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AXISYMMETRIC PROBLEMS IN NONLINEAR ELASTICITY: EXISTENCE AND GLOBAL INJECTIVITY OF ENERGY MINIMIZERS AND NEW CLASSES OF EXACT SOLUTIONS

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ABSTRACT

AXISYMMETRIC PROBLEMS IN NONLINEAR ELASTICITY: EXISTENCE AND GLOBAL INJECTIVITY OF ENERGY MINIMIZERS AND NEW CLASSES OF EXACT SOLUTIONS

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Axisymmetric problems in nonlinear elasticity are investigated from two different perspectives. In the first part the existence theory for axisymmetric minimizers in Sobolev spaces, based on the approach suggested in the seminal paper by J. Ball (1977) and more recent results, is developed, and new classes of hyperelastic materials are included into the existence analysis. Under suitable assumptions, higher regularity properties and topological properties of openness and discreteness of the radial and axial components of the mappings are established. Global injectivity of axisymmetric minimizers is investigated, and stronger injectivity results are obtained compared with those known for full three-dimensional case. In the second part some classes of specialized three-dimensional axisymmetric motions in a neo-Hookean material under an internal constraint of incompressibility are examined. The original governing system of equations is found to reduce to a simpler unconstrained system of PDEs allowing for finding analytical solutions corresponding to various specialized motion classes. In certain particular cases these closed form solutions reduce to previously known results. A formal action functional whose Euler-Lagrange equations are given by the reduced system is also found.

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0.1 Notation

$\Omega \subset \mathbb{R}^3$: open and bounded domain occupied by a continuous material body in its reference (material, undeformed) configuration $\mathcal{B}_0$.

$\partial\Omega$: boundary of Ω that is assumed to be *strongly Lipschitz* ([39], Definition 3.4.1).

$|G|$: m-dimensional Lebesgue measure of m-dimensional set $G \subset \mathbb{R}^n$, $m \leq n$.

$B(a, R) \subset \mathbb{R}^n$: ball of radius R centered at the point a.

Function (deformation) $\mathbf{u} : \Omega \to \mathbb{R}^3$, $\mathbf{u} = (u^1, u^2, u^3)$, maps a material point $\mathbf{X} \in \Omega$ into corresponding point $\mathbf{x} = \mathbf{u}(\mathbf{X}) \in \mathbf{u}(\Omega)$ in the deformed configuration $\mathcal{B} = \mathbf{u}(\Omega)$.

$\mathbb{M}^{3\times 3}$: set of all 3×3 real matrices endowed with the usual Euclidian norm

$$|\mathbf{A}| = (A : A)^{1/2}.$$

$\mathbb{M}^{3\times 3}_{+}$: subset of matrices $\mathbf{A} \in \mathbb{M}^{3\times 3}$ such that $\det \mathbf{A} > 0$.

$\mathbf{F}(\mathbf{u}) := \nabla \mathbf{u} : \Omega \to \mathbb{M}^{3\times 3}_{+}$: differential (deformation gradient) of $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^3)$. In Cartesian coordinates it is represented by the matrix of partial derivatives of the components of $\mathbf{u}$

$$\mathbf{F}(\mathbf{u}) = (F_{ij}) = \left(\partial u^i / \partial X_j\right).$$

$\operatorname{cof} \mathbf{F} : \Omega \to \mathbb{M}^{3\times 3}$: matrix of cofactors of the deformation gradient. The adjugate matrix is the transpose of the matrix of cofactors, $\operatorname{adj} \mathbf{F} = \operatorname{cof} \mathbf{F}^T$.

$\mathbf{C}(\mathbf{u}) = \mathbf{F}^T(\mathbf{u})\mathbf{F}(\mathbf{u})$: right Cauchy-Green deformation tensor. Positive square roots of its eigenvalues are called *singular values* (principal stretches) of the deformation gradient $\mathbf{F}(\mathbf{u})$.

$I_1,\ I_2,\ I_3$: principal invariants of $\mathbf{C}$,

$$I_1 = \operatorname{tr}\mathbf{C} = |\mathbf{F}|^2, \qquad I_2 = \operatorname{tr}\operatorname{cof}\ \mathbf{C}, \qquad I_3 = \det\mathbf{C}.$$

$\operatorname{Div} D$: divergence operator on Ω.

$C(\Omega)$: space of continuous functions in Ω.

$\mathcal{D}(\Omega)$: space of C^∞ functions having compact support in Ω with the standard topology defined by uniform convergence on compact subsets.

$\mathcal{D}'(\Omega)$: space of Schwarz distributions (the dual space to $\mathcal{D}(\Omega)$).

$D_i,\ i = 1,2,3$: *distributional* derivative with respect to i-th coordinate, i.e., for $f \in \mathcal{D}',\ \phi \in \mathcal{D}$

$$< D_i f,\, \phi >= - < f,\, \phi_{,i} > .$$

$W^{1,p}(\Omega, \mathbb{R}^3)$ (more generally, $W^{1,p}(\Omega, \mathbb{R}^m)$, $\Omega \subset \mathbb{R}^n$): a triple ($m$-ple) of functions from Sobolev space $W^{1,p}(\Omega)$. For $p = 3$ ($p = n$) the latter is called Sobolev space with natural exponent.

$\operatorname{Cof}\ \mathbf{u},\ \operatorname{Det}\mathbf{u}$: matrix of distributional cofactors and the distributional determinant, respectively, defined by

$$(\operatorname{Cof}\ \mathbf{u})_{ij} = D_{i+2}(u^{j+2}u^{j+1}{}_{,i+1}) - D_{i+1}(u^{j+2}u^{j+1}{}_{,i+2}), \qquad \operatorname{Det}\mathbf{u} = D_j\left[u^1(\operatorname{Cof}\ \mathbf{F})_{j1}\right],$$

where $i, j = 1,2,3$. In the first equation the indices are to be taken modulo 3.

A function $f : U \to \mathbb{R}$, where U is a subset of a Banach space V, is said to be weakly lower semicontinuous (w.l.s.c.) if for any sequence $u_k \in U$ converging weakly to u, $u_k \rightharpoonup u$, the inequality

$$f(u) \leq \lim_{k \to \infty} f(u_k)$$

holds.

$N(f, \cdot) : Y \to \mathbb{N} \cup \{0, \infty\}$: *multiplicity function* for a map $f : X \to Y$. For $y \in Y$ the value $N(f, y)$ is defined as the number of elements in the set $\{x \in X \ : \ f(x) = y\}$.

Henceforth in this work, the conventions of Ogden [44] for tensor calculus are used. In particular, the divergence of a tensor $\mathbf{S}$ in Cartesian coordinates (X_1, X_2, X_3) reads

$$\operatorname{Div} \mathbf{S} = \partial S_{ij} / \partial X_i.$$

Cartesian coordinates of a tensor $\partial W / \partial \mathbf{F}$, $W = W(\mathbf{F})$ being a scalar function of $\mathbf{F}$, are written in the component form as

$$(\partial W / \partial \mathbf{F})_{\alpha i} = \partial W / \partial F_{i\alpha}.$$

Summation over repeating indices is assumed.

Chapter 1

Introduction

This thesis is concerned with the mathematical theory of nonlinear elasticity [44], [16], [37]. Specifically, the *hyperelastic* version is regarded as a useful model for solids undergoing large deformations without energy dissipation. This endows corresponding mathematical problems with a strong variational structure that makes it possible to use modern powerful machinery of the calculus of variations.

In general, a static variational problem in continuum mechanics, when the thermodynamic variables, such as temperature or entropy, are not under consideration, is to find *deformation(s)*

$$\mathbf{u}^* : \mathcal{B}0 \to \mathbb{R}^3$$

that render absolute minimum to the potential energy $E(\cdot)$ of the medium under consideration

$$\inf_{\mathbf{u}\in\mathcal{A}} E(\mathbf{u}) = E(\mathbf{u}^*) = \min_{\mathbf{u}\in\mathcal{A}} E(\mathbf{u}). \tag{1.0.1}$$

Here the set $\mathcal{A}$ of *admissible deformations* is usually a subset of an appropriate Banach space (e.g., Sobolev space $W^{1,p}(\Omega, \mathbb{R}^3)$) faithful to physical restrictions of the problem.

Rigorous mathematical approach to variational theory for general three-dimensional problems in nonlinear elastostatics was started in 1977 with a seminal paper by J.M. Ball [4]. He employed the direct method in the calculus of variations to state and prove his theorems on existence of absolute minimizers for equilibrium problems in nonlinear elasticity.

More generally, a *motion* is a time parametrized family of deformations described by a function

$$\chi : \mathcal{B}0 \times [0, \infty) \to \mathbb{R}^3, \qquad \chi(\cdot, t) = \mathbf{x}(\cdot, t) \in \mathcal{B}(t),$$

where $\mathcal{B}(t)$ is the *current* (deformed) configuration. The corresponding general variational problem in elastodynamics is to find motion(s) that render absolute minimum to the action functional

$$L(\chi) = \int_0^T \left\{ \int_{\mathcal{B}0} \frac{1}{2}\rho|\boldsymbol{V}|^2 d\mathbf{X} - E(\chi(\cdot, t)) \right\} dt$$

over curves in a set of admissible deformations. Here $\rho = \rho(\mathbf{X})$ is the inertial mass density in the reference configuration, $\boldsymbol{V} := \dot{\mathbf{x}}(\cdot, t)$, where dot stands for time derivative, and the first term in the integrand represents the kinetic energy. It is usually assumed that the initial deformation $\chi(\cdot, 0)$ and the velocity field $\boldsymbol{V}(\cdot, 0)$ are prescribed, and deformations $\chi(\cdot, t)$ belong to an admissible set satisfying appropriate physical requirements. When a problem admits a variational formulation, the equations of motion, which represent in a differential form the fundamental Balance of Linear Momentum Principle in continuum mechanics, can be obtained as Euler-Lagrange equations of the action functional. In the static theory, when the inertia effect is not an issue, the equations of motion become the equilibrium equations, and they can be interpreted as necessary condition for minimizers of the potential energy.

To specify the potential energy for a material under consideration a *constitutive relation* describing a mechanical response of the material should be included into the macroscopic model. Mathematical formulation of the constitutive laws must be consistent with available experimental data and satisfy certain physical restrictions such as *frame indifference* and (possibly) *material symmetry* requirements. The formulation must also satisfy mathematical restrictions related to such issues as existence and uniqueness of solutions to the balance equations. Other simplifying restrictions are introduced to make rigorous mathematical approach tractable.

A *hyperelastic* material is assumed to support a *strain (stored) energy density* $\hat{W} : \Omega \times \mathbb{M}_+^{3\times 3} \to \mathbb{R}$ so that $\hat{W}(\mathbf{X}, \mathbf{F})$ represents the stored energy per unit volume at a material point $\mathbf{X}$ when the elastic body is subjected to deformation $\mathbf{u}$ with deformation gradient $\mathbf{F} = \nabla \mathbf{u}$ at this point. The total stored energy in the deformed volume $\mathbf{u}(\Omega)$ is then [1]

$$E(\mathbf{u}) = \int_\Omega \hat{W}(\mathbf{X}, \nabla \mathbf{u}) dV. \tag{1.0.2}$$

Additional physically meaningful assumptions as outlined next simplify the functional

[1] If tractions (external surface forces) are exerted on (part of) the boundary in the reference configuration, an appropriate surface integral is added to the right hand side.

form of constitutive function.

Frame indifference is the assumption that physical laws are invariant with respect to observer orientation in space. In terms of the stored energy functions frame indifference translates into the requirement

$$\hat{W}(\mathbf{X}, \boldsymbol{Q}\mathbf{F}) = \hat{W}(\mathbf{X}, \mathbf{F})$$

for all $\mathbf{F} \in \mathbb{M}_+^{3\times3}$ and all proper orthogonal matrices $\boldsymbol{Q}$.

Material symmetry refers to a linear isometry $\boldsymbol{P} : \mathbb{R}^3 \to \mathbb{R}^3$ such that a material response is unaffected if the material orientation changes from $\mathcal{B}$ to $\boldsymbol{P}\mathcal{B}$. In terms of the stored energy functions this translates into the requirement

$$\hat{W}(\mathbf{X}, \mathbf{F}\boldsymbol{P}) = \hat{W}(\mathbf{X}, \mathbf{F}).$$

A set of all material symmetries $G_{\mathcal{B}}$ of the body is called the material symmetry group. Here we consider only *isotropic* materials with $G_{\mathcal{B}} = SO(3)$ (i.e., all orientations equivalent).

It can be shown (see, e.g., [16]) that a hyperelastic material is frame indifferent, isotropic, and homogeneous [2] if and only if $\hat{W}$ is a function only of the principal invariants of the right Cauchy deformation tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$,

$$\hat{W} = \Phi(I_1, I_2, I_3). \tag{1.0.3}$$

Some materials exhibit volume preservation property; they are termed *incompressible*.

[2]i.e., $\hat{W}$ is independent on $\mathbf{X}$

The deformations possible for such materials must satisfy the constraint

$$\det \mathbf{F} = 1 \tag{1.0.4}$$

and are called *isochoric.* In particular, many isotropic rubber-like materials are considered to be incompressible, and they are often modeled by *Mooney-Rivlin* stored energy

$$\hat{W} = \Phi(I_1, I_2) = \alpha(I_1 - 3) + \beta(I_2 - 3), \tag{1.0.5}$$

where α, $\beta \geq 0$ are material constants. In the limiting case $\beta = 0$ the material is called neo-Hookean and its stored density is usually written as

$$\hat{W} = \frac{\mu}{2}(I_1 - 3). \tag{1.0.6}$$

The material response function, the *nominal stress tensor* $\mathbf{S}$, in the incompressible case is given via the strain energy function by [44]

$$\mathbf{S} = \frac{\partial \hat{W}}{\partial \mathbf{F}} - p\mathbf{F}^{-1}, \tag{1.0.7}$$

$p = p(\mathbf{X})$ being the hydrostatic pressure (Lagrange multiplier) associated with the constraint of incompressibility (1.0.4).

For compressible materials there is no restriction (1.0.4), and the term $p\mathbf{F}^{-1}$ does not appear in (1.0.7). In all cases the Balance of Linear Momentum in the absence of body forces requires

$$\text{Div } \mathbf{S} = \rho\ddot{\mathbf{x}}, \tag{1.0.8}$$

which is the Euler-Lagrange equation associated with minimization of the action functional.

Ball's theory was based on the new notion of *polyconvexity* that ensures weak lower semicontinuity of the energy functional in an appropriate space of functions. Ball also showed that polyconvexity implies quasiconvexity. The latter concept was introduced by C.B. Morrey [36] who showed that, modulo some technical assumptions, quasiconvexity is a necessary and sufficient condition for w.l.s.c. of a multiple integral. Policonvexity can be effectively characterized in terms of the integrand (i.e., strain energy function in the setting of nonlinear elasticity), as opposed to quasiconvexity whose characterization is still an open question.

Ball has illustrated his theory by applying it to a wide class of isotropic stored energy functions commonly used in nonlinear elasticity and referred to as *Ogden materials.* These functions can be written in the form

$$\hat{W}(\mathbf{F}) = \sum_{i=1}^{M} a_i \left(\sum_{k=1}^{3} \lambda_k^{\gamma_i} \right) + \sum_{j=1}^{N} b_j \left(\sum_{k,l=1 \ k \neq l}^{3} (\lambda_k \lambda_l)^{\delta_j} \right) + h(\det \mathbf{F}), \tag{1.0.9}$$

where:

$$\begin{cases} \lambda_k = \lambda_k(\mathbf{F}) \text{ are the singular values of } \mathbf{F} \in \mathbb{M}_+^{3\times 3}; \\ a_i > 0, \ \gamma_i \geq 1, \ 1 \leq i \leq M; \ b_j > 0, \ \delta_j \geq 1, \ 1 \leq j \leq \ N; \\ h \ : \ (0, +\infty) \to \mathbb{R} \text{ is a convex function.} \end{cases}$$

Functions (1.0.9) satisfy the hypotheses of Ball's theory under appropriate choices of the *growth exponents* $p = \max_i \gamma_i, \ q = \max_j \delta_j$.

Ball's contribution stimulated investigations into existence theory through a variety of theoretical lenses. Materials with strain energy density with growth exponents below the values allowed by Ball's theory received significant attention in connection with cavitation [6], [29] and other phenomena involving singularities [7]. Regularity issues for function classes of mappings $\mathbf{u}$ that involve information on both the gradient $\mathbf{F} = D\mathbf{u}$ and its adjugate matrix were studied by Šverák [51]. The degree formula used in [51] was generalized by Müller, Qi, and Yan [42] and used for weakening Ball's constraints on the growth exponents needed for existence of a minimizer in Sobolev space.

Ball's conjecture about identicity of pointwise and distributional cofactors and determinants was proved by Müller [41] under the assumption that those *null Lagrangians* (quasiaffine functions) defined in the sense of distributions are functions. This development allowed one to reformulate the existence theorems by Ball in terms of pointwise null Lagrangians rather than the distributional ones. Variational problems with non-convex (non-quasiconvex, non-polyconvex) integrands are currently intensively studied. The relation between quasiconvexity and relaxation was discovered by Dacorogna (see [18] and references therein). Explicit formulations of relaxed problems are not in abundance, but when they are available and represented by a multiple integral, Ball's existence theory applies if the integrand meets appropriate requirements. More recent references can be found, e.g., in [8].

Classes of admissible functions introduced by Ball stimulated new developments in geometric function theory. A class of functions having *finite dilatation (distortion)*, which includes Ball's admissible functions, has been defined and investigated from various points of view (see [31] and references therein). Theory of mappings of finite distortion is a natural generalization of the theory of *quasiregular* mappings [48]. Well known topological properties of openness and discreetness of nonconstant quasiregular mappings were recently carried over to two-dimensional mappings of finite distortion by Iwaniec and Šverák [30], provided that the *dilatation quotient* K (see Definition 4.1.1) is an integrable function. For $n > 2$ the openness and discreetness of n-dimensional mappings of finite distortion was investigated by Heinionen and Koskela [26], and Kauhanen, Koskela, Malý [32].

Results in geometric function theory and degree theory allowed one to state conditions ensuring more realistic properties of admissible functions , e.g., global injectivity. This issue was investigated in different settings by Ball [5], Ciarlet and Nečas [15], and Tang [52], while Fonseca and Gangbo [22] studied local invertibility properties of Sobolev classes with natural exponent.

Despite the important contributions discussed above, genuine three-dimensional deformations of some commonly used material models are not covered by Ball's theory. In particular, as was noted by Ball himself, restrictions imposed by his theory rule out three-dimensional deformations of neo-Hookean materials (1.0.6). These materials were the object of numerous investigations (see, e.g., [43] and references therein). They can also serve as a good source for testing numerical methods. The neo-Hookean strain energy density has also been suggested as a useful form for modeling the base matrix material response in composite materials, subject to additional reinforcing, and used for analysis of different aspects of the theory of composites by a number of authors (see, e.g., [46] and references therein). On the other hand, more complicated expressions for the strain energy density function provide more flexibility for correlation with experimentally observed deformation behavior, and development of more sophisticated hyperelastic constitutive models is an active subject (see, e.g., [10], where fairly general constitutive relations for the *shape memory materials* are developed). Recently, Holzapfel, Gasser and Ogden [28] presented the analysis of the biomechanics of blood vessels, employing the neo-Hookean strain energy function for modeling the behavior of the matrix material. This example from biomechanics, along with other applications using nonlinearly elastic models for bodies of tubular geometries, motivated investigation of axisymmetric problems in nonlinear elasticity in this thesis.

In Chapters 2-4 the existence issues for variational formulation of axisymmetric problem are studied. It seems natural to expect that restrictions on growth exponents in the framework of Ball's theory will be milder if one confines analysis to a subclass of three-dimensional deformations. This is true for plane deformations that can be viewed as a subclass of three-dimensional deformations. We introduce a general class of axisymmetric deformations of the form

$$r = r\,(R, Z)\,, \qquad \theta = \omega + \tau\,(R, Z)\,, \quad \text{and} \quad z = z\,(R, Z)\,, \tag{1.0.10}$$

where (R, ω, Z), (r, θ, z) are cylindrical coordinates in the reference and deformed configurations, respectively. Reduced restrictions on the growth exponents for the strain energy densities of Ogden materials (1.0.9) subjected to deformations (1.0.10) are then obtained in the spirit of Ball's existence theory. Although not necessary for essential conclusions, for simplicity attention is restricted to isochoric deformations with boundary condition of place on a subset Γ of the boundary $\partial\Omega$ in the absence of external surface forces on the remainder of the boundary. To ensure the coerciveness inequality in axisymmetric setting, the body in the reference configuration is assumed to be cylindrically hollow.

Under a natural assumption that the radial component is nonnegative almost everywhere, improved regularity of two-dimensional mapping determined by the radial and axial components of isochoric deformation,

$$\mathbf{v} = (r, z) \in W^{1,p}(D, \mathbb{R}^2) \tag{1.0.11}$$

with $p = 2$, is established. Here D is half of the axial cross-section of the undeformed body. It is also found that the two-dimensional mapping $\mathbf{v}$ is open and discrete. This is one of the most novel and original result in this development, and it does not have an analogue in three-dimensional existence theory. Based on improved regularity and the topological property of openness of the mapping $\mathbf{v}$, injectivity of minimizers is established in a stronger form than that stated in [5] and [15]. For technical reasons, if the angular deformation function τ is present, a stronger restriction on the radial deformation is imposed, namely, it is assumed that originally hollow cylindrical body remains hollow after deformation.

Beginning with Chapters 5, we turn from existence issues for static axisymmetric problems in variational formulation to analysis of axisymmetric motions in differential form. Specifically, a time dependent version of (1.0.10) subject to additional specialization is considered. Attention is restricted to neo-Hookean material response.

Less is known about existence and uniqueness for elastodynamics than for elastostatics (see, e.g., [37] on some aspects of what is currently known). Closed form three-dimensional solutions to the equations of motion are rare, and most such solutions involve both specialized material response and *a priori* symmetry assumptions that impose severe structural restrictions on the unknown functions. Known explicit dynamical solutions for incompressible materials include the radial oscillatory solutions due to Knowles [33] and the circularly polarized finite amplitude wave motions studied by Carroll [13, 14]. For a Mooney–Rivlin material, detailed analysis of finite amplitude plane wave motions is given by Boulanger and Hayes [11, 12]. More references on exact solutions in finite elastodynamics can be found in [43].

The focus in chapters 5-7 is on deriving the governing differential equations for the specialized forms of three-dimensional motions in neo-Hookean material and obtaining new *physically meaningful* explicit solutions. The motions presented here give both space and time variation in all three principal stretches and naturally describe various wave forms in tubular geometries. In certain particular cases they reduce to previously known results.

Here is an outline of the content of the thesis.

In Section 2.1 an outline of Ball's theory is given providing a framework for following existence analysis. In Section 2.2 the axisymmetric variational problem is described. By a straightforward computation it is shown that the Euler-Lagrange equations for the reduced variational problem are equivalent to the equilibrium equations for the physical problem under consideration. New dependent variables that simplify the description and allow one to apply the direct method of the calculus of variations in the spirit of Ball's theory are introduced.

Two existence theorems for isotropic strain energy densities with and without dependence on the cofactor matrix are stated and proved in Chapter 3. Here cases with and without assumption that the distributional cofactor matrix and determinant are functions are examined, and we employ the result of [41]. Although, as expected, the restrictions on the growth parameters are significantly reduced due to the axial symmetry, materials with neo-Hookean rate of growth ($p = 2$) represent a marginal case for the existence theorem in the admissible set *without* restriction on the distributional determinant.

The goal of Chapter 4 is twofold: to extend the existence results to integrands with rate of growth $p = 2$ and *without* conditions on the cofactor matrix, and to examine injectivity of admissible mappings. The cylindrical description of admissible mappings is used. Under a natural assumption that the radial component of deformation is nonnegative, some remarkable properties of two-dimensional mapping $\mathbf{v}$, defined in (1.0.11) for a mapping $\mathbf{u} \in W^{1,2}(D, \mathbb{R}^3)$ from a set of admissible functions, are presented in Section 4.1. Firstly, it has been proven that $\mathbf{v}$ has finite dilatation. Furthermore, it is also shown that for functions of finite distortion in Sobolev space with natural exponent the mapping

$$\iota : W^{1,n}(\Omega, \mathbb{R}^n) \to C(G, \mathbb{R}^n), \ \iota(f) = f|_G, \tag{1.0.12}$$

where $G \subset \Omega \subset \mathbb{R}^n$ is a relatively compact domain in Ω, is compact. Consequently, for any relatively compact $G \subset\subset D$ weak convergence of a sequence of admissible functions $\mathbf{u}_k = (r_k, \tau_k, z_k)$ in $W^{1,2}(D, \mathbb{R}^3)$ implies uniform convergence of the corresponding sequence (up to a subsequence) $\mathbf{v}_k = (r_k, z_k)$ in G. It is worth noting that the general fact of compactness of the embedding (1.0.12) is of interest in its own right. The most remarkable properties of the two-dimensional mapping $\mathbf{v}$, established in this section, are openness and discreteness that follow from the result in [30] on *Stoilow type factorization.*

In Section 4.2 two additional conditions are introduced for admissible sets. The values of the radial component are assumed to be separated away from zero, that is,

$$r(R, Z) \geq \alpha > 0 \tag{1.0.13}$$

for a fixed positive number α and almost all $(R, Z) \in D$, and an axisymmetric counterpart of the well-known injectivity condition of [15] is imposed. For this smaller set of admissible mappings, existence of minimizers with $p = 2$ is proved, and the injectivity of minimizers almost everywhere is established for $p \geq 2$ along the lines of [15]. In the border case $p = 2$ the argument of [15] needs to be modified, and we use some results from geometric function theory. Furthermore, making use of the openness of the two-dimensional mappings (1.0.11), one concludes that under the injectivity condition these mappings are in fact homeomorphic. For $p > 2$ this implies that corresponding axisymmetric deformation is a homeomorphism too, which represents a substantial improvement compared with relevant results known for the three-dimensional case.

For a two-dimensional isochoric deformation from appropriate Sobolev class, the Stoilow type factorization is also readily available. This observation allows one to sharpen previously known results on injectivity a.e. when they apply to this class of mappings, although two-dimensional deformations are not a focus in this work.

Global injectivity for Dirichlet problem presented in this section relies on the result in [5]. For the two-dimensional case, the condition on the adjugate matrix introduced in [5] is found to be automatic for the mapping (1.0.11). Therefore, except for that condition, the statement about global injectivity for the Dirichlet problem in this section is otherwise identical to that in [5].

In Chapter 5 a specialized class of motions is considered. In cylindrical coordinates it is given in terms of axially varying twist function $\tau(Z,t)$, radial inflation/deflation function $s(Z,t)$, and axial contraction/elongation function $z(Z,t)$ by the following ansatz

$$r = Rs(Z,t), \qquad \theta = \omega + \tau(Z,t), \qquad z = z(Z,t). \tag{1.0.14}$$

(These motions are referred to as TIE motions.)

The general governing equations for axisymmetric motion of neo-Hookean material, derived in Section 2.2, are reduced here for the specialized ansatz (1.0.14) to a second order system of two coupled nonlinear partial differential equations for functions s and τ. This system contains two material constants, the inertial mass density ρ and the neo-Hookean shear modulus μ, as well as an arbitrary function of time $C(t)$ that results from a general integration. The neo-Hookean shear wave speed is given by $c_* = \sqrt{\mu/\rho}$, a parameter that has special significance with respect to the various motions described herein.

Chapter 6 presents four various classes of specialized solutions to the governing system for TIE motions.

- Equilibrium deformation solutions of three different forms depending on the neo-Hookean shear wave speed c_*. One of the well known universal deformations for incompressible hyperelasticity emerges as a special case.

- Travelling wave solutions of the same three forms as above at arbitrary wave speed. Further, at the neo-Hookean shear wave speed c_* additional travelling wave solutions are also available.

- Motions with specialized forms of the twist function. For the special case of zero twist the governing equations reduce to a single linear partial differential equation which can be treated by standard means.

- Motions for which both the twist function τ and the inflation/deflation function s are constant on rays $Z/t = constant$. Although we are unable to obtain explicit solution in this general case, an analytic expression is given for a special case when one of the parameters in the governing system of ODE vanishes.

In Chapter 7 Cartesian descriptions of TIE and TIES motions are derived. It is shown that the reduced governing system of PDE for the radial and angular components of TIE motion, found in Chapter 5, admits a variational formulation. Formal change of dependent variables transforms the Lagrangian of this variational problem into quadratic expression with respect to new dependent variables therefore leading to a linear system of Euler-Lagrange equations. For a particular case, when the function C(t) involved in these equations is a constant, the system reduces to two identical telegraphy equations, which can be treated by standard means. In Section 7.2 more general class of motions, termed TIES, are considered. The motions include *transverse shear* in addition to twist, inflation/deflation, and contraction/elongation, describing TIE motion. Although two unknown functions accounting for the in-plane shear are introduced, the governing system for TIES motion is shown to decompose into four identical decoupled linear equations of the same type as for TIE motion. In Section 7.3 the governing system for general axisymmetric motions in neo-Hookean solid in Cartesian coordinates is derived, and it seems to be more convenient for further analysis than the original one, derived in terms of cylindrical coordinates.

To the best of my knowledge, presented in this thesis results on existence, injectivity, and regularity for axisymmetric minimizers, as well as the development on specialized elastodynamic equations of motion and their explicit analytic solutions, are new and have not been discussed in the literature up to now.

Chapter 2

Setting axisymmetric variational problem

2.1 Overview of Ball's existence theory

Let a material body $\mathcal{B}_0$ in its reference configuration occupy an open and bounded domain $\Omega \subset \mathbb{R}^3$ with strongly Lipschitz boundary $\partial\Omega$. Given a material point $\mathbf{X} \in \Omega$, a mapping $\mathbf{u} : \Omega \to \mathbb{R}^3$ describes the material deformation with $\mathbf{x} = \mathbf{u}(\mathbf{X}) \in \mathbf{u}(\Omega)$ the corresponding point in the deformed configuration, and $\mathbf{F} := \nabla \mathbf{u}$ the deformation gradient.

The material of the body $\mathcal{B}_0$ is assumed to be hyperelastic with the stored energy function $\hat{W}$ satisfying the requirements of frame indifference and, unless stated otherwise, isotropy. Thus the total stored energy in the deformed volume $\mathbf{u}(\Omega)$ is defined by (1.0.2), and corresponding minimization problem is then given by (1.0.1), where a set of admissible deformations $\mathcal{A}$ is a subset of Sobolev space $W^{1,p}(\Omega,\mathbb{R}^3)$ satisfying appropriate physical restrictions of the problem, e.g., boundary condition of place $\mathbf{u}=\mathbf{u}_0$ on $\Gamma\subset\partial\Omega$, $|\Gamma|>0$, ($\mathbf{u}_0\in W^{1,p}(\Omega,\mathbb{R}^3)$ being a given function), specified traction values on the remainder of the boundary, and the incompressibility constraint (1.0.4).

For the successful application of the direct method of the calculus of variations to problems in nonlinear elasticity, one needs to formulate physically realistic hypotheses on both the stored energy density $\hat{W}$ and the admissible set $\mathcal{A}$ so that the following major argument can be realized:

Step 1. Ensure finiteness of the infimum of the total energy functional $E(\cdot)$ over the admissible set $\mathcal{A}$ and show existence of a minimizing sequence $\mathbf{u}_n\in\mathcal{A}$ that converges weakly to some $\bar{\boldsymbol{u}}$ for a suitable choice of p.

Step 2. Show that weak limits of minimizing sequences belong to the admissible set $\mathcal{A}$.

Step 3. Verify that the total energy functional $E(\cdot)$ is w.l.s.c.

Then the inequality

$$E(\bar{\boldsymbol{u}})\leq\lim_{n\to\infty}E(\mathbf{u}_n)=\inf_{\mathbf{u}\in\mathcal{A}}E(\mathbf{u})\leq E(\bar{\boldsymbol{u}}) \tag{2.1.1}$$

implies that $\bar{\boldsymbol{u}}$ is a minimizer.

To ensure the application of the above three steps to three-dimensional problems in nonlinear elasticity, J. Ball [4] assembled the following hypotheses (with appropriate modifications of $\hat{W}$ and $\mathcal{A}$ in different settings) on the stored energy function :

$(H1)$ Polyconvexity: For almost all $\mathbf{X} \in \Omega$ there exists a continuous convex function $W(\mathbf{X}, \cdot, \cdot, \cdot) : \mathbb{M}^{3\times3} \times \mathbb{M}^{3\times3} \times \mathbb{R} \to \mathbb{R}$ such that

$$\hat{W}(\mathbf{X}, \mathbf{F}) = W(\mathbf{X}, \mathbf{F}, \text{cof } \mathbf{F}, \det \mathbf{F}) \text{ for all } \mathbf{F} \in \mathbb{M}^{3\times3}_{+},$$

and $W(\cdot, \mathbf{F}, \mathbf{H}, \delta)$ is measurable over Ω for every $(\mathbf{F}, \mathbf{H}, \delta) \in \mathbb{M}^{3\times3} \times \mathbb{M}^{3\times3} \times \mathbb{R}$.

(H2) Coercivity: There exist real numbers $\alpha > 0,\ \beta,\ p > 1,\ q > 1,\ s > 1$ such that for almost all $\mathbf{X} \in \Omega$

$$W(\mathbf{X}, \mathbf{F}, \text{cof } \mathbf{F}, \det \mathbf{F}) \geq \alpha \left(|\mathbf{F}|^p + |\text{cof } \mathbf{F}|^q + (\det \mathbf{F})^s\right) + \beta. \tag{2.1.2}$$

$(H3)$ Finiteness: There exists an admissible deformation $\mathbf{u} \in \mathcal{A}$ such that $E(\mathbf{u}) < \infty$.

It was also shown that Ogden materials (1.0.9) satisfy the hypotheses (H1) and (H2) with $p = \max_i \gamma_i$ and $q = \max_j \delta_j$. A typical existence result obtained by employing Ball's theory within the context of the above hypotheses is given by the following

Theorem 2.1.1 *Let a stored energy function $\hat{W}$ satisfy (H1)-(H3) with* [1]

$$p \geq 2 \quad \textit{and} \quad q \geq \frac{p}{p-1}.$$

Let

$$\lim_{\det \mathbf{F} \to 0^+} \hat{W} = \infty.$$

[1] In [42] it was shown that the right hand side of the inequality for parameter q can be replaced by $3/2$.

Then the total energy (1.0.2) assumes its minimum in the admissible set $\mathcal{A}$ given by

$$\begin{aligned}\mathcal{A} \;:=\; & \{\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^3) \,:\, \mathbf{u} = \mathbf{u}_0 \;\text{ a.e. in } \Gamma, \operatorname{cof} \mathbf{F} \in L^q(\Omega, \mathbb{M}^{3\times 3}), \\ & \det \mathbf{F} \in L^s(\Omega),\; \det \mathbf{F} > 0 \;\text{ a.e. in } \Omega\},\end{aligned}$$

where $\Gamma \subset \partial\Omega$, $|\Gamma| > 0$, and $\mathbf{u}_0$ is a specified function in $W^{1,p}(\Omega, \mathbb{R}^3)$.

Proof (Sketch). Existence of a minimizing sequence $\mathbf{u}_k$ and boundedness of the corresponding gradients in $L^p(\Omega, \mathbb{M}^{3\times 3})$ follow from the finiteness and the coercivity hypotheses, respectively. Boundedness of the sequence $\mathbf{u}_k$ in $L^p(\Omega, \mathbb{R}^3)$ is proven via the generalized Poincaré inequality (see Theorem 6.1.8 (b) in [16])

$$\int_\Omega |f|^p \, dx \leq c \left(\int_\Omega |\nabla f|^p \, dx + \left| \int_\Gamma f \, da \right|^p \right). \tag{2.1.3}$$

applied to the components of $\mathbf{u}_k$. The displacement boundary condition imposed on admissible functions is needed here to ensure that the second term on the right hand side is bounded (in fact, it is a constant). Thus $\mathbf{u}_k$ is bounded in $W^{1,p}(\Omega, \mathbb{R}^3)$, and the existence of a weakly convergent subsequence follows from the reflexivity of Sobolev space $W^{1,p}$ with $p > 1$.

The most technical part of the proof is establishing that the weak limit resides in the admissible set. It includes proof of weak continuity of the minors of the deformation gradient, and this dictates the restrictions on the growth exponents p, q. Satisfaction of the boundary condition for the weak limit relies on the compactness of the trace operator $\operatorname{tr} \in \mathcal{L}\left(W^{1,p}(D); L^p(\Gamma)\right)$ and is proved in a standard manner by extracting a subsequence converging almost everywhere on Γ.

The polyconvexity of the integrand implies its quasiconvexity [4], which is essentially equivalent to the weak lower semi-continuity of the energy functional [36]. This completes the proof. ■

As was pointed out in [4], for three-dimensional deformation of an incompressible hyperelastic material with stored energy function independent on cof $\mathbf{F}$, the bound on the growth exponent needed for the weak continuity of the determinant (distributional determinant) is $p \geq 3$ ($p > 9/4$). The optimality of these bounds has been demonstrated in [19]. Consequently, any of these restrictions rules out neo-Hookean materials.

In Chapters 3 and 4 new existence theorems for variational formulation for isochoric, axially symmetric deformations of hyperelastic materials will be presented, and some regularity and injectivity properties of minimizers of energy will be established. It is always assumed that certain boundary conditions of place are prescribed on the part of the boundary and the rest of the boundary is traction free. It is also assumed that the following conditions are satisfied.

- $\hat{W}(\mathbf{X}, \mathbf{F})$ is frame indifferent and, unless stated otherwise, isotropic;
- $\hat{W}(\mathbf{X}, \mathbf{F})$ satisfies the hypotheses (H1), (H2);
- Function $\mathbf{u}_0$ prescribing boundary condition of place for an admissible set $\mathcal{A}$ belongs to this set, and $I(\mathbf{u}_0) < \infty$.

The existence theorems cover some classes of hyperelastic incompressible materials with stored energy functions that do not satisfy growth conditions of Ball's existence theory in genuine three-dimensional case, in particular, the class of neo-Hookean materials. The detail will be provided only for proving the fact that weak limits belong to appropriate admissible sets, since the rest of the argument sketched above is standard (see [4], [16], [18], and references therein). Parameters p, q always denote the growth exponents in the coercivity hypothesis (2.1.2) for the stored energy function under consideration.

2.2 Description of the axisymmetric problem

In this section we describe the axisymmetric setting in both cylindrical and Cartesian coordinates, adjust the total energy functional to this setting, and prove the equivalency of the equilibrium equations for the physical problem under consideration and the Euler - Lagrange equations of the reduced minimization problem.

Let a hyperelastic body in its reference configuration occupy a domain $\Omega \in \mathbb{R}^3$ given in cylindrical coordinates $\mathbf{X} = (R, \omega, Z)$ by

$$\Omega := \{(R, \omega, Z) : \mathbf{X}' := (R, Z) \in D, \omega \in [0, 2\pi)\}, \tag{2.2.1}$$

where $D \subset \mathbb{R}^2$ is an open domain with strongly Lipschitz boundary ∂D such that

$$\min_{\mathbf{X}' \in \bar{D}} R = R_i > 0. \tag{2.2.2}$$

Introduce a class $Axi(\Omega)$ of almost everywhere isochoric, axisymmetric deformations $\hat{\mathbf{u}} : \Omega \to \mathbb{R}^3$ with components (r, θ, z) of the form given by (1.0.10).

The deformation gradient of $\hat{\mathbf{u}} \in Axi(\Omega)$ takes the form

$$\mathbf{F} = \begin{bmatrix} r_{,1} & 0 & r_{,3} \\ r\tau_{,1} & r/R & r\tau_{,3} \\ z_{,1} & 0 & z_{,3} \end{bmatrix} \tag{2.2.3}$$

with the corresponding right Cauchy-Green deformation tensor given by

$$\mathbf{C} = \begin{bmatrix} r_{,1}{}^2 + (r\tau_{,1})^2 + z_{,1}{}^2 & r^2\tau_{,1}/R & r_{,1} r_{,3} + r^2\tau_{,1}\tau_{,3} + z_{,1} z_{,3} \\ \dots & (r/R)^2 & r^2\tau_{,3}/R \\ \dots & \dots & r_{,3}{}^2 + (r\tau_{,3})^2 + z_{,3}{}^2 \end{bmatrix}, \tag{2.2.4}$$

where the ellipses stand for appropriate symmetric expressions. Here and throughout this work we adopt the notation

$$f_{,1} := \partial f/\partial R \qquad f_{,2} := \partial f/\partial \omega \quad \text{and} \quad f_{,3} := \partial f/\partial Z$$

for any scalar function $f(R, \omega, Z)$.

The incompressibility condition (1.0.4) takes the form

$$\frac{r}{R}(r_{,1} z_{,3} - r_{,3} z_{,1}) = 1. \tag{2.2.5}$$

In the three-dimensional Cartesian setting, the first invariant $I_1 = \operatorname{tr}\mathbf{C} = |\mathbf{F}|^2$ of the Cauchy-Green strain tensor $\mathbf{C}$ is a sum of the squares of all partial derivatives of the Cartesian components of deformation. Consequently, if the total energy is finite, those partial derivatives belong to L^p due to the coercivity inequality (2.1.2). However, as follows from (2.2.3), in cylindrical coordinates the first invariant has the form

$$I_1 = |\mathbf{F}|^2 = r_{,1}{}^2 + (r\tau_{,1})^2 + z_{,1}{}^2 + (r/R)^2 + r_{,3}{}^2 + (r\tau_{,3})^2 + z_{,3}{}^2, \tag{2.2.6}$$

so that the derivatives of τ in this expression are directly coupled with r. Hence, for a minimizing sequence of deformations $\hat{\mathbf{u}}^k \in Axi(\Omega)$, one can only conclude from the coercivity inequality that the functions $r,_i^k$, $z,_i^k$, $r^k\tau,_i^k$, $i = 1, 3$, converge weakly in the space $L^p(D)$ thereby preventing determination of appropriate Sobolev space for the limiting angular deformation function τ. One way of resolving this problem is to assume that the radial component r in the deformed configuration is uniformly bounded below away from zero, $r(R, Z) \geq \alpha > 0$, for almost all $(R, Z) \in D$. This possibility will be explored in Chapter 4.

The problem of decoupling functions τ and r can be also eliminated through the introduction of the new dependent variables

$$\xi = \xi(R, Z) := r\cos\tau \quad \text{and} \quad \eta = \eta(R, Z) := r\sin\tau. \tag{2.2.7}$$

In fact, then corresponding right Cauchy-Green strain tensor and its first invariant $I_1(\mathbf{C})$ in terms of $\mathbf{u} = (\xi,\ \eta,\ z)$ take the forms

$$\mathbf{C} = \begin{bmatrix} \xi,_1{}^2 + \eta,_1{}^2 + z,_1{}^2 & (\eta,_1 \xi - \eta\xi,_1)/R & \xi,_1 \xi,_3 + \eta,_1 \eta,_3 + z,_1 z,_3 \\ \cdots & (\xi^2 + \eta^2)/R^2 & (\eta,_3 \xi - \eta\xi,_3)/R \\ \cdots & \cdots & \xi,_3^2 + \eta,_3^2 + z,_3^2 \end{bmatrix} \tag{2.2.8}$$

and

$$I_1 = \sum_{m=1,3} (\xi,_m^2 + \eta,_m^2 + z,_m^2) + (\xi^2 + \eta^2)/R^2. \tag{2.2.9}$$

Remark. It should be noted at this point that nothing prevents the radial component r from taking negative values.[2] Therefore, the unique determination of the cylindrical

[2]The existence theorems of Ball [4] also assert only that under certain assumptions a minimizer

coordinates r, θ, z of the image of a point (R, ω, Z) in terms of $(\xi, \ \eta, z)$ is not possible. However, using (1.0.10), (2.2.7), and the standard relations

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z(R, Z)$$

with $\theta = \omega + \tau$, the corresponding image can be described in Cartesian coordinates (x, y, z) by the formulae

$$x = \frac{X}{R}\xi - \frac{Y}{R}\eta, \qquad y = \frac{Y}{R}\xi + \frac{X}{R}\eta, \qquad z = z(R, Z) \tag{2.2.10}$$

with $X = R\cos\omega$, $Y = R\sin\omega$. These equations will be referred to as Cartesian description of deformation. It follows from the equations (2.2.10) that the new dependent variables ξ and η have clear physical meaning: these are the first two Cartesian coordinates of the image of the axial cross-section $\omega = 0$ in the deformed configuration,

$$\xi = x(R, 0, Z), \qquad \eta = y(R, 0, Z).$$

If one defines a matrix valued function

$$\mathbf{F}_0(\mathbf{u}) = \begin{bmatrix} \xi_{,1} & -\eta/R & \xi_{,3} \\ \eta_{,1} & \xi/R & \eta_{,3} \\ z_{,1} & 0 & z_{,3} \end{bmatrix} \tag{2.2.11}$$

corresponding to an axisymmetric deformation $\hat{\mathbf{u}} \in Axi(\Omega)$, then a direct computation shows that

$$\mathbf{C} = \boldsymbol{F}^T\mathbf{F} = \mathbf{F}_0^{\mathrm{T}}\mathbf{F}_0,$$

u *exists.* Injectivity of a minimizer is another problem that was later stated and investigated in different settings, *cf.* . [5], [15], [52].

so that $\mathbf{F}$ and $\mathbf{F}_0$ have the same singular values. Using the chain rule, it is easy to show that the deformation gradient $\mathbf{F}(\hat{\mathbf{u}})$ *in Cartesian coordinates* satisfies the equation

$$\mathbf{F} = \boldsymbol{Q}\,\mathbf{F}_0\,\boldsymbol{Q}^T,$$

where

$$\mathbf{Q} = \begin{bmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This representation implies that the polyconvexity hypothesis (H1) and the coercivity hypothesis (H2) hold for axisymmetric deformations with $\mathbf{F}$ replaced by $\mathbf{F}_0$ due to the frame indifference and the isotropy assumptions. It also follows from the above representation that $\mathbf{F}_0$ is the deformation gradient in Cartesian coordinates restricted to the section $\omega = 0$ of the cylindrical body Ω, $\mathbf{F}_0 = \mathbf{F}(R, 0, Z)$.

The incompressibility condition (2.2.5) in terms of ξ, η, z reads

$$(\xi^2 + \eta^2)_{,1}\, z_{,3} - (\xi^2 + \eta^2)_{,3}\, z_{,1} = 2R. \tag{2.2.12}$$

For $\hat{\mathbf{u}} \in Axi(\Omega)$, the three-dimensional minimization problem (1.0.1) formally reduces to

$$\inf_{\hat{\mathbf{u}} \in \mathcal{A}} \int_\Omega R\,\hat{W}\,(\mathbf{X}, \mathbf{F}(\mathbf{u}))\, dR\, d\omega\, dZ,$$

where $\mathbf{X} = \mathbf{X}(R, \omega, Z)$. To reduce the dimensionality of the underlying space we have to assume that $\hat{W}$ depends on X, Y only through the variable $R = (X^2 + Y^2)^{1/2}$.

Then the energy functional of the problem (1.0.1) takes the form

$$I(\mathbf{u}) := \inf_{\mathbf{u}\in\mathcal{A}} \int_D R\hat{W}(\mathbf{X}', \mathbf{F}(\mathbf{u}))\, da \tag{2.2.13}$$

with $\mathbf{u} = (r, \tau, z)$, $\mathbf{F}$ given by (2.2.3), da the area element in D, and $\mathbf{X} = (R, Z)$. For Cartesian description of deformation,

$$I(\mathbf{u}) := \inf_{\mathbf{u}\in\mathcal{A}} \int_D R\hat{W}(\mathbf{X}', \mathbf{F}_0(\mathbf{u}))\, da. \tag{2.2.14}$$

with $\mathbf{u} = (\xi, \eta, z)$ and $\mathbf{F}_0$ given by (2.2.11). In each case $\mathcal{A}$ represents a set of admissible ordered triplets of functions that is assumed to be a subset of appropriate Sobolev space $W^{1,p}(D, \mathbb{R}^3)$ faithful to physical restrictions of the problem including the incompressibility constraint and boundary conditions to be specified later. Note that by virtue of (2.2.2) the coercivity hypothesis (2.1.2) holds for the integrand in (2.2.14), provided it is true for the strain energy density $W(\mathbf{X}, \mathbf{F}, \operatorname{cof} \mathbf{F})$.

Before proceeding with the existence analysis, the reduced variational formulation needs to be justified from mechanical point of view. Specifically, one must show that the equilibrium equations for the problem under consideration coincide with (more exactly, are equivalent to) the Euler-Lagrange equations for the reduced functional to which an appropriate term accounting for the incompressibility constraint must be added. In the absence of body forces, the equation of motion (1.0.8) transforms into the equilibrium equation

$$\operatorname{Div} \mathbf{S} = \mathbf{0}, \tag{2.2.15}$$

where $\mathbf{S}$ is the nominal stress tensor (1.0.7).

In the next lemma the equivalency between the Euler-Lagrange equations for the reduced functional (2.2.13) and equilibrium equation (2.2.15) is shown. For simplicity, it is assumed that the strain energy density function $\hat{W}$ does not depend explicitly on the spacial variables.

Lemma 2.2.1 *Let*

1. $\mathbf{u} = (r, \theta, z) : \Omega \to \mathbb{R}^3$ *be the triplet of functions corresponding to a deformation* $\hat{\mathbf{u}} \in Axi(\Omega)$, *where* Ω *is defined by (2.2.1), (2.2.2);*

2. *The strain energy density* $\hat{W} = \hat{W}(\mathbf{F})$ *is frame invariant and isotropic;*

3. *The Lagrange multiplier does not depend on the angular variable, i.e.,* $p = p(R, Z)$.

Then the Euler-Lagrange equations for the functional in (2.2.13) differ from the equilibrium equations by a factor R *and therefore are equivalent to the equilibrium equations.*

Proof. For a material satisfying the requirements of isotropy, frame indifference, homogeneity, and incompressibility the strain energy density can be written in the form $\hat{W} = W(I_1, I_2)$, where I_1, I_2 are the principle invariants of the right Cauchy-Green deformation tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$, and the formula (1.0.7) becomes [45]

$$\mathbf{S} = 2\left(\frac{\partial W}{\partial I_1} + I_1\frac{\partial W}{\partial I_2}\right)\mathbf{F}^T - 2\frac{\partial W}{\partial I_2}\mathbf{C}\mathbf{F}^T - p\mathbf{F}^{-1}. \tag{2.2.16}$$

Assume temporarily that $W = W(I_1)$, and let the prime denote differentiation with respect to I_1. It follows from (2.2.15), (2.2.16) that the equilibrium equation under the assumptions of lemma becomes

$$2W' \operatorname{Div} \mathbf{F}^T + 2W''\mathbf{F}\, \nabla I_1 - \mathbf{F}^{-T}\, \nabla p = 0, \tag{2.2.17}$$

where we used the gradient operator ∇ in cylindrical coordinates

$$\nabla = \mathbf{E}_R \frac{\partial}{\partial R} + \mathbf{E}_\omega \frac{1}{R}\frac{\partial}{\partial \omega} + \mathbf{E}_Z \frac{\partial}{\partial Z},$$

and Piola identity [16], Pg. 39, which for isochoric deformation takes the form $\operatorname{Div} \mathbf{F}^{-1} = 0$.

To compute the first term in (2.2.17) note that the transpose deformation gradient can be written in the form

$$\begin{aligned} \mathbf{F}^T &= r_{,1}\, \boldsymbol{E}_R \otimes \boldsymbol{e}_r + r_{,3}\, \boldsymbol{E}_Z \otimes \boldsymbol{e}_r + r\tau_{,1}\, \boldsymbol{E}_R \otimes \boldsymbol{e}_\theta + r/R \boldsymbol{E}_\omega \otimes \boldsymbol{e}_\theta \\ &+ \; r\tau_{,3}\, \boldsymbol{E}_Z \otimes \boldsymbol{e}_\theta + z_{,1}\, \boldsymbol{E}_R \otimes \boldsymbol{e}_z + z_{,3}\, \boldsymbol{E}_Z \otimes \boldsymbol{e}_z. \end{aligned}$$

Then direct calculation gives

$$\begin{aligned} \text{Div } \mathbf{F}^T &= \mathbf{e}_r\Big(\triangle r - r(\nabla\tau)^2 + (r/R)_{,1}\Big) + \mathbf{e}_\theta\Big(r\triangle\tau + 2\nabla r\cdot\nabla\tau + (r/R)\tau_{,1}\Big) \\ &+ \mathbf{e}_z\Big(\triangle z + z_{,1}/R\Big). \end{aligned}$$

The calculation uses the fact that the operator Div is the gradient operator followed by contraction [44] and the following elementary formulae for the derivatives of the basic vectors

$$(\mathbf{E}_R)_{,2} = \mathbf{E}_\omega, \quad (\mathbf{E}_\omega)_{,2} = -\mathbf{E}_R, \quad (\mathbf{e}_r)_{,2} = \mathbf{e}_\theta, \quad (\mathbf{e}_\theta)_{,2} = -\mathbf{e}_r,$$

$$(\mathbf{e}_r)_{,i} = \mathbf{e}_\theta \theta_{,i}\,, \quad (\mathbf{e}_\theta)_{,i} = -\mathbf{e}_r \theta_{,i} \;\; i = 1, 3.$$

It is easy to verify that the second term in (2.2.17) is given by the expression

$$\mathbf{e}_r \nabla r \cdot \nabla I_1 + \mathbf{e}_\theta r \nabla \tau \cdot \nabla I_1 + \mathbf{e}_z \nabla z \cdot \nabla I_1.$$

To compute the last term note that the inverse of $\mathbf{F}$ reads

$$\mathbf{F}^{-1} = \begin{bmatrix} rz_{,3}/R & 0 & -r\tau_{,3}/R \\ r(\tau_{,3} z_{,1} - \tau_{,1} z_{,3}) & r/R & r(r_{,3}\tau_{,1} - r_{,1}\tau_{,3}) \\ -rz_{,1}/R & 0 & r\tau_{,1}/R \end{bmatrix}$$

implying that

$$\mathbf{F}^{-T} \nabla p = \frac{r}{R}\Big(\mathbf{e}_r (p_{,1} z_{,3} - p_{,3} z_{,1}) - \mathbf{e}_z (p_{,1} r_{,3} - p_{,3} z_{,1})\Big).$$

Combining the above computations one obtains the following equilibrium equations

$$\begin{aligned} W'\left[\triangle r - r(\nabla\tau)^2 + (r/R)_{,1}\right] + W''\nabla I_1 \cdot \nabla r & \\ -(r/2R)(p_{,1} z_{,3} - p_{,3} z_{,1}) &= 0, \quad (2.2.18) \\ W'\left[r\triangle\tau + 2\nabla r \cdot \nabla\tau + (r/R)\tau_{,1}\right] + W'' r \nabla I_1 \cdot \nabla\tau &= 0, \quad (2.2.19) \\ W'\left[\triangle z + z_{,1}/R\right] + W''\nabla I_1 \cdot \nabla z + (r/2R)(p_{,1} r_{,3} - p_{,3} r_{,1}) &= 0. \quad (2.2.20) \end{aligned}$$

Derivation of the Euler-Lagrange equations is standard. Under the assumptions of the lemma the energy functional in (2.2.14) modified to incorporate the incompressibility constraint takes the form

$$I(\mathbf{u}) = \int_D R\left(W(I_1) - p\left[\frac{r}{R}(r_{,1} z_{,3} - r_{,3} z_{,1}) - 1\right]\right) dRdZ.$$

Then the first Euler-Lagrange equation is

$$\begin{aligned}\left[R\left(W'\frac{\partial I_1}{\partial(r_{,1})}-prz_{,3}/R\right)\right]_{,1} &+ R\left(W'\frac{\partial I_1}{\partial(r_{,3})}+prz_{,1}/R\right)_{,3}\\ &- R\left(W'\frac{\partial I_1}{\partial r}-p(r_{,1}z_{,3}-r_{,3}z_{,1})/R\right)=0.\end{aligned}$$

Using (2.2.6) one arrives after elementary computations at the first equilibrium equation (2.2.18) multiplied by $2R$. Derivation of the other two equations is similar.

The same argument applies when the strain energy density depends on the second invariant I_2, but it requires more technically involved computations. ■

Remark. For neo-Hookean stored energy density (1.0.6) the equilibrium equations stated in the lemma become

$$\mu(\triangle r - r(\nabla\tau)^2 + (r/R)_{,1}) - \frac{r}{R}(p_{,1}z_{,3} - p_{,3}z_{,1}) = 0, \tag{2.2.21}$$

$$r\triangle\tau + 2\nabla r\cdot\nabla\tau + (r/R)\tau_{,1} = 0, \tag{2.2.22}$$

$$\mu(\triangle z + z_{,1}/R) + \frac{r}{R}(p_{,1}r_{,3} - p_{,3}r_{,1}) = 0. \tag{2.2.23}$$

Relative simplicity of the system suggests that for *a priori* simplified forms of functions r, τ, z finding exact solutions could be possible. Some such possibilities will be explored in Chapters 5-7.

Chapter 3

Existence theorems

In this chapter admissible sets appropriate for the Cartesian description of the axisymmetric deformations defined in the previous section will be introduced, and the main existence results for stored energy densities with and without dependence on the cofactor matrix will be stated and proved.

To handle the incompressibility constraint (2.2.12), it is convenient to introduce the following expressions that are similar to the pointwise and the distributional determinants in genuine three-dimensional setting

$$\operatorname{del}(\mathbf{u}) = (\xi^2+\eta^2)_{,1}\, z_{,3} - (\xi^2+\eta^2)_{,3}\, z_{,1}\,, \tag{3.0.1}$$

$$\operatorname{Del}(\mathbf{u}) = D_1\left((\xi^2+\eta^2)z_{,3}\right) - D_3\left((\xi^2+\eta^2)z_{,1}\right). \tag{3.0.2}$$

Now (2.2.12) takes the form

$$\operatorname{del}(\mathbf{u}) = 2R \; a.e. \text{ in } D.$$

Next we define four classes of admissible ordered triplets of functions $\mathbf{u} = (\xi,\, \eta,\, z)$:

$$\begin{aligned}
\mathcal{A}^{p,q} &:= \{\mathbf{u} \in W^{1,p}(D, \mathbb{R}^3) : \mathbf{u} = \mathbf{u}_0 \ a.e. \text{ in } \Gamma,\ \operatorname{cof} \mathbf{F}_0 \in L^q(D, \mathbb{M}^{3\times 3}),\\
&\qquad \operatorname{del}(\mathbf{u}) = 2R \ a.e. \text{ in } D\}\\
\mathcal{A}^{p,q}_d &:= \{\mathbf{u} \in W^{1,p}(D, \mathbb{R}^3) : \mathbf{u} = \mathbf{u}_0 \ a.e. \text{ in } \Gamma,\ \operatorname{Cof} \mathbf{F}_0 \in L^q(D, \mathbb{M}^{3\times 3}),\\
&\qquad \operatorname{del}(\mathbf{u}) = 2R \ a.e. \text{ in } D\}\\
\mathcal{A}^{p} &:= \{\mathbf{u} \in W^{1,p}(D, \mathbb{R}^3) : \mathbf{u} = \mathbf{u}_0 \ a.e. \text{ in } \Gamma,\ \operatorname{del}(\mathbf{u}) = 2R \ a.e. \text{ in } D\}\\
\mathcal{A}^{p}_d &:= \{\mathbf{u} \in W^{1,p}(D, \mathbb{R}^3) : \mathbf{u} = \mathbf{u}_0 \ a.e. \text{ in } \Gamma,\ \operatorname{Del}(\mathbf{u}) = 2R \ a.e. \text{ in } D\}
\end{aligned}$$

where $\Gamma \subset \partial D$, $|\Gamma| > 0$, and $\mathbf{u}_0$ is a specified function in $W^{1,p}(D, \mathbb{R}^3)$.

Although the deformation (1.0.10) is in general three-dimensional, the integrand in the energy functional depends only on two variables. It is this reduction of the space dimension that allows for the relaxation of Ball's *a priori* restrictions on the growth exponents p and q in the coercivity hypothesis.

In the lemma below relations between certain pointwise and distributional null Lagrangians are established. The lemma relies on the following theorem from [41].

Theorem 3.0.1 *Let $\Omega \subset \mathbb{R}^n$ be open, $1 \le p < n$, $v \in W^{1,p}(\Omega)$, and $\boldsymbol{\sigma} \in L^q(\Omega; \mathbb{R}^n)$ for $1/p + 1/q - 1/n \le 1$. If the distributional divergences $\operatorname{Div} \boldsymbol{\sigma}$ and $\operatorname{Div}(v\, \boldsymbol{\sigma})$ belong to $L^1(\Omega)$, then $\operatorname{Div} v\, \boldsymbol{\sigma} = \nabla v(\mathbf{x}) \cdot \boldsymbol{\sigma}(\mathbf{x}) + v\,(\mathbf{x}) \operatorname{Div} \boldsymbol{\sigma}(\mathbf{x})$ a.e. in Ω.*

Lemma 3.0.1 *Let $\mathbf{u} \in W^{1,s}(D, \mathbb{R}^3)$.*

1. *If $s \ge 4/3$, then*

$$\operatorname{Cof} \mathbf{F}_0(\mathbf{u}) \in L^1 \quad \Rightarrow \operatorname{Cof} \mathbf{F}_0(\mathbf{u}) = \operatorname{cof} \mathbf{F}_0(\mathbf{u}).$$

2. If $s \geq 3/2$, then

$$\operatorname{Del}(\mathbf{u}) \in L^1(\Omega) \Rightarrow \operatorname{Del}(\mathbf{u}) = \operatorname{del}(\mathbf{u}).$$

The equalities hold a.e. *in D.*

Proof. Part 1 is proven in [41]. To prove Part 2, we set $v = \xi^2 + \eta^2$, $\boldsymbol{\sigma} = (z_{,3}\,, -z_{,1}\,)$, $s = 3/2$ and check the hypotheses of Theorem 3.0.1. Clearly, $q = 3/2$, $\operatorname{Div}\boldsymbol{\sigma} = 0$, and $\operatorname{Div}(v\,\boldsymbol{\sigma}) = \operatorname{Del}(\mathbf{u}) \in L^1(D)$. By Sobolev embedding theorem (continuous embeddings) $\xi,\ \eta \in L^{s^*}$, where $s^* := 2s/(2-s) = 6$ is the critical Sobolev exponent. Using Hölder's inequality, it is easy to verify that $v \in W^{1,6/5}$. In fact,

$$\int_D |\xi\,\xi_{,i}\,|^{6/5} da \leq \left(\int_D |\xi|^6\right)^{1/5} \left(\int_D |\xi_{,i}\,|^{3/2} da\right)^{4/5} < \infty,\ i = 1, 3.$$

The identical argument applies to $\eta\eta_{,i}\,$. Now the statement of Part 2 follows from Theorem 3.0.1 since $p = 6/5$ is a borderline case for the inequality relating parameters in his theorem. ■

Remark. Note from Lemma 3.0.1 that $\mathcal{A}_d^{p,q} \subset \mathcal{A}^{p,q}$ and $\mathcal{A}_d^{p} \subset \mathcal{A}^{p}$.

The following theorem gives sufficient conditions for the existence of a minimizer of the reduced stored energy functional (2.2.14) when the stored energy density W depends explicitly on both $\mathbf{F}$ and $\operatorname{cof}\mathbf{F}$. The first statement ensures the existence of a solution to the minimization problem under weaker restrictions on the growth exponents p and q, but in a smaller set of mappings for which the entries of the distributional cofactor matrices are functions.

Theorem 3.0.2 *Let $\hat{W}(\mathbf{F}) = W(\mathbf{F}, \mathrm{cof}\ \mathbf{F})$ and $q > 1$. Then the energy functional (2.2.14) assumes its minimum in admissible set $\mathcal{A}$ in each of the following cases:*

1. $\mathcal{A} = \mathcal{A}_d^{p,q}$, $p > 4/3$, *and* $p^{-1} + q^{-1} \leq 3/2$.

2. $\mathcal{A} = \mathcal{A}^{p,q}$, $p \geq 2$.

The only fact that needs to be proven is that the admissible sets are closed with respect to weak convergence of mappings $\mathbf{u}$ and corresponding cofactors. The rest of the argument is standard (*cf.* discussion in Section 2.2). To begin, we need the following lemma concerning weak continuity properties of $\mathrm{Cof}\,(\cdot)$, $\mathrm{Del}\,(\cdot)$, and $\mathrm{del}\,(\cdot)$.

Lemma 3.0.2 1. *Let $p > 4/3$. Then the mapping $\mathrm{Cof}_{\,0} : W^{1,p}(D, \mathbb{R}^3) \to \mathcal{D}'$ defined by $\mathrm{Cof}_{\,0}(\mathbf{u}) = \mathrm{Cof}\ \mathbf{F}_0(\mathbf{u})$ is weakly continuous, i.e.,*

$$\mathbf{u}_k \rightharpoonup \mathbf{u} \ \textit{in}\ W^{1,p}(D, \mathbb{R}^3) \Rightarrow [\mathrm{Cof}\ \mathbf{F}_0(\mathbf{u}_k)]_{ij} \to [\mathrm{Cof}\ \mathbf{F}_0(\mathbf{u})]_{ij} \ \textit{in}\ \mathcal{D}'(D).$$

2. *Let $p > 3/2$. Then the mapping $\mathrm{Del} : W^{1,p} \to \mathcal{D}'$ is weakly continuous, i.e.,*

$$\mathbf{u}_k \rightharpoonup \mathbf{u} \ \textit{in}\ W^{1,p}(D, \mathbb{R}^3) \Rightarrow \mathrm{Del}\,(\mathbf{u}_k) \to \mathrm{Del}\,(\mathbf{u}) \ \textit{in}\ \mathcal{D}'(D).$$

3. *Let $p > 4/3$, $q > 1$, and $p^{-1} + q^{-1} \leq 3/2$. Then*

$$\{\mathbf{u}_k \rightharpoonup \mathbf{u} \ \textit{in}\ W^{1,p}(D, \mathbb{R}^3) \quad \textit{and} \quad \mathrm{cof}\ \mathbf{F}_0(\mathbf{u}_k) \rightharpoonup \mathrm{cof}\ \mathbf{F}_0(\mathbf{u}) \ \textit{in}\ L^q(D, \mathbb{M}^{3\times 3})\}$$

$$\Rightarrow \mathrm{del}\,(\mathbf{u}_k) \to \mathrm{del}\,(\mathbf{u}) \ \textit{in}\ \mathcal{D}'(D).$$

Proof. 1. This fact is well known (*cf.* . [4], [16]) and is stated here for completeness.

2. From Relich-Kondrakov theorem (compact embeddings) we have that weak convergence of the sequence $\mathbf{u}_k$ in $W^{1,p}$ ensures strong convergence of some subsequence (not relabelled) $\mathbf{u}_k$ in L^q for any q such that $1 < q < p^*$, where p^* is the critical Sobolev exponent. In particular, for $p > 3/2$ this yields

$$(\xi^k)^2 \to \xi^2 \quad \text{and} \quad (\eta^k)^2 \to \eta^2 \text{ in } L^3.$$

Since $1/p + 1/3 < 1$, it follows from Hölder's inequality that products of the form $(\xi^k)^2 z^k_{,m}$ and $(\eta^k)^2 z^k_{,m}$ for $m = 1, 3$, are integrable in D. Consequently, $\mathrm{Del}\,(\mathbf{u}_k) \in \mathcal{D}'$ and therefore for any fixed $\phi \in \mathcal{D}(D)$

$$< \mathrm{Del}\,(\mathbf{u}_k), \phi >:= -\int_D \left((\xi^k)^2 + (\eta^k)^2\right) \left(z,^k_3 \phi,_1 - z,^k_1 \phi,_3\right) da \ \to < \mathrm{Del}\,(\mathbf{u}), \phi >,$$

thereby proving the second part of the lemma.

3. First note that

$$\mathrm{del}\,(\mathbf{u}_k) = \xi^k\,(\mathrm{cof}\ \mathbf{F}_0(\mathbf{u}_k))_{22} - \eta^k\,(\mathrm{cof}\ \mathbf{F}_0(\mathbf{u}_k))_{12}.$$

As in Part 2, it can be inferred that there exist subsequences ξ^k, η^k such that $\xi^k \to \xi$ and $\eta^k \to \eta$ in L^s for $1 < s < p^*$. Combining this observation with the assumed weak convergence of the cofactor matrix and bounds on the growth exponents concludes the proof. ■

Proof of Theorem 3.0.2 . 1. By Lemma 3.0.1, Part 1, $\mathrm{Cof}\ \mathbf{F}_0(\mathbf{u}) = \mathrm{cof}\ \mathbf{F}_0(\mathbf{u})$ for $\mathbf{u} \in \mathcal{A}_d^{p,q}$, and existence of a weakly convergent in $W^{1,p}(D, \mathbb{R}^3)$ minimizing sequence $\mathbf{u}_k = (\xi^k, \eta^k, z^k)$, as well as boundedness of $\mathrm{cof}\ \mathbf{F}_0(\mathbf{u}_k)$ in L^q are established in a standard way. (*cf.* discussion in Section 2.2). Hence, for some subsequence $\mathbf{u}_k$ (not relabelled) we have

$$\text{cof } \mathbf{F}_0(\mathbf{u}_k) \rightharpoonup \mathbf{H} \text{ in } L^q.$$

Now by Lemma 3.0.2, Part 1, one concludes that $\mathbf{H} = \text{Cof } \mathbf{F}_0(\mathbf{u})$, thus proving that Cof $\mathbf{F}_0(\mathbf{u}) \in L^q$.

Consequently, the assumptions of Lemma 3.0.2, Part 3, hold, thereby implying that $\text{del}\,(\mathbf{u}_k) \to \text{del}\,(\mathbf{u})$ in $\mathcal{D}'$. Weak convergence of $\text{del}\,(\mathbf{u}_k)$ to $2R$ in L^r for any $r > 1$ follows from the incompressibility constraint. Hence $\text{del}\,(\mathbf{u}) = 2R$ *a.e.* and therefore $\mathbf{u} \in \mathcal{A}_d^{p,q}$, proving Part 1 of the theorem.

2. If $\mathbf{u} \in W^{1,p}$, $p \geq 2$, the pointwise and distributional cofactor matrices of $\mathbf{F}_0(\mathbf{u})$ coincide, since for any fixed function $\phi \in \mathcal{D}(D)$ and for any fixed pair of indices i, j, $1 \leq i, j \leq 3$, the functionals

$$g_\phi(\mathbf{u}) = \int_D (\text{cof } \mathbf{F}_0(\mathbf{u}))_{ij}\, \phi da \quad \text{and} \quad f_\phi(\mathbf{u}) =< (\text{Cof } \mathbf{F}_0(\mathbf{u}))_{ij}\,, \phi >$$

coincide on the dense set $C^2(D, \mathbb{R}^3) \subset W^{1,p}(D, \mathbb{R}^3)$ and are continuous in $W^{1,p}(D, \mathbb{R}^3)$ norm. (cf. [16], Theorem 7.5.1.) Therefore Part 2 is a particular case of Part 1 of the theorem. ■

Remark. To compare the assertions of Theorem 3.0.2 with analogous results in the genuine three-dimensional case, recall that the restrictions on the growth exponents for the analog of the second statement of the theorem are $p \geq 2$, $q \geq 3/2$ [42]. If one seeks a minimizer in a set

$$\{\mathbf{u} \in W^{1,p} : \text{ Cof } \mathbf{F} \in L^q, \text{Det}\, \mathbf{F} \in L^1\},$$

the restrictions are $p \geq 3/2,\ p^{-1} + q^{-1} \leq 4/3$ [41].

The next theorem provides conditions for the existence of a minimizer when the stored energy density does *not* depend on cof $\mathbf{F}$. As in Theorem 3.0.2, the first statement ensures the existence of a solution to the minimization problem under weaker restrictions on the growth exponent p, but in a smaller set of mappings $\mathbf{u}$ for which the distributional counterpart (3.0.2) of the expression del ($\mathbf{u}$) is a function.

Theorem 3.0.3 *Let* $\hat{W} = W(\mathbf{F})$. *Then the energy functional (2.2.14) assumes its minimum in admissible set* $\mathcal{A}$ *in each of the following cases:*

1. $\mathcal{A} = \mathcal{A}^p_d, \qquad p > 3/2.$

2. $\mathcal{A} = \mathcal{A}^p, \qquad p > 2.$

Furthermore, for $p > 2$ *any minimizer* $\mathbf{u}$ *belongs to Hölder space* $C^{0,\alpha}(\bar{D})$ *with* $0 \leq \alpha \leq 2/p$, *and there exists a minimizing sequence* $\mathbf{u}_k$ *converging to* $\mathbf{u}$ *in* $C^{0,\alpha}(\bar{D})$*-norm for* $0 \leq \alpha < 2/p$.

Proof. 1. It follows from Lemma 3.0.2, Part 2, that for any minimizing sequence $\mathbf{u}_k$ converging weakly in $W^{1,p}$ to a function $\mathbf{u}$ one has

$$\mathrm{Del}\,(\mathbf{u}_k) \to \mathrm{Del}\,(\mathbf{u}) \text{ in } \mathcal{D}'.$$

On the other hand, the definition of the admissible set $\mathcal{A}^p_d$ implies

$$\mathrm{Del}\,(\mathbf{u}_k) \to 2R \quad \text{a.e. in } D$$

for some subsequence (not relabelled). Therefore $\mathrm{Del}\,(\mathbf{u}) = 2R$ a.e. in D. The rest of the proof is standard.

2. For $p > 2$ there exists a minimizing sequence $\mathbf{u}_k$ and some $q > 1$ such that

$$\operatorname{cof} \mathbf{F}_0(\mathbf{u}_k) \rightharpoonup \operatorname{cof} \mathbf{F}_0(\mathbf{u}) \text{ in } L^q.$$

Then the weak closedness of the admissible set follows immediately from Lemma 3.0.2, Part 3.

The statement about regularity and convergence in Hölder spaces follows from Sobolev-Relich-Kondrakov theorems. ■

Remark. Note that the first statement of the Theorem 3.0.3 ensures the existence of a minimizer for neo-Hookean materials in the admissible set $\mathcal{A}^2_d$. It is tempting to obtain existence result for the case $p = 2$ in the larger set $\mathcal{A}^2$. This would be possible if $W^{1,2}(D) \subset L^\infty_{loc}(D)$, but it is well known that in general a function $f \in W^{m,p}(\Omega)$, $\Omega \subset \mathbb{R}^n$ with $mp = n$, and $n > 1$, does not belong to $L^\infty(\Omega)$. An example is provided by a function $f = |\log|x||^{1-2/(n-1)}$, $n \geq 2$, defined in the ball $B(0,r)$, $r < 1$, [55]. However, as will be shown in the next chapter, the existence theorem can be extended to the marginal case $p = 2$ under an additional constraint on the radial component in *cylindrical* description of the deformation.

Chapter 4

Global injectivity of axisymmetric minimizers

The mere existence of a minimizer for a problem in nonlinear elasticity is not quite satisfactory. It is desirable to ensure some realistic properties of solutions, e.g., injectivity, which physically means that interpenetration of matter does not occur. For smooth mappings $\mathbf{u} \in C^1(\Omega)$ *local* invertibility follows from positivity of the determinant. However, this does not prevent overlapping of parts of the image $\mathbf{u}(\Omega)$. In this chapter the global injectivity of minimizers for cylindrical description of axisymmetric deformations is investigated. We make use of some properties of *mappings of finite distortion* that are collected in Section 4.1. The results on global injectivity of admissible functions as well as the extension of Theorem 3.0.3 to the case $p = 2$ are stated and proved in Section 4.2.

If a uniform positivity assumption (1.0.13) is imposed on the radial component, the direct method of the calculus of variations applies to axisymmetric deformations in cylindrical description, $\mathbf{u} = (r, \tau, z)$. Then existence of a minimizer for the problem (2.2.14) in the admissible set $\mathcal{A}^p$ for $p > 2$, stated in the second part of Theorem 3.0.3, can be obtained in exactly the same manner for the minimization problem (2.2.13) in the admissible set of triplets of functions $\mathbf{u} = (r, \tau, z)$ defined by

$$\mathcal{A}^p_\alpha := \{\mathbf{u} \in W^{1,p}(D, \mathbb{R}^3) : \mathbf{u} = \mathbf{u}_0 \ a.e. \text{ in } \Gamma, \ (2.2.5), (1.0.13) \text{ hold } \text{ a.e. in } D\}$$

with $\alpha > 0$.

4.1 Some properties of mappings of finite distortion

Ball's existence theory in nonlinear elasticity motivated introduction of a class of mappings of finite distortion since the admissible functions he introduced belong to this class. A class of mappings of finite distortion includes well known mappings of bounded distortion (or, equivalently, quasiregular mappings) [48]. The latter is a generalization of classical quasiconformal mappings [1], [25]. In this section some properties of functions of finite distortion needed in the sequel are stated. These properties will allow one to obtain essentially *sharper* injectivity results in axisymmetric setting compared with those presented in [5], [15], and [52].

Definition 4.1.1 *Let Ω be a bounded connected, open subset in $\mathbb{R}^n$.*

A mapping $f : \Omega \to \mathbb{R}^n$ is said to be a mapping of finite distortion (MFD) if

1. $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$

2. *The Jacobian $J(x, f)$ of f is locally integrable and does not change sign in Ω.*

3. *There is a measurable function $K : \Omega \to \mathbb{R}$ such that $K(x) \geq 1$, finite almost everywhere, and f satisfies the dilatation inequality*

$$|Df(x)|^n \leq K(x)|J(x, f)| \quad \text{a.e. in } \Omega. \tag{4.1.1}$$

The smallest of such functions $K(\cdot)$, $K(x, f)$, is called the dilatation, or distortion, quotient.

In the theorem below some of the properties of functions of finite distortion are listed. The theorem is similar to Theorem 1.3 in [26] (parts 1, 3, 5 in both theorems are identical.)

Theorem 4.1.1 *Let $f \in W^{1,n}(\Omega, \mathbb{R}^n)$ be a MFD. Then*

1. *f has a continuous representative.*

2. *The following estimate of the modulus of continuity holds*

$$|f(x) - f(y)| \leq C(n, B)||\nabla f||_{L^n(\Omega)} \left|\log \frac{|x-y|}{2R}\right|^{-1/n}. \tag{4.1.2}$$

with arbitrary $x,\ y \in B(a, R) \subset B(a, 2R) \subset \Omega$.

3. *f is differentiable a.e.*

4. For every measurable set $G \subset \Omega$ the inequality

$$|f(G)| \leq \int_G |J(x, f)| dx$$

holds.

5. f satisfies condition (N), i.e., $|f(E)| = 0$ whenever $E \subset \Omega$ and $|E| = 0$.

Proof. See Theorem 1.3 in [26] for references to proofs of assertions in Parts 1, 3, 5.

Part 2 is a particular case of Theorem 7.5.1 from [31]. The theorem is stated there for *weakly monotone* functions in *Orlich-Sobolev* spaces $W^P(\Omega)$ with Orlich function P satisfying the following conditions:

$$\int_1^\infty P(t) \frac{dt}{t^{n+1}} = \infty,$$

$$\text{the function } t \mapsto P\left(t^{(2n+1)/(2n^2)}\right) \text{ is convex.}$$

Clearly, function $P(t) = t^n$, corresponding to Sobolev space $W^{1,n}(\Omega)$, satisfies those conditions. Without introducing the notion of weak monotonicity, we refer to Theorem 7.3.1 in [31], which states that the coordinate functions of a mapping $f \in W^{1,n}(\Omega, \mathbb{R}^n)$ with finite dilatation has this property.

Part 4 follows from Theorem 1.4 in [25], Pg.274. The assumptions of the theorem are ensured by Part 3 and Part 5 of Theorem 4.1.1, and by local integrability of the Jacobian $J(x, f)$. ■

In the following lemma it is shown that for any relatively compact set $G \Subset \Omega$ a bounded set of MFD from Sobolev space with natural exponent is pre-compact in $C(\bar{G})$.

Lemma 4.1.3 *Let a sequence $g_k \in W^{1,n}(\Omega, \mathbb{R}^n)$ of mappings of finite distortion converge weakly in $W^{1,n}(\Omega, \mathbb{R}^n)$ to some mapping g. Then for any relatively compact domain $G \subset \Omega$ there exists a subsequence converging uniformly in $\bar{G}$.* [1]

Proof: We show that a sequence of i^{th} components g_k^i of mappings g_k is equicontinuous and uniformly bounded. To simplify notation introduce the scalar functions $f_k = g_k^i$, $f = g^i$. Clearly, it follows from (4.1.2) that given $\epsilon > 0$ one can find δ, $0 < \delta < dist(\bar{G}, \partial\Omega)/2$, such that for any pair $x, y, \in G$, $|x - y| < \delta$, the left hand side in (4.1.2) will be less than ϵ. This proves equicontinuity of the sequence f_k.

For a fixed δ as above, there exists a finite covering $B_j = B(x_0^j, \delta)$, $j = 1, \ldots, N$, of $\bar{G}$. By Relich-Kondrakov compact embedding theorem the sequence f_k is pre-compact in any space L^s, $s \geq 1$. Since for any sequence of functions converging strongly in L^s there exists a subsequence that converges almost everywhere, one can assume that for some subsequence, not relabeled, $f_k(x_0^j) \to f(x_0^j)$ for every j. Uniform boundedness of f_k in $\bar{G}$ then follows from the inequality

$$|f_k(x)| \leq |f_k(x) - f_k(x_0^j)| + |f_k(x_0^j) - f(x_0^j)| + \max_{1 \leq j \leq N} |f(x_0^j)|,$$

where $x \in B_j$. Note that the first term on the right is uniformly bounded due to equicontinuity. The statement of the lemma then follows from Arzela-Ascoli theorem.

■

[1] It was known to Lebesgue [34] that a family of continuous and *monotone* functions with bounded Dirichlet integral is equicontinuous.

If a nonconstant mapping f has finite distortion with integrable dilatation quotient (by definition f is called quasiregular in this case), then, by a fundamental result of Reshetnyak [48], the mapping is open and discrete. Here 'discrete' means that the preimage of a point $y \in f(\Omega)$ is a discrete subset of Ω, i.e., it does not have cluster points. These properties were recently carried over to two-dimensional MFD with *integrable* dilatation quotient [30] . The result is stated in the theorem below.

Theorem 4.1.2 *Let Ω be a bounded domain in the complex plane $(\mathbb{C}, \sigma(z))$, $\sigma(z)$ being the area element, and $f \in W^{1,2}(\Omega, \mathbb{C})$ with $J(z,f) \geq 0$ and $K(\cdot, f) \in L^1(\Omega)$. Then there exists a homeomorphism $h : \Omega' \to \Omega$, with $\Omega' = h^{-1}(\Omega)$ and a holomorphic function $\phi : \Omega' \to \mathbb{R}^2$ such that*

$$f = \phi \circ \boldsymbol{h}^{-1}.$$

Remark. If the conclusion of the Theorem 4.1.2 holds, function f is said to admit Stiolow's type factorization. Then, obviously, f is open and discrete.

We will need also a change of variables formula for functions of finite distortion from Sobolev space $W^{1,n}(\Omega, \mathbb{R}^n)$. The following fragment of Theorem 2.2 from [48], Pg. 99, will be sufficient for our purposes.

Theorem 4.1.3 *Let Ω be an open set in $\mathbb{R}^n$, and $f : \Omega \to \mathbb{R}^n$ a continuous mapping. Assume that*

1. *f has property N;*
2. *f is differentiable almost everywhere in Ω;*

3. Function $x \to J(x,f)$ is locally integrable in Ω.

Then for every nonnegative function $g : \Omega \to \mathbb{R}$ the function

$$y \to N(y,f,g), \qquad N(y,f,g) = \sum_{x=f^{-1}(y)} g(x)$$

is measurable in $\mathbb{R}^n$ and

$$\int_{\mathbb{R}^n} N(y,f,g)dy = \int_{\Omega} g(x)|J(x,f)|dx. \tag{4.1.3}$$

A particular version of this theorem with $g \equiv 1$ will be used in a sequel. Note that $N(y,f,1)$ is simply a multiplicity function.

Let $p \geq 2$. Introduce a set of triplets of functions

$$\mathcal{A}_0^p := \{\mathbf{u} \in W^{1,p}(D,\mathbb{R}^3) : r \geq 0,\ (2.2.5),\ \text{hold a.e. in } D\} \tag{4.1.4}$$

and recall the definition (1.0.11) of two-dimensional mapping corresponding to any $\mathbf{u} = (r,\tau,z)$,

$$\mathbf{v} :\ D \to \mathbb{R}^2, \qquad \mathbf{v}(\mathbf{X}') = (r(\mathbf{X}'),\ z(\mathbf{X}')),\ \mathbf{X}' = (R,Z) \in D.$$

In the lemma below two remarkable properties of the two-dimensional mapping $\mathbf{v}$ corresponding to $\mathbf{u} \in \mathcal{A}_0^p$ are stated.

Lemma 4.1.4 *Let $\mathbf{u} = (r,\tau,z) \in \mathcal{A}_0^p$. Then the mapping $\mathbf{v} = (r,z)$ is an open and discrete MFD.*

Proof. We show that $\mathbf{v}$ is MFD. The notation in Definition 4.1.1 translates according to

$$|Df(x)|^n = |\nabla \mathbf{v}|^2,\ J(x,f) = \det \nabla \mathbf{v},\ K(x,f) = K(\mathbf{X}',\mathbf{v}) = |\nabla \mathbf{v}|^2 / \det \nabla \mathbf{v}.$$

Now the definition of $\mathcal{A}_0^p$ implies that the Jacobian of $\mathbf{v}$, $\det \nabla \mathbf{v} = R/r$ is positive a.e. in D, and the dilatation quotient is finite almost everywhere.

To prove openness and discreteness we employ Theorem 4.1.2. Since D is strongly Lipschitz and lies in the half plane $\{(R,Z) : R > 0\}$, we can find a sequence of open relatively compact strongly Lipschitz subdomains $G_i \subset \{(R,Z) : R > \alpha_i > 0\}$ such that

$$G_1 \subset \bar{G}_1 \subset G_2 \subset \ldots, \qquad D = \bigcup_{j=1}^{\infty} G_j. \tag{4.1.5}$$

It suffices to show that for any relatively compact subset G in D the dilatation quotient of the mapping $f = \mathbf{v}|_G$ is integrable. In fact, due to the incompressibility constraint the dilatation quotient satisfies the inequality

$$K(X',f) \leq |D\mathbf{v}(X')|^2/\det \mathbf{v} \leq \frac{(r/R)}{(r/R)\det \mathbf{v}}|D\mathbf{v}(X')|^2 = (r/R)|D\mathbf{v}(X')|^2$$

almost everywhere in G. By Theorem 4.1.1, Part 1, the mapping f is continuous in $\bar{G}$ and therefore bounded. Recall that for all points (R,Z) of the domain D we have $R > R_i > 0$. Therefore there exists a positive constant C so that

$$K(X',f) \leq C|D\mathbf{v}(X')|^2,$$

and the integrability of the dilatation quotient in G follows. Since any open set $E \subset D$ can be represented as a union of open relatively compact sets, $E = \bigcup_{i=1}^{\infty} G_i \cap E$, this completes the proof. ■

Remark. In the compressible case, if $\mathbf{u} = (r, \tau, z) \in W^{1,2}(D, \mathbb{R}^3)$ and

$$\det \mathbf{F} = \frac{r}{R}(r_{,1}\, z_{,3} - r_{,3}\, z_{,1}) > 0 \text{ a.e. in } D,$$

then the two-dimensional mapping $\mathbf{v}$ still has finite distortion provided $r \geq 0$ a.e., and therefore Theorem 4.1.1 remains valid for the mapping. But openness and discreteness are not available for $\mathbf{v}$ in this case without additional assumption, for example, $\det \mathbf{F} \geq \beta > 0$ *a.e.* in D.

4.2 Global injectivity theorems

Now we are in a position to examine the injectivity of axisymmetric minimizers.

For Dirichlet boundary conditions, the main result in [5] on global invertibility can be applied to the two-dimensional mappings $\mathbf{v}$ corresponding to $\mathbf{u}$ from an appropriate admissible set *without* imposing any condition on the adjugate matrix. This is shown in the next theorem.

We recall that a domain $U \in \mathbb{R}^n$ is said to satisfy the *cone condition* if for all $x \in U$ a set $\{x + E(e(x))\}$ is a subset of U, where $E(e(x))$ is the right circular cone of fixed radius and height with vertex at the origin, and a vector $e(x)$ specifies the direction of the axis of the cone.

Theorem 4.2.4 *Let $p > 2$, $\mathcal{A} = \mathcal{A}^p_{\mathrm{cyl}}$ with $\Gamma = \partial D$. If $\mathbf{u}_0$, defining the boundary condition of place, is such that the corresponding $\mathbf{v}_0$ is continuous in $\bar{D}$, one-to-one in D, and $\mathbf{v}_0(D)$ satisfies the cone condition, then any $\mathbf{u} \in \mathcal{A}$ (in particular, any minimizer) is a homeomorphism of D onto $\mathbf{v}_0(D)$, and the inverse function $\mathbf{X}'(\cdot)$ belongs to $W^{1,p}(\mathbf{v}_0(D))$.*

If $\mathbf{v}_0(D)$ is strongly Lipschitz, then $\mathbf{v} : \bar{D} \to \mathbf{v}_0(\bar{D})$ is a homeomorphism.

Further, in the former (latter) case corresponding three-dimensional deformation

$$(R, \omega, Z) \to (r, \omega + \tau, z)$$

is also a homeomorphism of Ω ($\bar{\Omega}$) *onto its image.*

Proof. All assertions of the theorem, except for the last one, are identical to those in Theorem 2 of [5] with the only missing condition

$$\int_D |(\nabla \mathbf{v})^{-1}(\mathbf{X}')|^p \det \nabla \mathbf{v}(\mathbf{X}')\, d\mathbf{X}' < \infty.$$

This condition can be established in exactly the same manner as the integrability of the dilatation quotient in Lemma 4.1.4.

The last statement follows from the well-known general fact that a one-to-one, continuous, and open mapping $f : U \to V$ of topological space U into topological space V is a homeomorphism of U onto $f(U)$. ■

To examine the injectivity of minimizers when boundary condition is prescribed only on the part of the boundary, we need an analogue of the famous *injectivity condition* by Ciarlet and Nečas [15], viz.

$$\int_\Omega \det \mathbf{F}(\mathbf{u})\, dX \le |\mathbf{u}(\Omega)|. \tag{4.2.6}$$

For isochoric axisymmetric deformation this condition simplifies to

$$\int_D R\, d\mathbf{X}' \le \int_{\mathbf{v}(D)} r\, d\mathbf{x}', \qquad \mathbf{x}' = (r, z), \tag{4.2.7}$$

which can be recast into

$$\int_D \det \nabla \mathbf{w}\, d\mathbf{X}' \le |\mathbf{w}(D)|, \tag{4.2.8}$$

where the mapping $\mathbf{w}$ is defined as

$$\mathbf{w} : D \to \mathbb{R}^2 \quad \mathbf{w}(\mathbf{X}') = (\rho, z) := (r^2, z), \tag{4.2.9}$$

where the incompressibility constraint (2.2.5) was employed. Inequality (4.2.8) is the injectivity condition for axisymmetric setting.

Remark. Although for isochoric deformations (4.2.7) looks simpler, the injectivity condition in the form (4.2.8) is similar to commonly used three-dimensional version (4.2.6) and, more importantly, can be used for the compressible case as well.

Now we are ready to carry over the injectivity results obtained in [15] and [52] to the axisymmetric problem under consideration with essential sharpening due to higher regularity and the openness of the two-dimensional mapping $\mathbf{v}$. For a fixed $\alpha > 0$ introduce an admissible set

$$\mathcal{A}_I^p := \{\mathbf{u} \in \mathcal{A}_\alpha^p \text{ such that (4.2.8) holds}\}.$$

The next theorem is the main result of this section.

Theorem 4.2.5 *Let $\hat{W} = W(\mathbf{F})$ and $p \geq 2$. Then*

1. *The energy functional (2.2.14) assumes its minimum in $\mathcal{A}_I^p$.*

2. *For any $\mathbf{u} \in \mathcal{A}_I^p$ (in particular, any minimizer) the corresponding two - dimensional mapping $\mathbf{v} : D \to \mathbf{v}(D)$ is a homeomorphism, and $\mathbf{v}^{-1} \in W^{1,p}_{loc}(\mathbf{v}(D), \mathbb{R}^2)$.*

3. *If $p > 2$ then for any admissible function (in particular, any minimizer) $\mathbf{u} = (r, \tau, z) \in \mathcal{A}_I^p$ a mapping $\hat{\mathbf{u}} := (r, \omega + \tau, z)$ is a homeomorphism of Ω onto $\hat{\mathbf{u}}(\Omega)$.*

Proof. 1. It suffices to show that the incompressibility condition and the injectivity condition are preserved by weak limits of the elements from $\mathcal{A}_I^p$. Let $\mathbf{u}_k \in \mathcal{A}_I^p$, $\mathbf{u}_k \rightharpoonup \mathbf{u}$ in $W^{1,p}(D, \mathbb{R}^3)$.

Incompressibility condition for $\mathbf{u}$ will follows from weak continuity of the mapping

$$\text{del} \; : \; W^{1,p}(D, \mathbb{R}^3) \to \mathcal{D}'(D).$$

Note that condition $r(R, Z) \geq \alpha > 0$ combined with the incompressibility constraint implies that

$$\det \nabla \mathbf{v}_k = R/r \leq C \;\text{ a.e. in } D$$

with a constant C independent on k, $C \geq \max_{\bar{D}} R/\alpha$. Therefore for an arbitrary fixed s, $t > 1$, there exists a subsequence $\mathbf{u}_k$ (not relabelled) such that $r_k \to r$ in L^s and $\det \nabla \mathbf{v}_k \rightharpoonup \det \nabla \mathbf{v}$ in L^t. Since $\text{del}\,(\mathbf{u}_k) = 2r \det \nabla \mathbf{v}_k$, the weak continuity of the mapping del follows.

To prove that the injectivity condition (4.2.8) is preserved by weak limits, it suffices to show that the injectivity condition holds for any $G \subset\subset D$, i.e.,

$$\int_G \det \nabla \mathbf{w} \, d\mathbf{X}' \leq |\mathbf{w}(G)| \tag{4.2.10}$$

for any $\mathbf{w}$ corresponding to a weak limit of a sequence of admissible functions. Indeed, then writing this condition with $G = G_k$, where G_k is a subdomain from (4.1.5), and passing to the limit as $k \to \infty$, we obtain the injectivity condition in D.

Firstly, we prove (4.2.10) for any $\mathbf{w}$ corresponding to a mapping $\mathbf{u} \in \mathcal{A}_I^p$. By

Theorem 4.1.1, Part 4, for any measurable set $E \subset D$ the inequality

$$|\mathbf{w}(E)| \leq \int_E \det \mathbf{w} dR\, dZ \tag{4.2.11}$$

holds. If we assume that for some $G \subset D$ (4.2.10) does not hold, then, by virtue of injectivity condition (4.2.8), there must be a set $E \subset D$ of positive measure such that the inequality opposite to (4.2.11) must hold.

Note that this contradiction implies even more than was claimed. In fact, it has been proven that for any $G \subset D$ (4.2.10) holds with strict equality.

Secondly, we prove that (4.2.10) is preserved by weak limits. Note that Theorem 7.9.1 from [16] does not apply directly to two-dimensional mappings $\mathbf{w}_k$ in case $p = 2$ because these mappings belong to the space $W^{1,2}(G, \mathbb{R}^2)$ while $\mathbf{w}_k \in W^{1,p}$ with $p > 2$ is needed. However, the argument, used in the theorem, applies since it relies on the following facts:

- Functions $\mathbf{w}_k$, $\mathbf{w}$ have (N) property.
- Functions $\mathbf{w}_k$ are continuous, and the sequence converges to $\mathbf{w}$ uniformly in $\bar{G}$.
- $\det \nabla \mathbf{w}_k \rightharpoonup \det \nabla \mathbf{w}$ in L^q for some $q > 1$.

The last property is just the weak continuity of the function del, which is proved above, the other two follow from Theorem 4.1.1, Parts 1, 5, and Lemma 4.1.3. Using those facts we reproduce the argument of Theorem 7.9.1 [16] below with appropriate modifications.

Since the set $\mathbf{w}(\bar{G})$ is compact, whence measurable, there exists, by a classical property of the Lebesgue measure, an open set $O(\epsilon)$ such that

$$\mathbf{w}(\bar{G}) \subset O(\epsilon), \qquad |O(\epsilon) \setminus \mathbf{w}(\bar{G})| < \epsilon.$$

Then it is easy to show [16] that there exists a number $\delta(\epsilon) > 0$ such that

$$\bigcup B\left(y', \delta(\epsilon)\right) \subset O(\epsilon),$$

where the union is taken over all $y' \in \mathbf{w}(\bar{G})$. Hence there exists an integer $K = K(\epsilon)$ such that

$$\mathbf{w}_k(\bar{G}) \subset O(\epsilon), \text{ for all } k \geq K,$$

since $\mathbf{w}_k$ converges to $\mathbf{w}$ uniformly in $\bar{G}$. Employing (4.2.10) one obtains

$$\int_G \det \nabla \mathbf{w}_k \, d\mathbf{X}' \leq |\mathbf{w}_k(G)| = |\mathbf{w}_k(\bar{G})| \leq |O(\epsilon)|, \text{ for all } k \geq K,$$

where we used the fact that functions $\mathbf{w}_k$ have (N) property.

Passing to the limit as $k \to \infty$ one obtains

$$\int_G \det \nabla \mathbf{w} \, d\mathbf{X}' \leq |O(\epsilon)| = |\mathbf{w}(\bar{G})| + |O(\epsilon) \setminus \mathbf{w}(\bar{G})| \leq |\mathbf{w}(G)| + \epsilon,$$

where (N) property of the function $\mathbf{w}$ was used. Since ϵ is arbitrary, this proves (4.2.10) for weak limits and therefore completes the proof of Part 1.

2: Making use of the change of variables formula (4.1.3) with $g \equiv 1$

$$\int_D \det \nabla \mathbf{w} \, d\mathbf{X}' = \int_{\mathbf{w}(D)} N(\mathbf{w}, y') \, dy',$$

where $\boldsymbol{y}' = (\rho, z) = (r^2(\mathbf{X}'), z(\mathbf{X}'))$, and following the argument in [16] one derives

$$|\mathbf{w}(D)| = \int\limits_{\mathbf{w}(D)} d\rho\, dz \leq \int\limits_{\mathbf{w}(D)} N(\mathbf{w},\, \boldsymbol{y}')\, d\boldsymbol{y}' = \int\limits_{D} \det \nabla\mathbf{w}\, d\mathbf{X}' \leq |\mathbf{w}(D)|.$$

Since $N(\mathbf{w}, \boldsymbol{y}') = N(\mathbf{v}, \mathbf{x}')$, this implies that $N(\mathbf{v}, \mathbf{x}') = 1$ for *almost all* $\mathbf{x}' \in \mathbf{v}(D)$. Suppose $N(\mathbf{v}, \mathbf{x}') > 1$ for some $\mathbf{x}' \in \mathbf{v}(D)$. By Lemma 4.1.4 the mapping $\mathbf{v}$ is open. Therefore there exists a neighborhood $U \subset D$ of $\mathbf{x}'$ such that $N(\mathbf{v}, \mathbf{x}') > 1$, $\forall\, \mathbf{x}' \in U$. This contradiction implies that $N(\mathbf{v}, \mathbf{x}') = 1$ for all $\mathbf{x}' \in \mathbf{v}(D)$. Noting that a continuous, open, and bijective mapping is a homeomorphism, the proof of the first statement in Part 2 is complete.

The second statement follows from Theorem 3.1 in [22].

3. If $p > 2$ the tree-dimensional mapping $\hat{\mathbf{u}}$ is continuous, open and bijective. ■

Remark 1. A version of Theorem 4.2.5 can be proved in the compressible case with appropriate modifications.

The argument used for proving the weak continuity of the determinant in Part 1 of the theorem fails. However, this property can be established as follows.

Let $\mathbf{u}_k \rightharpoonup \mathbf{u}$ in $W^{1,2}(D,\, \mathbb{R}^3)$, $\operatorname{del}(\mathbf{u}) > 0$ *a.e.*, and $r \geq \alpha > 0$ *a.e.* Then corresponding two-dimensional mappings $\mathbf{v}_k$ are continuous, and for any relatively compact $G \subset\subset D$ we have (up to a subsequence, not relabeled)

$$r_k\, r_{k,m} \rightharpoonup r\, r_{,m} \ \text{ in } L^2(G).$$

Now the weak continuity of

$$\operatorname{del}(\mathbf{u}) = ({r^2}_{,1}\,,\ {r^2}_{,3}\,) \cdot (z_{,3}\,,\ -z_{,1}\,)$$

follows from the compensated compactness theorem [53].

Injectivity almost everywhere for weak limits in compressible case can be established via the injectivity condition (4.2.8) in the same manner as in Part 2 of Theorem 4.2.5, but without the openness of the mapping $\mathbf{v}$ stronger assertions of the theorem are not in general true for the compressible case.

Remark 2. Clearly, by Theorem 4.1.2, two-dimensional isochoric deformations from Sobolev space $W^{1,p}(\Omega, \mathbb{R}^2)$, $\Omega \subset \mathbb{R}^2$, $p \geq 2$, are open and discrete. This observation can be used in exactly the same manner as in the proof of Theorem 4.2.5, Part 2, for sharpening the injectivity results of [15], [52] for this class of deformations, but we do not pursue this issue here.

Chapter 5

Governing equations for TIE motion

Beginning with this chapter, more specialized axisymmetric deformations than those examined in the previous chapters are considered, but they are assumed to depend on the time variable. Here the governing elastodynamic equations for motions involving axially varying $\underline{t}$wist, radial $\underline{i}$nflation/deflation, and axial contraction/$\underline{e}$longation (TIE) and introduced by the equations (1.0.14),

$$r = Rs(Z,t), \qquad \theta = \omega + \tau(Z,t), \qquad z = z(Z,t),$$

are derived.

Attention is henceforth restricted to neo-Hookean materials whose strain energy density is given by (1.0.6).

The nominal stress tensor $\mathbf{S}$, given in the incompressible case by (1.0.7), reduces for neo-Hookean material to

$$\mathbf{S} = \mu \mathbf{F}^T - p\mathbf{F}^{-1}. \tag{5.0.1}$$

Direct computation shows that for TIE motion the isochoric constraint (1.0.4) becomes

$$\frac{\partial z}{\partial Z} = \frac{1}{s^2}. \tag{5.0.2}$$

Utilizing the standard expression of the acceleration vector in cylindrical coordinates and the equilibrium equations for neo-Hookean material (2.2.21)-(2.2.23) enables the linear momentum balance (1.0.8) to be written as

$$\begin{aligned}
\mu\left[\triangle r - r\,(\nabla\tau)^2 + (r/R)_{,1}\right] - (r/R)(p_{,1}\, z_{,3} - p_{,3}\, z_{,1}) &= \rho(\ddot{r} - r\dot{\tau}^2), & (5.0.3)\\
\mu\left[r\triangle\tau + 2\nabla r\cdot\nabla\tau + (r/R)\tau_{,1}\right] &= \rho(r\ddot{\tau} + 2\dot{r}\dot{\tau}), & (5.0.4)\\
\triangle z + z_{,1}/R + (r/\mu R)(p_{,1}\, r_{,3} - p_{,3}\, r_{,1}) &= \rho\ddot{z}. & (5.0.5)
\end{aligned}$$

For TIE motion these equations transform into

$$\begin{aligned}
\mu R(s_{,33} - s\tau_{,3}^2) - p_{,1} z_{,3}\, s &= \rho R(\ddot{s} - s\dot{\tau}^2), & (5.0.6)\\
\mu(s\tau_{,33} + 2s_{,3}\,\tau_{,3}) &= \rho(s\ddot{\tau} + 2\dot{s}\dot{\tau}), & (5.0.7)\\
\mu z_{,33} + Rss_{,3}\, p_{,1} - s^2 p_{,3} &= \rho\ddot{z}. & (5.0.8)
\end{aligned}$$

We seek to investigate controllable motions associated with this system. Here the term 'controllable' is used in the sense that the motion (or deformation) is sustainable by surface tractions alone. For our purposes, a suitably smooth set of functions $s(Z,t) > 0$, $\tau(Z,t)$, $z(Z,t)$ defined on $[Z_1, Z_2] \times [t_1, t_2]$ (where Z_1 and Z_2 may take infinite values) is said to define a controllable axially varying TIE motion via (1.0.14) for a neo-Hookean material if:

(a) the mapping in cylindrical coordinates defined by (1.0.14) at fixed t is one-to-one,

(b) the constraint (5.0.2) is satisfied,

(c) there exists $p(R, Z, t)$ such that (5.0.6)–(5.0.8) are satisfied.

Recall [44] that the surface traction per unit area on the vector area element $\mathbf{da}$ in $\mathcal{B}(t)$ is denoted by $\mathbf{t}$ and is given by

$$\mathbf{t}\, da = \mathbf{S}^T \mathbf{N}\, dA = \mathbf{S}^T \mathbf{dA}, \tag{5.0.9}$$

where $\mathbf{dA}$ is the corresponding vector area element in $\mathcal{B}_0$ with associated unit normal $\mathbf{N}$.

Note that $\mathbf{F} = \mathbf{F}(R, Z, t)$ and $\ddot{\mathbf{x}} = \ddot{\chi}(R, Z, t)$ implying that

$$p = p(R, Z, t). \tag{5.0.10}$$

Next we simplify the system (5.0.6)–(5.0.8) by eliminating the pressure. To this end equations (5.0.6) and (5.0.8) can be solved for $p_{,1}$ and $p_{,3}$:

$$\begin{aligned} p_{,1} &= \mu R s (s_{,33} - s\tau_{,3}^{2}) - \rho R s (\ddot{s} - s\dot{\tau}^2), && (5.0.11) \\ p_{,3} &= \mu \left(R^2 s_{,3} \left(s_{,33} - s\tau_{,3}^{2} \right) + z_{,3}\, z_{,33} \right) - \rho \left(R^2 s_{,3} \left(\ddot{s} - s\dot{\tau}^2 \right) + z_{,3}\, \ddot{z} \right), && (5.0.12) \end{aligned}$$

where (5.0.2) gives certain simplifications. Equating the cross-derivatives of p gives an equation

$$s \left(\mu (s_{,33} - s\tau_{,3}^{2}) - \rho (\ddot{s} - s\dot{\tau}^2) \right)_{,3} - s_{,3} \left(\mu (s_{,33} - s\tau_{,3}^{2}) - \rho (\ddot{s} - s\dot{\tau}^2) \right) = 0.$$

Integration then provides

$$\mu(s_{,33} - s\tau_{,3}^{2}) - \rho(\ddot{s} - s\dot{\tau}^{2}) - C(t)s = 0,$$

where $C(t)$ is, in general, an arbitrary function of time.

Thus the system (5.0.6) - (5.0.8) reduces to the following system of two coupled nonlinear PDE for r, τ:

$$\mu(s'' - s\tau'^{2}) - \rho(\ddot{s} - s\dot{\tau}^{2}) - C(t)s = 0, \tag{5.0.13}$$

$$\mu(s\tau'' + 2s'\tau') - \rho(s\ddot{\tau} + 2\dot{s}\dot{\tau}) = 0, \tag{5.0.14}$$

where prime stands for differentiation with respect to Z.

Note from (5.0.11) and (5.0.13) that $p_{,1} = Rs^2C(t)$ whereupon

$$p(R, Z, t) = \tfrac{1}{2}C(t)R^2s^2 + \hat{p}(Z, t), \tag{5.0.15}$$

with (5.0.12)–(5.0.15) providing

$$\hat{p}_{,3} = -2\mu s^{-5}s_{,3} - \rho s^{-2}\ddot{z} \iff \hat{p} = \frac{\mu}{2}s^{-4} - \rho\int s^{-2}\ddot{z}\,dZ + p_o(t). \tag{5.0.16}$$

The determination of controllable motions now reduces to the determination of suitable functions $s(Z, t)$ and $\tau(Z, t)$ that satisfy the governing system given by the two second order nonlinear partial differential equations (5.0.13) and (5.0.14). The function $C(t)$ may be arbitrarily chosen. The axial contraction/elongation $z(Z, t)$ then follows from integration of (5.0.2) and so is determined up to an arbitrary function of time that represents an axial rigid body displacement. The pressure $p(R, Z, t)$ follows from (5.0.15) and (5.0.16), and so is also only determined to within an arbitrary function of time. One finds that the principal stretches λ_1, λ_2, λ_3 are given in terms of functions s, τ by

$$\lambda_1^2 = \tfrac{1}{2}(\gamma_1 + \gamma_2)/s^2, \quad \lambda_2^2 = \frac{1}{2}t(\gamma_1 - \gamma_2)/s^2, \quad \text{and} \quad \lambda_3^2 = s^2, \tag{5.0.17}$$

with $\gamma_1 = 1 + \beta$, $\gamma_2 = \{-4s^6 + (1+\beta)^2\}^{1/2}$, $\beta = R^2 s^4 s_{,3}^2 + s^6(1 + R^2\tau_{,3}^2)$. Thus, in general, all three principal stretches vary in both space and time. For the case $\tau(Z,t) = \tau(t)$ it is seen that $\mathbf{e}_\theta$ is a principal direction on $\mathcal{B}(t)$. For the case $s(Z,t) = s(t)$ it is seen that $\mathbf{e}_r$ is a principal direction on $\mathcal{B}(t)$.

A case of some physical interest is that for which $\mathcal{B}_0$ is a cylinder whose cross-section is an annulus with inner radius $R1 \geq 0$ and outer radius $R2 > R1$. Notice that cross-sections of constant Z map into cross-sections of constant z. On the lateral surfaces $R = R1$ and $R = R2$, $\mathbf{N} = \pm\mathbf{E}_R$ giving

$$\mathbf{S}^T\mathbf{N} = \pm\Big(\mu s(Z,t) - p(R,Z,t)/s(Z,t)\Big)\mathbf{e}_r \pm Rp(R,Z,t)s(Z,t)s'(Z,t)\,\mathbf{e}_z \tag{5.0.18}$$

and thus determining sustaining surface tractions via (5.0.9). In addition, tractions associated with ends at any fixed $Z = \hat{Z}$ require a resultant axial force N and a resultant twisting moment M given by

$$\begin{aligned} N(Z,t) \;&:=\; 2\pi\int_{R1}^{R2} S_{Zz}(R,Z,t)R\,dR \\ &= \frac{\pi\mu}{2}\frac{((R2))^2 - ((R1)^2}{s^2(Z,t)} - \frac{\pi}{4}C(t)\,s^4(Z,t)\left((R2)^4 - (R1)^4\right) \\ &\quad - \pi\,s^2(Z,t)\left((R2)^2 - (R1)^2\right)\left(p_o(t) - \rho\int\frac{\ddot{z}(Z,t)}{s^2(Z,t)}dZ\right), \end{aligned}$$

$$
\begin{aligned}
M(Z,t) &:= 2\pi \int_{R1}^{(R2)} r(R,Z,t) S_{Z\theta}(R,Z,t) R\, dR \\
&= \frac{\pi}{2}\mu s^2(Z,t)\tau'(Z,t)\left((R2)^4 - (R1)^4\right).
\end{aligned}
$$

As a premliminary and very simple example consider the case $s(Z,t) = \text{constant}$. In order to restrict rigid body motion, take $z(0,t) = 0$ so that (5.0.2) gives $z(Z,t) = Z/s^2$. Then (5.0.14) requires $\tau(Z,t) = h_+(Z+\sqrt{\mu/\rho}t)+h_-(Z-\sqrt{\mu/\rho}t)$ for arbitrary functions h_+ and h_-, whereupon (5.0.13) gives that either $\tau(Z,t) = h_+(Z+\sqrt{\mu/\rho}t)$ or $\tau(Z,t) = h_-(Z-\sqrt{\mu/\rho}t)$. The motion is thus a single travelling wave. In addition, $C(t) = 0$ and the pressure $p = p_o(t)$. According to (5.0.18), this solution is supported by uniform normal traction on the lateral surfaces $R = R1$ and $R = R2$. This includes the case of traction free lateral surfaces obtained by taking $p_o(t) = \mu s^2$. The resultant axial force $N(\hat{Z},t)$ associated with this traction free solution is $N(\hat{Z},t) = \pi\mu/2((R2)^2 - (R1)^2)(s^{-2} - s^4)$, also a constant. In particular, $N < 0$ for axial contraction ($s > 1$) and $N > 0$ for axial elongation ($s < 1$). The twisting moment is given by $M(\hat{Z},t) = \pi\mu/2s^2((R2)^4 - (R1)^4)s^2 h'_\pm(\hat{Z} \pm \sqrt{\mu/\rho}t)$, where prime denotes derivative with respect to the argument. Thus M, unlike N, varies with the passage of the travelling wave. More general travelling wave motions wherein s is not necessarily constant are discussed in the next chapter, where we obtain closed form solutions for the following four classes of controllable motion:

- controllable deformation, which is the special case for which $s = s(Z)$, $\tau = \tau(Z)$, $z = z(Z)$;

- controllable travelling waves, which is the special case for which $s = s(Z - ct)$, $\tau = \tau(Z - ct)$, $z = z(Z - ct)$ where c is a constant;
- controllable simple twist motion, which is the special case for which $\tau = \alpha Z + \tau_0(t)$ where α is a constant;
- controllable motion with a Riemann type similarity variable, which is the special case for which $s = s(Z/t)$, $\tau = \tau(Z/t)$.

Chapter 6

Closed form solutions for TIE motion

6.1 Controllable deformations

Controllable deformations $s = s(Z)$, $\tau = \tau(Z)$, $z = z(Z)$ provide equilibrium solutions to the equations of motion (5.0.6)–(5.0.8). Then the equations (5.0.14) and (5.0.13) (under the replacement $C \to \mu C$) give

$$s\tau'' + 2s'\tau' = 0, \qquad s'' - s\tau'^2 - Cs = 0. \tag{6.1.1}$$

The first equation in (6.1.1) gives $\tau''/\tau' = -2s'/s$, which upon integration provides

$$\tau' = c_1/s^2, \tag{6.1.2}$$

where c_1 is a constant of integration. Substitution from (6.1.2) into $(6.1.1)_2$ gives

$$s'' - (c_1^2/s^3) - Cs = 0. \tag{6.1.3}$$

Introducing $q(s) := s'$ and using the relation $s'' = q(dq/ds)$ permits (6.1.3) to be rewritten as $q(dq/ds) = Cs + c_1^2/s^3$, so that integration provides

$$q^2 = Cs^2 - (c_1^2/s^2) + c_2,$$

where c_2 is another integration constant. Since $q = ds/dZ$, one obtains

$$Z = \pm \int \frac{s ds}{\sqrt{Cs^4 + c_2 s^2 - c_1^2}} := \pm J_C.$$

Evaluation of J_C is sensitive to the sign of C giving results as follows from subsequent elementary calculations:

(i) If $C < 0$ then

$$J_C = \frac{1}{2\sqrt{-C}} \arcsin\left(\frac{s^2 - B}{A}\right) - c_3,$$

where c_3 is an integration constant, and the new constants $A > 0,\ B > A$ take the place of $c_1,\ c_2$. Since c_1 is necessary for the determination of $\tau(Z)$ from (6.1.2), it is noted that $c_1 = \pm\sqrt{C(A^2 - B^2)}$. Hence in this case

$$s(Z) = \Big(B + A \sin 2a(\pm Z + c_3)\Big)^{1/2} \tag{6.1.4}$$

with $a = \sqrt{-C}$ and $c_1 = \pm a\sqrt{B^2 - A^2}$.

(ii) If $C > 0$ then a corresponding calculation yields

$$J_C = \frac{1}{2\sqrt{C}} \ln\left|s^2 - B + \sqrt{(s^2 - B)^2 - A^2}\right| - c_3,$$

where A, B with $A^2 > B^2$ replace $c_1,\ c_2$. In this case

$$s(Z) = \left(B + \frac{1}{2}\exp\Big(2a(\pm Z + c_3)\Big) + \frac{A^2}{2}\exp\Big(-2a(\pm Z + c_3)\Big)\right)^{1/2} \tag{6.1.5}$$

with $a = \sqrt{C}$ and $c_1 = \pm a\sqrt{A^2 - B^2}$.

(iii) If $C = 0$ then

$$s(Z) = (B^2 + A^2(Z + c_3)^2)^{1/2} \tag{6.1.6}$$

with $c_1 = \pm AB$.

The functions $s(Z)$ given in (6.1.4)–(6.1.6) provide the framework for a family of controllable deformations. Given any such $s(Z)$ the associated $\tau(Z)$ and $z(Z)$ follow respectively from (6.1.2) and (5.0.2) and so differ from the integral

$$H(Z) := \int \frac{dZ}{s^2(Z)}$$

only by (distinct) multiplicative factors and (distinct) constants of integration. If $s(z)$ is given by (6.1.4) then, to within an integration constant,

$$H(Z) = \frac{1}{a\sqrt{B^2 - A^2}} \arctan \frac{B \tan a(Z + c_3) \pm A}{\sqrt{B^2 - A^2}}.$$

If $s(z)$ is given by (6.1.6) then, to within an integration constant,

$$H(Z) = \frac{1}{AB} \arctan \frac{A(Z + c_3)}{B}.$$

If $s(z)$ is given by (6.1.5) then $H(Z)$ can again be expressed in terms of elementary functions, but the expression is rather cumbersome; the particular case $B = 0$ gives, to within an integration constant,

$$H(Z) = \frac{1}{aA} \arctan \frac{\exp 2a(Z + c_3)}{A}.$$

The following theorem summarizes this development.

Theorem 6.1.6 *For arbitrary constants a, c_3, c_4 and c_5, and constants A, B subject to the restrictions listed below, each of the following sets of functions represent controllable TIE deformation for a neo-Hookian solid on an appropriate interval in Z.*

(i) *For $|B| > |A|$ and any Z-interval of a length less than π/a:*

$$\begin{aligned} r(R,Z) &= R\Big(B + A\sin 2a(Z+c_3)\Big)^{1/2}, \\ \theta(\omega,Z) &= \omega \pm \arctan\frac{B\tan a(Z+c_3)+A}{\sqrt{B^2-A^2}} + c_4, \\ z(Z) &= \frac{1}{a\sqrt{B^2-A^2}}\arctan\frac{B\tan a(Z+c_3)+A}{\sqrt{B^2-A^2}} + c_5. \end{aligned}$$

(ii) *For $|A| > |B|$ and any Z-interval:*

$$\begin{aligned} r(R,Z) &= Rs(Z), \\ \theta(\omega,Z) &= \omega \pm a\sqrt{A^2-B^2}\int\frac{dZ}{s^2(Z)} + c_4, \\ z(Z) &= \int\frac{dZ}{s^2(Z)} + c_5, \end{aligned}$$

where $s(Z)$ is given by (6.1.5).

(iii) *For arbitrary A, B and any Z-interval:*

$$\begin{aligned} r(R,Z) &= R(B^2 + A^2(Z+c_3)^2)^{1/2}, \\ \theta(\omega,Z) &= \omega \pm \arctan\frac{A(Z+c_3)}{B} + c_4, \\ z(Z) &= \frac{1}{AB}\arctan\frac{A(Z+c_3)}{B} + c_5. \end{aligned}$$

In the above formulae, constants c_4, c_5 represent rigid body motion, whereas constant c_3 is a simple offset distance for the dependence on axial coordinate Z. Symmetry with respect to clockwise and counterclockwise twist is provided by the $\pm$ in the formulae for $\theta(\omega, Z)$. The associated pressure is determined directly from (5.0.11) and (5.0.12) to be

$$p(R, Z) = \frac{\mu}{2}\Big(\frac{R}{r(R, Z)}\Big)^4 - \frac{\mu}{2}(ar(R, Z))^2 + p_o$$

where p_o is an arbitrary constant and $a = 0$ for case (iii) above. Formally one may take $p_o = p_o(t)$ if desired.

It is worth noting that Theorem 6.1.6(i) with $A = 0$ gives the relatively simple deformation

$$r = \sqrt{B}R, \qquad \theta = \omega \pm a(Z + c_3) + c_4, \qquad z = (Z + c_3)/B + c_5$$

in which twist (characterized by the parameter a) decouples from the radial inflation/deflation and axial contraction/elongation (characterized by parameter B). This special case represents one of the universal deformations for an arbitrary homogeneous, isotropic, incompressible, hyperelastic material [21]. Other than this special case, the deformations described in this section do not correspond to a universal deformation for an arbitrary homogeneous, isotropic, incompressible hyperelastic material. This deformation also represents the only solution that is physically meaningful for the infinite Z-interval. For the periodic solution in Theorem 6.1.6(i), the Z-interval is restricted by the period of the solution; for solutions in Theorem 6.1.6(ii) and (iii), the radial deformation grows without bound as $Z \to \pm\infty$.

Figures 1 and 2 depict the deformation of the coordinate plane $\omega = 0$ for conditions representative of Theorem 6.1.6(i) and (ii), respectively. The left part of each figure

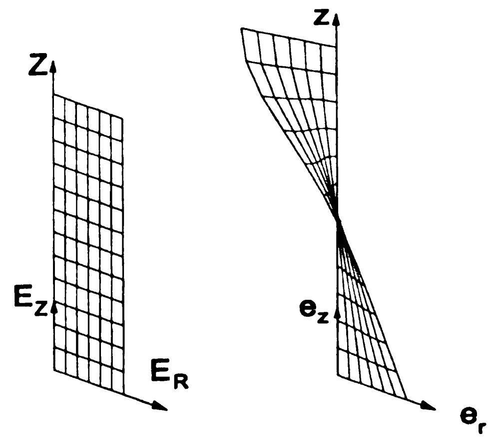

Figure 1: Deformation of the coordinate plane $\omega = 0$ for a case in Theorem 6.1.6 with $B > A$.

depicts the plane in the reference configuration for $Z_1 \leq Z \leq Z_2$, $0 \leq R \leq R_o$. In both cases the values of the parameters a, A, B were chosen a priori, and the other three parameters, c_3, c_4, c_5, found by precluding deformation of the section $Z = Z_1$ via $s(Z_1) = 1$, $\tau(Z_1) = 0$ and $z(Z_1) = Z_1$.

Figure 1 depicts the case $A = 0.5$, $B = 1$, $a = 0.3$ with $Z_1 = -4$, $Z_2 = 4$, $R_o = 2$. It illustrates radial inflation combined with axial contraction (when $-4 < Z < \pi/(2a) - c_3$) and radial deflation coupled with axial elongation (when $\pi/(2a) - c_3 < Z < 4$).

Figure 2 depicts the case $A = 1$, $B = 0$, $a = 1$ with $Z_1 = 0$, $Z_2 = 1$, $R_o = 1$. It illustrates radial inflation combined with axial contraction.

Notice that assigning the resultant force and moment for the section $Z = Z_2$

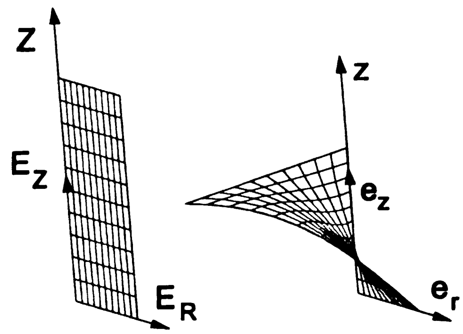

Figure 2: Deformation of the coordinate plane $\omega = 0$ for a case in Theorem 6.1.6 with $B < A$.

generates two additional conditions that can in principle be associated with the determination of A and B (or equivalently c_1 and c_2.) The constants p_o and a remain unrestricted in this assignment.

6.2 Traveling waves

Traveling waves in the axial direction are motions in which the dependence on Z and t is via the similarity variable $\eta := Z - ct$. The constant c is the traveling wave speed. Consequently, $s = s(Z - ct)$, $\tau = \tau(Z - ct)$, $z = z(Z - ct)$. The special case $c = 0$ retrieves the static deformations discussed in the previous section. For the purpose of the present section, the prime notation will denote differentiation with respect to the similarity variable argument $\eta = Z - ct$. Introducing $s = s(\eta), \tau = \tau(\eta)$ into

(5.0.14) and (5.0.13) gives

$$(\mu - \rho c^2)(s\tau'' + 2s'\tau') = 0, \qquad (\mu - \rho c^2)(s'' - s\tau'^2) - Cs = 0. \tag{6.2.7}$$

For any suitably smooth functions s and τ, this system is satisfied with $C = 0$ for traveling wave motion at the neo-Hookean shear wave velocity, $c_* := \pm\sqrt{\mu/\rho}$.

Alternatively, if s and τ satisfy (6.1.1), then they satisfy (6.2.7) under the replacements $Z \to \eta$ and $C \to (\mu - \rho c^2)\, C$. As in (6.1.1) it is necessary to take C independent of t in order to obtain solutions for this second alternative.

In summary, TIE traveling waves are supported in a neo-Hookean material. For propagation at the shear wave velocities $\pm c_*$, any appropriately invertible functions $s(\eta) > 0$, $\tau(\eta)$ define such a controllable traveling wave via

$$r(R, Z, t) = Rs(\eta), \quad \theta(Z, t) = \omega + \tau(\eta), \quad z(Z, t) = \int \frac{d\eta}{s^2(\eta)}, \quad \eta = Z \pm c_* t.$$

At all other traveling wave speeds $c \neq \pm c_*$, TIE traveling waves are supported in the forms defined in Theorem 6.1.6 provided that the independent variable Z is replaced by $Z \pm ct$ and the constant C is replaced by $(\mu - \rho c^2)\, C$. Notice for fixed C of the static deformation, that, under these replacements, the associated traveling waves change their form in transitioning from the subsonic case $c < c_*$ to the supersonic case $c > c_*$.

6.3 Simple twist motion

Controllable simple twist motion is here defined as $\tau = \alpha Z + \tau_0(t)$, where α is a constant, so that (5.0.14) and (5.0.13) become

$$2\mu\alpha s' - \rho(s\ddot{\tau}_0 + 2\dot{s}\dot{\tau}_0) = 0, \tag{6.3.8}$$

$$\mu(s'' - \alpha^2 s) - \rho(\ddot{s} - s\dot{\tau}_0{}^2) - C(t)s = 0. \tag{6.3.9}$$

For the purpose of the present section, the prime notation will denote differentiation with respect to Z. We consider three special cases.

(i) Suppose τ_0 is constant and $\alpha \neq 0$. Then (6.3.8) is satisfied if and only if $s = s(t)$ whereupon (6.3.9) gives

$$\ddot{s} + \rho^{-1}(C(t) + \mu\alpha^2)s = 0. \tag{6.3.10}$$

Hence any sufficiently smooth $s(t) > 0$ is consistent with such motion by taking $C(t) = -\mu\alpha^2 - \rho\ddot{s}/s$. Note that (6.3.10) is of the general and standard form

$$\ddot{y} + h(t)y = 0. \tag{6.3.11}$$

This equation has been intensively studied from a variety of perspectives. (See, for example, [9].) When $h(t)$ is periodic it is called Hill's equation, whereupon a formal series solution is readily obtained [40].

In our framework it is possible to reduce the order of the equation (6.3.11) using the positiveness of $s(t)$. The new dependent variable $u = \ln s$ transforms the equation into

$$u'' + (u')^2 + h(t) = 0.$$

One more change of the dependent variable $v = u'$ results in the first-order ordinary differential equation

$$v' = -v^2 - h(t),$$

which can be treated by appropriate standard methods.

(ii) More generally, suppose $s = s(t)$. Then (6.3.8) integrates so as to give

$$\dot{\tau}_0 = c_1/s^2 \tag{6.3.12}$$

with c_1 a constant. Substitution from (6.3.12) into (6.3.9) gives

$$\ddot{s} - (c_1^2/s^3) + \rho^{-1}(C(t) + \mu\alpha^2)s = 0. \tag{6.3.13}$$

Once again, any sufficiently smooth $s(t) > 0$ gives rise to controllable simple twist motion, now by taking $C(t) = -\mu\alpha^2 - \rho\ddot{s}/s + \rho c_1^2/s^4$. The choice $c_1 = 0$ retrieves the results for case (i). Alternatively, if s is not assigned and if $C(t) = C$, a constant, then, somewhat surprisingly, (6.3.13) is of the same form as (6.1.3) although the independent variable is now t instead of Z. Accordingly, the various solution forms (6.1.4)–(6.1.6) can be appropriated with simple modification. For an arbitrary $C = C(t)$, (6.3.13) can be examined by appropriate numerical or qualitative methods for ordinary differential equations, although we do not pursue this issue here.

(iii) Suppose τ_0 is constant and $\alpha = 0$. Then (6.3.8) is satisfied identically whereas (6.3.9) becomes a single linear partial differential equation,

$$\mu s'' - \rho\ddot{s} - C(t)s = 0. \tag{6.3.14}$$

Solving (6.3.14) for $C(t)$ and requiring $C' = 0$ gives

$$\mu(ss''' - s's'') - \rho(s\ddot{s}' - s'\ddot{s}) = 0 \tag{6.3.15}$$

as a necessary and sufficient condition for $s(Z,t)$ to satisfy (6.3.14) for some $C(t)$. The particular solution of (6.3.15) given by $s = s(t)$ retrieves a motion corresponding to (ii). Separation of variables on (6.3.14) formally gives two sets of solutions. The first set involves either

$$s = b(t)\sin(\sqrt{\beta/\mu}Z) \qquad \text{or} \qquad s = b(t)\cos(\sqrt{\beta/\mu}Z), \tag{6.3.16}$$

where $\beta > 0$ is an arbitrary constant and $b(t)$ satisfies

$$\ddot{b} + \rho^{-1}(C(t) + \beta)b = 0. \tag{6.3.17}$$

The second set involves either

$$s = b(t)\exp(\sqrt{\beta/\mu}Z) \qquad \text{or} \qquad s = b(t)\exp(-\sqrt{\beta/\mu}Z), \tag{6.3.18}$$

where $\beta > 0$ is an arbitrary constant and $b(t)$ satisfies

$$\ddot{b} + \rho^{-1}(C(t) - \beta)b = 0. \tag{6.3.19}$$

Both equations (6.3.17) and (6.3.19) are of the form (6.3.11).

In view of the linearity of (6.3.17) and the arbitrariness of $C(t)$, any superposition of solutions (6.3.16) and (6.3.18) with $\beta = \beta_k$, $b(t) = b_k(t)$, $k = 1, 2, ...$ will also provide a formal solution. At issue then is the positivity requirement on s. This requirement would provide restrictions on $\{\beta_k, b_k(t)\}$ that incorporate, for example, the length of the Z-interval on which the motion holds.

Note also for $C(t) = C$, a constant, that (6.3.14) is the classical telegraphy equation for which there are standard treatments (see, for example, [17]). In particular, integral representations for solutions of various initial/boundary value problems in terms of corresponding Green's functions are given in [54].

6.4 Motion with a Riemann type similarity variable

For wave propagation problems in one spatial dimension, solutions in terms of the similarity variable $\xi := Z/t$ are central to the analysis of initial value problems characterized by step function initial data [50]. Riemann's problem in gas dynamics, and shock tube problems in general, provide standard examples. In the present case (5.0.2) is inconsistent with non-trivial solutions such that both $s = s(\xi)$ and $z = z(\xi)$. However, solutions with $s = s(\xi)$ and $\tau = \tau(\xi)$ can be considered, whereupon, as discussed previously, $z = z(Z,t)$ and $p = p(R,Z,t)$ follow from (5.0.2), (5.0.11) and (5.0.12). For the purpose of the present section, the prime notation will denote differentiation with respect to $\xi = Z/t$. Introducing $s = s(\xi), \tau = \tau(\xi)$ into (5.0.14) and into (5.0.13) followed with multiplication by t^2/ρ gives, respectively,

$$(c_* - \xi^2)(s\tau'' + 2s'\tau') - 2\xi s\tau' \;=\; 0, \tag{6.4.20}$$

$$(c_* - \xi^2)(s'' - s\tau'^2) - 2\xi s' - \frac{C(t)}{\rho}t^2 s \;=\; 0. \tag{6.4.21}$$

Note from (6.4.21) that solutions consistent with this framework can be constructed only if

$$C(t)/\rho = k_0 t^{-2}, \tag{6.4.22}$$

where k_0 is a constant. Equation (6.4.20) can be written as

$$\left[(c_* - \xi^2)s\tau'\right]' + (c_* - \xi^2)s'\tau' = 0.$$

Dividing by the expression in brackets gives

$$\left[\ln\left((c_* - \xi^2)s\tau'\right)\right]' + (\ln s)' = 0,$$

whereupon integration and solving for τ' gives

$$\tau' = \frac{c_1}{(c_* - \xi^2)s^2}, \tag{6.4.23}$$

where c_1 is a constant of integration. Entering (6.4.21) with both (6.4.22) and (6.4.23) then leads to

$$(c_*^2 - \xi^2)^2 s'' - 2(c_*^2 - \xi^2)\xi s' - c_1^2 s^{-3} - k_0(c_*^2 - \xi^2)s = 0. \tag{6.4.24}$$

We now show that (6.4.24) can be recast so as to eliminate first derivatives. This recasting is somewhat different for subsonic waves ($|\xi| < c_*$) and supersonic waves ($|\xi| > c_*$):

(i) For $-c_* < \xi < c_*$, introduce the new independent variable

$$\zeta = \ln\frac{c_* + \xi}{c_* - \xi} \tag{6.4.25}$$

so that

$$\zeta' = \frac{2c_*}{c_*^2 - \xi^2}, \qquad \zeta'' = \frac{4c_*\xi}{(c_*^2 - \xi^2)^2}, \qquad c_*^2 - \xi^2 = \frac{4c_*^2 e^\zeta}{(e^\zeta + 1)^2}.$$

After standard manipulations equation (6.4.24) for $s = s(\zeta)$ becomes

$$\frac{d^2 s}{d\zeta^2} - k_1 s^{-3} - k_0 \frac{e^\zeta}{(e^\zeta + 1)^2} s = 0, \tag{6.4.26}$$

where $k_1 = c_1^2/(4c_*^2) \geq 0$.

(ii) For both $\xi < -c_*$ and $\xi > c_*$, introduce the new independent variable

$$\zeta = \ln \frac{\xi + c_*}{\xi - c_*} \tag{6.4.27}$$

so that

$$\zeta' = -\frac{2c_*}{c_*^2 - \xi^2}, \qquad \zeta'' = -\frac{4c_*\xi}{(c_*^2 - \xi^2)^2}, \qquad c_*^2 - \xi^2 = -\frac{4c_*^2 e^\zeta}{(e^\zeta - 1)^2}.$$

After standard manipulations equation (6.4.24) for $s = s(\zeta)$ becomes

$$\frac{d^2 s}{d\zeta^2} - k_1 s^{-3} + k_0 \frac{e^\zeta}{(e^\zeta - 1)^2} s = 0, \tag{6.4.28}$$

where again $k_1 = c_1^2/(4c_*^2)$.

Note that (6.4.26) and (6.4.28) are of the same form as (6.3.13) (with an appropriate choice of $C(t)$), although the independent variable is now ζ instead of t. Those equations are potentially more convenient than (6.4.24) for numerical computation. They are also convenient for further analysis as described next. Here we only consider the separate special cases of $k_1 = 0$ and $k_0 = 0$.

Case $k_1 = 0$ implies $c_1 = 0$ and it follows from (6.4.23) that the motion is twist-free in the sense of Section 5. The case $k_1 = 0$ can therefore be developed directly from (6.3.14) by requiring the additional specializations (6.4.22) and $s(Z,t) = s(Z/t)$. Both (6.4.26) and (6.4.28) are then of the general form (6.3.11), and the comments following that equation apply.

In the case $k_0 = 0$ both (6.4.26) and (6.4.28) reduce to a common form. This form is integrable after multiplication by $ds/d\zeta$ giving $(ds/d\zeta)^2 + k_1/s^2 = k_2$, where $k_2 \geq 0$ is the integration constant. Yet another integration yields

$$\zeta + \ln k_3 = \frac{1}{k_2}\sqrt{k_2^2 s^2 - k_1}, \tag{6.4.29}$$

where the integration constant is written as $\ln k_3$ with $k_3 > 0$. For subsonic waves, solving (6.4.29) for $s > 0$ and invoking (6.4.25) gives

$$s(\xi) = \left(k_4 + \left(\ln k_3 \frac{c_* + \xi}{c_* - \xi}\right)^2\right)^{1/2}, \tag{6.4.30}$$

where $k_4 = k_1/k_2^2 \geq 0$. It is readily verified that (6.4.30) applies also to the supersonic waves (where ζ is given by (6.4.27)) provided that $k_3 < 0$. It is to be noted that s as given by (6.4.30) is unbounded as $Z/t \to \pm c_*$.

In summary, TIE motions such that $s = s(Z/t)$, $\tau(Z/t)$ are supported in a neo-Hookean material. On the subsonic characteristic curves, the function s must satisfy the second-order ordinary differential equation (6.4.26) where $k_1 \geq 0$ and k_0 are otherwise arbitrary. On the supersonic characteristic curves, the function s must satisfy (6.4.28) where, again, $k_1 \geq 0$ and k_0 are otherwise arbitrary. In both cases, τ then follows from (6.4.23) using $c_1 = \pm 2c_*\sqrt{k_1}$, in general giving rise to a five parameter family of solutions for s and τ.

Chapter 7

Cartesian description of TIE and TIES motions

In this chapter we obtain Cartesian descriptions of TIE and TIES motions. It is found by trial that the reduced nonlinear system of PDE for the radial and angular components of TIE motion, derived in Chapter 5, admits a variational formulation. Formal change of dependent variables transforms the Lagrangian of the corresponding variational problem into quadratic expression with respect to new dependent variables therefore leading to a linear decoupled system of Euler-Lagrange equations. Governing equations for a more general class, called TIES motions, are also derived. Although in addition to twist, inflation/deflation, and contraction/elongation functions, describing TIE motion, two unknown functions accounting for in-plane shear are introduced into the ansatz for TIES motions, the governing system is shown to decompose into four identical decoupled linear equations of the same type as for TIE motions. The governing system for general axisymmetric motions of neo-Hookean body is also transformed in the same manner as TIE motion into a system that seems to be more convenient for further investigations than the original one.

7.1 TIE motion in Cartesian description

Consider the system of nonlinear PDE (5.0.13), (5.0.14) for functions s, τ which determine the radial and the angular components of TIE motion. The system can be recast under replacement $C \to \rho C$ into

$$\begin{aligned} \ddot{s} - c_*^2 s'' - s(\dot{\tau}^2 - c_*^2 \tau'^2) + C(t)s &= 0, \\ s(\ddot{\tau} - c_*^2 \tau'') + 2(\dot{s}\dot{\tau} - c_*^2 s'\tau') &= 0. \end{aligned}$$

Direct computation shows that these are Euler-Lagrange equations of variational problem with Lagrange density given by

$$\mathcal{L} = \frac{1}{2}\Big(\dot{s}^2 - c_*^2 s'^2 + (s\dot{\tau})^2 - c_*^2(s\tau')^2 - C(t)s^2\Big). \tag{7.1.1}$$

Introducing new dependent variables by formulae

$$\xi = s\cos\tau, \qquad \eta = s\sin\tau$$

the Lagrangian (7.1.1) becomes

$$\mathcal{L} = \frac{1}{2}\left(\dot{\xi}^2 + \dot{\eta}^2 - c_*^2(\xi'^2 + \eta'^2) - C(t)(\xi^2 + \eta^2)\right).$$

Indeed,

$$s'^2 + (s\tau')^2 = \frac{(\xi\xi' + \eta\eta')^2}{\xi^2+\eta^2} + \frac{(\eta'\xi - \eta\xi')^2}{\xi^2+\eta^2} = (\xi')^2 + (\eta')^2.$$

Transformation of the terms involving time derivatives is identical.

Euler-Lagrange equations corresponding to the transformed Lagrangian take the form

$$\ddot{\xi} - c_*^2\xi'' + C(t)\xi = 0, \qquad \ddot{\eta} - c_*^2\eta'' + C(t)\eta = 0. \tag{7.1.2}$$

These equations are of the form (6.3.14), and the comments after that equation apply, except for the one concerning positivity of solutions, since neither of functions ξ, η needs to be positive.

Axisymmetric description (1.0.14) of TIE motion translates in terms of functions ξ, η, z into the following Cartesian description

$$x = X\xi - Y\eta, \qquad y = Y\xi + X\eta, \qquad z = \int \frac{dZ}{\xi^2+\eta^2} + z_0(t), \tag{7.1.3}$$

where functions ξ, η satisfy equations (7.1.2), and z_0 is an arbitrary function of t.

7.2 TIES motion in Cartesian description

Here we consider more general than TIE class of motions for neo-Hookean body given by the ansatz

$$x = X\xi(Z,t) - Y\eta(Z,t) + f(Z,t),\ y = Y\xi + X\eta + g(z,T),\ z = z(Z,t). \tag{7.2.4}$$

For $f = g = 0$ (7.2.4) reduces to a Cartesian description of the previous TIE motion. The addition of nonzero f and g can be interpreted in terms of transverse shear.

Hence (7.2.4) will be referred to as TIES motion. The motions investigated in [47], [2] are particular cases of TIES. In [47] it is assumed that $\eta = 0$ (twist free motion) and $\xi = \xi(t)$, $\eta = \eta(t)$ (no dependence on the axial coordinate in ξ, η). The model in [2] does not include inflation/deflation, and therefore, due to incompressibility, elongation/contraction.

To simplify derivation of the governing equations for TIES motion the following simple technical lemma, aimed at eliminating the pressure terms in the governing equations, will be helpful.

Lemma 7.2.5 *Let a motion of a neo-Hookean solid be determined by functions*

$$x_i = x(X_1, X_2, X_3, t), \; i = 1, 2, 3,$$

that are three times continuously differentiable. Then

$$\nabla(\Box x_i) \times \nabla x_i = \mathbf{0}. \tag{7.2.5}$$

If the body is simply connected, corresponding pressure can be found from the equations

$$p_{,k} = x_{i,k} \Box x_i, \; i, k = 1, 2, 3. \tag{7.2.6}$$

Here and below notation $\Box := \mu\Delta - \rho\partial^2/\partial t$ is used for d'Alambertian operator.

Proof. Since the nominal stress of neo-Hookean material is given by (5.0.1), the general equation of isochoric motion for hyperelastic material (1.0.8) translates for this material into

$$\mathrm{Div}(\mu\mathbf{F}^T - p\mathbf{F}^{-1}) = \rho\dot{\mathbf{v}}.$$

Using Nanson's formula this can be rewritten as

$$\mu\,\mathrm{Div}(\mathbf{F}^T) - \mathbf{F}^{-T}\nabla p = \rho\dot{\mathbf{v}}. \tag{7.2.7}$$

Pre-multiplying (7.2.7) by $\mathbf{F}^T$ one obtains

$$\nabla p = \mu\mathbf{F}^T\,\mathrm{Div}(\mathbf{F}^T) - \rho\mathbf{F}^T\dot{\mathbf{v}}, \tag{7.2.8}$$

which is a vector form of (7.2.6).

Applying the cross product operation with the operator ∇ to both sides of (7.2.8) gives

$$\mu\nabla \times \Big(\mathbf{F}^T Div(\mathbf{F}^T)\Big) = \rho\nabla \times (\mathbf{F}^T\dot{\mathbf{v}}).$$

In tensorial notation this reads

$$\epsilon_{mnk}\Big(\mu x_{i,jj}x_{i,k} - \rho\ddot{x}_i x_{i,k}\Big)_{,n} = 0, \qquad \text{or} \qquad \epsilon_{mnk}\Big((\Box\, x_i)x_{i,k}\Big)_{,n} = 0.$$

Since $\epsilon_{mnk}x_{i,kn} = 0$ this implies

$$\epsilon_{mnk}(\Box\, x_i)_{,n}\; x_{i,k} = 0,$$

which is (7.2.5) in tensorial notation. ■

Next we apply the vector equation (7.2.5) to TIES motion (7.2.4). The equation will be shown to split into a system of five equations for unknown functions ξ, η, f, g, and $C = C(t)$, where C is an auxiliary function simplifying the structure of the system just as in the case of TIE motion.

To this end we need to compute the left hand side of the equation

$$\nabla(\Box\, x) \times \nabla x + \nabla(\Box\, y) \times \nabla y + \nabla(\Box\, z) \times \nabla z = 0. \tag{7.2.9}$$

Clearly, the last term on the left hand side is zero for the ansatz (7.2.4).

From (7.2.4) we derive

$$\nabla x = \left[\ \xi, \quad -\eta, \quad X\xi' - Y\eta' + f'\ \right], \ \Box\nabla x = \left[\ \Box\xi, \quad -\Box\eta, \quad X\Box\xi' - Y\Box\eta' + \Box f'\ \right].$$

Then

$$\Box\nabla x \times \nabla x = \begin{bmatrix} -\Box\eta(X\xi' - Y\eta' + f') + \eta(X\Box\xi' - Y\Box\eta' + \Box f'), \\ \xi(X\Box\xi' - Y\Box\eta' + \Box f') - \Box\xi(X\xi' - Y\eta' + f'), \\ \xi\Box\eta - \eta\Box\xi \end{bmatrix}.$$

Similarly,

$$\Box\nabla y \times \nabla y = \begin{bmatrix} \Box\xi(Y\xi' + X\eta' + g') - \xi(Y\Box\xi' + X\Box\eta' + \Box g'), \\ \eta(Y\Box\xi' + X\Box\eta' + \Box g') - \Box\eta(Y\xi' + X\eta' + g'), \\ \xi\Box\eta - \eta\Box\xi \end{bmatrix}.$$

Now equation (7.2.9) can be written in the following scalar form

$$\begin{aligned} A(\xi,\eta)X + B(\xi,\eta)Y + \eta\Box f' - \xi\Box g' - f'\Box\eta + g'\Box\xi &= 0, &\quad (7.2.10)\\ -B(\xi,\eta)X + A(\xi,\eta)Y + \xi\Box f' + \eta\Box g' - f'\Box\xi - g'\Box\eta &= 0, &\quad (7.2.11)\\ \xi\Box\eta - \eta\Box\xi &= 0, &\quad (7.2.12) \end{aligned}$$

where

$$\begin{aligned} A &= -\xi'\Box\eta + \eta\Box\xi' + \eta'\Box\xi - \xi\Box\eta' = (\eta\Box\xi)' - (\xi\Box\eta)', &\quad (7.2.13)\\ B &= \eta'\Box\eta - \eta\Box\eta' + \xi'\Box\xi - \xi\Box\xi' = -\Big(\frac{\Box\eta}{\eta}\Big)'\eta^2 - \Big(\frac{\Box\xi}{\xi}\Big)'\xi^2. &\quad (7.2.14) \end{aligned}$$

Equation (7.2.12) implies

$$\frac{\Box\eta}{\eta} = \frac{\Box\xi}{\xi},$$

which can be written as

$$\Box\xi - C\xi = 0 \qquad \Box\eta - C\eta = 0 \tag{7.2.15}$$

with an arbitrary function $C = C(Z, t)$. These equations are identical to (7.1.2).

Equations (7.2.10), (7.2.11) imply

$$\begin{aligned} A(\xi,\eta) = 0, \qquad B(\xi,\eta) &= 0, & (7.2.16)\\ \eta\Box f' - \xi\Box g' - f'\Box\eta + g'\Box\xi &= 0, & (7.2.17)\\ \xi\Box f' + \eta\Box g' - f'\Box\xi - g'\Box\eta &= 0, & (7.2.18)\end{aligned}$$

where A, B are defined by (7.2.13), (7.2.14). The first condition in (7.2.16) is true identically by virtue of (7.2.15), while the second implies

$$C'(\xi^2 + \eta^2) \equiv 0.$$

Assuming $\xi^2 + \eta^2 \not\equiv 0$, this means that function C does not depend on the axial coordinate, i.e., $C = C(t)$. From (7.2.17), (7.2.18) one obtains after some manipulations $\Box f' - Cf' = 0, \qquad \Box g' - Cg' = 0$, or, equivalently,

$$\Box f - Cf = D_1(t) \qquad \Box g - Cg = D_2(t) \tag{7.2.19}$$

with arbitrary functions D_1, D_2.

Next we compute the pressure for TIES motion. Equations (7.2.6) become

$$\begin{aligned} p_{,1} &= \xi(X\Box\xi - Y\Box\eta + \Box f) + \eta(X\Box\eta + Y\Box\xi + \Box g),\\ p_{,2} &= -\eta(X\Box\xi - Y\Box\eta + \Box f) + \xi(X\Box\eta + Y\Box\xi + \Box g),\\ p_{,3} &= x'(X\Box\xi - Y\Box\eta + \Box f) + y'(X\Box\eta + Y\Box\xi + \Box g) + z'\Box z.\end{aligned}$$

Making use of (7.2.4, (7.2.15), and (7.2.19) these equations simplify to

$$p_{,1} = CXs^2 + C(f\xi + g\eta) + D_1\xi + D_2\eta, \tag{7.2.20}$$

$$p_{,2} = CYs^2 + C(f\eta + g\xi) + D_1\eta + D_2\xi, \tag{7.2.21}$$

$$\begin{aligned} p_{,3} &= \frac{1}{2}C(X^2+Y^2)(s^2)' + X\Big(C(f\xi+g\eta) + D_1\xi + D_2\eta\Big)' \\ &+ Y\Big(C(g\xi - f\eta + f^2/2 + g^2/2) - D_1\eta + D_2\xi\Big)' + z'\Box z, \end{aligned} \tag{7.2.22}$$

where the notation $s^2 = \xi^2 + \eta^2$ is used for simplicity. Equations (7.2.20), (7.2.21) imply

$$\begin{aligned} p(X,Y,Z,t) = \frac{C}{2}(X^2+Y^2)(\xi^2+\eta^2) + X\Big(C(f\xi+g\eta) + D_1\xi + D_2\eta\Big) + \\ +Y\Big(C(g\xi - f\eta) - D_1\eta + D_2\xi\Big) + p_0(Z,t). \end{aligned} \tag{7.2.23}$$

Substituting this function into the equation (7.2.22) one obtains after simple manipulations

$$p_0 = \frac{C}{2}(f^2+g^2) + D_1(t)f + D_2(t)g + \int z'\Box z\, dZ + \hat{p}(t), \tag{7.2.24}$$

where $\hat{p}$ is an arbitrary function of t.

In summary, TIES motion (7.2.4) is supported in neo-Hookean body by the pressure given by (7.2.23), where functions $\xi(Z,t)$, $\eta(Z,t)$, $f(Z,t)$, $g(Z,t)$ solve the equations (7.2.15), (7.2.19), respectively, function $p_0(Z,t)$ is determined by (7.2.24), and functions C, D_1, D_2, $\hat{p}$ are arbitrary functions of the time variable.

7.3 Cartesian description of general axisymmetric motion of neo-Hookean body

Imitating derivation of the system of governing equations for TIE motion in Section 7.1, we substitute new dependent variables defined by the equations

$$\xi = r\cos\tau \qquad \eta = r\sin\tau$$

into equations of motion (5.0.3),(5.0.4) and after direct calculation obtain

$$\begin{aligned} \frac{\xi}{r}\Box\xi + \frac{\eta}{r}\Box\eta + \mu(\xi\xi_{,1} + \eta\eta_{,1})/(rR) - \mu\frac{r}{R^2} - \frac{r}{R}(p_{,1}z_{,3} - p_{,3}z_{,1}) &= 0, \\ \frac{\xi}{r}\Box\eta - \frac{\eta}{r}\Box\xi + \mu(\xi\eta_{,1} - \eta\xi_{,1})/(rR) &= 0. \end{aligned}$$

Solving for $\Box\xi$, $\Box\eta$ and simplifying, one obtains

$$\begin{aligned} \Box\xi + \frac{\mu}{R}\xi_{,1} - \frac{\mu}{R^2}\xi - \frac{1}{R}(p_{,1}z_{,3} - p_{,3}z_{,1})\xi &= 0, \\ \Box\eta + \frac{\mu}{R}\eta_{,1} - \frac{\mu}{R^2}\eta - \frac{1}{R}(p_{,1}z_{,3} - p_{,3}z_{,1})\eta &= 0. \end{aligned}$$

Introducing notation

$$C(R,Z) := \frac{1}{R}(p_{,1}z_{,3} - p_{,3}z_{,1}) + \frac{\mu}{R^2}, \tag{7.3.25}$$

the system (5.0.3)-(5.0.5) in terms of ξ, η, z takes the form

$$\Box\xi + \mu\xi_{,1}/R - C\xi = 0, \tag{7.3.26}$$

$$\Box\eta + \mu\eta_{,1}/R - C\eta = 0, \tag{7.3.27}$$

$$\Box z + \mu z_{,1}/R + \frac{1}{2R}\left(p_{,1}(\xi^2+\eta^2)_{,3} - p_{,3}(\xi^2+\eta^2)_{,1}\right) = 0. \tag{7.3.28}$$

Equations (7.3.25) and (7.3.26)-(7.3.28) together with the isochoric motion constraint (5.0.2) represent a system with *five* unknown functions. Due to one more unknown function C, added by the equation (7.3.25), the structure of this system seems to be more convenient for further, possibly, numerical investigation than the original system of four coupled nonlinear equations (5.0.3)-(5.0.5), (5.0.2). This will be the subject of future analysis.

Chapter 8

Conclusions and discussion

Axisymmetric problems arising in nonlinear elasticity were investigated from two different perspectives.

Under certain restrictions on the integrand, the existence problem for axisymmetric minimizers in Sobolev spaces was solved in the spirit of John Ball's theory. Reduced restrictions on the growth exponents of the integrand in the energy functional allowed new classes of hyperelastic isotropic materials, not covered by Ball's theory in the genuine three-dimensional case, to be included into the existence analysis. Under the assumption that the radial component of admissible mappings (deformations) is nonnegative almost everywhere, higher regularity properties of the radial and axial components of admissible mappings and topological properties of openness and discreteness were discovered. Using these properties and assuming in addition that an originally hollow cylindrical body remains hollow after deformation, global injectivity results were obtained in a stronger form compared with those known for the genuine three-dimensional case.

In the other part of this work more specialized but time-dependent axisymmetric deformations for a neo-Hookean materials were examined, and several classes of exact solutions were found.

Below some possibilities for further research relevant to the results presented here are discussed.

1. Regularity of minimizers is one of the key issues in the calculus of variations, and there is a significant number of works devoted to this problem (see [23] for up-to-date account and relevant references). In the multidimensional case the best one can expect is that a minimizer is of class $C^{1,\alpha}$ off a set of measure zero. All presently known variational approaches to regularity rely on the upper bound on the integrand of the form

$$|f(x, u, Du)| \leq C|Du|^p + b(x)|u|^\gamma + a(x), \tag{8.0.1}$$

where $1 < p \leq \gamma < p^*$. Consequently, this rules out problems of nonlinear elasticity since for compressible materials $f(x, u, Du) \to \infty$ when $\det \mathbf{F} \to 0$, and for incompressible ones the integrand takes infinite value when the incompressibility constraint is violated. Thus the limited partial regularity presented in this work (viz., $r, z \in C^1(D)$ a.e.) gives a nontrivial example of regularity problem solved for the integrand with physically relevant behavior. New approaches are needed for further advances in regularity analysis for variational problems in nonlinear elasticity.

2. Polyconvexity employed in this work implies quasiconvexity, which is essentially

equivalent to weak lower semicontinuity. Many important for applications integrands do not enjoy this property. Hence the problem of quasiconvexification (QC) arises. At the present no effective systematic QC technique is known, but for specific integrands some approaches have been developed (see, for example, [20]). The integrand of the reduced functional obtained in axisymmetric settings in this work depends on the *rectangular* matrix Du, $u = (\xi, \eta, z)$, and under suitable assumptions may be amenable to the method developed for such functions in [49]. In [49] the problem of constructing the *semiconvex* envelope [1] for so called invariant integrands[2] depending on $m \times n$, $m > n$, matrices is reduced to the same problem for an associated function defined on $n \times n$ matrices. If the strain energy density depends only on the first invariant I_1, it is easy to verify that the integrand for the reduced axisymmetric variational problem is invariant, so that the method in [49] applies with appropriate modifications.

The case of nonconvex *anisotropic* problem is more challenging. To show this, consider the following simple example of the stored energy function modeling so called fiber reinforced material (see [28] for more detail)

$$W = W_{iso}(I_1, I_2) + W_{aniso}(I_4), \tag{8.0.2}$$

where W_{iso} represents the stored energy due to deformation of the isotropic incompressible *matrix material*, and W_{aniso} accounts for the effect of reinforcing. A partic-

[1] this is a unifying term for convex, polyconvex, quasiconvex, and rank one convex envelopes

[2] A function $f : \mathbb{M}^{m\times n} \to \mathbb{R}$ is said to be invariant if $f(QAR) = f(A)$ for each $A \in \mathbb{M}^{m\times n}$, $Q \in SO(m)$, $R \in SO(n)$.

ularly simple form for the reinforcement term is

$$f(I_4) = \gamma(I_4 - 1)^2/2, \tag{8.0.3}$$

where $\gamma > 0$ is a positive constant depending on the fiber material, $I_4 = \mathbf{a}^{\mathbf{T}}\,\mathbf{C}\,\mathbf{a}$ is the *pseudo-invariant* of $\mathbf{C}$, and a unit vector $\mathbf{a}$ represents the preferred direction of the fibers. This form has been used by a number of authors to analyze different aspects of the theory of transversely isotropic materials (see, for example, [46] and references therein). Even if neo-Hookean response is chosen for a matrix material,

$$W_{iso} = \mu(I_1 - 3)/2, \tag{8.0.4}$$

constructing the QC envelope for the overall strain energy density (8.0.2) is a challenging task. The method developed in [20] does not apply since the reinforcement term cannot be represented as a function of the singular values (principal stretches) alone. Moreover, justification of the dimensionality reduction for isotropic integrands described in Section 2.2 needs to be modified for this anisotropic case. Yet another problem is to choose an appropriate functional space to accommodate different growth exponents for partial derivatives with respect to R and Z (the pseudo-invariant I_4 contains partial derivatives with respect to Z of power four that may be different from the growth exponents of the isotropic part). A new relaxation theorem that does not make use of the bound (8.0.1) is also needed for this and other physically relevant situations.

3. Dimensionality reduction of three-dimensional problems due to an assumption that the problems under consideration have certain types of symmetry is crucial in

this work. It is desirable to justify this reduction using a more elegant and systematic approach than the straightforward and tedious computation in Section 2.2 or *ad hoc* considerations in Chapters 5 and 7. The derivation of lower dimensional theories for domains which are thin in one or more directions has a long history (detailed account on this issue and further references can be found in [3]). Recently results rigorously justifying some classical theories for rods, plates, and membranes (see, for example, [38]) were obtained using Γ-convergence technique. Another approach, based on the principle of virtual power as a starting point, was developed by Antman [3], and its application for derivation of the governing equation for TIE motions in a rod-like bodies may be mathematically tractable. But the method needs modification to incorporate the incompressibility constraint.

4. Exact solutions for TIE motions of neo-Hookean body found in Chapter 6 can be used in perturbations methods for problems whose strain energy density includes neo-Hookean energy as the leading term. An example is provided by Mooney-Rivlin material (1.0.5) with $\beta < \alpha$.

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