This is to certify that the
thesis entitled

THEORETICAL DERIVATION AND NUMERICAL IMPLEMENTATION OF CONTINUUM DAMAGE BASED CONSTITUTIVE EQUATIONS

presented by

Nima Salajegheh

has been accepted towards fulfillment
of the requirements for the

M.S. degree in Mechanical Engineering

Major Professor's Signature

8/18/04

Date

THEORETICAL DERIVATION AND NUMERICAL IMPLEMENTATION OF CONTINUUM DAMAGE BASED CONSTITUTIVE EQUATIONS

By

Nima Salajegheh

A THESIS

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

MASTER OF SCIENCE

Department of Mechanical Engineering

2004

ABSTRACT

THEORETICAL DERIVATION AND NUMERICAL IMPLEMENTATION OF CONTINUUM DAMAGE BASED CONSTITUTIVE EQUATIONS

By

Nima Salajegheh

In this work, the isotropic damage is assumed and the theoretical formulations of damage-coupled material behavior are presented based on the internal variable approach. Additionally, several methods for damage measurement and specification of material damage parameters are introduced and an experiment is performed to measure damage parameters qualitatively. Finally, the constitutive and evolution equations are implemented into a commercial finite element code, Ls-Dyna, and the results are compared with the non-damage based analysis. Throughout this work, other common approaches to Continuum Damage Mechanics and their differences from the current approach are mentioned briefly.

To my Family

ACKNOWLEDGMENTS

I would like to thank my advisor, Dr. Farhang Pourboghrat, who supported me throughout my study. I would also like to thank other faculty members of the department of Mechanical Engineering at Michigan State University who were helpful and willing to help me with my research.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF ABBREVIATIONS

σ - Cauchy stress

$\tilde{\sigma}$ - Effective stress acting on the resisting area

σ^D - Deviatoric stress

σ_H - Hydrostatic stress

σ_y - Yield stress

σ_u - Ultimate stress

σ_f - Fatigue limit stress

σ_s - Yield stress for elastic perfectly plastic materials

σ_{eq} - von Mises equivalent stress

ε - Cauchy strain

ε^e - Elastic part of Cauchy strain

ε^p - Plastic part of Cauchy strain

p - Equivalent plastic strain

D - Damage parameter

D_c - Critical value of damage parameter in three dimensions

D_{1c} - Critical value of damage parameter in pure tension

S - Damage strength material parameter

ε_{pD} - Damage plastic strain threshold in pure tension

ε_R - Strain to rupture in pure tension

Y - Strain energy density release rate

$\bar{Y}$ - The associated variable of damage parameter

Y_c - Critical strain energy density release rate at rupture

ρ - Mass density

$\tilde{\rho}$ - Mass density for the damaged material

m - Mass

E - Young's modulus

$\tilde{E}$ - Young's modulus for the damaged material

υ - Poison's ratio

$\tilde{\upsilon}$ - Poison's ratio for the damaged material

λ, μ - Lame coefficient

$\dot{\lambda}$ - Plastic multiplier

α, R, R_∞, γ, - Nonlinear kinematic hardening material parameter

K, n - Nonlinear isotropic hardening material parameters

r, R, R_∞, b - Nonlinear isotropic hardening material parameters

w_e - Elastic strain energy density

w^*_e - Complementary elastic strain energy density

t - Time

T - Temperature

s - Entropy density

v - Wave speed

$\tilde{v}$ - Wave speed for the damaged material

V - Electrical voltage and volume

r - Resistivity

R_v - Triaxiality function

ψ - Helmholtz specific free energy

F - Function potential of dissipation

F_D - Damage potential function

N_R - Number of cycles to rupture

C - Elastic stiffness tensor and

$\tilde{C}$ - Effective elastic stiffness tensor for the damaged material

M - Damage effect tensor

H - Hardness

H - Step function

k' - Coefficient of proportionality for hardness test

R, r -Radius of cavities and radius of the yield surface

l - Length

i - Intensity of the electrical current

K_p, M - Material parameters for fatigue

K_v, N - Material parameters for creep

δ_{ij} - Kronecker delta

$\vec{q}$ - Heat flux vector

t_R - Time to failure

CHAPTER 1: Continuum Damage Mechanics

1.1 Introduction

A new branch of science usually develops thus. Somebody publishes the basic ideas. Hesitatingly at first, then little by little, other original contributions appear, until a certain threshold is reached. Then, overview articles are printed, conferences are held, and a first mention is made in textbooks, until specialized monographs are written. Continuum damage mechanics has reached that status now.

Professor Horst Lippmann
Lehrstuhl A fUr Mechanik
Technische Universitiit Miinchen,
Germany

Ductile fracture has been the subject of active research in the last decades because of its importance to practical engineering applications where the forming of a part introduces large deformations beyond the elastic region. Because of its success in the analysis of brittle fracture, a major effort has been devoted to the application of fracture mechanics to characterize ductile fracture. This effort has led to the introduction of different ductile fracture criteria which notably include

J-integral and COD (Crack opening displacement). These criteria offer a tool for predicting the crack progression at the macroscale but they fail to take into account the continuous deterioration of material properties as the effect of microcrack nucleation and accumulation. Moreover, fracture mechanics is based on the analysis of existing cracks which might be too late to prevent a disaster. Therefore, the nucleation stage, which consists of the evolution of internal damage before macrocracks become visible, caught attention. As a result it was highly appreciated by the scientific community when L.M. Kachanov published in 1958 a simple model of material damage which subsequently could be extended to brittle elastic, plastic or viscous materials under all conditions of uniaxial or multiaxial, simple or cyclic loadings. Recently, the theory of continuum damage mechanics has been developed to a state ready for engineering applications. The theory has been applied to solve a number of important engineering problems including low-cycle fatigue in metals, high cycle fatigue, creep-fatigue interaction, damage in composites, creep damage, and ductile rupture. Also, the theory has been extended by the development of both isotropic and anisotropic models of continuum damage mechanics.

1.2 Phenomenological Aspects of Damage

The damage of materials is the progressive process which degrades the material parameters continuously and causes them to fail. At the microscale level this is the breakage of bonds between atoms due to concentration of stress in the neighborhood of defects or interfaces. At the mesoscale level of the representative volume element (RVE), this is the growth and the coalescence of microcracks or microvoids which initiate a crack and at the macroscale level this is the growth of that crack. The two first stages may be studied by means of damage variables of the mechanics of continuous media defined at the mesoscale level which is the goal of Continuum Damage Mechanics approach. The third stage is usually studied using fracture mechanics with variables defined at the macroscale level. Continuum mechanics and the thermodynamics of irreversible processes model the material behavior without detailed reference to the complexity of their physical microstructures.

1.3 Atoms, Elasticity and Damage

All materials are composed of atoms, which are held together by bonds resulting from the interaction of electromagnetic fields. Elasticity is directly related to the relative movement of atoms. When debonding occurs, this is the beginning of the damage process. For example, metals are organized in crystals or grains, a regular array of atoms except on lines of dislocations where atoms are missing; if an adequate shear stress is applied, the dislocations will move, thus creating a plastic strain by slip without any debonding as shown in the Figure 1.

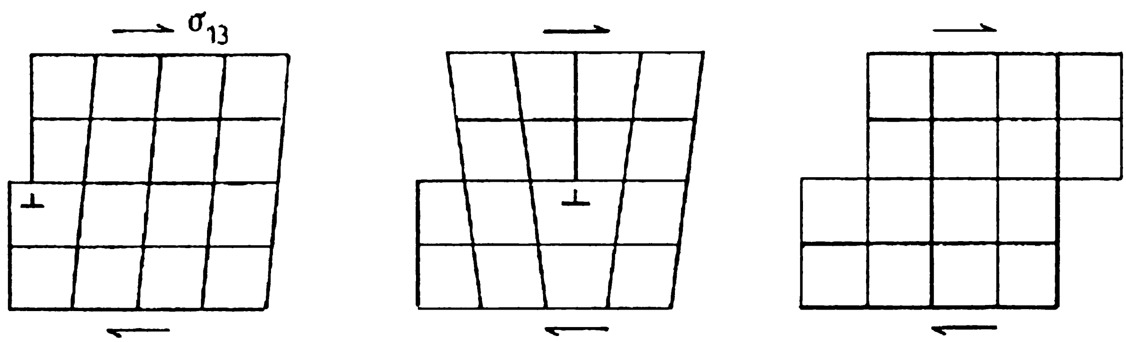


Figure 1 : Plastic strain due to dislocation movement

If the dislocation is stopped by a defect or a stress concentration site, it creates a zone in which another dislocation may be stopped. The accumulation of dislocations creates a debonding damage as shown in Figure 2 and eventually a microcrack will form.

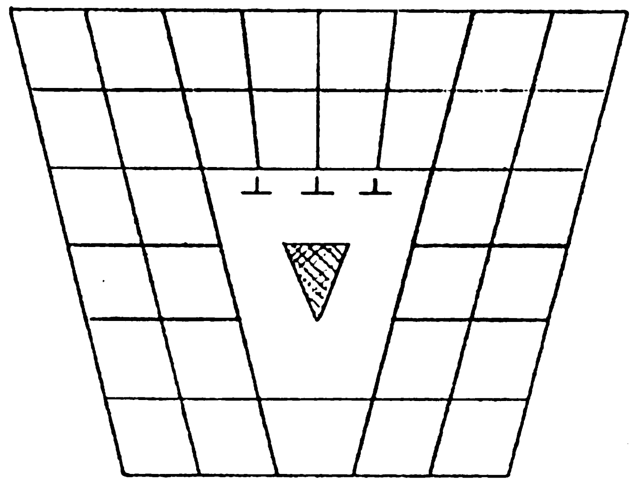


Figure 2 : Elementary damage by nucleation of microcrack due to an accumulation of dislocations (after D. Krajcinovic)

There are some other damage mechanisms in metals such as intergranular debonding and decohesion between inclusions and the matrix all of which create plastic strains.

In all cases, elasticity is directly influenced by the damage, since the number of atomic bonds responsible for elasticity decreases with damage. This coupling, which occurs at the level of the state of the material, is called a state coupling.

1.4 Plasticity and Irreversible Strains

Plasticity is directly related to slips due to the movement, climbing, or twinning of dislocations. Damage influences plastic (irreversible) strain only because the elementary area of resistance decreases as the number of bonds decreases. However, damage does not directly influence the mechanism of slip itself; that is, there is no state coupling. The indirect coupling owing to an increase in the effective stress arises only in the rate constitutive equations and it is called kinetic coupling.

1.5 Scale of the Phenomena of Strain and Damage

- Elasticity takes place at the level of atoms.
- Plasticity is governed by slips at the level of crystals or grains.
- Damage is debonding from the level of atoms to the mesoscale level for crack initiation and it can be viewed as a bridge between microscale and macroscale analysis.

Continuum mechanics deals with quantities defined at a geometrical point. From the physical point of view, these quantities represent averages on a certain volume, the Representative Volume Element (RVE). The RVE must be small enough to capture high gradients but large enough to represent an average of the processes at the microscale[15]. To summerize:

- The microscale is the scale of the mechanisms of strains and damage;
- The mesoscale is the scale at which the constitutive equations for numerical analysis are written;
- The macroscale is the scale of engineering structures;

1.6 Different Manifestations of Damage

Even though the damage at the microscale is governed by one general mechanism of debonding, at the mesoscale it can be seen in various ways depending upon the nature of the material, the type of loading, and the temperature [15]. Some of these manifestations are:

1.6.1 Brittle damage

The damage is called brittle when a crack is initiated at the mesoscale without a large amount of plastic strain. This means that the cleavage forces are below the forces that could produce slips but are higher than the debonding forces. The degree of localization is high.

1.6.2 Ductile damage

On the other hand, the damage is called ductile when it occurs simultaneously with plastic deformations, larger than a certain threshold p_D. Damage is due to the nucleation of cavities followed by their growth and their coalescence. As a consequence, the degree of localization of ductile damage is comparable to that of plastic strain.

1.6.3 Creep damage

When a metal is loaded at elevated temperatures, for instance a temperature above 1/3 of the melting temperature, the plastic strain occurs at a stress lower than the yield stress. When the strain is large enough, there are intergranular decohesions which produce damage and an increase of the strain rate through the period of tertiary creep [1]. Like ductile damage, the gradients of creep damage are similar to the viscoplastic strain gradients.

1.6.4 Low cycle fatigue damage

When a material is subjected to a cyclic loading at high values for stress or strain, damage develops together with cyclic plastic strain after a period of saturation preceding the phases of nucleation and propagation of microcracks. The degree of damage localization is higher than those of ductile or creep damage. Because of the high values for the stress, the low cycle fatigue is characterized by low values of the number of cycles to rupture, $N_R < 10000$. If the material is strain loaded, the damage induces a drop of the stress amplitude as shown in Figure 3 for two stress-strain loops corresponding to the initial cycles and a cycle close to the rupture.

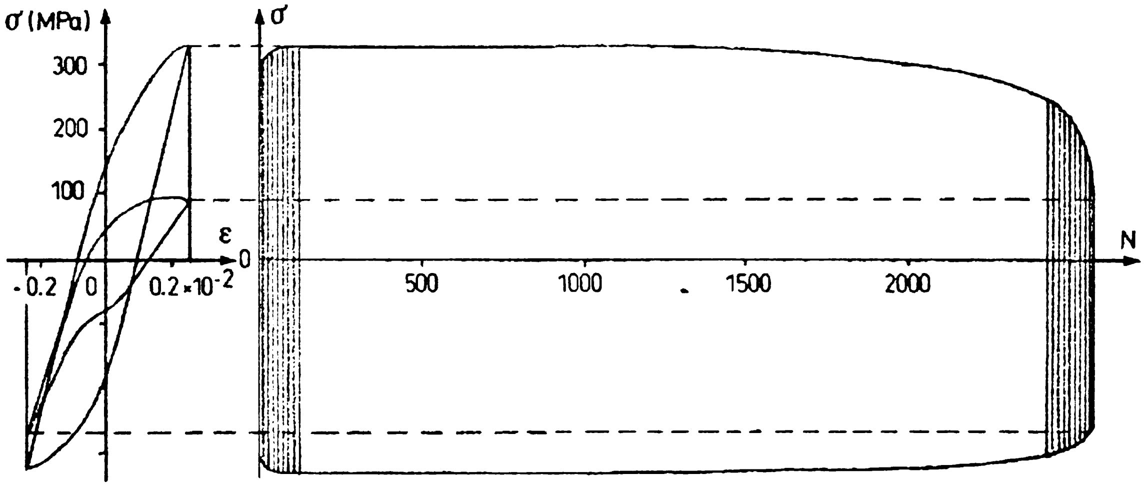


Figure 3 : Cyclic tension-compression curves for low cycle fatigue of stainless steel (after J. Duffaily)

1.6.5 High cycle fatigue

When a material is loaded with lower values for stress or strain, the plastic strain at the mesolevel remains small and negligible. The number of cycles to failure may be very large, $N_R > 100000$ and the localization of damage is high and similar to that of brittle fracture. Also, a drop of stress occurs similar to low cycle fatigue.

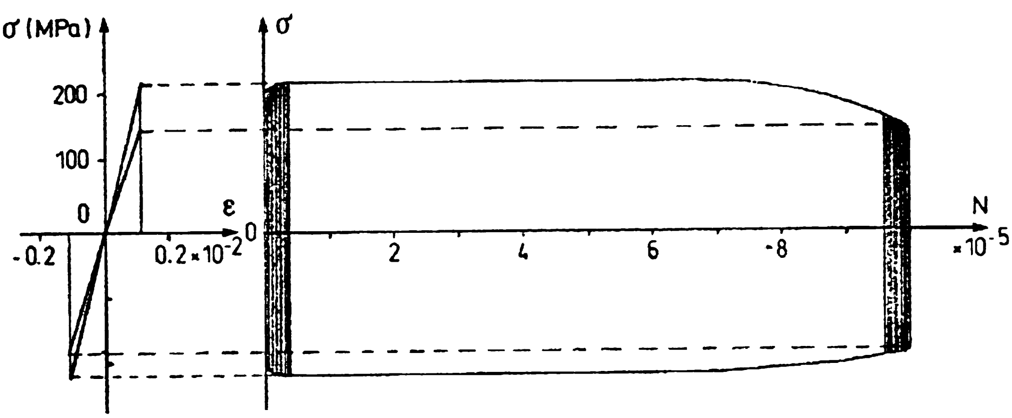


Figure 4 : Cyclic tension-compression curves for low cycle fatigue of stainless steel (after J. Duffaily)

Note that for brittle damage and high cycle fatigue damage where damage localization is high, a stress-strain curve obtained from a classical tension-compression test at the macroscale does not represent the true behavior for strain and damage because the localization induces plastic and damage zones much smaller than those of the specimen. Nevertheless, it is used because mechanical experiments at the microscale are difficult to perform; the results are averages of nonuniform quantities over a mesovolume. The microhardness test may help to characterize a microvolume as it involves a size of the order of microns but its state of stress is complex.

1.7 Mechanical representation of Damage

1.7.1 One-Dimensional Surface Damage Variable (L.M. Kachanov [1])

At the microscale, damage may be interpreted as the creation of discontinuous surfaces, breaking of atomic bonds, and enlargement of microcavities. At the mesoscale, the number of broken bonds may be approximated in any plane by the area of the intersections of all the flaws with that plane [1]. In order to work with a dimensionless quantity, this area is scaled by the size of the representative volume element, which is of great importance in the definition of a continuous variable in the sense of continuum mechanics. At one point, it must be the representative effect of microdefects over the mesoscale volume element. It is similar to plasticity where the plastic strain ε_P represents, at one point, the average of many slips. If we consider a Representative Volume Element (RVE) oriented along the direction $\vec{n}$ and a plane passing through the RVE with the same orientation (Figure 5), we can define:

- ∂S as the area of the intersection of the plane with the RVE;
- ∂S_D as the area of the intersections of all microcracks or microcavities which lie in ∂S;

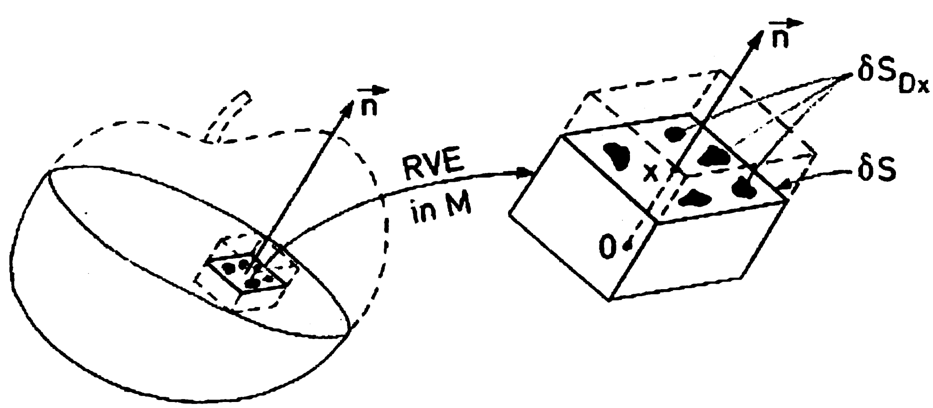


Figure 5 : Isotropic definition of damage parameter (Apple representation is after J. Lemaitre, 1975)

- The value of the damage $D(M,\vec{n})$ attached to the point M in the direction $\vec{n}$ is defined as:

$$D(M,\vec{n}) = \frac{\partial S_D}{\partial S}$$

The damage parameter at point M is the maximum value of $D(M,\vec{n})$ for all possible orientations, $\vec{n}$.

It follows from this definition that the value of the scalar variable *D* (which depends upon the point and the direction considered) is bounded by 0 and 1:

$D = 0 \rightarrow$ Undamaged RVE material;

$D = 1 \rightarrow$ Fully broken RVE material in two parts.

In fact, the failure occurs for $D < 1$ through a process of instability.

1.7.2 Three-Dimensional tensorial representation of the Damage parameter (Murakami and Ohno, 1978 [8]; Murakami, 1988)

The theory of continuum damage mechanics was first developed for material deterioration in the process of creep. Kachanov in 1958 first proposed a phenomenological theory of creep damage by introducing a scalar damage variable *D* to characterize the material degradation which is responsible for tertiary creep and rupture [1]. The underlying physical suggestion behind the scalar characterization is that creep damage in a continuum is assumed to be everywhere the same and independent of the specific orientation chosen. Thereafter, Rabotnov interpreted that the damage variable represents the fraction of damaged material, thus reducing the effective resisting area of the local material element from S to $S_D = (1-D)S$. In the one-dimensional case, the effective stress was thus defined as:

$$\tilde{\sigma} = \frac{F}{S_D} = \frac{F}{S(1-D)} = \frac{\sigma}{(1-D)}$$

Although the idea of scalar isotropic damage parameter is intuitively appealing and has been universally accepted, it needs careful re-examination. The main drawback of this isotropic model is that the model predicts a constant value of the Poisson's ratio not affected by material damage while the Young's modulus E is reduced to $E(1-D)$ after damage has occurred. However, experimental observations have shown a change in Poisson's ratio. ([9], [11], and [12])

On the other hand, despite the complexity of their mathematical structure, tensors of second ranks can describe the spatial distribution of microcavities more accurately, and hence, considerable efforts have already been made to develop damage models based on second order tensors [9], [10]. The second order damage field $\underline{D}(x)$ has the following form in the principal directions:

$$D(x) = \begin{bmatrix} D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & D_3 \end{bmatrix}$$

D_1, D_2, and D_3 are simply the ratios of the damaged surface area over the total surface area for any point in three orthogonal directions. It is assumed that the principal directions of $\underline{D}(x)$ coincide with the principal directions of stress field at the specified material element because the dislocations tend to line up and form the crack in the plane, which is perpendicular to the principal stress direction[11].

In addition to scalars and second order tensors, vectors and forth order tensors have also been used to reperesent the damage parameter. ([3], [4], [5], and [6])

However since the differentiation with respect to damage parameter will be necessary in the derivation of constitutive laws and tensors will result in additional terms and calculations; it is very beneficial to avoid tensorial representation whenever the material is not highly anisotropic. Besides the increase in the cost of analysis, anisotropic approaches introduce additional material properties which should be determined from classical experiments such as tensile loading-unloading test and measurements of Young's modulus and Poison's ratio along other orientations [11]. Throughout this work, the isotropic scalar representation of damage is used.

1.8 Effective Stress Concept (Y.N. Rabotnov, 1968)

Under the loading of F, the usual uniaxial stress is:

$$\sigma = \frac{F}{S}$$

If all the defects are open and no force is carried by the broken area represented by S_D, an effective stress $\tilde{\sigma}$ is introduced which is related to the surface that effectively resists the load, namely $(S - S_D)$:

$$\tilde{\sigma} = \frac{F}{S - S_D}$$

Introducing the isotropic damage variable $D(M,\vec{n}) = \frac{\partial S_D}{\partial S}$,

$$\tilde{\sigma} = \frac{F}{S - S_D} = \frac{F}{S(1 - \frac{S_D}{S})}$$

Or $\tilde{\sigma} = \frac{\sigma}{1 - D}$

This definition is the effective stress on the material in tension. In compression, some defects close and the surface that effectively resists the load is larger than $(S - S_D)$, In particular, if all the defects close, the effective stress in compression is equal to the usual stress. Moreover, the damage evolution is almost zero and the damage parameter remains constant. This effect is called the crack closure effect and is taken into account by introducing a factor in the numerical implementation of damage evolution equation. This factor equals to zero when the hydrostatic stress is negative corresponding to the material in compression and equals to one for positive values of hydrostatic stress. However since the damage evolution in compression may not be negligible for some materials and temperatures, one can introduce a material parameter ranging from zero to one which of course should be determined experimentally.

1.9 Equivalence Principles

A way to avoid micromechanical analysis for each type of defect and each type of mechanism of damage is to postulate a principle at the mesoscale. Equivalence principles are used to relate the damaged state to the undamaged state. Several equivalence principles have been postulated and their validity has been investigated [20]. The choice of an equivalence principle is the first step in the derivation of damage-coupled equations and somehow depends on the problem we are looking at. For instance, if the material is highly anisotropic, the tensorial representation of damage has to be considered and elastic strain energy equivalence criterion has shown better agreement with experimental observations. Whereas, strain equivalence criterion is theoretically more reasonable in the case of scalar representation of damage corresponding to isotropic materials [19].

Two widely used equivalence principles are elastic strain energy equivalence and strain equivalence. The elastic strain energy equivalence states that (Sidoroff, 1982; Cordebois, 1983; Chow and Wang, 1987; Ju, 1989):

There exists a pseudo-undamaged material made of the virgin material in the sense that its elastic strain energy is equal to that of the damaged material except that the stress is replaced by the effective stress in the elastic strain energy formulation.

The complementary elastic energy of an undamaged material $(D=0)$ is:

$$w_e(\sigma,0)=\frac{1}{2}\sigma^T:C^{-1}:\sigma$$

Then the complimentary energy of a damaged material $(D\neq 0)$ is expressed as:

$$w*_e(\sigma,D)=w*_e(\tilde{\sigma},0)=\frac{1}{2}\tilde{\sigma}^T:C^{-1}:\tilde{\sigma}=\frac{1}{2}\sigma^T:\left(M^T:C^{-1}:M\right):\sigma$$

Where C is the elastic stiffness tensor and $\tilde{C}$ can be defined as effective elastic stiffness tensor such that:

$$\tilde{C}^{-1}=\left(M^T:C^{-1}:M\right)$$

M is a fourth order tensor called the damage effect tensor and is defined such that:

$$\tilde{\sigma}=M:\sigma$$

For the general case of anisotropic damage, damage effect tensor is equal to [10]:

$$M = \begin{bmatrix} \frac{1}{1-D_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{1-D_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{1-D_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{(1-D_2)(1-D_3)}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{(1-D_1)(1-D_3)}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{(1-D_2)(1-D_1)}} \end{bmatrix}$$

Throughout this work, the strain equivalence postulate is used which is explained hereafter. Reader can find the details about elastic strain energy equivalence criterion and other equivalence criteria in [11] and [20].

1.10 Strain Equivalence Principle (J. Lemaitre, 1971)

At the microscale, this postulate assumes that the constitutive equations for the strain of an element are not modified by a neighboring element containing a microcrack [15]. At the mesoscale, this means that the strain constitutive equations written for the surface $(S - S_D)$ are not modified by the damage or that the true stress loading on the material is the effective stress $\tilde{\sigma}$ and no longer σ. Consequently, any strain constitutive equation for a damaged material may be derived in the same way as for a virgin material except that the usual stress is replaced by the effective stress [15].

$$\varepsilon(\sigma, D) = \varepsilon(\tilde{\sigma}, 0)$$
$$\tilde{\sigma} = \frac{\sigma}{1-D}$$

This principle is applied to both elasticity and plasticity.

1.11 Coupling between Strains and Damage

Applying the strain equivalence principle, we can write the uniaxial laws of elasticity and plasticity of a damaged material as following:

1.11.1 Elasticity law

Using the concept of effective stress:

$$\varepsilon^e = \frac{\tilde{\sigma}}{E}$$

And strain in other directions for isotropic damage:

$$\varepsilon_{22}{}^e = \varepsilon_{33}{}^e = -\upsilon \varepsilon_{11}{}^e$$

E and υ are the Young's modulus and the Poisson's ratio of the undamaged material. As a result, the elasticity modulus of the damaged material can be defined by the ratio $\frac{\sigma}{\varepsilon^e}$ which is $\tilde{E} = E(1-D)$

This inspires a method for evaluating the damage parameter based on the change in Young's modulus:

$$D = 1 - \frac{\tilde{E}}{E}$$

1.11.2 Plasticity Law

In order to model plasticity two kinds of strain hardening are usually considered:

-The isotropic hardening which indicates the size of the yield function.

-The kinematic hardening or the back stress which indicates the center of the yield function.

If σ_y is the yield stress, R the stress due to isotropic hardening and X the back stress, both functions of the plastic strain, the one-dimensional plasticity criterion defining the current threshold of yield limit is:

$$\sigma = \sigma_y + R + X$$

Or:

$$f = |\sigma - X| - \sigma_y - R = 0$$

f is the yield function from which the kinetic constitutive equation for plastic strain is derived.

$$\exists \dot{\varepsilon}^p \neq 0 if \begin{cases} f = 0 \\ and \\ \dot{f} = 0 \end{cases}$$

And:

$$\dot{\varepsilon}^p = 0 \text{ if } \begin{cases} f < 0 \\ or \\ \dot{f} < 0 \end{cases}$$

When damage occurs, according to the principle of equivalence, the yield function f must be written as:

$$f = \left| \frac{\sigma}{1-D} - X \right| - \sigma_y - R = 0$$

1.12 Rupture criterion:

The rupture at the mesoscale is a crack formation which occupies the whole surface of the RVE; that is, $D = 1$. In many cases this is caused by a process of instability which suddenly causes the separation of remaining area. This point of instability corresponds to a critical value of damage $D_c < 1$ which depends upon the material and the conditions of loading.

The final decohesion of atoms is characterized by a critical value of the effective stress acting on the resisting area. If the ultimate stress σ_u is taken as the critical value of the effective stress, we have:

$$\tilde{\sigma} = \frac{\sigma}{1-D_c} = \sigma_u$$

Then:

$$D_c \approx 1 - \frac{\sigma}{\sigma_u}$$

This gives the critical value of the damage at a mesocrack initiation occurring for the one-dimensional stress, σ. The ultimate stress σ_u being identified as a material characteristic, D_c may vary between $D_c \approx 0$ for pure brittle fracture to $D_c \approx 1$ for pure ductile fracture but usually D_c remains in the order of 0.2 to 0.5.

This relation, applied to the pure monotonic tension test, which is taken as a reference, defines the corresponding critical damage D_{1c} considered as a material characteristic:

$$D_{1c} \approx 1 - \frac{\sigma_R}{\sigma_u}$$

Where σ_R is the stress at rupture.

1.13 Damage initial threshold

Before the microcracks are initiated, creating the damage modeled by D, they must nucleate by the accumulation of dislocations. In the pure tension case, this corresponds to a certain value ε_{pD} of the plastic strain below which no damage occurs:

$$\varepsilon^p < \varepsilon_{pD} \rightarrow \dot{D} = 0$$

Finally, the four main relations which comprise the basis of Continuum Damage Mechanics are:

- $\varepsilon^e = \dfrac{\sigma}{E(1-D)}$ for elasticity

- $f = \left| \frac{\sigma}{1-D} - X \right| - \sigma_y - R = 0$ as the plastic yield criterion

- $\varepsilon^p < \varepsilon_{pD} \rightarrow \dot{D} = 0$ as the damage threshold

- $D = D_c \rightarrow$ crack initiation

CHAPTER 2: Measurement of Damage ([15] & [18])

2.1 Direct Measurements

This method consists in the evaluation of the total crack areas δS_D lying on a surface δS at mesoscale. This can be done by observing microscopic pictures. The main drawback of this method is that it is a destructive method and one sample is needed for each data point. Also, it is tedious to practice since the sample needs to be taken out at various strain values and observed under a microscope.

2.2 Variation of the Elasticity Modulus

This is a non-direct measurement based on the elasticity law:

$$\varepsilon^{e} = \frac{\sigma}{E(1-D)}$$

It assumes uniform homogeneous damage in the specimen gauge section.

If $\tilde{E} = E(1-D)$ is considered as the effective elasticity modulus of the damaged material, the values of the damage may be derived from measurements of $\tilde{E}$, provided that Young's modulus E is known:

$$D = 1 - \frac{\tilde{E}}{E}$$

This very useful method requires accurate strain measurements. Strain gauges are commonly used and $\tilde{E}$ is most accurately measured during unloading [18]. This technique may be used for any kind of damage as long as the damage is uniformly distributed in the volume on which the strain is measured which is the main limitation of the method. If the damage is greatly localized, as for high cycle fatigue of metals, for example, another method must be used.

At the beginning and at the end of the unloading paths in the plane (σ,ε), there are small nonlinearities, owing to viscous or hardening effects and also due to the experimental devices as well. It is best to ignore them and to identify $\tilde{E}$ from the slope of the middle points. Also, it is most important to always use the same procedure to evaluate E and the evolution of $\tilde{E}$.

For damage in metals, an early decrease of $\tilde{E}$ at low strain levels may happen which is due to movements of dislocations, and to texture development, but not to damage [15]. As this phenomenon rapidly saturates, it is common to consider a damage threshold such that:

$$\varepsilon^{p} < \varepsilon_{pD} \rightarrow \dot{D} = 0$$

Instead of strain gage, one may use grid patterns and observe the deformation of the grid pattern. After the deformation, a circular pattern may look like an oval whose major and minor diameters and their orientations can be used to obtain principal strains and their orientations respective to the loading direction. To avoid taking out the sample at each unloading increment in order to measure the strains, one may take pictures from the grid pattern while the sample is still in the MTS machine and analyze the pictures when the test is complete.

2.3 Ultrasonic wave propagation

This method is based on the effect of change in elasticity modulus on the speed of ultrasonic waves. For frequencies higher than 200 kHz the longitudinal wave speed v_L and the transversal wave speed v_T in a linear isotropic elastic cylindrical medium are [18]:

$$v^2{}_L = \frac{E}{\rho}\frac{1-\upsilon}{(1+\upsilon)(1-2\upsilon)}$$

$$v^2{}_T = \frac{E}{\rho}\frac{1}{2(1+\upsilon)}$$

Where E is the Young's modulus, ρ the density, and υ Poisson's ratio. A measurement of the longitudinal wave speed of a damaged material gives:

$$\tilde{v}^2{}_L = \frac{\tilde{E}}{\tilde{\rho}}\frac{1-\upsilon}{(1+\upsilon)(1-2\upsilon)}$$

Where $\tilde{E}$ and $\tilde{\rho}$ are the actual damaged elastic modulus and density. Poisson's ratio does not vary with damage if elasticity and damage are isotropic. If anisotropic damage is considered, the Poisson's ratio for the damaged material is different. The damage is calculated by:

$$D = 1 - \frac{\tilde{E}}{E} = 1 - \frac{\tilde{\rho}}{\rho}\frac{\tilde{v}^2{}_L}{v^2{}_L}$$

If the damage consists mainly of microcracks or if small cavitation is considered, $\frac{\tilde{\rho}}{\rho} \approx 1$ and:

$$D \approx 1 - \frac{\tilde{v}^2{}_L}{v^2{}_L}$$

To measure the speed v_L, ultrasonic transducers need to be placed on both sides of the sample and damage parameter is assumed to be uniform throughout the sample. Thus, the sample's size must be of an order of magnitude coherent with that of the RVE and for metals, this size is too small in comparison with ultrasonic transducers size and accuracy of time measurements. Also, this method is destructive because we need to cut the material into very tiny parts. Nevertheless, this method gives good results for materials with larger RVE size such as concrete [15].

2.4 Variation of the Micro hardness

This method is a non-direct measurement based on the influence of damage on the plasticity criterion. In the uniaxial state of stress, the yield function is written as:

$$f = \left| \frac{\sigma}{1 - D} - X \right| - \sigma_y - R = 0$$

As microhardness is a process of very small indentation, this method may be considered as practically nondestructive. The test consists in inserting a diamond indenter in the material and the hardness H is defined by the mean stress:

$$H = \sigma = \frac{F}{S}$$

The load F on the indenter is adjusted so as to obtain a projected indented area S of the same order of magnitude as that of the RVE.

Theoretical analyses and many experimental results prove a linear relationship between H and the plasticity threshold σ_s, k' being the coefficient of proportionality:

$$H = k'\sigma_s$$

This threshold corresponds to the actual yield stress,

$$\sigma_s = (\sigma_y + R + X)(1 - D)$$

Then:

$$H = k'(\sigma_y + R + X)(1 - D)$$

In fact, the microhardness test itself increases the strain hardening by an amount which corresponds to a plastic strain ε^p of the order of 5 to 8% [18]. H is then always related to $(\varepsilon^p + \varepsilon_H{}^p)$, ε^p being the current plastic strain.

If $H^* = k'(\sigma_y + R + X)$ is the microhardness of the material which would exist without any damage for $(\varepsilon^p + \varepsilon_H{}^p)$, and $H = k'(\sigma_y + R + X)(1 - D)$ the actual micro-hardness for $(\varepsilon^p + \varepsilon_H{}^p)$, then:

$$D = 1 - \frac{H}{H^*}$$

2.5 Other Methods

Several other methods have been introduced based on the influence of damage on some physical or mechanically measurable properties such as:

2.5.1 Variation of density

In the case of pure ductile damage, the defects are cavities which can be assumed to be roughly spherical. This means that the volume increases with damage and the corresponding decrease of density is measurable with devices based on the Archimedean principle.

If $\frac{(\tilde{\rho} - \rho)}{\rho}$ is the relative variation of density between the damaged state $\tilde{\rho}$ and the initial non-damaged state ρ, it is easy, considering a spherical cavity of radius r in a spherical RVE of initial radius R and mass m, to derive the following relationships between the surface damage D and the variation of density or porosity:

$$\rho = \frac{m}{\frac{4}{3}\pi R^3}$$

$$\tilde{\rho} = \frac{m}{\frac{4}{3}\pi (R^3 + r^3)}$$

$$\frac{\tilde{\rho} - \rho}{\rho} = \frac{R^3}{R^3 + r^3} - 1 = \frac{-r^3}{R^3 + r^3}$$

$$D = \frac{\delta S_D}{\delta S} \approx \frac{\pi r^2}{\pi \left(R^3 + r^3\right)^{2/3}} = \left(\frac{r^3}{R^3 + r^3}\right)^{2/3}$$

$$D = \left(1 - \frac{\tilde{\rho}}{\rho}\right)^{2/3}$$

2.5.2 Variation of electrical resistance (potential drop)

Ohm's law for a non-damaged element of length l, area S and resistivity r is written as:

$$V = r\frac{l}{S}i$$

Where i is the intensity of the electrical current. Whereas, for a damaged element of the same size:

$$\tilde{V} = \tilde{r}\frac{l}{S - S_D}i = \tilde{r}\frac{l}{S(1-D)}i = \tilde{r}\frac{l}{S}\tilde{i}$$

Where $\tilde{i} = \frac{i}{1-D}$ is the effective intensity of the electrical current defined in the same way as the effective stress was defined, using the surface definition of damage.

$\tilde{r}$ is the resistivity affected by the damage. Bridgman's law [18] gives $\tilde{r}$ as:

$$\tilde{r} = r\left(1 + K\frac{\tilde{\rho} - \rho}{\rho}\right) = r\left(1 + KD^{3/2}\right)$$

K is a coefficient which is approximately 2 for metals.

If the same intensity i is considered for the non damaged and the damaged elements, the damage D may be derived from the two expressions:

$$\frac{V}{\tilde{V}} = \frac{r}{\tilde{r}} \frac{\frac{l}{S} i}{\frac{l}{S}\frac{i}{1-D}}$$

$$D = 1 - \frac{V}{\tilde{V}}\frac{\tilde{r}}{r}$$

$$\frac{\tilde{r}}{r} = 1 + KD^{3/2}$$

For small values of D, the correction term $\frac{\tilde{r}}{r}$ due to the volume change is close to 1 (for instance, $D = 0.1 \rightarrow \frac{\tilde{r}}{r} = 1.064$); then:

$$D \approx 1 - \frac{V}{\tilde{V}}$$

This method is known as the "potential drop method".

2.5.3 Variation of the cyclic plasticity response (stress amplitude drop)

The influence of damage on plasticity may be used to measure the low cycle fatigue damage in which the material undergoes plastic cyclic loading.

The one-dimensional law of cyclic plasticity at stabilization may be written as a power relationship between the amplitude of stress $\Delta\sigma$ and the amplitude of plastic strain $\Delta\varepsilon^p$ as $\Delta\varepsilon^p = \left(\frac{\Delta\sigma}{K_p}\right)^M$ for a non-damaged material, and as $\Delta\varepsilon^p = \left(\frac{\Delta\sigma}{K_p(1-D)}\right)^M$ for a damaged material according to the principle of strain equivalence, where K_p and M are material parameters.

Considering a test at constant plastic strain amplitude, if $\Delta\sigma^*$ is the stress amplitude before the beginning of the damage process and $\Delta\sigma$ is the stress amplitude after the beginning of the damage process, we have:

$$\Delta\varepsilon^p = \left(\frac{\Delta\sigma^*}{K_p}\right)^M = \left(\frac{\Delta\sigma}{K_p(1-D)}\right)^M$$

From which it follows that:

$$D = 1 - \frac{\Delta\sigma}{\Delta\sigma^*}$$

This method successfully identifies the evolution of damage during low cycle fatigue of metals. An example is given in Figure 6.

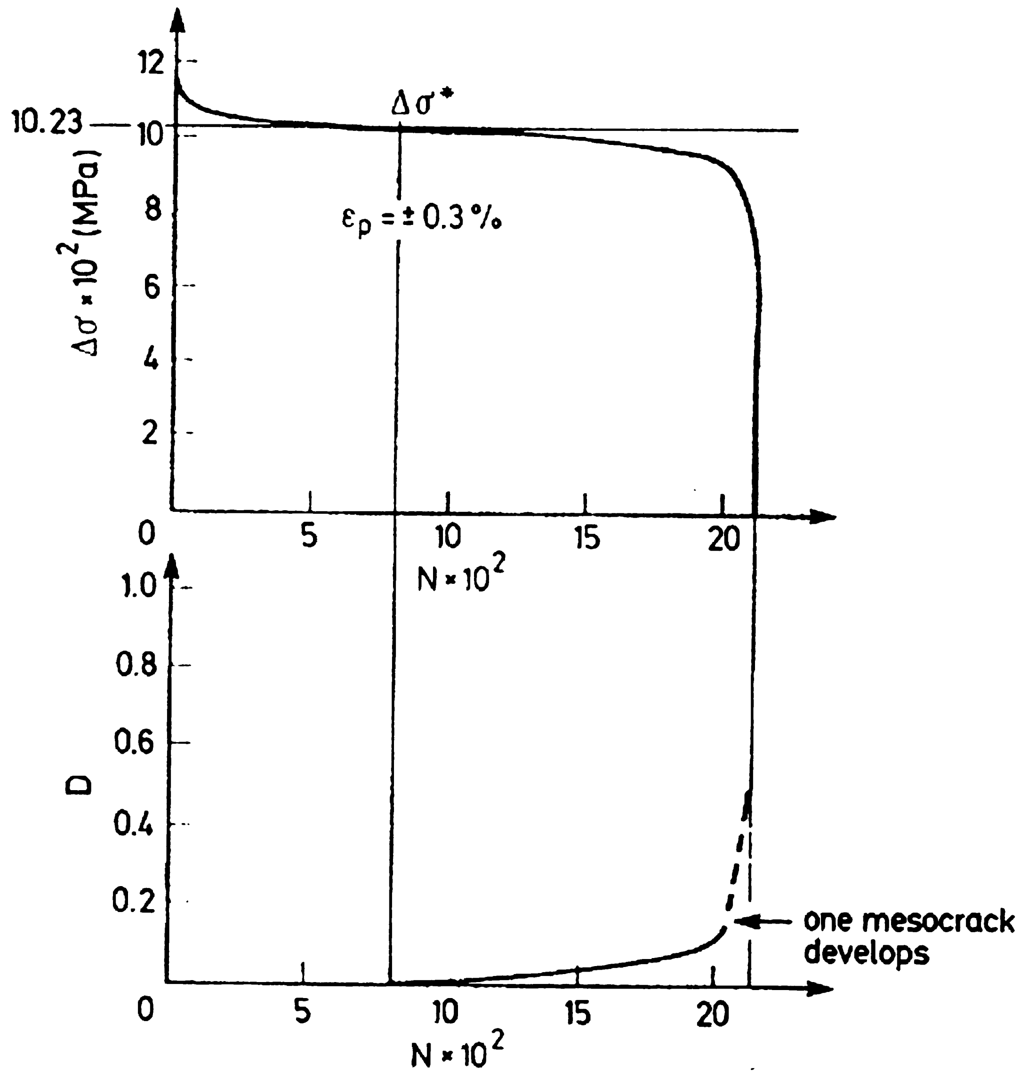


Figure 6 : Evolution of damage during low cycle fatigue

2.5.4 Tertiary creep response

Creep damage occurs in metals loaded at temperatures above approximately one third of the melting temperature. If we apply the principle of strain equivalence to Norton's law of tertiary creep.

$$\dot{\varepsilon}^{p*} = \left(\frac{\sigma}{K_v}\right)^N$$

K_v and N being temperature dependent material parameters.

Assuming that the damage process begins at the end of the secondary creep, during tertiary creep one may write:

$$\dot{\varepsilon}^p = \left(\frac{\sigma}{K_v (1 - D)} \right)^N$$

From which one derives:

$$D = 1 - \left(\frac{\dot{\varepsilon}^p *}{\dot{\varepsilon}^p} \right)^{1/N}$$

Where $\dot{\varepsilon}^p *$ is the minimum creep rate.

This method yields good results which are in accordance with those obtained by measuring the variation of the elasticity modulus. An example is given in Figure 7.

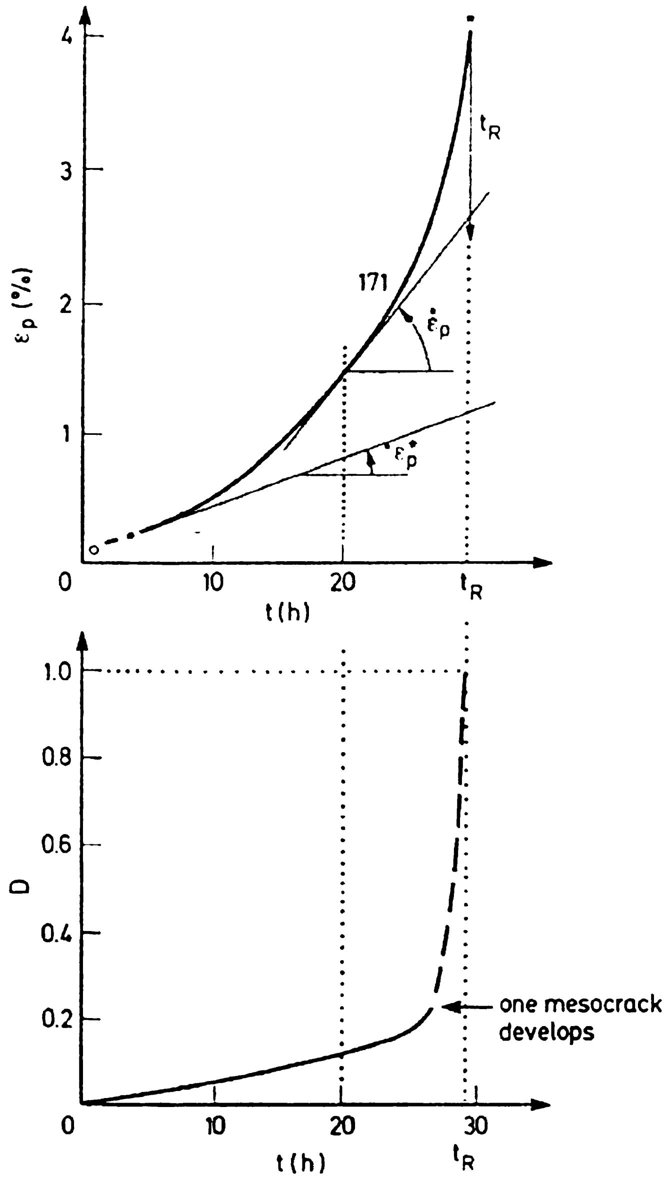


Figure 7 : Evolution of creep damage in IN 100 super alloy (after H. Poheella)

As a conclusion, Figure 8 offers some advices for choosing the proper method of damage measurement, depending on the kind of damage involved. [18]

Damage		Brittle	Ductile	Creep	Low cycle fatigue	High cycle fatigue
Micrography	$D = \frac{\partial S_D}{\partial S}$	*	**	**	*	*
Density	$D = \left(1 - \frac{\tilde{\rho}}{\rho}\right)^{2/3}$		**	*	*	
Elasticity modulus	$D = 1 - \frac{\tilde{E}}{E}$	**	***	***	***	
Ultrasonic waves	$D = 1 - \frac{\tilde{V}_L^2}{V_L^2}$	***	**	**	*	*
Cyclic stress amplitude	$D = 1 - \frac{\Delta\sigma}{\Delta\sigma^*}$		*	*	**	*
Tertiary creep	$D = 1 - \left(\frac{\dot{\varepsilon}_p^*}{\dot{\varepsilon}_p}\right)^{1/N}$		*	***	*	
Micro-hardness	$D = 1 - \frac{H}{H^*}$	**	***	**	***	*
Electrical resistance	$D = 1 - \frac{V}{\tilde{V}}$	*	**	**	*	*

Figure 8 : More number of stars specifies a better method for damage measurement

CHAPTER 3: Thermodynamics Approach

3.1 How Thermodynamics Approaches Damage

The thermodynamics approach to obtain damage equations in three dimensions is to postulate the existence of energy potentials from which one can derive the state laws and the kinetic constitutive equations. In the thermodynamics of irreversible processes, two potentials are introduced [7]:

1-The state potential, a function of the state variables, which defines the state laws and the variables associated with the state variables to define the power involved in each physical process. For the damage variable, an energy damage criterion is derived from the damaged elasticity potential.

2-The potential of dissipation, which is a function of the associated variables accounts for the kinetic laws of evolution of the state variables including damage. A constitutive equation for the damage gives the damage rate as a function of its associated variable.

3.2 Thermodynamical Variables, State Potential

The state variables can be categorized as follows:

- Observable variables such as:

ε - The strain tensor of components ε_{ij} associated with the Cauchy stress tensor σ

T - The temperature associated with the entropy density s

α - The back strain tensor, whose associated variable is the back stress X^D and X^D represents the kinematic hardening, translation of the center of the yield surface in the deviatoric space

r - Which is equal to effective plastic strain when damage is absent and its associated variable is R representing the isotropic hardening.

D - The damage variable. If the damage is considered to be isotropic, it has the same value in all directions, and the scalar $D = \dfrac{\partial S_D}{\partial S}$ characterizes completely the three-dimensional state of damage in the RVE at the point considered. $\bar{Y}$ is the associated variable of D which will be derived from the state potential. As D is dimensionless and since the product - $\bar{Y}\dot{D}$ is the power involved in the process of damage, it can be seen that $\bar{Y}$ is a volume energy density.

The following chart summarizes the variables involved in the continuous mechanics of elasticity, plasticity and damage:

State Variables		Associated Variables
Observable	Internal	
ε		σ
T		s
	r	R
	α	X^D
	D	$\overline{Y}$

Table 1 : State variable and their associated variables

It is postulated that the state laws are derived from a state potential; a continuous scalar function, concave with the temperature, convex with the other state variables and containing the origin [7]. Taking the Helmholtz free energy:

$$\Psi = \Psi(\varepsilon, T, \varepsilon^e, \varepsilon^p, r, \alpha, D)$$

The strains act only through their difference $\varepsilon^e = \varepsilon - \varepsilon^p$:

$$\Psi = \Psi(\varepsilon^e, T, r, \alpha, D)$$

The associated variables can be derived as:

$$\sigma = \rho \frac{\partial \Psi}{\partial \varepsilon^e}$$

$$s = -\frac{\partial \Psi}{\partial T}$$

$$R = \frac{\partial \Psi}{\partial r}$$

$$X^D = \rho \frac{\partial \Psi}{\partial \alpha}$$

$$\bar{Y} = \rho \frac{\partial \Psi}{\partial D}$$

The analytical expression for Ψ is chosen considering the experimental observations and:

- The linear isotropic elasticity;

- The state coupling of damage with elastic strain and the principle of strain equivalence together with the concept of effective stress written for the three dimensional case as:

$$\tilde{\sigma}_{ij}(t) = \frac{\sigma_{ij}(t)}{1 - D(t)}$$

For anisotropic damage, the damage variable is no longer a scalar and the effective stress needs the application of damage effect tensor. The difference of the behavior in tension and compression is not taken into account here either. For isothermal loadings:

$$\Psi = \frac{1}{\rho}\left[\frac{1}{2} C_{ijkl} \varepsilon^e{}_{ij} \varepsilon^e{}_{kl} (1 - D) + R_\infty \left[r + \frac{1}{b} e^{(-br)} \right] + \frac{X_\infty \gamma}{3} \alpha_{ij} \alpha_{ij} \right]$$

The law of elasticity coupled with damage is:

$$\sigma = \rho \frac{\partial \Psi}{\partial \varepsilon^e} = C_{ijkl} \varepsilon^e{}_{kl} (1 - D)$$

Where C is the fourth order elasticity stiffness tensor or, by inversion for the isotropic case,

$$\varepsilon^e{}_{ij} = \frac{1+\upsilon}{E}\frac{\sigma_{ij}}{1-D} - \frac{\upsilon}{E}\frac{\sigma_{kk}}{1-D}\delta_{ij}$$

E being Young's modulus, υ Poisson's ratio and δ_{ij} the Kronecker delta. The isotropic strain hardening scalar stress is expressed as:

$$R = \rho\frac{\partial\Psi}{\partial r} = R_\infty\left[1 - e^{(-br)}\right]$$

R_∞ and b are two parameters which characterize the isotropic strain hardening for each material. X_∞ and γ are two parameters of nonlinear kinematic hardening written with the factor $\frac{2}{3}$ to ensure a simple expression in one dimension. The associated variable for D is defined as:

$$\bar{Y} = \rho\frac{\partial\Psi}{\partial D} = -\frac{1}{2}C_{ijkl}\varepsilon^e{}_{ij}\varepsilon^e{}_{kl}$$

In order to work with a positive quantity, Y is introduced as:

$$Y = -\bar{Y}$$

A relationship can be found between Y and the elastic strain energy density w_e defined as:

$$dw_e = \sigma_{ij}d\varepsilon^e{}_{ij}$$

Integrating with the law of elasticity, and assuming no variation of damage, that is, $D = const.$, yields:

$$w_e = \int C_{ijkl}\varepsilon^e{}_{kl}(1-D)d\varepsilon^e{}_{ij} = \frac{1}{2}C_{ijkl}\varepsilon^e{}_{ij}\varepsilon^e{}_{kl}(1-D)$$

This shows that:

$$Y = \frac{w_e}{1-D}$$

Y is also equal to one half the variation of the strain energy density with damage at constant stress, $d\sigma_{ij} = 0$, [7]. Starting with the law of elasticity:

$$d\sigma_{ij} = C_{ijkl}\left[(1-D)d\varepsilon^e{}_{kl} - \varepsilon^e{}_{kl}dD\right] = 0$$

Or:

$$d\varepsilon^e{}_{kl} = \varepsilon^e{}_{kl}\frac{dD}{1-D}$$

Together with:

$$dw_{e(\sigma=const)} = \sigma_{ij}d\varepsilon^e{}_{ij} = \sigma_{ij}\varepsilon^e{}_{kl}\frac{dD}{1-D} = C_{ijkl}\varepsilon^e{}_{ij}(1-D)\varepsilon^e{}_{kl}\frac{dD}{1-D}$$

Or:

$$\frac{dw_{e(\sigma=const)}}{dD} = C_{ijkl}\varepsilon^e{}_{ij}\varepsilon^e{}_{kl}$$

And from the definition of $Y = -\overline{Y}$

$$Y = \frac{1}{2}\frac{dw_{e(\sigma=const)}}{dD}$$

Because of this equation, Y is called the strain energy density release rate. This is the energy released by loss of stiffness, damage softening, in the RVE when damage occurs.

3.3 Potential of Dissipation, Damage Evolution Equation

Having defined all the state and associated variables, a second potential is needed to describe the evolution of the phenomena. First we investigate the second principle of thermodynamics to ensure a non-negative dissipation. Starting from the Clausius-Duhem inequality [21],

$$\sigma_{ij}\dot{\varepsilon}_{ij} - \rho(\dot{\Psi} + s\dot{T}) - q_i \frac{T_{,i}}{T} \geq 0$$

$\vec{q}$ is the heat flux vector associated with the temperature gradient.

If we take the derivative of free energy with respect to the state variables, we have:

$$\dot{\Psi} = \frac{\partial \Psi}{\partial \varepsilon^e{}_{ij}}\dot{\varepsilon}^e{}_{ij} + \frac{\partial \Psi}{\partial T}\dot{T} + \frac{\partial \Psi}{\partial r}\dot{r} + \frac{\partial \Psi}{\partial \alpha_{ij}}\dot{\alpha}_{ij} + \frac{\partial \Psi}{\partial D}\dot{D}$$

Together with $\dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p$ it becomes:

$$\left(\sigma_{ij} - \rho\frac{\partial \Psi}{\partial \varepsilon^e{}_{ij}}\right)\dot{\varepsilon}^e{}_{ij} - \rho\left(s + \frac{\partial \Psi}{\partial T}\right)\dot{T} + \sigma_{ij}\dot{\varepsilon}^p{}_{ij} - \rho\frac{\partial \Psi}{\partial r}\dot{r} - \rho\frac{\partial \Psi}{\partial \alpha_{ij}}\dot{\alpha}_{ij} - \rho\frac{\partial \Psi}{\partial D}\dot{D} - q_i\frac{T_{,i}}{T} \geq 0$$

Or, with the definition of the associated variables,

$$\sigma_{ij}\dot{\varepsilon}^p{}_{ij} - R\dot{r} - X^D{}_{ij}\dot{\alpha}_{ij} - \overline{Y}\dot{D} - q_i\frac{T_{,i}}{T} \geq 0$$

To always satisfy this inequality of a positive dissipation and particularly for an isothermal process where the plastic dissipation, the first term, is negligible, we must have:

$$-\overline{Y}\dot{D} \geq 0$$

As $-\overline{Y}$ is a positive quadratic function, $-\overline{Y} \geq 0$, and the damage rate $\dot{D}$ is non-negative, the Clausius-Duhem inequality is satisfied. Notice that the dissipation inequality is the sum of the products of rate or flux variables (with the negative sign) multiplied by their dual variables as shown by Table 2:

flux variables	dual variables
$\dot{\varepsilon}^p$	σ
$-R\dot{r}$	R
$-\dot{\alpha}$	X^D
$-\dot{D}$	$-\overline{Y}$
$\vec{q}$	$-\frac{T_{,i}}{T}$

Table 2 : Flux and their dual variables

It is postulated that the kinetic laws are derived from a potential of dissipation, a scalar continuous and convex function F, of the dual variables and the state variables may act as parameters [15]. For the isothermal case:

$$F(\sigma, R, X^D, Y : (\varepsilon^e, r, \alpha, D))$$

The laws of plasticity coupled with damage are derived from this potential by means of a scalar multiplier which is always positive. This ensures the normality condition of yielding for plasticity.

$$\left.\begin{aligned}\dot{\varepsilon}^p_{ij} &= \frac{\partial F}{\partial \sigma}\dot{\lambda} \\ \dot{r} &= -\frac{\partial F}{\partial R}\dot{\lambda} \\ \dot{\alpha} &= -\frac{\partial F}{\partial X^D}\dot{\lambda}\end{aligned}\right\} \textit{plasticity constitutive equations}$$

And for damage:

$$\dot{D} = -\frac{\partial F}{\partial \overline{Y}}\dot{\lambda}$$

In this work, we use the von-Mises criterion in the yield function which identifies the equivalent stress as:

$$\sigma_{eq} = \left(\frac{3}{2}\sigma^D{}_{ij}\sigma^D{}_{ij}\right)^{1/2}$$

Together with kinematic hardening, the von-Mises criterion is applied to define the size of the yield surface regardless of the translation X^D and it acts upon the difference between $\sigma^D{}_{ij}$ and X^D.

Furthermore, In the presence of damage, the coupling between the damage and the plastic strain is written in accordance with the principle of strain equivalence. The yield criterion is written, in the same way as for a non-damaged material except that the stress is replaced by the effective stress, which, for isotropic damage is:

$$\frac{\sigma}{1-D} = \tilde{\sigma}$$

Then, the yield function f is written as:

$$f = \left|\sigma^D - X^D\right|_{eq} - R - \sigma_y = 0$$

With:

$$\left|\sigma^D - X^D\right|_{eq} = \left(\frac{3}{2}\left|\frac{\sigma^D{}_{ij}}{1-D} - X_{ij}{}^D\right|\left|\frac{\sigma^D{}_{ij}}{1-D} - X_{ij}{}^D\right|\right)^{1/2}$$

3.4 Plasticity

Plastic strain occurs when the yield criterion, $f = 0$, is satisfied. The plastic strain continues to grow if the yield criterion is continuously satisfied, that is, if $\dot{f} = 0$. Then plasticity deals with these two conditions, which define loading, or unloading with $f = 0$:

$$\left.\begin{array}{l} f = 0 \\ and \\ \dot{f} = 0 \end{array}\right\} \dot{\varepsilon}^p \neq 0$$

$$\left.\begin{array}{l} f < 0 \\ or \\ \dot{f} < 0 \end{array}\right\} \dot{\varepsilon}^p = 0$$

$\dot{\lambda}$ is deduced from these two conditions:

$f = 0$ and $\dot{f} = 0$.

3.5 Damage Coupled Constitutive Equations

The choice of an analytical expression for the two potentials and particularly for the potential of dissipation is of great importance. Thermodynamics provides the general framework and some restrictions on the functions that can be used, but only experiments can give the details and verify the model. As the constitutive equations must be as general as possible and cover a wide range of materials, the general trends of basic experiments are considered and only the value of a few parameters are taken to be as material parameters.

A schematic test in tension with some unloadings and compressive loadings is shown in Figure 9:

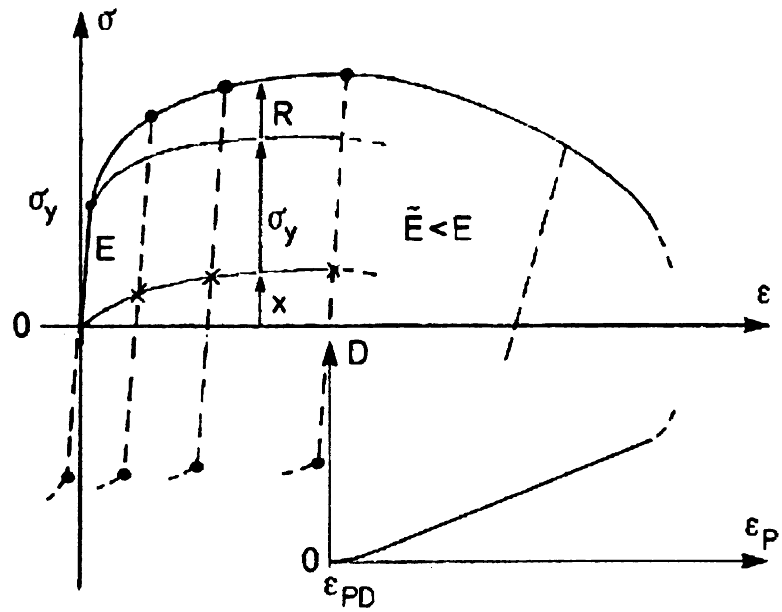


Figure 9 : Damage evolution and material degradation

Below a certain value of the plastic strain, a threshold ε_{pD}, no damage occurs. This is implemented in the code by comparing the equivalent plastic to ε_{pD} at the end of each increment. The kinematic back stress X, defined as the locus of the center of the elastic domain, increases with the plastic strain, is nonlinear with the plastic strain, and tends to saturate to some value X_∞. The evolution of X is chosen as:

$$X = X_\infty \left[1 - e^{(-\gamma \varepsilon_p)} \right]$$

X_∞ And γ are parameters to be identified for each material and each temperature. The isotropic hardening stress R, identified as:

$$R = \sigma - \sigma_y - X$$

- Increases with the plastic strain;
- is nonlinear with the plastic strain;
- tends to also saturate to some value R_∞

Like X The evolution of R is chosen as:

$$R = R_\infty \left[1 - e^{(-b\varepsilon_p)} \right]$$

If we use the exponential representation for hardening components, the potential of dissipation is proposed to be [15]:

$$F = \left| \sigma^D - X^D \right|_{eq} - R - \sigma_y + \frac{3}{4X_\infty} X_{ij}{}^D X_{ij}{}^D + F_{D(Y;(r,D))}$$

However, since the last term, F_D, is used to derive the damage evolution law, other hardening equations such as the power law can be used in the numerical procedure as well. The first three terms correspond to the yield function f. The fourth term $\frac{3}{4X_\infty}X_{ij}{}^D X_{ij}{}^D$ is responsible for the nonlinear kinematic hardening.

The choice of the function F_D is of course the key to representing the damage evolution and is based on physical observations and experimental data.

Damage is always related to some irreversible strain. This property is taken into account in the damage law by the multiplier $\dot{\lambda}$, which is proportional to the accumulated plastic strain.

$$\dot{D} = -\frac{\partial F_D}{\partial \overline{Y}}\dot{\lambda} = \frac{\partial F_D}{\partial Y}\dot{p}(1-D)$$

The variable $\dot{p}$ shows the irreversible nature of the damage, as $\dot{p}$ is always positive or zero.

As the accumulated plastic strain increases from zero the damage remains equal to zero during the nucleation of microcracks and accumulation of dislocations. A one-dimensional damage threshold related to the plastic strain ε_{pD} has already been introduced. As the equation of damage is governed by the accumulated plastic strain and as $p = \varepsilon^p$ in one-dimensional loading, it is needed to introduce a threshold on the variable p such that:

If $p \geq p_D$:

$$\dot{D} = -\frac{\partial F_D}{\partial \overline{Y}}\dot{\lambda} = \frac{\partial F_D}{\partial Y}\dot{p}(1-D)$$

If $p < p_D$:

$$\dot{D} = 0$$

This allows us to introduce a step function in the potential F_D:

$$H_{(p-p_D)} = \begin{cases} p \geq p_D \rightarrow 1 \\ p < p_D \rightarrow 0 \end{cases}$$

In the next section, an approach is introduced to identify p_D from the uniaxial damage threshold ε_{pD}.

On the basis of a thermodynamical analysis, the dual variable for the damage is the strain energy density release rate Y. Hence, F_D must be a function of Y:

$$F_D = F_D(Y,...)$$

Another important feature of fracture is the influence of the triaxiality ratio $\frac{\sigma_H}{\sigma_{eq}}$, ($\sigma_H$ is the hydrostatic stress, σ_{eq} and is the Von Mises equivalent stress). The modeling of this effect is contained in the expression of Y by the triaxialitv factor R_v:

$$R_v = \frac{2}{3}(1+\upsilon) + 3(1-2\upsilon)\left(\frac{\sigma_H}{\sigma_{eq}}\right)^2$$

$$Y = \frac{\tilde{\sigma}^2_{eq}}{2E} R_v$$

$$Y = \frac{\sigma^2_{eq}}{2E(1-D)^2} R_v$$

In order to choose the proper and simplest expression for F_D, a general trend in the damage evolution obtained by micromechanics for particular mechanisms is considered as a guide [15] which show:

$$\dot{D} \approx Y.\dot{p}$$

Thus:

$$F_D \approx Y^2$$

Here, as in every constitutive equation, a scale factor must be introduced. $S(1-D)$ is taken as the scale factor where S is a material constant. The term $(1-D)$ is considered here to be cancelled, with $(1-D)$ coming from $\dot{\lambda} = \dot{p}(1-D)$ because experiments show a non-decreasing damage rate when Y and $\dot{p}$ are constant:

$$F_D \approx \frac{Y^2}{S(1-D)}$$

Finally, according to the quantitative properties listed above, the damage potential is written as:

$$F_{D(Y;(r,D))} = \frac{Y^2}{2S(1-D)} H_{(p-p_D)}$$

The factor ½ is used here to avoid 2 in the derivation:

$$\dot{D} = -\frac{\partial F_D}{\partial \overline{Y}} \dot{\lambda} = \frac{\partial F_D}{\partial Y} \dot{\lambda} = \frac{Y}{S(1-D)} \dot{p}(1-D) H_{(p-p_D)}$$

Thus:

$$\dot{D} = \frac{Y}{S} \dot{p} H_{(p-p_D)}$$

With the rupture condition for crack initiation,

$$D = D_c$$

Three material parameters are introduced to characterize the damage evolution:

- S the damage strength,

- p_D the initial damage threshold,

- D_c the critical damage,

The effects of temperature can be taken into account by the variation of these coefficients and yield function coefficients with temperature.

3.6 Three Dimensional Damage Threshold

The damage threshold p_D in three dimensions which is equivalent to ε_{pD} in one dimension corresponds to the initial threshold of damage evolution. There has not been a universally acceptable way for derivation of p_D and in many cases ε_{pD} is used with very satisfactory results. Nonetheless, one of the most dependable criteria is developed by Lemaitre [17] and is cited in this section which is related to the amount of energy stored in the material as a result of plastic deformation.

The total plastic strain energy dissipated may reach tremendous values before failure but it is postulated that the stored energy remains constant at the beginning of damage evolution. This stored energy is the result of stress concentrations which develop in the neighborhood of dislocations in metals. For a unit volume, it is equal to the difference between the total plastic strain energy $\int_0^t \sigma_{ij}\dot{\varepsilon}^p{}_{ij}dt$ and the energy dissipated in heat given by the Clausius-Duhem inequality of the second principle of thermodynamics. The dissipated power for an isothermal deformation is:

$$\Phi = \sigma_{ij}\dot{\varepsilon}^p{}_{ij} - R\dot{p} - X^D{}_{ij}\dot{\alpha}_{ij} \geq 0$$

By replacing the rate variables with their definition from the potential of dissipation:

$$\dot{\alpha} = -\frac{\partial F}{\partial X^D}\dot{\lambda}$$

$$\dot{\varepsilon}^p{}_{ij} = \frac{\partial F}{\partial \sigma}\dot{\lambda}$$

$$\dot{p} = -\frac{\partial F}{\partial R}\dot{\lambda} = \dot{\lambda}$$

$$\Phi = \left(\sigma_{ij}\frac{\partial F}{\partial \sigma} + R\frac{\partial F}{\partial R} + X^D{}_{ij}\frac{\partial F}{\partial X^D}\right)\dot{p} \geq 0$$

$$\Phi = \left(\sigma_{ij} \frac{3}{2} \frac{\left| \frac{\sigma^D{}_{ij}}{1-D} - X_{ij}{}^D \right|}{\left| \frac{\sigma^D{}_{ij}}{1-D} - X_{ij}{}^D \right|_{eq}} - R + X^D{}_{ij} \left(-\frac{3}{2} \frac{\left| \frac{\sigma^D{}_{ij}}{1-D} - X_{ij}{}^D \right|}{\left| \frac{\sigma^D{}_{ij}}{1-D} - X_{ij}{}^D \right|_{eq}} + \frac{3}{2X_\infty} X_{ij}{}^D \right) \right) \dot{p}$$

$$\Phi = \left(\left| \frac{\sigma^D{}_{ij}}{1-D} - X_{ij}{}^D \right|_{eq} - R + \frac{3}{2X_\infty} X_{ij}{}^D X_{ij}{}^D \right) \dot{p}$$

$$\Phi = \left(\sigma_y + \frac{3}{2X_\infty} X_{ij}{}^D X_{ij}{}^D \right) \dot{p}$$

Then, the stored energy w_s as a function of time is:

$$w_{s(t)} = \int_0^t \sigma_{ij} \dot{\varepsilon}^p{}_{ij} dt - \int_0^t \left(\sigma_y + \frac{3}{2X_\infty} X_{ij}{}^D X_{ij}{}^D \right) \dot{p} dt$$

If we neglect the effect of the kinematic hardening,

$$X^D{}_{ij} = 0 \rightarrow \dot{\varepsilon}^p{}_{ij} = \frac{\partial F}{\partial \sigma} \dot{\lambda} = \frac{3\sigma^D{}_{ij}}{2\sigma_{eq}} \dot{p}$$

$$w_{s(t)} = \int_0^t \frac{3\sigma^D{}_{ij}\sigma^D{}_{ij}}{2\sigma_{eq}} \dot{p} dt - \int_0^t \sigma_y \dot{p} dt$$

Or with:

$$\sigma_{eq} = \left(\frac{3}{2} \sigma^D{}_{ij} \sigma^D{}_{ij} \right)^{1/2}$$

$$w_{s(t)} = \int_0^p (\sigma_{eq} - \sigma_y) dp$$

It is assumed that this energy is equal to that of a perfectly plastic material of plastic threshold $\sigma_s \geq \sigma_y$ whose stored energy is a function of the difference between σ_s and the fatigue limit σ_f [17]. Then, as

$\sigma_{eq} = const.$

$$w_s = (\sigma_{eq} - \sigma_f)p$$

If we put this energy equal to that of pure tension reference case having a damage threshold ε_{pD} and a plastic threshold σ_u, the ultimate stress:

$$(\sigma_{eq} - \sigma_f)p_D = (\sigma_u - \sigma_f)\varepsilon_{pD}$$

$$p_D = \varepsilon_{pD} \frac{\sigma_u - \sigma_f}{\sigma_{eq} - \sigma_f}$$

We have to remember that in this formula the material is considered to be perfectly plastic. Then for a varying loading $\sigma_{eq}(t)$, a more accurate calculation consists in performing the integration for w_s. After all these assumptions, it is fair to say that the derivation of three dimensional damage threshold calls for more investigation. Nevertheless, since σ_{eq} is close to σ_u but slightly smaller, taking $p_D = \varepsilon_{pD}$ is a good approximation and will be used in the numerical analysis.

3.7 Three-Dimensional Rupture Criterion

Similar to the three dimensional damage threshold, the expansion of one dimensional critical value of damage to three dimensions is subjective and there is no universally acceptable method to obtain it. A one-dimensional rupture criterion is obtained from the tensile loading-unloading test. The damage parameter is calculated from the change in the elasticity modulus and the critical value of damage is the final value of damage parameter. For the general case of loading, a critical three dimensional damage value is needed for numerical prediction of rupture. In so many numerical analyses, the critical value of damage is taken to be one in both one dimensional and three dimensional loading situations. Nevertheless, experimental data show that the material fails at the critical value much less than one and this value decreases as the ductility decreases. One of the approaches for obtaining a reasonable value for three-dimensional critical damage has been proposed by Lemaitre [15] and is cited hereafter.

This approach assumes that the rupture moment corresponding to the critical value of damage is governed by the amount of energy dissipated in damage growth:

$$\int_0^{D_C} Y dD = const.$$

For the case of perfect plasticity in proportional loading considered here for simplicity:

$$\int_0^{D_C} Y dD = \int_0^{D_C} \frac{\sigma^2{}_{eq}}{2E(1-D)^2} R_v dD = \int_0^{D_C} \frac{\sigma^2{}_s}{2E} R_v dD = \frac{\sigma^2{}_s}{2E} R_v D_c$$

This quantity should be equal to that of the uniaxial reference case which is:

$$\left.\begin{array}{l} \sigma_{eq} = \sigma_R \\ R_v = 1 \\ D = D_{1c} \end{array}\right\} Y = \frac{\sigma^2{}_R}{2E(1-D_{1c})^2} = \frac{\sigma^2{}_u}{2E}$$

$$\int_0^{D_{1c}} Y dD = \frac{\sigma^2_u}{2E} D_{1c}$$

$$\frac{\sigma^2_s}{2E} R_v D_c = \frac{\sigma^2_u}{2E} D_{1c}$$

$$\boxed{D_c = \frac{\sigma^2_u}{\sigma^2_s R_v} D_{1c} \leq 1}$$

And since $\sigma_s \approx \sigma_u$:

$$D_c = \frac{D_{1c}}{R_v}$$

This formula shows that the critical value of the damage in three dimensions decreases as the triaxiality ratio, $\frac{\sigma_H}{\sigma_{eq}}$ contained in the R_v expression, increases. Since, R_v is usually very close to one, taking $D_c = D_{1c}$ is a good approximation and will be used throughout the numerical part of this work.

The set of damage equations used in this work can be summarized as follows:

$$\dot{D} = \begin{cases} p \geq p_D \rightarrow \frac{Y}{S} \dot{p} \;\&\; p_D = \varepsilon_{pD} \\ p < p_D \rightarrow 0 \end{cases}$$

$$D = D_c = D_{1c} \rightarrow Crack$$

The material parameters are:

- S, ε_{pD} and D_{1c} which must be determined from damage measurements in tensile loading-unloading test;

- σ_u and σ_f which are classical characteristics and easy to find in handbooks or to identify by tensile and fatigue tests.

CHAPTER 4: Identification of Material Parameters

The material damage parameters S , ε_{pD} and D_{1c} which characterize the damage must be measured for each material and temperature by the means of tensile loading-unloading test. Let us assume that a good tensile test has been performed with measurement of the damage during unloading by elasticity change and the following curves have been obtained:

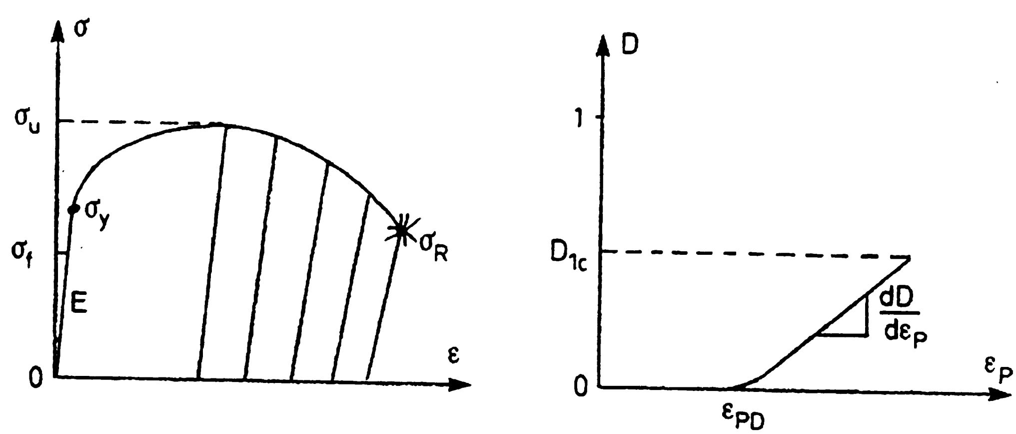


Figure 10 : Measurement of damage by elasticity modulus

For the one-dimensional case:

$$Y = \frac{\sigma^2}{2E(1-D)^2} \text{ as } R_v = 1$$

$$\dot{p} = \left(\frac{2}{3}\dot{\varepsilon}^p{}_{ij}\dot{\varepsilon}^p{}_{ij}\right)^{1/2} = \left|\dot{\varepsilon}^p\right|$$

Thus:

$$\dot{D} = \frac{\sigma^2}{2ES(1-D)^2}\left|\dot{\varepsilon}^p\right| H_{(\varepsilon^p - \varepsilon_{pD})}$$

The parameter S is determined from the slope of the curve, D versus the plastic strain ε_p:

$$\dot{D} = \frac{\sigma^2}{2ES(1-D)^2}\left|\dot{\varepsilon}^p\right|$$

Or:

$$\frac{dD}{d\varepsilon^p} = \frac{\sigma^2}{2ES(1-D)^2}$$

At each point of the curve, D is known, σ is known from the stress strain curve, $\frac{dD}{d\varepsilon^p}$ is estimated and E is known from a previous identification:

$$S = \frac{\sigma^2}{2E(1-D)^2 \frac{dD}{d\varepsilon^p}}$$

The main difficulty is obtaining a good stress strain curve in the softening range where necking occurs and strains must be measured locally by a small strain gauge.

4.1 Identification of the Material Parameters by standard tensile test for Al3003

In the following experiment, an extensometer was used to measure the strain values. Extensometers assume uniform distribution of strain as well as damage over their gage length and can not capture the localization and necking of the specimen during which almost the entire process of damage evolution occurs. To obtain accurate data, small strain gages should be used to obtain strains at the much localized points. Nonetheless, the method for derivation of the material damage parameters is explained hereafter using the data acquired from the extensometer keeping in mind that the calculated values are just for the purpose of describing the procedure and are not quantitatively accurate. For numerical analyses, more accurate parameters are used which have been obtained by Lemaitre [14].

4.1.1 Tensile-Test Specimen Specifications

The tensile loading-unloading test was carried out for Aluminum alloy 3003. Table 3 specifies the dimensions of the Dog-Bone shape specimen.

G – Gage Length	2.000 ±0.005 in
W – Width	0.500 ±0.010 in
T – Thickness	Thickness of Material
R – Radius of fillet	½ in
L – Overall length	8 in
A – Length of reduced section	2 ¼ in

TH

60.

B – Length of grip section	2 in
C – Width of grip section	¾ in

Table 3 : Tensile specimen dimensions in the English units

The unloading of the specimen was done at the strain increments of 5% and the following stress-strain curve was obtained:

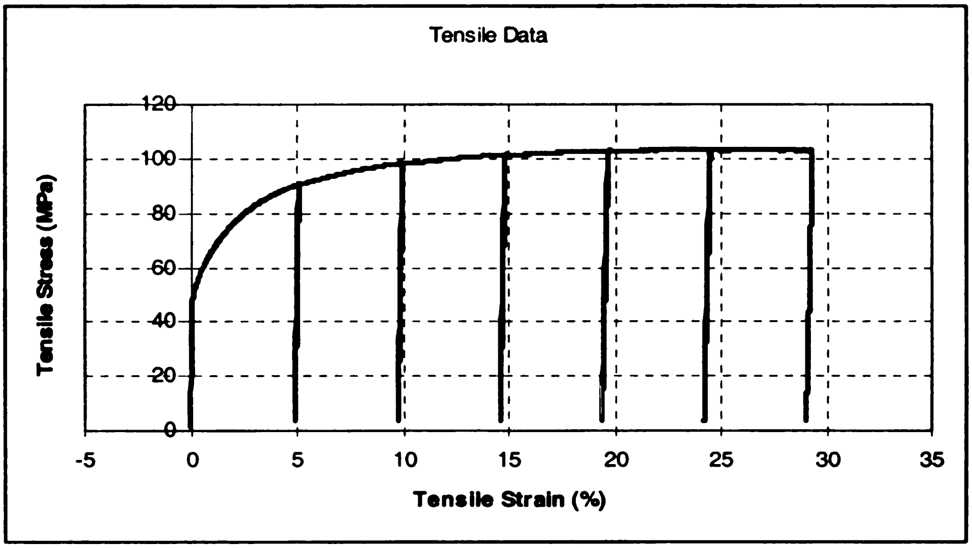


Figure 11 : Stress-strain curve for the loading-unloading test

The Young's modulus values are calculated using the linear portion of the loading part after each unloading. The reason for taking the loading portion is that we want to follow the same procedure as we used for measuring the initial Young's modulus. The damage parameter is calculated based on the strain equivalence principle by using the following formula:

$$D = 1 - \left(\frac{\tilde{E}}{E}\right)$$

Where $\tilde{E}$ is for the damaged material and measured at each unloading increment. E is for the very first loading step and its variation from the data indicated in the handbooks shows an initial damage in the specimen. The following data is obtained:

Unloading step	1	2	3	4	5	6	E for the input material
E(Gpa) for unloading data	47.771	42.85	39.72	35.29	34.29	33.1078	51.43552201
E(Gpa) for loading data	50.226	47	43.79	40.8	37.15		
Strain Eq.(Lemaitre)for loading	0.0235	0.086	0.149	0.207	0.278	***0.337***	$\leftarrow D_{1c}$
Strain %	5.07	9.97	14.84	19.6	24.48	29.3	

Table 4 : Young's modulus data

Since $\varepsilon^e << \varepsilon^p$, ε values can be used to plot D versus ε^p.

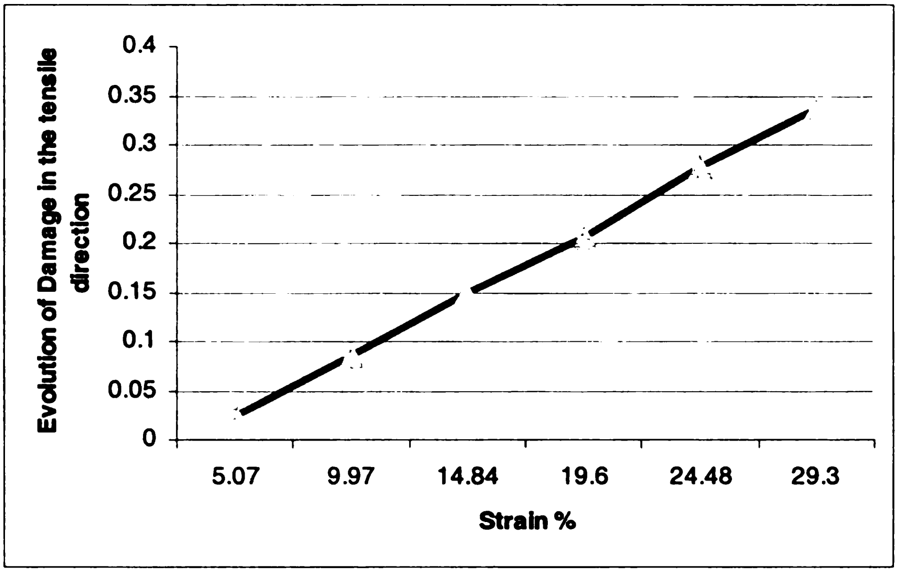


Figure 12 : Plot of Damage versus strain

This plot is used to calculate the material parameters. The variation of damage shows a linear behavior as obtained by Lemaitre[14]. S is determined by measuring the slope of the line using the following formula:

$$S = \frac{\sigma^2}{2E(1-D)^2 \dfrac{dD}{d\varepsilon^p}}$$

We also use the Kuhn-Tucker condition for damage-coupled plasticity which for perfectly plastic material is written as:

$$f = \frac{\sigma}{1-D} - \sigma_s = 0$$

Where σ_s is the yield value for the perfectly plastic material and is usually taken to be equal to ultimate stress, σ_u.

So:

$$\left(\frac{\sigma}{1-D}\right)^2 = \sigma_s^{\ 2}$$

Thus, the equation for S can be written as:

$$S = \frac{\sigma_u^{\ 2}}{2E \dfrac{dD}{d\varepsilon^p}} = \left(\frac{100^2}{2 \times 51435 \times 0.01298}\right) \approx 7.4891 MPa$$

ε_{pD} is obtained by intersecting the line with the horizontal axis:

$$\varepsilon_{pD} \approx 3.357$$

And finally, D_{1c} is obtained by finding the value of the fitted line at the failure strain:

$$D_{1c} \approx 0.337$$

The mechanical properties of the specimen can also be found from the tensile test. They are measures as:

$$\sigma_y \approx 40MPa$$
$$\sigma_u \approx 100MPa$$
$$E \approx 51436MPa$$

4.2 A Simpler representation of the Lemaitre's Damage Evolution Equation

Let's take the power law for hardening written as:

$$\tilde{\sigma} = \frac{\sigma}{1-D} = \sigma_y = K\left(p\right)^n$$

If we use this relation in the definition of damage evolution equation, we have:

$$\dot{D} = \frac{\sigma^2 R_v}{2ES(1-D)^2}\dot{p} = \frac{K^2\left(p\right)^{2n} R_v}{2ES}\dot{p}$$

In the case of radial loading where the directions of principal stresses remain unchanged, R_v is constant throughout the loading and we can integrate the above equation to obtain a relation between damage parameter and equivalent plastic strain:

$$D = \int_{p_D}^{p} \frac{K^2\left(p\right)^{2n} R_v}{2ES}\dot{p} = \frac{K^2 R_v}{2ES}\int_{p_D}^{p}\left(p\right)^{2n}\dot{p} = \frac{K^2 R_v\left(\left(p\right)^{2n+1} - \left(p_D\right)^{2n+1}\right)}{2ES\left(2n+1\right)}$$

$p = p_R$ corresponds to $D = D_c$ which is evaluated as:

$$D_c = \frac{K^2 R_v\left(\left(p_R\right)^{2n+1} - \left(p_D\right)^{2n+1}\right)}{2ES\left(2n+1\right)}$$

Since n is very small, $(2n+1)$ is of order one and the above equation can be written as:

$$D_c = \frac{K^2 R_v (p_R - p_D)}{2ES}$$

Or:

$$\frac{D_c}{(p_R - p_D)} = \frac{K^2 R_v}{2ES}$$

For the one-dimensional tensile case, $R_v = 1$:

$$\frac{D_{1c}}{\varepsilon_R - \varepsilon_{pD}} = \frac{K^2}{2ES}$$

If we replace $\frac{K^2}{2ES}$ in the damage evolution law by its relation from the above equation, we have:

$$\dot{D} = \left(\frac{D_{1c}}{\varepsilon_R - \varepsilon_{pD}}\right) R_v (p)^{2n} \dot{p}$$

This equation will be used in the next chapter for the coupled analysis of damage evolution. D_{1c}, ε_R, and ε_{pD} values for some materials can be found in [14].

CHAPTER 5: Numerical implementations

In previous chapters, the isotropic formulations of damage coupled constitutive equations were developed and the corresponding damage material parameters were introduced. These results can be implemented as a material model or in conjunction with the available material models as a failure criterion. In this chapter, the constitutive and rate equations are utilized in two different ways, uncoupled and coupled, to analyze two simple problems under uniaxial and biaxial loadings.

5.1 Uncoupled Analysis of Crack Initiation

Damage analysis of a model subjected to a given history of loading consists in the calculation of the evolution of the damage as a function of the time at the critical point(s) and the critical time at which the damage reaches its critical value corresponding to a crack initiation. An assumption which simplifies the analysis consists in neglecting the coupling between the damage and the strains. In a first step the stress and strain field histories are calculated by a commercial finite element software. Let us suppose that the results for point M are:

$$\varepsilon(M,t)$$
$$\sigma(M,t)$$

The second step consists in determining the point(s) where the damage has its maximum value. There is no exact way to select the critical point(s) except to calculate the evolution of the damage field in each point. However, an "intelligent" look at the evolution of σ as a function of space and time will restrict the number of dangerous areas where the damage must be calculated.

5.1.1 Integration of the Kinetic Damage Law

At the critical point, the structural calculation gives:

$$\varepsilon_{ij}(M^*,t)$$
$$\varepsilon^{e}{}_{ij}(M^*,t)$$
$$\varepsilon^{p}{}_{ij}(M^*,t)$$
$$\sigma_{ij}(M^*,t)$$

It is easy to deduce:

- The accumulated plastic strain rate, (It's automatically calculated in some finite element commercial software)

$$\dot{p} = \left(\frac{2}{3}\dot{\varepsilon}^{p}{}_{ij}\dot{\varepsilon}^{p}{}_{ij}\right)^{1/2}$$

- The strain energy density release rare,

$$Y_{(t)} = \frac{\sigma^{2}{}_{eq}}{2E(1-D)^{2}} R_{v}$$

From:

$$\sigma_{eq} = \left(\frac{3}{2}\sigma^{D}{}_{ij}\sigma^{D}{}_{ij}\right)^{1/2}$$

$$R_v = \frac{2}{3}(1+\upsilon) + 3(1-2\upsilon)\left(\frac{\sigma_H}{\sigma_{eq}}\right)^2$$

And to write the general kinetic damage law as:

$$\dot{D} = \frac{Y_{(t,D)}}{S}\dot{p}H_{(\varepsilon_p - \varepsilon_{pD})} = \frac{\sigma_{eq}(t)^2 R_v(t)}{2ES(1-D)^2}\dot{p}H_{(\varepsilon_p - \varepsilon_{pD})}$$

The initial condition for the damage evolution is the end of the period of accumulation of dislocations and the initiation of the first mesocrack [15] corresponding to the value p_D of the accumulated plastic strain. The initial conditions are:

$$t = 0 \rightarrow p = 0$$

$$t = t_0 \rightarrow \begin{cases} p = p_D = \varepsilon_{pD} \\ D = 0 \end{cases}$$

The damage evolution is given by the integral:

$$\int_0^D (1-D^2)dD = \frac{1}{2ES}\int_{t_0}^{t} \sigma_{eq}(t)^2 R_v(t)\dot{p}(t)dt$$

$$\frac{1}{3}(-(1-D)^3 + 1) = \frac{1}{2ES}\int_{t_0}^{t} \sigma_{eq}(t)^2 R_v(t)\dot{p}(t)dt$$

$$D = 1 - \left[1 - \frac{3}{2ES}\int_{t_0}^{t} \sigma_{eq}(t)^2 R_v(t)\dot{p}(t)dt\right]^{1/3}$$

The critical time t_R at which a crack is initiated is reached when the damage itself reaches its critical value given by the rupture criterion:

$$D_c = D_{1c} \leq 1$$

If the material is highly ductile:

$$D_c = 1$$

And t_R can be calculated by the expression:

$$1 = 1 - \left[1 - \frac{3}{2ES} \int_{t_0}^{t_R} \sigma_{eq}(t)^2 R_v(t) \dot{p}(t) dt \right]^{1/3}$$

Or:

$$\int_{t_0}^{t_R} \sigma_{eq}(t)^2 R_v(t) \dot{p}(t) dt = \frac{2}{3} ES$$

Where t_0 is given by the elastoplastic analysis: $t_0 = t(p = p_D)$

5.1.2 Uncoupled analysis gives more conservative results

Uncoupled analysis is not an accurate analysis for determination of the failure time. However, it gives more conservative results meaning that the design is on the safe side but not optimized as opposed to the strain-damage coupled calculation. In order to demonstrate this analytically, we assume the material to be elastic-perfectly plastic and the loading to be proportional:

$$\sigma_{eq} = \sigma_s = const$$
$$R_v = 1$$

The last expression becomes:

$$\int_{t_0}^{t_R} \dot{p}(t) dt = \frac{2}{3} \frac{ES}{R_v \sigma^2{}_s}$$

The strain history is given and this allows us to take $\dot{p}(t)$ as a given function, which is particularly simple to calculate if the elastic strain is neglected:

$$\dot{p} = \left(\frac{2}{3}\dot{\varepsilon}_{ij}\dot{\varepsilon}_{ij}\right)^{1/2}$$

Then t_R may be determined.

For the case of damage-coupled analysis of the same material loaded under the same conditions:

- For $p < p_D$ no damage occurs; the same calculation gives the same result for t_0,

$$t(p = p_D) = t_0$$

- The material is perfectly plastic with the same threshold σ_s which allows us to write the coupled plasticity criterion as:

$$\frac{\sigma_{eq}}{(1-D)} = \sigma_s = const.$$

- The loading is proportional:

$$R_v = const$$

- The same strain history is imposed and the elastic strain is again neglected, thus the function $\dot{p}(t)$ is the same.

The critical time for crack initiation in the coupled case t_R is calculated from the same kinetic damage law:

$$\dot{D} = \frac{Y_{(t,D)}}{S}\dot{p}H_{(\varepsilon_p - \varepsilon_{pD})} = \frac{\sigma_{eq}(t)^2 R_v(t)}{2ES(1-D)^2}\dot{p}H_{(\varepsilon_p - \varepsilon_{pD})} = \frac{\sigma_s{}^2 R_v(t)}{2ES}\dot{p}H_{(\varepsilon_p - \varepsilon_{pD})}$$

Where $(1-D)^2$ disappears due to the coupling in the plasticity criterion.

The integration is obvious:

$$D = \frac{R_v \sigma^2{}_s}{2ES}\int_{t_0}^{t_R} \dot{p}(t)dt$$

Taking again $D_c = 1$ as the critical condition:

$$D = D_c = 1 \rightarrow t = t'_R$$

$$\int_{t_0}^{t'_R} \dot{p}(t)dt = \frac{2ES}{R_v \sigma^2{}_s}$$

Comparison with the uncoupled case shows that:

$$\int_{t_0}^{t'_R} \dot{p}(t)dt = 3\int_{t_0}^{t_R} \dot{p}(t)dt$$

Which implies that:

$$t_R < t'_R$$

In the particular case of loading where $\dot{p} = const$.

Uncoupled case	Coupled case
$t_R = t_0 + \frac{2ES}{3R_v \sigma^2{}_s \dot{p}}$	$t'_R = t_0 + \frac{2ES}{R_v \sigma^2{}_s \dot{p}}$

Table 5 : Comparison of the time to failure between uncoupled and coupled analysis for monotonic loading

To summarize, the uncoupled calculation is always conservative. It is a lower bound, but usually far from the coupled solution. In other terms, component design may benefit from coupled calculations, which can prove the enhanced safety of components or indicate how light-weight economical components could be built.

5.2 Coupled analysis:

In this section, two algorithms are developed for the strain coupled damage analysis and implemented in parallel with the finite element software. In each algorithm, a simplifying assumption or condition is taken into account to avoid excessive complexity in the formulation of the algorithm. In this work, first the strain components were obtained from Ls-Dyna and the first algorithm is utilized in Matlab and then the second algorithm was written in Fortran and coupled as a material model with Ls-Dyna.

5.2.1 Strain coupled algorithm with the assumption of perfect plasticity

By assuming the material to be perfectly plastic and by neglecting the microcrack closure effect. The following equations should be solved:

$$\varepsilon = \varepsilon^e + \varepsilon^p$$

$$\varepsilon^e{}_{ij} = \frac{1+\upsilon}{E}\frac{\sigma_{ij}}{1-D} - \frac{\upsilon}{E}\frac{\sigma_{kk}}{1-D}\delta_{ij}$$

$$\dot{\varepsilon}_{ij}^{p} = \frac{3}{2}\frac{\tilde{\sigma}^{D}}{\sigma_s}\dot{p}\begin{cases} f = \frac{\sigma_{eq}}{1-D} - \sigma_s = 0 \\ \& \\ \dot{f} = 0 \end{cases}$$

$\dot{\varepsilon}^{p}{}_{ij} = 0$ Otherwise.

If $p > pD$, then $\dot{D} = \frac{\sigma_s^{\ 2}}{2ES} R_v \dot{p}$ and if $p < pD$, then $\dot{D} = 0$.

Crack initiation if $D_c = D_{1c}$ is reached.

These equations may be used for piecewise perfect plasticity by considering several values of the plastic threshold σ_s as the loading or the time like parameter vary. Then, the material parameters must be considered as follows:

- E and v for elasticity;

- σ_f , σ_y and σ_u for plasticity

- $\sigma_s = \sigma_u$ for perfect plasticity. Since, in reality, no material has the perfect plasticity behavior, the choice of σ_s is somehow subjective. However, choosing σ_u is a conservative choice which is taken in this work.

- $S, \varepsilon_{PD}, D_{1C}$ for damage

The input of the calculation is the time history of the strain components $\varepsilon_{ij}(t)$ which may come from the result of a finite element structural calculation. The outputs are:

- The time like parameter t,

- The damage evolution $D(t)$, the last point corresponding to $D = D_c$, that is, crack initiation,

- The accumulated plastic strain evolution $p(t)$;

- The evolution of the von-Mises equivalent stress $\sigma_{eq}(t)$,

The numerical procedure is a strain-driven incremental time like algorithm using an elastic predictor and a plastic corrector. The hypothesis of perfect plasticity allows us to explicitly formulate this plastic correction and helps us see the softening effect of damage. The constitutive equations are written in an incremental form corresponding to a fully implicit integration scheme having unconditional stability; but the damage parameter is updated explicitly.

After an elastic prediction obtained from the law of elasticity, the plasticity criterion $f \leq 0$ is checked. If $f > 0$, the plasticity corrections are obtained by Newton's iterative procedure applied to a system of two equations deduced from the constitutive equations.

At each time increment, the values of the stress and the other state variables at the beginning of the increment (t_n) are known and they have to be updated at the end of the increment (t_{n+1}). In the following formulation, tensorial notation is used for convenience.

We first assume that the increment is entirely elastic. So, we have:

$$\tilde{\sigma}^{s}_{new} = \left(1 - D_{old}\right)\sigma_{old} + \lambda tr(\Delta\varepsilon)I + 2\mu(\Delta\varepsilon)$$

Where λ and μ are the Lame coefficients and I is the second order identity tensor.

$$\lambda = \frac{\upsilon E}{(1+\upsilon)(1-2\upsilon)}$$

$$\mu = G = \frac{E}{2(1+\upsilon)}$$

All the other plastic variables are equal to their values at (t_n), if this elastic predictor satisfies the yield condition $f \leq 0$, then the elastic assumption is correct and the computation for this time increment ends. If $f > 0$, this elastic state is corrected as explained in the following in order to find the plastic solution.

The solution at (t_{n+1}) has to satisfy the following equations:

$$f = \frac{\left(\sigma_{n+1}\right)_{eq}}{1 - D_{n+1}} - \sigma_s = 0$$

$$\tilde{\sigma}_{n+1} = \frac{\sigma_{old}}{\left(1 - D_{old}\right)} + \lambda tr(\Delta\varepsilon)I + 2\mu(\Delta\varepsilon) - 2\mu(\Delta\varepsilon^{p})$$

$$\Delta\varepsilon^{p} = N\Delta p$$

$$\Delta D = \frac{Y}{S}\Delta p$$

Where $\Delta(x) = x_{n+1} - x_n$ and $N = \frac{3}{2}\frac{\sigma^{D}{}_{n+1}}{\left(\sigma_{n+1}\right)_{eq}}$. If we replace $\Delta\varepsilon^{p}$ by its expression in the second equation, we see that the problem is reduced to the first two equations for the two unknowns, $\tilde{\sigma}_{n+1}$ and Δp.

We need to find $\tilde{\sigma}_{n+1}$ and Δp which satisfy:

$$f = \left(\tilde{\sigma}_{n+1}\right)_{eq} - \sigma_s = 0$$

$$h = \tilde{\sigma}_{n+1} - \frac{\sigma_n}{\left(1-D_n\right)} - \lambda tr(\Delta\varepsilon)I - 2\mu(\Delta\varepsilon) + 2\mu N \Delta p = 0$$

This nonlinear system is solved iteratively by Newton's method. For each iteration (s), we have:

$$f(\tilde{\sigma}_{n+1}{}^{s} + C_{\tilde{\sigma}}) = (\tilde{\sigma}_{n+1}{}^{s} + C_{\tilde{\sigma}})_{eq} - \sigma_s = 0$$

$$h(\tilde{\sigma}_{n+1}{}^{s} + C_{\tilde{\sigma}}, \Delta p + C_p) = 0$$

So:

$$f + \frac{\partial f}{\partial \tilde{\sigma}} : C_{\tilde{\sigma}} = 0$$

$$h + \frac{\partial h}{\partial \tilde{\sigma}} : C_{\tilde{\sigma}} + \frac{\partial h}{\partial p} C_p = 0$$

Where f, h and their partial derivatives are taken at (t_{n+1}) and at the iteration (s). (The starting point is the elastic predictor)

In other words, the integration is done explicitly over the time and implicitly at each time over $\tilde{\sigma}$ and p. The corrections $C_{\tilde{\sigma}}$ and C_p are defined by:

$$C_{\tilde{\sigma}} = \tilde{\sigma}^{s+1} - \tilde{\sigma}^{s}$$
$$C_p = p^{s+1} - p^{s}$$

The starting iteration $(s=0)$ corresponds to the elastic predictor. The set of equations for f and h may be rewritten as:

$$f + \frac{\partial f}{\partial \tilde{\sigma}} : C_{\tilde{\sigma}} = 0$$

$$h + \frac{\partial h}{\partial \tilde{\sigma}} : C_{\tilde{\sigma}} + \frac{\partial h}{\partial p} C_p = 0$$

$$f + N : C_{\tilde{\sigma}} = 0$$
$$h + \left[\Pi + 2\mu\Delta p \frac{\partial N}{\partial \tilde{\sigma}} \right] : C_{\tilde{\sigma}} + 2\mu N C_p = 0$$

Because:

$$\frac{\partial h}{\partial p} = 2\mu N$$

And:

$$\frac{\partial h}{\partial \tilde{\sigma}} = \left[\Pi + 2\mu\Delta p \frac{\partial N}{\partial \tilde{\sigma}} \right]$$

Where Π is the fourth order identity tensor and: $N = \frac{3}{2} \frac{\tilde{\sigma}^{s\,D}_{n+1}}{\left(\tilde{\sigma}^{s}_{n+1}\right)_{eq}}$

$$\tilde{\sigma}_{eq} = \left(\frac{3}{2} \tilde{\sigma}^{D}{}_{ij} \tilde{\sigma}^{D}{}_{ij} \right)^{1/2} = \frac{E(1-D)}{1+\upsilon} \left(\frac{3}{2} \varepsilon^{e}{}_{ij}{}^{D} \varepsilon^{e}{}_{ij}{}^{D} \right)^{1/2}$$

$$\frac{\partial N}{\partial \tilde{\sigma}} = \frac{1}{\tilde{\sigma}_{eq}} \left[\frac{3}{2} (\Pi - \frac{1}{3} I \otimes I) - N \otimes N \right]$$

$N : N = \frac{3}{2}$ and $N : \frac{\partial N}{\partial \tilde{\sigma}} = 0$ because $(\Pi - \frac{1}{3} I \otimes I)$ is a deviatoric projector because if you take its inner product with a 2nd-rank deviatoric tensor N, the results is still N:

$$(\Pi - \frac{1}{3} I \otimes I) : N = N$$

So:

$$N:\frac{\partial N}{\partial \tilde{\sigma}} = N:\frac{1}{\tilde{\sigma}_{eq}}\left[\frac{3}{2}(\Pi - \frac{1}{3}I \otimes I) - N \otimes N\right] = \frac{1}{\tilde{\sigma}_{eq}}\left[\frac{3}{2}N - (N \otimes N):N\right]$$
$$= \frac{1}{\tilde{\sigma}_{eq}}\left[\frac{3}{2}N - (N:N)N\right] = \frac{1}{\tilde{\sigma}_{eq}}\left[\frac{3}{2}N - \frac{3}{2}N\right] = 0$$

If we multiply the second equation of the system, $h + \left[\Pi + 2\mu\Delta p\frac{\partial N}{\partial \tilde{\sigma}}\right]:C_{\tilde{\sigma}} + 2\mu N C_p = 0$, by N, the system gives explicitly the correction for p:

$$C_p = \frac{f - N:h}{3\mu}$$

Now, the correction for $\tilde{\sigma}$ should also be found explicitly. First we prove that $C_{\tilde{\sigma}}$ is a deviatoric tensor. From the second equation of the system and using the expression of $\frac{\partial N}{\partial \tilde{\sigma}}$, one obtains:

$$\frac{\partial N}{\partial \tilde{\sigma}} = \frac{1}{\tilde{\sigma}_{eq}}\left[\frac{3}{2}(\Pi - \frac{1}{3}I \otimes I) - N \otimes N\right]$$

$(\Pi - \frac{1}{3}I \otimes I)$ is a deviatoric projector because if you take its inner product with a 2nd-rank tensor (a) you find the deviator of (a):

$$(\Pi - \frac{1}{3}I \otimes I):a = a^D$$

$(N \otimes N):a = (N:a)N$ this is collinear with N and since N is deviatoric, the result is also deviatoric:

$$\frac{\partial N}{\partial \tilde{\sigma}}:a = deviatoric$$

Therefore taking the trace on the second equation of the system gives:

$$h+\left[\Pi+2\mu\Delta p\frac{\partial N}{\partial\tilde{\sigma}}\right]:C_{\tilde{\sigma}}+2\mu NC_p=0$$

$$tr(h)+tr(C_{\tilde{\sigma}})+tr(2\mu\Delta p\frac{\partial N}{\partial\tilde{\sigma}}:C_{\tilde{\sigma}})+tr(2\mu NC_p)=0$$

$tr(2\mu\Delta p\frac{\partial N}{\partial\tilde{\sigma}}:C_{\tilde{\sigma}})=0$ because $\frac{\partial N}{\partial\tilde{\sigma}}:C_{\tilde{\sigma}}$ is deviatoric.

$tr(2\mu NC_p)=0$ because N is deviatoric. So:

$$tr(h)+tr(C_{\tilde{\sigma}})=0$$

Using the definition of $\tilde{\sigma}_{n+1}$, and noting that:

$$\tilde{\sigma}_{n+1}=\tilde{\sigma}_n+\lambda tr(\Delta\varepsilon)I+2\mu(\Delta\varepsilon)-2\mu(\Delta\varepsilon^p)$$

$$tr(\varepsilon^p{}_n)=0$$
$$tr(\Delta\varepsilon^p)=0$$
$$tr(I)=3$$

We have:

$$tr(\tilde{\sigma}_{n+1})=(3\lambda+2\mu)tr(\varepsilon_{n+1})$$

Therefore using the definition of h:

$$h=\tilde{\sigma}^s_{new}-\tilde{\sigma}_{old}-\lambda tr(\Delta\varepsilon)I-2\mu(\Delta\varepsilon)+2\mu N\Delta p$$

And noting that $tr(N)=0$, we find that:

$$tr(C_{\tilde{\sigma}})=-tr(h)=0$$

These relations being true for any iteration(s), then $tr(C_{\tilde{\sigma}})=0$ and the Proof is achieved. Using this last result together with the expressions of $\frac{\partial N}{\partial\tilde{\sigma}}$ and C_p yields for the second equation:

$$f + N : C_{\tilde{\sigma}} = 0$$

$$h + \left[\Pi + 2\mu\Delta p \frac{\partial N}{\partial \tilde{\sigma}}\right] : C_{\tilde{\sigma}} + 2\mu N C_p = 0$$

$$\frac{\partial N}{\partial \tilde{\sigma}} = \frac{1}{\tilde{\sigma}_{eq}}\left[\frac{3}{2}(\Pi - \frac{1}{3} I \otimes I) - N \otimes N\right]$$

$$C_p = \frac{f - N : h}{3\mu}$$

$$A : C_{\tilde{\sigma}} = -h - \frac{2}{3}(f - N : h)N$$

Where:

$$A = \left[\Pi + 2\mu\Delta p \frac{\partial N}{\partial \tilde{\sigma}}\right]$$
$$= \left[\Pi + 2\mu\Delta p \frac{1}{\tilde{\sigma}_{eq}}\left[\frac{3}{2}(\Pi - \frac{1}{3} I \otimes I) - N \otimes N\right]\right]$$

In calculating $A : C_{\tilde{\sigma}}$, the term $\left(\frac{1}{3} I \otimes I\right) : C_{\tilde{\sigma}} = \left(\frac{1}{3} I : C_{\tilde{\sigma}}\right) I = \left(\frac{1}{3} tr(C_{\tilde{\sigma}})\right) I = 0$

because $C_{\tilde{\sigma}}$ is deviatoric.

Therefore:

$$A : C_{\tilde{\sigma}} = \left[(1 + \frac{3\mu}{\tilde{\sigma}_{eq}}\Delta p)\Pi - \frac{2\mu}{\tilde{\sigma}_{eq}}\Delta p N \otimes N\right] : C_{\tilde{\sigma}}$$

Although:

$$A \neq \left[(1 + \frac{3\mu}{\tilde{\sigma}_{eq}}\Delta p)\Pi - \frac{2\mu}{\tilde{\sigma}_{eq}}\Delta p N \otimes N\right]$$

So if we call $B = \left[(1+\frac{3\mu}{\tilde{\sigma}_{eq}}\Delta p)\Pi - \frac{2\mu}{\tilde{\sigma}_{eq}}\Delta p N \otimes N\right]$, the tensor B is invertible if and only if $\tilde{\sigma}_{eq} \neq 0$, and in this case:

$$B^{-1} = \frac{1}{1+\frac{3\mu}{\tilde{\sigma}_{eq}}\Delta p}\left[\Pi + \frac{2\mu}{\tilde{\sigma}_{eq}}\Delta p N \otimes N\right]$$

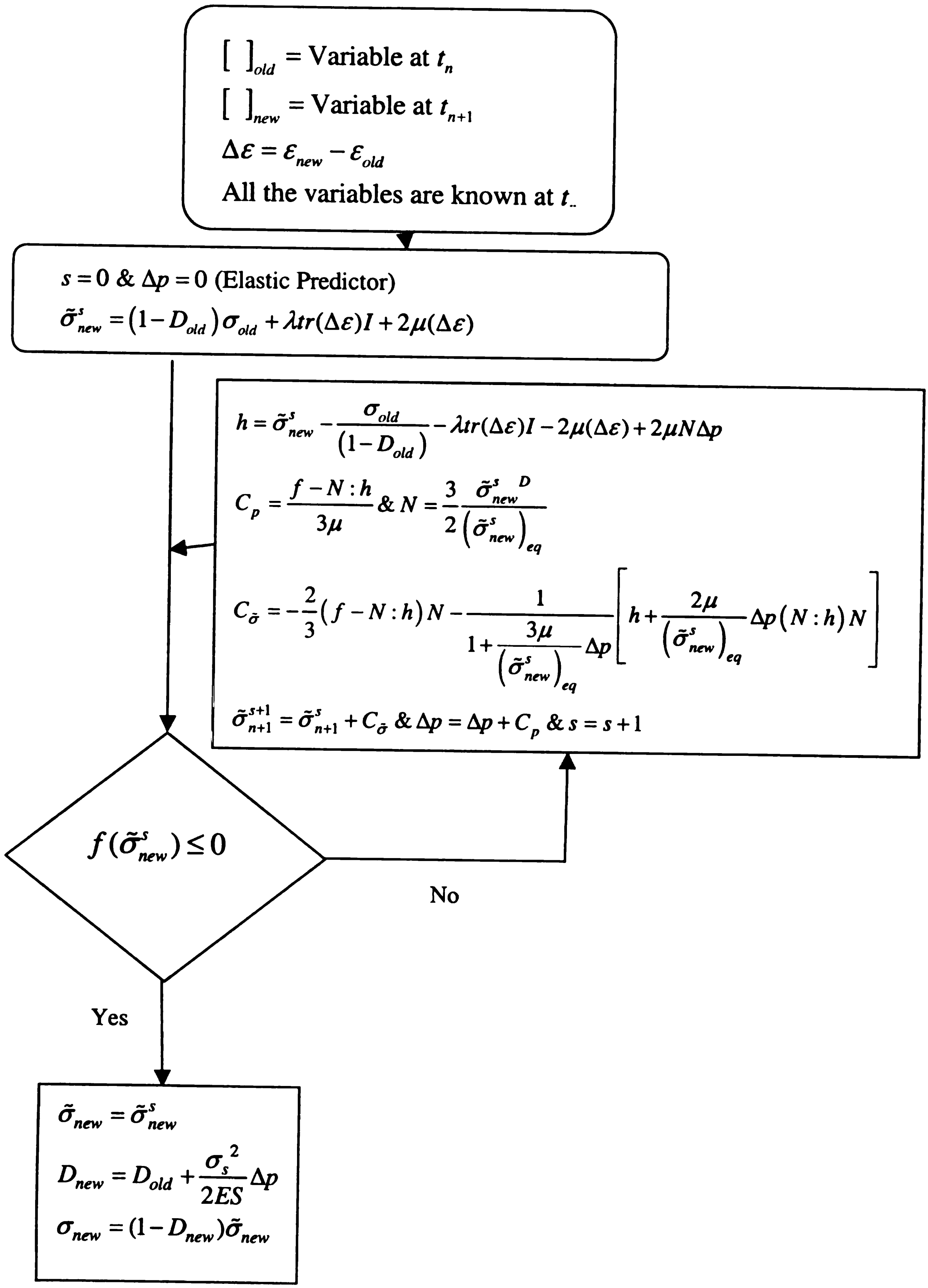
Using this expression gives explicitly the correction for $\tilde{\sigma}$:

$$C_{\tilde{\sigma}} = -\frac{2}{3}(f - N:h)N - \frac{1}{1+\frac{3\mu}{\tilde{\sigma}_{eq}}\Delta p}\left[h + \frac{2\mu}{\tilde{\sigma}_{eq}}\Delta p(N:h)N\right]$$

Once p and $\tilde{\sigma}$ are found, ε^p and D are calculated from their discretized constitutive equations and the stress components are given by:

$$\sigma_{ij} = (1-D)\tilde{\sigma}_{ij}$$

The following flow chart summarizes the above-mentioned procedure.


$[\ ]_{old}$ = Variable at t_n
$[\ ]_{new}$ = Variable at t_{n+1}
$\Delta\varepsilon = \varepsilon_{new} - \varepsilon_{old}$
All the variables are known at t.
$s = 0$ & $\Delta p = 0$ (Elastic Predictor)
$\tilde{\sigma}^{s}_{new} = (1 - D_{old})\sigma_{old} + \lambda tr(\Delta\varepsilon)I + 2\mu(\Delta\varepsilon)$
$h = \tilde{\sigma}^{s}_{new} - \frac{\sigma_{old}}{(1 - D_{old})} - \lambda tr(\Delta\varepsilon)I - 2\mu(\Delta\varepsilon) + 2\mu N \Delta p$
$C_p = \frac{f - N:h}{3\mu}$ & $N = \frac{3}{2}\frac{\tilde{\sigma}^{s\ D}_{new}}{(\tilde{\sigma}^{s}_{new})_{eq}}$
$C_{\tilde{\sigma}} = -\frac{2}{3}(f - N:h)N - \frac{1}{1 + \frac{3\mu}{(\tilde{\sigma}^{s}_{new})_{eq}}\Delta p}\left[h + \frac{2\mu}{(\tilde{\sigma}^{s}_{new})_{eq}}\Delta p (N:h) N\right]$
$\tilde{\sigma}^{s+1}_{n+1} = \tilde{\sigma}^{s}_{n+1} + C_{\tilde{\sigma}}$ & $\Delta p = \Delta p + C_p$ & $s = s + 1$
$f(\tilde{\sigma}^{s}_{new}) \leq 0$
No
Yes
$\tilde{\sigma}_{new} = \tilde{\sigma}^{s}_{new}$
$D_{new} = D_{old} + \frac{\sigma_s^{\ 2}}{2ES}\Delta p$
$\sigma_{new} = (1 - D_{new})\tilde{\sigma}_{new}$

This algorithm was then written in Matlab and Fortran and paralleled with Ls-Dyna as a user defined material to give stress distribution.

5.2.2 Strain coupled algorithm with the assumption of using the damage value at the beginning of the increment

This time, we write the yield function in terms of the Cauchy stress instead of the effective stress and we take into account the damage effects into the yield stress. Thus, the von-Mises yield function is written as:

$$f\left(\sigma^D\right)=\frac{3}{2}\sigma^D:\sigma^D-(1-D)^2\sigma_y{}^2\leq 0$$

Radius of this yield surface is:

$$R=\sqrt{\frac{2}{3}}(1-D)\sigma_y$$

And the normal unit vector of the yield surface is:

$$N=\frac{\sigma^D}{\left|\sigma^D\right|}=\frac{\sigma^D}{\sqrt{\sigma^D:\sigma^D}}=\frac{\sigma^D}{R}=\sqrt{\frac{3}{2}}\frac{\sigma^D}{(1-D)\sigma_y}$$

The other equations used in this section are:

Total strain rate in its additive form:

$$\dot{\varepsilon}=\dot{\varepsilon}^e+\dot{\varepsilon}^p$$

Flow rule:

$$\dot{\varepsilon}^p=\dot{\gamma}N$$

Equivalent plastic strain rate:

$$\dot{p}=\sqrt{\frac{2}{3}\dot{\varepsilon}^p:\dot{\varepsilon}^p}=\sqrt{\frac{2}{3}}\dot{\gamma}$$

Stress rate:

$$\dot{\sigma}=(1-D)\left(\lambda\text{trace}\left(\dot{\varepsilon}^e\right)\underset{\sim}{I}+2\mu\dot{\varepsilon}^e\right)$$

Equivalent von-Mises stress:

$$\sigma_{eq}=\sqrt{\frac{3}{2}\sigma^D:\sigma^D}$$

Isotropic hardening law:

$$\sigma_y=K\left(\varepsilon_0+p\right)^n$$
$$h=nK\left(\varepsilon_0+p\right)^{n-1}$$

Lemaitre's damage law:

$$\dot{D}=\left(\frac{D_{1c}}{\varepsilon_R-\varepsilon_{pD}}\right)\left[\frac{2}{3}(1+\nu)+3(1-2\nu)\left(\frac{\sigma_H}{\sigma_{eq}}\right)^2\right]\left(\varepsilon_0+p\right)^{2n}\dot{p}$$

The Computational procedure is as follows:

First, we assume that the increment is totally elastic and we calculate the elastic predictor as the trial stress for the new increment:

$$\sigma_{new}^{trial}=\sigma_{old}+\left(1-D_{old}\right)\left(\lambda\text{trace}(\Delta\varepsilon)I+2\mu\Delta\varepsilon\right)$$

Then, if the elastic predictor, σ_{new}^{trial}, is within the current yield surface, the elastic predictor is the actual value of stress for the current increment:

$$\sigma_{new}=\sigma_{new}^{trial}$$

If not, we need to subtract a plastic correction written as following:

$$\sigma_{new} = \sigma_{new}^{trial} - 2\mu\Delta\varepsilon^p (1 - D_{old}) = \sigma_{new}^{trial} - 2\mu\Delta\gamma N (1 - D_{old})$$

Since the increment is no longer totally elastic, we need to update the value of total equivalent plastic strain, p :

$$p_{new} = p_{old} + \sqrt{\frac{2}{3}}\Delta\gamma$$

And the new yield stress considering the plastic hardening is calculated as:

$$\sigma_{ynew} = \sigma_{yold} + h\Delta p = \sigma_{yold} + nK(\varepsilon_0 + p)^{n-1}\sqrt{\frac{2}{3}}\Delta\gamma$$

If $p \geq \varepsilon_{pD}$, the damage parameter has to be updated using the Lemaitre's equation:

$$D_{new} = D_{old} + \left(\frac{D_{1c}}{\varepsilon_R - \varepsilon_{pD}}\right)\left[\frac{2}{3}(1+\nu) + 3(1-2\nu)\left(\frac{\sigma_H}{\sigma_{eq}}\right)^2\right](\varepsilon_0 + p)^{2n}\sqrt{\frac{2}{3}}\Delta\gamma$$

Then, $\Delta\gamma$ is determined considering constant *D* during one step which is justified in the explicit calculation because of very small increments. At the end of new step:

$$\sigma^{D}{}_{new} = R_{new}N = \sqrt{\frac{2}{3}}(1 - D_{old})\sigma_{ynew}N$$

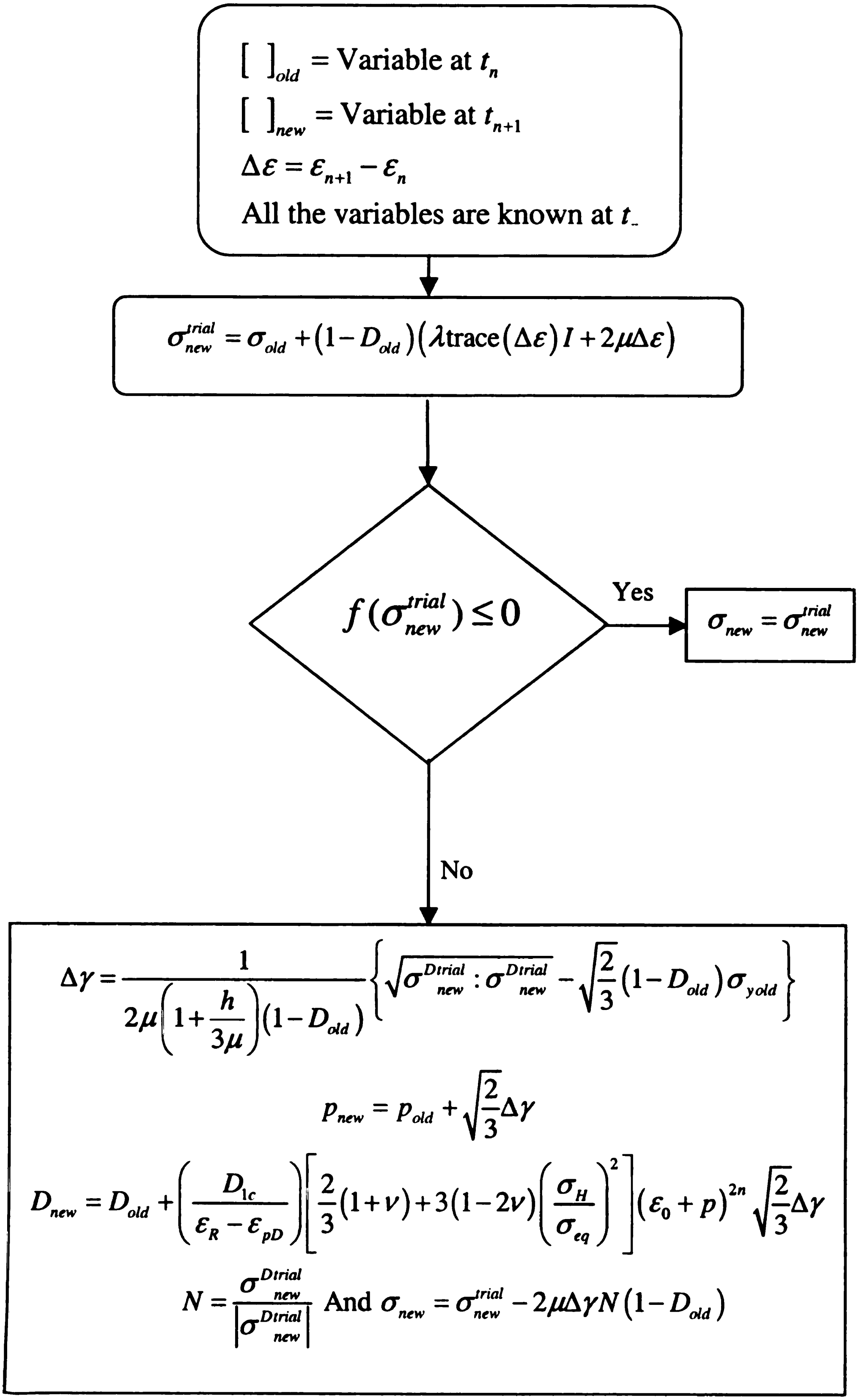
$$\sigma^{Dtrial}_{new} - 2\mu\Delta\gamma(1 - D_{old})N = \sqrt{\frac{2}{3}}(1 - D_{old})\left(\sigma_{yold} + \sqrt{\frac{2}{3}}h\Delta\gamma\right)N$$

$$\sigma^{Dtrial}_{new} = \left\{\sqrt{\frac{2}{3}}\left(\sigma_{yold} + \sqrt{\frac{2}{3}}h\Delta\gamma\right) + 2\mu\Delta\gamma\right\}(1 - D_{old})N = \left\{\sqrt{\frac{2}{3}}\sigma_{yold} + \left(\frac{2}{3}h + 2\mu\right)\Delta\gamma\right\}(1 - D_{old})N$$

$$\sqrt{\sigma_{new}^{Dtrial} : \sigma_{new}^{Dtrial}} = (1 - D_{old}) \left\{ \sqrt{\frac{2}{3}} \sigma_{yold} + \left(\frac{2}{3} h + 2\mu \right) \Delta\gamma \right\}$$

$$\therefore \Delta\gamma = \frac{1}{2\mu \left(1 + \frac{h}{3\mu} \right) (1 - D_{old})} \left\{ \sqrt{\sigma_{new}^{Dtrial} : \sigma_{new}^{Dtrial}} - \sqrt{\frac{2}{3}} (1 - D_{old}) \sigma_{yold} \right\}$$

The following flow chart summarizes the above-mentioned computational procedure.


$[\ ]_{old}$ = Variable at t_n
$[\ ]_{new}$ = Variable at t_{n+1}
$\Delta\varepsilon = \varepsilon_{n+1} - \varepsilon_n$
All the variables are known at t_{-}
$\sigma_{new}^{trial} = \sigma_{old} + (1 - D_{old})(\lambda \text{trace}(\Delta\varepsilon) I + 2\mu\Delta\varepsilon)$
$f(\sigma_{new}^{trial}) \leq 0$
Yes
$\sigma_{new} = \sigma_{new}^{trial}$
No
$\Delta\gamma = \frac{1}{2\mu\left(1+\frac{h}{3\mu}\right)(1-D_{old})}\left\{\sqrt{\sigma_{new}^{Dtrial} : \sigma_{new}^{Dtrial}} - \sqrt{\frac{2}{3}}(1-D_{old})\sigma_{yold}\right\}$
$p_{new} = p_{old} + \sqrt{\frac{2}{3}}\Delta\gamma$
$D_{new} = D_{old} + \left(\frac{D_{1c}}{\varepsilon_R - \varepsilon_{pD}}\right)\left[\frac{2}{3}(1+\nu) + 3(1-2\nu)\left(\frac{\sigma_H}{\sigma_{eq}}\right)^2\right](\varepsilon_0 + p)^{2n}\sqrt{\frac{2}{3}}\Delta\gamma$
$N = \frac{\sigma_{new}^{Dtrial}}{\left|\sigma_{new}^{Dtrial}\right|}$ And $\sigma_{new} = \sigma_{new}^{trial} - 2\mu\Delta\gamma N(1-D_{old})$

5.3 Numerical Results

In what follows, the results of numerical analysis are presented and compared for three analysis cases of without damage, uncoupled damage, and coupled damage material model. The case problems are uniaxial and biaxial tensions and the material properties are those of copper 99.9% obtained from [14]:

Material	Copper 99.9%
Young's modulus	98990 MPa
Poison's ratio	0.34
Yield stress	90 MPa
Ultimate stress	500 MPa
K (Hardening coefficient)	491.3 MPa
n (Hardening exponent)	0.2459
Initial damage threshold	0.35
Final damage threshold	1.07
Critical damage	0.85
Damage strength	1.0696

Table 6 : Material properties for numerical calculations

The geometry of the models is shown in the following. The thicknesses are 1mm for the uniaxial case and 10mm for the biaxial case. In each case, some kind of defect or stress concentrator is needed to induce localization. This is done in the uniaxial case by decreasing the width at the center by 2mm and in the biaxial case by considering a hole in the center. Due to symmetry, just a quarter of the model is considered for the biaxial case and the symmetry boundary conditions are applied to the edges adjacent to the hole.

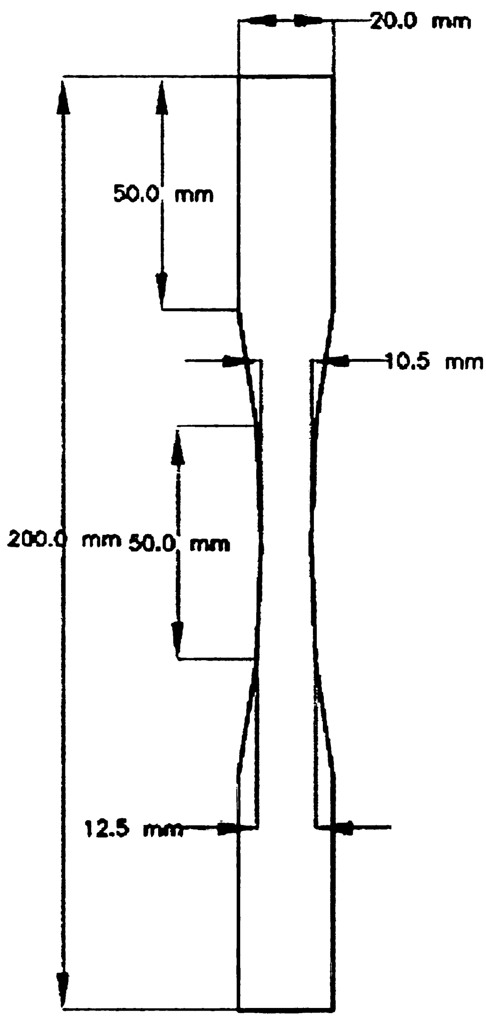


Figure 13 : The front view of the uniaxial test model

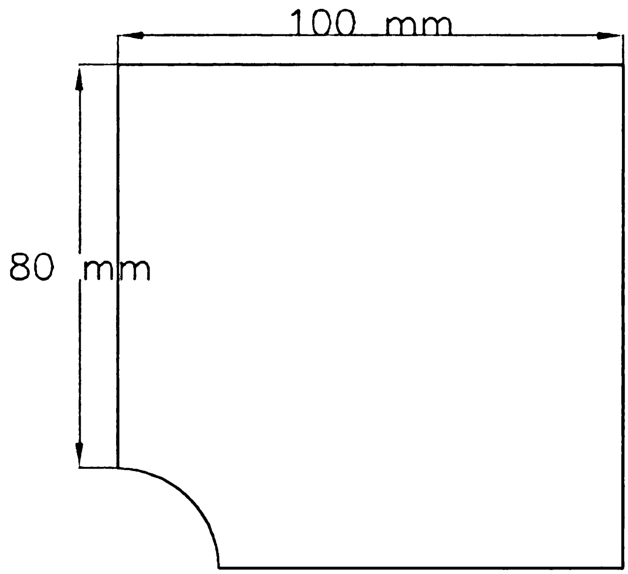


Figure 14 : The front view of the biaxial test model

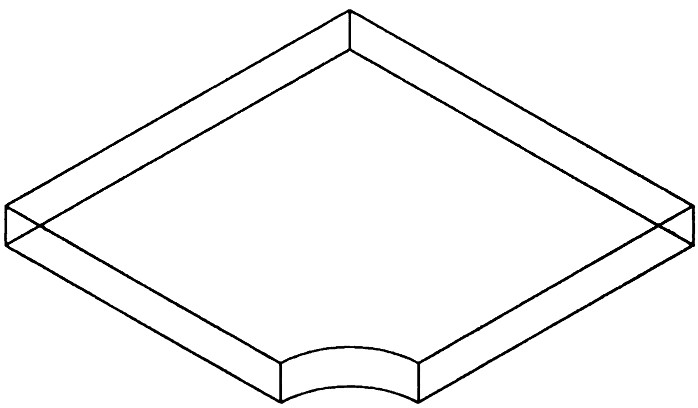


Figure 15 : The isoperimetric view of the biaxial test model

Since the code is developed for the general case, three dimensional brick elements are used to mesh the model avoiding any plain strain or plain stress assumption. The preprocessing is carried out in Hypermesh and the model is then solved by Ls-Dyna.

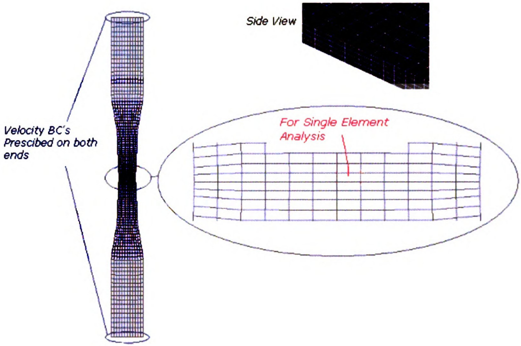


Figure 16 : Mesh configuration and location of the critical element for single element analysis in the uniaxial test model

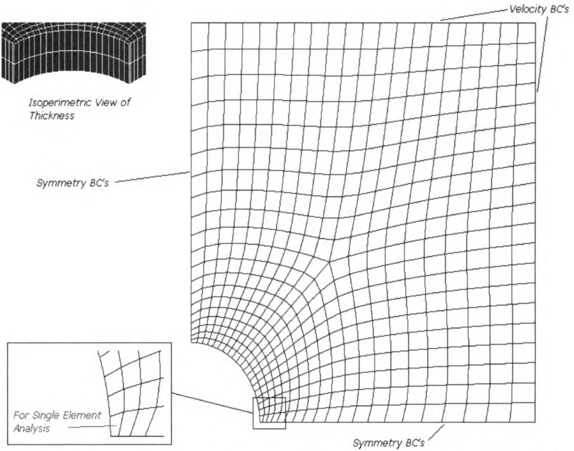


Figure 17 : Mesh configuration and location of the critical element for single element analysis in the biaxial test model

First, the algorithm mentioned in section 5.2.1 was written in Matlab and the strain components for the element number 700 of the plate specimen which is shown in Figure 17 were obtained from Ls-Dyna with the assumption of perfect plasticity. The following plots show the comparison of von-Mises stress, effective plastic strain, and damage for three analysis cases of without damage, uncoupled damage, and coupled damage.

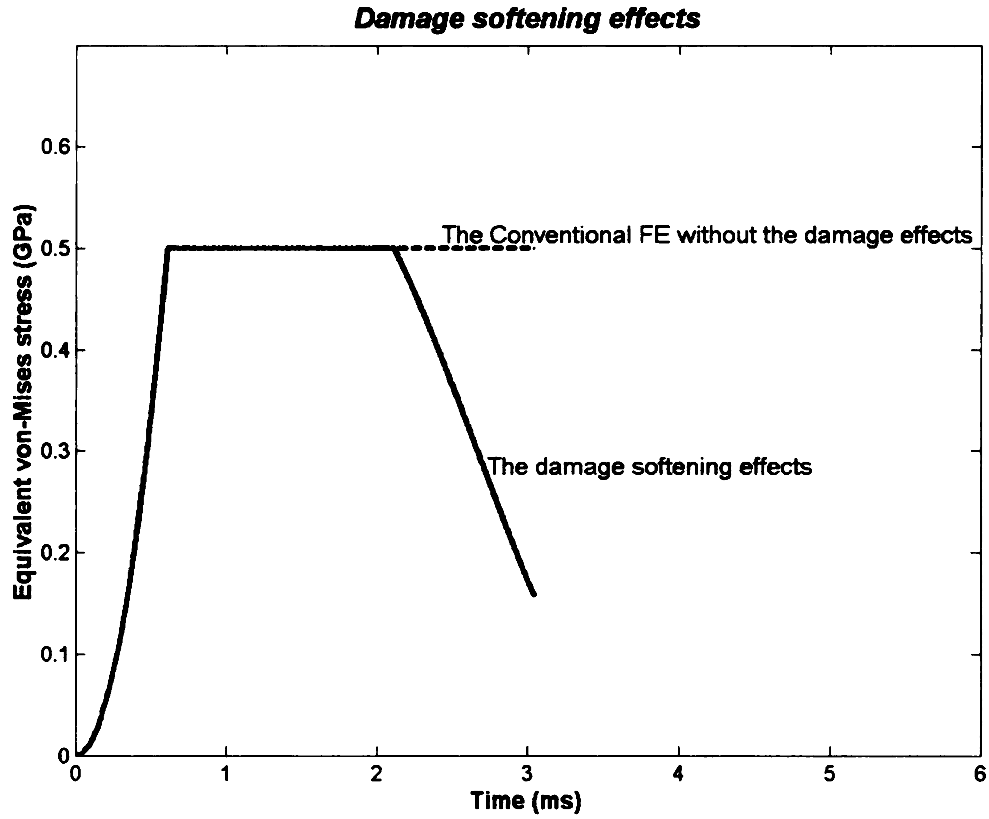


Figure 18 : Comparison of von-Mises stress for without damage and coupled damage analyses for element number 700

Damage softening effects are noticeable and indicate a lower load bearing capability than what is calculated with a non-damage analysis.

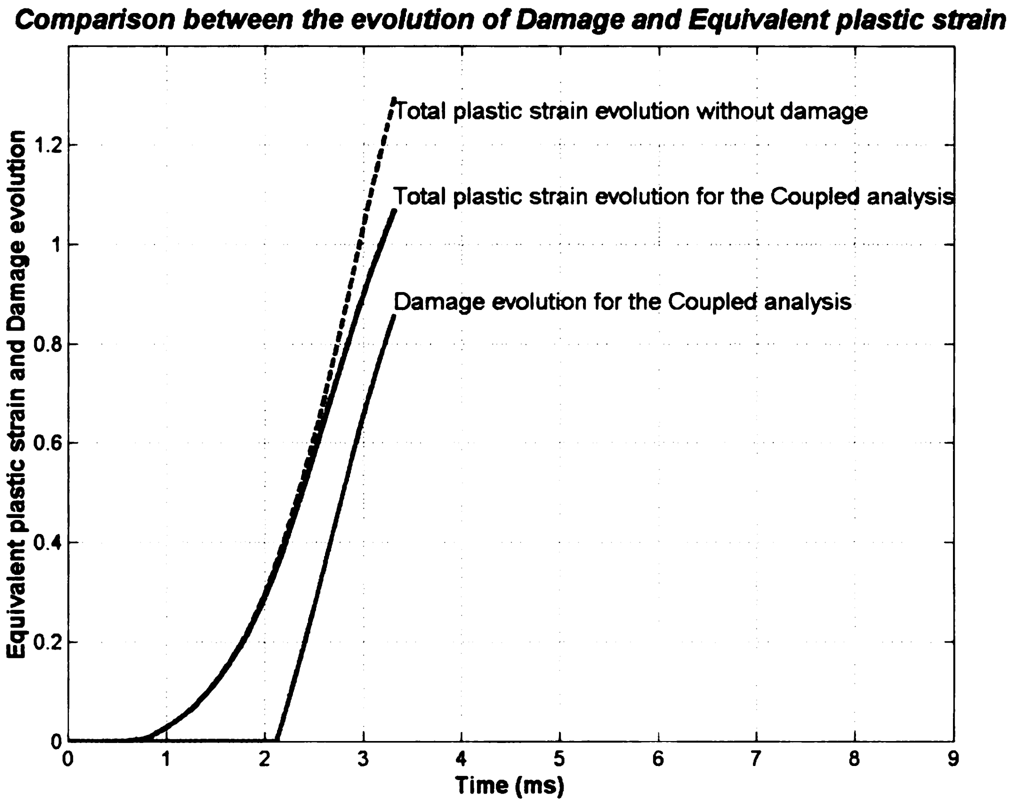


Figure 19 : Damage evolution and comparison of effective (total) plastic strain for analysis cases of without damage and coupled damage

Effective or total plastic strain is one of the most common failure criteria and is commonly used in element deletion or node release processes in the finite element codes. The above mentioned plot shows that the damage parameter not only takes into account the continuous degradation of the material properties but also the failure prediction based on the critical value of damage is in good agreement with the critical effective plastic strain criterion. That is, the damage parameter reaches its critical value, 0.85, when the effective plastic strain is approximately equal to its critical value, 1.07.

The following plot shows conservative prediction of failure from uncoupled analysis, which was analytically derived in section 5.1.2. The rate of damage evolution increases rapidly as damage increases since there is a $(1-D)^2$ term in the denominator of damage rate equation.

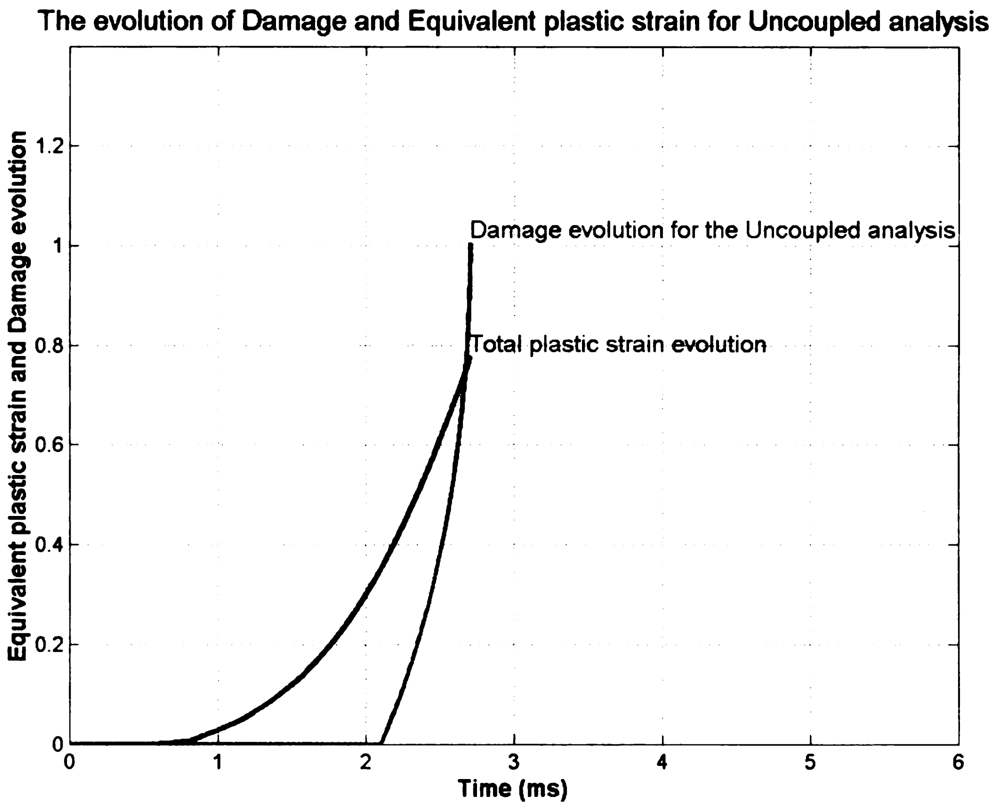
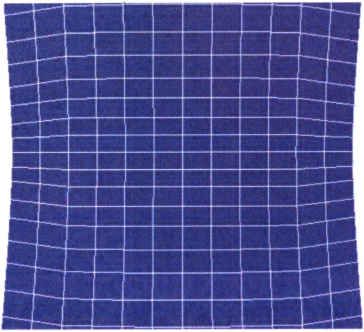


Figure 20 : Conservative prediction of failure from uncoupled analysis

Figure 18, Figure 19, and Figure 20 are results for the biaxial model and for a single element (The element is shown in Figure 17). In what follows, the algorithm mentioned in section 5.2.2 is written in Fortran and implemented as a user defined material model with Ls-Dyna. The showing the continuous degradation of the material. Another point of interest is the comparison between the prediction of failure by the critical value of damage and by the critical value of effective plastic strain. As shown in Figure 21 and Figure 22, damage occurs when the effective plastic strain reaches the initial damage threshold:



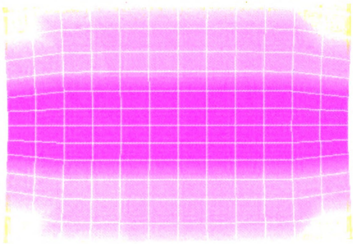


Figure 21 : Initiation of Damage for the uniaxial model. (The bottom figure shows the effective plastic strain)

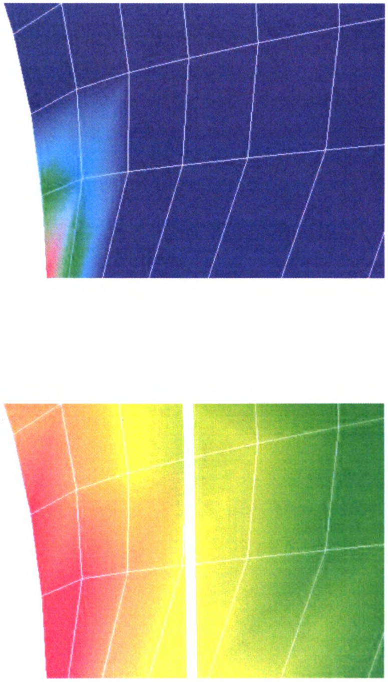


Figure 22 : The Initiation of Damage for the biaxial model. (The bottom figure shows the effective plastic strain)

In the uniaxial model, both damage and effective plastic strain start to grow from the edges where the three dimensionality thus the triaxiality ratio are maximum. This is shown in Figure 23.

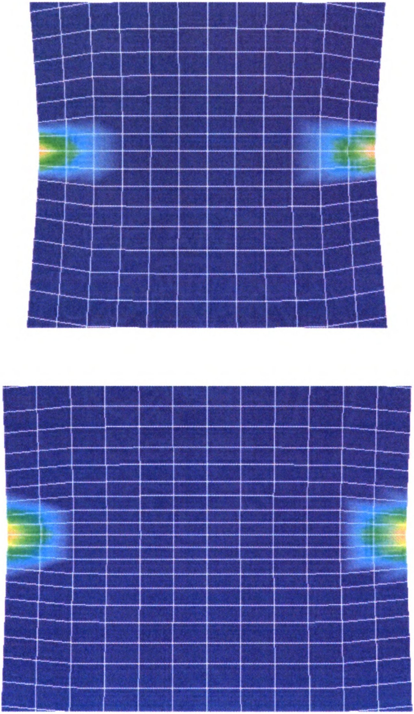


Figure 23 : Damage and effective plastic strain start to grow from the edges (The top picture shows the effective plastic strain)

Then, damage reaches its critical value when the effective plastic strain is approximately equal to its critical value both at the same location. This indicates that the critical value of damage can satisfactorily predict failure. In Figure 24 and Figure 25, a comparison is made between contours of damage and effective plastic strain when damage reaches its critical value.

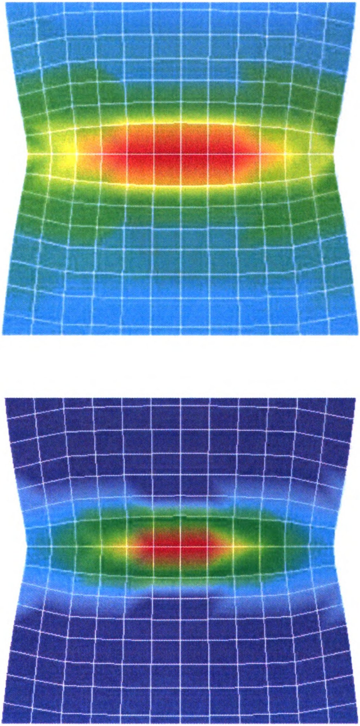


Figure 24 : Localization of damage and its comparison with effective plastic strain for the uniaxial model (The top picture shows the effective plastic strain)

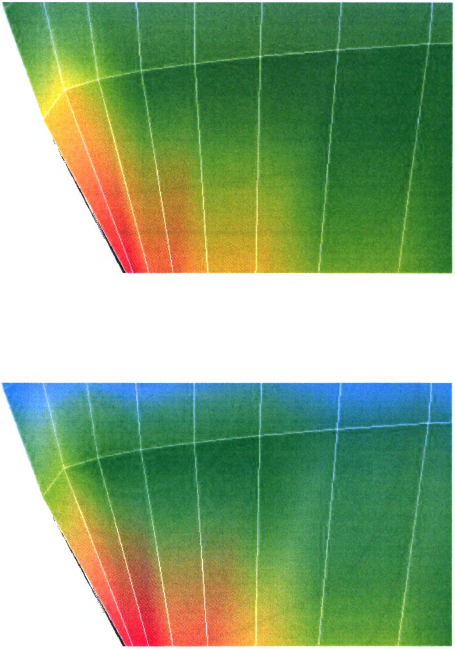


Figure 25 : Localization of damage and its comparison with effective plastic strain for the biaxial model (The top picture shows the effective plastic strain)

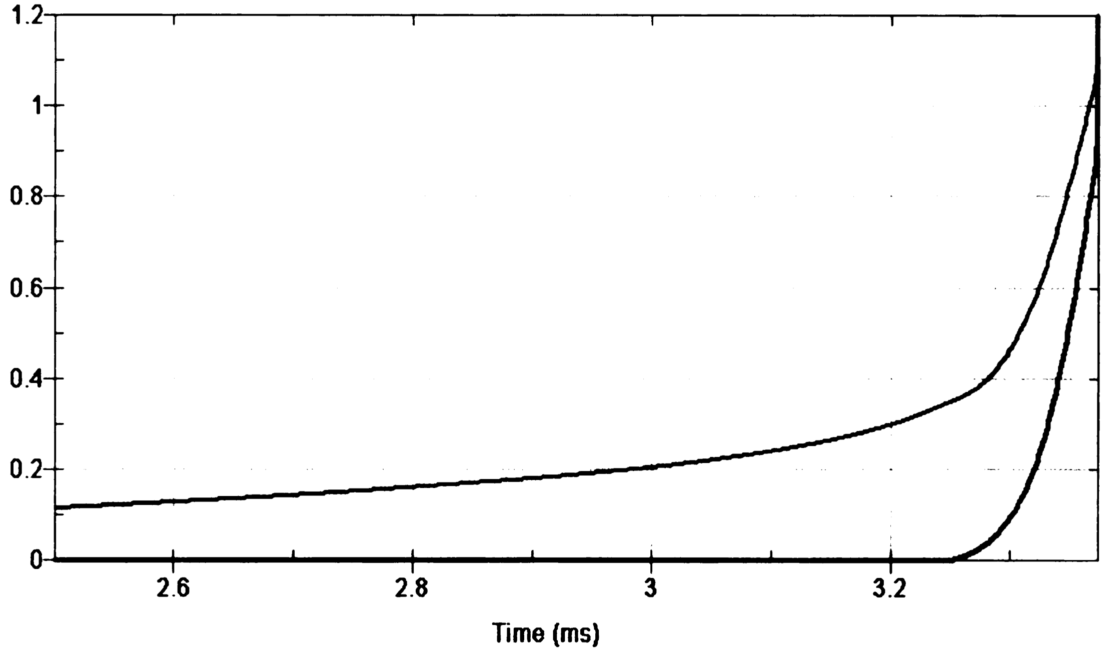


Figure 26 : Evolution of damage (black) and effective plastic strain (blue) of the element shown in Figure 16 for the uniaxial model

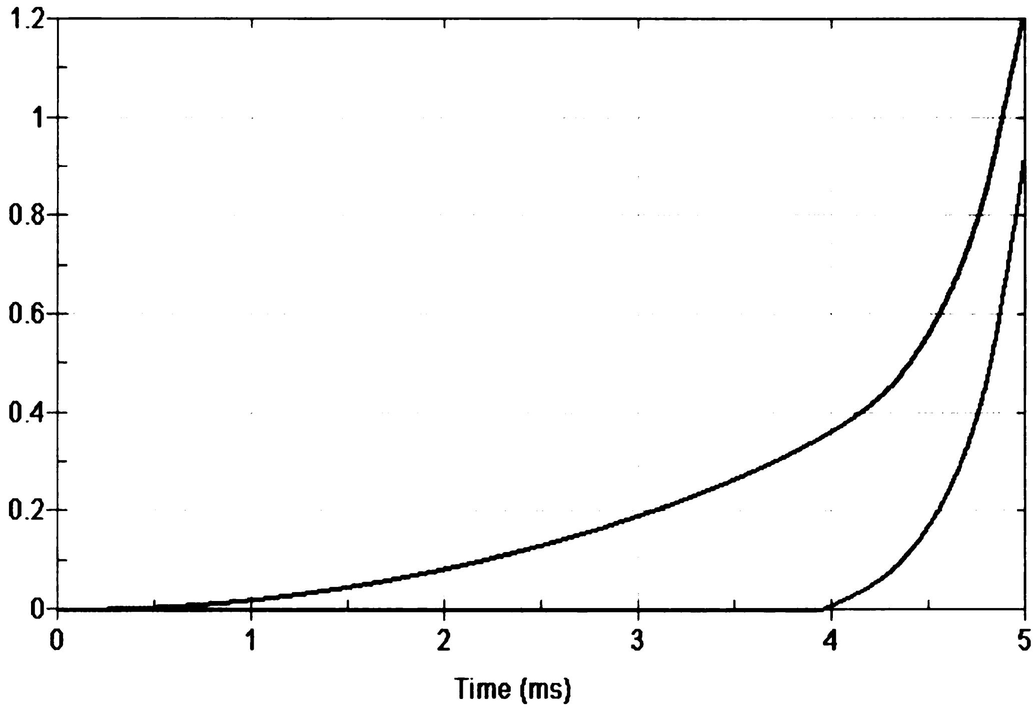


Figure 27 : Evolution of damage (black) and effective plastic strain (blue) of the element shown in Figure 17 for the biaxial model

And finally, the softening effects of damage can be seen in the following figures and plots. The von-Mises equivalent stress is compared between the damage model results and results from a model with no damage consideration.

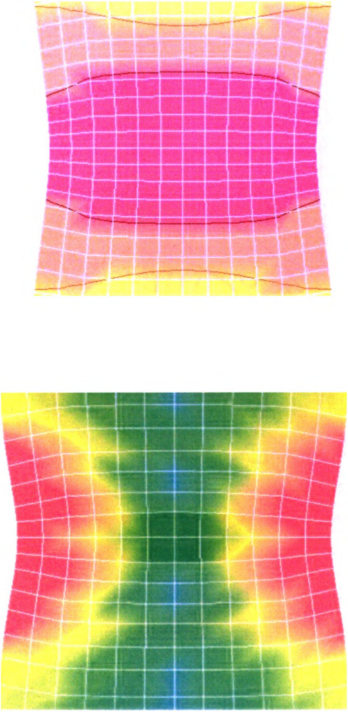


Figure 28 : Distribution of the von-Mises stress for the uniaxial model. (The bottom figure shows the damage model result and the top figure shows the non-damage model result)

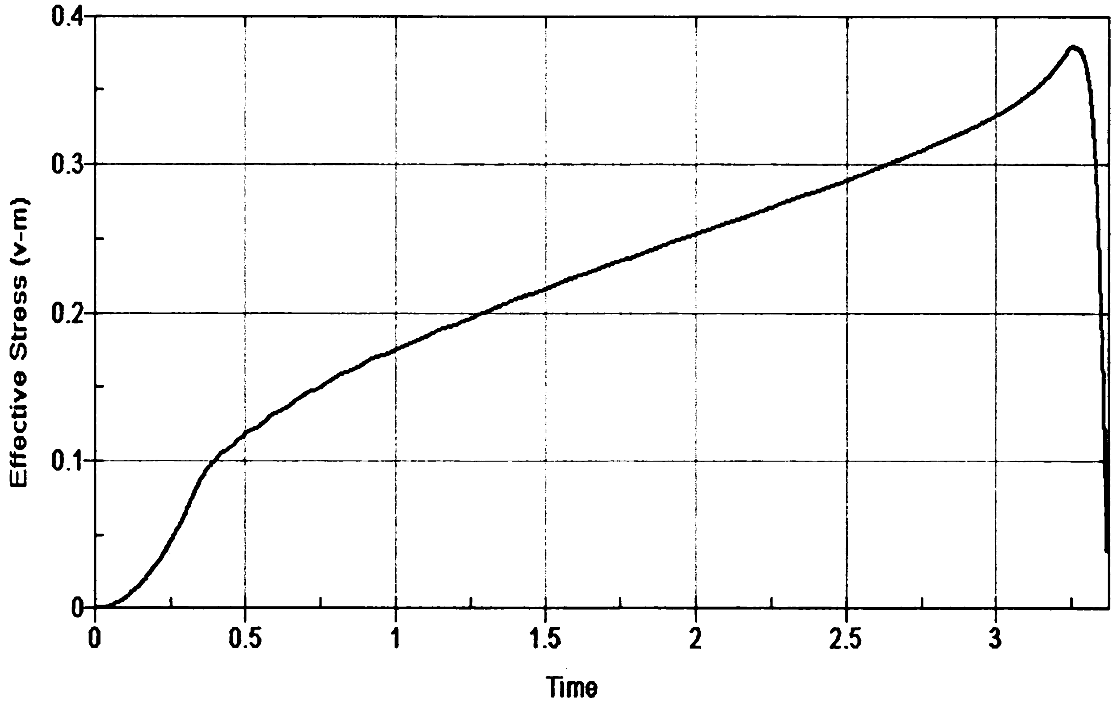


Figure 30 : Damage model result for von-Mises stress in element shown in Figure 16 for the uniaxial model

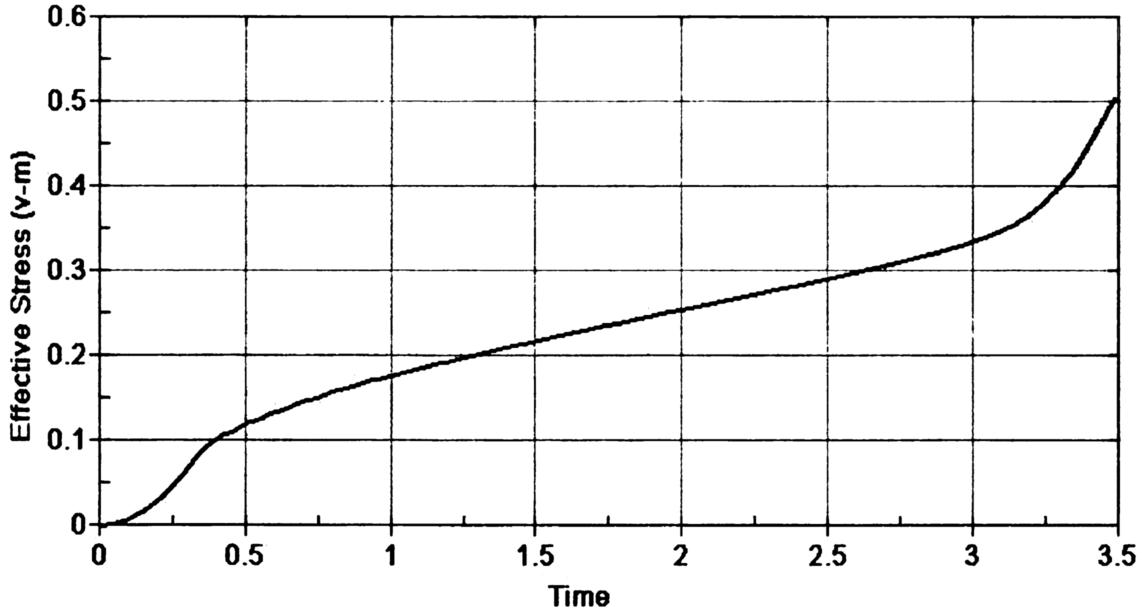


Figure 31 : Non-Damage model result for von-Mises stress in element shown in Figure 16 for the uniaxial model

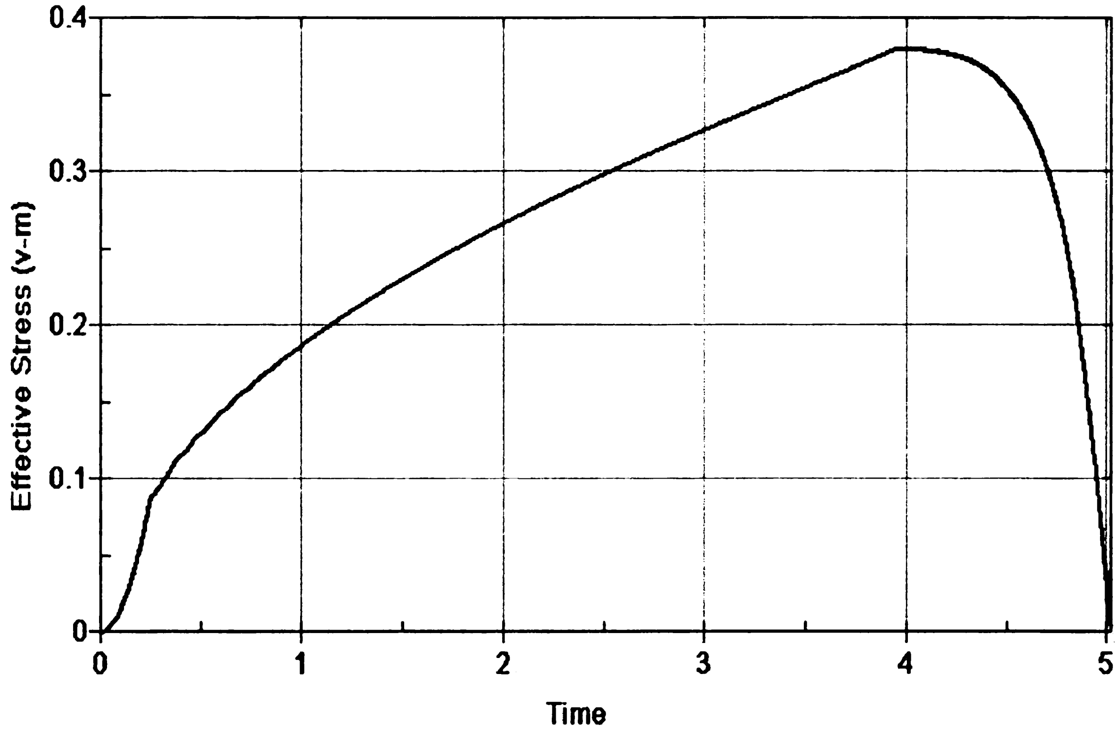


Figure 32 : Damage model result for von-Mises stress in element shown in Figure 17 for the biaxial model

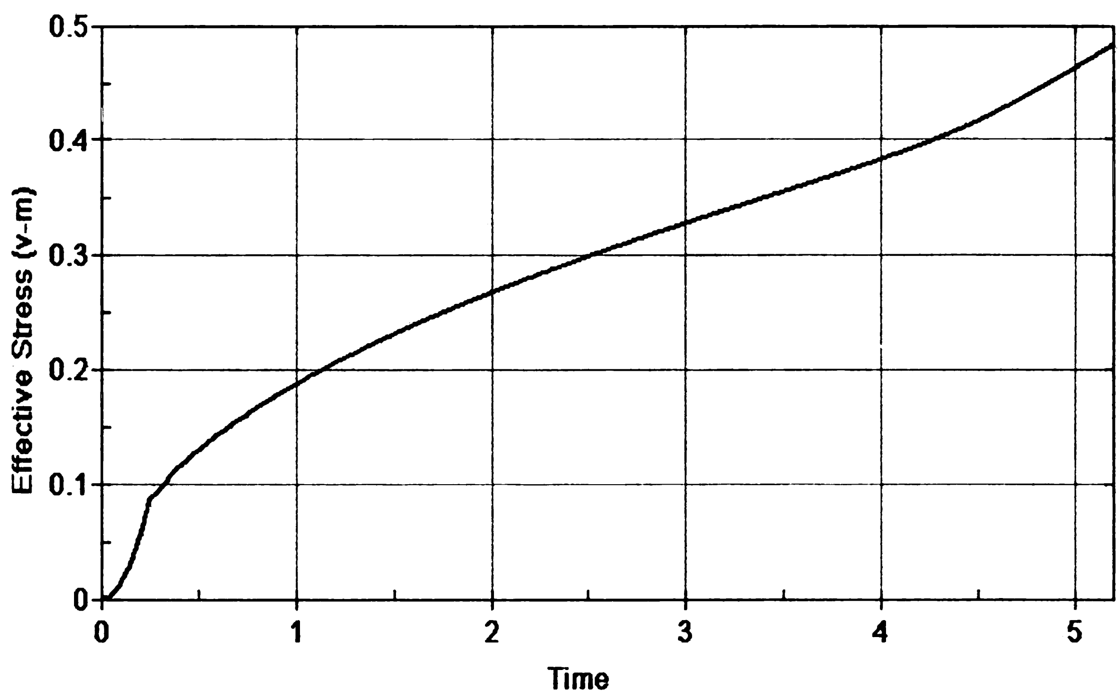


Figure 33 : Non-Damage model result for von-Mises stress in element shown in Figure 17 for the biaxial model

References

[1] L.M. Kachanov, Introduction to continuum damage mechanics (Dordrecht ; Boston : M. Nijhoff, 1986).

[2] D. Krajcinovic, G.U. Fonseka, The continuous damage theory of brittle materials-parts I & II, Journal of Applied Mechanics, 48 (1981) 809-824.

[3] D. Krajcinovic, Continuous damage mechanics revisited: basic concepts and definitions Journal of Applied Mechanics, 52 (1985) 829-834.

[4] F.A Leckie, E.T. Onat, Tensorial nature of damage measuring internal variables, Proc. IUTAM Symp. on Physical Nonlinearities in Structures, (1987) 547-558.

[5] E.T. Onat, F.A. Leckie, Representation of mechanical behavior in the presence of changing internal structure, Journal of Applied Mechanics, 55 (1988) 1-10.

[6] J.L. Chaboche, Anisotropic creep damage in the framework of continuum damage mechanics, Nuclear Engineering and Design, 79 (1984) 309-319.

[7] J.L. Chaboche, Thermodynamically founded CDM models for creep and other conditions, CISM Courses, "Creep and Damage in Materials and Structures", H. Altenbach and J.J. Skrzypek eds., Springer, 209-283, 1999.

[8] S. Murakami, Mechanical modeling of material damage, Journal of Applied Mechanics, 55 (1988) 280-286.

[9] C.L. Chow, J. Wang, An anisotropic theory of elasticity for continuum damage mechanics, International Journal of Fracture, 33 (1987) 3-16.

[10] C.L. Chow, J. Wang, An anisotropic theory of continuum damage mechanics for ductile fracture, Engineering Fracture Mechanics, 27 (1987) 547-558.

[11] C.L. Chow, T.J. Lu, On evolution laws of anisotropic

damage, Engineering Fracture Mechanics, 34 (1989) 679-701.

[12] C.L. Chow, J. Wang, Ductile fracture characterization with an anisotropic continuum damage theory, Engineering Fracture Mechanics, 30 (1988) 547-563.

[13] C.L. Chow, J. Wang, A finite element analysis of continuum damage mechanics for ductile fracture, International Journal of Fracture, 38 (1988) 83-102.

[14] J. Lemaitre, A continuous damage mechanics model for ductile fracture. Journal of Engineering Materials and Technology, 107 (1985) 83-89.

[15] J. Lemaitre, A Course on Damage Mechanics, Second Edition (Springer: New York, 1996)

[16] J. Lemaitre, J.L. Chaboche, Mechanics of Solid Materials (Cambridge University Press, 1990).

[17] J. Lemaitre, I. Doghri, Damage 90: a post processor for crack initiation, Computer methods in applied mechanics and engineering, 115 (1994) 197-232

[18] J.Lemaitre, J. Dufailly, Damage Measurements, Engineering Fracture Mechanics, 28 (1987) 643-661

[19] J. Lemaitre, R. Desmorat, M. Sauzay, Anisotropic damage law of evolution, European Journal of Mechanics - A/Solids , 19 (2000) 187-208

[20] J.J. Skryzpek, Material damage models for creep failure analysis and design of structures

[21] G.T. Mase, G.E. Mase, Continuum Mechanics for Engineers, Second Edition (CRC Press, 1999)