{U}^2)$ may be viewed as a subspace of $L^p(\mathbf{T}^2, m)$.

For f in $L^p(\mathbf{T}^2) = L^p(\mathbf{T}^2, m_2)$, S_1 and S_2 will denote the operators of multiplication by the first and second coordinate functions respectively. That is,

$$S_1(f)(z_1, z_2) = z_1 f(z_1, z_2)$$

and

$$S_2(f)(z_1, z_2) = z_2 f(z_1, z_2).$$

A good part of the remaining portion of this dissertation is spent studying vector subspaces of $L^p(\mathbf{T}^2)$ invariant under S_1 and S_2.

1.3 Hilbert Spaces Contained in Banach Spaces

We say a Hilbert space $\mathcal{H}$ is contained in a Banach space $\mathcal{X}$ if $\mathcal{H}$ is a vector subspace of $\mathcal{X}$. If further, there exists a $0 < C < \infty$ such that

$$||x||_{\mathcal{X}} \leq C||x||_{\mathcal{H}}$$

for all x in $\mathcal{H}$, we call $\mathcal{H}$ **boundedly contained** in $\mathcal{X}$. If $C = 1$, we say that $\mathcal{H}$ is **contractively contained** in $\mathcal{X}$.

We give some examples to illustrate the concept.

Example 1 *$L^2(\mathbf{T})$ is contractively contained in $L^p(\mathbf{T})$ for $1 \leq p \leq 2$. Similarly, for the Lebesgue spaces on $\mathbf{T}^2$.* $\triangle$

Example 2 *If $\mathcal{X}$ is a Hilbert space, then any subspace of $\mathcal{X}$ is contractively contained in $\mathcal{X}$.* $\triangle$

Example 3 *For perhaps a more interesting example take $\mathcal{X} = H^2(\mathbf{T})$ and $\mathcal{H} = gH^2(\mathbf{T})$ where g is in $H^\infty(\mathbf{T})$ with norm on $\mathcal{H}$ given by*

$$||gf||_{\mathcal{H}} = ||f||_2$$

for all f in $H^2(\mathbf{T})$. By the definition of the norm on $\mathcal{H}$, $\mathcal{H}$ is clearly a Hilbert space. Since g is in $H^\infty(\mathbf{T})$, $\mathcal{H}$ is clearly a vector subspace of $H^2(\mathbf{T})$. Further we point out that

$$\|gf\|_2 \leq \|g\|_\infty \|f\|_2 = \|g\|_\infty \|gf\|_{\mathcal{H}}.$$

So, $\mathcal{H}$ is always boundedly contained in $H^2(\mathbf{T})$ and is actually contractively contained in $H^2(\mathbf{T})$ if $\|g\|_\infty \leq 1$. $\triangle$

Alternatively, we can describe this idea using operator theoretic terminology.

Let A be a bounded operator from a Hilbert space $\mathcal{K}$ into a Banach space $\mathcal{X}$. Define $\mathcal{H}_A$ to be the range of A in $\mathcal{X}$ and equip $\mathcal{H}_A$ with the inner product given by

$$(Ax, Ay)_{\mathcal{H}_A} = (x, y)_{\mathcal{K}}$$

with at least one of x, y orthogonal to the $Ker(A)$. Then $\mathcal{H}_A$ is boundedly contained in $\mathcal{X}$ since

$$\|Ax\|_{\mathcal{X}} \leq \|A\| \|x\|_{\mathcal{K}} = \|A\| \|Ax\|_{\mathcal{H}_A}.$$

Further, $\mathcal{H}_A$ is contractively contained in $\mathcal{X}$, if A is a contraction. Every Hilbert space $\mathcal{H}$ boundedly contained in $\mathcal{X}$ is such an operator range; it is the range of the inclusion map of $\mathcal{H}$ into $\mathcal{X}$. We note that if $\mathcal{X}$ is a Hilbert space and A is a partial isometry, then $\mathcal{H}_A$ is an ordinary closed subspace of $\mathcal{X}$ since for all x orthogonal to the $Ker(A)$ we have

$$\|Ax\|_{\mathcal{X}} = \|x\|_{\mathcal{K}} = \|Ax\|_{\mathcal{H}_A}.$$

Conversely, if $\mathcal{H}_A$ is an ordinary subspace of $\mathcal{X}$, then A is a partial isometry.

In our above examples, for

Example 1 $\mathcal{X} = L^p(\mathbf{T})$, $\mathcal{H} = L^2(\mathbf{T})$ *and* $A = I$. $\triangle$

Example 2 $\mathcal{X}$ *is any Hilbert space,* $\mathcal{H}$ *is any subspace and* $A = I$. $\triangle$

Example 3 $\mathcal{X} = H^2(\mathbf{T})$, $\mathcal{H} = H^2(\mathbf{T})$ *and* $A = \mathbf{M}_g$, *where* $\mathbf{M}_g$ *is the operator of multiplication by* g. $\triangle$

Chapter 2

Invariant Vector Subspaces of $L^2(\mathbf{T})$

2.1 Known Results

We begin by discussing vector subspaces of $H^2(\mathbf{T})$ that are invariant under S. The first result is due to Beurling. He characterized all invariant subspaces of $H^2(\mathbf{T})$. Before we give his characterization we need a definition.

Definition 7 *A function ϕ in $H^\infty(\mathbf{T})$ is called **inner** if $|\phi| = 1$ a.e.* on $\mathbf{T}$.

We now give Beurling's characterization of subspaces of $H^2(\mathbf{T})$ invariant under S.

Theorem 2 (Beurling [1]) *A subspace $\mathcal{M}$ of $H^2(\mathbf{T})$ is invariant under S if and only if $\mathcal{M} = \phi H^2(\mathbf{T})$ where ϕ is an inner function.*

We won't go into any details, but we point out that Beurling's Theorem can be proved using the following decomposition.

Theorem 3 (Halmos-Wold Decomposition, see [8]) *Let $V \in \mathcal{B}(\mathcal{H})$ be an isometry.*

1. *There is a unique decomposition*

$$\mathcal{H} = \mathcal{G} \oplus \mathcal{L}$$

such that $\mathcal{G}$ and $\mathcal{L}$ are reducing subspaces for V, $\mathsf{S} = V|_{\mathcal{G}}$ is a shift operator on $\mathcal{G}$, and $\mathsf{U} = V|_{\mathcal{L}}$ is unitary on $\mathcal{L}$.

2. *Define $\mathcal{K} = \mathcal{H} \ominus V(\mathcal{H})$. Then $\{V^n(\mathcal{K})\}_{n=0}^{\infty}$ is an orthogonal family of subspaces of $\mathcal{H}$ satisfying*

$$\mathcal{G} = \sum_{n=0}^{\infty} \oplus V^n(\mathcal{K})$$

and

$$\mathcal{L} = \mathcal{G}^{\perp} = \bigcap_{n=0}^{\infty} V^n(\mathcal{H}).$$

We call S and U the *shift* and *unitary parts* of V, respectively.

Many of the remaining theorems from this section can also be proved using the Halmos-Wold decomposition. Some of our results also rely heavily on this decomposition.

An important corollary of Theorem 2 (p. 12) is used later. The proof can be found in [9]. Before we state it, we need another definition.

Definition 8 *A function g in $H^p(\mathbf{T})$ is called* ***outer*** *if the linear combination of functions*

$$g(\xi),\ \xi g(\xi),\ \xi^2 g(\xi),\ \ldots$$

are dense in $H^p(\mathbf{T})$.

With this terminology, we get

Corollary 2 *Each function f in $H^2(\mathbf{T})$ has a factorization*

$$f = \phi g$$

where ϕ is inner or constant and g is outer. This factorization is unique up to a constant factor of modulus 1.

We point out that this corollary is true for $H^p(\mathbf{T})$, $1 \leq p \leq \infty$, not just $H^2(\mathbf{T})$, see Rudin, [19] for details.

We now turn our attention to a generalization of Theorem 2 (p. 12) due to de Branges. He proved the following theorem.

Theorem 4 (de Branges, see [20]) *$\mathcal{M} \neq \{0\}$ is a Hilbert space contractively contained in $H^2(\mathbf{T})$ invariant under S and S acts as an isometry on $\mathcal{M}$ if and only if $\mathcal{M} = gH^2(\mathbf{T})$ for some g in the unit ball of $H^\infty(\mathbf{T})$ unique up to a constant multiple of modulus 1 with $\|gf\|_{\mathcal{M}} = \|f\|_2$ for all $f \in H^2(\mathbf{T})$.*

We recall that every subspace of $H^2(\mathbf{T})$ is contractively contained in $H^2(\mathbf{T})$. We also point out that S acts as an isometry on every subspace of $H^2(\mathbf{T})$. So, this is a nice and reasonable generalization of Theorem 2 (p. 12).

It was pointed out later by U. N. Singh and Dinesh Singh that de Branges' contractively contained condition could be relaxed. Their result is stated here.

Theorem 5 (U. N. Singh & Dinesh Singh [23]) *$\mathcal{M} \neq \{0\}$ is a Hilbert space that is a vector subspace of $H^2(\mathbf{T})$ invariant under S and S acts as an isometry on $\mathcal{M}$ if and only if $\mathcal{M} = gH^2(\mathbf{T})$ for some $g \in H^\infty(\mathbf{T})$ unique up to a constant multiple of modulus 1 with $\|gf\|_{\mathcal{M}} = \|f\|_2$ for all $f \in H^2(\mathbf{T})$.*

We now turn our attention to the results known in $L^2(\mathbf{T})$. We start with a result due to Helson and Lowdenslager. They were able to characterize the subspaces of

$L^2(\mathbf{T})$ that are simply invariant under S. We note that every subspace of $H^2(\mathbf{T})$ that is invariant under S is in fact simply invariant under S. Their work not only extended but also generalized the work of Beurling. We point out that Beurling's original proof of Theorem 2 (p. 12) weighed heavily on analytic function theory. Recall, $H^2(\mathbf{T})$ is just $H^2(\mathbf{U})$ in disguise. So Beurling's techniques could not be applied to characterize the simply invariant subspaces of $L^2(\mathbf{T})$. Helson and Lowdenslager used a Hilbert space approach to solve the problem in $L^2(\mathbf{T})$.

Theorem 6 (Helson & Lowdenslager, see [9]) *A subspace $\mathcal{M}$ of $L^2(\mathbf{T})$ is simply invariant under S if and only if $\mathcal{M} = gH^2(\mathbf{T})$ where g is in $L^\infty(\mathbf{T})$ and $|g| = 1$ a.e. on $\mathbf{T}$.*

In contrast to the situation in $H^2(\mathbf{T})$, in $L^2(\mathbf{T})$ there are subspaces invariant under S that are not simply invariant. Weiner was able to characterize these. In the following theorem, we use the term **doubly invariant** to mean invariant, but not simply invariant.

Theorem 7 (Weiner, see [9]) *A subspace $\mathcal{M}$ of $L^2(\mathbf{T})$ is doubly invariant under S if and only if $\mathcal{M} = \mathbf{1_E}L^2(\mathbf{T})$ where $\mathbf{E}$ is a measurable subset of $\mathbf{T}$.*

In recent work [15], contractively contained Hilbert spaces in $L^2(\mathbf{T})$ were studied. There were examples given to show that there are contractively contained Hilbert spaces in $L^2(\mathbf{T})$ satisfying the conditions of Theorem 4 (p. 14), but not of the form $gH^2(\mathbf{T})$ with g in $L^\infty(\mathbf{T})$ nonzero *a.e.* These examples show that a direct generalization to $L^2(\mathbf{T})$ is not possible. Additional conditions are required for these Hilbert spaces to have the form $gH^2(\mathbf{T})$ with g in $L^\infty(\mathbf{T})$ nonzero *a.e.* In [15], Paulsen and Singh gave an additional condition, namely a continuity condition on the norm of

$\mathcal{M}$ in addition to its contractive containment. Their motivation was to generalize Theorem 6 (p. 15) along the lines of de Branges' generalization of Theorem 2 (p. 12).

Theorem 8 (Paulsen & Singh [15]) *Let $\mathcal{M} \neq \{0\}$ be a Hilbert space contractively contained in $L^2(\mathbf{T})$ simply invariant under S and on which S acts as an isometry. Further, suppose there are p, $2 \leq p \leq \infty$ and $\delta > 0$ such that*

$$\|f\|_{\mathcal{M}} \leq \delta \|f\|_p \qquad \text{for all } f \text{ in } \mathcal{M} \cap L^p(\mathbf{T}). \tag{2.1}$$

Then there exists a unique b (up to a scalar multiple of modulus 1) in the unit ball of $L^\infty(\mathbf{T})$, which is non-zero a.e. such that

1. *$\mathcal{M} = bH^2(\mathbf{T})$ with $\|bf\|_{\mathcal{M}} = \|f\|_2$ for all f in $H^2(\mathbf{T})$,*
2. *$b^{-1} \in L^s(\mathbf{T})$ and $\|b^{-1}\|_s \leq \delta$, where*

$$s = \begin{cases} \infty, & p = 2 \\ \frac{2p}{p-2} & 2 < p < \infty \\ 2, & p = \infty. \end{cases}$$

(Note: We do not assume in (2.1) that $\mathcal{M} \cap L^p(\mathbf{T}) \neq \{0\}$ for $p > 2$.)

Paulson and Singh also had a doubly invariant result. Before we state that we need a little terminology. If $T \in \mathcal{B}(\mathcal{K})$ where $\mathcal{K}$ is a Hilbert space, we denote by $\mathcal{R}(T)$ the range of T which is a vector subspace of $\mathcal{K}$. If we endow $\mathcal{R}(T)$ with the norm

$$\|h\|_{\mathcal{R}(T)} = \inf \left\{ \|k\| : Tk = h \right\},$$

then $\mathcal{R}(T)$ is a Hilbert space in this norm called the *range space* of T and it is boundedly contained in $\mathcal{K}$. If T is a contraction, then the range space of T is contractively contained in $\mathcal{K}$.

We use $\mathbf{M}_\phi$ to denote the operator of multiplication by ϕ on $L^2(\mathbf{T})$.

Theorem 9 (Paulsen & Singh [15]) *Let $\mathcal{H}$ be boundedly contained in $L^2(\mathbf{T})$. Then S acts unitarily on $\mathcal{H}$ if and only if there exists a function ϕ in $L^\infty(\mathbf{T})$ such that $\mathcal{H} = \mathcal{R}(\mathbf{M}_\phi)$ isometrically; i.e., $\|h\|_{\mathcal{H}} = \|h\|_{\mathcal{R}(\mathbf{M}_\phi)}$ for all h in $\mathcal{H}$. When $\mathcal{H}$ is contractively contained in $L^2(\mathbf{T})$, we have $\|\phi\|_\infty \leq 1$.*

2.2 Hilbert Spaces Boundedly Contained in L^2

In this section, rather than finding conditions so that our Hilbert space is of the form $gH^2(\mathbf{T})$ with g in $L^\infty(\mathbf{T})$ nonzero *a.e.*, we describe all Hilbert spaces boundedly contained in $L^2(\mathbf{T})$ which are invariant under S and for which S acts as an isometry.

Theorem 10 *If $\mathcal{M}$ is a Hilbert space which is boundedly contained in $L^2(\mathbf{T})$, invariant under S and S acts as an isometry on $\mathcal{M}$, then there exists $\phi, g_1, g_2, \ldots \in L^\infty(\mathbf{T})$ such that $\mathcal{M} = \phi L^2(\mathbf{T}) + \sum_i g_i H^2(\mathbf{T})$ with norm $\left\|\phi p + \sum_i g_i f_i\right\|_{\mathcal{M}} = \left(\inf\left\{\|q\|_2^2 : \phi q = \phi p\right\} + \sum_i \|f_i\|_2^2\right)^{1/2}$. Note, we do not assume that $\phi, g_1, g_2, \ldots \in L^\infty(\mathbf{T})$ are nonzero.*

Proof: By Theorem 3 (p. 13), we get

$$\mathcal{M} = \mathcal{H} \oplus \sum_{n=0}^{\infty} \oplus S^n(\mathcal{N}) \tag{2.2}$$

where S is unitary on $\mathcal{H}$ and $\mathcal{N} = \mathcal{M} \ominus S(\mathcal{M})$. If $\mathcal{H} \neq \{0\}$, then by Theorem 9 (p. 17) we get that there exists $\phi \in L^\infty(\mathbf{T})$ such that

$$\mathcal{H} = \phi L^2(\mathbf{T})$$

with norm

$$\|\phi p\|_{\mathcal{M}} = \inf\left\{\|q\|_2 : \phi q = \phi p\right\}.$$

If $\mathcal{N} \neq \{0\}$, let $g_1 \in \mathcal{N}$ with $\|g_1\|_{\mathcal{M}} = 1$. Then, $\left\{g_1 e^{in\theta}\right\}_{n\geq 0}$ is an orthonormal sequence in $\mathcal{M}$. Let $f \in H^2(\mathbf{T})$. Then $f(e^{i\theta}) = \sum_{k=0}^{\infty} \hat{f}(k) e^{ik\theta}$. Let $f_n = \sum_{k=0}^{n} \hat{f}(k) e^{ik\theta}$.

Then f_n converges to f in $L^2(\mathbf{T})$ and *a.e.* Consider the following computation.

$$\begin{aligned}
\|f_n\|_2^2 &= \sum_{k=0}^{n} |\hat{f}(k)|^2 \\
&= \sum_{k=0}^{n} |\hat{f}(k)|^2 \|g_1 e^{ik\theta}\|_{\mathcal{M}}^2 \\
&= \sum_{k=0}^{n} \|\hat{f}(k) g_1 e^{ik\theta}\|_{\mathcal{M}}^2 \\
&= \left\| \sum_{k=0}^{n} \hat{f}(k) g_1 e^{ik\theta} \right\|_{\mathcal{M}}^2 .
\end{aligned}$$

Since $\left(f_n\right)_n$ is Cauchy in $L^2(\mathbf{T})$, $\left(\sum_{k=0}^{n} \hat{f}(k) g_1 e^{ik\theta}\right)_n$ is Cauchy in $\mathcal{M}$. Since $\mathcal{M}$ is a Hilbert space, there exits a h in $\mathcal{M}$ such that

$$\left\| \sum_{k=0}^{n} \hat{f}(k) g_1 e^{ik\theta} - h \right\|_{\mathcal{M}} \longrightarrow 0 \qquad \text{as } n \to \infty.$$

Since $\mathcal{M}$ is boundedly contained in $L^2(\mathbf{T})$, we get that

$$\left\| \sum_{k=0}^{n} \hat{f}(k) g_1 e^{ik\theta} - h \right\|_{2} \longrightarrow 0 \qquad \text{as } n \to \infty.$$

So a subsequence converges almost everywhere. But

$$\sum_{k=0}^{n} \hat{f}(k) e^{ik\theta} \longrightarrow f \ a.e. \qquad \text{as } n \to \infty.$$

So

$$\sum_{k=0}^{n} \hat{f}(k) g_1 e^{ik\theta} \longrightarrow g_1 f \ a.e. \qquad \text{as } n \to \infty.$$

So $h = g_1 f$. Therefore,

$$\|g_1 f\|_{\mathcal{M}} = \|f\|_2.$$

Since $\mathcal{M}$ is boundedly contained in $L^2(\mathbf{T})$, we get

$$\|g_1 f\|_2 \le C \|g_1 f\|_{\mathcal{M}} = C \|f\|_2.$$

Since f was an arbitrary element of $H^2(\mathbf{T})$, we get that g_1 multiplies $H^2(\mathbf{T})$ into $L^2(\mathbf{T})$. Since $L^2(\mathbf{T}) = H^2(\mathbf{T}) \oplus \overline{e^{i\theta} H^2(\mathbf{T})}$, we conclude that g_1 multiplies $L^2(\mathbf{T})$

into $L^2(\mathbf{T})$. Since g_1 was an arbitrary element of $\mathcal{N}$, we conclude that $\mathcal{N}$ must be contained in $L^\infty(\mathbf{T})$. Now fix an orthonormal basis $\{g_i\}$ in $\mathcal{N}$. So, we have

$$\left\| \sum_k g_k f_k \right\|_{\mathcal{M}}^2 = \sum_k \|f_k\|_2^2.$$

Now putting this altogether we get for $\phi p + \sum_k g_k f_k \in \mathcal{M}$ that

$$\left\| \phi p + \sum_k g_k f_k \right\|_{\mathcal{M}}^2 = \|\phi p\|_{\mathcal{M}}^2 + \left\| \sum_k g_k f_k \right\|_{\mathcal{M}}^2 = \inf\left\{\|q\|_2^2 : \phi q = \phi p\right\} + \sum_k \|f_k\|_2^2.$$

Therefore,

$$\left\| \phi p + \sum_k g_k f_k \right\|_{\mathcal{M}} = \left(\inf\{\|q\|_2^2 : \phi q = \phi p\} + \sum_k \|f_k\|_2^2 \right)^{1/2}$$

as desired. $\triangle$

The above result explains the examples given in [15].

2.3 Hilbert Spaces having L^2-Closures that are Simply Invariant

In this section, we characterize all Hilbert spaces contained in $L^2(\mathbf{T})$ which are invariant under S, for which S acts as an isometry and whose $L^2(\mathbf{T})$-closure is a simply invariant subspace of $L^2(\mathbf{T})$.

Our motivation for this section comes from Theorem 5 (p. 14), Corollary 2 (p. 14) and the fact that all subspaces of $H^2(\mathbf{T})$ are simply invariant.

Let g be any element of $L^\infty(\mathbf{T})$ having the modulus of an outer function *a.e.* Consider $\mathcal{M} = gH^2(\mathbf{T})$ with $\|gf\|_{\mathcal{M}} = \|f\|_2$ for all f in $H^2(\mathbf{T})$. Then it easily follows that $\mathcal{M}$ is a Hilbert space invariant under S and S acts as an isometry on $\mathcal{M}$. Further note that $\overline{\mathcal{M}}^{L^2(\mathbf{T})}$ is a simply invariant subspace of $L^2(\mathbf{T})$, (see [2], p. 142). Our result gives the converse.

Theorem 11 *If $\mathcal{M} \neq \{0\}$ is a Hilbert space that is a vector subspace of $L^2(\mathbf{T})$ such that $\mathcal{M}$ is invariant under S, S acts as an isometry on $\mathcal{M}$ and $\overline{\mathcal{M}}^{L^2(\mathbf{T})}$ is a simply invariant subspace of $L^2(\mathbf{T})$, then $\mathcal{M} = gH^2(\mathbf{T})$ with $g \in L^\infty(\mathbf{T})$ which has the modulus of an outer function a.e. and is unique up to a constant multiple of modulus 1 with $\|gf\|_{\mathcal{M}} = \|f\|_2$ for all $f \in H^2(\mathbf{T})$.*

Originally, we formulated and proved this result directly. The proof below was suggested to us by an unknown reviewer.

Proof: By Theorem 6 (p. 15) we have that $\overline{\mathcal{M}}^{L^2(\mathbf{T})} = \phi H^2(\mathbf{T})$ for some unimodular function ϕ. Then $\mathcal{M}' = \overline{\phi}\mathcal{M}$ is contained in $H^2(\mathbf{T})$ and with norm $\|\overline{\phi}p\|_{\mathcal{M}'} = \|p\|_{\mathcal{M}}$ for all p in $\mathcal{M}$ is a Hilbert space invariant under S and S acts as an isometry on $\mathcal{M}'$. So by Theorem 5 (p. 14), we get that $\mathcal{M}' = bH^2(\mathbf{T})$ with $b \in H^\infty(\mathbf{T})$ with norm $\|bf\|_{\mathcal{M}'} = \|f\|_2$ for all f in $H^2(\mathbf{T})$. So $\mathcal{M} = \phi\mathcal{M}' = \phi b H^2(\mathbf{T})$ with norm $\|\phi b f\|_{\mathcal{M}} = \|bf\|_{\mathcal{M}'} = \|f\|_2$ for all f in $H^2(\mathbf{T})$. $\triangle$

Chapter 3

Invariant Vector Subspaces of $L^p(\mathbf{T})$

3.1 Known Results

We begin by discussing vector subspaces of $H^p(\mathbf{T})$ that are invariant under S. For the first result we give credit to de Leeuw and Rudin, who first proved this result for $H^1(\mathbf{T})$. Here we give the characterization of all subspaces of $H^p(\mathbf{T})$ invariant under S.

Theorem 12 (de Leeuw & Rudin, see [9]) *A subspace $\mathcal{M}$ of $H^p(\mathbf{T})$ is invariant under S if and only if $\mathcal{M} = \phi H^p(\mathbf{T})$ where ϕ is an inner function.*

Remark 1 *We point out that even though this theorem is a strict Banach space result for $p \neq 2$, a proof found in [9] shows that this is really a corollary of Beurling's $H^2(\mathbf{T})$ result. See Appendix A and B for an alternative approach.*

At this point one might expect a generalization as done in the $H^2(\mathbf{T})$ case. Perhaps one would expect a vector subspace $\mathcal{X}$ of $H^p(\mathbf{T})$ invariant under S which is not closed in the $H^p(\mathbf{T})$-norm but is able to support a new norm that makes it complete. Considering what we know in the $H^2(\mathbf{T})$ case, we may think of $\mathcal{X}$ as being of the form $gH^p(\mathbf{T})$ where g is in $H^\infty(\mathbf{T})$ with norm

$$\|gf\|_{\mathcal{X}} = \|f\|_p$$

for all f in $H^p(\mathbf{T})$. $\mathcal{X}$ is a Banach space in this norm and invariant under S. Further, S acts as an isometry on $\mathcal{X}$. However, given the complicated structure of a Banach space, that is not a Hilbert space, this seems like quite a task. We will slightly modify this idea to get a better handle on the problem. Rather then allowing vector subspaces of $H^p(\mathbf{T})$ that support any norm, we will only allow those vectors subspaces of $H^p(\mathbf{T})$ that can support a norm, that will make them a Hilbert space. Such ideas have already been studied by Dinesh Singh and Sanjeev Agrawal [22] who characterized certain Hilbert spaces contained in some Banach spaces of analytic functions. In particular, they proved the following $H^p(\mathbf{T})$ result.

Theorem 13 (Dinesh Singh and Sanjeev Agrawal [22]) *If $\mathcal{M}$ is a Hilbert space contained in $H^p(\mathbf{T})$, invariant under S and S acts as an isometry on $\mathcal{M}$, then*

$$\mathcal{M} = bH^2(\mathbf{T})$$

for a unique b:

1. *If $1 \leq p \leq 2$, $b \in H^{\frac{2p}{2-p}}(\mathbf{T})$. When $p = 2$, we mean $H^\infty(\mathbf{T})$.*

2. *If $p > 2$, $b = 0$.*

Further, $\|bf\|_{\mathcal{M}} = \|f\|_2$ for all f in $H^2(\mathbf{T})$ $(1 \leq p \leq 2)$.

We now turn our attention to $L^p(\mathbf{T})$. We start with a generalization of Theorem 12 (p. 21). This is due to Forelli, who characterized the subspaces of $L^p(\mathbf{T})$ that are simply invariant under S.

Theorem 14 (Forelli, see [9]) *A subspace $\mathcal{M}$ of $L^p(\mathbf{T})$ is simply invariant under S if and only if $\mathcal{M} = gH^p(\mathbf{T})$ where g is in $L^\infty(\mathbf{T})$ and $|g| = 1$ a.e. on $\mathbf{T}$.*

As one might expect, there is also a doubly invariant result. We give credit for it to Weiner.

Theorem 15 (Weiner, see [9]) *A subspace $\mathcal{M}$ of $L^p(\mathbf{T})$ is doubly invariant under S if and only if $\mathcal{M} = \mathbf{1_E}L^p(\mathbf{T})$ where $\mathbf{E}$ is a measurable subset of $\mathbf{T}$.*

The same complications arise in $L^p(\mathbf{T})$, as did in $H^p(\mathbf{T})$, so here too, we only consider vector subspaces of $L^p(\mathbf{T})$ that are Hilbert spaces. Paulsen and Singh [15], in addition to their aforementioned $L^2(\mathbf{T})$ result, showed the following.

Theorem 16 (Paulsen & Singh [15]) *Let $\mathcal{M}$ be a simply invariant Hilbert space contractively contained in $L^r(\mathbf{T})$ for some $r > 2$ and on which S acts as an isometry. Further, suppose there are p, $2 \leq p \leq \infty$ and $\delta > 0$ such that*

$$\|f\|_{\mathcal{M}} \leq \delta \|f\|_p \qquad \text{for all } f \text{ in } \mathcal{M} \cap L^p(\mathbf{T}). \tag{3.1}$$

Then $\mathcal{M} = \{0\}$. (Note: We do not assume in (3.1) that $\mathcal{M} \cap L^p(\mathbf{T}) \neq \{0\}$ for $p > 2$.)

Later, we investigate the situation in $L^q(\mathbf{T})$ for $1 \leq q < 2$ and give conditions so that the Hilbert space is of "Beurling type".

3.2 An Extension of a Result of Singh and Agrawal

Here, we give an extension of Theorem 13 (p. 22). It is easy to see from Theorem 11 (p. 20) that this result can be extended to certain vector subspaces of $L^p(\mathbf{T})$ in the following way.

Theorem 17 *If $\mathcal{M}$ is a Hilbert space contained in $L^p(\mathbf{T})$, invariant under S, S acts as an isometry on $\mathcal{M}$ and $\overline{\mathcal{M}}^{L^p(\mathbf{T})}$ is a simply invariant subspace of $L^p(\mathbf{T})$, then*

$$\mathcal{M} = bH^2(\mathbf{T})$$

for a unique b:

1. *If $1 \leq p \leq 2$, $b \in L^{\frac{2p}{2-p}}(\mathbf{T})$ and has the modulus of an outer function a.e. When $p = 2$, we mean $L^\infty(\mathbf{T})$.*

2. *If $p > 2$, $b = 0$.*

Further, $\|bf\|_{\mathcal{M}} = \|f\|_2$ for all f in $H^2(\mathbf{T})$ $(1 \leq p \leq 2)$.

Proof: By Theorem 14 (p. 23) we have that $\overline{\mathcal{M}}^{L^p(\mathbf{T})} = \phi H^p(\mathbf{T})$ for some unimodular function ϕ. Then, $\mathcal{M}' = \overline{\phi}\mathcal{M}$ is contained in $H^p(\mathbf{T})$ and with norm $\|\overline{\phi}p\|_{\mathcal{M}'} = \|p\|_{\mathcal{M}}$ for all p in $\mathcal{M}$ is a Hilbert space invariant under S and S acts as an isometry on $\mathcal{M}'$. So by Theorem 13 (p. 22), we get for $p > 2$ that $\mathcal{M}' = \{0\}$ and hence $\mathcal{M} = \{0\}$ and for $1 \leq p \leq 2$, $\mathcal{M}' = gH^2(\mathbf{T})$ with $g \in H^{\frac{2p}{2-p}}(\mathbf{T})$ with norm $\|gf\|_{\mathcal{M}'} = \|f\|_2$ for all f in $H^2(\mathbf{T})$. So $\mathcal{M} = \phi\mathcal{M}' = \phi g H^2(\mathbf{T})$ with norm $\|\phi g f\|_{\mathcal{M}} = \|gf\|_{\mathcal{M}'} = \|f\|_2$ for all f in $H^2(\mathbf{T})$. $\triangle$

The converse of the theorem is clear except for the fact that the closure is simply invariant. This follows from a straight forward variation of a theorem in [2], page 142.

3.3 Hilbert Spaces Contractively Contained in L^q for $1 \leq q < 2$

In this section, we give conditions so a contractively contained Hilbert space in $L^q(\mathbf{T})$ for $1 \leq q < 2$ is of "Beurling Type".

Before we give our result, we consider the following situation. Fix $1 \leq q < 2$ and let b be an element of the unit ball of $L^{\frac{2q}{2-q}}(\mathbf{T})$ with $b \neq 0$ *a.e.* Then b multiplies $H^2(\mathbf{T})$ into $L^q(\mathbf{T})$. Let's call $\mathcal{M}$ the range of such a multiplication; i.e., $\mathcal{M} = bH^2(\mathbf{T})$ and endow $\mathcal{M}$ with the norm

$$||bf||_{\mathcal{M}} = ||f||_2.$$

Then $\mathcal{M}$ becomes a Hilbert space contained in $L^q(\mathbf{T})$. In fact, $\mathcal{M}$ is contractively contained in $L^q(\mathbf{T})$ since

$$\begin{aligned} ||bf||_q &\leq ||b||_{\frac{2q}{2-q}} ||f||_2 && \text{by Hölder's Inequality} \\ &\leq ||bf||_{\mathcal{M}}. \end{aligned}$$

The last inequality follows from the definition of the norm on $\mathcal{M}$ and the fact that b is in the unit ball in $L^{\frac{2q}{2-q}}(\mathbf{T})$. We further point out that $\mathcal{M}$ is simply invariant and that S acts as an isometry on $\mathcal{M}$. Unfortunately, this is not enough to hope for a characterization as pointed out by Paulsen and Singh in [15]. We consider the following extra condition, which is similar to the condition found in Theorem 8 (p. 16), but slightly modified to fit the $L^q(\mathbf{T})$ setting. Suppose that b^{-1} is in $L^s(\mathbf{T})$ where

$$s = \begin{cases} \infty & \text{if } p = 2 \\ \frac{2p}{p-2} & \text{if } 2 < p \leq \frac{2q}{2-q}. \end{cases}$$

Let $\delta = ||b^{-1}||_s$ which we point out is strictly greater then zero. We now make the following calculation.

$$\begin{aligned} ||bf||_{\mathcal{M}} &= ||f||_2 \\ &= ||b^{-1}bf||_2 \\ &\leq ||b^{-1}||_{\frac{2p}{p-2}} ||bf||_p && \text{by Hölder's Inequality} \\ &= \delta ||bf||_p. \end{aligned}$$

So we get

$$\|bf\|_{\mathcal{M}} \leq \delta \|bf\|_p.$$

Our theorem gives the converse.

Theorem 18 *Let $\mathcal{M} \neq \{0\}$ be a simply invariant Hilbert space contractively contained in $L^q(\mathbf{T})$ $(1 \leq q < 2)$ and on which S acts as an isometry. Further, suppose there are p, $2 \leq p \leq \frac{2q}{2-q}$ and $\delta > 0$ such that*

$$\|f\|_{\mathcal{M}} \leq \delta \|f\|_p \qquad \text{for all } f \text{ in } \mathcal{M} \cap L^p(\mathbf{T}) \tag{3.2}$$

and if the unitary part of $\mathcal{M}$ is nonzero then so is its intersection with $L^p(\mathbf{T})$. Then there exists a unique b (up to a scalar multiple of modulus 1) in the unit ball of $L^{\frac{2q}{2-q}}(\mathbf{T})$, which is non-zero a.e. such that

1. *$\mathcal{M} = bH^2(\mathbf{T})$ with $\|bf\|_{\mathcal{M}} = \|f\|_2$ for all f in $H^2(\mathbf{T})$,*

2. *$b^{-1} \in L^s(\mathbf{T})$ and $\|b^{-1}\|_s \leq \delta$, where*

$$s = \begin{cases} \infty & \text{if } p = 2 \\ \frac{2p}{p-2} & \text{if } 2 < p \leq \frac{2q}{2-q}. \end{cases}$$

To prove our theorem we need a lemma.

Let $\mathbf{M_g}$ denote the linear transformation of multiplication by g.

Lemma 1 *If $\mathbf{M_g} : L^2(\mathbf{T}) \longrightarrow L^q(\mathbf{T})$ $(1 \leq q < 2)$ is bounded, then $\|\mathbf{M_g}\| = \|g\|_{\frac{2q}{2-q}}$. So, in particular, $g \in L^{\frac{2q}{2-q}}(\mathbf{T})$.*

Proof: Let f be any element of $L^2(\mathbf{T})$. Then by Hölder's Inequality we get that $\|gf\|_q \leq \|g\|_{\frac{2q}{2-q}} \|f\|_2$. Therefore, we get

$$\begin{aligned} \|\mathbf{M_g}\| &= \sup\left\{\|gf\|_q : \|f\|_2 \leq 1\right\} \\ &\leq \sup\left\{\|g\|_{\frac{2q}{2-q}} \|f\|_2 : \|f\|_2 \leq 1\right\} \\ &\leq \|g\|_{\frac{2q}{2-q}}. \end{aligned}$$

For the other inequality we note that there exists a measurable function α with $|\alpha| = 1$ such that $\alpha g = |g|$. Now let $E_n = \{x : |g(x)| < n\}$ and define $f = \chi_{E_n}|g|^{\frac{q}{2-q}}\alpha$. Then $|f|^2 = \chi_{E_n}|g|^{\frac{2q}{2-q}}$. So, $f \in L^\infty(\mathbf{T})$. Also, $fg = \chi_{E_n}|g|^{\frac{2}{2-q}}$. So we have

$$\begin{aligned}\left(\int_{E_n}|g|^{\frac{2q}{2-q}}dm\right)^{1/q} &= \left(\int_{\mathbf{T}}|gf|^q dm\right)^{1/q}\\ &= \|\mathbf{M_g}(f)\|_q\\ &\leq \|\mathbf{M_g}\|\|f\|_2\\ &= \|\mathbf{M_g}\|\left(\int_{E_n}|g|^{\frac{2q}{2-q}}dm\right)^{1/2}.\end{aligned}$$

Dividing through by $\left(\int_{E_n}|g|^{\frac{2q}{2-q}}dm\right)^{1/2}$ which is finite we get $\left(\int_{\mathbf{T}}\chi_{E_n}|g|^{\frac{2q}{2-q}}dm\right)^{1/q-1/2} \leq \|\mathbf{M_g}\|$. Noticing that $1/q - 1/2 = \frac{2-q}{2q}$ and applying the monotone convergence theorem we get $\|g\|_{\frac{2q}{2-q}} \leq \|\mathbf{M_g}\|$. Putting these two inequalities together gives our desired result. $\triangle$

Proof of Theorem 18 (p. 26): By Theorem 3 (p. 13) we may write $\mathcal{M}$ as

$$\mathcal{M} = \mathcal{H} \oplus \sum_{n=0}^{\infty} \oplus S^n(\mathcal{N}),$$

where S is unitary on $\mathcal{H}$ and $\mathcal{N} = \mathcal{M} \ominus S(\mathcal{M})$. Let $\mathcal{M}_1 = \mathcal{M} \ominus \mathcal{H}$. We point out that $\mathcal{M}_1 \neq \{0\}$, otherwise, we contradict the fact that $\mathcal{M}$ is simply invariant. So we have that $\mathcal{N} \neq \{0\}$. Therefore, we may choose an arbitrary element b of $\mathcal{N}$ with unit norm in $\mathcal{M}$. Then, $\left\{be^{in\theta}\right\}_{n\geq 0}$ forms an orthonormal sequence in $\mathcal{M}$. Let $f \in H^2(\mathbf{T})$. Then $f(e^{i\theta}) = \sum_{k=0}^{\infty}\hat{f}(k)e^{ik\theta}$. Let $f_n = \sum_{k=0}^{n}\hat{f}(k)e^{ik\theta}$. Then f_n converges to f in $L^2(\mathbf{T})$ and *a.e.* We also have

$$\begin{aligned}\|f_n\|_2^2 &= \sum_{k=0}^{n}|\hat{f}(k)|^2\\ &= \sum_{k=0}^{n}|\hat{f}(k)|^2\|be^{ik\theta}\|_{\mathcal{M}}^2\\ &= \sum_{k=0}^{n}\|\hat{f}(k)be^{ik\theta}\|_{\mathcal{M}}^2\end{aligned}$$

$$= \left\| \sum_{k=0}^{n} \hat{f}(k) b e^{ik\theta} \right\|_{\mathcal{M}}^2 .$$

Since $\left(f_n\right)_n$ is Cauchy in $L^2(\mathbf{T})$, $\left(\sum_{k=0}^{n} \hat{f}(k) b e^{ik\theta}\right)_n$ is Cauchy in $\mathcal{M}$. Since $\mathcal{M}$ is a Hilbert space, there exists an h in $\mathcal{M}$ such that

$$\left\| \sum_{k=0}^{n} \hat{f}(k) b e^{ik\theta} - h \right\|_{\mathcal{M}} \longrightarrow 0 \qquad \text{as } n \to \infty.$$

Since $\mathcal{M}$ is contractively contained in $L^q(\mathbf{T})$, we get that

$$\left\| \sum_{k=0}^{n} \hat{f}(k) b e^{ik\theta} - h \right\|_{q} \longrightarrow 0 \qquad \text{as } n \to \infty.$$

So a subsequence converges almost everywhere. But

$$\sum_{k=0}^{n} \hat{f}(k) e^{ik\theta} \longrightarrow f \ a.e. \qquad \text{as } n \to \infty.$$

So

$$\sum_{k=0}^{n} \hat{f}(k) b e^{ik\theta} \longrightarrow bf \ a.e. \qquad \text{as } n \to \infty.$$

So $h = bf$. Therefore,

$$\|bf\|_{\mathcal{M}} = \|f\|_2.$$

Since $\mathcal{M}$ is contractively contained in $L^q(\mathbf{T})$, we get

$$\|bf\|_q \leq \|bf\|_{\mathcal{M}} = \|f\|_2. \tag{3.3}$$

Since f was an arbitrary element of $H^2(\mathbf{T})$, we get that b multiplies $H^2(\mathbf{T})$ into $L^q(\mathbf{T})$. Since $L^2(\mathbf{T}) = H^2(\mathbf{T}) \oplus \overline{e^{i\theta} H^2(\mathbf{T})}$, we conclude that b multiplies $L^2(\mathbf{T})$ into $L^q(\mathbf{T})$. Using inequality (3.3) (p. 28), we see that $\mathbf{M}_b$ is a bounded transformation from $L^2(\mathbf{T})$ to $L^q(\mathbf{T})$; that is a contraction. So by Lemma 1 (p. 26) we conclude that b is in the unit ball of $L^{\frac{2q}{2-q}}(\mathbf{T})$. Since b was an arbitrary normalized element of $\mathcal{N}$, we conclude that $\mathcal{N}$ must be contained in $L^{\frac{2q}{2-q}}(\mathbf{T})$.

Now we show that no element of $\mathcal{N}$ can vanish on a set of positive measure unless it is identically zero. Choose any nonzero element d in $\mathcal{N}$. Suppose that d is identically zero on a set of positive measure; call the set E. Let

$$k_n = \begin{cases} n & \text{on } E \\ 1 & \text{on } E^c. \end{cases}$$

Then k_n is in $L^\infty(\mathbf{T})$ for all n. Let

$$h_n = \exp(k_n + i\tilde{k_n})$$

where $\tilde{k_n}$ denotes the harmonic conjugate of k_n. So h_n is in $H^\infty(\mathbf{T})$. Replacing b by d in the above computations gives

$$\|h_n\|_2 = \|dh_n\|_{\mathcal{M}} \leq \delta \|dh_n\|_p.$$

First note that $dh_n \in L^{\frac{2q}{2-q}}(\mathbf{T}) \subseteq L^p(\mathbf{T})$. By the construction of h_n, the right hand side of the above inequality is bounded by a fixed constant independent of n whereas the left hand side goes to infinity as $n \to \infty$. This contradiction shows our supposition must be incorrect. So no element of $\mathcal{N}$ can vanish on a set on positive measure unless it is identically zero.

Next we show that $\mathcal{N}$ is one dimensional. To do this, we suppose not. So we can find a b_1 in $\mathcal{N}$ with unit norm orthogonal to b in $\mathcal{M}$. By our decomposition, we get that $bH^2(\mathbf{T})$ is orthogonal to $b_1H^2(\mathbf{T})$ in $\mathcal{M}$. Since b and b_1 are in $L^{\frac{2q}{2-q}}(\mathbf{T})$ and in $\mathcal{M}_1$, we see that $\{b, b_1\} \subset \mathcal{M}_1 \cap L^p(\mathbf{T})$. Further, $\mathcal{M}_1 \cap L^p(\mathbf{T})$ is invariant under S. Let $A(\mathcal{M}_1 \cap L^p(\mathbf{T}))$ denote the annihilator of $\mathcal{M}_1 \cap L^p(\mathbf{T})$, which is a subspace of $L^{\frac{p}{p-1}}(\mathbf{T})$. By the definition of the annihilator we get that the annihilator is invariant under S since $\mathcal{M}_1 \cap L^p(\mathbf{T})$ is invariant under S. We now show that $A(\mathcal{M}_1 \cap L^p(\mathbf{T}))$ contains an element that multiplies b and b_1 in to $L^\infty(\mathbf{T})$. Let f be any non-zero element of $A(\mathcal{M}_1 \cap L^p(\mathbf{T}))$. Let $\{p_n(z)\}$ be a sequence of analytic polynomials that converges boundedly and pointwise to

$$\exp[-(|f|+i|f|\tilde{\ })]\exp[-(|b|+i|b|\tilde{\ })]\exp[-(|b_1|+i|b_1|\tilde{\ })].$$

Clearly, $p_n f$ converges to

$$\exp[-(|f|+i|f|\tilde{\ })]\exp[-(|b|+i|b|\tilde{\ })]\exp[-(|b_1|+i|b_1|\tilde{\ })]f$$

in $L^{\frac{p}{p-1}}$; so $\exp[-(|f|+i|f|\tilde{\ })]\exp[-(|b|+i|b|\tilde{\ })]\exp[-(|b_1|+i|b_1|\tilde{\ })]f$ is in $A(\mathcal{M}_1 \cap L^p(\mathbf{T}))$ and clearly multiplies b and b_1 into $L^\infty(\mathbf{T})$. So now let g be any non-zero element of $A(\mathcal{M}_1 \cap L^p(\mathbf{T}))$ that multiplies b and b_1 into $L^\infty(\mathbf{T})$. Note then that

$$\int_{-\pi}^{\pi} g(e^{i\theta})b(e^{i\theta})e^{in\theta}\,d\theta = 0 \qquad n = 0, 1, 2, \ldots$$

and

$$\int_{-\pi}^{\pi} g(e^{i\theta})b_1(e^{i\theta})e^{in\theta}\,d\theta = 0 \qquad n = 0, 1, 2, \ldots .$$

Let's write $k = gb$ and $k_1 = gb_1$. Then k and k_1 are in $H^\infty(\mathbf{T})$. Since b and k do not vanish on a set of positive measure, we conclude that g does not vanish on a set of positive measure. Now consider the function

$$\frac{kk_1}{g} = \begin{cases} kb_1 & \in b_1H^2(\mathbf{T}) \\ bk_1 & \in bH^2(\mathbf{T}). \end{cases}$$

Since $bH^2(\mathbf{T}) \cap b_1H^2(\mathbf{T}) = \{0\}$, we get that $\frac{kk_1}{g} = 0$, but this is a contradiction since $\frac{kk_1}{g}$ does not vanish on a set of positive measure. From this contradiction we conclude that our supposition must be incorrect. So $\mathcal{N}$ must be one dimensional.

Note that if we use $\exp[-(|f|+i|f|\tilde{\ })]\exp[-(|b|+i|b|\tilde{\ })]\exp[-(|b_1|+i|b_1|\tilde{\ })]f$ instead of g in the above calculations, then we get that f does not vanish on a set of positive measure. This follows from the reasoning employed above concerning the k and k_1 and the observation that $\exp[-(|f|+i|f|\tilde{\ })]\exp[-(|b|+i|b|\tilde{\ })]\exp[-(|b_1|+i|b_1|\tilde{\ })]$ is a bounded analytic function. So every nonzero member of $A(\mathcal{M}_1 \cap L^p(\mathbf{T}))$ does not vanish on a set of positive measure.

Now we show that $\mathcal{H} = \{0\}$. By hypothesis, we need only show that $\mathcal{H} \cap L^p(\mathbf{T}) = \{0\}$. To do this, we suppose not. Let ϕ be an element of $\mathcal{H} \cap L^p(\mathbf{T})$. Let f be a nonzero element of $A(\mathcal{M} \cap L^p(\mathbf{T}))$. Let $\{p_n(z)\}$ be a sequence of analytic polynomials that converges boundedly and pointwise to

$$\exp[-(|f| + i|f|\tilde{\ })] \exp[-(|\phi| + i|\phi|\tilde{\ })].$$

Clearly, $p_n f$ converges to $\exp[-(|f| + i|f|\tilde{\ })] \exp[-(|\phi| + i|\phi|\tilde{\ })]f$ in $L^{\frac{p}{p-1}}$; so

$$u \equiv \exp[-(|f| + i|f|\tilde{\ })] \exp[-(|\phi| + i|\phi|\tilde{\ })]f$$

is in $A(\mathcal{M} \cap L^p(\mathbf{T}))$. Since $A(\mathcal{M} \cap L^p(\mathbf{T})) \subset A(\mathcal{M}_1 \cap L^p(\mathbf{T}))$, the above observation shows us that u does not vanish on a set of positive measure and by the construction of u, u multiplies ϕ in to $L^\infty(\mathbf{T})$. Now ϕz^n is in $\mathcal{H} \cap L^p(\mathbf{T})$ for all integers n. Therefore,

$$\int_{-\pi}^{\pi} u(e^{i\theta})\phi(e^{i\theta})e^{in\theta}d\theta = 0 \qquad \text{for all integers } n.$$

Therefore, $u\phi = 0$, but u does not vanish on a set of positive measure. Therefore, $\phi = 0$. So $\mathcal{H} \cap L^p(\mathbf{T}) = \{0\}$.

To finish the proof we need to establish our conditions on b^{-1}. To do this, we first consider the case when $2 < p \leq \frac{2q}{2-q}$. Let

$$E_n = \left\{ e^{i\theta} : \frac{1}{n} < |b(e^{i\theta})| < n \right\}.$$

Now define

$$k_n = \begin{cases} \frac{p}{2-p} \log|b| & \text{on } E_n \\ 0 & \text{on } E_n^c \end{cases}$$

and

$$h_n = \exp(k_n + i\tilde{k_n}).$$

Note that h_n is $H^\infty(\mathbf{T})$ for all n in $\mathbf{Z}_+$. We make the following computation.

$$\begin{aligned}\left(\frac{1}{2\pi}\int_{E_n} |b|^{\frac{2p}{2-p}}\, d\theta\right)^{1/2} &\le \|h_n\|_2 = \|bh_n\|_{\mathcal{M}} \\ &\le \delta\left(\frac{1}{2\pi}\int_{\mathbf{T}} |b|^p |h_n|^p\, d\theta\right)^{1/p}.\end{aligned}$$

The last inequality holds because bh_n is in $\mathcal{M}$ and b is in $L^{\frac{2q}{2-q}}(\mathbf{T})$ and h_n is in $H^\infty(\mathbf{T})$; so bh_n is in $L^{\frac{2q}{2-q}}(\mathbf{T})$ which is contained in $L^p(\mathbf{T})$. So bh_n is in $\mathcal{M} \cap L^p(\mathbf{T})$. Further the above inequality holds for all n. So the quotient is bounded by δ for all n. Letting n go to infinity, we get that

$$\|b^{-1}\|_s \le \delta$$

as desired. Now, for the case where $p = 2$, we note that for h a trigonometric polynomial

$$\|h\|_2 = \|bh\|_{\mathcal{M}} \le \delta\|bh\|_p.$$

Therefore,

$$\frac{1}{2\pi}\int_{\mathbf{T}} (\delta^2|b|^2 - 1)|h|^2 d\theta \ge 0$$

for all trigonometric polynomials h, from which it follows that

$$\|b^{-1}\|_\infty \le \delta. \quad \triangle$$

Chapter 4

Invariant Vector Subspaces of $L^p(\mathbf{T}^2)$

In this chapter, we begin the second part of this dissertation. Here, we consider vector subspaces of $L^p(\mathbf{T}^2)$ that are invariant under S_1 and S_2. We start by considering vector subspaces of $H^p(\mathbf{T}^2)$. In fact, let's start with the case $p = 2$. Naturally, one might start with the subspace $\mathcal{M} = \phi H^2(\mathbf{T}^2)$ and hope, as in the $H^2(\mathbf{T})$ case, that these are all of the subspaces of $H^2(\mathbf{T}^2)$ invariant under both S_1 and S_2. Unfortunately, that is not the case. Rudin [18] showed that, unlike $H^2(\mathbf{T})$ where all subspaces invariant under S are generated by a single inner function, there are subspaces of $H^2(\mathbf{T}^2)$ invariant under S_1 and S_2 that are not generated by a single function. In fact, there are subspaces of $H^2(\mathbf{T}^2)$ invariant under S_1 and S_2 that are not even finitely generated. Further, he showed that there are subspaces of $H^2(\mathbf{T}^2)$ invariant under S_1 and S_2 that contain no bounded elements, again in contrast with the $H^2(\mathbf{T})$ case where every subspace invariant under S contains a bounded function, in fact an inner function. To my knowledge, the description of all the subspaces of $H^2(\mathbf{T}^2)$ invariant under S_1 and S_2 is still unknown. However, some work has been done to that end. The first result is due to Mandrekar.

Theorem 19 (Mandrekar [13]) *Let $\mathcal{M} \neq \{0\}$ be a subspace of $H^2(\mathbf{T}^2)$ invariant under S_1 and S_2. Then, $\mathcal{M} = qH^2(\mathbf{T}^2)$ with q inner if and only if S_1 and S_2 are doubly commuting on $\mathcal{M}$.*

We want to extend Mandrekar's result to $H^p(\mathbf{T}^2)$, $1 \leq p \leq \infty$. Before we do this we give a result of Ghatage and Mandrekar in $L^2(\mathbf{T}^2)$ to prevent proving a similar result twice.

Theorem 20 (Ghatage & Mandrekar [5]) *Let $\mathcal{M} \neq \{0\}$ be a subspace of $L^2(\mathbf{T}^2)$ invariant under S_1 and S_2. Then, $\mathcal{M} = qH^2(\mathbf{T}^2)$ with q unimodular if and only if S_1 and S_2 are doubly commuting shifts on $\mathcal{M}$.*

Here the extra condition **shifts** is very important.

The above two theorems can be shown by exploiting the following decomposition.

Theorem 21 (Halmos-Wold Four-Fold Decomposition, [11], [24]) *Let $V_1, V_2 \in \mathcal{B}(\mathcal{H})$ be isometries with V_1 and V_2 doubly commuting on $\mathcal{H}$.*

1. *There is a unique decomposition*

$$\mathcal{H} \;=\; \mathcal{H}_{ss} \oplus \mathcal{H}_{su} \oplus \mathcal{H}_{us} \oplus \mathcal{H}_{uu}$$

such that

(a) *$V_i(\mathcal{H}_{ss}) \subset \mathcal{H}_{ss}$ and $V_i|_{\mathcal{H}_{ss}}$ is a shift for $i = 1, 2$.*

(b) *$V_1(\mathcal{H}_{su}) \subset \mathcal{H}_{su}$ and $V_1|_{\mathcal{H}_{su}}$ is a shift.*

(c) *$V_2(\mathcal{H}_{su}) = \mathcal{H}_{su}$ and $V_2|_{\mathcal{H}_{su}}$ is unitary.*

(d) *$V_2(\mathcal{H}_{us}) \subset \mathcal{H}_{us}$ and $V_2|_{\mathcal{H}_{us}}$ is a shift.*

(e) $V_1(\mathcal{H}_{us}) = \mathcal{H}_{us}$ *and* $V_1|_{\mathcal{H}_{us}}$ *is unitary.*

(f) $V_i(\mathcal{H}_{uu}) = \mathcal{H}_{uu}$ *and* $V_i|_{\mathcal{H}_{uu}}$ *is unitary for* $i = 1, 2$.

2. *Define* $\mathcal{K}_1 = \mathcal{H} \ominus V_1(\mathcal{H})$, $\mathcal{K}_2 = \mathcal{H} \ominus V_2(\mathcal{H})$ *and* $\mathcal{K} = (\mathcal{H} \ominus V_1(\mathcal{H})) \cap (\mathcal{H} \ominus V_2(\mathcal{H}))$.

   Then we have

$$\begin{aligned}
\mathcal{H}_{ss} &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \oplus V_1^n V_2^m(\mathcal{K}), \\
\mathcal{H}_{su} &= \sum_{m=0}^{\infty} \oplus V_1^m [\cap_{n=0}^{\infty} V_2^n(\mathcal{K}_1)], \\
\mathcal{H}_{us} &= \sum_{m=0}^{\infty} \oplus V_2^m [\cap_{n=0}^{\infty} V_1^n(\mathcal{K}_2)], \\
\mathcal{H}_{uu} &= \bigcap_{n,m \geq 0} V_1^n V_2^m(\mathcal{H}).
\end{aligned}$$

Notice, this is just Theorem 3 (p. 13) applied to two operators. The double commuting condition is required to make everything work out as we expect.

Now, we extend this result to $L^p(\mathbf{T}^2)$, $1 \leq p \leq \infty$. As a corollary we will get the $H^p(\mathbf{T}^2)$ extension. Before we do this, we need some terminology. We let $RP(\mathbf{U}^2)$ denote the class of all functions in $\mathbf{U}^2$ which are the real parts of holomorphic functions. We point out here, in contrast with functions analytic in $\mathbf{U}$, not all real harmonic functions in $\mathbf{U}^2$ are the real parts of holomorphic functions in $\mathbf{U}^2$, see [18] for more details. We also recall that,

$$f : \mathbf{T}^2 \longrightarrow (-\infty, \infty]$$

is called **lower semicontinuous, (l.s.c.)** if

$$\left\{(e^{i\theta_1}, e^{i\theta_2}) : f(e^{i\theta_1}, e^{i\theta_2}) > \alpha\right\}$$

is open for all real α.

The proof of the following lemma is found in [18] (p. 34).

Lemma 2 *Suppose f is a l.s.c. positive function on $\mathbf{T}^2$ and $f \in L^1(\mathbf{T}^2)$. Then there exists a singular (complex Borel) measure σ on $\mathbf{T}^2$, $\sigma \geq 0$, such that $P[f - d\sigma] \in RP(\mathbf{U}^2)$.*

Before we give our next lemma we make some observations. First of all, if f is continuous, then f is *l.s.c.*. If f and g are two continuous functions, then so is $\big(f \vee g\big)(x) = \max\big\{f(x), g(x)\big\}$. This follow from a straight forward $\epsilon - \delta$ argument once you note that $\big(f \vee g\big)(x) = \frac{1}{2}\big[f(x) + g(x) + |f(x) - g(x)|\big]$. Finally, if $\big(f_n\big)_{n=1}^{\infty}$ is a sequence of *l.s.c.* functions, then $f(x) = \sup_n f_n(x)$ is also *l.s.c.* This follows from the definition of *l.s.c.* and the fact that

$$\big\{x : f(x) > \alpha\big\} = \bigcup_{n=1}^{\infty} \big\{x : f_n(x) > \alpha\big\}.$$

Lemma 3 *Suppose f is real-valued on $\mathbf{T}^2$ and $f \in L^p(\mathbf{T}^2)$ for $1 \leq p < \infty$. Then there exists two positive l.s.c. functions g_1 and g_2 in $L^p(\mathbf{T}^2)$ such that $f = g_1 - g_2$ a.e. on $\mathbf{T}^2$.*

Presently, we only need this result for $p = 1$, but later we will need this result for other values of p.

Proof: Since f is real-valued on $\mathbf{T}^2$, $f \in L^p(\mathbf{T}^2)$ and continuous functions are dense in $L^p(\mathbf{T}^2)$ there exists ϕ_1 continuous such that

$$\|f - \phi_1\|_p < 2^{-1}$$

and by the reverse triangle inequality we get

$$\|\phi_1\|_p < \big(1 + 2\|f\|_p\big) \cdot 2^{-1}.$$

Now we can find ϕ_2 continuous such that

$$\Big\|(f - \phi_1) - \phi_2\Big\|_p < 2^{-2}$$

and by the reverse triangle inequality we get

$$\|\phi_2\|_p < 2^{-2} + \|f - \phi_1\|_p < 3 \cdot 2^{-2}.$$

Continuing in the manner we get the existence of a sequence of real-valued continuous functions $(\phi_n)_n$ such that

$$f = \sum_{n=1}^{\infty} \phi_n$$

in $L^p(\mathbf{T}^2)$ and

$$\|\phi_n\|_p < C \cdot 2^{-n} \qquad \text{for all } n, \text{ where } C = \max\left\{1 + 2\|f\|_p \,,\ 3\right\}.$$

Now, for $\epsilon > 0$, define

$$\psi_n^+ = (\phi_n \vee 0) + \epsilon \cdot 2^{-n}$$

and

$$\psi_n^- = (-\phi_n \vee 0) + \epsilon \cdot 2^{-n}.$$

Then ψ_n^+ and ψ_n^- are positive continuous functions with $\phi_n = \psi_n^+ - \psi_n^-$. So

$$f = \sum_{n=1}^{\infty} (\psi_n^+ - \psi_n^-) = \sum_{n=1}^{\infty} \psi_n^+ - \sum_{n=1}^{\infty} \psi_n^- \qquad \text{in } L^p(\mathbf{T}^2).$$

Since

$$\begin{aligned} \sum_{n=1}^{\infty} \|\psi_n^+\|_p \ &\leq\ \sum_{n=1}^{\infty} (\|\phi_n \vee 0\|_p + \epsilon \cdot 2^{-n}) \ \leq\ \sum_{n=1}^{\infty} (\|\phi_n\|_p + \epsilon \cdot 2^{-n}) \\ &<\ \sum_{n=1}^{\infty} (C \cdot 2^{-n} + \epsilon \cdot 2^{-n}) \ <\ \infty \end{aligned}$$

we get that there exists a g_1 in $L^p(\mathbf{T}^2)$ such that

$$g_1 = \sum_{n=1}^{\infty} \psi_n^+ \qquad \text{in } L^p(\mathbf{T}^2).$$

Similarly, we get that there exists a g_2 in $L^p(\mathbf{T}^2)$ such that

$$g_2 = \sum_{n=1}^{\infty} \psi_n^- \qquad \text{in } L^p(\mathbf{T}^2).$$

So we have that

$$f = g_1 - g_2 \quad \text{in } L^p(\mathbf{T}^2).$$

It is left to show that g_1 and g_2 are equal to positive *l.s.c.* functions *a.e.* Let

$$s_n = \sum_{k=1}^{n} \psi_k^+.$$

Since s_n converges to g_1 in $L^p(\mathbf{T}^2)$, there exists a subsequence that converges to g_1 *a.e.* But since s_n is monotone increasing, we get that s_n converges to g_1 *a.e.* and further that $\sup s_n = \lim s_n$. By our above observation, we conclude that $\sup s_n$ is *l.s.c.* It is clear that $\sup s_n$ is positive. Therefore, g_1 is equal to a positive *l.s.c.* function *a.e.* Similarly, we get that g_2 is equal to a positive *l.s.c.* function *a.e.* So f is equal *a.e.* to the difference of two positive *l.s.c.* functions. $\triangle$

Lemma 4 *Suppose f is real-valued on $\mathbf{T}^2$ and $f \in L^1(\mathbf{T}^2)$. Then there exists a singular (complex Borel) measure σ on $\mathbf{T}^2$, such that $P[f - d\sigma] \in RP(\mathbf{U}^2)$.*

Proof: If f is real-valued on $\mathbf{T}^2$ and $f \in L^1(\mathbf{T}^2)$, then Lemma 3 (p. 36) asserts the existence of two positive *l.s.c.* functions g_1 and g_2 in $L^1(\mathbf{T}^2)$ such that $f = g_1 - g_2$ *a.e.* By Lemma 2 (p. 36) there exists nonnegative singular measures σ_1 and σ_2 such that $P[g_1 - d\sigma_1]$ and $P[g_2 - d\sigma_2]$ are in $RP(\mathbf{U}^2)$. Letting $\sigma = \sigma_1 - \sigma_2$ we get a singular measure such that

$$\begin{aligned} P[f - d\sigma] &= P[(g_1 - g_2) - d(\sigma_1 - \sigma_2)] \\ &= P[(g_1 - d\sigma_1) - (g_2 - d\sigma_2)] \\ &= P[g_1 - d\sigma_1] - P[g_2 - d\sigma_2]. \end{aligned}$$

So, $P[f - d\sigma]$ is in $RP(\mathbf{U}^2)$. $\triangle$

Theorem 22 *Let $\mathcal{M} \neq \{0\}$ be a subspace of $L^p(\mathbf{T}^2)$, $1 \leq p < 2$, invariant under S_1 and S_2. Then $\mathcal{M} = qH^p(\mathbf{T}^2)$ where q is a unimodular function if and only if S_1 and S_2 are doubly commuting shifts on $\mathcal{M} \cap L^2(\mathbf{T}^2)$.*

Proof: Let N denote $\mathcal{M} \cap L^2(\mathbf{T}^2)$. Then N is a closed invariant subspace of $L^2(\mathbf{T}^2)$ and by hypothesis S_1 and S_2 are doubly commuting shifts on N. Therefore, by Theorem 20 (p. 34), $N = qH^2(\mathbf{T}^2)$ where q is a unimodular function. Now since N is contained in $\mathcal{M}$ and $\mathcal{M}$ is closed, the closure of N in $L^p(\mathbf{T}^2)$, which is $qH^p(\mathbf{T}^2)$, is contained in $\mathcal{M}$. So we need to show that N is dense in $\mathcal{M}$. To do this, let $f \in \mathcal{M}$, f not identically zero. Then define

$$u_n = \begin{cases} 0, & |f| \leq n, \\ \log |f|^{-1}, & |f| > n. \end{cases}$$

Note that $u_n \in L^p(\mathbf{T}^2)$ for all n since

$$\begin{aligned} \int |u_n|^p \, dm &= \int_{|f|>n} |\log |f|^{-1}|^p \, dm = \int_{|f|>n} |\log |f||^p \, dm \\ &\leq \int_{|f|>n} |f|^p \, dm \leq \|f\|_p^p < \infty. \end{aligned}$$

So in particular, $u_n \in L^1(\mathbf{T}^2)$ and real valued for all n. So by Lemma 4 (p. 38), there exists a sequence $\{\sigma_n\}_{n\geq 0}$ of singular measures such that $P[u_n - d\sigma_n] \in RP(U^2)$ for all n. So there exists a sequence of analytic functions $(F_n)_n$ such that $Re(F_n) = P[u_n - d\sigma_n]$. By the M. Riesz theorem, which holds on the polydisc (see [17]), we have $\|F_n\|_p \leq C_p \|u_n\|_p$ for all n. Now since $u_n \in L^p(\mathbf{T}^2)$ and u_n converges to 0 in $L^p(\mathbf{T}^2)$, we get F_n converges to 0 in $L^p(\mathbf{T}^2)$ and hence at least a subsequence converges to zero *a.e.* Let $\phi_n = \exp\{F_n\}$. Then

$$|\phi_n| = \begin{cases} 1, & |f| \leq n, \\ |f|^{-1}, & |f| > n \end{cases}$$

and ϕ_n tends to the constant function 1. By construction, $\phi_n f$ is a bounded function dominated by f for all n. Also, $\phi_n f \in \mathcal{M}$ because ϕ_n is bounded analytic and hence is boundedly the limit of analytic trigonometric polynomials. Since $\phi_n f$ is bounded, it is in N. As n goes to infinity $\phi_n f$ converges to f in $L^p(\mathbf{T}^2)$ by the dominated convergence theorem. So each f in $\mathcal{M}$ is the limit of functions from N. So N is dense in $\mathcal{M}$ as desired.

Conversely, if $\mathcal{M} = qH^p(\mathbf{T}^2)$ with q unimodular, then $\mathcal{M} \cap L^2(\mathbf{T}^2) = qH^2(\mathbf{T}^2)$. So S_1 and S_2 are doubly commuting shifts on $\mathcal{M} \cap L^2(\mathbf{T}^2)$ by Theorem 20 (p. 34). $\triangle$

Corollary 3 *Let $\mathcal{M} \neq \{0\}$ be a subspace of $H^p(\mathbf{T}^2)$, $1 \le p < 2$, invariant under S_1 and S_2. Then $\mathcal{M} = qH^p(\mathbf{T}^2)$ where q is an inner function if and only if S_1 and S_2 are doubly commuting on $\mathcal{M} \cap H^2(\mathbf{T}^2)$.*

Proof: $H^p(\mathbf{T}^2)$ is a subspace of $L^p(\mathbf{T}^2)$; so $\mathcal{M}$ is a subspace of $L^p(\mathbf{T}^2)$. Note that $\mathcal{M} \cap H^2(\mathbf{T}^2) = \mathcal{M} \cap L^2(\mathbf{T}^2)$ since $\mathcal{M} \subset H^p(\mathbf{T}^2)$. Since S_1 and S_2 are shifts on all subspaces of $H^p(\mathbf{T}^2)$, we get $\mathcal{M} = qH^p(\mathbf{T}^2)$ where q is unimodular by the previous theorem. Since $q \in qH^p(\mathbf{T}^2) \subset H^p(\mathbf{T}^2)$, we see that q is holomorphic, and hence inner. The converse is just a special case of the above theorem. $\triangle$

We use the notation $H^p_o(\mathbf{T}^2) = \left\{ f \in H^p(\mathbf{T}^2) : \hat{f}(0,0) = 0 \right\}$ in the next theorem.

Theorem 23 *Let $\mathcal{M} \neq \{0\}$ be a subspace*[1] *of $L^p(\mathbf{T}^2)$, $2 < p \le \infty$, invariant under S_1 and S_2. Then $\mathcal{M} = qH^p_o(\mathbf{T}^2)$ where q is a unimodular function if and only if S_1 and S_2 are doubly commuting shifts on $A(\mathcal{M}) \cap L^2(\mathbf{T}^2)$.*

Proof: If $\mathcal{M} = qH^p_o(\mathbf{T}^2)$ where q is a unimodular function, then $A(\mathcal{M}) = \overline{q}H^{\frac{p}{p-1}}(\mathbf{T}^2)$. Therefore, $A(\mathcal{M}) \cap L^2(\mathbf{T}^2) = \overline{q}H^2(\mathbf{T}^2)$. It then follows from Theorem 20 (p. 34) that S_1 and S_2 are doubly commuting shifts on $A(\mathcal{M}) \cap L^2(\mathbf{T}^2)$.

[1]Assume further star-closed when $p = \infty$.

Conversely, if S_1 and S_2 are doubly commuting shifts on $A(\mathcal{M}) \cap L^2(\mathbf{T}^2)$, then by Theorem 22 (p. 39) we get that $A(\mathcal{M}) = qH^{\frac{p}{p-1}}(\mathbf{T}^2)$ where q is a unimodular function. Therefore, $\mathcal{M} = \overline{q}H^p_o(\mathbf{T}^2)$ where q is a unimodular function. When $p = \infty$ we need that $\mathcal{M}$ is star-closed to make our final conclusion. $\triangle$

Corollary 4 *Let $\mathcal{M} \neq \{0\}$ be a subspace[2] of $H^p(\mathbf{T}^2)$, $2 < p \leq \infty$, invariant under S_1 and S_2. Then $\mathcal{M} = qH^p_o(\mathbf{T}^2)$ where q is an inner function if and only if S_1 and S_2 are doubly commuting shifts on $A(\mathcal{M}) \cap L^2(\mathbf{T}^2)$.*

Proof: A similar argument as used in the above corollary gives the result. $\triangle$

We now consider the ideas from the first part of this dissertation; namely, the idea of our vector subspaces being Hilbert spaces. Our first result is due to Dinesh Singh. He proved a generalization of Theorem 19 (p. 34).

Theorem 24 (Singh) *$\mathcal{N}$ is a Hilbert space which is a vector subspace of $H^2(\mathbf{T}^2)$ such that $\mathcal{N}$ is invariant under S_1 and S_2 and for which S_1 and S_2 are doubly commuting isometries on $\mathcal{N}$ if and only if there exists g in $H^\infty(\mathbf{T}^2)$ unique up to a factor of modulus one such that $\mathcal{N} = gH^2(\mathbf{T}^2)$ with norm $\|gf\|_{\mathcal{N}} = \|f\|_2$ for all f in $H^2(\mathbf{T}^2)$.*

We now slightly modify the proof of this theorem to prove a general $H^p(\mathbf{T}^2)$ result. The following theorem is just a two-variable analogue of Theorem 13 (p. 22).

Theorem 25 *If $\mathcal{M}$ is a Hilbert space contained in $H^p(\mathbf{T}^2)$, invariant under S_1 and S_2 and if S_1 and S_2 are doubly commuting isometries on $\mathcal{M}$, then*

$$\mathcal{M} = bH^2(\mathbf{T}^2)$$

for a unique b:

[2] Assume further star-closed when $p = \infty$.

1. If $1 \le p \le 2$, $b \in H^{\frac{2p}{2-p}}(\mathbf{T}^2)$. When $p = 2$, we mean $H^\infty(\mathbf{T}^2)$.

2. If $p > 2$, $b = 0$.

Further, $\|bf\|_{\mathcal{M}} = \|f\|_2$ *for all* f *in* $H^2(\mathbf{T}^2)$ $(1 \le p \le 2)$.

Note that the converse of this theorem is also true. Before we prove this theorem, we give several lemmas. The first two lemmas are due to Slocinski [24].

Lemma 5 (Slocinski [24]) *Suppose that* V_1 *and* V_2 *are commuting isometries on a Hilbert space* $\mathcal{H}$ *and write* $R_i^\perp = \mathcal{H} \ominus V_i(\mathcal{H})$ $(i = 1, 2.)$. *Then the following are equivalent:*

1. *There is a wandering subspace* $\mathcal{L}$ *for the semigroup* $\left\{V_1^n V_2^m\right\}_{n,m\ge 0}$ *such that*

$$\mathcal{H} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \oplus V_1^n V_2^m(\mathcal{L}).$$

2. V_1 *and* V_2 *are doubly commuting shifts.*

3. $R_1^\perp \cap R_2^\perp$ *is a wandering subspace for the semi-group* $\left\{V_1^n V_2^m\right\}_{n,m\ge 0}$ *and*

$$\mathcal{H} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \oplus V_1^n V_2^m(R_1^\perp \cap R_2^\perp).$$

Lemma 6 (Slocinski [24]) *Suppose* V_1 *and* V_2 *are commuting isometries on the Hilbert space* $\mathcal{H} \ne \{0\}$. *If* $R_1^\perp \cap R_2^\perp = \{0\}$ *where* $R_i^\perp = \mathcal{H} \ominus V_i(\mathcal{H})$ $(i = 1, 2.)$, *then*

$$\mathcal{H} \ne \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \oplus V_1^n V_2^m(R_1^\perp \cap R_2^\perp).$$

Lemma 7 *If* ψ *is positive and l.s.c. on* $\mathbf{T}^2$ *and* $\psi \in L^p(\mathbf{T}^2)$, *then* $\psi = |f|$ *a.e for some* $f \in H^p(\mathbf{T}^2)$.

Proof: Since *l.s.c.* functions attain their minimum on compact sets, (for a proof see [16]), we may assume without loss of generality that $\psi > 1$. Applying Lemma 2 (p. 36) to $\log \psi$ asserts the existence of a singular measure $\sigma \geq 0$ and a holomorphic function g in $\mathbf{U}^2$ such that $Re(g) = P[\log \psi - d\sigma]$. Put $f = \exp(g)$. Then f is holomorphic in $\mathbf{U}^2$ and

$$|f| = |\exp(g)| = \exp(\log \psi) = \psi$$

on $\mathbf{T}^2$. Since $\psi \in L^p(\mathbf{T}^2)$, $f \in H^p(\mathbf{T}^2)$ as desired. $\triangle$

Lemma 8 *For all $h \in L^p(\mathbf{T}^2)$ with $1 \leq p < \infty$, there exists a positive, l.s.c. $\phi \in L^p(\mathbf{T}^2)$ such that $\phi \geq |h|$ a.e. on $\mathbf{T}^2$.*

Proof: If $h \in L^p(\mathbf{T}^2)$, then $|h| \in L^p(\mathbf{T}^2)$ and real-valued. So, by Lemma 3 (p. 36), there exists two positive *l.s.c.* functions ϕ and ψ in $L^p(\mathbf{T}^2)$ such that

$$|h| = \phi - \psi \qquad a.e. \text{ on } \mathbf{T}^2.$$

So

$$|h| \leq \phi \qquad a.e. \text{ on } \mathbf{T}^2. \qquad \triangle$$

Lemma 9 *Let f be an element of $H^p(\mathbf{T}^2)$ that multiplies $H^2(\mathbf{T}^2)$ into $H^p(\mathbf{T}^2)$. Then f multiplies $L^2(\mathbf{T}^2)$ into $L^p(\mathbf{T}^2)$.*

Proof: Let g be an element of $L^2(\mathbf{T}^2)$. Then by Lemma 8 (p. 43) there exists a positive *l.s.c.* function ϕ in $L^2(\mathbf{T}^2)$ such that $|g| \leq \phi$ *a.e.* on $\mathbf{T}^2$. Then by Lemma 7 (p. 42) there exists an h in $H^2(\mathbf{T}^2)$ such that $|h| = \phi$ *a.e.* on $\mathbf{T}^2$. Now consider

$$\begin{aligned}
\int_{\mathbf{T}^2} |fg|^p \, dm_2 &= \int_{\mathbf{T}^2} |f|^p |g|^p \, dm_2 \\
&\leq \int_{\mathbf{T}^2} |f|^p |h|^p \, dm_2 \qquad \text{since } |g| \leq \phi = |h| \ a.e. \text{ on } \mathbf{T}^2 \\
&= \int_{\mathbf{T}^2} |fh|^p \, dm_2 \ < \ \infty \qquad \text{by hypothesis.} \qquad \triangle
\end{aligned}$$

This next lemma is a straight forward calculation found in [22]. We include it for completeness.

Lemma 10 *If g is a measurable on $\mathbf{T}^2$ that multiplies $L^2(\mathbf{T}^2)$ into $L^q(\mathbf{T}^2)$ where $1 \leq q \leq 2$ then $g \in L^{\frac{2q}{2-q}}(\mathbf{T}^2)$. When $q = 2$, we mean $L^\infty(\mathbf{T}^2)$.*

Proof: So

$$\int |fg|^q \, dm < \infty \qquad \text{for all } f \in L^2(\mathbf{T}^2).$$

That is,

$$\int |f|^q |g|^q \, dm < \infty \qquad \text{for all } |f|^q \in L^{2/q}(\mathbf{T}^2).$$

Hence,

$$\int |g|^q h \, dm < \infty \qquad \text{for all } h \in L^{2/q}(\mathbf{T}^2) \text{ with } h \geq 0.$$

Since every h in $L^{2/q}(\mathbf{T}^2)$ is equal to $(h_1 - h_2) + i(h_3 - h_4)$ where h_j is in $L^{2/q}(\mathbf{T}^2)$ and $h_j \geq 0$ for $j = 1, 2, 3, 4$, we have

$$|g|^q h \in L^1(\mathbf{T}^2) \qquad \text{for all } h \in L^{2/q}(\mathbf{T}^2).$$

Now by an inverse of Hölder's Inequality found in [26], we may conclude that $|g|^q$ is in the dual of $L^{2/q}(\mathbf{T}^2)$; that is,

$$|g|^q \in L^{\frac{2}{2-q}}(\mathbf{T}^2).$$

Hence,

$$g \in L^{\frac{2q}{2-q}}(\mathbf{T}^2).$$

So the set of multipliers of $L^2(\mathbf{T}^2)$ into $L^q(\mathbf{T}^2)$ $(1 \leq q \leq 2)$ is the space $L^{\frac{2q}{2-q}}(\mathbf{T}^2)$. $\triangle$

Proof of Theorem 25 (p. 41): We first consider the case $1 \leq p \leq 2$. Observe that $\cap_{n=0}^{\infty} S_i^n(\mathcal{M}) = \{0\}$ $(i = 1, 2.)$. This observation and our doubly commuting hypothesis give us that

$$\mathcal{M} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \oplus V_1^n V_2^m (R_1^\perp \cap R_2^\perp) \qquad \text{by Lemma 5 (p. 42)}$$

and that

$$R_1^\perp \cap R_2^\perp \neq \{0\} \qquad \text{by Lemma 6 (p. 42)}$$

where $R_i^\perp = \mathcal{H} \ominus V_i(\mathcal{H})$ $(i = 1, 2.)$. So we may take g from $R_1^\perp \cap R_2^\perp$, with $\|g\|_{\mathcal{M}} = 1$. Then $\left\{ g e^{in\theta_1} e^{im\theta_2} \right\}_{n,\ m \geq 0}$ is an orthonormal sequence in $\mathcal{M}$. Let f be an arbitrary element of $H^2(\mathbf{T}^2)$. Then $f(e^{i\theta_1}, e^{i\theta_2}) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \hat{f}(n,\ m) e^{in\theta_1} e^{im\theta_2}$. Let $f_{nm}(e^{i\theta_1}, e^{i\theta_2}) = \sum_{k=0}^{n} \sum_{l=0}^{m} \hat{f}(k, l) e^{ik\theta_1} e^{il\theta_2}$. Then f_{nm} converges to f in $L^2(\mathbf{T}^2)$ and *a.e.* along rectangles. We make the following computation.

$$\begin{aligned} \|f_{nm}\|_2^2 &= \sum_{k=0}^{n} \sum_{l=0}^{m} |\hat{f}(k, l)|^2 \\ &= \sum_{k=0}^{n} \sum_{l=0}^{m} |\hat{f}(k, l)|^2 \|g e^{ik\theta_1} e^{il\theta_2}\|_{\mathcal{M}}^2 \qquad (4.1) \\ &= \sum_{k=0}^{n} \sum_{l=0}^{m} \|\hat{f}(k, l) g e^{ik\theta_1} e^{il\theta_2}\|_{\mathcal{M}}^2 \\ &= \left\| \sum_{k=0}^{n} \sum_{l=0}^{m} \hat{f}(k, l) g e^{ik\theta_1} e^{il\theta_2} \right\|_{\mathcal{M}}^2 . \end{aligned}$$

Since $\left(f_{nm}\right)_{(n,m)}$ is Cauchy in $L^2(\mathbf{T}^2)$, $\left(\sum_{k=0}^{n} \sum_{l=0}^{m} \hat{f}(k, l) g e^{ik\theta_1} e^{il\theta_2}\right)_{(n,m)}$ is Cauchy in $\mathcal{M}$. Since $\mathcal{M}$ is a Hilbert space, there exits a h in $\mathcal{M}$ such that

$$\left\| \sum_{k=0}^{n} \sum_{l=0}^{m} \hat{f}(k, l) g e^{ik\theta_1} e^{il\theta_2} - h \right\|_{\mathcal{M}} \longrightarrow 0 \qquad \text{as } (n, m) \longrightarrow \infty \text{ along rectangles.}$$

Thus,

$$h = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \hat{f}(k, l) g e^{ik\theta_1} e^{il\theta_2}$$

and since

$$g = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \hat{g}(k, l) e^{ik\theta_1} e^{il\theta_2},$$

we have for fixed m and n,

$$
\begin{aligned}
h &= \hat{f}(0,0)g + \hat{f}(0,1)ge^{i\theta_2} + \hat{f}(1,0)ge^{i\theta_1} + \cdots + \\
&\quad + \hat{f}(m,n)ge^{im\theta_1}e^{in\theta_2} + h_1 e^{i(m+1)\theta_1} + h_2 e^{i(n+1)\theta_2}
\end{aligned} \tag{4.2}
$$

where

$$h_1 = \hat{f}(m+1,0)g + \hat{f}(m+1,1)ge^{i\theta_2} + \hat{f}(m+2,0)ge^{i\theta_1} + \cdots$$

and

$$h_2 = \hat{f}(0,n+1)g + \hat{f}(0,n+2)ge^{i\theta_2} + \hat{f}(1,n+1)ge^{i\theta_1} + \cdots .$$

It's clear that h_1 and h_2 are in $\mathcal{M}$ and hence in $H^p(\mathbf{T}^2)$. Thus from equation (4.2) (p. 46), we see that the (m,n)-th Fourier coefficients of h are the same as the (m,n)-th Fourier coefficients of the formal product of the series of g and f. This means that $h = gf$ in $H^p(\mathbf{T}^2)$ and hence in $\mathcal{M}$. This observation along with equation (4.1) (p. 45) gives us that

$$\|gf\|_{\mathcal{M}} = \|f\|_2.$$

Since f was an arbitrary element of $H^2(\mathbf{T}^2)$, we see that g multiplies $H^2(\mathbf{T}^2)$ into $\mathcal{M} \subseteq H^p(\mathbf{T}^2)$. By Lemma 9 (p. 43) we conclude that g multiplies $L^2(\mathbf{T}^2)$ into $L^p(\mathbf{T}^2)$. Lemma 10 (p. 44) shows us that any d that multiplies $L^2(\mathbf{T}^2)$ into $L^p(\mathbf{T}^2)$ must be a member of $L^{\frac{2p}{2-p}}(\mathbf{T}^2)$. Thus g must be in $H^{\frac{2p}{2-p}}(\mathbf{T}^2)$.

Note that $\frac{2p}{2-p} \geq 2$ when $1 \leq p \leq 2$, so g is in $H^2(\mathbf{T}^2)$.

It's left to show that $R_1^\perp \cap R_2^\perp$ is one dimensional. Suppose there is a g_1 in $R_1^\perp \cap R_2^\perp$ with unit norm and $g \perp g_1$ in $\mathcal{M}$. Then by the same computations above we get that $g_1 H^2(\mathbf{T}^2)$ is also contained in $\mathcal{M}$ and by our decomposition we get that $gH^2(\mathbf{T}^2) \perp g_1 H^2(\mathbf{T}^2)$ in $\mathcal{M}$. Further, $gg_1 = g_1 g$ is in $gH^2(\mathbf{T}^2)$ as well as $g_1 H^2(\mathbf{T}^2)$. So, $gg_1 = 0$. As g and g_1 do not vanish on a set of positive Lebesgue measure unless they are identically zero we get a contradiction. Hence $R_1^\perp \cap R_2^\perp$ is one dimensional as desired.

Now we consider the case $p > 2$. Suppose $\mathcal{M} \neq \{0\}$. Proceeding as in the previous case we get that g multiples $L^2(\mathbf{T}^2)$ into $L^p(\mathbf{T}^2) \subset L^2(\mathbf{T}^2)$ and hence g is in $H^\infty(\mathbf{T}^2)$. Choosing an appropriate $\epsilon > 0$ such that

$$E = \left\{(e^{i\theta_1}, e^{i\theta_2}) : |g(e^{i\theta_1}, e^{i\theta_2})| > \epsilon\right\}$$

has positive measure, let b be a function that vanishes on the complement of E which is in $L^2(\mathbf{T}^2)$ but not $L^p(\mathbf{T}^2)$. But then, gb is in $L^p(\mathbf{T}^2)$ and so b will lie in $L^p(\mathbf{T}^2)$ since g is invertible on E. Hence a contradiction. So our supposition must be incorrect. So, $\mathcal{M} = \{0\}$. $\triangle$

Before we give a corollary of this result, we need a definition.

Definition 9 *A function g in $H^p(\mathbf{T}^2)$ is called* ***outer*** *if the linear combination of functions*

$$g(e^{i\theta_1}, e^{i\theta_2}), e^{i\theta_1} g(e^{i\theta_1}, e^{i\theta_2}), e^{i\theta_2} g(e^{i\theta_1}, e^{i\theta_2}), e^{i\theta_1} e^{i\theta_2} g(e^{i\theta_1}, e^{i\theta_2}), \ldots$$

are dense in $H^p(\mathbf{T}^2)$.

Before we proceed, we make an observation. In $H^p(\mathbf{T})$, a function f being outer is equivalent to

$$\log|f(0)| = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log|f(e^{i\theta})|\, d\theta.$$

In $H^p(\mathbf{T}^2)$, this is not the case. Let's call the functions in $H^p(\mathbf{T}^2)$ that satisfy

$$\log|f(0)| = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \log|f(e^{i\theta_1}, e^{i\theta_2})|\, d\theta_1 d\theta_2.$$

weakly outer. It is known that outer implies weakly outer but weakly outer does not imply outer. See Rudin, [18] for more details.

Corollary 5 *Suppose $\mathcal{M}$ is a Hilbert space contained in $L^p(\mathbf{T}^2)$, invariant under S_1 and S_2, S_1 and S_2 are doubly commuting isometries on $\mathcal{M}$.*

Case 1: *For $1 \leq p \leq 2$, if S_1 and S_2 are doubly commuting shifts on $\overline{\mathcal{M}}^{L^p} \cap L^2(\mathbf{T}^2)$, then*

$$\mathcal{M} = bH^2(\mathbf{T}^2)$$

for a unique $b \in L^{\frac{2p}{2-p}}(\mathbf{T}^2)$ having the modulus of an outer function a.e. When $p = 2$, we mean $L^\infty(\mathbf{T}^2)$.

Case 2: *For $p > 2$, if S_1 and S_2 are doubly commuting shifts on $Ann(\overline{\mathcal{M}}^{L^p}) \cap L^2(\mathbf{T}^2)$, then $\mathcal{M} = \{0\}$.*

Further, $||bf||_{\mathcal{M}} = ||f||_2$ for all f in $H^2(\mathbf{T}^2)$ $(1 \leq p \leq 2)$.

Note that the converse of this theorem is also true.

Proof: Case 1: By Theorem 22 (p. 39), we have that $\overline{\mathcal{M}}^{L^p} = \phi H^p(\mathbf{T}^2)$ for some unimodular function ϕ. Then, $\mathcal{M}' = \overline{\phi}\mathcal{M}$ is contained in $H^p(\mathbf{T}^2)$ and with norm $||\overline{\phi}p||_{\mathcal{M}'} = ||p||_{\mathcal{M}}$ is a Hilbert space invariant under S_1 and S_2, S_1 and S_2 act as isometries on $\mathcal{M}'$. We also see that S_1 and S_2 doubly commute on $\mathcal{M}'$. So then by the above result we get that $\mathcal{M}' = gH^2(\mathbf{T}^2)$ with $g \in H^{\frac{2p}{2-p}}(\mathbf{T}^2)$ with norm $||gf||_{\mathcal{M}'} = ||f||_2$. So $\mathcal{M} = \phi\mathcal{M}' = \phi g H^2(\mathbf{T}^2)$ with norm $||\phi g f||_{\mathcal{M}} = ||gf||_{\mathcal{M}'} = ||f||_2$. Further, since the closure of $\mathcal{M}$ in $L^p(\mathbf{T}^2)$ is $\phi H^2(\mathbf{T}^2)$, we have that g must have the modulus of an outer function *a.e.*

Case 2: By Theorem 23 (p. 40), we get that $\overline{\mathcal{M}}^{L^p} = \phi H^p_o(\mathbf{T}^2)$ for some unimodular function ϕ. Then, $\mathcal{M}' = \overline{\phi}\mathcal{M}$ is contained in $H^p(\mathbf{T}^2)$ and with norm $||\overline{\phi}g||_{\mathcal{M}'} = ||g||_{\mathcal{M}}$ is a Hilbert space invariant under S_1 and S_2, S_1 and S_2 act as isometries on $\mathcal{M}'$. We also see that S_1 and S_2 doubly commute on $\mathcal{M}'$. So then by the above result we get that $\mathcal{M}' = \{0\}$. Therefore, $\mathcal{M} = \{0\}$. $\triangle$

Before we give another corollary we recall some definitions. Let $BMO(\mathbf{T}^2)$ be the class of all $L^1(\mathbf{T}^2)$ functions f such that

$$\|f\|_* = \sup \frac{1}{|I|}\int_I |f - \frac{1}{|I|}\int_I f| < \infty$$

where the supremum is taken over all squares of $\mathbf{T}^2$ and $|I|$ denotes the normalized Lebesgue measure of I.

$BMO(\mathbf{T}^2)$ is a Banach space under the norm

$$\|f\| = \|f\|_* + |\hat{f}(0)|.$$

$VMO(\mathbf{T}^2)$ is the closure of the continuous functions in $BMO(\mathbf{T}^2)$. $BMOA(\mathbf{T}^2) = BMO(\mathbf{T}^2) \cap H^1(\mathbf{T}^2)$ and $VMOA(\mathbf{T}^2) = VMO(\mathbf{T}^2) \cap H^1(\mathbf{T}^2)$. By the John-Nirenberg theorem [25], we get that $BMOA(\mathbf{T}^2) \subset H^p(\mathbf{T}^2)$ for $p < \infty$. We are now ready to state our corollary.

Corollary 6 *If $\mathcal{M}$ is a Hilbert space contained in $BMOA(\mathbf{T}^2)$ ($VMOA(\mathbf{T}^2)$), invariant under S_1 and S_2 and if S_1 and S_2 are doubly commuting isometries on $\mathcal{M}$, then $\mathcal{M} = \{0\}$.*

Proof: By the John-Nirenberg Theorem mentioned above we get that $BMOA(\mathbf{T}^2) \subset H^p(\mathbf{T}^2)$ for $p < \infty$. So in particular, $BMOA(\mathbf{T}^2) \subset H^p(\mathbf{T}^2)$ for $p > 2$. So by Theorem 25 (p. 41), $\mathcal{M} = \{0\}$. $\triangle$

Chapter 5

Random Fields

5.1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space (measure space with $P(\Omega) = 1$) and $\left\{X_{m,n} : (m, n) \in \mathbf{Z}^2\right\}$ be a family of random variables (complex measurable functions) on $(\Omega, \mathcal{F}, P)$ such that $E|X_{m,n}|^2 < \infty$ $\left(\int_\Omega |X_{m,n}|^2 \, dP < \infty\right)$ for all $(m, n) \in \mathbf{Z}^2$. We assume that $EX_{m,n} = 0$ for all $(m, n) \in \mathbf{Z}^2$. The $\left\{X_{m,n} : (m, n) \in \mathbf{Z}^2\right\}$ is called a **second order random field**. For $(m, n), (m', n') \in \mathbf{Z}^2$, we define

$$C\Big((m, n), (m', n')\Big) = EX_{m,n}\overline{X_{m',n'}}$$

the **covariance function** of $\left\{X_{m,n} : (m, n) \in \mathbf{Z}^2\right\}$. We call a second order random field **(weakly) stationary** if

$$C\Big((m, n), (m', n')\Big) = r\Big(m - m', n - n'\Big).$$

One can prove using the Bochner Theorem that

$$r\Big(m, n\Big) = \int_{\mathbf{T}^2} e^{-(im\lambda + in\theta)} F(d\lambda, d\theta).$$

The measure F on the torus is called the **spectral distribution** of the weakly stationary random field $\left\{X_{m,n} : (m,n) \in \mathbf{Z}^2\right\}$. One can use the extension of Stone's Theorem, to show that

$$X_{m,n} = \int_{\mathbf{T}^2} e^{-(im\lambda+in\theta)} \mathcal{Z}(d\lambda, d\theta) \tag{5.1}$$

where

$$\mathcal{Z} : \mathcal{B}(\mathbf{T}^2) \to L(X)$$

is an (orthogonal scattered) measure with

$$L(X) = \overline{span}^{L^2(\Omega)}\left\{X_{m,n} : (m,n) \in \mathbf{Z}^2\right\}$$

and $\mathcal{B}(\mathbf{T}^2)$ denotes the collection of all Borel sets of the torus. Here $E\mathcal{Z}(\Delta)\overline{\mathcal{Z}(\Delta')} = F(\Delta \cap \Delta')$. It is easy to check that the map

$$X_{m,n} \to e^{im\cdot+in\cdot}$$

is an isometry from $L(X)$ onto $L^2(\mathbf{T}^2, F)$. This isometry can be used to study analytic properties of F corresponding to prediction questions related to the stationary random field. The "prediction" of the "future" from the "past" observations is defined by giving an order on $\mathbf{Z}^2$. For an ordering induced by a semi-group, this problem was studied by Helson and Lowdenslager. In their context the semigroup $\mathcal{S}$ satisfied $\mathcal{S} \cup (-\mathcal{S}) = \mathbf{Z}^2$ and $\mathcal{S} \cap (-\mathcal{S}) = \{(0,0)\}$. The "analyticity" was defined by using functions of the form

$$f(\lambda, \theta) = \sum_{(m,n)\in\mathcal{S}} a_{(m,n)} e^{im\lambda+in\theta}$$

where $\sum |a_{(m,n)}|^2 < \infty$. Using the above isometry one can show that the stationary random field satisfies $\bigcap_{(m,n)\in\mathbf{Z}^2} \overline{span}\left\{X_{k,l} : (k,l) <<_{\mathcal{S}} (m,n)\right\} = \{0\}$ if and only if

$$X_{m,n} = \sum_{(k,l)\in\mathcal{S}} a_{m-k,n-l}\xi_{k,l} \tag{5.2}$$

where $\left\{\xi_{k,l}\right\}$ are orthogonal random variables and $\sum |a_{(m,n)}|^2 < \infty$. The representation (5.2) (p. 51) is called the **moving average** (MA) representation. For all different orderings, the problem is studied in [14], where also the MA representation for the semigroup given by the quarter plane $(m \geq 0, n \geq 0)$ is studied.

Recently, based on data from finance, insurance and hydrology, it is found that one needs to study the stationary random fields which are not second order $\left(\text{i.e., } E|X_{m,n}|^2 = \infty\right)$. For this, one needs to study the models given by so called **stable random fields**. In this case, we show that the class of random fields of type (5.1) (p. 51) and (5.2) (p. 51) are disjoint following the work of [12]. We then generalize some other results of [12] for some half-space ordering for stable random fields contained in random fields of the form (5.1) (p. 51). Presently, we have not completed the project. However, we indicate at the end the open problems for our future research which connect with the invariant subspaces of weighted spaces $L^p(\mathbf{T}^2, \omega)$ $(1 \leq p < 2)$ and an analogue of a result of Zygmund [27].

5.2 Some Probability Background

Let $(\Omega, \mathcal{F})$ be a measurable space. A positive measure P on $(\Omega, \mathcal{F})$ is called a **probability measure** if $P(\Omega) = 1$. In what follows, P will always denote a probability measure, the term **event** is used to mean a member of $\mathcal{F}$ and **random variable** is used to mean a complex measurable function on $(\Omega, \mathcal{F})$.

Definition 10 *Two events A and B are said to be* ***independent*** *if $P[A \cap B] = P[A]P[B]$.*

Definition 11 *Two random variables X and Y are* ***independent*** *if the events $X \in A$ and $Y \in B$ are independent for any two Borel sets A and B on the line; i.e.,*

$$P\Big[[X \in A] \cap [Y \in B]\Big] = P[X \in A]P[Y \in B].$$

Definition 12 *A finite collection* $\Big\{X_j : 1 \leq j \leq n\Big\}$ *of random variables is said to be* ***independent*** *if for any n Borel sets* $A_1, A_2, \ldots, A_n$ *on the line*

$$P\Bigg[\bigcap_{1\leq j\leq n} [X_j \in A_j]\Bigg] = \prod_{1\leq j\leq n} P\Big[X_j \in A_j\Big].$$

Definition 13 *A collection of random variables indexed by a parameter is called a* ***random process****; e.g.,* $\Big\{X_n : n \in \mathbf{Z}\Big\}$*. When the index set is a collection of ordered pairs one calls the random process a* ***random field****; e.g.,* $\Big\{X_{n,m} : (n, m) \in \mathbf{Z}^2\Big\}$*.*

Definition 14 *For a k-dimensional random vector* $X = (X_1, \ldots, X_k)$*, the* ***distribution*** μ *(a probability measure on* $\mathbf{R}^k$*) and the* ***distribution function*** F *(a real function on* $\mathbf{R}^k$*) are defined by*

$$\mu(A) = P[(X_1, \ldots, X_k) \in A], \qquad A \in \mathcal{B}$$

and

$$F(x_1, \ldots, x_k) = P[X_1 \leq x_1, \ldots, X_k \leq x_k] = \mu(S_x)$$

where $S_x = [y : y_i \leq x_i, i = 1, \ldots, k]$*.*

Definition 15 *By a field of* ***i.i.d.*** *random variables we mean a field of random variables that are independent and identically distributed. Thus, if* $\{X_{n,m} : (n, m) \in \mathbf{Z}^2\}$ *is a field of i.i.d. random variables, then the* $X_{m,n}$*'s are independent and all have the same distribution function F (say):*

$$P(X_{m,n} \leq x) = F(x) \qquad \text{for all } (m, n) \in \mathbf{Z}^2 \text{ and for all } x.$$

Definition 16 *A distribution function F is* ***stable*** *if for each n there exists constants* $a_n > 0$ *and* b_n*, such that, if* $X_1, \ldots, X_n$ *are independent and have distribution function F, then* $\frac{X_1+\cdots+X_n}{a_n} + b_n$ *also has distribution function F.*

Remark 2 *If* $0 < \alpha \le 2$, *then* $\exp(-|t|^\alpha)$ *is the characteristic function of a symmetric stable distribution; it is called the* ***symmetric stable law of exponent*** α. *The case* $\alpha = 2$ *is the normal law, and* $\alpha = 1$ *is the Cauchy law.*

5.3 Strongly Harmonizable Stable Fields

By an $S\alpha S$ field we will mean a family of complex random variables $\left\{X_{n,m} : (n,m) \in \mathbf{Z}^2\right\}$, such that for every $(n_1, m_1), \ldots, (n_k, m_k) \in \mathbf{Z}^2$, the joint distribution of the $2k$-dimensional random vector ReX_{n_1,m_1}, ImX_{n_1,m_1}, $\ldots$, ReX_{n_k,m_k}, ImX_{n_k,m_k} is symmetric stable with parameter α. For each real $S\alpha S$ random variable X there exists a number $|X|_\alpha \ge 0$ such that

$$E\exp(itX) = \exp\{-|X|_\alpha^\alpha |t|^\alpha\}$$

for all $t \in \mathbf{R}$. It is well know that an $S\alpha S$ process is a p^{th} order process for any p satisfying $1 < p < \alpha$. For a linear space of $S\alpha S$ random variables, the function $X \to |X|_\alpha$ defines a norm for $1 < \alpha < 2$. The norm is related to the usual $L^p(\Omega)$ norm by

$$\|X\|_p = C(p,\alpha)|X|_\alpha$$

where $C(p,\alpha)$ is the following constant depending on α and p, $1 \le p < \alpha \le 2$:

$$C(p,\alpha) = \left[\frac{2^{p-1}\int_0^\infty s^{-p/(\alpha-1)}(1-e^{-s})\,ds}{\alpha\int_0^\infty v^{-p-1} sin^2 v\,dv}\right]^{1/p}.$$

If $\left\{X_{n,m} : (n,m) \in \mathbf{Z}^2\right\}$ is a complex $S\alpha S$ field, then $L(X)$ we will denote the closure in probability of the set of all linear combinations of $\left\{X_{n,m} : (n,m) \in \mathbf{Z}^2\right\}$. The Schilder norm of a complex $S\alpha S$ random variable $Z = X + iY$, is defined by

$$\|Z\|_\alpha = \begin{cases} \left(F_{X,Y}(\mathbf{T})\right)^{1/\alpha}, & 1 \le \alpha \le 2 \\ F_{X,Y}(\mathbf{T}), & 0 < \alpha < 1 \end{cases}$$

where $F_{X,Y}$ is the unique symmetric measure on the circle $\mathbf{T}$ such that

$$E\exp\Big(itX+isY\Big)=\exp\Big(-\int_{\mathbf{T}}|tx+sy|^{\alpha}F_{X,Y}(dx,dy)\Big). \tag{5.3}$$

If $\alpha=2$, then $2\|Z\|_2^2=E|Z|^2$. If $1<\alpha<2$, then $\|\cdot\|_\alpha$ is a norm on $L(X)$ which is equivalent to any L^p-norm for $1<p<\alpha$. (If $0<\alpha<1$, then $\|\ \|_\alpha$ gives merely a metric.)

Definition 17 *A complex $S\alpha S$ field $\Big\{X_{n,m}:(n,m)\in\mathbf{Z}^2\Big\}$ is said to be* ***harmonizable*** *if there exists an $L(X)$-valued Borel measure $\mathcal{Z}$ on $\mathbf{T}^2$ such that for every $(n,m)\in\mathbf{Z}^2$,*

$$X_{n,m}=\int_{\mathbf{T}^2}e^{-(im\lambda+in\theta)}\mathcal{Z}(d\lambda,d\theta). \tag{5.4}$$

If, in addition, the measure $\mathcal{Z}$ takes independent values on disjoint Borel sets, then $\Big\{X_{n,m}:(n,m)\in\mathbf{Z}^2\Big\}$ is called ***strongly harmonizable****.*

If $\Big\{X_{n,m}:(n,m)\in\mathbf{Z}^2\Big\}$ is strongly harmonizable $S\alpha S$ field, then the mapping

$$f\to\int_{\mathbf{T}^2}f\,d\mathcal{Z}$$

is an isometry from $L^{\alpha}(\mathbf{T}^2,\mu)$ onto $L(\Delta\mathcal{Z}):=\overline{span}\Big\{\mathcal{Z}(B):B\in\mathcal{B}(\mathbf{T}^2)\Big\}$. Here μ is a nonnegative measure related to $\mathcal{Z}$ by the formula

$$\mu(B)=\begin{cases}\|\mathcal{Z}(B)\|_\alpha^\alpha, & 1\le\alpha\le 2\\ \|\mathcal{Z}(B)\|_\alpha, & 0<\alpha<1\end{cases}$$

for $B\in\mathcal{B}(\mathbf{T}^2)$ where $\mathcal{B}(\mathbf{T}^2)$ denotes the collection of all Borel sets of the torus.

Definition 18 *A complex $S\alpha S$ field $\Big\{X_{n,m}:(n,m)\in\mathbf{Z}^2\Big\}$ is said to have a* ***moving average*** *(MA) representation if there exists a sequence $(a_{m,n})\in\ell^{\alpha}(\mathbf{Z}^2)$ and a field of i.i.d. $S\alpha S$ random variables $\Big\{Y_{n,m}:(n,m)\in\mathbf{Z}^2\Big\}$ such that*

$$X_{m,n}=\sum_{(k,l)\in\mathbf{Z}^2}a_{m-k,n-l}Y_{k,l}\qquad(m,n)\in\mathbf{Z}^2.$$

It is know that mapping

$$\{a_{k,l}\} \in \ell^{\alpha}(\mathbf{Z}^2) \to \sum_{(k,l)\in\mathbf{Z}^2} a_{m-k,n-l}Y_{k,l} \in L(Y)$$

is an isometry.

In what follows, $\hat{\mu}$ means the Fourier transform of the measure μ. We write $\hat{f}$ to mean $\widehat{f dm_2}$. Also, $\widehat{L^p}$ means the collection of all functions which are the Fourier transform of a member of L^p. Recall that the Fourier transform is defined as follows. For a measure on $\mathbf{T}^2$,

$$\hat{\mu}(m,n) = \int_{\mathbf{T}^2} e^{-(im\lambda+in\theta)} d\mu(\lambda,\theta)$$

and for a measure on $\mathbf{Z}^2$,

$$\hat{\mu}(\lambda,\theta) = \sum_{(m,n)\in\mathbf{Z}^2} e^{-(im\lambda+in\theta)}\mu(m,n).$$

If μ is a measure on $\mathbf{T}^2$ which takes values in $\ell^{\alpha}(\mathbf{Z}^2)$, then for $B \in \mathcal{B}(\mathbf{T}^2)$

$$\hat{\mu}(B)(\lambda,\theta) = \sum_{(k,l)\in\mathbf{Z}^2} e^{-(ik\lambda+il\theta)}\mu(B)(k,l),$$

where $\mu(B)(k,l)$ is the (k,l)-th coordinate of $\mu(B)$.

Theorem 26 ([4], Theorem 2.7) *If F is a function on $\mathbf{T}^2$ with the property that $fF \in \cup_{1\leq p<2}\widehat{\ell^p(\mathbf{Z}^2)}$ for each $f \in C(\mathbf{T}^2)$, then $F = 0$ a.e. $[m_2]$.*

Theorem 27 *Let $1 < \alpha < 2$ and let $X_{m,n} = \sum_{(k,l)\in\mathbf{Z}^2} a_{m-k,n-l}Y_{k,l}$, where $(a_{m,n}) \in \ell^{\alpha}(\mathbf{Z}^2)$ and $\left\{Y_{n,m} : (n,m) \in \mathbf{Z}^2\right\}$ is a field of i.i.d. $S\alpha S$ random variables. Suppose there exists an $L(Y)$-valued measure $\mathcal{Z}$ such that for each $(m,n) \in \mathbf{Z}^2$,*

$$X_{n,m} = \int_{\mathbf{T}^2} e^{-(im\lambda+in\theta)} \mathcal{Z}(d\lambda, d\theta). \tag{5.5}$$

Then $X_{m,n} = 0$ for all $(m,n) \in \mathbf{Z}^2$.

Proof: One can assume that $\|Y_{k,l}\|_\alpha = 1$, $(k,l) \in \mathbf{Z}^2$. Since $L(Y)$ is isometric to $\ell^\alpha(\mathbf{Z}^2)$, the formula (5.5) (p. 56) implies that there exists an $\ell^\alpha(\mathbf{Z}^2)$-valued measure μ such that for all $(m,n),(k,l) \in \mathbf{Z}^2$

$$a_{k-m,l-n} = \int_{\mathbf{T}^2} e^{-(im\lambda+in\theta)}\mu(d\lambda,d\theta)(k,l)$$

where $\mu(B)(k,l)$ is the (k,l)-th coordinate of $\mu(B)$. Note that $(a_{k,l}) \in \ell^\alpha(\mathbf{Z}^2) \subset \ell^2(\mathbf{Z}^2)$. Thus, $\hat{a} \in L^2(\mathbf{T}^2)$ and

$$\int_{\mathbf{T}^2} e^{i(k-m)\lambda+i(l-n)\theta}\hat{a}(\lambda,\theta)\,dm_2(\lambda,\theta) = a_{k-m,l-n} \qquad (m,n),(k,l) \in \mathbf{Z}^2.$$

Thus $\mu(d\lambda,d\theta)(k,l) = \exp(ik\lambda+il\theta)\hat{a}(\lambda,\theta)\,dm_2(\lambda,\theta)$, $(k,l) \in \mathbf{Z}^2$ and for each Borel set

$$\begin{aligned}
\int_{\mathbf{T}^2} e^{ik\lambda+il\theta}\hat{\mu}(B)(\lambda,\theta)\,dm_2(\lambda,\theta) &= \mu(B)(k,l) \\
&= \int_B \mu(d\lambda,d\theta)(k,l) \\
&= \int_{\mathbf{T}^2} e^{ik\lambda+il\theta}1_B(\lambda,\theta)\hat{a}(\lambda,\theta)\,dm_2(\lambda,\theta) \qquad (k,l) \in \mathbf{Z}^2.
\end{aligned}$$

Hence $\hat{\mu}(B) = 1_B\hat{a}$ for each Borel $B \in \mathcal{B}(\mathbf{T}^2)$. Since μ is a measure and the Fourier transform is continuous from $\ell^\alpha(\mathbf{Z}^2)$ into $L^{\alpha'}(\mathbf{T}^2)$ with $\frac{1}{\alpha}+\frac{1}{\alpha'} = 1$, we conclude that $(\int g\,d\mu)\hat{} = g\hat{a}$ for each continuous g. Therefore, by Theorem 26 (p. 56), $(a_{m,n}) \equiv 0$. $\triangle$

Definition 19 *Let*

$$M_k^{(\alpha,1)} = \overline{span}\left\{e^{in\theta_1+im\theta_2} : n \le k, m \in \mathbf{Z}\right\}$$

and

$$M_k^{(\alpha,2)} = \overline{span}\left\{e^{in\theta_1+im\theta_2} : n \in \mathbf{Z}, m \le k\right\}$$

*where the closure is in $L^\alpha(\mathbf{T}^2,\mu)$. A strongly harmonizable $S\alpha S$ field is $(\alpha,1)$-**regular** if $M_{-\infty}^{(\alpha,1)} := \bigcap_k M_k^{(\alpha,1)} = \{0\}$ and $(\alpha,2)$-**regular** if $M_{-\infty}^{(\alpha,2)} := \bigcap_k M_k^{(\alpha,2)} = \{0\}$.*

In what follows, μ_j $j = 1, 2$, is the marginal of μ. That is, for all $B \in \mathcal{B}(\mathbf{T})$,

$$\mu_1(B) = \mu(B \times \mathbf{T})$$

and

$$\mu_2(B) = \mu(\mathbf{T} \times B).$$

This is in contrast with terminology used before. Recall that m_2 meant normalized Lebesgue measure on $\mathbf{T}^2$.

Theorem 28 *$M_{-\infty}^{(\alpha,1)} = \{0\}$ if and only if each nonzero $f \in M_0^{(\alpha,1)}$ is different from zero a.e. $[m \otimes \mu_2]$. If $M_{-\infty}^{(\alpha,1)} = \{0\}$, then $\mu << m \otimes \mu_2$ and there exists a $B \in \mathcal{B}(\mathbf{T})$ with $\mu_2(B) = 0$ such that $\int_{\mathbf{T}} \log\left(\frac{d\mu}{d(m\otimes\mu_2)}(\theta_1, \theta_2)\right) dm(\theta_1) > -\infty$ for $\theta_2 \notin B$.*

Proof: ($\Leftarrow$) Let's suppose that $M_{-\infty}^{(\alpha,1)} \neq \{0\}$. Let $f \in M_{-\infty}^{(\alpha,1)}$ with $f \neq 0$. Fix $\theta_2 \in \mathbf{T}$, write $f_{\theta_2}(\cdot)$ for $f(\cdot, \theta_2)$ and define

$$\mathcal{M}_{f,\theta_2} = \overline{span}^{L^\alpha(\mathbf{T},d\mu)}\left\{e^{in\cdot} f_{\theta_2}(\cdot) : (n, m) \in \mathbf{Z}^2\right\}$$

and

$$\mathcal{N} = \overline{span}^{L^\alpha(\mathbf{T},|f_{\theta_2}|^\alpha d\mu)}\left\{e^{in\cdot} : (n, m) \in \mathbf{Z}^2\right\}.$$

Finally, defining $T_{f,\theta_2} : \mathcal{N} \to \mathcal{M}_{f,\theta_2}$ by $\phi \mapsto \phi f_{\theta_2}$. T_{f,θ_2} is an onto isometry. Since continuous functions are dense in $L^\alpha(\mathbf{T}, |f_{\theta_2}|^\alpha d\mu)$, we get that $\mathcal{N} = L^\alpha(\mathbf{T}, |f_{\theta_2}|^\alpha d\mu)$. Therefore, $1_B f_{\theta_2} \in \mathcal{M}_{f,\theta_2} \subset M_{-\infty}^{(\alpha,1)} \subset M_0^{(\alpha,1)}$ for all $B \in \mathcal{B}(\mathbf{T})$. Thus contradicting our hypothesis.

($\Rightarrow$) Note that $M_{-\infty}^{(2,1)} \subseteq M_{-\infty}^{(\alpha,1)}$. Therefore, $M_{-\infty}^{(2,1)} = \{0\}$. So by Theorem 2.6, p. 18 of [14] $\mu << m \otimes \mu_2$ and there exists a set $B \in \mathcal{B}(\mathbf{T})$ such that $\mu_2(B) = 0$ and for $\theta_2 \notin B$, $\int_{\mathbf{T}} \log\left(\frac{d\mu}{d(m\otimes\mu_2)}(\theta_1, \theta_2)\right) dm(\theta_1) > -\infty$. By Theorem 7.33 of [27], one can find a $\phi_{\theta_2} \in H^\alpha$ such that $|\phi_{\theta_2}|^\alpha = \frac{d\mu_{\theta_2}}{d(m\otimes\mu_2)}$, for all $\theta_2 \notin B$. The mapping $e^{in\cdot} \to e^{in\cdot}\phi_{\theta_2}$,

$(n \geq 0)$ extends to an isometry from $M_0^{(\alpha,1)}$ to H^α. Since each non-zero function in H^α is different from zero *a.e.* $[m]$ and in particular $\phi_{\theta_2} \neq 0$ *a.e.* $[m]$ for all $\theta_2 \notin B$, every function in $M_0^{(\alpha,1)}$ has the same property. So, every nonzero member of $M_0^{(\alpha,1)}$ is different from zero *a.e.* $[m \otimes \mu_2]$. $\triangle$

Theorem 29 *An $S\alpha S$ field is $(\alpha,1)$-regular $(0 < \alpha \leq 2)$ if and only if $\mu << m \otimes \mu_2$ and there exists a $B \in \mathcal{B}(\mathbf{T})$ with $\mu_2(B) = 0$ such that $\int_{\mathbf{T}} \log\left(\frac{d\mu}{d(m\otimes\mu_2)}(\theta_1,\theta_2)\right) dm(\theta_1) > -\infty$ for $\theta_2 \notin B$.*

Proof: It suffices to prove sufficiency. As proved in Theorem 28 (p. 58), the existence of a set $B \in \mathcal{B}(\mathbf{T})$ such that $\mu_2(B) = 0$ and $\int_{\mathbf{T}} \log\left(\frac{d\mu}{d(m\otimes\mu_2)}(\theta_1,\theta_2)\right) dm(\theta_1) > -\infty$ for $\theta_2 \notin B$ implies that no member of $M_0^{(\alpha,1)}$ is different from zero *a.e.* $[m \otimes \mu_2]$. Now using Theorem 28 (p. 58), we get $M_{-\infty}^{(\alpha,1)} = \{0\}$. $\triangle$

Theorem 30 *If $\int_{\mathbf{T}} \log\left(\frac{d\mu}{d(m\otimes\mu_2)}(\theta_1,\theta_2)\right)dm(\theta_1) = -\infty$ a.e. $[\mu_2]$, then $\left\{X_{n,m} : (n,m) \in \mathbf{Z}^2\right\}$ is $(\alpha,1)$-singular (i.e., $M_n^{(\alpha,1)} = L^\alpha(\mathbf{T}^2,\mu)$ for all n)* $(0 < \alpha \leq 2)$.

Proof: $M_n^{(2,1)} \subseteq M_n^{(\alpha,1)}$ for all n. From the assumption and Lemma 2.7, p. 19 of [14], it follows that $M_n^{(2,1)} = L^2(\mathbf{T}^2,\mu)$. Now $M_n^{(\alpha,1)}$ is the closure of $M_n^{(2,1)}$ in $L^\alpha(\mathbf{T}^2,\mu)$ giving $M_n^{(\alpha,1)} = L^\alpha(\mathbf{T}^2,\mu)$. $\triangle$

We now state open problems which are needed to be solved in order to obtain complete generalizations of the work in [12].

Problem 1 *Extension of the result of Guadalupe [7] to the case of the Torus under appropriate assumptions.*

Problem 2 *Extension of the result of Zygmund [Theorem 7.33, [27]] to the case of the torus.*

Appendix A

Orthogonal Decomposition of Isometries in a Banach Space

The following results are found in [3]. Let $\mathcal{X}$ be a Banach space. For $x, y \in \mathcal{X}$, we write $x \perp y$ if for all $\alpha \in \mathbf{C}$,

$$\|x\| \leq \|x + \alpha y\|. \tag{A.1}$$

Remark A.1 *This is a nonsymmetric notion of orthogonality but it is equivalent to the usual concept of orthogonality in Hilbert space.*

We write $\mathcal{M} \perp \mathcal{N}$ for $\mathcal{M}, \mathcal{N} \subseteq \mathcal{X}$ if $x \in \mathcal{M}$ and $y \in \mathcal{N}$ implies $x \perp y$.

Definition A.1 *A **semi-inner-product** (s.i.p.) on $\mathcal{X}$ is a function $[\cdot,\cdot]$ from $\mathcal{X} \times \mathcal{X}$ into $\mathbf{C}$ with the following properties:*

1. *$[\cdot, y]$ is linear for each $y \in \mathcal{X}$*
2. *$|[x, y]| \leq \|x\|\|y\|$ for $x, y \in \mathcal{X}$*
3. *$[x, x] = \|x\|^2$ for all $x \in \mathcal{X}$*

4. $[x, \alpha y] = \overline{\alpha}[x, y]$ *for all* $x, y \in \mathcal{X}$ *and* $\alpha \in \mathbf{C}$.

Remark A.2 *A particular Banach space may have many s.i.p.'s consistent with the norm and the notion of orthogonality will be dependent on the s.i.p.*

For $\mathcal{M}, \mathcal{N} \subseteq \mathcal{X}$, we write $[\mathcal{M}, \mathcal{N}]$ to mean $\Big\{[x, y] : x\mathcal{M}, y \in \mathcal{N}\Big\}$.

Theorem A.1 *Let* $\mathcal{X}$ *be a normed linear space and* $\mathcal{M}$ *and* $\mathcal{N}$ *be subspaces of* $\mathcal{X}$ *with* $\mathcal{M} \perp \mathcal{N}$. *Then there exists a s.i.p.* $[\cdot, \cdot]$ *such that* $[\mathcal{N}, \mathcal{M}] = \{0\}$.

Lemma A.1 *Let* $\mathcal{X} = \mathcal{M} \oplus \mathcal{N}$ *where* $\mathcal{M}$ *and* $\mathcal{N}$ *are subspaces of* $\mathcal{X}$ *with* $\mathcal{M} \perp \mathcal{N}$. *Then* $\mathcal{N} = \{x \in \mathcal{X} : [x, \mathcal{M}] = 0\}$ *for some s.i.p.* $[\cdot, \cdot]$.

Lemma A.2 *Let* $\mathcal{M}$ *and* $\mathcal{N}$ *be closed subspaces of* $\mathcal{X}$ *with* $\mathcal{M} \perp \mathcal{N}$. *Then* $\mathcal{M} \oplus \mathcal{N}$ *is closed.*

By a smooth Banach space, we will mean a Banach space that is uniformly Fréchet differentiable. That is, for all x and y in the unit sphere $\mathcal{S}$ of $\mathcal{X}$ and λ real,

$$\lim_{\lambda \to 0} \frac{||x + \lambda y|| - ||x||}{\lambda} \quad \text{exists}$$

and this limit is approached uniformly for $(x, y) \in \mathcal{S} \times \mathcal{S}$.

Remark A.3 *In a smooth Banach space, the s.i.p. is unique, so we may write* $\mathcal{M}^{\perp}$ *for* $\{x \in \mathcal{X} : [x, \mathcal{M}] = 0\}$.

Lemma A.3 *Let* $\mathcal{X}$ *be a smooth, reflexive Banach space and suppose that* $\{\mathcal{M}_k\}$ *and* $\{\mathcal{N}_k\}$ *are a sequence of closed subspaces such that*

1. $\mathcal{X} = \mathcal{M}_k \oplus \mathcal{N}_k$

2. $\mathcal{N}_k \perp \mathcal{M}_k$

3. $\mathcal{N}_k \subseteq \mathcal{N}_{k-1}$ *and* $\mathcal{M}_{k-1} \subseteq \mathcal{M}_k$

Let $\mathcal{M} = \overline{[\cup_{k=1}^{\infty}\mathcal{M}_k]}$ *and* $\mathcal{N} = \cap_{k=1}^{\infty}\mathcal{N}_k$; *then* $\mathcal{X} = \mathcal{M} \oplus \mathcal{N}$ *and* $\mathcal{N} \perp \mathcal{M}$.

Definition A.2 *Let V be an isometry on the normed linear space $\mathcal{X}$. V is said to be* ***orthogonally complemented (o.c.)*** *provided there exists a closed subspace $\mathcal{M}$ of $\mathcal{X}$ such that $\mathcal{X} = \mathcal{M} \oplus V(\mathcal{X})$ and $V(\mathcal{X}) \perp \mathcal{M}$.*

Remark A.4

1. *V is o.c. if and only if there exists a projection $P : \mathcal{X} \to V(\mathcal{X})$ of norm 1.*

2. *In a Hilbert space, each isometry is orthogonally complemented.*

3. *Every isometry of L^p $(1 \le p < \infty)$ is orthogonally complemented.*

Definition A.3 *An isometry $V : \mathcal{X} \to \mathcal{X}$ will be called a* ***unilateral shift*** *if there exists a subspace $\mathcal{L} \subseteq \mathcal{X}$ such that*

1. $V^n(\mathcal{L}) \perp V^m(\mathcal{L})$ *for* $n > m$

2. $\mathcal{X} = \bigoplus_{n=0}^{\infty} V^n(\mathcal{L})$.

Theorem A.2 (Generalized Wold Decomposition) *Let V be an isometry on a smooth, reflexive Banach space $\mathcal{X}$. If V is o.c., then there exist closed subspaces $\mathcal{X}_1$ and $\mathcal{X}_2$ such that*

1. *$\mathcal{X}_1$ and $\mathcal{X}_2$ are invariant under V,*

2. *$V|_{\mathcal{X}_1}$ is unitary (surjective),*

3. *$V|_{\mathcal{X}_2}$ is a unilateral shift,*

4. $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$.

Corollary A.1 *If V is an isometry on a smooth, reflexive Banach space which satisfies:*

1. *V is o.c.*

2. *$\bigcap_{n=0}^{\infty} V^n(\mathcal{X}) = \{0\}$,*

then V is a unilateral shift.

Appendix B

A "Beurling Type" Theorem in $H^p(\mathbf{T})$

In this section, we turn our attention back to S-invariant subspaces of $H^p(\mathbf{T})$. Recall that the problem was first solved by Beurling for the case $p = 2$. Later, de Leeuw and Rudin solved the problem for $p = 1$. A duality argument gives $p = \infty$. We solve a weaker version of the problem for $p \in \mathcal{P} := \{p : 1 < p < \infty, p \neq 2\}$, using the tools just developed. Before we give our theorem, we point out that all subspaces of $L^p(\mathbf{T})$ are smooth, reflexive Banach spaces for $p \in \mathcal{P}$. So, the s.i.p. on $L^p(\mathbf{T})$ is unique. It is given by

$$[g, f] = \|f\|_p^{2-p} \int_{\mathbf{T}} g\bar{f}|f|^{p-2}\, dm.$$

We now give our result.

Theorem B.1 *$\mathcal{M}$ is an S-invariant subspace of $H^p(\mathbf{T})$, $p \in \mathcal{P}$ with $\mathcal{M}$ S o.c. if and only if $\mathcal{M} = \phi H^p(\mathbf{T})$ with ϕ inner.*

Proof: ($\Leftarrow$) If $\mathcal{M} = \phi H^p(\mathbf{T})$ with ϕ inner, then clearly, $\mathcal{M}$ is an S-invariant subspace of $H^p(\mathbf{T})$ and S is o.c. on $\mathcal{M}$ since

$$\mathcal{M} = \mathcal{L} \oplus \bigoplus_{n=1}^{\infty} S^n(\mathcal{L}) = \mathcal{L} \oplus S(\mathcal{M})$$

with $S(\mathcal{M}) \perp \mathcal{L}$ where $\mathcal{L} = \left\{\alpha\phi : \alpha \in \mathbf{C}\right\}$.

($\Rightarrow$) We start by noting that $\mathcal{M}$ satisfies all the conditions of Corollary A.1 (p. 63). Therefore, S is a unilateral shift. That is, there exists a subspace $\mathcal{L} \subseteq \mathcal{M}$ namely, $\mathcal{L} = S(\mathcal{M})^{\perp}$ such that

$$\mathcal{M} = \bigoplus_{n=0}^{\infty} S^n(\mathcal{L})$$

with $S^n(\mathcal{L}) \perp S^m(\mathcal{L})$ for all $n > m$. We need only show that $\mathcal{L}$ is one dimensional and spanned by an inner function. Let $\phi \in \mathcal{L}$ with $||\phi||_p = 1$. Then by our decomposition of $\mathcal{M}$ we get that $\phi z^n \perp \phi$. That is,

$$\int_{\mathbf{T}} \phi z^n \overline{\phi} |\phi|^{p-2}\, dm = 0 \qquad \text{for all } n \geq 1.$$

That is,

$$\int_{\mathbf{T}} z^n |\phi|^p\, dm = 0 \qquad \text{for all } n \geq 1.$$

Taking complex conjugates of both sides we get

$$\int_{\mathbf{T}} z^n |\phi|^p\, dm = 0 \qquad \text{for all } n \neq 0.$$

Therefore, $|\phi|^p$ is constant and hence $|\phi|$ is constant. Since $||\phi||_p = 1$, we get that $|\phi| = 1$ *a.e.* So, ϕ is inner. It is left to show that $\mathcal{L}$ is one dimensional. To do this, let's suppose not. Then from Lemma 4, p. 440 of [6] there exists $\psi \neq 0$ in $\mathcal{L}$ of norm one with $\psi \perp \phi$. By our decomposition we get that $\psi z^n \perp \phi$ and $\phi z^n \perp \psi$. Also doing that same calculations above with ψ in place of ϕ we see that ψ is also inner. That is, both ϕ and ψ have constant modulus one a.e. on $\mathbf{T}$. Writing our above orthogonal relations in term of the s.i.p. we get

$$\int_{\mathbf{T}} \psi z^n \overline{\phi} |\phi|^{p-2}\, dm = 0 \qquad \text{for all } n \geq 0$$

and

$$\int_{\mathbf{T}} \phi z^n \overline{\psi} |\psi|^{p-2}\, dm = 0 \qquad \text{for all } n \geq 1.$$

Since ϕ and ψ are unimodular we get

$$\int_{\mathbf{T}} \psi z^n \overline{\phi}\, dm = 0 \qquad \text{for all } n \geq 0 \tag{B.1}$$

and

$$\int_{\mathbf{T}} \phi z^n \overline{\psi}\, dm = 0 \qquad \text{for all } n \geq 1.$$

Taking complex conjugates of (B.1) (p. 66), we get

$$\int_{\mathbf{T}} \phi z^{-n} \overline{\psi}\, dm = 0 \qquad \text{for all } n \geq 0.$$

Therefore, $\phi\overline{\psi} = 0$ a.e., but that is a contradiction since ϕ and ψ are unimodular. So our supposition must be incorrect. That is, $\mathcal{L}$ is one dimensional as desired. $\triangle$

Although the above result is weaker than Theorem 12 (p. 21), it shows a property of the subspaces of the form $\phi H^p(\mathbf{T})$. In the future, we want to examine in the question of a Generalized Wold Decomposition for two isometries in this context and get the analogue of Corollary 3 (p. 40) with conditions on the isometries acting on the subspaces of $H^p(\mathbf{T}^2)$ directly.

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