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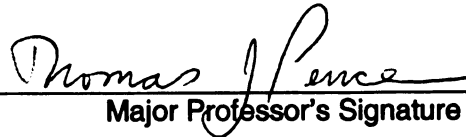
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NUMERICAL INTEGRATION OF CONSTITUTIVE MODELS

APPLICABLE TO METAL FOAMS

By

Abhishek Modi

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ABSTRACT

NUMERICAL INTEGRATION OF CONSTITUTIVE MODELS

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By

Abhishek Modi

Metal foams have unique combinations of physical and mechanical properties and are becoming increasingly popular among researchers and designers. Four elasto-plastic material models for metal foams are studied in detail. All four material models considered are rate independent elastoplastic models with isotropic hardening rules. A return mapping algorithm using the Newton-Raphson closest point projection is discussed for integrating the constitutive equations of the models. A way to do the return mapping along the spectral directions of the stress tensor is described so as to reduce the dimensionality of the problem. An algorithm applicable to all metal foam models, both with associative and non-associative hardening, is presented. A user subroutine is written in FORTRAN for implementing the algorithm. Numerical simulation of the diagonal crushing of metal foam cube is done in FEM program ABAQUS to verify the material model algorithm. The result of the numerical simulation using user defined subroutine, is compared with the result using the inbuilt metal foam model in ABAQUS. The energy absorption capacity of metal foams is demonstrated by comparing the crushing process of a metal foam cube with a solid aluminum cube.

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Chapter 1

Introduction to Metal Foams and their properties

1.1 Introduction

Foamed metals have become increasingly important to both researchers and designers in the past two decades owing to the unique and interesting combinations of physical and mechanical properties of these materials. Increased interest in these materials in recent times is also due to improved production methodologies [6, 10] which result in easier availability of foamed metals of better and consistent properties at affordable prices. Metal foams have unique combinations of desired properties like impact energy absorption capacity, air and liquid permeability, good acoustic properties as well as low thermal conductivity and high electrical conductivity. One reason why cellular metals have not been more popular in engineering design till recently is the poor manufacturing processes available in the past. New manufacturing techniques developed in the last decade have now ensured that foamed metals have a greater range and control of physical properties. Increased process control over the manufacturing parameters has improved the quality of the metal foams produced. The micro mechanical structure of these new metal foams is more consistent leading to better mechanical properties which can be predicted with more accuracy. In this chapter we have listed some of the applications for metallic foams, classification of cellular metals and the commonly available metal foams with their market names.

1.2 Applications of metal foams

Metallic Foams are important materials in engineering because of the interesting combinations of physical and mechanical properties. The existing applications for metal foams cover a wide range of fields [10]. The applications for metal foams can be broadly divided into three categories based upon the structure and properties of these foams.

Foams with predominantly closed cells are useful for shock and impact absorption applications. In energy absorption applications their structure allows these closed cell foams to absorb large amounts of mechanical energy when they are deformed. Closed cell foams have a high compressive strength and can undergo large plastic deformations because of their porous structure. The automotive industry is looking at metal foams for use in front bumpers to use their high impact absorption capacity. Some other suggested uses for these materials are as protective envelope for air borne safety equipment like aircraft black boxes. Disposable landing feet for spacecrafts and disposable clamping fixtures are other places these materials can be used.

Open cell foams, because of their air and fluid permeability, can be used for functional applications like filtration, damping etc. Metal foams have good thermal and acoustic properties which can be used to dampen vibrations and absorb sounds in certain conditions. Dust and fluid filters, engine exhaust mufflers are some areas these properties are useful. Inert metal foams (ex. gold, silver) are used for the filtration of corrosive liquids. Other useful properties like low thermal conductivity in conjunction with high structural strength is desirable for applications like high temperature gaskets, flame

arrestors etc. Metal foams are good electric conductors which make them good candidates for application as porous electrodes. Foamed metals have also been used advantageously in heaters and heat exchangers.

Foamed aluminum is being used as core material in sandwich panels which are used for structural, load bearing purposes. When used as the filler in sandwich panels they take up a lot of volume without considerably increasing the overall weight of the structure. These three dimensional panels which are structurally very stiff can be used effectively in the aircraft industry as a low cost replacement of honeycomb panels. The automotive industry is also looking at these panels to use in the body of the car. The high structural strength of these panels coupled with the thermal properties of the porous metals has found application as wall and floor heating tiles. Open cell metal foams when used as the core of reinforcing structural struts allow the flow of liquids through them making them useful for heating or cooling purposes.

The above list of applications for cellular metals is not a complete list of all the places these remarkable materials with a combination of useful properties have found application. As the manufacturing processes for metallic foams improve leading to better quality foams at lower prices, designers will use them more widely.

1.3 Classification of metal foams

Cellular metals are defined as metals with solid and gaseous phases in their morphology. To properly identify metallic foams which are a subset of the larger class of cellular metals, proper classification has to be made for all kinds of cellular metals [6].

Cellular metals- This is the most general classification and refers to a metallic body in which any kind of gaseous voids are dispersed.

Porous Metals- These are a special kind of cellular metal. The voids are pores which are round and isolated from each other.

Metal Foams- These are a kind of cellular metal with restricted morphology. The cells are closed and are separated from each other by thin films.

Metal Sponges- These are open cell cellular metals with interconnected voids.

The subject of this study will be limited to Metal Foams and Metal Sponges.

New and improved manufacturing techniques have resulted in the availability of better quality metal foams at lower prices. Banhart [6] and Davies [10] have listed the new manufacturing techniques being used by leading manufacturers of metal foams in the industry. Alcan, Alporas and Alulight are the trade names of the most commonly available closed cell aluminum foams. The manufacturing processes of these common types of aluminum foams are described below along with the range of their physical properties.

Trade name: ALCAN

Manufacturers: *Hydro Aluminum in Norway & Cymat Aluminum Corporation in Canada*

Manufacturing process: This is a foaming method for obtaining metal foam from aluminum melt. The foaming of aluminum melt is done by gas injection. Silicon-carbide, aluminum-oxide or magnesium-oxide particles are used to increase the viscosity of the melt. This helps in the stability of the foam when it is being formed. This step requires sophisticated mixing techniques to ensure a uniform distribution of particles. The melt is formed in the second step by injecting gas. The foaming is done in a continuous process in which the mixture of gases and solid metal floats to the top while the liquid metal drains out. Because of the gravitationally induced drainage, a gradient in density, pore size and pore elongation is observed in the resulting foam. By this process the foams obtained have the following average properties:

Average densities $0.069 - 0.54 \frac{g}{cm^3}$

Average pore size 3mm-25mm

Trade name: ALPORAS

Manufacturers: *Shinko Wire Company in Japan*

Manufacturing process: Metal/Aluminum melts can also be foamed by directly adding a blowing agent in the melt instead of injecting gas into it. First some 1.5% wt Ca metal is added to the melt for increasing the viscosity by the formation of calcium oxide and calcium-aluminum oxide. After the viscosity has reached the desired level, titanium hydride is added (1.6% wt), serving as a blowing agent by releasing hydrogen gas in the hot viscous liquid. After cooling the vessel below the melting point of the alloy, the

liquid foam turns into solid aluminum foam. By this process the foams obtained have the following average properties:

Average densities $0.18 - 0.24 \frac{g}{cm^3}$

Average pore size 2mm-10mm

Trade name: ALULIGHT

Manufacturers: *Fraunhofer Institute in Germany*

Manufacturing process: Foamed aluminum can also be prepared from metal powder. Aluminum powder or alloy powder are mixed with the right amount of blowing agent, after which the mix is compacted to yield a dense product. Next, this semi-finished product is heat treated to temperatures close to the melting point of the matrix material. This leads the blowing agent to decompose and release gas, which in turn forces the precursor material to expand and foam. An advantage of this process is that the expansion during foaming can be controlled inside a mold which results in the formation of near net shapes of the foam material.

1.4 Issues in manufacturing and applications of metal foam

There has been relatively limited use of metal foams in engineering design despite their impressive combinations of properties. Notwithstanding the recent developments in manufacturing of metal foams, there is a lack of sophistication in manufacturing processes when compared to polymeric foams. Some of the important issues hindering in the manufacturing and application of metal foams are listed below.

Stability of foams: foams are unstable systems because their large surface area causes energy to be far from a minimum value. Foams can at best be meta-stable, constantly decaying at a certain rate. One way to increase the stability of foams is to introduce solid particles in the melt to increase the viscosity. The action of foam stabilization by the addition of particle additives is not entirely clear. Some recent work has been done to study the stability of metal foams [9].

Consistent quality of foam in production: availability of consistent foams in a wide range of densities is important for wide scale industrial applications. There is a lack of sufficient knowledge of the mechanisms of foam formation to be able to produce foams of consistent quality with pre-defined parameters.

Expense of metal foams: There have been some recent improvements in the field of manufacturing, but there is still need for setting up efficient industrial processes to reduce the cost and improve the quality of foams. Metal foams are nevertheless less expensive than honeycomb panels that are popular in the design of weight bearing structures in the aircraft industry.

Physical properties of foam: In recent years there has been considerable research interest in experimental studies of the physical and mechanical properties of available metal foams [2, 12, 17, 18]. There is still a need to establish a wide database for the physical properties like density, average cell size, cell wall irregularities, and cell wall thickness

and mechanical properties like compressive strength, tensile strength, hydrostatic strength, elastic modulus and energy absorption capacity.

Modeling of the mechanical response of foams- It is increasingly important for design engineers to be able to predict the large strain response of materials they use in design. Efficient and accurate material modeling of metal foams is required for use in FEM packages. There has been some research which treats metal foams as solid continuum and applies the equations of elastoplasticity theory [11, 19].

The focus of this study is to provide an algorithm for the numerical integration of the elastoplastic constitutive equations of metal foams.

Chapter 2

Experimental studies on metal foams

2.1 Introduction

There is a sizable and growing amount of literature on the experimental studies done on metal foams to record their physical and mechanical properties. Several researchers have done experimental studies on the uniaxial and biaxial response of foamed metals [17, 18]. A majority of these experimental works have focused on the uniaxial stress strain response of metal foams both in compression and in tension. Some work has been done to summarize the results of these tests in simple theoretical equations [2]. Closed cell foams exhibit remarkably lower values of strength in experiments than the theoretical predictions. The micro structure of foams has been studied in some detail to explain the apparent drop in the physical properties of closed cell metal foams. Apart from uniaxial experiments, some biaxial loading studies [11, 12] have been performed to measure the performance of foamed metals under various stress situations. Such experiments are useful in determining the initial yield surface of foams. A presentation of results from the experimental literature, most pertinent to the present study, is given in this chapter.

2.2 Scaled equations for strength and modulus of foamed metals

Andrews et al [2] have attempted to use simple dimensional arguments to model the yield strength and elastic stiffness of closed cell and open cell foams in terms of simple equations. The most important physical property which determines the mechanical

properties of foams is the density of the foam relative to the density of the cell wall material. Dimensional arguments can be used to relate the Young's Modulus and yield strength of the foam to the relative density with respect to the cell wall material.

The mechanical properties of foams can be modeled by considering the mechanisms by which the cells deform and fail. Open cell foams and closed cell foams fail by different mechanisms. Open cell foams under uniaxial stress deform by the formation of plastic hinges in the cell walls. In Closed cell foams on the other hand, bending of cell walls is accompanied by stretching of cell faces.

A structural mechanical analysis of a low density, open cell, Kelvin foam with a tetrakaidecahedral structure leads to the following equation for the modulus and yield strength of open cell foams [2].

$$\frac{E^*}{E} = 0.98 \left(\frac{\rho^*}{\rho} \right)^2 \quad (2.1)$$

$$\frac{\sigma_c^*}{\sigma_c} = 0.3 \left(\frac{\rho^*}{\rho} \right)^{\frac{3}{2}} \quad (2.2)$$

where

E^* - elastic modulus of open cell foam

σ_c^* - uniaxial compressive yield stress of open cell foam.

E - elastic modulus of cell wall metal (aluminum)

σ_c - uniaxial compressive yield stress of cell wall metal (aluminum).

$\frac{\rho^*}{\rho}$ - relative density of the foamed metal with respect to the base metal.

A similar analysis for ideal tetrakaidecahedral closed cell structure leads to the following set of equations [2].

$$\frac{E^*}{E} = 0.32 \left(\frac{\rho^*}{\rho} \right)^2 + 0.32 \left(\frac{\rho^*}{\rho} \right) \quad (2.3)$$

$$\frac{\sigma_c^*}{\sigma_c} = 0.33 \left(\frac{\rho^*}{\rho} \right)^2 + 0.44 \left(\frac{\rho^*}{\rho} \right) \quad (2.4)$$

where now

E^* - elastic modulus of closed cell foam

σ_c^* - uniaxial compressive yield stress of closed cell foam.

Comparison of the predicted values of modulus and strengths from these equations to the experimental results from studies on the stiffness and strengths of the open cell foam ERG and the closed cell foams Alcan, Alporas & Alulight show interesting results [2]. The theoretical equations for the open cell foam (Eq 2.1 and 2.2) are quite close to the experimental results for ERG, but the theoretical equation for closed cell foam (Eq 2.3 and 2.4) over-predicts the experimental results from Alcan, Alporas and Alulight quite appreciably. The reason for the observed difference between the model and experimental values is the presence of micro-mechanical imperfections in the morphology of closed cell foams.

2.3 Effect of micro-structural irregularities on strength and modulus

Micro-structural observations of the closed cell foams (Alcan, Alporas and Alulight) reveal a number of defects in the structure which is not taken into account in the theoretical model [15]. Some of these defects are listed below

Cell wall curvature: The walls of some of the cells are not plane but have some initial curvature which is not taken into account by the model

Cell wall corrugations: The walls of some cells have corrugations and chinks.

Density gradient: The density of the foam varies throughout the thickness of the foam due to the processing conditions. This is pronounced in Alcan which is cast under the influence of gravity.

Defects in cell walls: some of the cell walls are missing and others have voids.

The effect of cell wall curvature and corrugations has been modeled using FEM and compared to the strength of a periodic tetrakaidecahedral cell [15]. The observed cell wall imperfections can account for up to a 70% drop in the mechanical properties of foams. The presence of these micro structural defects is a direct result of the production methods for foams. One of the important challenges for the manufacturers of foamed metals is to develop production methodologies that increase the uniformity of the microstructure of foams.

2.4 Uniaxial compressive stress strain response of metal foams

Experimental studies on the uniaxial compressive response of various types of closed and open cell metal foams reveal certain similarities [7]. All foams irrespective of the cell wall material exhibit the schematic response under uniaxial loading. Initial loading results in an increase in stress magnitude which appears to be proportional to strain. But the unloading modulus is greater than the loading modulus suggesting that local yielding takes place almost immediately on loading. In a macro sense the material can be considered elastic up to a certain strain. After the stress reaches the yield point, there is a regime of strong plastic deformation with very little strain hardening. Sometimes a separate upper and lower yield point is observed. After an extended plateau the stress strain curve enters a regime of rapid strain hardening. This part of the stress strain curve is characterized by densification of cell walls as they collapse on one another, accompanied by a rapid increase in stress. The strain at which the stress strain regime enters densification is called the densification strain. Fig. 5.1 shows a typical uniaxial compressive response of closed cell aluminum foam.

2.5 Initial yield surface of foams

The initial yield surface for metal foams is important in the mathematical model of the elastoplastic response of foams. Some phenomenological yield criteria have been suggested by Deshpande-Fleck [11], Miller [19] and Doyoyo [12] among others, for foamed metals. Experimental studies for biaxial and multiaxial loading of foams are limited when compared to the large experimental literature available for uniaxial loading.

Nevertheless attempts have been made in this regard by Deshpande-Fleck [11] and Doyoyo et al [12] among others.

Doyoyo et al [12] have investigated the yield surface of closed cell aluminum foam using the modified Arcan apparatus. Butterfly shaped test specimens of the foam were used in the Arcan apparatus which ensures the yielding of the foam is at the center of the specimen.

The Yield surface was found to be pressure dependant and quadratic in the meridian plane. A phenomenological yield surface has been proposed in the principal stress space which depends on the first invariant of principal stresses (I_1) and second invariant of deviatoric stresses (J_2).

$$I_1 = \sigma_{ii} \quad (2.5)$$

$$J_2 = \frac{1}{2} S_{ij} S_{ij} \quad (2.6)$$

Here

$$\sigma_{ij} = \frac{1}{3} \sigma_{kk} \delta_{ij} + S_{ij} \quad (2.7)$$

σ_{ij} - stress tensor

S_{ij} - deviatoric stress tensor

Deshpande & Fleck [11] have performed biaxial loading experiments for the yield surface and have come up with a similar pressure dependant yield surface in the principal stress space. They used triaxial system consisting of a pressure cell and a screw driven

piston for axial loading. The probing of the yield surface suggests a quadratic shape in the stress space of mean stress (σ_m) and Von Mises effective stress (σ_e).

where

$$\sigma_m = \frac{1}{3} \sigma_{ii} \quad (2.7)$$

$$\sigma_e = \sqrt{\frac{3}{2} S_{ij} S_{ij}} \quad (2.8)$$

Chapter 3

Elastoplastic material models of metal foam

3.1 Introduction

Metal foams are heterogeneous because of the gaseous voids present in the morphology. Metal foams are therefore not continuous materials in the strict sense of the definition. The elastoplastic theory applicable to continuous materials however, can be applied to metal foam specimens which are considerably larger in dimension than the average cell dimension. The rule of thumb followed in some references [2, 12] is that the smallest specimen dimension should be at least 10 times the average cell diameter for continuum theory to be applicable. Several attempts have been made to model the mechanical response of metal foams under the broad theory of elastoplastic materials. Commercially available FEM programs LS-DYNA [13] and ABAQUS [1] have material subroutines for metallic foams incorporated in the program. Two other material models in literature by Deshpande-Fleck [11] and Miller [19] are also studied in detail. In this chapter we present the general equations of elastoplastic theory as applicable to metal foams and also the constitutive equations of the four models.

3.2 General equations of elastoplastic theory as applicable to metal foams

Uniaxial experiments [18] performed on metal foams have shown that the unloading elastic modulus of foams is higher than the loading modulus even for small strains. This is explained by the fact that metal foams undergo local plastic yielding almost immediately on loading. The local yielding is due to the presence of irregularities at the

cellular level. The overall specimen can still be treated as elastic at the macroscopic level up to the yield stress. All the four models considered assume isotropic elastic behavior in the elastic region with a constant elasticity tensor.

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl}^e \quad (3.1)$$

where

C_{ijkl} - elasticity tensor

ε_{ij}^e - elastic strain

since isotropic elastic behavior is assumed, C_{ijkl} is completely defined by any two elastic parameters like modulus of elasticity (E) and elastic Poisson's ratio (ν).

Elastoplastic theories in general assume the strain rates to decompose additively. So we can write

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^e + \dot{\varepsilon}_{ij}^p \quad (3.2)$$

where

ε_{ij}^p - plastic strain

Isotropic elastic behavior is assumed until yielding is reached. Once yielding is reached, the inelastic response is assumed to be strain rate independent by all four material models. A yield surface in stress space is used to indicate yielding.

$$f(\sigma_{ij}, \varepsilon_{ij}^p) = 0 \quad (3.3)$$

The presence of ε_{ij}^p in the yield rule is to take into account the strain hardening of the material. As explained later, the yield surface of metal foams can be conveniently

represented in the stress space by the Von-Mises equivalent stress (q) and hydrostatic pressure stress (p). where p and q are defined as follows

$$p = -\frac{1}{3} \sigma_{ii} \quad (3.4)$$

$$q = \sqrt{\frac{3}{2} S_{ij} S_{ij}} \quad (3.5)$$

The hardening of the yield surface depends on the amount of inelastic strain in the material. Three of the models (ABAQUS, Deshpande-Fleck, and Miller) assume an isotropic hardening rule for strain hardening. LS-DYNA uses a hardening rule that allows for change in the shape of the yield surface. The hardening for all four models can be given in terms of two scalar quantities that are used to track the growth of plastic strain.

The following strain-like scalar internal variables are introduced

$\hat{\epsilon}^P$ - equivalent plastic strain

ϵ_v^P - volumetric plastic strain

The equivalent plastic strain is defined later (Eq 4.7). The volumetric plastic strain can be written as

$$\epsilon_v^P = \epsilon_{ii}^P \quad (3.6)$$

Now we can write the yield surface in terms of four scalars in the following manner

$$f(p, q, \hat{\epsilon}^P, \epsilon_v^P) = 0 \quad (3.7)$$

Elastoplastic theories in general define the rate of inelastic strain in terms of a flow potential function

$$\dot{\epsilon}_{ij}^P = \gamma \frac{\partial \phi}{\partial \sigma_{ij}} \quad (3.8)$$

where

γ - plastic multiplier

ϕ - flow potential function

The flow potential function (ϕ) is defined in the stress space and can evolve with accumulation of plastic strain just like the yield potential function (f). In fact a lot of elastoplastic models assume the flow potential to be the same as the yield potential. If this is the case then the flow rule is said to be associative, otherwise it is said to be non-associative. The flow rule is associative for two of the models (Deshpande-Fleck & Miller) while the other two models (LS-DYNA & ABAQUS) use a non-associative flow rule.

The strain hardening of the yield function and flow potential is incorporated in the models through the input of experimental data. The four models considered use the data from uniaxial compressive test and/or hydrostatic compressive test as input. The equivalent plastic strain ($\hat{\epsilon}^P$) is used as a measure of the uniaxial plastic strain for all models except the Miller model (see Sec 3.3.4)

$$p_c = \bar{p}_c(\epsilon_v^P) \quad (3.9)$$

$$\sigma_c = \bar{\sigma}_c(\hat{\epsilon}^P) \quad (3.10)$$

where

p_c - hydrostatic compressive yield stress

σ_c - uniaxial compressive yield stress

The yield surfaces of elastoplastic materials are most conveniently represented in the principal stress space. The complete representation of the yield surface in principal stress

space requires the description of cross-sectional shape in the deviatoric plane and its trace on the meridian (p-q) plane. All four models considered here assume the material to remain isotropic in the inelastic region leading to circular cross-sections in the deviatoric plane. Therefore the models can be compared in the meridian plane generated by Von Mises equivalent stress (q) and hydrostatic pressure stress (p). The yield surface for all the four material models studied here are pressure dependant and quadratic in the p-q plane.

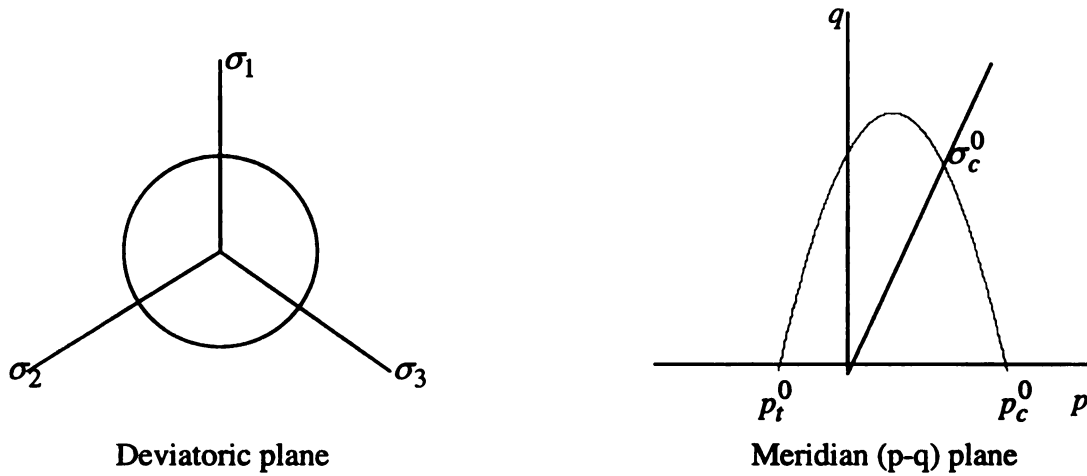


Fig 3.1 metal foam yield surface in deviatoric and meridian plane

Experiments done on probing the initial yield surface of metal foams reveal that all metal foams share some important characteristics [16]. All metal foams have a pressure dependant yield surface that is quadratic in the p-q plane. The yield functions of all four material models share this property. The other characteristics of the yield surface of foam

can be described in terms of some experimental parameters. We define these parameters below

σ_c^0 - initial compressive uniaxial stress

p_c^0 - initial compressive hydrostatic stress

p_t^0 - initial tensile hydrostatic stress

ν^P - plastic Poisson's ratio

$\beta = \frac{\sigma_c^0}{\sigma_t^0}$ - ratio of the initial compressive and tensile uniaxial stress

The four material models studied, have yield surface functions that can incorporate one or more of these additional experimental parameters. The complexity of the yield surface depends on how many parameters it can accommodate. Miller model [19] and ABAQUS [1] are both three parameter models while DYNA [13] and Deshpande-Fleck model [11] are two parameter models.

3.3 Description of selected constitutive models

Four elastoplastic material models for metal foam are studied in detail and the description of their yield surfaces, hardening rule and flow rule are given in this section.

3.3.1 LS DYNA model (material model # 75) [13]

LS-DYNA has an inbuilt material model for metal foams as material model # 75.

The yield function is given by

$$f(p, q, \hat{\varepsilon}^p, \varepsilon_v^p) = 0 \quad (\text{Eq. 3.7})$$

where

$$f = \sqrt{\left(\frac{p - \frac{1}{2}(p_c - p_t)}{a} \right)^2 + \left(\frac{q}{b} \right)^2} - 1 \quad (3.11)$$

$$p_t = \frac{p_c}{10} \quad (3.12)$$

$$a = \frac{1}{2}(p_t + p_c) \quad (3.13)$$

$$M = \sigma_c / \sqrt{(p_c p_t \frac{1}{3} \sigma_c (p_c - p_t) - \frac{1}{9} \sigma_c^2)} \quad (3.14)$$

$$b = Ma \quad (3.15)$$

As can be seen from the above equations, the initial calibration of the yield surface requires the knowledge of σ_c^0 and p_c^0 . The material model can therefore be described as a two parameter model.

The strain hardening uses two sets of experimental data as input. The values of σ_c and p_c in the above equations are obtained from the results of uniaxial compressive test (Eq. 3.10) and hydrostatic compressive stress (Eq. 3.9) respectively.

The ratio of the half axes of the ellipse (M) is not a constant and so a and b can evolve independently from each other. Hence the shape of the yield surface can change during

loading. The yield surface remains an ellipse in p-q plane and no corner development takes place (Fig 3.2)

The flow rule is not associative and is given by

$$\dot{\epsilon}_{ij}^p = \gamma \frac{\partial \phi}{\partial \sigma_{ij}} \quad (\text{Eq 3.8})$$

where

$$\phi(p, q) = \sqrt{q^2 + \frac{9}{2}p^2} \quad (3.16)$$

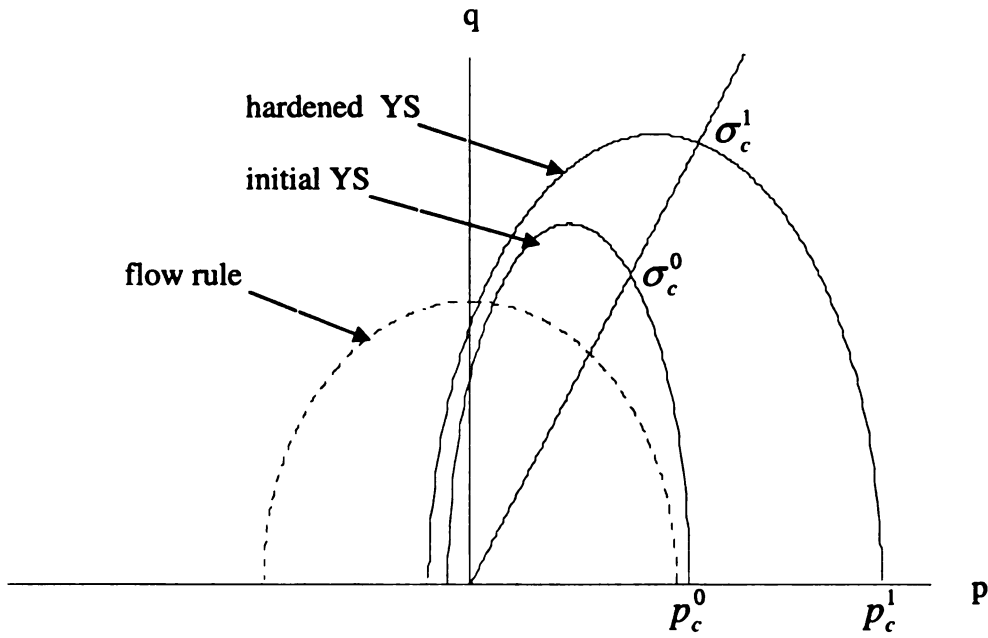


Fig 3.2 evolution of yield surface and flow potential for LS-DYNA model # 75

3.3.2 ABAQUS metal foam model (metal foam plasticity with volumetric hardening) [1]

ABAQUS has two inbuilt material models: metal foam plasticity with volumetric hardening and metal foam plasticity with isotropic hardening. We will consider metal foam plasticity with volumetric hardening.

The yield function is the same as the one in DYNA and is given by

$$f(p, q, \hat{\epsilon}^P, \epsilon_v^P) = 0 \quad (\text{Eq 3.7})$$

where

$$f = \sqrt{\left(\frac{p - \frac{1}{2}(p_c - p_t)}{a} \right)^2 + \left(\frac{q}{b} \right)^2} - 1 \quad (3.17)$$

$$a = \frac{1}{2}(p_t^0 + p_c) \quad (3.18)$$

$$M = \sigma_c^0 / \sqrt{(p_c^0 p_t^0 \frac{1}{3} \sigma_c^0 (p_c^0 - p_t^0) - \frac{1}{9} \sigma_c^{02}} \quad (3.19)$$

$$b = Ma \quad (3.20)$$

Tensile hydrostatic yield stress (p_t^0) is a constant and is independent of the compressive hydrostatic yield stress (p_c) for this model. The initial calibration of the yield surface requires the knowledge of σ_c^0 , p_c^0 and p_t^0 . The material model can therefore be described as a three parameter model.

The hardening rule for this model is more simplistic than LS-DYNA and is governed completely by the hydrostatic compressive yield stress - volumetric strain curve.

$$p_c = \bar{p}_c(\epsilon_v^P) \quad (\text{Eq 3.9})$$

The ratio of the half axes, M is a constant during the plastic flow which conserves the shape of the ellipse. Thus the hardening in this case can be considered isotropic (Fig 3.3)

The flow rule is not associative and is the same as the one in LS-DYNA.

$$\dot{\epsilon}_{ij}^p = \gamma \frac{\partial \phi}{\partial \sigma_{ij}}$$

where

$$\phi(p, q) = \sqrt{q^2 + \frac{9}{2}p^2} \quad (3.21)$$

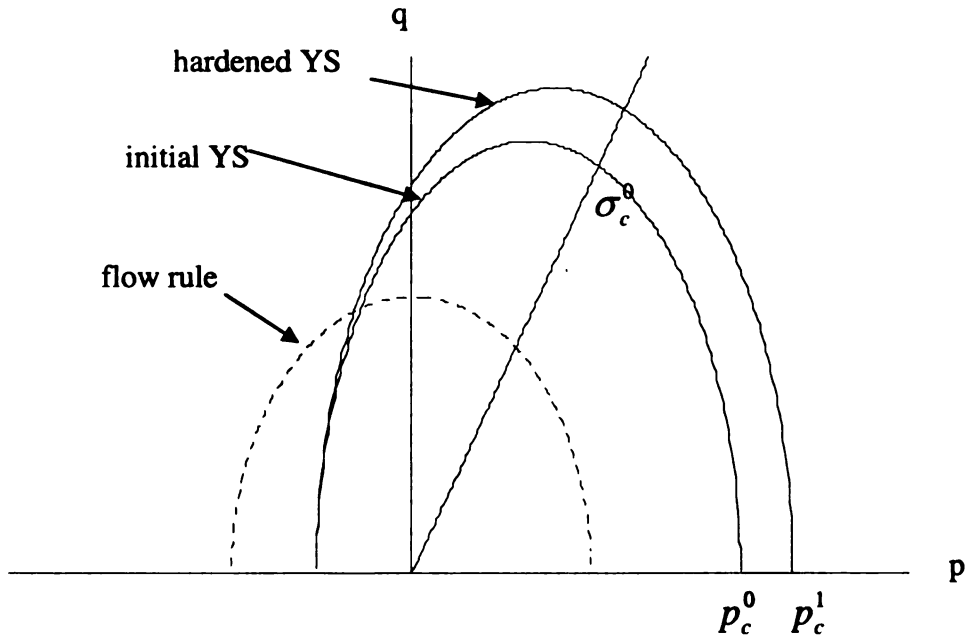


Fig 3.3 evolution of yield surface and flow potential for ABAQUS metal foam model

3.3.3 Deshpande & Fleck model [11]

Unlike metals, metal foams undergo volume reduction under hydrostatic compressive pressure. The plastic Poisson's ratio (ν^p) can be taken as a measure of the compressibility of materials. Solid metals which do not undergo volume reduction under pure hydrostatic pressure have a constant plastic Poisson's ratio ($\nu^p = \frac{1}{2}$). Experimental studies done by Deshpande & Fleck indicate that in metal foams the plastic Poisson's ratio is an important parameter that can be directly related to the yield surface ellipticity. They introduced a shape factor α that is related to plastic Poisson's ratio (ν^p) by

$$\alpha^2 = \frac{9(1-2\nu_p)}{2(1+\nu_p)} \quad (3.22)$$

They further suggested a two parameter yield surface that is quadratic in p-q plane. Initial calibration of the yield surface requires knowledge of shape factor α and initial uniaxial compressive stress Y^0 . The yield function is given by

$$f(p, q, \hat{\epsilon}^p, \epsilon_v^p) = 0 \quad (\text{Eq 3.7})$$

where

$$f = \sqrt{\frac{1}{1 + \left(\frac{\alpha}{3}\right)^2} (q^2 + \alpha^2 p^2)} - \sigma_c \quad (3.23)$$

Note that $\nu^p = 0.5$ gives elliptic shape factor $\alpha = 0$ which reduces the Deshpande-Fleck yield criterion to the Von Mises yield criterion for metals

$$q - \sigma_c = 0 \quad (3.24)$$

Deshpande-Fleck have suggested two different hardening models in their paper. Differential hardening model allows the yield surface to change shape whereas self-similar hardening preserves the shape of the yield surface during plastic deformation. The differential hardening model has a yield function of the form

$$\phi = \left(\frac{q}{Q} \right)^2 + \left(\frac{p}{P} \right)^2 - 1 \quad (3.25)$$

Q and P are the yield strengths under deviatoric and hydrostatic stress respectively. Since these can evolve independently of each other, the shape of the ellipse can change while always remaining symmetric in tension and compression. Different yield strengths under tension and compression are therefore not incorporated in either of the Deshpande-Fleck models.

We have considered only the more simple case of self-similar hardening that is sufficient for most purposes and is governed by the uniaxial compressive stress and equivalent plastic strain curve.

$$\sigma_c = \bar{\sigma}_c(\hat{\varepsilon}^P) \quad (\text{Eq 3.10})$$

The flow rule is associative

$$\varepsilon_{ij}^P = \gamma \frac{\partial \phi}{\partial \sigma_{ij}} \quad (\text{Eq 3.8})$$

where

$$\phi(p, q, \hat{\varepsilon}^P, \varepsilon_v^P) = \sqrt{\frac{1}{1 + \left(\frac{\alpha}{3}\right)^2} (q^2 + \alpha^2 p^2)} - \sigma_c \quad (3.26)$$

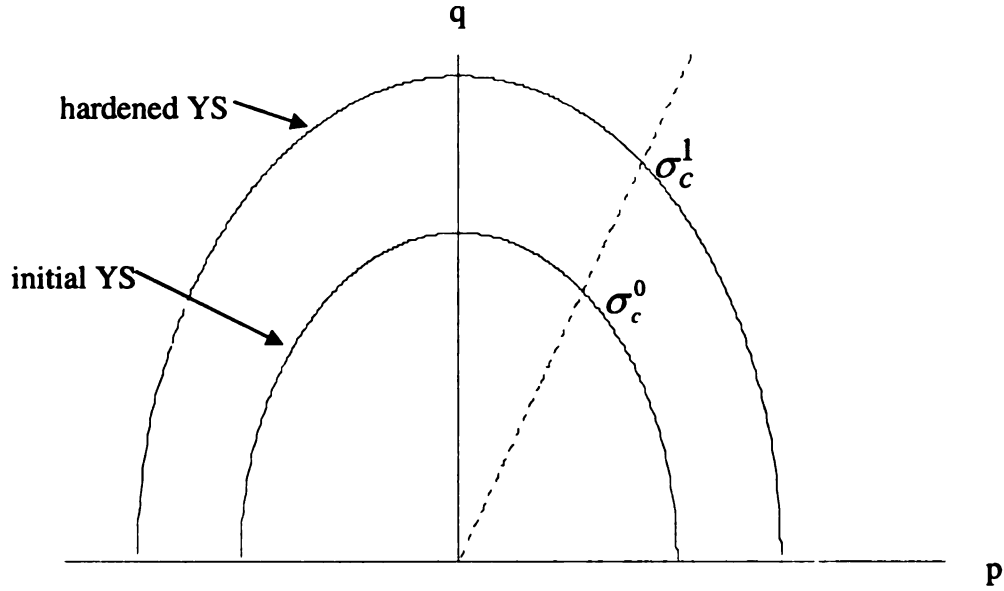


Fig 3.4 evolution of yield surface and flow potential for Deshpande-Fleck model

3.3.4 Miller model [19]

Ron Miller has proposed a yield surface to describe the plastic flow of metal foams by starting with the Drucker-Prager material model originally proposed for frictional materials like soil.

The Drucker-Prager yield surface is

$$q - \gamma p - \sigma_c = 0 \quad (3.27)$$

By adding a quadratic pressure term to the Drucker-Prager model one can include several important features of metal foam plasticity. The new model requires three parameters for

the initial calibration which can be obtained by the results of uniaxial compression and tension experiments. The yield surface is now given by

$$f(p, q, \hat{\varepsilon}^P, \varepsilon_v^P) = 0 \quad (\text{Eq 3.7})$$

where

$$f = q - \gamma p + \frac{\alpha}{d_0 \sigma_c} p^2 - d_0 \sigma_c \quad (3.28)$$

The model has a three parameter yield surface. Apart from the initial compressive uniaxial yield stress (σ_c^0), the model incorporates the ratio of the uniaxial compressive and uniaxial tensile stresses (β) and the plastic Poisson's ratio (ν_p).

The coefficients γ, α and d_0 are the functions of β and ν_p

$$\gamma = \frac{\left(6\beta^2 - 12\beta + 6 + 9 \frac{(\beta^2 - 1)}{(1 + \nu_p)} \right)}{2(\beta + 1)^2} \quad (3.29)$$

$$\alpha = \frac{45 + 24\gamma - 4\gamma^2 + 4\nu_p(2 + \nu_p)(6\gamma - \gamma^2 - 9)}{16(1 + \nu_p)^2} \quad (3.30)$$

$$d_0 = \frac{1}{2} \left(1 - \frac{\gamma}{3} + \sqrt{\left(1 - \frac{\gamma}{3} \right)^2 + \frac{4\alpha}{9}} \right) \quad (3.31)$$

The model incorporates hardening through a function that is explicitly entered as input. The uniaxial stress strain curve is used for the input data. A novelty in this model is the multiplicative decomposition of the uniaxial stress-strain curve into hardening caused by strain hardening of base material and hardening caused by cell compaction.

$$\sigma_c = \bar{\sigma}_c(\hat{\varepsilon}^P) \bar{V}(\varepsilon_v^P) \quad (3.32)$$

The advantage of this method of modeling hardening is that the large densification hardening due to cell collapse can be reached only through large values of volumetric plastic strain (ε_v^P). This is unlike the other models like Deshpande-Fleck where densification stress levels can be reached even in a case where volumetric plastic strain (ε_v^P) is low but equivalent plastic strain ($\hat{\varepsilon}^P$) is high.

The flow rule is associative

$$\dot{\varepsilon}_{ij}^P = \gamma \frac{\partial \phi}{\partial \sigma_{ij}} \quad (\text{Eq 3.8})$$

where

$$\phi(p, q, \hat{\varepsilon}^P, \varepsilon_v^P) = q - \gamma p + \frac{\alpha}{d_0 \sigma_c} p^2 - d_0 \sigma_c \quad (3.33)$$

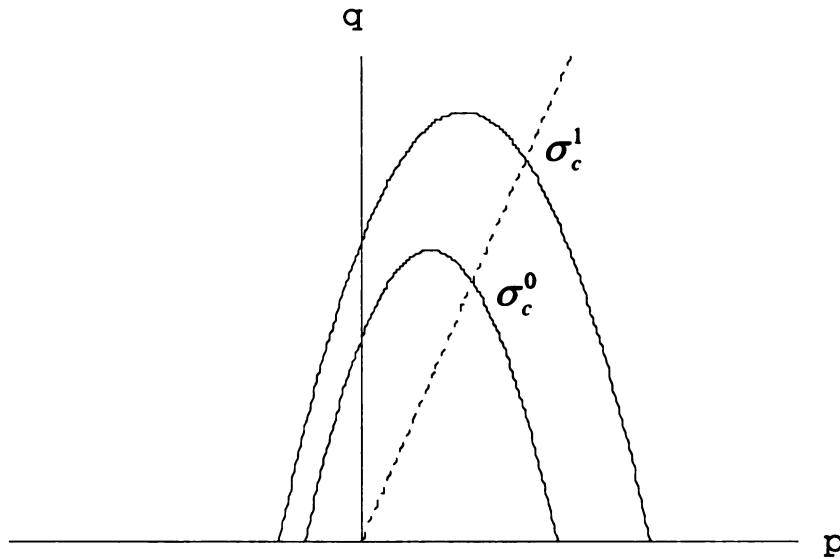


Fig 3.5 evolution of yield surface and flow potential for Miller model

3.4 Comparison of Yield Surfaces

The complexity of the yield surface depends on the number of experimental parameters that are used for calibration. Two of the four models (ABAQUS and Miller) are three parameter models while the other two (DYNA and Deshpande-Fleck) are two parameter models. Moreover the experimental parameters that these models take as input are different. For example, ABAQUS uses the values of compressive uniaxial stress (σ_c^0), compressive hydrostatic stress (p_c^0) and tensile hydrostatic stress (p_t^0). On the other hand, Miller model which is also a three parameter model uses the values of compressive uniaxial stress (σ_c^0), ratio of compressive to tensile uniaxial stress ($\beta = \frac{\sigma_c^0}{\sigma_t^0}$) and plastic Poisson's ratio (ν^P).

In order to compare the shape of the yield surface of these four models we have calibrated all of the yield surfaces for the same parameters. The values we choose are

$$\sigma_c^0 = 1180 \text{ kPa}$$

$$p_c^0 = 690 \text{ kPa}$$

$$p_t^0 = 200 \text{ kPa}$$

ABAQUS and Miller, being three parameter models, pass through all three points while DYNA and Deshpande-Fleck can accommodate only the first two.

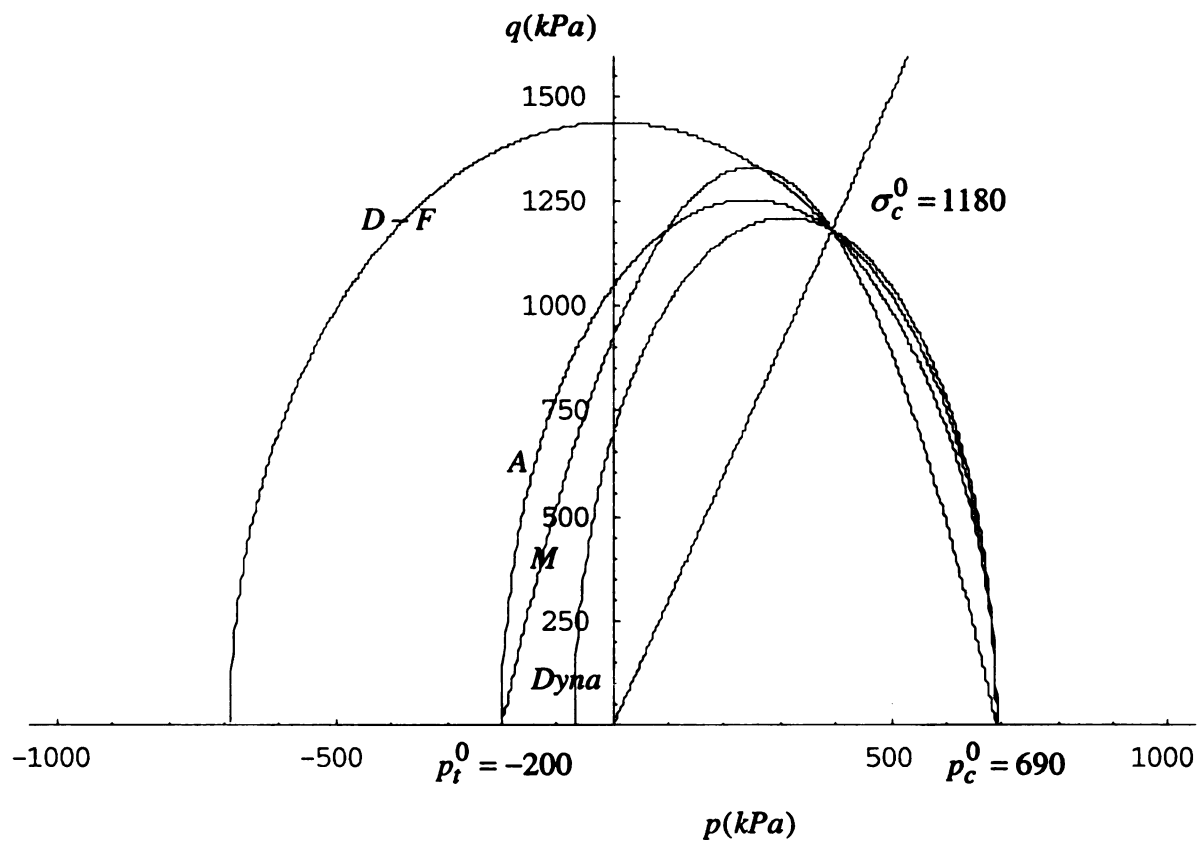


Fig 3.6 comparison of initial yield surface in p - q plane

Chapter 4

Integration of the constitutive equations

4.1 Introduction

Integration of the constitutive equations of the inelastic material models is an important requirement in the finite element analysis of mechanical structures. All four material models presented in the previous chapter fall in the category of rate independent elastoplastic models. Typically rate independent elastoplasticity models define the rate of inelastic strain in terms of stress and a set of internal state variables (Eq. 3.8). The solution of the nonlinear differential constitutive equations is typically performed incrementally and numerical integration becomes necessary. In a finite element method for solving boundary value problems the integration of the constitutive equations is performed at each quadrature point for every time increment in a strain driven framework. The mathematical problem then reduces to a local one in which the updated stresses and state variables are sought for a given strain increment and initial values of the internal variables. In problems of finite deformations, where strain increments are two or three times the yield strains, the stability of the numerical integration scheme becomes important. An unconditionally stable algorithm specific to the material models of metal foams is described in this chapter.

4.2 Constitutive equations in continuous form

As seen in the previous chapter, the yield potentials (f) and flow potentials (ϕ) of metal foams are conveniently represented in terms of the deviatoric stress (q) and hydrostatic stress (p). It is therefore instructive to write the stress (σ_{ij}) in terms of p and q .

$$\sigma_{ij} = -p\delta_{ij} + S_{ij} \quad (4.1)$$

where

S_{ij} - deviatoric stress tensor

we can then write

$$\sigma_{ij} = -p\delta_{ij} + \frac{2q}{3}n_{ij} \quad (4.2)$$

where n_{ij} is the unit tensor that gives the direction of the stress point in the deviatoric plane

$$n_{ij} = \frac{3}{2q}S_{ij} \quad (4.3)$$

The track of the plastic strain (ϵ_{ij}^P) in the material is kept by the scalar quantities $\hat{\epsilon}^P$ and ϵ_v^P as mentioned in chapter 3. These quantities along with p and q are used to define the yield rule

$$f(p, q, \hat{\epsilon}^P, \epsilon_v^P) = 0 \quad (\text{Eq 3.7})$$

and flow potential

$$\phi(p, q, \hat{\epsilon}^P, \epsilon_v^P) \quad (\text{Eq 3.8})$$

The complete description of the state of a material point therefore requires the knowledge of σ_{ij} , $\hat{\epsilon}^P$ and ϵ_v^P .

The flow rule gives the rate of plastic strain (Eq 3.8)

$$\dot{\epsilon}_{ij}^P = \gamma \frac{\partial \phi}{\partial \sigma_{ij}}$$

using Eq. (4.2) we can break it down along p and q directions

$$\begin{aligned} \dot{\epsilon}_{ij}^P &= \gamma \left(\frac{\partial \phi}{\partial p} \frac{\partial p}{\partial \sigma_{ij}} + \frac{\partial \phi}{\partial q} \frac{\partial q}{\partial \sigma_{ij}} \right) \\ \dot{\epsilon}_{ij}^P &= \gamma \left(-\frac{1}{3} \frac{\partial \phi}{\partial p} \delta_{ij} + \frac{\partial \phi}{\partial q} n_{ij} \right) \end{aligned} \quad (4.4)$$

Now we will define the equivalent plastic strain $(\hat{\epsilon}^P)$ in terms of the plastic multiplier (γ). Consider a complementary equivalent stress ($\hat{\sigma}$) corresponding to the equivalent plastic strain $(\hat{\epsilon}^P)$. Then we get

$$\begin{aligned} \hat{\sigma} \dot{\hat{\epsilon}}^P &= \sigma_{ij} \dot{\epsilon}_{ij}^P \\ &= \gamma \sigma_{ij} \frac{\partial \phi}{\partial \sigma_{ij}} \end{aligned}$$

using Eq. (4.2) and Eq. (4.4) we get

$$\begin{aligned} \hat{\sigma} \dot{\hat{\epsilon}}^P &= \gamma \left(-p \delta_{ij} + \frac{2}{3} q n_{ij} \right) \left(-\frac{1}{3} \frac{\partial \phi}{\partial p} \delta_{ij} + \frac{\partial \phi}{\partial q} n_{ij} \right) \\ &= \gamma \left(p \frac{\partial \phi}{\partial p} + \frac{2}{3} q \frac{\partial \phi}{\partial q} n_{ij} n_{ij} \right) \\ \hat{\sigma} \dot{\hat{\epsilon}}^P &= \gamma \left(p \frac{\partial \phi}{\partial p} + q \frac{\partial \phi}{\partial q} \right) \end{aligned} \quad (4.5)$$

the term $p \frac{\partial \phi}{\partial p} + q \frac{\partial \phi}{\partial q}$ has the dimensions of stress and can be considered as a measure of

the equivalent stress. Lets define

$$\hat{\sigma} = p \frac{\partial \phi}{\partial p} + q \frac{\partial \phi}{\partial q} \quad (4.6)$$

then we get

$$\dot{\hat{\epsilon}}^P = \gamma \quad (4.7)$$

Eq (3.8), Eq (4.7) and Eq (3.6) give the rate of plastic strain in terms of the plastic multiplier (γ).

Kuhn-Tucker condition is used to maintain the consistency of the yield function in the elastic region and the plastic region. The condition is given by

$$\gamma f = 0 \quad (4.8)$$

such that

$$f \leq 0 \quad (\text{Elastic region})$$

$$\gamma \geq 0 \quad (\text{Plastic region})$$

When the loading is in the elastic region, the stress is within the yield envelope ($f \leq 0$) and there is no plastic growth ($\gamma = 0$). In the plastic region, $\gamma \geq 0$ and the stress point is on the yield envelope ($f = 0$)

These equations can be taken together to state the mathematical problem. In a strain driven framework the mathematical problem of integrating the constitutive equations can be stated in continuous form as follows

Given $\dot{\epsilon}_{ij}$

find

σ_{ij} , $\hat{\epsilon}^P$ and ϵ_v^P

subject to

$$\dot{\epsilon}_{ij}^P = \gamma \frac{\partial \phi}{\partial \sigma_{ij}} \quad (\text{Eq. 3.8})$$

$$\dot{\hat{\epsilon}}^P = \gamma \quad (\text{Eq. 4.7})$$

$$\epsilon_v^P = \epsilon_{ii}^P \quad (\text{Eq. 3.6})$$

$$\gamma f = 0 \quad (\text{Eq. 4.8})$$

such That

$$\sigma_{ij} = C_{ijkl} \left(\epsilon_{kl} - \epsilon_{kl}^P \right) \quad (\text{Eq. 3.1})$$

$$\gamma \geq 0$$

$$f \leq 0$$

4.3 Time discret form of the mathematical problem

4.3.1 Closest point projection algorithm

The continuous differential equations in sec 4.2 cannot be solved in a closed form and they need to be integrated numerically in time. For ensuring the stability of the algorithms in dealing with large strains, we choose backward Euler time integration. In the following equations all the quantities, unless explicitly mentioned are calculated at time $t + \Delta t$.

The mathematical problem for the time discrete form now takes the following form

Given

$$\Delta \varepsilon_{ij}$$

find

$$\sigma_{ij}, \hat{\varepsilon}^p \text{ and } \varepsilon_v^p$$

subject to

$$\varepsilon_{ij}^p = \varepsilon_{ij}^p \Big|_t + \Delta \gamma \frac{\partial \phi}{\partial \sigma_{ij}} \quad (4.9)$$

$$\hat{\varepsilon}^p = \hat{\varepsilon}^p \Big|_t + \Delta \gamma \quad (4.10)$$

$$\varepsilon_v^p = \varepsilon_{ii}^p \quad (4.11)$$

$$\Delta \gamma f = 0 \quad (4.12)$$

such that

$$\sigma_{ij} = C_{ijkl} \left(\varepsilon_{kl} - \varepsilon_{kl}^p \right) \quad (\text{Eq 3.1})$$

$$\Delta \gamma \geq 0$$

$$f \leq 0$$

4.3.2 Predictor-Corrector algorithms

The numerical solution of the mathematical problem defined above is usually accomplished following a predictor/corrector strategy. A common consideration for the predictor is the elastic trial state defined as follows

$$\varepsilon_{ij}^{p,trial} = \varepsilon_{ij}^p \Big|_t \quad (4.13)$$

$$\hat{\varepsilon}^{p,trial} = \hat{\varepsilon}^p \Big|_t \quad (4.14)$$

$$\varepsilon_v^{p,trial} = \varepsilon_v^p \Big|_t \quad (4.15)$$

Then we have

$$\sigma_{ij}^{trial} = C_{ijkl} \left(\varepsilon_{kl} - \varepsilon_{kl}^{p,trial} \right) \quad (4.16)$$

$$f^{trial} = f(\sigma_{ij}^{trial}, \hat{\varepsilon}^{p,trial}, \varepsilon_v^{p,trial}) \quad (4.17)$$

The advantage of this elastic trial state is that we can determine if the strain increment is elastic or plastic simply by considering the trial state.

Elastic loading

If $f^{trial} \leq 0$ then $\Delta\gamma = 0$

In this case the strain increment is elastic and the trial state is the final state

$$\varepsilon_{ij}^p = \varepsilon_{ij}^{p,trial} \quad (4.13)$$

$$\hat{\varepsilon}^p = \hat{\varepsilon}^{p,trial} \quad (4.14)$$

$$\varepsilon_v^p = \varepsilon_v^{p,trial} \quad (4.15)$$

Plastic loading

If $f_{t+\Delta t}^T \geq 0$ then $\Delta\gamma \geq 0$

In this case the increment has a plastic component and the plastic corrector is required to restore the consistency of the yield surface. The mathematical problem is now of the

same form as in section 4.3.1 except that there is no need of the Kuhn-Tucker condition.

Using Eq. (4.9) – Eq. (4.11), the problem is restated as

Given

$$\Delta \varepsilon_{ij}$$

find

$$\sigma_{ij}, \hat{\varepsilon}^P \text{ and } \varepsilon_v^P$$

subject to

$$\varepsilon_{ij}^P = \varepsilon_{ij}^P \Big|_t + \Delta \gamma \frac{\partial \phi}{\partial \sigma_{ij}} \quad (\text{Eq. 4.9})$$

$$\hat{\varepsilon}^P = \hat{\varepsilon}^P \Big|_t + \Delta \gamma \quad (\text{Eq. 4.10})$$

$$\varepsilon_v^P = \varepsilon_{ii}^P \quad (\text{Eq. 4.11})$$

$$f = 0$$

such that

$$\sigma_{ij} = C_{ijkl} \left(\varepsilon_{kl} - \varepsilon_{kl}^P \right) \quad (\text{Eq 3.1})$$

4.4 Stability and complexity of the numerical algorithms

Return mapping algorithms employing the predictor/corrector operator split strategy are the most common methods for the integration of the elastoplastic constitutive equations.

Ortiz & Simo [21] have studied the accuracy and stability of the generalized trapezoidal and generalized midpoint rules for time integration of the constitutive equations. They have found that for finite deformations, where the strain increments are the order of a few

percent, backward Euler time integration provides unconditional stability. If backward Euler method is used to discretize the constitutive equations in time, the algorithm is called closest point projection algorithm. The mathematical problem statement in the previous section (4.3.2) is the general set of constitutive equations employing backward Euler method and the predictor/corrector method. This set of implicit equations can be solved using Newton-Raphson iteration at each time step. If Newton-Raphson iterations are employed for solving the backward Euler time integrated set of constitutive equations, the algorithm is referred to as Newton-Raphson closest point projection algorithm.

Newton-Raphson closest point projection is the preferred method to solve constitutive equations because of unconditional stability of the algorithm. However a complex yield surface with corners is extremely unwieldy to implement using the Newton-Raphson iterations. Yield surfaces are most conveniently represented in principal stress space. If the yield surface is a function of all the three stress invariants, they have an irregular cross-sectional shape in deviatoric plane and a general trace in the meridian plane when represented in principal stress space. R. Borja et al [8] have proposed a return mapping algorithm in the principal stress directions appropriate for three invariant plasticity models. This reduces the dimension of the stress space for the return mapping from six to three. However in describing the evolution of the plastic response the rotation of the principal stress directions must be accounted for. This introduces numerical complexity in addition to the complexity of the yield surface and flow potential.

Since metal foam models are only two invariant elastoplastic models, (the cross-sections in deviatoric plane are circular) the return mapping can be done in the p-q plane. An elegant method to do this is described in the next section.

4.5 Algorithm for two invariant plasticity models

Metal foams have a pressure dependant yield surface with circular cross-sections in the deviatoric plane. The yield surface is not dependant on the third stress invariant (I_3) and in this sense it can be considered a two invariant yield surface. As seen in chapter 3, metal foam yield surfaces (f) and flow potentials (ϕ) are conveniently represented in terms of hydrostatic pressure stress (p) and Von-Mises deviatoric stress (q). Aravas [3] has proposed an algorithm for return mapping in the p-q plane which we shall implement for metal foams.

We can decompose the plastic strain rate in the directions corresponding to p and q.

Using Eq. 4.4

$$\dot{\varepsilon}_{ij}^p = \gamma \left(-\frac{1}{3} \frac{\partial \phi}{\partial p} \delta_{ij} + \frac{\partial \phi}{\partial q} n_{ij} \right)$$

after time discretization this equation becomes

$$\Delta \varepsilon_{ij}^p = \Delta \gamma \left(-\frac{1}{3} \frac{\partial \phi}{\partial p} \delta_{ij} + \frac{\partial \phi}{\partial q} n_{ij} \right) \quad (4.16)$$

Let us define

$$-\Delta \gamma \frac{\partial \phi}{\partial p} = \Delta \varepsilon_p \quad (4.17)$$

$$\Delta \gamma \frac{\partial \phi}{\partial q} = \Delta \varepsilon_q \quad (4.18)$$

Then we have

$$\Delta \varepsilon_{ij}^p = \frac{1}{3} \Delta \varepsilon_p \delta_{ij} + \Delta \varepsilon_q n_{ij} \quad (4.19)$$

Thus the plastic strain increment is decomposed in the spectral directions of the stress tensor (δ_{ij}, n_{ij}) . This is possible because the flow rule (ϕ) is independent of the third invariant which in turn is a consequence of circular cross-sections of the yield surface in deviatoric plane.

Now consider the trial stress (Eq. 4.4)

$$\sigma_{ij}^{trial} = C_{ijkl} (\varepsilon_{kl} - \varepsilon_{kl}^{p,trial})$$

The updated stress (σ_{ij}) can be found from the trial stress and plastic strain increment

$$\sigma_{ij} = \sigma_{ij}^{trial} - C_{ijkl} \Delta \varepsilon_{kl}^p \quad (4.20)$$

use the definition of the isotropic elastic tensor

$$C_{ijkl} = 2G \delta_{ik} \delta_{jl} - \left(K - \frac{2}{3} G \right) \delta_{ij} \delta_{kl} \quad (4.21)$$

where

G- elastic shear modulus

K- elastic bulk modulus

Using Eq (4.21) and Eq (4.19) in Eq (4.20) we get

$$\sigma_{ij} = \sigma_{ij}^{trial} - K \Delta \varepsilon_p \delta_{ij} - 2G \Delta \varepsilon_q n_{ij} \quad (4.22)$$

use Eq 4.2 to decompose σ_{ij} and σ_{ij}^{trial}

$$-p \delta_{ij} + \frac{2}{3} q n_{ij} = -p^{trial} \delta_{ij} + \frac{2}{3} q^{trial} n_{ij}^{trial} - K \Delta \varepsilon_p \delta_{ij} - 2G \Delta \varepsilon_q n_{ij} \quad (4.23)$$

comparing the coefficients in the spectral directions (δ_{ij}, n_{ij}) we get

$$n_{ij}^{trial} = n_{ij} \quad (4.24)$$

This means that the stress direction in the deviatoric plane does not change during return mapping. So the return mapping can be done in the p-q plane.

using Eq. (4.23), the updated stress components (p, q) can then be written as

$$p = p^{trial} + K\Delta\epsilon_p \quad (4.25)$$

$$q = q^{trial} - 3G\Delta\epsilon_q \quad (4.26)$$

Eq 4.25 and 4.26 can replace Eq 3.1 to describe the evolution of stress in terms of two scalars $\Delta\epsilon_p$ and $\Delta\epsilon_q$

Moreover Eq 4.17 – Eq 4.19 can be used to describe the evolution of the plastic strain in terms of three scalars $(\Delta\epsilon_p, \Delta\epsilon_q, \Delta\gamma)$

The mathematical problem statement in sec 4.3.2 can therefore be now written in terms of these three scalar quantities $(\Delta\epsilon_p, \Delta\epsilon_q, \Delta\gamma)$. We have further reduced the dimension of the problem by eliminating $\Delta\gamma$ from Eq (4.17) and Eq (4.18). The constitutive equations can now be written as

Find

$$p_{t+\Delta t}, q_{t+\Delta t}$$

subject to

$$\Delta\epsilon_p \frac{\partial \phi}{\partial q} + \Delta\epsilon_q \frac{\partial \phi}{\partial p} = 0 \quad (4.27)$$

$$f(p, q, \hat{\epsilon}^p, \epsilon_v^p) = 0 \quad (\text{Eq 3.7})$$

such that

$$p = p^{trial} + K \Delta \varepsilon_p \quad (\text{Eq 4.25})$$

$$q = q^{trial} - 3G \Delta \varepsilon_q \quad (\text{Eq 4.26})$$

$$\hat{\varepsilon}^p = \hat{\varepsilon}^{p,trial} - \frac{\Delta \varepsilon_p}{\frac{\partial \phi}{\partial p}} = \hat{\varepsilon}^{p,trial} + \frac{\Delta \varepsilon_q}{\frac{\partial \phi}{\partial q}} \quad (4.28)$$

$$\varepsilon_v^p = \varepsilon_v^{p,trial} + \Delta \varepsilon_p \quad (4.29)$$

The dimension of the problem has been effectively reduced from seven to two. This greatly simplifies the numerical implementation of Newton-Raphson to solve the explicit equations.

Implementation of the N-R iterations for the above set of equations is done as follows

Let

$$\bar{X} = \begin{bmatrix} \Delta \varepsilon_p \\ \Delta \varepsilon_q \end{bmatrix} \quad (4.30)$$

$$F(\bar{X}) = \begin{bmatrix} \Delta \varepsilon_p \frac{\partial \phi}{\partial q} + \Delta \varepsilon_q \frac{\partial \phi}{\partial p} \\ f(p, q, \hat{\varepsilon}^p, \varepsilon_v^p) \end{bmatrix} \quad (4.31)$$

let

$$\bar{X}^{k+1} = \bar{X}^k + \bar{c} \quad (4.32)$$

where $\bar{X}^{k+1}$, $\bar{X}^k$ are the $(k+1)^{st}$ and $(k)^{th}$ iterations and $\bar{c}$ is defined as the correction vector.

$$\bar{c} = \begin{bmatrix} c_p \\ c_q \end{bmatrix} \quad (4.33)$$

$\bar{c}$ is the correction for $\bar{X}$ vector. By applying Newton-Raphson method to solve

$F(\bar{X}) = 0$ we get

$$\frac{\partial F(\bar{X}^k)}{\partial \bar{X}} \cdot \bar{c} = -F(\bar{X}^k) \quad (4.34)$$

we can write

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} c_p \\ c_q \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad (4.35)$$

where

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \frac{\partial F(\bar{X}^k)}{\partial \bar{X}} \quad (4.36)$$

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} -F_1^k \\ -F_2^k \end{bmatrix} \quad (4.37)$$

The derivatives of the flow rule and the yield function can be expressed as

$$A_{11} = \frac{\partial \phi}{\partial q} + \Delta \epsilon_p \frac{\partial^2 \phi}{\partial q \partial \Delta \epsilon_p} + \Delta \epsilon_q \frac{\partial^2 \phi}{\partial p \partial \Delta \epsilon_p} \quad (4.38)$$

$$A_{12} = \frac{\partial \phi}{\partial p} + \Delta \epsilon_q \frac{\partial^2 \phi}{\partial p \partial \Delta \epsilon_q} + \Delta \epsilon_p \frac{\partial^2 \phi}{\partial q \partial \Delta \epsilon_q} \quad (4.39)$$

$$A_{21} = K \frac{\partial f}{\partial p} + \frac{\partial f}{\partial \hat{\epsilon}^p} \frac{\partial \hat{\epsilon}_p}{\partial \Delta \epsilon_p} + \frac{\partial f}{\partial \epsilon_v^p} \quad (4.40)$$

$$A_{22} = -3G \frac{\partial f}{\partial q} + \frac{\partial f}{\partial \hat{\epsilon}^p} \frac{\partial \hat{\epsilon}_p}{\partial \Delta \epsilon_q} \quad (4.41)$$

where

$$\frac{\partial^2 \phi}{\partial q \partial \Delta \varepsilon_p} = \left(1 + \frac{\frac{\partial^2 \phi}{\partial q \partial \hat{\varepsilon}^p} \Delta \varepsilon_q}{\left(\frac{\partial \phi}{\partial q} \right)^2} \right)^{-1} \left(K \frac{\partial^2 \phi}{\partial q \partial p} + \frac{\partial^2 \phi}{\partial q \partial \varepsilon_v^p} \right) \quad (4.42)$$

$$\frac{\partial^2 \phi}{\partial p \partial \Delta \varepsilon_p} = \left(1 - \frac{\frac{\partial^2 \phi}{\partial p \partial \hat{\varepsilon}^p} \Delta \varepsilon_p}{\left(\frac{\partial \phi}{\partial p} \right)^2} \right)^{-1} \left(K \frac{\partial^2 \phi}{\partial p \partial p} - \frac{\frac{\partial^2 \phi}{\partial p \partial \hat{\varepsilon}^p}}{\frac{\partial \phi}{\partial p}} + \frac{\partial^2 \phi}{\partial p \partial \varepsilon_v^p} \right) \quad (4.43)$$

$$\frac{\partial^2 \phi}{\partial p \partial \Delta \varepsilon_q} = \left(1 + \frac{\frac{\partial^2 \phi}{\partial p \partial \hat{\varepsilon}^p} \Delta \varepsilon_p}{\left(\frac{\partial \phi}{\partial p} \right)^2} \right)^{-1} \left(-3G \frac{\partial^2 \phi}{\partial q \partial p} \right) \quad (4.44)$$

$$\frac{\partial^2 \phi}{\partial q \partial \Delta \varepsilon_q} = \left(1 - \frac{\frac{\partial^2 \phi}{\partial q \partial \hat{\varepsilon}^p} \Delta \varepsilon_q}{\left(\frac{\partial \phi}{\partial q} \right)^2} \right)^{-1} \left(-3G \frac{\partial^2 \phi}{\partial q \partial q} + \frac{\frac{\partial^2 \phi}{\partial q \partial \hat{\varepsilon}^p}}{\frac{\partial \phi}{\partial q}} \right) \quad (4.45)$$

$$\frac{\partial \hat{\varepsilon}^p}{\partial \Delta \varepsilon_p} = -\Delta \varepsilon_q \frac{1}{\left(\frac{\partial \phi}{\partial q} \right)^2} \frac{\partial^2 \phi}{\partial q \partial \Delta \varepsilon_p} \quad (4.46)$$

$$\frac{\partial \hat{\varepsilon}^p}{\partial \Delta \varepsilon_q} = \Delta \varepsilon_p \frac{1}{\left(\frac{\partial \phi}{\partial p} \right)^2} \frac{\partial^2 \phi}{\partial p \partial \Delta \varepsilon_q} \quad (4.47)$$

Eq (4.38) to Eq (4.47) can be calculated for the four material models discussed in chapter3 to implement the algorithm for these models.

4.6 implementation of the algorithm for selected material models

Implementing the above scheme for a particular material model requires the following derivatives.

Flow rule $\phi(p, q, \hat{\varepsilon}^p, \varepsilon_v^p)$

$$\frac{\partial \phi}{\partial p}, \frac{\partial \phi}{\partial q}$$

$$\frac{\partial^2 \phi}{\partial p \partial p}, \frac{\partial^2 \phi}{\partial p \partial q}, \frac{\partial^2 \phi}{\partial q \partial q}$$

$$\frac{\partial^2 \phi}{\partial p \partial \hat{\varepsilon}^p}, \frac{\partial^2 \phi}{\partial q \partial \hat{\varepsilon}^p}, \frac{\partial^2 \phi}{\partial p \partial \varepsilon_v^p}, \frac{\partial^2 \phi}{\partial q \partial \varepsilon_v^p}$$

Yield function $f(p, q, \hat{\varepsilon}^p, \varepsilon_v^p)$

$$\frac{\partial f}{\partial p}, \frac{\partial f}{\partial q}$$

$$\frac{\partial f}{\partial \hat{\varepsilon}^p}, \frac{\partial f}{\partial \varepsilon_v^p}$$

The derivatives of the four material models described in chapter 3 have been calculated and are presented in appendix I.

Chapter 5

Implementation of the algorithm

5.1 Introduction

The Newton-Raphson closest point projection algorithm described in the last chapter can be implemented for all the material models of metal foam. FEM software package ABAQUS has the functionality of a user defined subroutine built in. A subroutine was written in FORTRAN for implementing the algorithm (Appendix II). The subroutine can be used for all the four material models by the input of the appropriate derivatives. A numerical simulation was done in ABAQUS for diagonal crushing of a cube. The subroutine was implemented for the Deshpande-Fleck model. In this chapter we present the results of the numerical simulation.

5.2 Input data for the model

The Deshpande-Fleck model requires the experimental data of uniaxial stress and uniaxial strain as input for incorporating hardening rule. Due to lack of any experimental data an input curve has been manufactured for closed cell aluminum foam. To ensure that the manufactured data is close to the actual data, elastic modulus and the initial uniaxial yield stress were calculated using the scaling equations introduced in chapter 2. (Eq 2.3 and Eq 2.4) The cell wall material properties were taken to be that of Aluminum alloy 2014.

$$\rho = 2800 \text{ kg/m}^3$$

$$E = 70 \times 10^9 \text{ N/m}^2$$

$$\sigma_c = 260 \text{ MPa}$$

We will consider closed cell aluminum foam with 5% relative density for the simulation.

$$\frac{\rho^*}{\rho} = 0.05$$

In order to account for the reduction of observed values of the modulus and yield stress due to the presence of irregularities in the microstructure (Sec. 2.3), 30% of the equation values were taken.

We get

$$E^* = 0.35 \text{ GPa}$$

$$\sigma_c^* = 1.18 \text{ MPa}$$

For the uniaxial stress strain data we have assumed shape similar to that typical of metal foams. The plateau stress is taken as 1.25Mpa. Densification strain of 200% is taken where the material undergoes rapid strain hardening due to the collapse of cells.

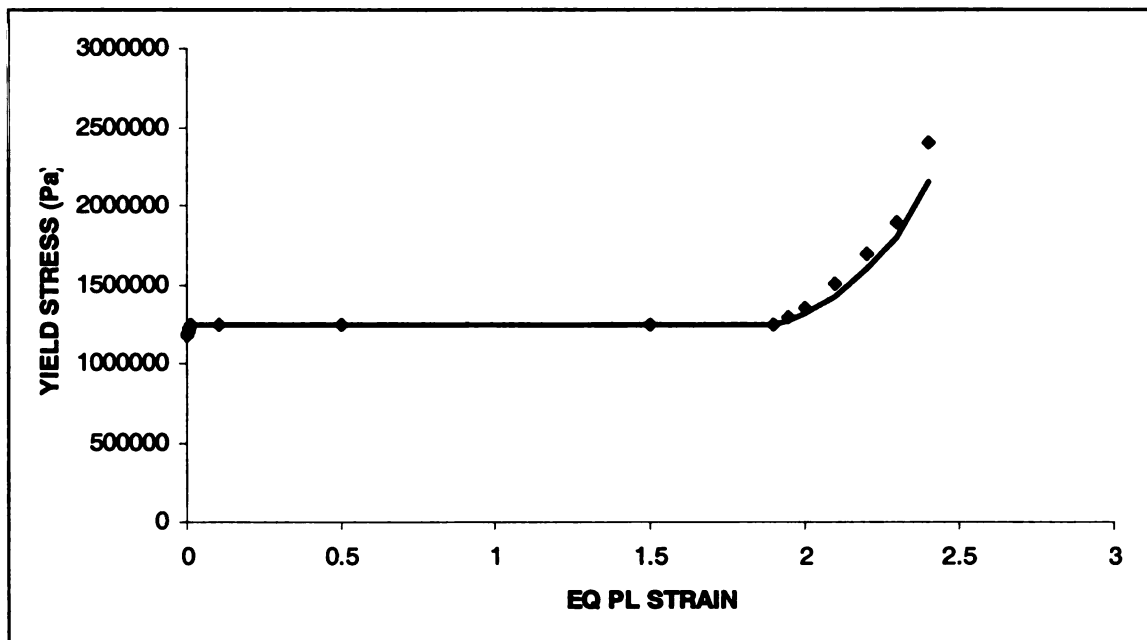


Fig 5.1 typical compressive uniaxial response of closed cell foam

5.3 Numerical simulation

Numerical simulation of the diagonal crushing of a cube has been done in ABAQUS. The motivation for selecting the diagonal crushing process is to provide biaxial stress loading as against uniaxial loading while at the same time keeping the load condition simple. At the same time the diagonal crushing process demonstrates the energy absorption of metal foam due to the reduction in volume.

Solid elements with reduced integration were used to model both the plate and the cube. Perfectly elastic steel is used for the material in the plate. Material models using Deshpande-Fleck and ABAQUS metal foam were used for defining the foam material of the cube. Simulation was also done for the case of solid aluminum cube in order to compare the energy absorption capacity of the metal foam.

Two of the bottom faces of the cube are rigidly constrained. Displacement boundary conditions are applied on the bottom face of the steel plate to come down and again go up in two steps. The displacement boundary conditions are applied in a ramp fashion to ensure that there is no discontinuity in the displacements at any time.

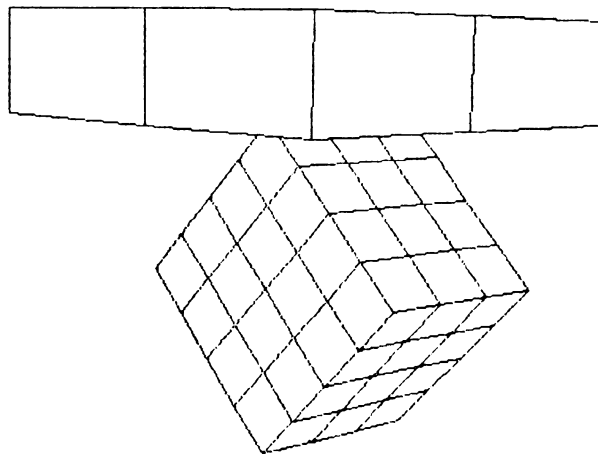
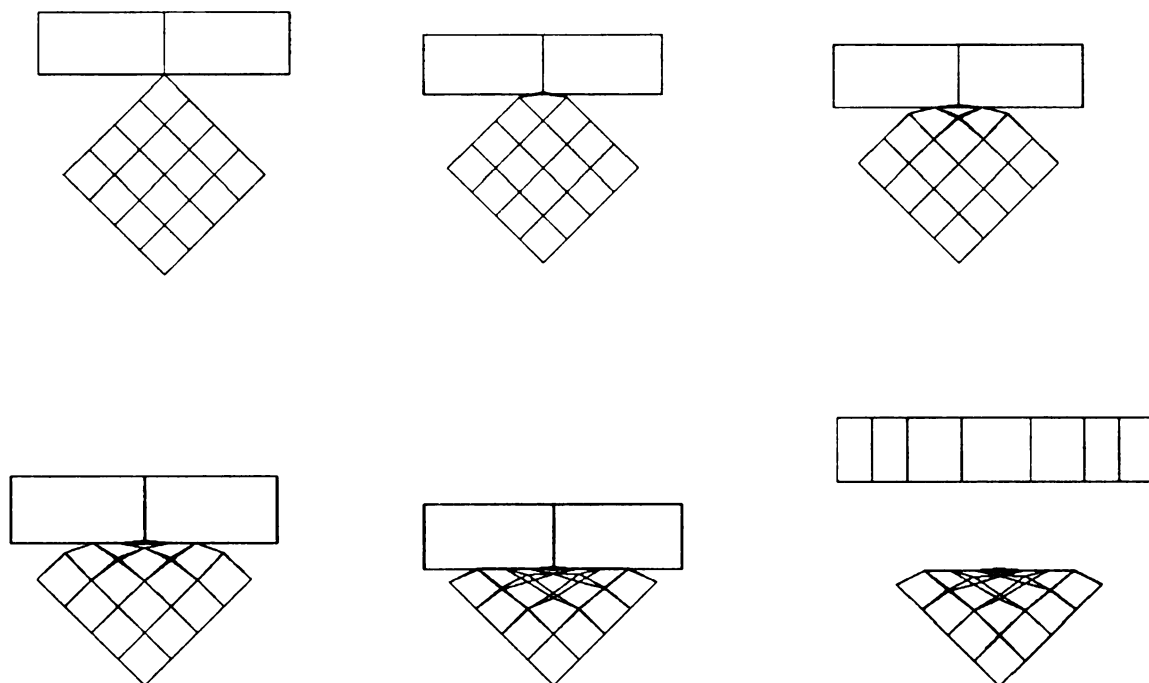
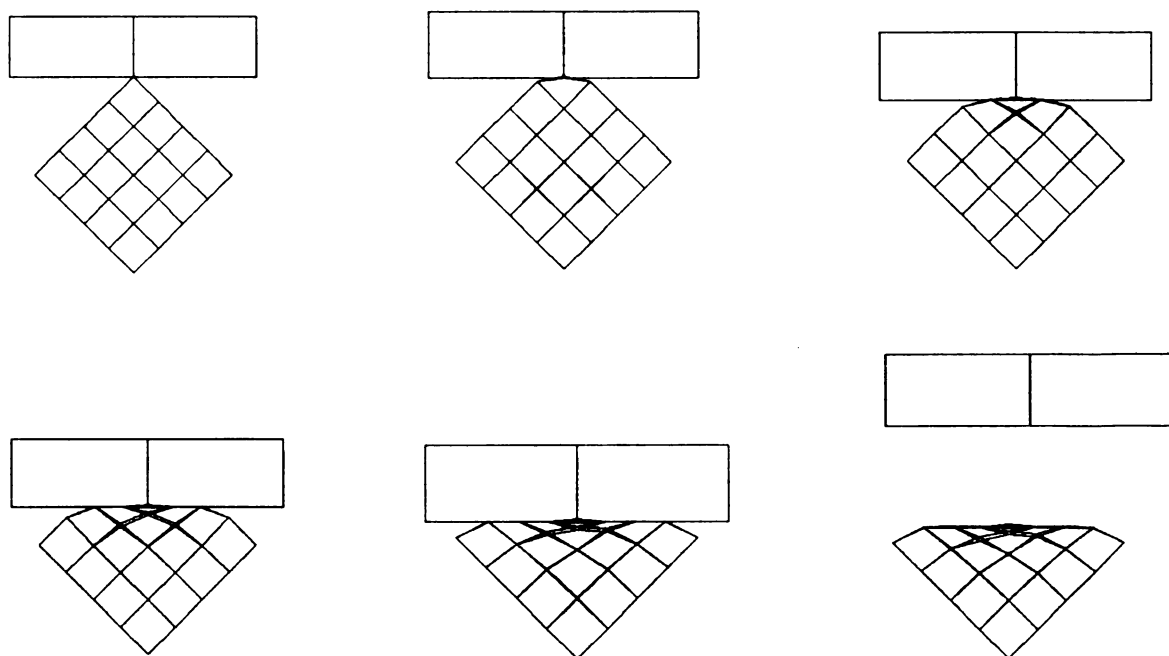


Fig 5.2 model used for simulation of diagonal crushing



**Fig 5.3 diagonal crushing of cube with user defined material model
(Deshpande-Fleck model)**



**Fig 5.4 diagonal crushing of cube with metal foam model
included with ABAQUS**

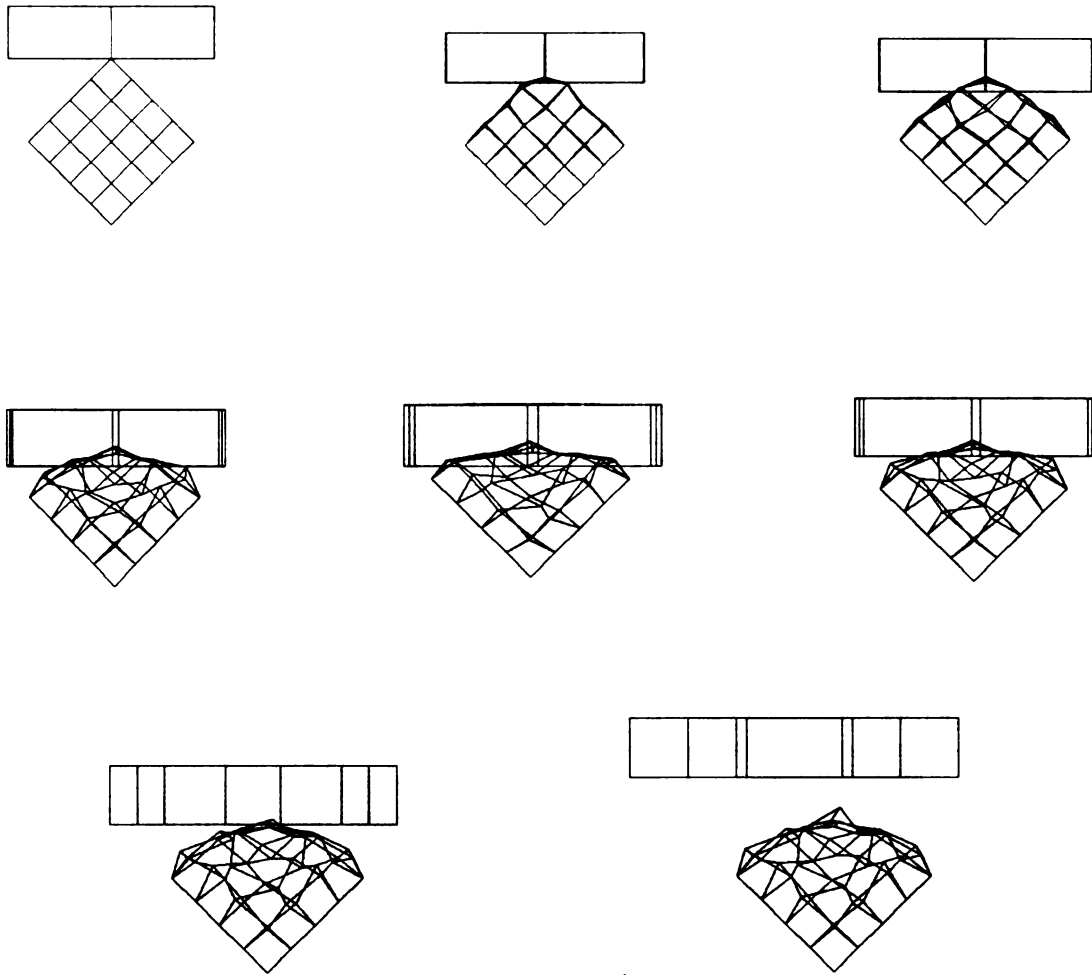


Fig 5.5 diagonal crushing of a solid aluminum cube

5.4 Comparison of results

The results of the simulation for both the user defined model (Deshpande-Fleck) and the metal foam model (ABAQUS) are quite similar to each other (Fig 5.3 and Fig 5.4). The collapse of the elements as the plate comes down can be seen clearly. This demonstrates the cell wall collapse in actual metal foams and also shows the volume reduction capacity of the metal foam. The plastic Poissons ratio is close to zero as can be seen by the negligible flow of material under the plate. When the plate is taken up in step 2 there is very little elastic springback. This is due to the small elastic region of the metal foams.

It is instructive to compare the crushing process of foam cube with an aluminum cube (Fig 5.5). The lateral expansion of the cube in this case is a result of the fact the the volume of aluminum must remain constant during plastic flow. In metal foam most of the internal strain energy (Fig 5.6) is in the form of plastic strain energy. The elastic strain energy stored in step 1 is negligible. This means that most of the energy absorbed during step 1 is used up in the plastic volume reduction and very little is stored in the form of elastic energy. On the other hand, solid aluminum stores more than 50% of its internal energy as elastic strain energy and the rest is dissipated as plastic strain energy (Fig 5.7). In step 2 all the stored elastic energy is released resulting in springback action.

Due to the different amounts of external work done in the two crushing processes, the total energy absorbed by the two materials cannot be compared quantitatively. This type of comparison is useful for making a statement about the the capacity of metal foams to absorb plastic energy during a large range of deformation.

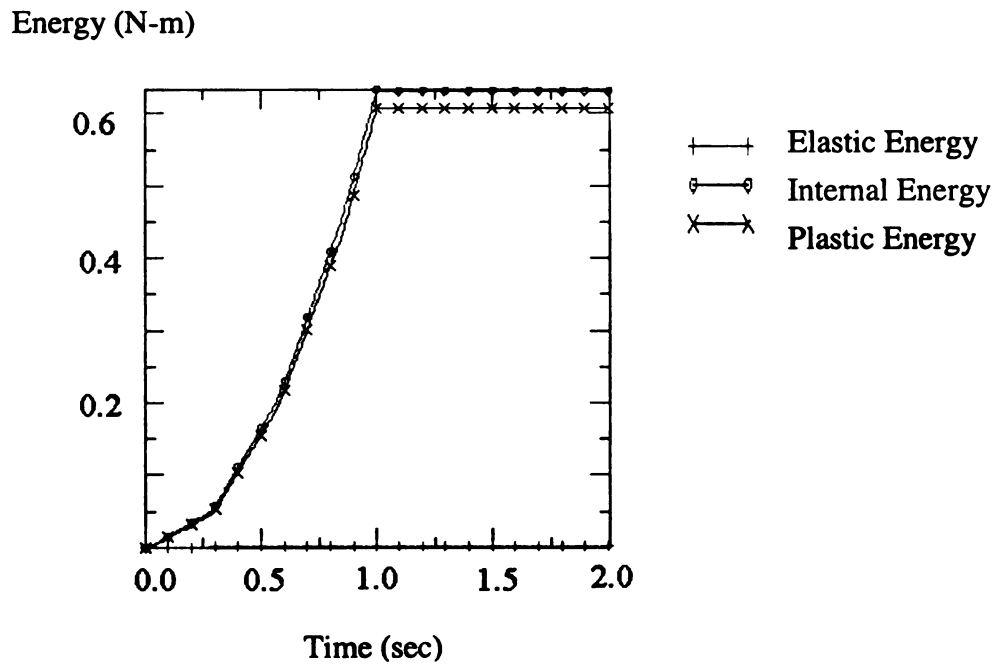


Fig 5.6 composition of internal energy in metal foam cube during step1 and 2

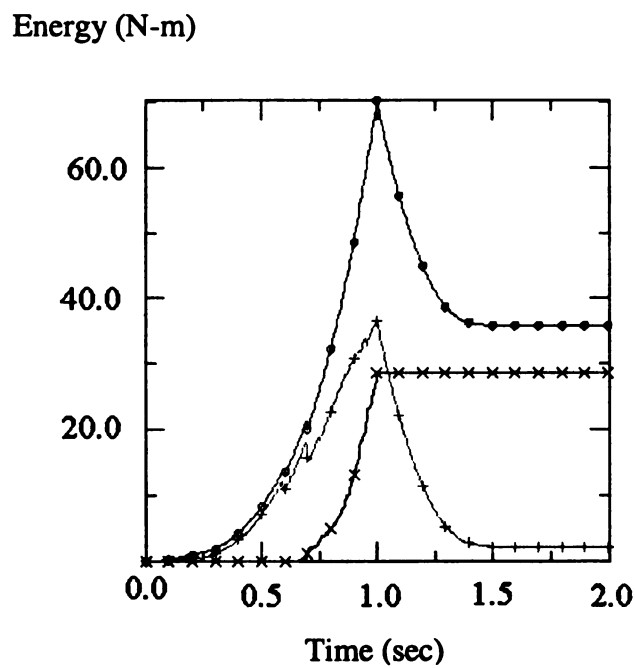


Fig 5.7 composition of internal energy in solid aluminum cube during step1 and 2

5.5 Conclusion

An elegant method of integrating the constitutive equations of two invariant constitutive models specific to metal foams is presented. Backward Euler time integration is chosen for deriving the constitutive equations of the closest point projection algorithm. This ensures the convergence of the solution even for large strain increments. Newton-Raphson iterations are used for solving the closest point projection algorithm because of their global convergence property which makes the algorithm stable.

The other advantage of using Newton-Raphson iterations is that the tangent modulus of the algorithm can be derived [3, 21]. The algorithmic tangent modulus is essential in ensuring the optimal rate of global convergence of the FEM equations. Since only quasi-static loading was used in this work, we have used explicit solution of FEM equations and so calculation of the tangent modulus is not necessary.

The present algorithm can be extended to include the derivation of the algorithmic tangent modulus. In tension, metal foams are susceptible to fracture. A further suggestion for additional work would therefore be to incorporate failure of material in the model.

APPENDIX I

Derivatives of the yield function and flow potential for all four models

LS-DYNA material model # 75

$$f = \sqrt{\left(\frac{p - \frac{1}{2}(p_c - p_t)}{a} \right)^2 + \left(\frac{q}{b} \right)^2} - 1$$

$$\phi = \sqrt{q^2 + \frac{9}{2}p^2}$$

$$\frac{\partial \phi}{\partial p} = \frac{9}{2} \frac{p}{\sqrt{q^2 + \frac{9}{2}p^2}}$$

$$\frac{\partial \phi}{\partial q} = \frac{q}{\sqrt{q^2 + \frac{9}{2}p^2}}$$

$$\frac{\partial^2 \phi}{\partial p \partial p} = \frac{9}{2} \frac{1}{\sqrt{q^2 + \frac{9}{2}p^2}} + \frac{81}{2} \frac{p^2}{\left(q^2 + \frac{9}{2}p^2 \right)^{\frac{3}{2}}}$$

$$\frac{\partial^2 \phi}{\partial q \partial q} = \frac{1}{\sqrt{q^2 + \frac{9}{2}p^2}} + \frac{q^2}{\left(q^2 + \frac{9}{2}p^2 \right)^{\frac{3}{2}}}$$

$$\frac{\partial^2 \phi}{\partial p \partial q} = \frac{9}{2} \frac{q}{\left(q^2 + \frac{9}{2}p^2 \right)^{\frac{3}{2}}} + \frac{81}{2} \frac{p^2 q}{\left(q^2 + \frac{9}{2}p^2 \right)^{\frac{5}{2}}}$$

$$\frac{\partial^2 \phi}{\partial p \partial \hat{\varepsilon}^p} = 0$$

$$\frac{\partial^2 \phi}{\partial q \partial \hat{\epsilon}^P} = 0$$

$$\frac{\partial^2 \phi}{\partial p \partial \epsilon_v^P} = 0$$

$$\frac{\partial^2 \phi}{\partial q \partial \epsilon_v^P} = 0$$

$$\frac{\partial f}{\partial p} = \left(\left(\frac{p - \frac{1}{2}(p_c - p_t)}{a} \right)^2 + \left(\frac{q}{b} \right)^2 \right)^{-\frac{1}{2}} \left(\frac{p - \frac{1}{2}(p_c - p_t)}{a^2} \right)$$

$$\frac{\partial f}{\partial q} = \left(\left(\frac{p - \frac{1}{2}(p_c - p_t)}{a} \right)^2 + \left(\frac{q}{b} \right)^2 \right)^{-\frac{1}{2}} \left(\frac{q}{b^2} \right)$$

$$\frac{\partial f}{\partial \hat{\epsilon}^P} = \frac{1}{2} \left(\left(\frac{p - \frac{1}{2}(p_c - p_t)}{a} \right)^2 + \left(\frac{q}{b} \right)^2 \right)^{-\frac{1}{2}} \left(\frac{-2q}{b^3} \frac{\partial b}{\partial \hat{\epsilon}^P} \right)$$

$$\frac{\partial f}{\partial \epsilon_v^P} = \frac{1}{2} \left(\left(\frac{p - \frac{1}{2}(p_c - p_t)}{a} \right)^2 + \left(\frac{q}{b} \right)^2 \right)^{-\frac{1}{2}} \left(\frac{-9 \left(p - \frac{9}{20} p_c \right)}{10a^2} \frac{\partial p_c}{\partial \epsilon_v^P} + \frac{-2 \left(p - \frac{9}{20} p_c \right)^2}{a^3} \frac{\partial a}{\partial \epsilon_v^P} + \frac{-2q^2}{b^3} \frac{\partial b}{\partial \epsilon_v^P} \right)$$

where

$$\frac{\partial b}{\partial \hat{\varepsilon}_p} = \frac{11 p_c}{4\sqrt{10}} \left(\frac{2 p_c^2 + 3 \sigma_c p_c}{\left(p_c^2 + 3 \sigma_c p_c - \frac{10}{9} \sigma_c^2 \right)^{\frac{3}{2}}} \right) \frac{\partial \sigma_c}{\partial \hat{\varepsilon}^p}$$

$$\frac{\partial a}{\partial \varepsilon_v^p} = \frac{11}{20} \frac{\partial p_c}{\partial \varepsilon_v^p}$$

$$\frac{\partial b}{\partial \varepsilon_v^p} = \frac{11 \sigma_c}{4\sqrt{10}} \left(\frac{3 \sigma_c p_c - \frac{20}{9} \sigma_c^2}{\left(p_c^2 + 3 \sigma_c p_c - \frac{10}{9} \sigma_c^2 \right)^{\frac{3}{2}}} \right) \frac{\partial p_c}{\partial \varepsilon_v^p}$$

ABAQUS volumetric hardening metal foam model

$$f = \sqrt{\left(\frac{p - \frac{1}{2}(p_c - p_t)}{a} \right)^2 + \left(\frac{q}{b} \right)^2} - 1$$

$$\phi = \sqrt{q^2 + \frac{9}{2}p^2}$$

$$\frac{\partial \phi}{\partial p} = \frac{9}{2} \frac{p}{\sqrt{q^2 + \frac{9}{2}p^2}}$$

$$\frac{\partial \phi}{\partial q} = \frac{q}{\sqrt{q^2 + \frac{9}{2}p^2}}$$

$$\frac{\partial^2 \phi}{\partial p \partial p} = \frac{9}{2} \frac{1}{\sqrt{q^2 + \frac{9}{2}p^2}} + \frac{81}{2} \frac{p^2}{\left(q^2 + \frac{9}{2}p^2 \right)^{\frac{3}{2}}}$$

$$\frac{\partial^2 \phi}{\partial q \partial q} = \frac{1}{\sqrt{q^2 + \frac{9}{2}p^2}} + \frac{q^2}{\left(q^2 + \frac{9}{2}p^2 \right)^{\frac{3}{2}}}$$

$$\frac{\partial^2 \phi}{\partial p \partial q} = \frac{9}{2} \frac{q}{\left(q^2 + \frac{9}{2}p^2 \right)^{\frac{3}{2}}} + \frac{81}{2} \frac{p^2 q}{\left(q^2 + \frac{9}{2}p^2 \right)^{\frac{5}{2}}}$$

$$\frac{\partial^2 \phi}{\partial p \partial \varepsilon^p} = 0$$

ABAQUS volumetric hardening metal foam model

$$f = \sqrt{\left(\frac{p - \frac{1}{2}(p_c - p_t)}{a} \right)^2 + \left(\frac{q}{b} \right)^2} - 1$$

$$\phi = \sqrt{q^2 + \frac{9}{2}p^2}$$

$$\frac{\partial \phi}{\partial p} = \frac{9}{2} \frac{p}{\sqrt{q^2 + \frac{9}{2}p^2}}$$

$$\frac{\partial \phi}{\partial q} = \frac{q}{\sqrt{q^2 + \frac{9}{2}p^2}}$$

$$\frac{\partial^2 \phi}{\partial p \partial p} = \frac{9}{2} \frac{1}{\sqrt{q^2 + \frac{9}{2}p^2}} + \frac{81}{2} \frac{p^2}{\left(q^2 + \frac{9}{2}p^2 \right)^{\frac{3}{2}}}$$

$$\frac{\partial^2 \phi}{\partial q \partial q} = \frac{1}{\sqrt{q^2 + \frac{9}{2}p^2}} + \frac{q^2}{\left(q^2 + \frac{9}{2}p^2 \right)^{\frac{3}{2}}}$$

$$\frac{\partial^2 \phi}{\partial p \partial q} = \frac{9}{2} \frac{q}{\left(q^2 + \frac{9}{2}p^2 \right)^{\frac{3}{2}}} + \frac{81}{2} \frac{p^2 q}{\left(q^2 + \frac{9}{2}p^2 \right)^{\frac{5}{2}}}$$

$$\frac{\partial^2 \phi}{\partial p \partial \varepsilon^p} = 0$$

$$\frac{\partial^2 \phi}{\partial q \partial \hat{\varepsilon}^P} = 0$$

$$\frac{\partial^2 \phi}{\partial p \partial \varepsilon_v^P} = 0$$

$$\frac{\partial^2 \phi}{\partial q \partial \varepsilon_v^P} = 0$$

$$\frac{\partial f}{\partial p} = \left(\left(\frac{p - \frac{1}{2}(p_c - p_t)}{a} \right)^2 + \left(\frac{q}{b} \right)^2 \right)^{-\frac{1}{2}} \left(\frac{p - \frac{1}{2}(p_c - p_t)}{a^2} \right)$$

$$\frac{\partial f}{\partial q} = \left(\left(\frac{p - \frac{1}{2}(p_c - p_t)}{a} \right)^2 + \left(\frac{q}{b} \right)^2 \right)^{-\frac{1}{2}} \left(\frac{q}{b^2} \right)$$

$$\frac{\partial f}{\partial \hat{\varepsilon}^P} = 0$$

$$\frac{\partial f}{\partial \varepsilon_v^P} = \frac{1}{2} \left(\left(\frac{p - \frac{1}{2}(p_c - p_t)}{a} \right)^2 + \left(\frac{q}{b} \right)^2 \right)^{-\frac{1}{2}} \left(\frac{-\left(p - \frac{1}{2}(p_c - p_t) \right)}{a^2} + \frac{-\left(p - \frac{1}{2}(p_c - p_t) \right)^2}{a^3} + \frac{-q^2}{b^3} \right) \frac{\partial p_c}{\partial \varepsilon_v^P}$$

$$\phi = \frac{1}{\sqrt{1 + \left(\frac{\alpha}{3}\right)^2}} \sqrt{q^2 + \alpha^2 p^2} - \sigma_c$$

$$f = \frac{1}{\sqrt{1 + \left(\frac{\alpha}{3}\right)^2}} \sqrt{q^2 + \alpha^2 p^2} - \sigma_c$$

$$\frac{\partial \phi}{\partial p} = \frac{1}{\sqrt{1 + \left(\frac{\alpha}{3}\right)^2}} \frac{\alpha^2 p}{\sqrt{q^2 + \alpha^2 p^2}}$$

$$\frac{\partial \phi}{\partial q} = \frac{1}{\sqrt{1 + \left(\frac{\alpha}{3}\right)^2}} \frac{q}{\sqrt{q^2 + \alpha^2 p^2}}$$

$$\frac{\partial^2 \phi}{\partial p \partial p} = \frac{1}{\sqrt{1 + \left(\frac{\alpha}{3}\right)^2}} \left(\alpha^2 \sqrt{q^2 + \alpha^2 p^2} - \frac{\alpha^4 p^2}{\sqrt{q^2 + \alpha^2 p^2}} \right) \frac{1}{\sqrt{q^2 + \alpha^2 p^2}}$$

$$\frac{\partial^2 \phi}{\partial p \partial q} = \frac{-1}{\sqrt{1 + \left(\frac{\alpha}{3}\right)^2}} \frac{\alpha^2 p q}{\left(q^2 + \alpha^2 p^2\right)^{\frac{3}{2}}}$$

$$\frac{\partial^2 \phi}{\partial q \partial q} = \frac{1}{\sqrt{1 + \left(\frac{\alpha}{3}\right)^2}} \left(\sqrt{q^2 + \alpha^2 p^2} - \frac{q^2}{\sqrt{q^2 + \alpha^2 p^2}} \right) \frac{1}{\sqrt{q^2 + \alpha^2 p^2}}$$

$$\frac{\partial^2 \phi}{\partial p \partial \hat{\varepsilon}^P} = 0$$

$$\frac{\partial^2 \phi}{\partial q \partial \hat{\varepsilon}^P} = 0$$

$$\frac{\partial^2 \phi}{\partial p \partial \varepsilon_v^P} = 0$$

$$\frac{\partial^2 \phi}{\partial q \partial \varepsilon_v^P} = 0$$

$$\frac{\partial f}{\partial p} = \frac{1}{\sqrt{1 + \left(\frac{\alpha}{3}\right)^2}} \frac{\alpha^2 p}{\sqrt{q^2 + \alpha^2 p^2}}$$

$$\frac{\partial f}{\partial q} = \frac{1}{\sqrt{1 + \left(\frac{\alpha}{3}\right)^2}} \frac{q}{\sqrt{q^2 + \alpha^2 p^2}}$$

$$\frac{\partial f}{\partial \hat{\varepsilon}^P} = -\frac{\partial \sigma_c}{\partial \hat{\varepsilon}_p}$$

$$\frac{\partial f}{\partial \varepsilon_v^P} = 0$$

Miller model [18]

$$\phi = q - \gamma p + \frac{\alpha}{d_0 \sigma_c} p^2 - d_0 \sigma_c$$

$$f = q - \gamma p + \frac{\alpha}{d_0 \sigma_c} p^2 - d_0 \sigma_c$$

$$\frac{\partial \phi}{\partial p} = -\gamma + \frac{2\alpha}{d_0 \sigma_c} p$$

$$\frac{\partial \phi}{\partial q} = 1$$

$$\frac{\partial^2 \phi}{\partial p \partial p} = \frac{2\alpha}{d_0 \sigma_c}$$

$$\frac{\partial^2 \phi}{\partial p \partial q} = 0$$

$$\frac{\partial^2 \phi}{\partial q \partial q} = 0$$

$$\frac{\partial^2 \phi}{\partial p \partial \hat{\varepsilon}^p} = -\frac{2\alpha p}{d_0 \sigma_c^2} \frac{\partial \sigma_c}{\partial \hat{\varepsilon}^p}$$

$$\frac{\partial^2 \phi}{\partial q \partial \hat{\varepsilon}^p} = 0$$

$$\frac{\partial^2 \phi}{\partial p \partial \varepsilon_v^p} = -\frac{2\alpha p}{d_0 \sigma_c^2} \frac{\partial \sigma_c}{\partial \varepsilon_v^p}$$

$$\frac{\partial^2 \phi}{\partial q \partial \varepsilon_v^p} = 0$$

$$\frac{\partial f}{\partial p} = -\gamma + \frac{2\alpha}{d_0 \sigma_c} p$$

$$\frac{\partial f}{\partial q}=1$$

$$\frac{\partial f}{\partial \hat{\epsilon}^p}=-\left(\frac{\alpha p^2}{d_0\sigma_c^2}+d_0\right)\frac{\partial \sigma_c}{\partial \hat{\epsilon}^p}$$

$$\frac{\partial f}{\partial \epsilon_v^p}=-\left(\frac{\alpha p^2}{d_0\sigma_c^2}+d_0\right)\frac{\partial \sigma_c}{\partial \epsilon_v^p}$$

APPENDIX II

Fortran Subroutines for implementing algorithm in ABAQUS

Subroutine for user material in FORTRAN for Deshpande-Fleck model

```
subroutine vumat(
c Read only (unmodifiable)variables -
  1 nblock, ndir, nshr, nstatev, nfieldv, nprops, lanneal,
  2 stepTime, totalTime, dt, cmname, coordMp, charLength,
  3 props, density, strainInc, relSpinInc,
  4 tempOld, stretchOld, defgradOld, fieldOld,
  5 stressOld, stateOld, enerInternOld, enerInelasOld,
  6 tempNew, stretchNew, defgradNew, fieldNew,
c Write only (modifiable) variables -
  7 stressNew, stateNew, enerInternNew, enerInelasNew )
c
  include 'vaba_param.inc'
  character*80 cmname
c
  dimension props(nprops), density(nblock), coordMp(nblock,*),
  1 charLength(nblock), strainInc(nblock,ndir+nshr),
  2 relSpinInc(nblock,nshr), tempOld(nblock),
  3 stretchOld(nblock,ndir+nshr),
  4 defgradOld(nblock,ndir+nshr+nshr),
  5 fieldOld(nblock,nfieldv), stressOld(nblock,ndir+nshr),
  6 stateOld(nblock,nstatev), enerInternOld(nblock),
  7 enerInelasOld(nblock), tempNew(nblock),
  8 stretchNew(nblock,ndir+nshr),
  8 defgradNew(nblock,ndir+nshr+nshr),
  9 fieldNew(nblock,nfieldv),
  1 stressNew(nblock,ndir+nshr), stateNew(nblock,nstatev),
  2 enerInternNew(nblock), enerInelasNew(nblock)
c
c-----Local Variables
  dimension
  1 xx(50),yy(50),
  2 C(6,6),
  3 strainI(6,1),stressInc(6,1),
  4 stresstrial(6,1),devstress(6,1),xntrial(6,1),
  5 Amat(2,2),cor(2,1)
c
  real
```

```

1 lamda,mu,
2 a,fracturestress,fractureenergy,
3 kmod,gmod,
4 alphatrial,xptrial,xqtrial,
5 deltaep,deltaeq,
6 xp,xq,alpha,
7 R1,R2,norm1,norm2,
8 Y,H,deltaalpha,
9 dfdp,dfdq,dfdpdp,dfdqpq,dfdpdq,dadep,dadeq
c
  integer
  1 npoints,mpoints,kstart,
  2 i,j,k
  3 nloop
c
c   OPEN(UNIT=17,FILE='/egr/research/helmholtz/abhishek/debug.dat')
c-----input properties
  i=1
  do 5 j=9,nprops,8
    xx(i)=props(j)
    yy(i)=props(j+1)
    i=i+1
  5 continue
  npoints = i-1
  mpoints = 4
  kstart = 1
c
  lamda = props(1)
  mu = props(2)
  a = props(3)
  fracturestress = props(4)
  fractureenergy = props(5)
c
  do 11 i = 1,6
    do 10 j = 1,6
  10 C(i,j) = 0
  11 continue
    do 16 i = 1,3
      do 15 j = 1,3
  15 C(i,j) = lamda
  16 continue
    C(1,1) = C(1,1) + 2*mu
    C(2,2) = C(2,2) + 2*mu
    C(3,3) = C(3,3) + 2*mu
    C(4,4) = C(4,4) + 2*mu
    C(5,5) = C(5,5) + 2*mu

```

```

C(6,6) = C(6,6) + 2*mu
c
kmod = (3*lamda + 2*mu)/3
gmod = mu
c
c-----loop over nblock material points-----
do 100 i = 1,nblock
c
  alphasrial = stateOld(i,1)
  do 20 j=1,6
    strainI(j,1)=strainInc(i,j)
  20 continue
  call matmul(C,strainI,stressInc,6,1,6)
  do 25 j=1,6
    stresstrial(j,1)=stressOld(i,j)+stressInc(j,1)
  25 continue
c
  if (stepTime.eq.0.0) then
    stateNew(i,1) = stateOld(i,1)
    do 27 j = 1,6
      stressNew(i,j) = stresstrial(j,1)
    27 continue
    go to 100
  endif
c
  xptrial = -(1.0/3.0)*(stresstrial(1,1)+stresstrial(2,1)+
1stresstrial(3,1))
  devstress(1,1) = stresstrial(1,1) + xptrial
  devstress(2,1) = stresstrial(2,1) + xptrial
  devstress(3,1) = stresstrial(3,1) + xptrial
  devstress(4,1) = stresstrial(4,1)
  devstress(5,1) = stresstrial(5,1)
  devstress(6,1) = stresstrial(6,1)
  xqtrial = sqrt(1.5)*sqrt(devstress(1,1)**2+devstress(2,1)**2
1+ devstress(3,1)**2+2*devstress(4,1)**2
1+ 2*devstress(5,1)**2 + 2*devstress(6,1)**2)
  if (xqtrial.eq.0.0) then
    do 29 j =1,6
      xntrial(j,1) = 0.0
    29 continue
  else
    do 30 j =1,6
      xntrial(j,1) = 3.0/(2.0*xqtrial)*devstress(j,1)
    30continue
  endif
c

```

```

deltaep = 0
deltaeq = 0
xp = xptrial
xq = xqtrial
alpha = alphaserial
c
R1 = 0
if (alpha.eq.0.0) then
  Y=yy(1)
  H=(yy(2)-yy(1))/xx(2)
else
  call startint(xx,yy,npoints,mpoints,
  lkstart,alpha)
  call polint(xx(kstart),yy(kstart),
  lmpoints,alpha,Y,H)
endif
R2 = sqrt(1/(1+a**2/9))*sqrt(xq**2+a**2*xp**2) - Y
norm1 = sqrt(R1**2)
norm2 = sqrt(R2**2)
c
if (R2.LE.0.0) go to 35
c
nloop = 0
do while (norm1.GT.1.OR.norm2.GT.100)
  nloop = nloop + 1
  dfdp = sqrt(1/(1+a**2/9))*(a**2*xp)/sqrt(xq**2+a**2*xp**2)
  dfdq = sqrt(1/(1+a**2/9))*(xq)/sqrt(xq**2+a**2*xp**2)
  dfdpdp = sqrt(1/(1+a**2/9))*(a**2*sqrt(xq**2+a**2*xp**2)-
  1a**4*xp**2/sqrt(xq**2+a**2*xp**2))*(1/(xq**2+a**2*xp**2))
  dfdqdq = sqrt(1/(1+a**2/9))*(sqrt(xq**2+a**2*xp**2) -
  1xq**2/sqrt(xq**2+a**2*xp**2))*(1/(xq**2+a**2*xp**2))
  dfdpdq = -sqrt(1/(1+a**2/9))*(a**2*xp*xq/
  1sqrt(xq**2+a**2*xp**2))*(1/(xq**2+a**2*xp**2))
  dadep = -1/dfdp + deltaep*(1/dfdp**2)*kmod*dfdpdp
  dadeq = -deltaep*(1/dfdp**2)*3*gmod*dfdpdq
c
  Amat(1,1) = dfdq+deltaep*kmod*dfdpdq+deltaeq*kmod*dfdpdp
  Amat(1,2) = dfdp-deltaeq*3*gmod*dfdpdq-deltaeq*3*gmod*dfdqdq
  Amat(2,1) = kmod*dfdp - H*dadep
  Amat(2,2) = -3*gmod*dfdq - H*dadeq
  cor(1,1) = -R1
  cor(2,1) = -R2
  call gaussj(Amat,2,2,cor,1,1)
c
  deltaep = deltaep + cor(1,1)
  deltaeq = deltaeq + cor(2,1)

```



```

c
xp = xptrial + kmod*deltaep
xq = xqtrial - 3*gmod*deltaeq
dfdp = sqrt(1/(1+a**2/9))*(a**2*xp)/sqrt(xq**2+a**2*xp**2)
dfdq = sqrt(1/(1+a**2/9))*(xq)/sqrt(xq**2+a**2*xp**2)
alpha = alphatrial - deltaep/dfdp
c
R1 = deltaep*dfdq + deltaeq*dfdp
if (alpha.LT.alphatrial) then
  alpha = alphatrial
  deltaalpha = 0
else
  deltaalpha = -deltaep/dfdp
endif
R2=sqrt(1/(1+a**2/9))*sqrt(xq**2+a**2*xp**2)-(Y+H*deltaalpha)
norm1 = sqrt(R1**2)
norm2 = sqrt(R2**2)
if (nloop.GT.100) pause 'divergent N-R'
enddo
c
35 stateNew(i,1) = alpha
  do 40 j = 1,3
    stressNew(i,j) = -xp + (2.0/3.0)*xq*xntrial(j,1)
40 continue
  do 45 j = 4,6
    stressNew(i,j) = (2.0/3.0)*xq*xntrial(j,1)
45 continue
c
c
100 continue

101 return
end

```

Subroutine for multiplying matrices

```
subroutine matmul(A, B, res, p, q, r)
integer p,q,r
dimension A(p,r),B(r,q),res(p,q)
c
do 105 i = 1,p
do 110 j = 1,q
res(i,j) = 0
do 115 k = 1,r
res(i,j) = res(i,j) + A(i,k)*B(k,j)
115 continue
110 continue
105 continue
return
end
```

Subroutine for inverting a matrix

```
subroutine gaussj(a,n,np,b,m,mp)
integer m,mp,n,np,NMAX
real a(np,np),b(np,mp)
parameter (NMAX = 50)
c
integer i,j,k,l,ll,irow,icol,indxc(NMAX),indxr(NMAX),ipiv(NMAX)
real big,dum,pivinv
c
do 211 j = 1,n
ipiv(j) = 0
211 continue
c
do 222 i = 1,n
c
big = 0
do 213 j = 1,n
if (ipiv(j).ne.1) then
do 212 k = 1,n
if (ipiv(k).eq.0) then
if (abs(a(j,k)).ge.big) then
big = abs(a(j,k))
irow = j
icol = k
endif
endif
212 continue
endif
213 continue
ipiv(icol) = ipiv(icol) + 1
c
if (irow.ne.icol) then
do 214 l = 1,n
dum = a(irow,l)
a(irow,l) = a(icol,l)
a(icol,l) = dum
214 continue
do 215 l = 1,m
dum = b(irow,l)
b(irow,l) = b(icol,l)
b(icol,l) = dum
215 continue
endif
c
```

```

indxr(i) = irow
indxc(i) = icol
c
  if (a(icol,icol).eq.0) pause 'singular matrix'
  pivinv = 1/a(icol,icol)
  a(icol,icol) = 1
  do 216 l = 1,n
    a(icol,l) = a(icol,l)*pivinv
216 continue
  do 217 l = 1,m
    b(icol,l) = b(icol,l)*pivinv
217 continue
c
  do 221 ll = 1,n
    if (ll.ne.icol) then
      dum = a(ll,icol)
      a(ll,icol) = 0
      do 218 l = 1,n
        a(ll,l) = a(ll,l) - a(icol,l)*dum
218 continue
      do 219 l = 1,m
        b(ll,l) = b(ll,l) - b(icol,l)*dum
219 continue
      endif
221 continue
c
  222 continue
c
  do 224 l = n,1,-1
    if (indxr(l).ne.indxc(l)) then
      do 223 k = 1,n
        dum = a(k,indxr(l))
        a(k,indxr(l))= dum
223 continue
      endif
224 continue
c
  return
end

```

Subroutine for interpolating the hardness data

```
subroutine polint(xa,ya,n,x,y,ydash)
integer n,NMAX
real x,y,xa(n),ya(n),ydash
parameter (NMAX=15)
integer i,m,ns
real dy,den,dif,dift,ho,hp,w,c(NMAX),d(NMAX)
real p(NMAX),pdash(NMAX)
ns=1
dif=abs(x-xa(1))
do 231 i=1,n
  dift=abs(x-xa(i))
  if (dift.lt.dif) then
    ns=i
    dif=dift
  endif
  c(i)=ya(i)
  d(i)=ya(i)
  p(i)=ya(i)
  pdash(i)=0
231 continue
y=ya(ns)
ns=ns-1
do 233 m=1,n-1
  do 232 i=1,n-m
    ho=xa(i)-x
    hp=xa(i+m)-x
    w=c(i+1)-d(i)
    den=ho-hp
    if(den.eq.0)pause 'failure in polint'
    den=w/den
    d(i)=hp*den
    c(i)=ho*den
    pdash(i)=((x-xa(i+m))*pdash(i)+(xa(i)-x)*pdash(i+1))
    1/(xa(i)-xa(i+m))+(p(i)-p(i+1))/(xa(i)-xa(i+m))
    p(i)=((x-xa(i+m))*p(i)+(xa(i)-x)*p(i+1))
    1/(xa(i)-xa(i+m))
  232 continue
  if (2*ns.lt.n-m)then
    dy=c(ns+1)
  else
    dy=d(ns)
    ns=ns-1
  endif
endif
```

```

    y=y+dy
233 continue
    ydash=pdash(1)
    return
end
c
c-----subroutine for finding start of table
    subroutine startint(xa,ya,n,m,kstart,x)
    integer n,m,kstart
    real x,xa(n),ya(n)
    integer i,ns
    real dif,dift
    ns=1
    dif=abs(x-xa(1))
    do 241 i=1,n
        dift=abs(x-xa(i))
        if (dift.lt.dif) then
            ns=i
            dif=dift
        endif
    241 continue
    kstart=min(max(ns-(m-1)/2,1),n+1-m)
    return
end

```

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