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ABSTRACT

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By

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We study two problems of constrained lower semicontinuity for a functional on Sobolev space. The first problem is motivated by certain models of microstructures and phase transitions which are distinguished by the fact that the associated Young measure is supported on a certain set K. We study the case when $K = \mathcal{L}$ is a linear subspace and we prove that the weak lower semicontinuity of a functional on a Sobolev space restricted to sequences whose gradients approach the linear subspace $\mathcal{L}$ satisfying a constant dimension condition is equivalent to a generalized version of quasiconvexity. The second problem is motivated by the Ekeland variational principle. We study a restricted weak lower semicontinuity for a given smooth functional on Sobolev space along all its weakly convergent Palais-Smale sequences. This type of constrained weak lower semicontinuity replaces the usual lower semicontinuity condition required for the direct method in the calculus of variations, and suffices for the existence of minimizers under the usual coercivity assumption. Although, in general, this condition is not equivalent to the usual weak lower semicontinuity condition, we show that, in certain cases, these two conditions are equivalent and reduce to the usual convexity or quasiconvexity conditions in the calculus of variations.

To Oana, for her special presence and support

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Introduction

A problem of significant importance in the Calculus of Variations is to find among all functions $u \in W^{1,p}(\Omega, \mathbb{R}^m)$, with certain prescribed constraints, those which minimize a given functional

$$I(u) = \int_{\Omega} f(x, u(x), Du(x))dx \tag{1}$$

where $f : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}$, $\Omega \subset \mathbb{R}^n$ a bounded domain and Du denotes the gradient of u in the sense of distributions. A direct method of proving existence of minimizers is to find minimizing sequences converging in some topology and check that the functional I is lower semicontinuous in that topology; then in this case the limit would be a minimizer. Therefore it is a special interest in finding necessary and sufficient conditions for the function f such that I defined by (1) is weakly lower semicontinuous on certain Sobolev space. One "right" candidate for such condition is the concept of *quasiconvexity* first introduced by Morrey in the early '50s [Mo 1].

According to Morrey a function $f : \mathbb{M}^{m\times n} \to \mathbb{R}$ is *quasiconvex* if

$$\int_{\Omega} f(A + Du(x))dx \geq |\Omega| f(A)$$

for all $A \in \mathbb{M}^{m\times n}$ and all $u \in C_0^\infty(\Omega, \mathbb{R}^m)$.

Acerbi and Fusco [AF] proved that under some proper growth condition the weak lower semicontinuity of the functional I given by (1) is equivalent to the quasiconvexity condition of f with respect to variable ξ.

The quasiconvexity condition is generally difficult to verify. As a major contribution in understanding this condition we distinguish the work of Ball [Ba 1]. He developed the concepts of *rank-one convexity* and *polyconvexity* along with the quasiconvexity emphasizing many interesting facts in the attempt to establish a useful sufficient condition for the weak lower semicontinuity. It turns out that rank-one convexity (see definition below), although easier to check, is the weakest among all three conditions. In general rank-one convexity does not imply quasiconvexity (Šverák [Sv 1]) but vice versa is always true. However there are particular cases when rank-one convexity is equivalent to quasiconvexity, for example when f is a quadratic form.

An efficient way to study weakly convergent sequences and the weak lower semicontinuity property for the functional (1) is to use the concept of Young measures developed by Tartar [Ta] following the original idea of L.C.Young [Yo]. Kinderlehrer and Pedregal [KP] showed that the homogeneous gradient Young measures (i.e. $x \to \nu_x$ is the constant map almost every x) are exactly those probability measures that satisfy Jensen's inequality for all quasiconvex functions f i.e.

$$\int_{\mathbb{M}^{m\times n}} f(\lambda)d\nu_x(\lambda) \geq f(\int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda)).$$

Using the techniques of Young measures, Fonseca and Müller [FM] studied the so called $\mathcal{A}$-quasiconvexity problem and Müller [Mu 3] also studied a similar problem without the constant rank condition.

For problems relevant to solid-solid phase transitions in the Material Science [BJ, Mu 2] one can model the so called microstructure through Young measures. In these situations it is very important to study the sequences satisfying

$$\operatorname{dist}(Du_k(x), K) \to 0 \tag{2}$$

for almost every $x \in \Omega$ where $\Omega \subset \mathbb{R}^n$ and $K \subset \mathbb{M}^{n\times n}$, which is a so called a *N-energy well* of the form $K = \cup_{i=1}^N SO(n)H_i$. In terms of Young measure this condition (2) is equivalent to the Young measure being supported on the set K.

It is as well very useful in practice to study the weak lower semicontinuity of functionals I given by (1) along sequences u_k satisfying constraint like (2) for a given set K. In the first part of this thesis we studied this problem with the set K being a linear subspace. In this case we also study the constrained rank-one and quasiconvexity. Let $K = \mathcal{L}$ be a linear subspace of $\mathbb{M}^{m\times n}$. We say that a function $f : \mathcal{L} \to \mathbb{R}$ is *$\mathcal{L}$-rank one convex* if for any $\lambda \in [0, 1]$ and A, $B \in \mathcal{L}$ such that rank$(A-B) \leq 1$ we have

$$f(\lambda A+(1-\lambda)B) \leq \lambda f(A)+(1-\lambda)f(B).$$

Also we say that f is *$\mathcal{L}$-quasiconvex* if

$$f(A) \leq \frac{1}{|Q|}\int\limits_{Q} f(A+Du(x))dx$$

for every cube $Q \subset \mathbb{R}^n$, any $A \in \mathcal{L}$ and every $u \in W^{1,\infty}(Q;\mathbb{R}^m)$, Q-periodic with $Du(x) \in \mathcal{L}$ for almost every x. We remark that if $\mathcal{L} = \mathbb{M}^{m\times n}$ we get the usual rank one convexity and quasiconvexity condition and thus the new conditions generalize the classical ones.

Let $f : \mathbb{M}^{m\times n} \to \mathbb{R}$ and define $I(u) = \int_\Omega f(Du)dx$. We say I is *$\mathcal{L}$-weakly lower semicontinuous* on $W^{1,p}$ if

$$I(u) \leq \liminf_{k\to\infty} I(u_k)$$

whenever $u_k \rightharpoonup u$ and $\text{dist}(Du_k, \mathcal{L}) \to 0$ as $k \to \infty$.

The main result of the first part is that if the subspace $\mathcal{L}$ satisfies the *constant dimension condition* (see definition below) then $\mathcal{L}$-quasiconvexity is equivalent to the $\mathcal{L}$-weak lower semicontinuity of the functional I.

In the second part of the thesis we assume the functional I defined above is C^1 on $W^{1,p}(\Omega;\mathbb{R}^m)$. This requires that f be C^1 in (s,ξ) and satisfy certain growth conditions. As in many problems in application, I is often also bounded below.

When minimizing bounded-below C^1 functionals over a Banach space, an important variational principle discovered by Ekeland [Ek] (see also [AE]) can provide more special minimizing sequences. For our functional I minimized over a Dirichlet class $\mathcal{A}_g$ in $W^{1,p}(\Omega;\mathbb{R}^m)$, we can always obtain a minimizing sequence $\{u_k\}$ in $\mathcal{A}_g$ which satisfies $I'(u_k) \to 0$ in $W^{-1,p'}(\Omega;\mathbb{R}^m)$. Here, we assume $p > 1$ and $p' = \frac{p}{p-1}$, and $W^{-1,p'}(\Omega;\mathbb{R}^m)$ denotes the dual space of $W_0^{1,p}(\Omega;\mathbb{R}^m)$. Consequently, the weak limit (if exists) of any such minimizing sequence will be an energy minimizer provided that $I(u)$ only satisfies the condition:

$$I(u) \le \liminf_{k\to\infty} I(u_k) \text{ whenever } \begin{cases} u_k \rightharpoonup u \text{ in } W^{1,p}(\Omega;\mathbb{R}^m) \text{ and} \\ \\ I'(u_k) \to 0 \text{ in } W^{-1,p'}(\Omega;\mathbb{R}^m). \end{cases} \tag{3}$$

Certainly the usual weak lower semicontinuity condition implies the condition (3). We shall say the functional $I(u)$ is *restricted weakly lower semicontinuous* on $W^{1,p}(\Omega;\mathbb{R}^m)$ if it satisfies condition (3). If the condition holds only for all u_k, u in the Dirichlet class $\mathcal{A}_g$, we then say I is restricted weakly lower semicontinuous on $\mathcal{A}_g$. Note that in nonlinear analysis [AE, Ra] the sequences $\{u_k\}$ with bounded $I(u_k)$ satisfying $I'(u_k) \to 0$ are usually called *the Palais-Smale sequences* of the functional $I(u)$. Therefore, in the following, we shall say a sequence $\{u_k\}$ *(PS) weakly converges to* u (with respect to I) and denote by $u_k \stackrel{\text{ps}}{\rightharpoonup} u$ in $W^{1,p}$ if it satisfies $u_k \rightharpoonup u$ in $W^{1,p}$ and $I'(u_k) \to 0$ in $W^{-1,p'}$.

As we shall see later, this restricted weak lower semicontinuity imposes some intrinsic property on the function f. Such a condition has also been mentioned in

[Mu 2] as a point of view to replace Morrey's quasiconvexity condition. In general, as shown in the paper (see Proposition 3.7), the restricted weak lower semicontinuity is not equivalent to the usual weak lower semicontinuity even for one dimensional scalar problems. However, the main results of the paper deal with certain cases where the restricted weak lower semicontinuity is actually equivalent to the usual weak lower semicontinuity of the functional (hence the convexity or quasiconvexity of f). In general cases, we do not know the necessary and sufficient condition for the restricted weak lower semicontinuity. We point out that the major difficulty in handling this type of restricted weak lower semicontinuity lies in that the test sequences $\{u_k\}$ in the usual techniques [AF, Da, Mo 1] do not satisfy the condition $I'(u_k) \to 0$ in $W^{-1,p'}(\Omega; \mathbb{R}^m)$.

A closely related problem to the restricted weak lower semicontinuity of functional I is to characterize all the gradient Young measures [KP] generated by weakly convergent (PS) sequences in $W^{1,p}(\Omega; \mathbb{R}^m)$. This problem is associated with the theory of compensated compactness [CLMS, Ta]. The difficulty lies in that in this case the strong convergence $I'(u_k) \to 0$ in $W^{-1,p'}(\Omega; \mathbb{R}^m)$ can not be realized by the Young measure of $\{Du_k\}$ in the dimension $n \geq 2$. Recently the weak lower semicontinuity of functionals under certain linear differential constraints has been studied using the Young measure theory [FM, Sa]. These linear constraints $\mathcal{A}(u)$ are independent of the functional and usually have large kernel. Then the constrained lower semicontinuity of functionals may be characterized through the Jensen's inequality of the integrant with the associated Young measures supported on the kernel of $\mathcal{A}$;

this is the so-called $\mathcal{A}$-quasiconvexity [FM]. In this paper, we do not pursue the Young measure method for our restricted weak lower semicontinuity studied here mainly because it does not realize the strong convergence $I'(u_k) \to 0$..

To put our restricted weak lower semicontinuity in another perspective similar to the linear constraint cases, one could study the lower semicontinuity of any given functional $J(u)$ under the (PS) weak convergence defined above. For example, one could define J to be restricted weakly lower semicontinuous on $W^{1,p}(\Omega;\mathbb{R}^m)$ (with respect to I) if

$$J(u) \leq \liminf_{k\to\infty} J(u_k), \quad \forall\, u_k \overset{\text{ps}}{\rightharpoonup} u \quad \text{(with respect to } I\text{)},$$

and study the relaxation of J under this lower semicontinuity if it is not restricted weakly lower semicontinuous. The study in such a direction seems interesting, but difficult in view of the nonlinear constraints. As in the linear constraint case, one might consider the certain convexity property of J on the kernel of $I'(u)$ consisting of all critical points of I, which may not be closed under weak convergence.

Chapter 1

Preliminaries and Notations

Let $\mathbb{R}^n$ the usual n-dimensional euclidean space with points $x = (x_1, x_2, ..., x_n)$, $x_i \in \mathbb{R}$ (real numbers). Let Ω be a bounded domain in $\mathbb{R}^n$ and $Q_0 = [0,1]^n$ the unit cube in $\mathbb{R}^n$. Let $\mathbb{M}^{m\times n}$ be the set of $m \times n$ matrices. For vectors $a, b \in \mathbb{R}^n$ and matrices $\xi, \eta \in \mathbb{M}^{m\times n}$, we define the inner products by

$$a \cdot b = \sum_{j=1}^{n} a_i b_i, \quad \xi : \eta = \langle \xi, \eta \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} \xi_{ij} \eta_{ij}$$

with the corresponding Euclidean norms denoted both by $|\cdot|$. For vectors $q \in \mathbb{R}^m$, $a \in \mathbb{R}^n$, we denote by $q \otimes a$ the rank-one $m \times n$ matrix $(q_i a_j)$ and also define $0 = 0_{m\times n}$ where $0_{m\times n}$ is the $m \times n$ matrix having 0 in all entries.

A cube in $\mathbb{R}^n$ is a set

$$Q = \{x \in \mathbb{R}^n \,|\, x = \sum_{i=1}^{n} c_i l_i,\ 0 \le c_i \le 1\}$$

where $\{l_1, l_2, ... l_n\}$ is an orthonormal basis of $\mathbb{R}^n$.

Denoting $\mu(\Omega)$ or $|\Omega|$ the Lebesgue measure of a measurable set Ω we have that $\mu(Q) = |Q| = 1$. A function u defined on $\mathbb{R}^n$ is called *Q-periodic* if

$$u(x) = u(x + \sum_{i=1}^{n} c_i l_i)$$

for any $x \in \mathbb{R}^n$ and any $c_i \in \mathbb{Z}$.

Let $W^{1,p}(\Omega)$ be the usual Sobolev space of scalar functions on Ω, and define $W^{1,p}(\Omega; \mathbb{R}^m)$ to be the space of vector functions $u \colon \Omega \to \mathbb{R}^m$ with each component $u^i \in W^{1,p}(\Omega)$ and we denote by Du the Jacobi matrix of u defined by

$$Du(x) = (\partial u^i / \partial x_j)_{i=1,\cdots,m}^{j=1,\cdots,n}.$$

Let $1 \leq p < \infty$. We make $W^{1,p}(\Omega; \mathbb{R}^m)$ a Banach space with the norm

$$\|u\|_{W^{1,p}(\Omega;\mathbb{R}^m)} = \left(\int_\Omega (|u|^p + |Du|^p) \, dx \right)^{\frac{1}{p}}.$$

Let $C_0^\infty(\Omega; \mathbb{R}^m)$ be the set of infinitely differentiable vector functions with compact support in Ω, and let $W_0^{1,p}(\Omega; \mathbb{R}^m)$ be the closure of $C_0^\infty(\Omega; \mathbb{R}^m)$ in $W^{1,p}(\Omega; \mathbb{R}^m)$. Then $W_0^{1,p}(\Omega; \mathbb{R}^m)$ is itself a Banach space and has an equivalent norm defined by $\||Du|\|_{L^p(\Omega)}$. We also recall the following version of Sobolev embedding:

Theorem 1.1. *If Ω is a bounded Lipschitz domain then the embedding*

$$W^{1,p}(\Omega;\mathbb{R}^m) \to L^p(\Omega;\mathbb{R}^m)$$

is compact for any $1 \leq p \leq \infty$.

By $C_0(\mathbb{R}^n)$ we denote the closure of continuous functions on $\mathbb{R}^n$ with compact support. The dual of $C_0(\mathbb{R}^n)$ can be identified with the space $\mathcal{M}(\mathbb{R}^n)$ of signed Radon measures with finite mass via the pairing

$$\langle \nu, f \rangle = \int_{\mathbb{R}^n} f d\nu$$

A map $\nu : E \to \mathcal{M}(\mathbb{R}^n)$ is called weak* measurable if the functions $x \to \langle \nu(x), f \rangle$ are measurable for all $f \in C_0(\mathbb{R}^n)$. We shall write ν_x instead of $\nu(x)$.

Let $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$ a measurable function such that $v \to f(x, v)$ is continuous for all $x \in \Omega$ (a function with this properties is called Carathéodory function). The following result represents the fundamental theorem of Young measures:

Theorem 1.2 ([Ba 2]). *Let $E \subset \mathbb{R}^n$ be a measurable set of finite measure and let $u_k : E \to \mathbb{R}^m$ be a sequence of measurable functions. Then there exists a subsequence u_{k_j} and a weak* measurable map $\nu : E \to \mathcal{M}(\mathbb{R}^m)$ such that the following hold.*

(i) $\nu_x \geq 0$, $\|\nu_x\|_{\mathcal{M}(\mathbb{R}^m)} = \int_{\mathbb{R}^m} d\nu_x \leq 1$, for almost every $x \in E$.

(ii) we have $\|\nu_x\|_{\mathcal{M}(\mathbb{R}^m)} = 1$ if and only if the sequence does not escape to infinity,i.e. if $\lim\limits_{r\to\infty} \sup\limits_j |\{|u_{k_j}|\}| \geq r| = 0$.

(iii) Let $A \subset E$ measurable and $f \in C(\mathbb{R}^m)$. If $\|\nu_x\|_{\mathcal{M}(\mathbb{R}^m)} = 1$ for almost every $x \in E$ and if $f(u_{k_j})$ is relatively compact in $L^1(A)$ then

$$f(u_{k_j}) \rightharpoonup \langle \nu_x, f \rangle = \int_{\mathbb{R}^m} f d\nu_x.$$

(iv) If f is Carathéodory and bounded from below then

$$\lim_{n\to\infty} \int_\Omega f(x, u_{k_j})(x))dx = \int_\Omega \langle \nu_x, f(x, u_{k_j}(x)) \rangle dx < \infty$$

if and only if $\{f(\cdot, u_{k_j}(\cdot))\}$ is equi-integrable.

The measures $(\nu_x)_{x\in\Omega}$ are called the *Young measures generated by the sequence* $\{u_{k_j}\}$. The Young measure is said to be *homogeneous* if there is a Radon measure $\nu_0 \in \mathcal{M}(\mathbb{R}^m)$ such that $\nu_x = \nu_0$ for almost every $x \in \Omega$.

Theorem 1.3 ([Pe]). *If $\{u_k\}$ is a sequence of measurable functions with associated Young measure $\nu = \{\nu_x\}_{x\in\Omega}$, then*

$$\liminf_{k\to\infty} \int_E f(x, u_k(x))dx \geq \int_E \int_{\mathbb{R}^m} f(x, \lambda) d\nu_x(\lambda) dx, \tag{1.1}$$

for every Carathéodory function f, bounded from below, and every measurable subset $E \subset \Omega$.

A Young measure (ν_x) is called a *gradient Young measure* if it is generated by a sequence of gradients. We say that (ν_x) is a $W^{1,p}$ gradient Young measure if it is generated by $\{Du_k\}$ and $u_k \rightharpoonup u$ in $W^{1,p}(\Omega, \mathbb{R}^m)$. The following result refers to the localization of the gradient Young measures.

Theorem 1.4 ([KP]). *Let (ν_x) be a gradient Young measure generated by a sequence of gradients of functions in $W^{1,p}(\Omega)$. Then for almost every $a \in \Omega$ there exists a sequence of gradients of functions in $W^{1,p}(\Omega)$ that generates the homogeneous Young measure (ν_a).*

We also provide the definitions of convexity, rank one convexity and quasiconvexity.

Definition 1.1. Let $h : \mathbb{M}^{m\times n} \to \mathbb{R}$. We say that h is *convex* on $\mathbb{M}^{m\times n}$ if the inequality

$$h(\lambda\xi + (1-\lambda)\eta) \leq \lambda h(\xi) + (1-\lambda)h(\eta) \tag{1.2}$$

holds for all $0 < \lambda < 1$ and $\xi,\ \eta \in \mathbb{M}^{m\times n}$.

Note also that h is convex if and only if $g(t) = h(\xi + t\eta)$ is a convex function of t on $\mathbb{R}$ for all $\xi,\ \eta \in \mathbb{M}^{m\times n}$. For C^1 functions h, the convexity condition is equivalent to the condition

$$h(\eta) \geq h(\xi) + D_\xi h(\xi) : (\eta - \xi), \quad \forall\ \eta,\ \xi \in \mathbb{M}^{m\times n}. \tag{1.3}$$

Furthermore, a C^1 function h on $\mathbb{R}$ is convex if and only if h' is nondecreasing, or equivalently, the following condition holds:

$$(h'(a) - h'(b))(a - b) \geq 0, \quad \forall\ a,\ b \in \mathbb{R}. \tag{1.4}$$

Definition 1.2. A function $f : \mathbb{M}^{m\times n} \to \mathbb{R}$ is called *rank one convex* if

$$f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda) f(B)$$

for all $\lambda \in [0,1]$ and any matrices A and B such that $\operatorname{rank}(A-B) \leq 1$.

Definition 1.3. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *separately convex* if $g_i(t) = f(x_1, ...x_{i-1}, t, x_{i+1}, ...x_n)$ is convex in t for all $1 \leq i \leq n$.

Definition 1.4. A function $f : \mathbb{M}^{m\times n} \to \mathbb{R}$ is said to be *quasiconvex* if

$$\int_{Q_0} f(A + Du(x))dx \geq f(A)$$

for any $A \in \mathbb{M}^{m\times n}$ and $u \in W_0^{1,\infty}(Q_0; \mathbb{R}^m)$.

If f is quasiconvex then one can show [Sv 1] that

$$f(A) = \inf_{u\in W^{1,\infty}_{per}(Q_0;\mathbb{R}^m)} \int_{Q_0} f(A + Du(x))dx$$

where $W^{1,\infty}_{per}(Q_0; \mathbb{R}^m)$ is the class of periodic functions in $W^{1,\infty}(Q_0; \mathbb{R}^m)$.

Let $\Lambda := \mathbb{Z}^N$ be the unit lattice, i.e. the additive group of points in $\mathbb{R}^n$ with integer coordinates. We say that $f : \mathbb{R}^n \to \mathbb{R}^m$ is Λ-periodic if

$$f(x + \lambda) = f(x) \text{ for all } x \in \mathbb{R}^n,\ \lambda \in \Lambda.$$

A $\Lambda-$ *periodic* function f may be identified with a function f_T on the n-torus

$$T_n := \{(e^{2\pi i x_1}, e^{2\pi i x_2}, ..., e^{2\pi i x_n}) \in \mathbb{C}^n : (x_1, x_2, ...x_n) \in \mathbb{R}^n\}$$

through the relation

$$f_T(e^{2\pi i x_1}, e^{2\pi i x_2}, ..., e^{2\pi i x_n}) := f(x_1, x_2, ...x_n)$$

The space $L^p(T_n)$ is identified with $L^p(Q_0)$ and $C(T_n)$ is the set of Λ-periodic continuous functions on $\bar{Q}_0$. We recall some results on Fourier transform for periodic functions. If $f \in L^1(T_n)$, then its Fourier coefficients are defined as:

$$\hat{f}(\lambda) := \int_{T_n} f(x) e^{-2\pi i x\cdot\lambda} dx, \ \lambda \in \Lambda.$$

Theorem 1.5. *We have the following:*

(i) The trigonometric polynomials

$$R(x) := \sum_{\lambda\in\Lambda'} a_\lambda e^{-2\pi i x\cdot\lambda}, \ \Lambda' \text{ all finite subsets of } \Lambda, \ a_\lambda \in \mathbb{C}$$

are dense in $C(T_n)$ and in $L^p(T_n)$ for all $1 \le p < \infty$.

(ii) If $f \in L^2(T_n)$ then

$$f(x) = \sum_{\lambda\in\Lambda} \hat{f}(\lambda) e^{-2\pi i x\cdot\lambda}, \ \sum_{\lambda\in\Lambda} |\hat{f}(\lambda)|^2 = \|f\|_{L^2}$$

Let $f \colon \Omega \times \mathbb{R}^n \times \mathbb{M}^{m\times n} \to \mathbb{R}$. We say f is *Carathéodory* if $f(x, s, \xi)$ is measurable in $x \in \Omega$ for all $(s, \xi) \in \mathbb{R}^n \times \mathbb{M}^{m\times n}$ and continuous in $(s, \xi) \in \mathbb{R}^n \times \mathbb{M}^{m\times n}$ for almost every $x \in \Omega$. Define the multiple integral functional I on $W^{1,p}(\Omega; \mathbb{R}^m)$ by

$$I(u) = \int_\Omega f(x, u(x), Du(x))\, dx, \quad u \in W^{1,p}(\Omega; \mathbb{R}^m).$$

If $f(x, s, \xi)$ is measurable in $x \in \Omega$ for all $(s, \xi) \in \mathbb{R}^n \times \mathbb{M}^{m\times n}$ and is C^1 in $(s, \xi) \in \mathbb{R}^n \times \mathbb{M}^{m\times n}$ for almost every $x \in \Omega$, we shall use the following notation to denote the derivatives of f on s and ξ:

$$D_s f(x, s, \xi) = (\frac{\partial f}{\partial s_1}, \cdots, \frac{\partial f}{\partial s_n}), \quad D_\xi f(x, s, \xi) = (\partial f / \partial \xi_{ij})_{i=1,\cdots,m}^{j=1,\cdots,n}.$$

Definition 1.5. A functional I is said to be (sequentially) *weakly lower semicontinuous* on $W^{1,p}(\Omega; \mathbb{R}^m)$ provided

$$I(u) \le \liminf_{k\to\infty} I(u_k) \quad \text{whenever } u_k \rightharpoonup u \text{ in } W^{1,p}(\Omega; \mathbb{R}^m). \tag{1.5}$$

The following important result has been proved by Acerbi and Fusco [AF].

Theorem 1.6. *Assume f is Carathéodory and satisfies*

$$0 \le f(x, s, \xi) \le c_1(|\xi|^p + |s|^p) + A(x),$$

where $c_1 > 0$ and $A \in L^1(\Omega)$. Then functional I defined above is weakly lower *semicontinuous on* $W^{1,p}(\Omega;\mathbb{R}^m)$ *if and only if* $f(x,s,\cdot)$ *is quasiconvex for almost every* $x \in \Omega$ *and all* $s \in \mathbb{R}^n$*; that is, the inequality*

$$f(x,s,\xi) \le \frac{1}{|\Omega|}\int_\Omega f(x,s,\xi + D\varphi(y))\,dy$$

holds for a.e. $x \in \Omega$*, all* $s \in \mathbb{R}^n$*,* $\xi \in \mathbb{M}^{m\times n}$ *and all* $\varphi \in C_0^\infty(\Omega;\mathbb{R}^m)$*.*

Finally we quote the following theorems which are known as the Ekeland variational principle. See [De, Ek] for proof and more on these principles.

Theorem 1.7. *Let* (X,d) *be a complete metric space and let* $\Phi\colon X \to \mathbb{R}\cup\{+\infty\}$ *be a lower semicontinuous function which is bounded below. Let* $\epsilon > 0$ *and* $\bar{u} \in X$ *be given such that*

$$\Phi(\bar{u}) \le \inf_X \Phi + \frac{\epsilon}{2}.$$

Then given any $\lambda > 0$ *there exists* $u_\lambda \in X$ *such that*

$$\Phi(u_\lambda) \le \Phi(\bar{u}), \quad d(u_\lambda,\bar{u}) \le \lambda \tag{1.6}$$

$$\Phi(u_\lambda) < \Phi(u) + \tfrac{\epsilon}{\lambda} d(u,u_\lambda) \quad \forall\, u \ne u_\lambda. \tag{1.7}$$

The following version, which follows from the general Ekeland principle above, is very useful for establishing certain results in chapter 3.

Theorem 1.8. *Let* X *be a Banach space and* X^* *its dual space, and let* $\Phi\colon X \to \mathbb{R}$ *be a* C^1 *functional which is bounded below. Then for each* $\epsilon > 0$ *there exists* $u_\epsilon \in X$

such that

$$\Phi(u_\epsilon) \leq \inf_X \Phi + \epsilon \tag{1.8}$$

$$\|\Phi'(u_\epsilon)\|_{X^*} \leq \epsilon. \tag{1.9}$$

Therefore, there exists a minimizing sequence $\{u_k\}$ *in* X *such that*

$$\lim_{k\to\infty} \Phi(u_k) = \inf_X \Phi, \quad \lim_{k\to\infty} \|\Phi'(u_k)\|_{X^*} = 0.$$

Chapter 2

Linear Restrictions with Constant Dimension

An interesting and motivating problem is to study necessary and sufficient conditions for the weak lower semicontinuity of the operator I restricted only to a class of functions that satisfy certain linear constraints, i.e. their gradients in the sense of distributions approach a preset target linear subspace of $\mathbb{M}^{m\times n}$ by means of L^2 convergence. When the linear subspace satisfies some special condition we prove that the restricted weak lower semicontinuity is equivalent to a generalized version of quasiconvexity.

2.1 $\mathcal{L}$-rank one convexity and $\mathcal{L}$-quasiconvexity

Let $\mathcal{L}$ be a linear subspace of $\mathbb{M}^{m\times n}$ and $P : \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ the linear map such that $PA = 0$ if and only if $A \in \mathcal{L}$, which is actually the orthogonal projection onto the orthogonal complement of $\mathcal{L}$.

Definition 2.1. We say that a function $f : \mathcal{L} \to \mathbb{R}$ is *$\mathcal{L}$-rank one convex* if for any $\lambda \in [0,1]$ and A, $B \in \mathcal{L}$ such that $\operatorname{rank}(A - B) \leq 1$ we have

$$f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda)f(B).$$

Definition 2.2. Given a cube $Q \subset \mathbb{R}^n$ we say that a function $f : \mathcal{L} \to \mathbb{R}$ is *$Q-\mathcal{L}$-quasiconvex* if

$$f(A) \leq \frac{1}{|Q|}\int\limits_{Q} f(A + Du(x))dx$$

for any $A \in \mathcal{L}$ and every $u \in W^{1,\infty}(Q;\mathbb{R}^m)$, Q-periodic with $Du \in \mathcal{L}$.

Definition 2.3. We say that a function $f : \mathcal{L} \to \mathbb{R}$ is *$\mathcal{L}$-quasiconvex* if it is Q-$\mathcal{L}$-quasiconvex for every cube Q, that is

$$f(A) \leq \frac{1}{|Q|}\int\limits_{Q} f(A + Du(x))dx$$

for any cube $Q \subset \mathbb{R}^n$, any $A \in \mathcal{L}$ and every $u \in W^{1,\infty}(Q;\mathbb{R}^m)$, Q-periodic with $Du \in \mathcal{L}$.

Theorem 2.1. *If a function $f : \mathcal{L} \to \mathbb{R}$ is $\mathcal{L}$-quasiconvex then it is also $\mathcal{L}$-rank one* *convex.*

Proof. Let $\lambda \in [0,1]$ and A, B two elements in the subspace $\mathcal{L}$ such that $\text{rank}(A - B) \leq 1$. Let $Q_0 = [0,1]^n$ a unit cube in $\mathbb{R}^n$. Since $\text{rank}(A-B) \leq 1$ there exist two vectors $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$ such that

$$A - B = a \cdot b^{\mathrm{T}}$$

and it exists a rotation, a matrix $R \in \mathbb{R}^{n \times n}$ such that $RR^{\mathrm{T}} = I_n$ and $\left(R^{\mathrm{T}} \cdot b\right)^{\mathrm{T}} = e_1$ where $e_1 \in \mathbb{R}^n$ with $e_1 = (1,0,0,...,0)$. Thus $(A-B)R = a \cdot \left(R^{\mathrm{T}} \cdot b\right)^{\mathrm{T}} = a \otimes e_1$. Let $Q = RQ_0$. Since f is assumed to be $\mathcal{L}$-quasiconvex we have

$$\int\limits_{Q} f(C + D\varphi(x))dx \geq f(C) \tag{2.1}$$

for all $C \in \mathcal{L}$ and $\varphi \in W^{1,\infty}(Q, \mathbb{R}^m)$ such that is Q periodic and $D\varphi(x) \in \mathcal{L}$ a.e. x. Let $\tilde{f}(A) = f(AR^{\mathrm{T}})$ and also denote $\tilde{A} = AR$ and $\tilde{\varphi}(x) = \varphi(Rx)$. Notice that $\tilde{\varphi}$ is Q_0 periodic,

$$D\tilde{\varphi}(x) \in \tilde{\mathcal{L}} = \{\tilde{M} | \tilde{M} = MR, M \in \mathcal{L}\}$$

almost every x and $\tilde{\varphi} \in W^{1,\infty}(Q_0, \mathbb{R}^m)$. By the change of variable under the integral we obtain

$$\int\limits_{Q_0} \tilde{f}(\tilde{C} + D\tilde{\varphi}(x))dx \geq \tilde{f}(\tilde{C}) \tag{2.2}$$

for all $\tilde{C} \in \tilde{\mathcal{L}}$ and all $\tilde{\varphi} \in W^{1,\infty}(Q_0, \mathbb{R}^m)$, Q_0 periodic and $D\tilde{\varphi(x)} \in \tilde{\mathcal{L}}$. Also we have

$$\tilde{A} - \tilde{B} = a \otimes e_1.$$

Let $\eta : [0,1] \to \mathbb{R}$ such that $\eta'(t) = \begin{cases} (1-\lambda) & \text{if } t \in [0,\lambda] \\ -\lambda & \text{if } t \in [\lambda, 1] \end{cases}$ and let $\tilde{\varphi}(x) = \eta(x_1)a$ where $x = (x_1, x_2, ...x_n)$. Thus we obtain that $\tilde{\varphi}$ is Q_0 periodic and we can extend by this periodicity to $\mathbb{R}^n$ and $D\tilde{\varphi}(x) \in \tilde{\mathcal{L}}$ a.e. x. Also notice that $\tilde{\varphi} \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^m)$ and

$$D\tilde{\varphi}(x) = \begin{cases} (1-\lambda)(\tilde{A} - \tilde{B}) & \text{if } x_1 \in [0,\lambda] \\ -\lambda(\tilde{A} - \tilde{B}) & \text{if } x_1 \in [\lambda, 1] \end{cases}$$

Thus we have that $\int\limits_{Q_0} \tilde{f}(\lambda\tilde{A} + (1-\lambda)\tilde{B} + D\tilde{\varphi}(x))dx = \lambda\tilde{f}(\tilde{A}) + (1-\lambda)\tilde{f}(\tilde{B})$ and $\int\limits_{Q_0} \tilde{f}(\lambda\tilde{A} + (1-\lambda)\tilde{B} + D\tilde{\varphi}(x))dx \geq \tilde{f}(\lambda\tilde{A}+(1-\lambda)\tilde{B})$ and obtain $\tilde{f}(\lambda\tilde{A}+(1-\lambda)\tilde{B}) \leq \lambda\tilde{f}(\tilde{A}) + (1-\lambda)\tilde{f}(\tilde{B})$ hence $f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda)f(B)$.

□

Proposition 2.2. *If the subspace $\mathcal{L}$ does not contain rank one matrices and a function $u \in W^{1,2}(Q; \mathbb{R}^m)$, Q-periodic has the property that $Du(x) \in \mathcal{L}$ almost every x then $u = const$.*

Proof. Assume first Q=Q_0. Since $\mathcal{L}$ does not contain rank one matrices we have that

$$\min_{|a|=1, |\lambda|=1} |P(a \otimes \lambda)| > 0 \tag{2.3}$$

and it follows that

$$|P(a \otimes \lambda)| > c|a||\lambda| \tag{2.4}$$

for any $a \in \mathbb{R}^m \setminus \{0_m\}$ and $\lambda \in \mathbb{R}^n \setminus \{0_n\}$. We consider now the Fourier transform of PDu which is $P(\hat{u}(\lambda) \otimes \lambda)$. Since $\mathcal{L}$ does not contain rank one matrices we have that

$$P(\hat{u}(\lambda) \otimes \lambda) = 0 \tag{2.5}$$

for all $\lambda \in \Lambda \setminus \{0_n\}$. Thus, using (2.4) we get that $\hat{u}(\lambda) = 0$ for all $\lambda \in \mathbb{R}^n \setminus \{0_n\}$ which proves that u must be a constant.

Now, if $Q = RQ_0$ for a rotation R and $u \in W^{1,2}(Q; \mathbb{R}^m)$, Q-periodic with $Du(x) \in \mathcal{L}$ we have that $\tilde{u}(x) = u(Rx)$ is in $W^{1,2}(Q_0; \mathbb{R}^m)$, Q_0-periodic. Also $D\tilde{u} = Du(Rx)R$ so $D\tilde{u} \in \tilde{\mathcal{L}}$ where $\tilde{\mathcal{L}} = \{\tilde{A} \in \mathbb{M}^{m\times n} \,|\, \tilde{A} = AR,\ A \in \mathcal{L}\}$. Since $\mathcal{L}$ doesn't contain rank one matrices it follows that $\tilde{\mathcal{L}}$ has the same property. Thus $\tilde{u}$ must be constant and therefore u is constant as well. $\square$

2.2 Examples

In this section we are going to discuss particular cases of linear subspaces $\mathcal{L}$ and some aspects related to the restricted rank one convexity and quasiconvexity.

Example 1. Consider $\mathcal{L} = \{\left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right) \,|\, a, b \in \mathbb{R}\}$ and let $f : \mathcal{L} \to \mathbb{R}$ a $\mathcal{L}$-rank one convex function. We show that f must be Q_0-$\mathcal{L}$-quasiconvex.

Given $u \in W^{1,\infty}(\Omega;\mathbb{R}^m)$, $u(x,y) = (u^1(x,y), u^2(x,y)$ with $Du \in \mathcal{L}$ it implies

$$\partial_x u^1 = \partial_y u^2.$$

Thus u^1 and u^2 satisfy the wave equation i.e.

$$\partial_{xx} u^1 - \partial_{yy} u^1 = 0$$

$$\partial_{xx} u^2 - \partial_{yy} u^2 = 0$$

and we get

$$u^1(x,y) = h(x+y) - g(x-y)$$

$$u^2(x,y) = h(x+y) + g(x-y)$$

where $h, g : \mathbb{R} \to \mathbb{R}$, absolutely continuous. If u is assumed to be Q_0 periodic it folows that h and g are periodic of period 1. Indeed, $u^1(x,y) = u^1(x+1,y)$ so $h(x+y+1) - h(x+y) = g(x-y) - g(x-y+1)$ for any $x, y \in \mathbb{R}$. It implies that $g(t) - g(t+1) = g(0) - g(1)$ for any $t \in \mathbb{R}$ since if two absolutely continuous functions α and β verify $\alpha(x+y) = \beta(x-y)$ for any x, y it follows that they must be constant. Thus we get that

$$g(1) - g(k+1) = (g(0) - g(1))k$$

for any positive integer k. Since g has to be bounded, we get $g(0) - g(1) = 0$ and thus $g(t) - g(t+1) = 0$ for any $t \in \mathbb{R}$.

Let $F : \mathbb{R}^2 \to \mathbb{R}$ defined as $F(a,b) = f\left(\begin{smallmatrix} a+b & a-b \\ a-b & a+b \end{smallmatrix}\right)$. Since f is $\mathcal{L}$-rank one convex, we have that F is separately convex in each variable and

$$f\left(\begin{pmatrix} c & d \\ d & c \end{pmatrix} + \begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix}\right) = F\left(\frac{c+d}{2} + a, \frac{c-d}{2} + b\right) \tag{2.6}$$

Now we prove that f is Q_0-$\mathcal{L}$-quasiconvex. Making the substitution $\xi = x + y$ and $\eta = x - y$ we get

$$\iint_{Q_0} f\left(\begin{pmatrix} c & d \\ d & c \end{pmatrix} + Du\right) dx dy = \frac{1}{2}\int_0^1 \left(\int_{-\xi}^{\xi} F\left(\frac{c+d}{2} + h'(\xi), \frac{c-d}{2} + g'(\eta)\right) d\eta \right) d\xi + \\ + \frac{1}{2}\int_1^2 \left(\int_{2-\xi}^{\xi-2} F\left(\frac{c+d}{2} + h'(\xi), \frac{c-d}{2} + g'(\eta)\right) d\eta \right) d\xi$$

Now using the fact that F is separately convex and Jenssen's inequality we get:

$$\int_0^1 \left(\xi F\left(h'(\xi), \frac{g(\xi) - g(-\xi)}{2\xi}\right) + (1-\xi) F\left(h'(\xi), \frac{g(-\xi) - g(\xi)}{2\xi}\right)\right) d\xi \geq \\ \int_0^1 F(h'(\xi), 0) d\xi \geq F\left(\frac{c+d}{2}, \frac{c-d}{2}\right) = f\left(\begin{pmatrix} c & d \\ d & c \end{pmatrix}\right)$$

and it follows

$$\iint_{Q_0} f\left(\begin{pmatrix} c & d \\ d & c \end{pmatrix} + Du\right) dx dy \geq f\left(\begin{pmatrix} c & d \\ d & c \end{pmatrix}\right)$$

hence f is Q_0-$\mathcal{L}$-quasiconvex.

Example 2. We show that Q_0-$\mathcal{L}$-quasiconvexity might not imply $\mathcal{L}$-rank one convexity.

Let $\mathcal{L} = \{\left(\begin{smallmatrix} a & b\sqrt{2} \\ b & a\sqrt{2}\end{smallmatrix}\right) | \, a, b \in \mathbb{R}\}$ a linear subspace of $\mathbb{R}^{2\times 2}$. If $u \in W^{1,\infty}(Q_0; \mathbb{R}^m)$ satisfies $Du \in \mathcal{L}$ it follows that

$$2\partial_{xx}u^1 - \partial_{yy}u^1 = 0 \tag{2.7}$$

which implies that there exist $h, g : \mathbb{R} \to \mathbb{R}$ such that

$$u^1(x, y) = h(x + y\sqrt{2}) + g(x - y\sqrt{2}) \tag{2.8}$$

Also, $u^1(x, y)$ is Q_0 periodic so we get, by reasoning as before, that h and g are periodic with periods 1 and $\sqrt{2}$. Since $\sqrt{2}$ is irrational and the set $\{k\sqrt{2}+p | \, k, p \in \mathbb{Z}\}$ is dense in $\mathbb{R}$ it follows that h and g must be constant [La]. Therefore, by definition, every function is Q_0-$\mathcal{L}$-quasiconvex, but not necessarily $\mathcal{L}$-rank one convex (see Example 1).

Example 3. We show that $\mathcal{L}$-rank one convexity does not imply $\mathcal{L}$-quasiconvexity. The following famous example belongs to Šverák [Sv 1].

Let

$$\mathcal{L} = \{ \begin{pmatrix} a & 0 \\ 0 & b \\ c & c \end{pmatrix}, \; a, b, c \in \mathbb{R}\} \tag{2.9}$$

a linear subspace of $\mathbb{M}^{3\times 2}$. Also let $f : \mathcal{L} \to \mathbb{R}$ be defined by

$$f\left(\begin{pmatrix} a & 0 \\ 0 & b \\ c & c \end{pmatrix}\right) = -abc \tag{2.10}$$

We notice that the only rank-one directions in $\mathcal{L}$ are given by

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

and the function f is convex on each rank-one line contained in $\mathcal{L}$. Consider the function $u : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$u(x, y) = \frac{1}{2\pi}\begin{pmatrix} \sin(2\pi x) \\ \sin(2\pi y) \\ \sin(2\pi(x+y)) \end{pmatrix}$$

We have that $u \in W^{1,\infty}(Q_0; \mathbb{R}^3)$ where $Q_0 = [0,1]^2$, u is Q_0-periodic and $Du \in \mathcal{L}$ since

$$Du(x, y) = \begin{pmatrix} \cos(2\pi x) & 0 \\ 0 & \cos(2\pi y) \\ \cos(2\pi(x+y)) & \cos(2\pi(x+y)) \end{pmatrix}$$

Thus we get

$$\iint_{Q_0} f(Du(x,y))dxdy = -\iint_{Q_0} (\cos(2\pi x))^2(cos(2\pi y))^2 dxdy < 0 = f(0_{3\times 2}) \quad (2.11)$$

which shows that f is not $\mathcal{L}$-quasiconvex.

Now we generalize the Example 3 to the case where some function $f : \mathcal{L} \to \mathbb{R}$ which is $\mathcal{L}$-rank one convex but not Q_0-$\mathcal{L}$-quasiconvex can be extended to the entire space $\mathbb{M}^{\mathrm{m}\times\mathrm{n}}$ and preserve this property.

Theorem 2.3. *Let $f : \mathcal{L} \to \mathbb{R}$ be a function which is $\mathcal{L}$-rank one but it is not $\mathcal{L}$-quasiconvex. Also assume that f is C^2 and for some $p \geq 2$:*

$$|f(A)| \leq c(1 + |A|^p) \quad (2.12)$$

$$|D^2 f(A)| \leq c(1 + |A|^{p-2}). \quad (2.13)$$

forall $A \in \mathcal{L}$. Then there exists an function $F : \mathbb{M}^{\mathrm{m}\times\mathrm{n}} \to \mathbb{R}$ which is rank one convex but not quasiconvex on $\mathbb{M}^{\mathrm{m}\times\mathrm{n}}$.

Proof. Since f is not $\mathcal{L}$-quasiconvex it exists a cube $Q = RQ_0$ and $u \in W^{1,\infty}(Q;\mathbb{R}^m)$, Q-periodic with $Du(x) \in \mathcal{L}$ such that

$$f(0) > \int_Q f(Du(x))dx \quad (2.14)$$

Let $F_{\epsilon,k} : \mathbb{M}^{m\times n} \to \mathbb{R}$ with

$$F_{\epsilon,k}(X) = f(PX) + \epsilon|X|^2 + \epsilon|X|^{p+1} + k|X - PX|^2. \tag{2.15}$$

Here P is the projection onto $\mathcal{L}$. Let $A, Y \in \mathbb{M}^{m\times n}$ arbitrary such that $\operatorname{rank} Y = 1$, $|Y| = 1$ and let $h_{\epsilon,k} = F_{\epsilon,k}(A + tY)$. We are going to prove that for every $\epsilon > 0$ it exists k such that $F_{\epsilon,k}$ is $\mathcal{L}$-rank one convex. To show this it is enough to prove that $h''_{\epsilon,k} \geq 0$.

Thus, now we prove that:

$$\frac{d^2}{dt^2} F_{\epsilon,k}(A + tY)\bigg|_{t=0} \geq 0 \tag{2.16}$$

for any matrices $A, Y \in \mathbb{M}^{m\times n}$ with $\operatorname{rank} Y = 1$, $|Y| = 1$. We have:

$$|A + tY|^{p+1} = (|A + tY|^2)^{\frac{p+1}{2}} = (|A|^2 + 2t < Y, A > +t^2)^{\frac{p+1}{2}} \tag{2.17}$$

$$\frac{d}{dt}|A + tY|^{p+1} = (p+1)(|A|^2 + 2t < Y, A > +t^2)^{\frac{p+1}{2}}(< Y, A > +t) \tag{2.18}$$

Thus we get

$$\frac{d^2}{dt^2}|A + tY|^{p+1}\bigg|_{t=0} = (p+1)(p-1)|A|^{p-3} < Y, A >^2 + (p+1)|A|^{p-1} \tag{2.19}$$

and

$$\frac{d^2}{dt^2}F_{\epsilon,k}(A+tY)\bigg|_{t=0} = \frac{d^2}{dt^2}f(PA+tPY)\bigg|_{t=0} + 2\epsilon + \epsilon(p+1)|A|^{p-1}$$
$$+ \ \epsilon(p+1)(p-1)|A|^{p-3} < Y, A >^2 + k|Y - PY|^2$$

Now, from (2.15), we have

$$\frac{d^2}{dt^2}f(PA+tPY)\bigg|_{t=0} \geq -c(1+|A|^{p-2}) \tag{2.20}$$

and

$$\frac{d^2}{dt^2}F_{\epsilon,k}(A+tY)\bigg|_{t=0} \geq -c(1+|A|^{p-2}) + \epsilon(p+1)|A|^{p-1} + 2\epsilon + 2k|Y-PY|^2 \tag{2.21}$$

Assume by contradiction that it exists ϵ_0 such that for every positive integer k we get A^k, Y^k satisfying

$$0 > \frac{d^2}{dt^2}F_{\epsilon,k}(A+tY)\bigg|_{t=0} \tag{2.22}$$

From (2.21) it follows that A^k is bounded and by extracting a subsequence we have $A^k \to \bar{A}$ and $Y^k \to \bar{Y} = P\bar{Y}$ as $k \to \infty$. Thus, passing to the limit in (2.21),

$$-\epsilon > \frac{d^2}{dt^2}f(\bar{A}+t\bar{Y})\bigg|_{t=0} \tag{2.23}$$

a contradiction with the fact that f is $\mathcal{L}$-rank one convex.

Now we can also choose ϵ small such that

$$F_{\epsilon,k}(0) > \int_Q F_{\epsilon,k}(Du(x))dx \tag{2.24}$$

where u is given in (2.14). Hence $F_{\epsilon,k}$ is not $\mathcal{L}$-quasiconvex. $\square$

2.3 Constant dimension condition

Let $\lambda \in \mathbb{R}^n$ and $R_{\mathcal{L}}^{\lambda} = \{w \in \mathbb{R}^m | w \otimes \lambda \in \mathcal{L}\}$. We notice that $R_{\mathcal{L}}^{\lambda}$ is a linear subspace of $\mathbb{R}^m$.

Definition 2.4. We say that the subspace $\mathcal{L}$ satisfies the *constant dimension condition* if the related subspace $R_{\mathcal{L}}^{\lambda}$ has the same dimension for all $\lambda \in \mathbb{R}^n \setminus \{0\}$.

If $\mathcal{L}$ satisfies the constant dimension condition we shall prove the equivalence between Q_0-$\mathcal{L}$-quasiconvexity and the weak lower semicontinuity of the functional

$$I_{\Omega}(u) = \int_{\Omega} f(Du)dx$$

along sequences satisfying the linear restriction $PDu_k(x) \to 0$ almost every x.

Remark 1. If $m = n = 2$ and $\mathcal{L}$ is the linear subspace of 2×2 symmetric matrices then the dimension of R_L^{λ} is constantly 1 for all $\lambda \in \mathbb{R}^2 \setminus \{0_2\}$.

Proof. We have that $\mathcal{L} = \{\left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right) |\ a, b, c \in \mathbb{R}\}$ and

$$R_L^{\lambda} = \{w = (w_1, w_2) \in \mathbb{R}^2 |\ w_1\lambda_2 = w_2\lambda_1\}.$$

Clearly the dimension of $R_{\mathcal{L}}^{\lambda}$ is 1 for any $\lambda \in \mathbb{R}^2 \setminus \{0_2\}$ $\square$

Lemma 2.4. *If $\mathcal{L}$ satisfies the constant dimension condition there exists $\gamma > 0$ such that for any $a \in (R_{\mathcal{L}}^{\lambda})^{\perp}$ and $\lambda \in \mathbb{R}^n \setminus \{0\}$ we have:*

$$|P(\lambda \otimes a)| \geq \gamma |\lambda \otimes a| \tag{2.25}$$

Proof. Assume by contradiction that

$$\min_{|\lambda|=1, |a|=1} |P(\lambda \otimes a)| = 0. \tag{2.26}$$

Then there exists a minimizing sequence $\lambda_j \to \bar{\lambda}$ and $a_j \to \bar{a}$. Let $k = \dim R_{\mathcal{L}}^{\lambda}$. For ϵ small enough and any λ such that $|\lambda - \bar{\lambda}| < \epsilon$ there exists a set $w_1(\lambda), w_2(\lambda), ...w_k(\lambda)$ of linearly independent vectors of $R_{\mathcal{L}}^{\lambda}$ and $\lim_{\lambda \to \bar{\lambda}} w_i(\lambda) = w_i(\bar{\lambda})$, for all i, $1 \leq i \leq k$. Since $a_j \in (R_{\mathcal{L}}^{\lambda})^{\perp}$, it implies that $\langle a_j, w_i(\lambda_j) \rangle = 0$ for all i, $1 \leq i \leq k$. We get $\langle \bar{a}, w_i(\bar{\lambda}) \rangle = 0$ so $\bar{a} \in (R_{\mathcal{L}}^{\lambda})^{\perp}$. Also, since $P(\bar{\lambda} \otimes \bar{a}) = 0$ it implies $\bar{a} \in R_{\mathcal{L}}^{\lambda}$. Thus $\bar{a} = 0$, in contradiction with $|\bar{a}| = 1$.

$\square$

First we shall prove the selection theorem:

Theorem 2.5. *Let Q a cube in $\mathbb{R}^n$ and $u \in W^{1,2}(Q; \mathbb{R}^m)$ a Q-periodic function. If the linear subspace $\mathcal{L}$ satisfies the constant dimension condition then for every $\epsilon > 0$ there exists a selection v_ϵ, $v_\epsilon \in C^\infty$ a Q-periodic function such that $Dv_\epsilon(x) \in L$ a.e.*

$x \in Q$ and

$$\|Du - Dv_\epsilon\|_{L^2(Q)} \leq \|PDu\|_{L^2(Q)} + \epsilon \tag{2.27}$$

Proof. First we assume that $Q = Q_0$. Let $\Lambda = \mathbb{Z}^n$ be the unit lattice, i.e. the additive group of points in $\mathbb{R}^n$ with integer coordinates. Since u is Q-periodic we can expand u as a Fourier series,

$$u(x) = \sum_{\lambda \in \Lambda} \hat{u}(\lambda) e^{2\pi i \lambda x}.$$

Thus $Du(x) = \sum_{\lambda \in \Lambda} \hat{u}(\lambda) \otimes \lambda e^{2\pi i \lambda x}$. Let $\hat{v}(\lambda) = \mathbb{P}_{R^\lambda_\mathcal{L}} \hat{u}(\lambda)$, projection of both real part and imaginary part of $\hat{u}(\lambda)$ onto $R^\lambda_\mathcal{L}$. By Riesz-Fischer theorem we have that

$$v(x) = \sum_{\lambda \in \Lambda} \hat{v}(\lambda) e^{2\pi i \lambda x} \tag{2.28}$$

is a function in $W^{1,2}(Q)$, Q-periodic and its gradient belong to $\mathcal{L}$ almost every x. Applying Lemma (2.4) for $a = \hat{u}(\lambda) - \hat{v}(\lambda)$ we get $\|Du - Dv\|_2 \leq \|PDu\|_2$.

Now we can consider $v_\epsilon(x)$ as the real part of $\sum_{\lambda \in \Lambda'} \hat{v}(\lambda) e^{2\pi i \lambda x}$ where is Λ' is a finite subset of Λ such that

$$\|Dv_\epsilon - Dv\|_{L^2} < \epsilon \tag{2.29}$$

since the imaginary part of $\sum_{\lambda \in \Lambda'} \hat{v}(\lambda) \otimes \lambda e^{2\pi i \lambda x}$ converges to $0_{m \times n}$ as $\Lambda' \nearrow \Lambda$.

Now if the cube Q is arbitrary then $Q = SQ_0$ for some $a \in \mathbb{R}^n$ and a rotation S. Let $\tilde{\mathcal{L}} = \{\tilde{A} \in \mathbb{M}^{m \times n} \,|\, \tilde{A} = AS,\ A \in \mathcal{L}\}$ and $\tilde{P}$ the orthogonal projection onto $\tilde{\mathcal{L}}$.

Define $\tilde{u} : Q_0 \to \mathbb{R}^m$ by $\tilde{u}(x) := u(Sx)$. Also notice that

$$R^{\lambda}_{\tilde{\mathcal{L}}} = R^{S\lambda}_{\mathcal{L}}$$

and therefore $R^{\lambda}_{\tilde{\mathcal{L}}}$ has constant dimension for any $\lambda \in \mathbb{R}^n$. Thus we can select $\tilde{v}$ such that $\tilde{v} \in C^{\infty}(Q_0)$, Q_0-periodic and

$$\|D\tilde{u} - D\tilde{v}_{\epsilon}\|_{L^2(Q_0)} \leq \|\tilde{P}D\tilde{u}\|_{L^2(Q_0)} + \epsilon \tag{2.30}$$

For each $x \in Q$ there exists a unique $\bar{x} \in Q_0$ such that $x = S\bar{x}$. Let $v : Q \to \mathbb{R}^m$ with $v_{\epsilon}(x) = \tilde{v}_{\epsilon}(S^T(x))$. We notice that v_{ϵ} satisfies the requirement of the lemma.

$\square$

2.4 $\mathcal{L}$-weak lower semicontinuity

Let $f : \mathbb{M}^{m\times n} \to \mathbb{R}$ satisfy the growth condition

$$|f(A)| \leq c(1 + |A|^2) \tag{2.31}$$

for any matrix $A \in \mathbb{M}^{m\times n}$ and consider the integral operator

$$I_{\Omega}(u) = \int_{\Omega} f(Du)dx \tag{2.32}$$

where Ω is open bounded domain with Lipschitz boundary and $u \in W^{1,2}(\Omega;\mathbb{R}^m)$.

In contrast to Example 2 in section 2.2 we show that under the constant dimension condition Q_0-$\mathcal{L}$-quasiconvexity implies $\mathcal{L}$-rank one convexity.

Theorem 2.6. *Assume that the linear subspace $\mathcal{L}$ satisfies the constant dimension condition. If a continuous function $f : \mathbb{M}^{m\times n} \longrightarrow \mathbb{R}$ satisfies the growth condition* (2.31) *and is Q_0-$\mathcal{L}$-quasiconvex then it is also $\mathcal{L}$-rank one convex.*

Proof. Let $A, B \in \mathcal{L}$ be such that $\operatorname{rank}(A - B) \leq 1$ and $\lambda \in [0,1]$. For any integer k there exists $Q_1^k, Q_2^k \subset Q_0$, $Q_1^k \cap Q_2^k = \emptyset$ and $\varphi_k \in W_0^{1,\infty}(Q_0, \mathbb{R}^m)$ such that

$$\begin{cases} |\mu(Q_1^k) - \lambda| \leq \frac{1}{k} \\ |\mu(Q_2^k) - (1-\lambda)| \leq \frac{1}{k} \\ D\varphi_k(x) = \begin{cases} (1-\lambda)(A-B) & \text{if } x \in Q_1^k \\ -\lambda(A-B) & \text{if } x \in Q_2^k \end{cases} \\ \|D\varphi_k\|_\infty \leq const(A,B) \end{cases}$$

since $\mu(Q_0) = 1$. (See [Da]). We extend the φ_k to be Q_0-periodic on $\mathbb{R}^n$. From these properties we also have that $PD\varphi_k \longrightarrow 0$ in $L^2(Q_0)$. Thus, by Theorem 2.5, for any ϵ we can find a selection $u_{k,\epsilon} \in W^{1,\infty}(Q_0, \mathbb{R}^m)$, Q_0-periodic such that $Du_{k,\epsilon} \in \mathcal{L}$ and

$$\|Du_{k,\epsilon} - D\varphi_k\|_{L^2(Q_0)} \longrightarrow 0$$

as $\epsilon \longrightarrow 0$ and it follows

$$
\begin{aligned}
\liminf_{k\to\infty,\epsilon\to 0}\int_{Q_0} f(\lambda A+(1-\lambda B)+Du_{k,\epsilon})dx &= \liminf_{k\to\infty}\int_{Q_0} f(\lambda A+(1-\lambda B)+D\varphi_k)dx \\
&= \lambda f(A)+(1-\lambda)f(B)
\end{aligned}
$$

Since f is Q_0-$\mathcal{L}$-quasiconvex we have

$$\int_{Q_0} f(\lambda A+(1-\lambda B)+Du_{k,\epsilon})dx \geq f(\lambda A+(1-\lambda B)) \tag{2.33}$$

for any k and ϵ. Taking $\liminf$ over k and ϵ for the left hand side of the previous inequality we obtain

$$\lambda f(A)+(1-\lambda)f(B) \geq f(\lambda A+(1-\lambda B))$$

which proves that f is $\mathcal{L}$-rank one convex. □

Definition 2.5. Let f and I_Ω be defined as above. We say that the functional I_Ω is *$\mathcal{L}$-weakly lower semicontinuous* on $W^{1,2}(\Omega;\mathbb{R}^m)$ if for any sequence $u_k \rightharpoonup u$ in $W^{1,2}(\Omega;\mathbb{R}^m)$ with $\|PDu_k\|_{L^2(\Omega)} \to 0$ as $k\to\infty$, we have

$$I_\Omega(u) \leq \liminf_{k\to\infty} I_\Omega(u_k) \tag{2.34}$$

Theorem 2.7. *If the functional I_Ω is $\mathcal{L}$-weakly lower semicontinuous then the functional f is $\mathcal{L}$-quasiconvex.*

Proof. Let $Q = RQ_0$, $A \in \mathcal{L}$ arbitrary and $u \in W^{1,\infty}(Q;\mathbb{R}^m)$, Q-periodic with $Du(x) \in \mathcal{L}$ for almost every x. We show that

$$\int_Q f(A + Du(x))dx \geq f(A) \tag{2.35}$$

assuming that I is $\mathcal{L}$-weakly lower semicontinuous. For any test function φ we have

$$\int_Q Du(kx)\varphi(x)dx = \int_{Q_0} Du(kR\tilde{x})\varphi(\tilde{x})d\tilde{x}$$

Thus, by Riemann-Lebesgue theorem, we have that

$$\lim_{k\to\infty}\int_Q Du(kx)\varphi(x)dx = \int_{Q_0} Du(R\tilde{x} = \int_Q Du(x)dx \tag{2.36}$$

Let $u_k(x) = \frac{1}{k}u(kx) + Ax$. We notice that $Du_k(x) = Du(kx) + A$ and $Du_k(x) \in \mathcal{L}$ for any k and almost every x. We have that

$$Du_k \overset{*}{\rightharpoonup} A \tag{2.37}$$

and also

$$\int_Q f(A + Du(x))dx = k^n \int_{\frac{1}{k}Q} f(A + Du(kx))dx. \tag{2.38}$$

For k sufficiently large there exists p_k cubes, $Q_1, Q_2, ...Q_{p_k}$, which are translates of

$\frac{1}{k}Q$ by multiples of $\frac{1}{k}$, muttually disjoint, such that

$$\bigcup_{i=1}^{p_k} Q_i \subset \Omega \text{ and } \mu(\Omega \setminus \bigcup_{i=1}^{p_k} Q_i) < \epsilon_k \tag{2.39}$$

where $\epsilon_k \to 0$ as $k \to \infty$. Thus we also get that $\frac{p_k}{k^n} \to \mu(\Omega)$ as $k \to \infty$.

Since I is $\mathcal{L}$-weakly lower semicontinuous it follows:

$$\liminf_{k\to\infty} \int_\Omega f(Du_k(x))dx \geq f(A)\mu(\Omega) \tag{2.40}$$

Also, from (2.38) we get

$$\begin{aligned} \int_\Omega f(Du_k(x))dx &= p_k \int_{\frac{1}{k}Q} f(A + Du(kx))dx + \int_{\Omega\setminus\bigcup_{i=1}^{p_k} Q_i} f(A + Du(kx))dx \\ &= \frac{p_k}{k^n} \int_Q f(A + Du(x))dx + \epsilon_k C \end{aligned}$$

Letting $k \to \infty$ we have $\mu(\Omega) \int_Q f(A + Du(x))dx \geq f(A)\mu(\Omega)$ and after dividing by $\mu(\Omega)$ we obtain what we had to prove. $\square$

Next we show under the constant dimension condition the $\mathcal{L}$-quasiconvexity is always sufficient for the $\mathcal{L}$-weak lower semicontinuity.

Theorem 2.8. *If the linear subspace $\mathcal{L}$ satisfies the constant dimension condition and if the function f is bounded from below, satisfies the growth condition 2.31 and is Q_0-$\mathcal{L}$-quasiconvex then functional I_Ω is L-weakly lower semicontinuous on $W^{1,2}(\Omega; \mathbb{R}^m)$.*

Proof. Let $u_k \in W^{1,2}(\Omega, \mathbb{R}^m)$ such that $u_k \rightharpoonup u$ in $W^{1,2}$ and $PDu_k \to 0$ in L^2. We assume that Du_k generates a parametrized Young measure $(\nu_x)_{x\in\Omega}$. Then

$$Du(x) = \int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda)$$

By Theorem 1.3 we also have that

$$\liminf_k \int_\Omega f(Du_k(x))dx \geq \int_\Omega \int_{\mathbb{M}^{m\times n}} f(\lambda)d\nu_x(\lambda)dx \tag{2.41}$$

For our purpose it would be sufficient to show

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} f(\lambda)d\nu_x(\lambda)dx \geq \int_\Omega f(Du(x))dx \tag{2.42}$$

Now we actually prove

$$\int_{\mathbb{M}^{m\times n}} f(\lambda)d\nu_a(\lambda) \geq f\Big(\int_{\mathbb{M}^{m\times n}} \lambda d\nu_a(\lambda)\Big) = f(Du(a)). \tag{2.43}$$

for almost every $a \in \Omega$.

By Theorem 1.4 we have that ν_a is also a gradient Young measure for almost every $a \in \Omega$. Consider a cube $Q \subset \Omega$ such that $a \in Q$. There exists $w_k \in W^{1,2}(Q)$ such that Dw_k generates ν_a and $w_k \to \bar{w}$ in L^2, by the Sobolev embedding. Also we get that $Dw_k \rightharpoonup Du(a) = D\bar{w}$ and by the fundamental theorem of Young measures $PDw_k \to 0$ in $L^2(Q)$.

Let $\varphi_j \in C_0^\infty(Q)$ such that $\varphi_j \nearrow 1$ uniformly and $v_{k,j} = \varphi_j(w_k - \bar{w})$. Since

$w_k \to \bar{w}$ in L^2 for each j there exists k_j such that

$$\|D\varphi_j \otimes (w_{k_j} - \bar{w})\|_{L^2(Q)} < \frac{1}{j}.$$

Thus we can select a subsequence of $v_{k,j}$ which we can conveniently denote by v_k and we have $v_k \in W_0^{1,2}(Q)$ and

$$\|Dv_k - D(w_k - \bar{w})\|_{L^2(Q)} \to 0 \tag{2.44}$$

By using Theorem 2.5, we can select $\tilde{v}_k \in C^\infty(Q)$, Q periodic such that $\|D\tilde{v}_k - Dv_k\|_{L^2(Q)} \to 0$ in $L^2(Q)$ and $D\tilde{v}_k(x) \in \mathcal{L}$ almost every x. So we have

$$\liminf_k \int_Q f(Du(a) + D(w_k(x) - \bar{w}(x))dx = \liminf_k \int_Q f(Du(a) + D\tilde{v}_k(x))dx$$

Also since f is $\mathcal{L}$-quasiconvex

$$\frac{1}{|Q|}\int_Q f(Du(a) + D\tilde{v}_k(x))dx \geq f(Du(a))dx$$

Thus it follows that

$$\frac{1}{|Q|}\liminf_k \int_Q f(Du(a) + D(w_k(x) - \bar{w}(x))dx = \int_{\mathbb{M}^{m\times n}} f(\lambda)d\nu_a(\lambda) \geq f(Du(a))$$

This completes the proof. $\square$

2.5 Particular case without the constant dimension condition

Consider the linear subspace $\mathcal{L} = \{ \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) | \, a, b \in \mathbb{R}\}$. We notice that the subspace

$$R^{\lambda}_{\mathcal{L}} = \{w \in \mathbb{R}^2 | w \otimes \lambda \in \mathcal{L}\}$$

does not have constant dimension for all $\lambda \in \mathbb{R}^2 \setminus \{0\}$. Therefore this space $\mathcal{L}$ does not satisfy the constant dimension condition defined above.

Let $f : \mathbb{M}^{2\times 2} \to \mathbb{R}$ be a C^1 function satisfying

$$0 \leq f(\xi) \leq c(1 + |\xi|^2) \tag{2.45}$$

$$|Df(\xi)| \leq c(1 + |\xi|) \tag{2.46}$$

Also, as above, define

$$I_{\Omega}(u) = \int_{\Omega} f(Du)dx \tag{2.47}$$

Theorem 2.9. *If $f : \mathbb{M}^{2\times 2} \to \mathbb{R}$ satisfies* (2.45) *and* (2.46) *and is $\mathcal{L}$-rank one convex then I_{Ω} is $\mathcal{L}$-weakly lower semicontinuous on $W^{1,2}(\Omega; \mathbb{R}^2)$.*

The following result by Müller is going to be essential in the course of the proof.

Theorem 2.10 ([Mu 3]). *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a separately convex function that satisfies*

$$0 \leq f(\xi) \leq C(1 + |\xi|^2).$$

Let $\Omega \subset \mathbb{R}^2$ be open and suppose that

$$u_k \rightharpoonup u, \quad v_k \rightharpoonup v \ in\, L^2_{loc}(\Omega) \tag{2.48}$$

$$\partial_y u_k \to \partial_y u, \quad \partial_x v_k \to \partial_x v, \ in\, H^{-1}_{loc}(\Omega). \tag{2.49}$$

Then we have

$$\liminf_{k\to\infty} \int_\Omega f(u_k, v_k)dz \geq \int_\Omega f(u,v)dz. \tag{2.50}$$

Now we are going to prove the Theorem 2.9.

Proof. Let $u_k \in W^{1,2}(\Omega;\mathbb{R}^2)$ with $u_k \rightharpoonup u$ and $PDu_k \to 0$ almost everywhere. Thus we have that $\partial_y u_k^1 \to 0$ and $\partial_x u_k^2 \to 0$ so $\partial_x(\partial_y u_k^1) \to 0$ and $\partial_y(\partial_x u_k^2) \to 0$ in $H^{-1}(\Omega)$.

Let $F : \mathcal{L} \to \mathbb{R}$ given by $F(a,b) = f(\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right))$. Since f is $\mathcal{L}$-rank one convex it follows that F is separately convex and satisfies the growth condition from Theorem 2.10. From (2.45) and (2.46) we also have that

$$|f(\xi) - f(\eta)| \leq c(1 + |\xi| + |\eta|)(\xi - \eta) \tag{2.51}$$

By using this inequality with $\xi = Du_k$ and $\eta = \left(\begin{smallmatrix} \partial_x u_k^1 & 0 \\ 0 & \partial_y u_k^2 \end{smallmatrix}\right)$ we get

$$\liminf_{k\to\infty} \int_\Omega f(Du_k)dz = \liminf_{k\to\infty} \int_\Omega f(\begin{pmatrix} \partial_x u_k^1 & 0 \\ 0 & \partial_y u_k^2 \end{pmatrix})dz. \tag{2.52}$$

From the Theorem 2.10 we obtain

$$\int_\Omega F(\partial_x u^1, \partial_y u^2)dz \leq \liminf_{k\to\infty} \int_\Omega F(\partial_x u_k^1, \partial_y u_k^2)dz \tag{2.53}$$

and since $\partial_y u^1 = 0$ and $\partial_x u^2 = 0$ we finally get

$$\int_\Omega f(Du)dz \leq \liminf_{k\to\infty} \int_\Omega f(Du_k)dz$$

□

Remark 2. From Theorem 2.9 and Theorem 2.7 for this $\mathcal{L}$, every $\mathcal{L}$-rank one convex function is $\mathcal{L}$-quasiconvex but it may very difficult to prove this directly.

Chapter 3

Nonlinear Restrictions of Palais-Smale Type

3.1 Introduction

In this chapter we investigate the weak lower semicontinuity for C^1 functionals defined as

$$I(u) = \int_\Omega f(x, u(x), Du(x))dx$$

from the perspective of a nonlinear constraint of Palais-Smale type. This requires that f be C^1 in (s, ξ) and the derivatives satisfy some growth conditions. When minimizing a smooth and bounded-below functional I over a Banach space, an important variational principle was discovered by Ekeland [Ek] in the 1970's. Applying this principle to the minimization problem for our functional I over a Dirichlet class $\mathcal{A}_g$ in $W^{1,p}(\Omega; \mathbb{R}^m)$, we can always obtain a minimizing sequence $\{u_k\}$ in $\mathcal{A}_g$ which satisfies $I'(u_k) \to 0$ in $W^{-1,p'}(\Omega; \mathbb{R}^m)$. Here, we assume $p > 1$ and $p' = \frac{p}{p-1}$, and $W^{-1,p'}(\Omega; \mathbb{R}^m)$ denotes the dual space of $W_0^{1,p}(\Omega; \mathbb{R}^m)$. Consequently, the weak limit (if exists) of any such minimizing sequence will be an energy minimizer provided that $I(u)$ only satisfies the condition:

$$I(u) \leq \liminf_{k\to\infty} I(u_k) \text{ whenever } \begin{cases} u_k \rightharpoonup u \text{ in } W^{1,p}(\Omega; \mathbb{R}^m) \text{ and} \\ \\ I'(u_k) \to 0 \text{ in } W^{-1,p'}(\Omega; \mathbb{R}^m). \end{cases} \tag{3.1}$$

In this case, we say that the functional $I(u)$ is *restricted weakly lower semicontinuous* on $W^{1,p}(\Omega; \mathbb{R}^m)$. If the restricted lower semicontinuity condition (3.1) holds only for all u_k, u in the Dirichlet class $\mathcal{A}_g$, we then say I is restricted weakly lower semicontinuous on $\mathcal{A}_g$. Since the sequences $\{u_k\}$ with bounded $I(u_k)$ satisfying $I'(u_k) \to 0$ are usually called the Palais-Smale sequences [Ra] for the given functional $I(u)$, we shall say that a sequence $\{u_k\}$ *(PS) weakly converges to* u (with respect to I) and denote by $u_k \overset{\text{ps}}{\rightharpoonup} u$ in $W^{1,p}$ if it satisfies $u_k \rightharpoonup u$ in $W^{1,p}$ and $I'(u_k) \to 0$ in $W^{-1,p'}$.

As we shall see later, this restricted weak lower semicontinuity imposes some intrinsic property on the function f. Although in the certain cases as presented in this paper the restricted weak lower semicontinuity is equivalent to the usual weak lower semicontinuity of the functional, in general, when f depends on x and s, we also give some examples to show that the restricted weak lower semicontinuity of I is be equivalent to the usual weak lower semicontinuity (see Proposition 3.7).

3.2 The (PS)-weak lower semicontinuity

In this section, we assume $f(x, s, \xi)$ is measurable in $x \in \Omega$ for all $(s, \xi) \in \mathbb{R}^n \times \mathbb{M}^{m\times n}$ and is C^1 in $(s, \xi) \in \mathbb{R}^n \times \mathbb{M}^{m\times n}$ for almost every $x \in \Omega$. We also assume $1 < p < \infty$ and f satisfies the growth conditions

$$|f(x, s, \xi)| \leq c_1(|s|^p + |\xi|^p) + A(x), \tag{3.2}$$

$$|D_s f(x, s, \xi)| + |D_\xi f(x, s, \xi)| \leq c_2(|s|^{p-1} + |\xi|^{p-1}) + B(x) \tag{3.3}$$

for almost every $x \in \Omega$ and for all $s \in \mathbb{R}^n$, $\xi \in \mathbb{M}^{m\times n}$, where c_1, c_2 are positive constants and A, B are positive functions with $A \in L^1(\Omega)$, $B \in L^{\frac{p}{p-1}}(\Omega)$.

From these assumptions, we can obtain the following result:

Proposition 3.1. *Under the above conditions, the functional I defined above is a C^1 functional on $W^{1,p}(\Omega; \mathbb{R}^m)$ and for each u the Fréchet derivative $I'(u)$ is given by*

$$\langle I'(u), v\rangle = \int_\Omega \big[D_s f(x, u, Du) \cdot v + D_\xi f(x, u, Du) \colon Dv\big]\, dx$$

for all $v \in W^{1,p}(\Omega; \mathbb{R}^m)$.

When minimizing the functional I on a Dirichlet class $\mathcal{A}_g$, one can shift the class to the Banach space $X = W_0^{1,p}(\Omega; \mathbb{R}^m)$ since

$$\inf_{u\in\mathcal{A}_g} I(u) = \inf_{w\in X} \Phi(w), \tag{3.4}$$

where $\Phi(w) = I(w + g)$. We easily have the following result.

Proposition 3.2. *Let $X = W_0^{1,p}(\Omega; \mathbb{R}^m)$. For any $g \in W^{1,p}(\Omega; \mathbb{R}^m)$, the functional $\Phi\colon X \to \mathbb{R}$ defined by $\Phi(w) = I(w+g)$ is C^1 and $\Phi'(w) = I'(w+g)$ as elements in X^*, the dual space of X.*

In the following we write $X^* = W^{-1,p'}(\Omega; \mathbb{R}^m)$, where $p' = \frac{p}{p-1}$. As usual, we define

$$\|I'(u)\|_{W^{-1,p'}} = \sup\{\langle I'(u), v\rangle \mid v \in W_0^{1,p}(\Omega; \mathbb{R}^m),\ \|v\|_{W_0^{1,p}} \leq 1\}. \tag{3.5}$$

Note that, given a smooth functional I on $X = W_0^{1,p}(\Omega; \mathbb{R}^m)$, the sequences $\{u_k\}$ in X satisfying

$$|I(u_k)| \leq M, \quad I'(u_k) \to 0 \quad \text{in } W^{-1,p'}(\Omega; \mathbb{R}^m)$$

are usually called the *Palais-Smale sequences* or *(PS) sequences* for the functional I. Therefore, for simplicity, we use the following definition.

Definition 3.1. A sequence $\{u_k\}$ is said to *(PS)-weakly converges to* u (with respect to I) in $W^{1,p}(\Omega; \mathbb{R}^m)$ and denoted by $u_k \overset{\text{ps}}{\rightharpoonup} u$ provided that $u_k \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^m)$ and $I'(u_k) \to 0$ in $W^{-1,p'}(\Omega; \mathbb{R}^m)$. Define the set of all (PS)-weak limits to be

$$\mathcal{S} = \{u \in W^{1,p}(\Omega; \mathbb{R}^m) \mid \exists\ u_k \in W^{1,p}(\Omega; \mathbb{R}^m) \text{ such that } u_k \overset{\text{ps}}{\rightharpoonup} u\}. \tag{3.6}$$

Let $\mathcal{C} = \{u \in W^{1,p}(\Omega; \mathbb{R}^m) \mid \|I'(u)\|_{W^{-1,p'}} = 0\}$. Then clearly $\mathcal{C} \subseteq \mathcal{S}$, and hence $\mathcal{S}$ can be viewed as a relaxation of $\mathcal{C}$ under the (PS)-weak convergence. However, for certain functionals I the set $\mathcal{S}$ may be empty.

Example 4. Let $f(x,\xi) = \chi_E(x)h(\xi)$, where χ_E is the characteristic function of a measurable set E in $(0,1)$ with $0 < |E| < 1$ and $h(\xi) = \frac{\pi}{2} + \arctan \xi$. Define

$$I(u) = \int_0^1 f(x, u'(x))dx, \quad u \in W^{1,2}(0,1).$$

We claim that for the functional I the (PS)-weak limit set $\mathcal{S} = \emptyset$. Suppose to the contrary $u_k \overset{\text{ps}}{\rightharpoonup} u$ in $W^{1,2}(0,1)$. Let $g_k(x) = \chi_E(x)h'(u_k'(x))$. Then, by Proposition 3.6 below, there exists a subsequence $g_{k_j} \to L$ strongly in $L^2(0,1)$ for some constant L. We also assume $g_{k_j}(x) \to L$ for almost every $x \in (0,1)$. Hence we must have $L = 0$ and $g_{k_j}(x) = h'(u_{k_j}'(x)) \to 0$ for almost every $x \in E$. By Egoroff's theorem, it follows that $|u_{k_j}'(x)| \to \infty$ almost uniformly on E, which implies $\|u_{k_j}'\|_{L^2(E)} \to \infty$, a contradiction.

Definition 3.2. Given any nonempty family $\mathcal{A} \subseteq W^{1,p}(\Omega; \mathbb{R}^m)$, we say that I is *(PS)-weakly lower semicontinuous on $\mathcal{A}$* provided that

$$I(u) \le \liminf_{k\to\infty} I(u_k) \quad \text{whenever} \quad u_k,\ u \in \mathcal{A}, \quad u_k \overset{\text{ps}}{\rightharpoonup} u. \tag{3.7}$$

We shall technically assume this property if $\mathcal{A} \cap \mathcal{S} = \emptyset$.

The following result shows that if $f = f(x,\xi)$ is convex in ξ then the functional I is in fact *(PS)-weakly continuous* on all Dirichlet classes.

Proposition 3.3. *Assume $f = f(x,\xi)$ satisfies the corresponding growth conditions as* (3.2) *and* (3.3) *above. Suppose $f(x,\xi)$ is convex in ξ for almost every $x \in \Omega$.*

Then both I and $-I$ are (PS)-weakly lower semicontinuous on all Dirichlet classes $\mathcal{A}_g$ with $g \in W^{1,p}(\Omega; \mathbb{R}^m)$. Therefore the functional I is (PS)-weakly continuous on $\mathcal{A}_g$ in the sense that

$$I(u) = \lim_{k\to\infty} I(u_k) \quad \forall\ u_k,\, u \in \mathcal{A}_g, \quad u_k \overset{\text{ps}}{\rightharpoonup} u. \tag{3.8}$$

Proof. For any u_k, $u \in W^{1,p}(\Omega; \mathbb{R}^m)$, by the convexity of f, it follows from (1.3) that

$$f(x, Du_k) \geq f(x, Du) + D_\xi f(x, Du) \colon (Du_k - Du), \tag{3.9}$$

$$f(x, Du) \geq f(x, Du_k) + D_\xi f(x, Du_k) \colon (Du - Du_k) \tag{3.10}$$

for almost every $x \in \Omega$. If $u_k \overset{\text{ps}}{\rightharpoonup} u$, and $u - u_k \in W_0^{1,p}(\Omega; \mathbb{R}^m)$, then integrating the above inequalities, we have

$$\liminf_{k\to\infty} I(u_k) \geq I(u) \geq \limsup_{k\to\infty} I(u_k),$$

and hence (3.8) follows. □

We show that in general the (PS)-weak lower semicontinuity on all Dirichlet classes does not imply the (PS)-weak lower semicontinuity on the whole space $W^{1,p}(\Omega; \mathbb{R}^m)$ (without the fixed boundary conditions).

Proposition 3.4. *Let Ω be the unit disc in $\mathbb{R}^2$ and $I(u) = -\int_\Omega |Du|^2 dx$ for $u \colon \Omega \to \mathbb{R}$. Then I is (PS)-weakly lower semicontinuous on all Dirichlet classes of $W^{1,2}(\Omega)$*

but not (PS)-weakly lower semicontinuous on $W^{1,2}(\Omega)$.

Proof. By the preceding proposition, I is (PS)-weakly lower semicontinuous on all Dirichlet classes of $W^{1,2}(\Omega)$. We now show it is not (PS)-weakly lower semicontinuous on $W^{1,2}(\Omega)$ (without the fixed boundary conditions). We identify $\mathbb{R}^2 \cong \mathbf{C}^1$. For $z = x_1 + ix_2 \in \Omega$ and $k = 1, 2, \cdots$, we define $u_k(x_1, x_2) = \frac{1}{\sqrt{\pi k}}\mathrm{Re}(z^k)$. Then u_k is harmonic in Ω and $\partial_{x_1} u_k - i\partial_{x_2} u_k = \sqrt{\frac{k}{\pi}} z^{k-1}$. Hence $|Du_k(x)| = \sqrt{\frac{k}{\pi}}|z|^{k-1}$ and thus we have $\|Du_k\|_{L^2(\Omega)} = 1$. So u_k is bounded in $W^{1,2}(\Omega)$. It is easy to see $u_k \to 0$ uniformly on $\bar{\Omega}$ and hence $u_k \rightharpoonup 0$ in $W^{1,2}(\Omega)$. Since u_k is harmonic in Ω, it also follows that $Du_k \to 0$ in $W^{-1,2}(\Omega)$. Therefore, for functional $I(u) = -\int_\Omega |Du|^2 dx$, we have $u_k \overset{\mathrm{ps}}{\rightharpoonup} 0$, but $I(0) = 0$ and $\liminf_k I(u_k) = -1$. Hence I is not (PS)-weakly lower semicontinuous on $W^{1,2}(\Omega)$. $\square$

As we mentioned in the introduction, the (PS)-weak lower semicontinuity has been motivated by using the Ekeland variational principle in the direct method for the minimization problem. We have the following existence result.

Theorem 3.5. *Assume f satisfies, in addition to* (3.2) *and* (3.3), *the following* coercivity *condition*

$$c_0|\xi|^p - a(x) \le f(x, s, \xi) \le c_1(|\xi|^p + |s|^p) + A(x), \tag{3.11}$$

where $c_0 > 0$ is a positive constant, $a \in L^1(\Omega)$ is a function. Given $g \in W^{1,p}(\Omega; \mathbb{R}^m)$, assume the functional I defined above is (PS)-weakly lower semicontinuous on $\mathcal{A}_g$. Then the minimization problem $\inf\limits_{u \in \mathcal{A}_g} I(u)$ has at least one solution $u \in \mathcal{A}_g$.

Proof. The proof uses a standard direct method of the calculus of variations. Let $X = W_0^{1,p}(\Omega;\mathbb{R}^m)$. Define $\Phi\colon X \to \mathbb{R}$ by

$$\Phi(u) = I(u+g) = \int_\Omega f(x, u(x) + g(x), Du(x) + Dg(x))dx.$$

Then Φ is C^1 and bounded below on X, and $\Phi'(u) = I'(u+g)$ in X^*. By Theorem 1.8, there exists a sequence $\{u_k\}$ in X such that

$$\Phi(u_k) \to \inf_X \Phi, \quad \|\Phi'(u_k)\|_{X^*} \to 0.$$

Let $w_k = u_k + g \in \mathcal{A}_g$. Then

$$I(w_k) \to \inf_{w\in\mathcal{A}_g} I(w), \quad \|I'(w_k)\|_{W^{-1,p'}} \to 0. \tag{3.12}$$

Under the condition $c_0 > 0$ the sequence $\{w_k\}$ determined by (3.12) above is bounded in $W^{1,p}(\Omega;\mathbb{R}^m)$ and, since $1 < p < \infty$, has a weakly convergence subsequence, relabeled $\{w_k\}$ again. Let u be the weak limit. Then $u \in \mathcal{A}_g$ and $w_k \overset{\text{ps}}{\rightharpoonup} u$; hence the (PS)-weak lower semicontinuity on $\mathcal{A}_g$ implies

$$I(u) \le \lim_{k\to\infty} I(w_k) = \inf_{w\in\mathcal{A}_g} I(w).$$

Hence $I(u) = \inf_{w\in\mathcal{A}_g} I(w)$. $\square$

Remark 3. Under the growth assumptions (3.2) and (3.3), any minimizer u of I

over $\mathcal{A}_g$ is a weak solution to the Dirichlet problem of the Euler-Lagrange equation of functional I; that is,

$$\begin{cases} -\operatorname{div} D_\xi f(x,u,Du) + D_s f(x,u,Du) = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{3.13}$$

3.3 One dimensional scalar cases

In this section we study the (PS)-weak lower semicontinuity in some special one dimensional scalar cases.

We first consider the Sobolev space $H^1(0,1) = W^{1,2}(0,1)$ and functions $f(x,\xi)$ satisfying

$$0 \le f(x,\xi) \le C|\xi|^2 + A(x), \quad |f_\xi(x,\xi)| \le C|\xi| + B(x), \tag{3.14}$$

with $A \in L^1(0,1)$, $B \in L^2(0,1)$. Define

$$I(u) = \int_0^1 f(x,u'(x))dx, \quad \forall\ u \in H^1(0,1).$$

Proposition 3.6. *If $u_k \overset{\text{ps}}{\rightharpoonup} u$ in $H^1(0,1)$, then there exists a subsequence $\{u_{k_j}\}$ such that $f_\xi(x,u'_{k_j}(x)) \to L$ strongly in $L^2(0,1)$ as $j \to \infty$, where L is a constant.*

Proof. Let $g_k(x) = f_\xi(x,u'_k(x))$ and $L_k = \int_0^1 g_k(x)dx$. Since $\{g_k\}$ is bounded in $L^2(0,1)$, we assume for a subsequence $g_{k_j} \rightharpoonup g$ in $L^2(0,1)$ as $j \to \infty$, where $g \in$

$L^2(0,1)$. We define v_k on $[0,1]$ by

$$v_k(x) = \int_0^x (g_k(t) - L_k)\, dt, \quad x \in [0,1].$$

Then it is easily seen that $v_k \in H_0^1(0,1) = W_0^{1,2}(0,1)$ and $v_k' = g_k - L_k$. Moreover, $\{v_k\}$ is bounded in $H_0^1(0,1)$ and hence

$$\langle I'(u_k), v_k \rangle = \int_0^1 f_\xi(x, u_k'(x))v_k'(x)dx = \int_0^1 g_k^2(x)dx - L_k^2 \to 0$$

as $k \to \infty$. Since $g_{k_j} \rightharpoonup g$ in $L^2(0,1)$, we have $L_{k_j} \to L = \int_0^1 g dx$ and

$$\int_0^1 g^2(x)dx \leq \liminf_{j\to\infty} \int_0^1 g_{k_j}^2 dx = \liminf_{j\to\infty} L_{k_j}^2 = \left(\int_0^1 g(x)dx\right)^2.$$

This implies $g(x) = L$ a.e. on $[0,1]$ and $g_{k_j} \to L$ strongly in $L^2(0,1)$. □

In contrast to the theorem of Acerbi and Fusco (Theorem 1.6), we show below by an example that the (PS)-weak lower semicontinuity of I may not imply f being quasiconvex in ξ even for smooth functions $f(x,\xi)$ in the scalar case.

Proposition 3.7. *There exists a C^1 function $f(x,\xi)$ satisfying condition* (3.14) *above for which the corresponding functional I is (PS)-weakly, but not (unrestricted) weakly, lower semicontinuous on $H^1(0,1)$.*

Proof. Assume $f(x,\xi) = a(x)h(\xi)$ with a, $h \geq 0$, both C^1 and satisfying the following conditions:

$$a(x) = 0 \quad x \in [0, \theta], \quad a(x) > 0 \quad x \in (\theta, 1], \tag{3.15}$$

$$h \geq 0, \quad (h')^{-1}(0) = \{0\}, \quad \liminf_{|\xi| \to \infty} |h'(\xi)| > 0., \tag{3.16}$$

where $\theta \in (0,1)$ is a constant. Note that the condition (3.16) implies $h(0) < h(\xi)$ for all $\xi \in \mathbb{R}$. Given any $u_k \overset{\text{ps}}{\rightharpoonup} u$ in $H^1(0,1)$, using subsequence if necessary, we assume $\lim_{k\to\infty} I(u_k)$ exists. By Proposition 3.6 above, there exists a subsequence $\{u_{k_j}\}$ such that $f_\xi(x, u'_k) = a(x)h'(u'_{k_j}) \to L$ strongly in $L^2(0,1)$ for some constant L. Since $a = 0$ on $[0, \theta)$, one must have the limit $L = 0$; this also implies the whole sequence $a(x)h'(u'_k) \to 0$ strongly in $L^2(0,1)$. Therefore $h'(u'_k) \to 0$ strongly in $L^2(\theta', 1)$ for any $\theta' \in (\theta, 1)$. Hence, for a subsequence it follows that $h'(u'_{k_j}(x)) \to 0$ for almost every $x \in (\theta', 1)$. By (3.16), we have that $u'_{k_j}(x) \to 0$ for almost every $x \in (\theta', 1)$. Therefore the weak limit $u' = 0$ on $(\theta', 1)$ for all $\theta' \in (\theta, 1)$. This implies $u' = 0$ in $(\theta, 1)$. Since $h(\xi) \geq h(0)$ for all ξ, we have

$$\lim_{k\to\infty} I(u_k) = \lim_{k\to\infty} \int_\theta^1 a(x) h(u'_k(x))\, dx \geq \int_\theta^1 a(x)h(0)dx = I(u).$$

Hence I satisfies the (PS)-weak lower semicontinuity on $H^1(0,1)$. Note that the condition (3.16) does not imply that h is convex. (See, e.g., condition (1.4).) Hence I may not be weakly lower semicontinuous on $H^1(0,1)$ by Theorem 1.6 above. □

Remark 4. For the functional I defined by a function $f(x, \xi) = a(x)h(\xi)$ as above, the minimization problem

$$\inf_{\substack{u\in H^1(0,1)\\ u(0)=a,u(1)=b}} I(u)$$

has as only minimizers any functions $u \in H^1(0,1)$ with $u(0) = a$ and $u \equiv b$ on $[\theta, 1]$ and the minimum equals $h(0)\int_\theta^1 a(x)dx$, for any constants $a, b \in \mathbb{R}$. These minimizers are exactly those functions u in the Dirichlet class for which there exists a sequence $\{u_k\}$ in the class such that $u_k \overset{\text{ps}}{\rightharpoonup} u$.

Despite of the result above, we shall show that the (PS)-weak lower semicontinuity is equivalent to the usual weak lower semicontinuity if $f(x,\xi)$ satisfies a coercivity condition. In this case, both conditions reduce to the convexity of f in ξ. For the technical reason of using the following Sard's theorem [Mi], we assume f is sufficiently smooth in both x and ξ.

Lemma 3.8. *Let $h\colon \mathbb{R} \to \mathbb{R}$ be C^1 and $S = \{y \in \mathbb{R} \mid \exists\, x \in \mathbb{R},\ y = h(x),\ h'(x) = 0\}$. Then the Lebesgue measure $|S| = 0$ and, in particular, the set of regular values of h, $\mathbb{R} \setminus S$, is dense in $\mathbb{R}$.*

In the following, for $\beta \in \mathbb{R}$, let $W^{1,p}_\beta(0,1)$ be the Dirichlet class of functions u in $W^{1,p}(0,1)$ with $u(0) = 0$, $u(1) = \beta$.

Theorem 3.9. *Assume $f(x,\xi)$ and $f_\xi(x,\xi)$ are both C^1 on $[0,1] \times \mathbb{R}$ and satisfy, for some $p > 1$,*

$$|\xi|^p \le f(x,\xi) \le c_1(|\xi|^p + 1), \quad |f_\xi(x,\xi)| \le c_2(|\xi|^{p-1} + 1)$$

for all x and ξ. If the functional I defined by f is (PS)-weakly lower semicontinuous on $W_\beta^{1,p}(0,1)$ for all $\beta \in \mathbb{R}$, then $f(x,\xi)$ is convex in ξ for all $x \in (0,1)$.

We proceed with several lemmas before proving this theorem. First of all, for $\beta \in \mathbb{R}$, we define $m(\beta) = \inf\{I(u) \mid u \in W_\beta^{1,p}(0,1)\}$. It follows easily that

$$|\beta|^p \leq m(\beta) \leq c_1(|\beta|^p + 1). \tag{3.17}$$

From Theorem 3.5 above, it follows that, if I is (PS)-weakly lower semicontinuous on $W_\beta^{1,p}(0,1)$, then there exists at least one minimizer $u_\beta \in W_\beta^{1,p}(0,1)$ such that $I(u_\beta) = m(\beta)$. Hence $I'(u_\beta) = 0$ in $W^{-1,p'}(0,1)$. This implies $f_\xi(x, u'_\beta(x))$ is constant in $(0,1)$. Let $\mu(\beta)$ be this constant. Note that $\mu(\beta)$ depends also on the minimizer u_β.

Lemma 3.10. *It follows that*

$$\limsup_{\beta\to+\infty} \limsup_{\epsilon\to 0^+} \frac{m(\beta+\epsilon) - m(\beta)}{\epsilon} = +\infty, \tag{3.18}$$

$$\liminf_{\beta\to-\infty} \liminf_{\epsilon\to 0^-} \frac{m(\beta+\epsilon) - m(\beta)}{\epsilon} = -\infty. \tag{3.19}$$

Proof. We only prove (3.18); the other follows similarly. By contradiction, suppose the limit is finite. Then there exist positive constants β_0, ϵ_0 and M such that

$$\frac{m(\beta+\epsilon) - m(\beta)}{\epsilon} \leq M, \quad \forall\, \beta \geq \beta_0, \ \epsilon \in (0, \epsilon_0].$$

For any positive integer k we get $m(\beta + k\epsilon) - m(\beta) \leq Mk\epsilon$ and from 3.17 we obtain

$(\beta + k\epsilon)^p \le Mk\epsilon + m(\beta)$ which is false for k sufficiently large. $\square$

Lemma 3.11. *For any $\beta \in \mathbb{R}$, it follows that*

$$\limsup_{\epsilon \to 0^+} \frac{m(\beta + \epsilon) - m(\beta)}{\epsilon} \le \mu(\beta) \le \liminf_{\epsilon \to 0^-} \frac{m(\beta + \epsilon) - m(\beta)}{\epsilon}. \tag{3.20}$$

Proof. For $0 < \delta < 1$ we define w to be the linear function with $w(1 - \delta) = 0$, $w(1) = \epsilon$. Hence $w'(x) = \epsilon/\delta$. Let u_β be a minimizer for $m(\beta)$ and let $v(x) = u_\beta(x)$ on $[0, 1 - \delta]$ and $v(x) = u_\beta(x) + w(x)$ on $[1 - \delta, 1]$. Then $v \in W^{1,p}(0, 1)$ satisfies $v(0) = 0$, $v(1) = \beta + \epsilon$. Hence

$$m(\beta + \epsilon) \le I(v) = I(u_\beta) + \int_{1-\delta}^{1} [f(x, v') - f(x, u'_\beta)]dx.$$

Since $f(x, v') - f(x, u'_\beta) = f_\xi(x, u'_\beta)\epsilon/\delta + o(\epsilon/\delta)$ for $\epsilon/\delta \to 0$, we have

$$m(\beta + \epsilon) \le m(\beta) + \mu(\beta)\epsilon + o(\frac{\epsilon}{\delta})\delta \le m(\beta) + \mu(\beta)\epsilon + o(\epsilon),$$

as $\epsilon \to 0$. From this the lemma follows. $\square$

The lemmas above imply

$$\limsup_{\beta \to +\infty} \mu(\beta) = +\infty, \quad \liminf_{\beta \to -\infty} \mu(\beta) = -\infty. \tag{3.21}$$

Lemma 3.12. *Let $h \colon \mathbb{R} \to \mathbb{R}$ be C^1 and $h \ge 0$. Then the following statements are equivalent.*

(i) h is convex.

(ii) For all $0 < \lambda < 1$, a, $b \in \mathbb{R}$ with $h'(a) = h'(b)$, it follows that

$$h(\lambda a + (1 - \lambda)b) \leq \lambda h(a) + (1 - \lambda)h(b). \tag{3.22}$$

(iii) There exist no numbers $\alpha < \beta$ satisfying $h'(\alpha) = h'(\beta) \neq h'(t)$ for all $t \in (\alpha, \beta)$.

Proof. It is easily seen that (i) implies (ii). To show that (ii) implies (iii), we use a contradiction proof. Suppose (iii) fails. Then there exist numbers α, $\beta \in \mathbb{R}$ satisfying

$$\alpha < \beta, \quad h'(\alpha) = h'(\beta), \quad h'(t) \neq h'(\alpha) \quad \forall\, t \in (\alpha, \beta). \tag{3.23}$$

Using (3.22) with $a = \alpha, b = \beta$ we have, for all $0 < \lambda < 1$ and $t_\lambda = \lambda\alpha + (1 - \lambda)\beta$,

$$\frac{h(t_\lambda) - h(\alpha)}{t_\lambda - \alpha} \leq \frac{h(\beta) - h(\alpha)}{\beta - \alpha} \leq \frac{h(t_\lambda) - h(\beta)}{t_\lambda - \beta}. \tag{3.24}$$

Letting $\lambda \to 1^-$ and 0^+ in (3.24) respectively, we have $h'(\alpha) \leq \frac{h(\beta)-h(\alpha)}{\beta-\alpha} \leq h'(\beta)$. Hence $h'(\alpha) = h'(\beta) = \frac{h(\beta)-h(\alpha)}{\beta-\alpha}$. However, by the mean value property, $\frac{h(\beta)-h(\alpha)}{\beta-\alpha} = h'(t)$ for some $t \in (\alpha, \beta)$, and hence we have arrived at a contradiction with (3.23).

Finally, we prove that (iii) implies (i). Again, by contradiction, suppose h is not convex. Then there exist $a < b$ such that $h'(a) > h'(b)$. We consider only the case when $h'(a) > 0$; otherwise, consider $\bar{h}(t) = h(-t)$, $\bar{a} = -b$ and $\bar{b} = -a$. We claim there exist $c < d \leq a$ such that $h'(c) < h'(d)$. If not, h' would be nonincreasing on $(-\infty, a]$ and hence h would be concave on $(-\infty, a]$. Therefore we would have $h(t) \leq h(a) + h'(a)(t-a)$ for all $t < a$. Since $h'(a) > 0$, letting $t \to -\infty$, we would have $h(t) \to -\infty$, a contradiction with $h \geq 0$. Let $c < d \leq a$ be any points as above. Let $m = \max_{[c,b]} h'$. Define $S = \{t \in [c, b] \mid h'(t) = m\}$, $s^- = \min S$, and $s^+ = \max S$. Then $s^-, s^+ \in S$ and $c < s^- \leq s^+ < b$. Hence $h'(c) < m$, $h'(b) < m$. We define $\alpha' < \beta'$ as follows: If $h'(c) = h'(b)$, define $\alpha' = c$, $\beta' = b$. If $h'(c) > h'(b)$, then $h'(c) \in (h'(s^+), h'(b))$ and hence by the intermediate value property of h', define $\beta' \in (s^+, b)$ so that $h'(\beta') = h'(c)$, and define $\alpha' = c$. If $h'(c) < h'(b)$, then $h'(b) \in (h'(c), h'(s^-))$ and hence again by the intermediate value property of h', we define $\alpha' \in (c, s^-)$ so that $h'(\alpha') = h'(b)$, and define $\beta' = b$. The points $\alpha' < \beta'$ defined this way will satisfy $\alpha' < s^- \leq s^+ < \beta'$ and $h'(\alpha') = h'(\beta) < h'(s^-)$. Let $G = \{t \in (\alpha', \beta') \mid h'(t) > h'(\alpha')\}$. Then G is an open set and $s^- \in G$. Let (α, β) be the component of G containing s^-. Then, for this pair of α, β, we have (3.23), a contradiction with (iii); hence h is convex.

This completes the proof of lemma. □

Lemma 3.13. *For any constant $\theta \in \mathbb{R}$, there exists a function $q_\theta \in L^p(0,1)$ such that $f_\xi(x, q_\theta(x)) = \theta$ for almost every $x \in (0,1)$.*

Proof. In view of (3.21) above, there exist $\beta_1 < \beta_2$ such that $\mu(\beta_1) < \theta < \mu(\beta_2)$. Hence for almost every $x \in (0,1)$ we have $f_\xi(x, u'_{\beta_1}(x)) < \theta < f_\xi(x, u'_{\beta_2}(x))$. Let

$$q^-(x) = \min\{u'_{\beta_1}(x), u'_{\beta_2}(x)\}, \quad q^+(x) = \max\{u'_{\beta_1}(x), u'_{\beta_2}(x)\}.$$

Then $q^{\pm} \in L^p(0,1)$. By the intermediate value property of $f_\xi(x,\cdot)$, there exists $q \in (q^-(x), q^+(x))$ such that $f_\xi(x,q) = \theta$. Let $q_\theta(x)$ be the infimum of all such q's. Then $f_\xi(x, q_\theta(x)) = \theta$, $q_\theta(x)$ is lower semicontinuous and $q^-(x) \le q_\theta(x) \le q^+(x)$ at almost every $x \in (0,1)$ and hence $q_\theta \in L^p(0,1)$. □

Proof of Theorem 3.9. Given any $x_0 \in (0,1)$, we prove $f(x_0,\cdot)$ is convex. By Lemma 3.12, it suffices to show that there exist no numbers $\xi_1 < \xi_2$ such that

$$f_\xi(x_0,\xi_1) = f_\xi(x_0,\xi_2), \quad f_\xi(x_0,t) \neq f_\xi(x_0,\xi_1) \quad \forall\, t \in (\xi_1,\xi_2). \tag{3.25}$$

We prove this by contradiction. Suppose $\xi_1 < \xi_2$ satisfy (3.25). We will derive a contradiction by showing such ξ_i's must satisfy

$$f(x_0, \lambda\xi_1 + (1-\lambda)\xi_2) \le \lambda f(x_0,\xi_1) + (1-\lambda) f(x_0,\xi_2) \tag{3.26}$$

for all $\lambda \in (0,1)$, which gives a desired contradiction as in the step 2 of the proof of Lemma 3.12.

To this end, assume $f_\xi(x_0,\xi_1) = f_\xi(x_0,\xi_2) = \theta_0$. Without loss of generality, assume $f_\xi(x_0,t) > f_\xi(x_0,\xi_1)$ for all $t \in (\xi_1,\xi_2)$. Let

$$\max_{[\xi_1,\xi_2]} f_\xi(x_0,\cdot) = f_\xi(x_0,\bar\xi) = \bar\theta > \theta_0.$$

To proceed, we need the following lemma, which is the only place we use the smooth assumption of $f_\xi(x,\xi)$ on (x,ξ).

Lemma 3.14. *There exist a sequence $\theta_n \in (\theta_0, \bar{\theta})$ with $\theta_n \to \theta_0$ as $n \to \infty$, a closed interval $J_n = [a_n, b_n] \subset (0,1)$ containing x_0, and two continuous functions $q_n^\pm \colon J_n \to (\xi_1, \xi_2)$ such that $q_n^-(x) < q_n^+(x)$ and $f_\xi(x, q_n^\pm(x)) = \theta_n$ for all $x \in J_n$. Moreover, $q_n^{-,+}(x_0) \to \xi_{1,2}$ as $n \to \infty$.*

Proof. The proof is based on a use of Sard's theorem. By Lemma 3.8 above with $h(\xi) = f_\xi(x_0, \xi)$, the set of regular values of $f_\xi(x_0, \cdot)$ is dense. Hence there exists a sequence of regular values θ_n of $f_\xi(x_0, \cdot)$ in $(\theta_0, \bar{\theta})$ such that $\theta_n \to \theta_0$ as $n \to \infty$. Since $f_\xi(x_0, \xi_{1,2}) = \theta_0 < \theta_n < \bar{\theta} = f_\xi(x_0, \bar{\xi})$, by intermediate value property, there exist $\xi_n^- \in (\xi_1, \bar{\xi})$ and $\xi_n^+ \in (\bar{\xi}, \xi_2)$ such that $f_\xi(x_0, \xi_n^\pm) = \theta_n$. The assumption (3.25) implies $\xi_n^- \to \xi_1$ and $\xi_n^+ \to \xi_2$ as $n \to \infty$. Since θ_n is a regular value of $f_\xi(x_0, \cdot)$, it follows that $f_{\xi\xi}(x_0, \xi_n^\pm) \neq 0$. By the implicit function theorem, we have interval $J_n = [a_n, b_n] \subset (0,1)$ containing x_0 and two differentiable functions $q_n^\pm \colon J_n \to (\xi_1, \xi_2)$ such that

$$q_n^\pm(x_0) = \xi_n^\pm, \quad f_\xi(x, q_n^\pm(x)) = \theta_n \quad \forall\, x \in J_n. \tag{3.27}$$

Then the functions $q_n^\pm(x)$ satisfy the requirements of the lemma. □

We continue the proof of the theorem. Let $\theta_n \in (\theta_0, \bar{\theta})$, $J_n = [a_n, b_n]$ and $q_n^\pm \colon J_n \to (\xi_1, \xi_2)$ be given as in the lemma above. Let $J = [a, b] \subset J_n$ be any interval containing x_0. Let $q_n \in L^p(0,1)$ be the function q_θ determined by Lemma 3.13 with $\theta = \theta_n$. In what follows, we fix n. For each $k = 1, 2, \cdots$, we define function

$u_k(x)$ by $u_k(x) = \int_0^x w_k(t)dt$, where $w_k(t)$ is defined as follows:

$$w_k(t) = \begin{cases} q_n(t) & t \in [0,1] \setminus [a,b], \\ q_n^-(t) & t \in \cup_{j=1}^k (a + \frac{(j-1)}{k}(b-a), a + \frac{(j-1+\lambda)}{k}(b-a)), \\ q_n^+(t) & t \in \cup_{j=1}^k (a + \frac{(j-1+\lambda)}{k}(b-a), a + \frac{j}{k}(b-a)). \end{cases} \tag{3.28}$$

It is easily seen that $u_k \in W^{1,p}(0,1)$ and $\{u_k\}$ is bounded in $W^{1,p}(0,1)$.

Lemma 3.15. *For all continuous functions $\Phi(x,\xi)$, it follows that*

$$\lim_{k\to\infty} \int_a^b \Phi(x, u_k'(x))dx = \int_a^b [\lambda\Phi(x, q_n^-(x)) + (1-\lambda)\Phi(x, q_n^+(x))]dx.$$

Proof. It is easy to see

$$\begin{aligned} \int_a^b \Phi(x, u_k'(x))dx &= \sum_{j=1}^k \int_{a+\frac{(j-1)}{k}(b-a)}^{a+\frac{(j-1+\lambda)}{k}(b-a)} \Phi(x, q_n^-(x))dx & (3.29) \\ &+ \sum_{j=1}^k \int_{a+\frac{(j-1+\lambda)}{k}(b-a)}^{a+\frac{j}{k}(b-a)} \Phi(x, q_n^+(x))dx & (3.30) \\ &= \lambda \sum_{j=1}^k \Phi(c_j, q_n^-(c_j)) \frac{(b-a)}{k} & (3.31) \\ &+ (1-\lambda) \sum_{j=1}^k \Phi(d_j, q_n^+(d_j)) \frac{(b-a)}{k}, & (3.32) \end{aligned}$$

where $a + \frac{(j-1)}{k}(b-a) \le c_j \le a + \frac{(j-1+\lambda)}{k}(b-a) \le d_j \le a + \frac{j}{k}(b-a)$ are some points. Hence the sums in (3.31) and (3.32) are Riemann sums; therefore, as $k \to \infty$, the lemma follows. □

Let $\bar{u} \in W^{1,p}(0,1)$ be defined by $\bar{u}(x) = \int_0^x \bar{w}(t)dt$, where

$$\bar{w}(t) = \begin{cases} q_n(t) & t \in [0,1] \setminus [a,b], \\ \lambda q_n^-(t) + (1-\lambda)q_n^+(t) & t \in [a,b]. \end{cases}$$

From the lemma above, it easily follows that $u_k \rightharpoonup \bar{u}$ in $W^{1,p}(0,1)$. In particular, $\epsilon_k = \bar{u}(1) - u_k(1) \to 0$ as $k \to \infty$. By the definition of u_k it follows easily that $f_\xi(x, u_k'(x)) = \theta_n$ for almost every $x \in (0,1)$; hence $I'(u_k) = 0$ in $W^{-1,p'}(0,1)$. We now modify u_k to a function $\tilde{u}_k \in W_\beta^{1,p}(0,1)$ with $\beta = \bar{u}(1)$. For $0 < \delta < 1-b$ to be selected later, we define $\tilde{u}_k(x) = u_k(x)$ for $x \in [0, 1-\delta]$, and $\tilde{u}_k(x) = u_k(x) + v_k(x)$ for $x \in [1-\delta, 1]$, where v_k is a linear function on $[1-\delta, 1]$ with $v_k(1-\delta) = 0$, $v_k(1) = \epsilon_k$. Hence $\tilde{u}_k \in W^{1,p}(0,1)$ with $\tilde{u}_k(0) = 0$, $\tilde{u}_k(1) = \bar{u}(1) = \beta$. Note that $v_k'(x) = \frac{\epsilon_k}{\delta}$. Hence we select $\delta = \delta_k = |\epsilon_k|^{1/2}$ for all sufficiently large k. For this choice of δ, it is easily shown that the function $\tilde{u}_k \in W_\beta^{1,p}(0,1)$ satisfies $u_k - \tilde{u}_k \to 0$ in $W^{1,p}(0,1)$, and hence it follows that $I'(\tilde{u}_k) \to 0$ in $W^{-1,p'}(0,1)$ and $I(u_k) - I(\tilde{u}_k) \to 0$ as $k \to \infty$. In particular, $\tilde{u}_k \overset{\text{ps}}{\rightharpoonup} \bar{u}$ in $W_\beta^{1,p}(0,1)$. Therefore, by the (PS)-weak lower semicontinuity of I on $W_\beta^{1,p}(0,1)$, we have $I(\bar{u}) \le \liminf_k I(\tilde{u}_k) = \liminf_k I(u_k)$. Using Lemma 3.15, after easy computations, this implies

$$\begin{aligned} &\int_a^b f(x, \lambda q_n^-(x) + (1-\lambda)q_n^+(x))dx \\ &\le \int_a^b [\lambda f(x, q_n^-(x)) + (1-\lambda) f(x, q_n^+(x))]dx. \end{aligned}$$

This holds for all intervals $[a,b] \subset J_n$ containing x_0 and hence, letting $[a,b]$ shrink to $\{x_0\}$, we have

$$f(x_0, \lambda q_n^-(x_0) + (1-\lambda)q_n^+(x_0)) \leq \lambda f(x_0, q_n^-(x_0)) + (1-\lambda) f(x_0, q_n^+(x_0)).$$

Finally letting $n \to \infty$, by Lemma 3.14, we have

$$f(x_0, \lambda\xi_1 + (1-\lambda)\xi_2) \leq \lambda f(x_0, \xi_1) + (1-\lambda) f(x_0, \xi_2),$$

as desired by (3.26).

The proof of the theorem is now completed. □

3.4 Special cases with $f = f(\xi)$

In this section, we study some special cases with function $f = f(\xi)$, where $f\colon \mathbb{M}^{m\times n} \to \mathbb{R}$ is a C^1 function satisfying the following growth conditions:

$$c_0|\xi|^p \leq f(\xi) \leq c_1(|\xi|^p + 1), \tag{3.33}$$

$$|D_\xi f(\xi)| \leq c_2(|\xi|^{p-1} + 1), \tag{3.34}$$

where $1 < p < \infty$ and $c_0 \geq 0$, $c_1 > 0, c_2 > 0$ are constants. In this case, we shall also use the simplified notation $D_\xi f(\xi) = Df(\xi) = f'(\xi)$. As before, let I be the functional associated with f:

$$I(u) = \int_\Omega f(Du(x))\,dx \quad u \in W^{1,p}(\Omega; \mathbb{R}^m).$$

We first have the following result when $m = 1$ (the scalar case) with $c_0 = 0$ in (3.33), which is in contrast to Proposition 3.7 above

Theorem 3.16. *Let $m = 1$ and let $f \colon \mathbb{R}^n \to \mathbb{R}$ satisfy the conditions* (3.33) *and* (3.34) *above with $c_0 = 0$. Then the functional I is (PS)-weakly lower semicontinuous on the Dirichlet classes $W_A^{1,p}$ for all $A \in \mathbb{R}^n$ if and only if f is convex on $\mathbb{R}^n$.*

Proof. By Theorem 1.6, we only need to show the necessary part of the theorem. Thus assume I is (PS)-weakly lower semicontinuous on the Dirichlet classes $W_A^{1,p}$ for all $A \in \mathbb{R}^n$. We prove that f is convex on $\mathbb{R}^n$. To this end, let $\xi, \eta \in \mathbb{R}^n$ and $|\eta| = 1$ be given and let $h(t) = f(\xi + t\eta)$. We show that h is a convex function of $t \in \mathbb{R}$; this implies f is convex on $\mathbb{R}^n$. By virtue of Lemma 3.12 above, to show h is convex, it suffices to establish the inequality (iii) in that lemma for all $a, b \in \mathbb{R}$ with $a < b$ and $h'(a) = h'(b)$. Note that $h'(t) = f'(\xi + t\eta) \cdot \eta$. Given such a, b, let $\alpha = \xi + a\eta$, $\beta = \xi + b\eta$. Then $h(a) = f(\alpha)$, $h(b) = f(\beta)$ and hence

$$h'(a) - h'(b) = (f'(\alpha) - f'(\beta)) \cdot \eta = 0. \tag{3.35}$$

Given any $\lambda \in (0,1)$, let $\theta(t)$ be the periodic function on $\mathbb{R}$ of period 1 satisfying $\theta = 0$ on $[0, \lambda)$ and $\theta = 1$ on $[\lambda, 1)$. Let $\rho(t)$ be the Lipschitz function on $\mathbb{R}$ with $\rho(0) = 0$ and $\rho'(t) = \theta(t)$ for almost every $t \in \mathbb{R}$. For $k = 1, 2, \cdots$, we define functions

$$u_k(x) = \alpha x + \frac{b-a}{k}\rho(kx \cdot \eta) \quad x \in \mathbb{R}^n. \tag{3.36}$$

Then $Du_k(x) = \alpha + (b-a)\theta(kx\cdot\eta)\eta$ and hence

$$Du_k(x) = \begin{cases} \alpha & \text{if } x\cdot\eta \in \cup_{j=-\infty}^{\infty}(\frac{j}{k}, \frac{j+\lambda}{k}) \\ \beta & \text{if } x\cdot\eta \in \cup_{j=-\infty}^{\infty}(\frac{j+\lambda}{k}, \frac{j+1}{k}). \end{cases} \tag{3.37}$$

Let $\{\eta_1, \eta_2, \cdots, \eta_n\}$ be an orthonormal basis of $\mathbb{R}^n$ with $\eta_1 = \eta$. For each $x \in \mathbb{R}^n$, we write $x = \sum_{i=1}^n t_i\eta_i$ and define

$$A_k^j = \left\{x \in \mathbb{R}^n \,\middle|\, t_1 \in \left(\frac{j}{k}, \frac{j+\lambda}{k}\right)\right\}, \quad B_k^j = \left\{x \in \mathbb{R}^n \,\middle|\, t_1 \in \left(\frac{j+\lambda}{k}, \frac{j+1}{k}\right)\right\}.$$

Let $\Omega_\alpha^k = \Omega \cap (\cup_j A_k^j)$, $\Omega_\beta^k = \Omega \cap (\cup_j B_k^j)$. Then one can easily show that

$$\lim_{k\to\infty} |\Omega_\alpha^k| = \lambda|\Omega|, \quad \lim_{k\to\infty} |\Omega_\beta^k| = (1-\lambda)|\Omega|. \tag{3.38}$$

For any $1 \le p < \infty$, the sequence $\{u_k\}$ defined by (3.36) above satisfies $u_k \rightharpoonup \bar{u}$ in $W^{1,p}(\Omega)$ as $k \to \infty$, where $\bar{u}(x) = [\lambda\alpha + (1-\lambda)\beta]x$. In fact, one can show that $u_k \to \bar{u}$ uniformly on $\bar{\Omega}$. We leave the proof of these facts to the interested reader.

Lemma 3.17. *$I'(u_k) = 0$ in $W^{-1,p'}(\Omega)$.*

Proof. Given any $v \in W_0^{1,p}(\Omega)$, we extend v to be zero outside Ω. Let Q_N be the cube

$$Q_N = \{x \in \mathbb{R}^n \mid x = \sum_{i=1}^n t_i\eta_i, \ |t_i| < N, \ \forall\, i = 1, 2, \cdots, n\}.$$

Assume N is large enough so that $\bar{\Omega} \subset Q_N$. Then

$$\int_\Omega f'(Du_k) \cdot Dv = \int_{Q_N} f'(Du_k) \cdot Dv = \sum_{j=-kN}^{kN-1} \int_{Q_k^j} f'(Du_k) \cdot Dv,$$

where $Q_k^j = \{x \in Q_N \,|\, x \cdot \eta \in (\frac{j}{k}, \frac{j+1}{k})\}$. We write $Q_k^j = A^j \cup B^j \cup \Gamma^j$, where $A^j = Q_k^j \cap A_k^j$, $B^j = Q_k^j \cap B_k^j$ and $\Gamma^j = \{x \in Q_k^j \,|\, x \cdot \eta = \frac{j+\lambda}{k}\}$. We also define $F^j = \{x \in Q_N \,|\, x \cdot \eta = \frac{j}{k}\}$. Note that $Du_k = \alpha$ on A^j and $Du_k = \beta$ on B^j and hence, by the divergence theorem and (3.35) above as well, we have

$$\begin{aligned}
\int_{Q_k^j} f'(Du_k) \cdot Dv &= \int_{A^j} f'(Du_k) \cdot Dv + \int_{B^j} f'(Du_k) \cdot Dv \\
&= f'(\alpha) \cdot \int_{A^j} Dv + f'(\beta) \cdot \int_{B^j} Dv \\
&= f'(\alpha) \cdot \left(\int_{F^j} vdS\right)(-\eta) + f'(\alpha) \cdot \left(\int_{\Gamma^j} vdS\right)\eta \\
&\quad + f'(\beta) \cdot \left(\int_{\Gamma^j} vdS\right)(-\eta) + f'(\beta) \cdot \left(\int_{F^{j+1}} vdS\right)\eta \\
&= f'(\alpha) \cdot \eta \left(\int_{F^{j+1}} vdS - \int_{F^j} vdS\right).
\end{aligned}$$

Hence, since $F^{\pm kN}$ lies in $\mathbb{R}^n \setminus \bar{\Omega}$, where $v = 0$, it follows that

$$\int_\Omega f'(Du_k(x)) \cdot Dv(x)\,dx = f'(\alpha) \cdot \eta \left(\int_{F^{kN}} vdS - \int_{F^{-kN}} vdS\right) = 0.$$

This proves $I'(u_k) = 0$ in $W^{-1,p'}(\Omega)$. □

To continue the proof of the theorem, we now modify the sequence $\{u_k\}$ above into a sequence in $W_A^{1,p}(\Omega)$, where $A = \lambda\alpha + (1-\lambda)\beta$. For all sufficiently large j, say $j \geq j_0$, consider nonempty open sets

$$\Omega_j = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > 1/j\}.$$

Note that the measure $|\Omega \setminus \Omega_j| \to 0$ as $j \to \infty$. Let $\varphi_j \in C_0^\infty(\Omega)$ be the cut-off functions such that $\varphi_j = 1$ on Ω_j and $0 \leq \varphi_j \leq 1$ in Ω. Since $u_k \to \bar{u}$ uniformly on $\bar{\Omega}$, we have that, for each $j \geq j_0$, there exists $k_j > j$ satisfying

$$\|(u_{k_j} - \bar{u}) D\varphi_j\|_{L^p(\Omega)} < \frac{1}{j}. \tag{3.39}$$

Let $\tilde{u}_j = \varphi_j u_{k_j} + (1 - \varphi_j)\bar{u}$. Then $\tilde{u}_j \in W_{\bar{u}}^{1,p}(\Omega) = W_A^{1,p}(\Omega)$ and $D\tilde{u}_j = \varphi_j D u_{k_j} + (1 - \varphi_j) D\bar{u} + (u_{k_j} - \bar{u}) D\varphi_j$. Hence, by (3.39) and also since $Du_{k_j}, D\bar{u}$ are bounded, it follows that

$$\lim_{j \to \infty} \|D\tilde{u}_j\|_{L^p(\Omega \setminus \Omega_j)} = 0. \tag{3.40}$$

Therefore $\tilde{u}_j \rightharpoonup \bar{u}$ in $W^{1,p}(\Omega)$ as $j \to \infty$. Since $\tilde{u}_j = u_{k_j}$ on Ω_j, by (3.40) and the growth conditions (3.33)-(3.34), it easily follows that

$$\lim_{j \to \infty} \|I'(\tilde{u}_j) - I'(u_{k_j})\|_{W^{-1,p'}(\Omega)} = 0, \quad \lim_{j \to \infty} |I(\tilde{u}_j) - I(u_{k_j})| = 0.$$

Hence $\tilde{u}_j, \bar{u} \in W_A^{1,p}(\Omega)$, and $\tilde{u}_j \overset{\text{ps}}{\rightharpoonup} \bar{u}$ since $I'(u_{k_j}) = 0$. By the (PS)-weak lower semicontinuity of I on $W_A^{1,p}(\Omega)$, we have $I(\bar{u}) \leq \liminf_j I(\tilde{u}_j) = \liminf_j I(u_{k_j})$.

Using (3.38), we easily see that this implies

$$h(\lambda a + (1-\lambda)b) \leq \lambda h(a) + (1-\lambda)h(b).$$

Hence by Lemma 3.12 above, $h(t) = f(\xi + t\eta)$ is convex for all ξ, η with $|\eta| = 1$. This proves f is convex on $\mathbb{R}^n$. $\square$

We now study the general case with $m \geq 2$. Under the coercivity condition that $c_0 > 0$ in (3.33), we have the following result:

Theorem 3.18. *Let $n, m \geq 1$ and let $f\colon \mathbb{M}^{m\times n} \to \mathbb{R}$ satisfy the conditions* (3.33) *and* (3.34) *above with $c_0 > 0$. Then the following statements are equivalent:*

(i) I is weakly lower semicontinuous on $W^{1,p}(\Omega; \mathbb{R}^m)$.

(ii) I is (PS)-weakly lower semicontinuous on $W^{1,p}(\Omega; \mathbb{R}^m)$.

(iii) I is (PS)-weakly lower semicontinuous on all $W^{1,p}_A(\Omega; \mathbb{R}^m)$.

(iv) f is quasiconvex.

Proof. By the theorem of Acerbi-Fusco (Theorem 1.6), $(i) \Leftrightarrow (iv)$ even when $c_0 = 0$. Moreover, by the definition of quasiconvexity and using approximation, if f is quasiconvex and only satisfies (3.33) with $c_0 \in \mathbb{R}$, then it readily follows that $I(g_A) \leq I(u)$ for all $u \in W^{1,p}_A(\Omega; \mathbb{R}^m)$. It is also clear that $(i) \Rightarrow (ii) \Rightarrow (iii)$ in general cases.

Therefore, to prove the theorem, it suffices to show that $(iii) \Rightarrow (iv)$. We prove this as separate result in the lemma below. $\square$

Lemma 3.19. *Under the assumptions* (3.33) *with $c_0 > 0$ and* (3.34), *$(iii) \Rightarrow (iv)$.*

Proof. Given $A \in \mathbb{M}^{m\times n}$, by Theorem 3.5 (note that $c_0 > 0$ is needed here), there exists $\bar{u} \in W_A^{1,p}(\Omega;\mathbb{R}^m)$ which is a minimizer of $I(u)$ on $W_A^{1,p}(\Omega;\mathbb{R}^m)$. We now apply the standard technique of Vitali covering [DM] to construct a sequence $\{u_k\}$ in $W_A^{1,p}(\Omega;\mathbb{R}^m)$ satisfying

$$I(u_k) = I(\bar{u}) = \inf_{u\in W_A^{1,p}} I(u); \tag{3.41}$$

$$u_k \rightharpoonup g_A \quad \text{in } W^{1,p}(\Omega;\mathbb{R}^m) \text{ as } k \to \infty; \tag{3.42}$$

$$I'(u_k) = 0 \quad \text{in } W^{-1,p'}(\Omega;\mathbb{R}^m). \tag{3.43}$$

Note that (3.43) will follow from (3.41) since $u_k \in W_A^{1,p}(\Omega;\mathbb{R}^m)$ is also a minimizer of $I(u)$ on $W_A^{1,p}(\Omega;\mathbb{R}^m)$. Once we have constructed such a sequence $\{u_k\}$, which certainly satisfies $u_k \overset{\text{ps}}{\rightharpoonup} g_A$, the (PS)-weak lower semicontinuity condition (*iii*) will imply

$$I(g_A) \le \liminf_{k\to\infty} I(u_k) = I(\bar{u}) = \inf_{u\in W_A^{1,p}} I(u),$$

for all $A \in \mathbb{M}^{m\times n}$, which is exactly the quasiconvexity condition of f, and hence the result follows. Assume, without loss of generality, $0 \in \Omega$ and then we use the Vitali covering theorem to decompose Ω as follows:

$$\Omega = \cup_{j=1}^{\infty} \bar{\Omega}_j \cup N; \quad \bar{\Omega}_i \cap \bar{\Omega}_j = \emptyset \quad (i \neq j),$$

where $\Omega_j = a_j + \epsilon_j \Omega \subset\subset \Omega$ with $a_j \in \Omega$, $0 < \epsilon_j < 1/k$, and $|N| = 0$. Let $\bar{u} = g_A + \bar{v}$, where $\bar{v} \in W_0^{1,p}(\Omega; \mathbb{R}^m)$. We define

$$u_k(x) = \begin{cases} Ax + \epsilon_j \bar{v}(\frac{x-a_j}{\epsilon_j}) & \text{if } x \in \Omega_j, \\ \\ Ax & \text{otherwise.} \end{cases} \tag{3.44}$$

Then one can easily check that u_k belongs to $W_A^{1,p}(\Omega; \mathbb{R}^m)$ and satisfies

$$\int_\Omega \psi(Du_k(x))\, dx = \int_\Omega \psi(D\bar{u}(x))\, dx$$

for all continuous functions $\psi\colon\ \mathbb{M}^{m\times n} \to \mathbb{R}$ satisfying $|\psi(\xi)| \leq C(|\xi|^p + 1)$. Certainly this implies (3.41). Furthermore, it is easy to see

$$\|u_k - g_A\|_{L^p(\Omega)} \leq \frac{1}{k}\, \|\bar{u} - g_A\|_{L^p(\Omega)}.$$

Hence $u_k \rightharpoonup g_A$ as $k \to \infty$. As mentioned above, condition (3.43) follows from (3.41). This completes the construction of $\{u_k\}$ and thus the proof of the lemma. □

Remark 5. For any $u \in W_A^{1,p}(\Omega; \mathbb{R}^m)$, we write $u = g_A + v$ with $v \in W_0^{1,p}(\Omega; \mathbb{R}^m)$ and define sequence $u_k \in W_A^{1,p}(\Omega; \mathbb{R}^m)$ as in (3.44) above with $\bar{v} = v$. Then, if u is not a minimizer of I over $W_A^{1,p}(\Omega; \mathbb{R}^m)$, one only has $I'(u_k) \rightharpoonup 0$, but not strongly, in $W^{-1,p'}(\Omega; \mathbb{R}^m)$, as $k \to \infty$, even when $I'(u) = 0$; hence the (PS)-weak lower semicontinuity can not be applied to this sequence.

Finally we show that without the coercivity $c_0 > 0$ in (3.33) the results of Theorem 3.18 may fail, at least in the case $n = 1, m = 2$.

Theorem 3.20. *Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a C^1 function satisfying the conditions* (3.33) *and* (3.34) *with $c_0 = 0$. Assume the derivative map $Df = f'\colon \mathbb{R}^2 \to \mathbb{R}^2$ is one-to-one. Then the functional I defined on $X = W^{1,p}((0,1);\mathbb{R}^2)$ by*

$$I(u) = \int_0^1 f(u'(x))\,dx = \int_0^1 f(u_1'(x), u_2'(x))dx, \quad u = (u_1, u_2) \in X,$$

is (PS)-weakly lower semicontinuous on X.

Proof. Let $u \in X = W^{1,p}((0,1);\mathbb{R}^2)$. Then

$$\langle I'(u), v\rangle = \int_0^1 [f_{\xi_1}(u'(x))v_1' + f_{\xi_2}(u'(x))v_2']\,dx, \quad \forall\, v = (v_1, v_2) \in X,$$

and hence it can be shown that

$$\|I'(u)\|_{W^{-1,p'}} \cong \|f_{\xi_1}(u') - C_1(u)\|_{L^{p'}(0,1)} + \|f_{\xi_2}(u') - C_2(u)\|_{L^{p'}(0,1)},$$

where $C_1(u), C_2(u)$ are two constants depending *boundedly* on $u \in X$.

Assume $u_k \overset{\text{ps}}{\rightharpoonup} u$ in X. We also assume $\lim I(u_k)$ exists as $k \to \infty$. Then there exists a subsequence $\{u_{k_j}\}$ such that

$$\|f_{\xi_1}(u_{k_j}') - C_1\|_{L^{p'}(0,1)} + \|f_{\xi_2}(u_{k_j}') - C_2\|_{L^{p'}(0,1)} \to 0$$

as $j \to \infty$, where C_1, C_2 are some constants. We also assume there exists a measurable set $E \subset (0,1)$ such that

$$|E| = 1, \quad \lim_{j\to\infty} f_{\xi_\nu}(u'_{k_j}(x)) = C_\nu \quad \forall\, x \in E \quad (\nu = 1, 2). \tag{3.45}$$

Note that for all $M > 0$ the measure $|\{x \in E \,|\, |u'_k(x)| > M\}| \leq \frac{C}{M^p}$ for all k, where C is a constant; hence there exists a sufficiently large $M > 0$ such that $|\{x \in E \,|\, |u'_k(x)| \leq M\}| > \frac{1}{2}$ for all $k = 1, 2, \cdots$. Therefore there exists $x_0 \in E$ such that $|u'_{k_j}(x_0)| \leq M$. By taking another subsequence, assume $u'_{k_{j_s}}(x_0) \to \alpha \in \mathbb{R}^2$ as $j_s \to \infty$. Therefore by (3.45), $f'(\alpha) = (C_1, C_2)$. Since f' is one-to-one, from (3.45), it must follow that $u'_{k_j}(x) \to \alpha$ as $j \to \infty$ for all $x \in E$. Hence $u'(x) = \alpha$ is constant, and therefore we have

$$I(u) = \lim_{j\to\infty} I(u_{k_j}) = \lim_{k\to\infty} I(u_k),$$

which proves the (PS)-weak lower semicontinuity of I (in fact I is *(PS)-weakly continuous*) on X.

Remark 6. Note that the function $f(\xi_1, \xi_2) = \varphi(\xi_1 - \xi_2^2)$ on $\mathbb{R}^2$, where $\varphi \geq 0$ is any C^1 function with a strictly increasing derivative $\varphi' > 0$ on $\mathbb{R}$, has an one-to-one derivative $f'\colon \mathbb{R}^2 \to \mathbb{R}^2$, but f is not convex on $\mathbb{R}^2$; this also shows that Lemma 3.12

above fails for functions $h\colon \mathbb{R}^2 \to \mathbb{R}$. An example of such a φ is given by

$$\varphi(t) = \begin{cases} e^t & t \leq 0; \\ t^2 + t + 1 & t > 0. \end{cases}$$

Note that the function $f(\xi) = f(\xi_1, \xi_2) = \varphi(\xi_1 - \xi_2^2)$ then satisfies the conditions (3.33) and (3.34) with $c_0 = 0$ and $p = 4$.

$\square$

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