HANDLEBODY STRUCTURES OF RATIONAL BALLS

By

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ABSTRACT

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It is known that for coprime integers $p > q \geq 1$, the lens space $L(p^2, pq-1)$ bounds a rational ball, $B_{p,q}$, arising as the 2-fold branched cover of a (smooth) surface in B^4 bounding the associated 2-bridge knot or link. Lekilli and Maydanskiy [32] give handle decompositions for each $B_{p,q}$. Whereas, Yamada [59] gives an alternative definition of rational balls, $A_{m,n}$, bounding $L(p^2, pq-1)$ by their handlebody decompositions alone. We show that these two families coincide - answering a question of Kadokami and Yamada. To that end, we show that each $A_{m,n}$ admits a Stein filling of the universally tight contact structure, $\bar{\xi}_{st}$, on $L(p^2, pq-1)$ investigated by Lisca. Furthermore, we construct boundary diffeomorphisms between these families. Using the carving process, pioneered by Akbulut, we show that these boundary maps can be extended to diffeomorphisms between the spaces $B_{p,q}$ and $A_{m,n}$.

To my wife Liz.

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Background

Classifying the homeomorphism types of closed simply-connected smooth 4-manifolds is determined entirely by their intersection pairings on the second homology groups [17]. Whereas, the question of determining the diffeomorphism type of a given smooth 4-manifold is much more subtle and far from being fully understood. With the addition of gauge theory to the subject, many examples of "exotic" smooth 4-manifolds have been discovered. Two 4-manifolds X and X' are *exotic copies* of each other if X is homeomorphic, but not diffeomorphic, to X'. The first such example was an exotic $\mathbb{C}P^2\#9\overline{\mathbb{C}P}^2$ discovered by Donaldson [9]. A plethora of examples have been constructed since. Less is known about 4-manifolds with boundary. In some cases, intersection pairings on simply connected 4-manifolds with boundary are still enough to pin down the homeomorphism type [5].

Many invariants have been developed to detect differences in smooth structures on homeomorphic 4-manifolds. Arguably, one of the most successful has been the set of Sieberg-Witten invariants (SW-invariants) [58] - which count solutions to certain PDE's on a given 4-manifold equipped with added structure that depends upon the diffeomorphism type of the manifold. Although difficult to compute in general, there are many constructions which allow the SW-invariants to be calculated efficiently. For instance, Taubes proves that there is a non-trivial SW-invariant for each closed symplectic 4-manifold [57]. Moreover, Fintushel and Stern provide cut-and-paste techniques to construct new closed 4-manifolds from old while tracing the effect on the set of SW-invariants - namely knot surgery [15] and the rational blow-down [14].

This thesis is concerned with aspects of the rational blow-down which is a topological generalization of the (honest) blow-down from algebraic-geometry. From a topologist's perspective, the honest blow-down can be described as splitting off a summand of $\overline{\mathbb{C}P}^2$ from a 4-manifold[1]. Fintushel and Stern [14] and later Park [47] note that there are other configurations of spheres whose neighborhoods have boundaries which are diffeomorphic to particular lens spaces; such lens spaces are known to bound rational balls [6]. Here, a 4-manifold X is a *rational ball* if its singular homology groups computed with $\mathbb{Q}$-coefficients agree with those of the 4-ball. That is, for each i we have

$$\widetilde{H}_i(X;\mathbb{Q}) = \widetilde{H}_i(B^4;\mathbb{Q}) = 0.$$

Removing a neighborhood of one of these configurations of spheres and gluing in the appropriate rational ball in its place is known as the rational blow-down of the 4-manifold along the configuration. As with the honest blow-down, this operation kills elements of second homology (now at the possible expense of increasing the fundamental group). Symington proves that these surgeries can be performed in the symplectic category (provided the given configuration consists of symplectic spheres inside a symplectic 4-manifold) [55, 56]. Moreover, under mild assumptions [14, 47], one can compute the SW-invariants of the rationally blown-down manifold from those of the original manifold.

Rational blow-downs have been effective in producing "small" exotic 4-manifolds: Using this technique, Park constructs an exotic $\mathbb{C}P^2\#7\overline{\mathbb{C}P}^2$ [48]. Also using a (generalized) rational blow down, Stipcisz and Szabó construct an exotic $\mathbb{C}P^2\#6\overline{\mathbb{C}P}^2$ [54]. By employing a

[1]Noting that S^3 is both the boundary of a punctured $\overline{\mathbb{C}P}^2$ (since S^3 is diffeomorphic to $L(1,1)$), as well as the boundary of the 4-ball B^4, if a neighborhood of a sphere of self-intersection -1 is located within a 4-manifold, it can be replaced with a copy of the 4-ball - thereby killing the -1 sphere in the second homology of the original 4-manifold.

variant of knot-surgery along with rational blow-downs, Fintushel and Stern provide infinite families of pairwise exotic $\mathbb{C}P^2\#k\overline{\mathbb{C}P}^2$ for $k = 6, 7, 8$ [16]. Using similar techniques, Park, Stipsicz and Szabó provide the same result for $\mathbb{C}P^2\#5\overline{\mathbb{C}P}^2$ [49]. Together, these examples demonstrate the power of the generalized rational blow-down.

All of these manifolds are shown to be exotic by examining their SW-invariants. It is worth noting that the rational blow-down is constructed using a specific rational 4-ball for a given sphere configuration. We arrive at a natural question. Do the lens spaces involved in the rational blow-down construction bound other rational balls? Yamada produces a family of rational balls [59] that, a priori, could have settled this question in the positive. The main results of this thesis revolve around showing that, in fact, Yamada's family coincides with the family of balls originally used in the rational blow-down procedure.

We employ two techniques to identify these families. Both techniques stand on their own, however, each has its merits. The first technique uses the classification of symplectic fillings of universally tight lens spaces [35] to quickly conclude that the families coincide. However, this hides a large amount of the details within the machinery developed by Lisca. That is, the actual diffeomorphisms ensured by this route go unseen. In order to rectify this, we employ a method pioneered by Akbulut known as carving [1]. Therein, one attacks the problem of building a diffeomorphism by first fixing a "well-behaved" diffeomorphism near the boundary. Here we can use the calculus of links developed by Kirby [28] and Fenn-Rourke [13] (also see Rolfsen [52]) to explicitly state such a diffeomorphism between the two given 3-manifold boundaries.

If the chosen diffeomorphism can be extended across the co-cores of each 2-handle, our required "well-behaved" condition, then one is left with only having to extend a self-diffeomorphism on $\#k(S^1 \times S^2)$ across $\natural k(S^1 \times B^3)$. A theorem of Laudenbach and Poénaru

[31] implies that this new extension problem *always* has a solution. We then specify such boundary diffeomorphisms on the two aforementioned families of rational balls and prove that the carving procedure goes through for our chosen diffeomorphisms.

1.1 Preliminaries and Assumptions

We assume that the reader is familiar with the theory of handlebody structures on 4-manifolds [2,22], the related theory of framed link surgery on three manifolds [2,13,22,28,52] and basic symplectic and contact geometry [7], especially as it relates to handle structures [8, 21, 41]. With this in mind, we will still recall some important definitions and relevant theorems for clarity. Since this thesis deals with rational balls bounding lens spaces, we start with lens spaces themselves.

Fix relatively prime integers p and q. Viewing S^3 as the unit complex numbers in $\mathbb{C}^2$, recall that $\mathbb{Z}_p$ acts on S^3 via

$$(z_1, z_2) \mapsto \left(z_1 e^{\frac{2\pi}{p} i}, z_2 e^{\frac{2\pi q}{p} i} \right).$$

By definition, the lens space $L(p,q) \doteq S^3/\mathbb{Z}_p$.

We adopt the standard convention that $L(p,q)$ is the result of $-p/q$-surgery on the unknot in S^3. It is well known that $L(p,q)$ is also given as the boundary of a linear plumbing of D^2-bundles over S^2 (see Figure 1.1) with Euler classes chosen according to a continued fraction associated to $-p/q$:

$$[c_1, \ldots, c_n] \doteq c_1 - \cfrac{1}{c_2 - \cfrac{1}{\ddots - \cfrac{1}{c_n}}} = -\frac{p}{q}$$

where, the c_i's are uniquely determined provided each $c_i \leq -2$. Taking advantage of this uniqueness, we can make the following standard definition:

Definition 1.1.1. Given $p > 0$ and q coprime, let $C_{p,q}$ be the 4-manifold bounding $L(p,q)$ obtained by plumbing D^2-bundles over S^2 according to a linear graph with weights $c_i \leq -2$ chosen so that $[c_1, \ldots, c_n] = -p/q$ (Figure 1.1). For conciseness, we denote $C_{p^2,pq-1}$ by $\mathcal{C}_{p,q}$.

We necessarily have that $C_{p,q}$ is negative definite [40]. However, we will often forgo the uniqueness of the c_i's in favor of more desirable continued fraction expansions and thus bounding 4-manifolds. It is immediate that any other linear plumbing of D^2-bundles over S^2 bounding $L(p,q)$ is related to $C_{p,q}$ via a sequence of blow-ups and blow-downs. It will be useful to understand the first homology of $L(p,q)$ when looking at it as a boundary of such a linear plumbing.

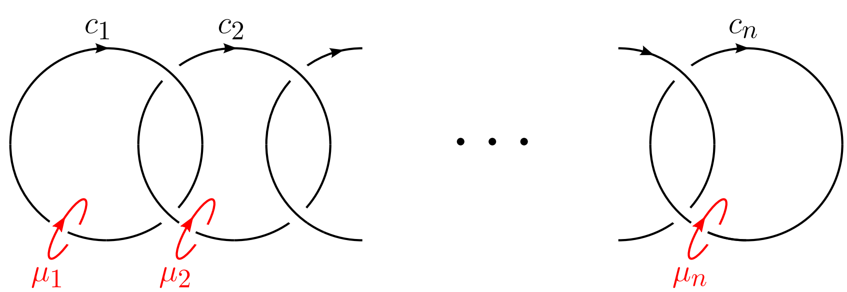

Figure 1.1: Preferred elements spanning $H_1(L(p,q))$ (in red) on a linear plumbing bounding $L(p,q)$. When each $c_i \leq -2$, this plumbing is denoted $C_{p,q}$.

Lemma 1.1.2. *Suppose that $L(p,q)$ is given by the linear plumbing in Figure 1.1 where the μ_i's are meridians spanning $H_1(L(p,q),\mathbb{Z})$. Then*

$$H_1(L(p,q),\mathbb{Z}) = \langle \mu_1 : (\det C_n)\mu_1 = 0 \rangle$$

where $C_i \doteq \begin{pmatrix} c_1 & 1 & \\ 1 & \ddots & 1 \\ & 1 & c_i \end{pmatrix}$ *and for* $i \in \{2, \ldots, n\}$, $\mu_i = (-1)^{i-1} (\det C_{i-1})\, \mu_1$.

Proof. Given a Dehn surgery description of a 3-manifold, one obtains a presentation for the first homology in terms of the right handed meridians of the (oriented) framed link [22]. In the above case, we find that

$$H_1(L(p,q), \mathbb{Z}) = \left\langle \mu_1, \ldots, \mu_n : \mu_2 = -c_1\mu_1, \{\mu_{i+1} = -c_i\mu_i - \mu_{i-1}\}_{i=2}^{n-1}, c_n\mu_n = -\mu_{n-1} \right\rangle$$

As $\mu_2 = -c_1\mu_1 = (-1)^{2-1}(\det C_{2-1})\mu_1$, the result follows by induction using that

$$\det C_k = c_k \det C_{k-1} - \det C_{k-2},$$

where, we are defining $\det C_{-1} = 1$. ☐

Remark 1.1.3. There is another important characterization of the lens space $L(p,q)$: given any continued fraction expansion $[c_1, \ldots, c_n]$ of $-p/q$, we can associate a 2-bridge knot (or link) K in S^3 such that $L(p,q)$ is the 2-fold cover of S^3 branched along K (see Montesinos [38] for details). If K happens to be smoothly slice (or bounds an appropriate ribbon immersion of a disk plus a Möbius band in the case that K is a 2-component link) then we can push the interior of that surface into the 4-ball and consider the 2-fold cover of B^4 branched along the surface. Such a ramified cover is necessarily a rational ball [27] and clearly bounds the given lens space. Interestingly, Lisca proves that every lens space $L(p,q)$ which bounds a rational ball, necessarily bounds a (possibly different) ball arising as such a covering of B^4 [34].

Casson and Harer show that for each pair $p > q > 0$ relatively prime, the 2-bridge knot or link associated to the fraction $-\dfrac{p^2}{pq-1}$ bounds such a surface Σ [6]. Moreover, they provide a method to obtain a handle decomposition for the 2-fold cover of B^4 along Σ.

Definition 1.1.4. For $p > q > 0$ coprime, let $B_{p,q}$ be the 2-fold cover of B^4 branched along Σ defined in [6] so that $\partial B_{p,q} \approx L(p^2, pq-1)$. Figure 1.2 gives a handle decomposition of $B_{p,q}$.

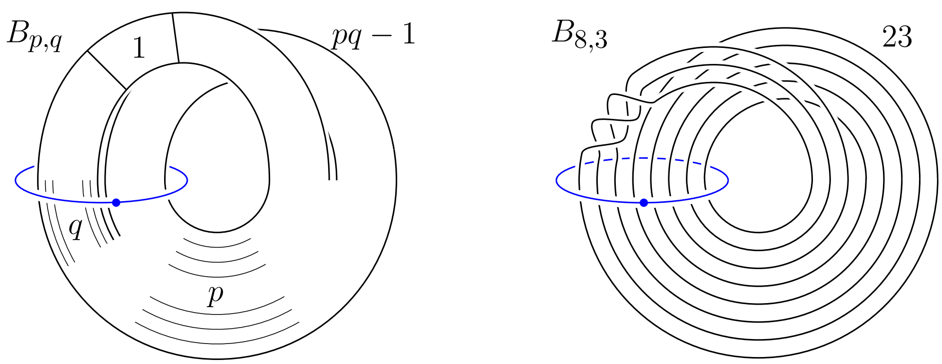

Figure 1.2: The rational ball $B_{p,q}$; e.g. $B_{8,3}$

Remark 1.1.5. It is worth noting that Casson and Harer do not explicitly give the handle decomposition of Figure 1.2. However, they do sketch a method which implies this decomposition. Lekili and Maydanskiy write down a (Stein) handle decomposition of $B_{p,q}$ [32] for general p and q coprime (see Figure 1.4 for this handle decomposition); the structure defined above is equivalent. This appears to be the first instance where $B_{p,q}$ is stated this way for all p and q. Prior to this, Fintushel and Stern express $B_{p,1} \approx B_{p,p-1}$ by the same handle decomposition [14]. Gompf gives a proof that certain Seifert fibered spaces are Stein fillable [21]. Viewing $L(p^2, pq-1)$ as Seifert fibered over S^2 with three "exceptional" fibers - two of which are honestly exceptional and the third being a regular fiber - Gompf's argument can be applied to give the Stein filling of $L(p^2, pq-1)$ that Lekili and Maydanskiy investigate (see the proof of Theorem 5.4(c) and Figure 43 of that paper [21]). At this point we have the necessary definitions to define the rational blow-down.

Definition 1.1.6 ([14,47]). Given X^4, a smooth 4-manifold containing $\mathcal{C}_{p,q}$ as a submanifold, the rational blow-down of X along $\mathcal{C}_{p,q}$ is the result of performing the codimension zero surgery of removing $\mathcal{C}_{p,q}$ and gluing $B_{p,q}$ in its place:

$$X_{p,q} = (X - \mathcal{C}_{p,q}) \cup B_{p,q}.$$

Since Definition 1.1.4 gives a handle decomposition of $B_{p,q}$, one can give the handle decomposition of $X_{p,q}$ given that of X. Viewing X as being built from $\mathcal{C}_{p,q}$ by attaching handles, then one can remove $\mathcal{C}_{p,q}$ and glue in $B_{p,q}$ by tracing the effect, on the belt-sphere of each 2-handle in $\mathcal{C}_{p,q}$, of an appropriate boundary diffeomorphism from $\partial\mathcal{C}_{p,q}$ to $\partial B_{p,q}$ (Akbulut gives general details of performing codimension zero surgeries at the handle level [2]). This type of surgery can be performed with *any* rational ball bounding $L(p^2, pq-1)$ in place of $B_{p,q}$. This leads to the question:

Question 1.1.7. Is the diffeomorphism type of a rational ball, with the same homotopy type as that of $B_{p,q}$, bounding $L(p^2, pq-1)$ unique?

Of course, the answer to this question is no, if we don't include some control on the homotopy type of such a ball - for instance consider the double D of a 2-handlebody with perfect (nontrivial) fundamental group and trivial second homology (one could choose a surgered $\Sigma(2,3,5) \times I$ for the 2-handlebody for instance), then $B_{p,q}\#D$ is a rational ball bounding $L(p^2, pq-1)$ with $\pi_1(B_{p,q}\#D) = \mathbb{Z}_p * \pi_1(D)$. Given the state of smooth 4-manifold theory, one wouldn't be unreasonable in thinking that the answer to Question 1.1.7 is still no. However, there is little technology available to deal with detecting exotic structures on manifolds without b_2^+ (let alone without b_2).

In spite of this, anytime one encounters a rational ball bounding $L(p^2, pq-1)$, it is natural to ask if it is diffeomorphic to $B_{p,q}$. To that end, consider the following family of handle decompositions of rational balls bounding lens spaces that appears in the literature. Yamada [59] defines this family directly via their handle decompositions as follows:

Definition 1.1.8. For $n, m \geq 1$ coprime, let $A_{m,n}$ be the 4-manifold obtained by attaching a 1-handle and a single 2-handle with framing mn to B^4 by attaching the 2-handle along a simple closed curve embedded on a once-punctured torus viewed in $S^1 \times S^2$ so that the attaching circle traverses the 1-handles of the torus m and n times respectively (Figure 1.3).

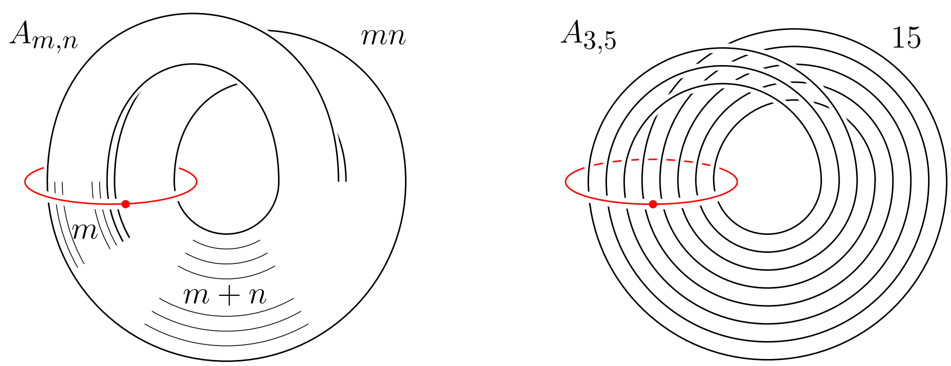

Figure 1.3: The rational ball $A_{m,n}$; e.g. $A_{3,5}$

Yamada goes on to define an involutive symmetric function, A, on the set of coprime pairs of positive integers such that if $A(p-q, q) = (m, n)$ then $\partial A_{m,n} \approx L(p^2, pq-1)$. Lemma 4.0.11 gives a formal definition of the function A; in the meantime, it suffices to know that in this case[2], $m+n = p$ and that $qm = \pm 1 \mod p$. We are led to a more tractable question than that of Question 1.1.7, posed by Kadokami and Yamada [25].

Question 1.1.9 ([25], Problem 1.9)**.** Supposing $A(p-q, q) = (m, n)$, so that $\partial A_{m,n} \approx \partial B_{p,q}$, when is $A_{m,n}$ diffeomorphic, homeomorphic, or even homotopic rel boundary to $B_{p,q}$?

[2]The sign ambiguity here arises because we will want to assume that $m < n$; obviously if $qm = -1 \mod p$ then $qn = 1 \mod p$.

To answer this question, we now turn to symplectic topology.

1.1.1 Symplectic and Contact Topology

We recall the relevant theory of syplectic 4-manifolds and contact 3-manifolds [7] - especially as it relates to handle decompositions of 4-manifolds [2, 21, 22, 41]. Recall that a smooth manifold X admits a *symplectic structure*, if there exists a 2-form $\omega \in \Omega^2(X)$ such that $\omega \wedge \omega$ is nowhere zero. The pair (X, ω) is a *symplectic manifold.* Similarly, a smooth 3-manifold Y admits a (coorientable) *contact structure*, if there exists a totally nonitegrable 2-plane field $\xi \subset TY$ such that $\xi = \ker \alpha$ for a 1-form $\alpha \in \Omega^1(Y)$ satisfying $\alpha \wedge d\alpha$ is nowhere zero. The pair (Y, ξ) is a *contact manifold.* Notice that ξ is well defined, but that *contact form* α can be scaled by any smooth nowhere zero function. Here totally nonitegrable means that for any embedding of a surface $\Sigma^2 \hookrightarrow (Y, \xi)$, the set of points $x \in \Sigma$ satisfying $\alpha(T_x\Sigma) = 0$ has positive codimension within Σ. That said, a 1-manifold L can be embedded in (Y, ξ) with $\alpha(T_xL) = 0$ for all $x \in L$ - in this case, L is a *Legendrian* submanifold of (Y, ξ). Notice that each Legendrian knot in (Y, ξ) inherits a well-defined trivialization of it's normal bundle from a transverse (with respect to the contact planes ξ_x) vector field along L. This trivialization is known as the contact (or Thurston-Bennequin) 0-framing of L. Unless specifically stated to the contrary, we will always consider Seifert framings (those measured against a framing specified by a Seifert surface) even when looking at Legendrian knots in a contact manifold.

Contact structures split into two types: tight and overtwisted. A contact manifold is *overtwisted* if it contains a disk bounding a Legendrian knot whose contact framing agrees with the Seifert framing induced by the disk. It is *tight* otherwise. For manifolds with simple enough fundamental groups (i.e. residually finite), the tight contact structures break further into two types: universally tight and virtually overtwisted determined by whether or not the tight contact structure pulls back to an overtwisted structure in a finite cover.

Example 1.1.10. S^3 admits a (unique) tight contact structure ξ_{st} arising as the set of complex tangencies in TS^3 by viewing $S^3 \subset \mathbb{C}^2$. Interestingly, the action by $\mathbb{Z}_p$ preserves these tangencies. Therefore, each lens space $L(p,q)$ inherits a "standard" contact structure $\bar{\xi}_{st}$ from ξ_{st}. As S^3 is the universal cover of $L(p,q)$, $(L(p,q), \bar{\xi}_{st})$ is universally tight.

One of the most successful ways of producing tight contact structures on a given 3-manifold is to realize that 3-manifold as the J-convex boundary of a Stein domain. A 4-manifold X is *Stein* if X admits a complex structure J so that equipped with this complex structure, X biholomorphically embeds in $\mathbb{C}^N$ for some N. Considering the distance from this embedding to a generic point in $\mathbb{C}^N$ gives a Morse function on X and each regular level set of this function becomes a (tight) contact 3-manifold Y where the contact structure arises as the set of complex tangencies in TY - i.e. $\xi = TY \cap JTY$. We'll refer to the compact codimension zero submanifold $W \subset X$, bounding Y as a *Stein domain.*

As there are natural Morse functions underlying any Stein structure, it is not surprising that Stein 4-manifolds have a handle theoretic characterization. In fact, the following theorem due to Eliashberg [11] and developed in the case of 4-manifolds by Gompf [21] allows us to recognize when a 4-manifold is a Stein domain via specific handle decompositions.

Theorem 1.1.11 ([11,21])**.** *A 4-manifold W admits the structure of a Stein domain if and only if W has a handle decomposition consisting of only handles of index less than or equal to two such that each 2-handle is attached along a Legendrian knot K in $\partial(\natural k(S^1 \times B^3)) = \# k(S^1 \times S^2)$ (equipped with the unique tight contact structure therein) with framing one less than the induced contact framing of K.*

Using Theorem 1.1.11, we see that $B_{p,q}$ admits a Stein structure.

Example 1.1.12. Lekili and Maydanskiy prove each that $B_{p,q}$ admits a Stein structure $(B_{p,q}, J_{p,q})$ specified by Figure 1.4 [32]. Indeed, by sliding the 2-handle of Figure 1.2 over the 1-handle q-times one arrives at Figure 1.4. It is immediate that the attaching circle is a Legendrian knot whose contact framing is $-pq$ as a Seifert framing. Therefore, the handle is attached with contact framing -1 and Theorem 1.1.11 gives that the unique Stein structure on $S^1 \times B^3$ extends across the 2-handle. In [32], the authors prove that $J_{p,q}$ fills the standard contact structure on $L(p^2, pq-1)$.

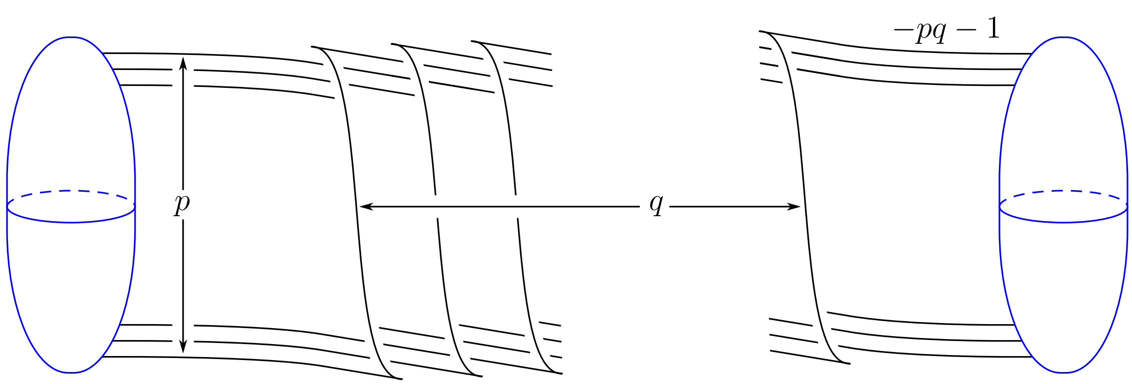

Figure 1.4: $(B_{p,q}, J_{p,q})$

A Stein structure equips W with an almost complex structure; it is natural to ask what $c_1(W,J)$ is for this almost complex structure. In the case of a Stein manifold presented as a handle decomposition as in Theorem 1.1.11, $c_1(W,J)$ can be computed combinatorially:

Proposition 1.1.13 ([21], Proposition 2.3)**.** *For a Stein structure J specified by an (oriented) Legendrian 2-handlebody, $c_1(X,J)$ is equal to a 2-cochain whose value on each $[K_i]$ evaluates to* $\mathrm{rot}(K_i)$.

Furthermore, as (W,J) imparts a contact structure ξ_J on ∂W, it is immediate that $c_1(\xi_J)$ is simply $c_1(W,J)$ restricted to ∂W. Thus for (W,J) as above, $PDc_1(\xi_J)$ is equal to a 1-chain satisfying that the coefficient on each right-handed meridian μ_i of K_i spanning $H_1(\partial W)$ is $\mathrm{rot}(K_i)$.

Returning to Question 1.1.9, we note that there is another characterization of the rational ball $B_{p,q}$ due to Lisca. To properly state this characterization we need the notion of a symplectic filling of a contact 3-manifold.

Definition 1.1.14. A *(weak) symplectic filling* of a contact 3-manifold (Y, ξ), is a symplectic 4-manifold (W, ω) together with an identification $\partial W \approx Y$ so that $\omega_{|\xi}$ is nonzero. (W, ω) is a *strong symplectic filling* of (Y, ξ) if we further require ω to be exact near ∂W so that its primitive is a contact form for ξ. A Stein domain (W, J) is a *Stein filling* of (Y, ξ) if (Y, ξ) is the J-convex boundary of W.

A considerable effort has been placed in determining which 3-manifolds are fillable in each sense, as well as classifying the smooth geography of such fillings. This geography can be extremely sparse; Eliashberg proves the tight contact structures on S^3 and $\#k(S^1 \times S^2)$ are uniquely Stein filled by B^4 and $\natural k(S^1 \times B^3) \approx B^4 \cup k$ 1-handles respectively [10]. It can also be quite complicated; for instance Akhmedov et. al. produce an infinite family of non-homeomorphic Stein fillings of a fixed contact 3-manifold [4]. Whereas, Akbulut and Yasui produce an infinite family of exotic fillings of a fixed contact 3-manifold [3].

Luckily, the geography of fillings of universally tight lens spaces don't admit such pathologies. McDuff proves that the diffeomorphism types of (weak) symplectic fillings of the lens space $(L(p, 1), \bar{\xi}_{st})$ are known to be unique upto smooth blow-up save for $L(4, 1)$ [37]. Each is filled by a manifold diffeomorphic to the Euler class $-p$ D^2-bundle over S^2. In the case of $L(4, 1)$, the rational ball $B_{2,1}$ gives the only other filling. (Plamenevskaya and Van Horn-Morris give a classification of the diffeomorphism types of all fillings of $L(p, 1)$ equipped with any tight contact structure [50].)

Furthering these results considerably, Lisca completely classifies the diffeomorphism types of symplectic fillings of $(L(p,q),\bar{\xi}_{st})$ [35]. In particular, Lisca defines 4-manifolds $W_{p,q}(\mathbf{n})$, such that

Theorem 1.1.15 ([35], Theorem 1.1). *Let $p > q \geq 1$ be relatively prime integers. Then each symplectic filling (W,ω) of $(L(p,q),\bar{\xi}_{st})$ is orientation preserving diffeomorphic to a smooth blowup of $W_{p,q}(\mathbf{n})$ for some $\mathbf{n} \in \mathbf{Z}_{p,q}$. Moreover, if $b_2(W) = 0$, then W is unique.*

We don't describe the spaces $W_{p,q}(\mathbf{n})$ in detail since we will only be interested in the case when a filling (W,ω) has $b_2(W) = 0$. In this case, the unique filling is $B_{p,q}$ (Figure 1.4). In light of Theorem 1.1.15, it is sufficient to prove that $A_{m,n}$ admits a symplectic structure that fills $(\partial A_{m,n},\bar{\xi}_{st})$ to conclude that $A_{m,n}$ is diffeomorphic to $B_{p,q}$ - thereby providing a complete answer to Question 1.1.9. Given a tight contact structure ξ on $L(p^2,pq-1)$, we need a means of determining when ξ and $\bar{\xi}_{st}$ specify the same contact structure on $L(p^2,pq-1)$ (up to contactomorphism). To answer this, we turn to homotopy invariants of the underlying 2-plane fields:

1.1.2 Homotopy Invariants of 2-Plane Fields

For identifying tight contact structures on lens spaces, it turns out to be enough to know that the two contact structures in question are homotopic as 2-plane fields. The following result of Honda and (independently) Giroux ensures this:

Theorem 1.1.16 ([23], Proposition 4.24; [19], Theorem 1.1). *The homotopy classes of the tight contact structures of $L(p,q)$ are all distinct. Moreover, if $q < p-1$, then all but exactly two tight contact structures on $L(p,q)$ are virtually overtwisted.*

Further, it is known for contact structures with c_1 torsion (which is always satisfied for 3-manifolds with $b_1 = 0$; e.g. lens spaces) that particular homotopy invariants completely determine their homotopy classes. Gompf defines two invariants [21], d_3 and Γ, and proves:

Theorem 1.1.17 ([21], Theorem 4.16)**.** *If* (Y^3, ξ_i) *for* $i = 1, 2$*, satisfies that* $c_1(\xi_1)$ *is torsion and* $\Gamma(\xi_1, \mathbf{s}) = \Gamma(\xi_2, \mathbf{s})$ *for some spin structure* $\mathbf{s}$*, then* ξ_1 *is homotopic to* ξ_2 *if and only if their* d_3 *invariants coincide.*

We recall the definitions of d_3 and Γ. For the three-dimensional invariant, d_3, we use the normalized definition [41] - but note that it is equivalent to the definition of θ originally defined by Gompf [21] which relies on the fact that each contact 3-manifold can be realized as the J-convex boundary of an almost complex 4-manifold as well as the fact that for (X^4, J), a closed almost complex 4-manifold, the quantity $c_1^2(X, J) - 3\sigma(X) - 2\chi(X) = 0$ where $\sigma(X)$ and $\chi(X)$ are the signature and Euler characteristic of X respectively.

Definition 1.1.18 ([21], Definition 4.2)**.** For a contact 3-manifold (M, ξ) with $c_1(\xi)$ torsion, the three-dimensional invariant

$$d_3(\xi) = \frac{1}{4}\left(c_1^2(X, J) - 3\sigma(X) - 2\chi(X)\right) \in \mathbb{Q}$$

for any almost complex 4-manifold (X, J) with $\partial X = M$ satisfying $TM \cap JTM = \xi$.

Γ associates to each spin structure on (M, ξ) an element of $H_1(M; \mathbb{Z})$. This is accomplished by noting that each spin structure on (M^3, ξ) provides a trivialization of TM, which, in turn, identifies $\mathrm{Spin}^{\mathbb{C}}(M)$ with $H^2(M; \mathbb{Z})$. Then, with respect to this identification, $\Gamma(\xi, \mathbf{s})$ is Poincaré dual to the spin$^{\mathbb{C}}$-structure induced by ξ. More concretely, noting that $\mathrm{Spin}^{\mathbb{C}}(Y)$ is an $H^2(Y)$-torsor, any two $\mathfrak{t}_0, \mathfrak{t}_1 \in \mathrm{Spin}^{\mathbb{C}}(Y)$, satisfy that their difference $\mathfrak{t}_1 - \mathfrak{t}_0$ is a well defined element of $H^2(Y)$. A spin structure on Y can be canonically viewed as a spin$^{\mathbb{C}}$-structure. Then $\Gamma(\xi, \mathbf{s})$ is Poincaré dual to the difference $\mathfrak{t}_\xi - \mathbf{s}$.

If $(M,\xi) = \partial(X,J)$, a Stein domain, there is the following characterization [21] of Γ that we make use of. Suppose that (X,J) is obtained by attaching 2-handles to a Legendrian link $K_1 \cup \ldots \cup K_k$ in $\partial(S^1 \times B^3 \natural \ldots \natural S^1 \times B^3)$ with Seifert framings given by $\mathrm{tb}(K_i) - 1$. Let $\tilde{X}$ be the result of surgering each 1-handle and let L_0 be the collection of 0-framed unknots, resulting from those surgeries.

Proposition 1.1.19 ([21], Theorem 4.12)**.** *Let (X,J) and $\tilde{X}$ be defined as above. Orient $K_1 \cup \ldots \cup K_k \cup L_0$ to obtain a spanning set for $H_2(\tilde{X};\mathbb{Z})$. Then $\Gamma(\xi,\mathbf{s}) \in H_1(\partial X;\mathbb{Z})$ is Poincaré dual to the restriction of the class $\rho \in H^2(X;\mathbb{Z})$ whose value on each $[K_i]$ is given by*

$$\rho([K_i]) = \frac{1}{2}\left(rot(K_i) + \ell k(K_i, L' + L_0)\right) \in \mathbb{Z}$$

where $rot(K) = 0$ for each $K \in L_0$ and where L' is the characteristic sublink (see Definition 3.2.2) associated to $\mathbf{s}$.

1.2 Statement of Results

The remainder of this thesis will provide a complete answer to Question 1.1.9 by proving the following theorems

Theorem 1.2.1. *For each pair of relatively prime positive integers, (m, n), $A_{m,n}$ carries a Stein structure, $\widetilde{J}_{m,n}$, filling a contact structure contactomorphic to the universally tight contact structure $\bar{\xi}_{st}$ on the lens space $\partial A_{m,n}$. In particular, $A_{m,n} \approx B_{p,q}$ if and only if $\partial A_{m,n} \approx \partial B_{p,q}$.*

The proof of Theorem 1.2.1 follows by first explicitly writing down a Stein structure on $A_{m,n}$ using Eliashberg and Gompf's [21] characterization of handle decompositions of Stein domains. Then, verifying that the homotopy invariants of the induced contact structures on the boundary agree with those of $(L(p^2, pq-1), \bar{\xi}_{\mathrm{st}})$ - thereby showing that the two structures are homotopic as 2-plane fields. Theorem 1.1.16 of Honda and Giroux shows that this is sufficient to conclude that these two contact structures are contactomorphic. Lisca's classification stated in Theorem 1.1.15 of the diffeomorphism types of symplectic fillings of $(L(p^2, pq-1), \bar{\xi}_{\mathrm{st}})$ then gives that $A_{m,n} \approx B_{p,q}$.

As the diffeomorphisms ensured in Theorem 1.2.1 rely on the nontrivial work of Lisca, Honda and Giroux, we go on to construct diffeomorphisms via handle theory alone. To do this, we first construct boundary diffeomorphisms, then show that these boundary diffeomorphisms can be extended to explicit diffeomorphisms between $B_{p,q}$ and $A_{m,n}$ through the carving process introduced by Akbulut [1]. In fact, we have:

Theorem 1.2.2. *Let $(m, n) = A(p-q, q)$ for some $p > q > 0$ relatively prime. Then there exists a diffeomorphism $f : \partial B_{p,q} \to \partial A_{m,n}$ such that f carries the belt sphere, μ_1, of the* *single 2-handle in $B_{p,q}$ to a slice knot in $\partial A_{m,n}$ (see Figure 1.5). Moreover, carving $A_{m,n}$ along the slice disk for $f(\mu_1)$ gives $S^1 \times B^3$.*

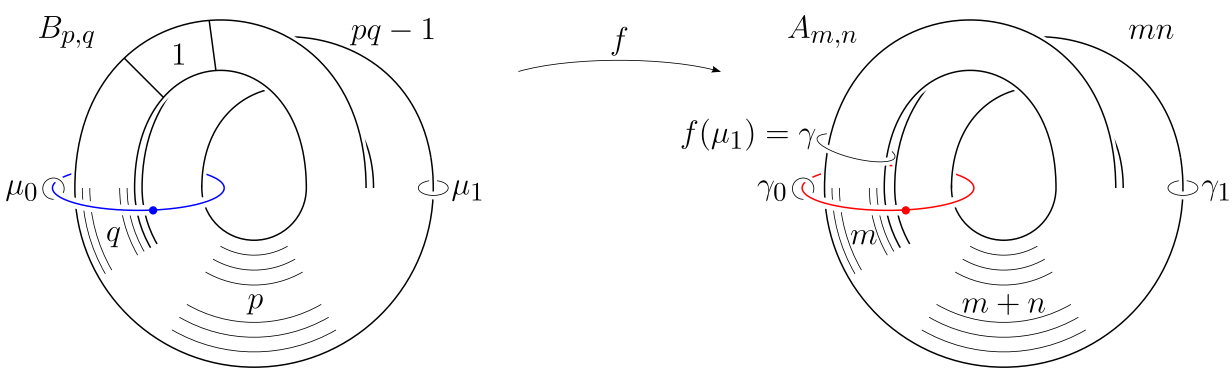

Figure 1.5: The boundary diffeomorphism $f : \partial B_{p,q} \to \partial A_{m,n}$.

Corollary 1.2.3. *f extends to a diffeomorphism $\tilde{f} : B_{p,q} \to A_{m,n}$.*

1.2.1 Conventions

Unless specifically stated to the contrary, throughout the paper, we assume $p - q > q \geq 1$, $n > m \geq 1$, and that both pairs are relatively prime. As $B_{p,q} \approx B_{p,p-q}$ and $A_{m,n} \approx A_{n,m}$, this assumption doesn't represent a restriction.

The continued fractions associated to $-p^2/(pq-1)$ involve the Euclidean algorithm [6,59]. Therefore, we use the Euclidean algorithm to define sequences of remainders and quotients of p and q as follows:

Definition 1.2.4. For $p > q \geq 1$, relatively prime, let $\{r_i\}_{i=-1}^{\ell+2}$ and $\{s_i\}_{i=0}^{\ell+1}$ be defined recursively by $r_{-1} \doteq p$, $r_0 \doteq q$ and

$$r_{i+1} = r_{i-1} \mod r_i, \qquad r_{i-1} = r_i s_i + r_{i+1}.$$

Let ℓ be the last index where $r_\ell > 1$ so that $r_{\ell+1} = 1$ and $r_{\ell+2} \doteq 0$.

For bookkeeping purposes, we'll differentiate between the above sequences for p and q and the analogously defined sequences $\{\rho_i\}_{i=-1}^{\ell+2}$ and $\{\sigma_i\}_{i=0}^{\ell+1}$ associated to $n > m \geq 1$. Furthermore, provided that $p-q > q$, ℓ agrees between the two sequences when $A(p-q, q) = (m, n)$ or (n, m) (see Remark 3.1.8 and Lemma 4.0.11).

1.2.2 Organization

The paper is organized as follows: In Chapter 2, we construct Stein structures on each $A_{m,n}$ using Eliashberg and Gompf's characterization of handle decompositions of Stein domains, proving Theorem 1.2.1. In Chapter 3, we outline the carving process and construct explicit diffeomoprhisms from $\partial B_{p,q}$ to $\partial A_{m,n}$ - proving Theorem 1.2.2. For clarity we relegate much of the required algebra to Chapter 4. Further, we provide a complete example, working out many of the handle-theoretic arguments of Chapters 2 and 3 in the Appendix.

Chapter 2

Identifying Rational Balls By Fillings

This chapter is devoted to proving that each rational ball $A_{m,n}$ admits a Stein structure filling a universally tight contact structure on the lens space $\partial A_{m,n}$ - thereby proving Theorem 1.2.1. Throughout the chapter, we will assume that we have fixed $n > m > 0$ and $p - q > q > 0$ so that $\partial A_{m,n} \approx L(p^2, pq-1)$. Ultimately we need to understand the classification of tight contact structures on $L(p^2, pq-1)$, so we handle this first.

2.1 Enumerating Tight Contact Structures

To begin, we determine the negative definite plumbing $\mathcal{C}_{p,q}$. Where convenient, we use a weighted tree Γ to represent a plumbing of disk-bundles over the sphere (see Nuemann [39]). Let $X(\Gamma)$ denote the resulting 4-manifold and let $Y(\Gamma) = \partial X(\Gamma)$. The following is proved in Chapter 3 (see Corollaries 3.1.3 and 3.1.7).

Proposition 2.1.1. *For $p > q > 0$ coprime, the lens space $L(p^2, pq-1)$ bounds $X(\Gamma)$ where Γ is the weighted graph of Figure 2.1 and where $\{r_i\}_{i=-1}^{\ell+2}$ and $\{s_i\}_{i=0}^{\ell+1}$ are defined as in*

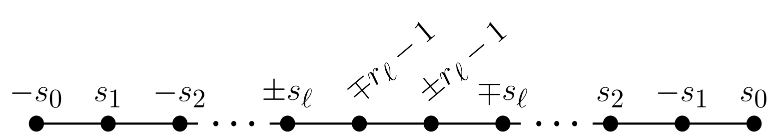

Figure 2.1: A linear plumbing bounding $L(p^2, pq-1)$.

Definition 1.2.4.

$X(\Gamma)$ defined in Proposition 2.1.1 has spheres of positive self intersection and is therefore *not* $\mathcal{C}_{p,q}$. We can alter the plumbing of Figure 2.1 through a series of blow-ups and blow-downs to repair this. With that goal in mind, consider the standard Lemma:

Lemma 2.1.2. *Suppose that* $Y^3 \approx Y(\Gamma)$ *is given as the boundary of a plumbing of* D^2*-bundles over* S^2 *plumbed according to a weighted tree* Γ*. If* $v \in \Gamma$ *has valence at most two and weight* $a_i > 0$*, then* $Y^3 \approx Y(\Gamma')$ *for the graph* Γ' *obtained from* Γ *by replacing the Euler-class* a_i D^2*-bundle specified by* v *with a chain of* $a_i - 1$ *Euler-class* -2 D^2*-bundles (with the framing of* v*'s neighbors changing accordingly) as in Figure 2.2.*

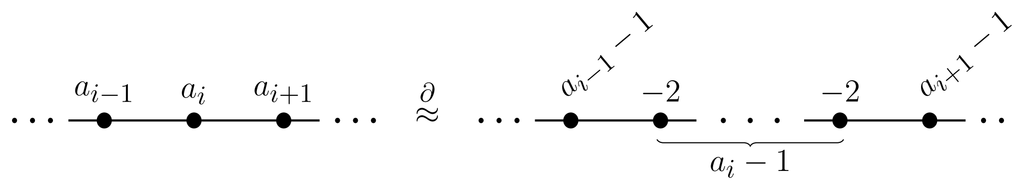

Figure 2.2: Removing spheres of positive self intersection.

Lemma 2.1.2 allows the exchange of each positive Euler-class disk bundle for, possibly many negative Euler-class bundles without altering the boundary. By applying it to Proposition 2.1.1, we immediately arrive at:

Corollary 2.1.3. *For* $p > q \geq 1$*, coprime, let* $\{s_i\}_{i=0}^{\ell}$ *and* $\{r_i\}_{i=-1}^{\ell+1}$ *be as defined in Definition 1.2.4, the space* $\mathcal{C}_{p,q}$ *is given by one of the linear plumbings of Figure 2.3 (depending upon the parity of* ℓ*).*

Remark 2.1.4. Notice that since each $s_i \geq 1$ we have that each weight in the graphs of Figure 2.3 are less than or equal to -2. By Definition 1.1.1, Figure 2.3 specifies $\mathcal{C}_{p,q}$. In general, $\mathcal{C}_{p,q}$ admits numerous Stein fillings. According to the classification of tight contact structure on lens spaces [19,23] each such contact structure arises as the boundary of a Stein

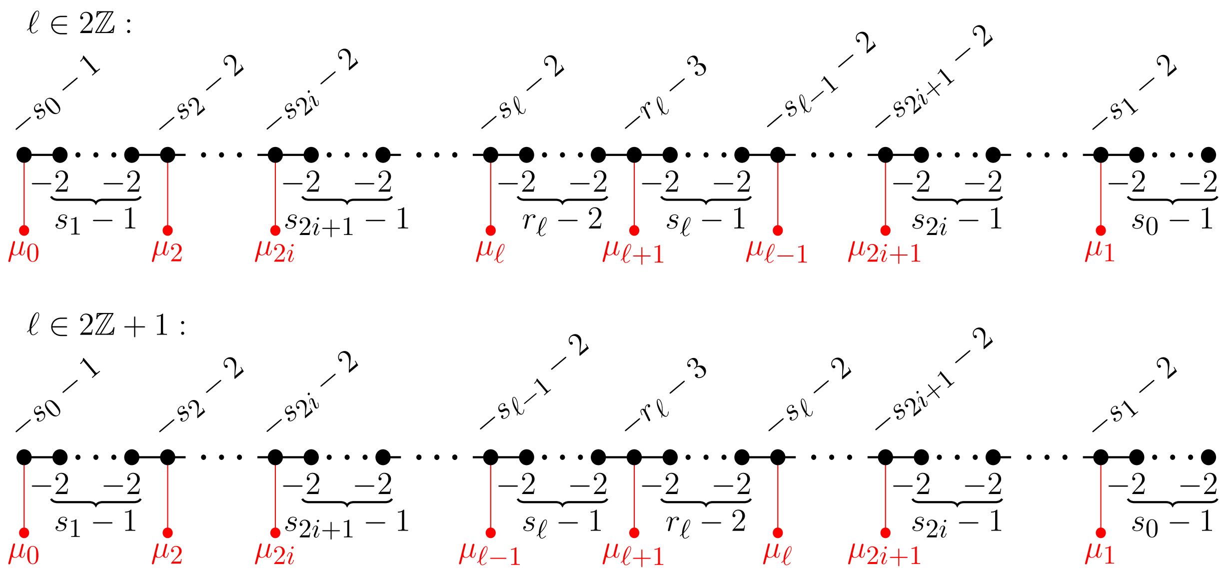

Figure 2.3: $\mathcal{C}_{p,q}$ when $\ell \in 2\mathbb{Z}$ and when $\ell \in 2\mathbb{Z}+1$ with relevant meridians used in homology calculations (in red).

structure on $\mathcal{C}_{p,q}$ obtained by attaching the 2-handles of $\mathcal{C}_{p,q}$ along Legendrian unknots whose Seifert framings are each one less than the their Thurston-Bennequin framings. For each $n < -1$, by stabilizing the standard Legendrian unknot positively and or negatively as needed, there are exactly $|n| - 1$ distinct rotation numbers for Legendrian unknots with Thurston-Bennequin framing equal to $n + 1$: namely $n + 2, n + 4, \ldots, -n - 2$ (see Figure 2.4). In particular, each unknot in the handle decomposition of $\mathcal{C}_{p,q}$ with Seifert framing -2

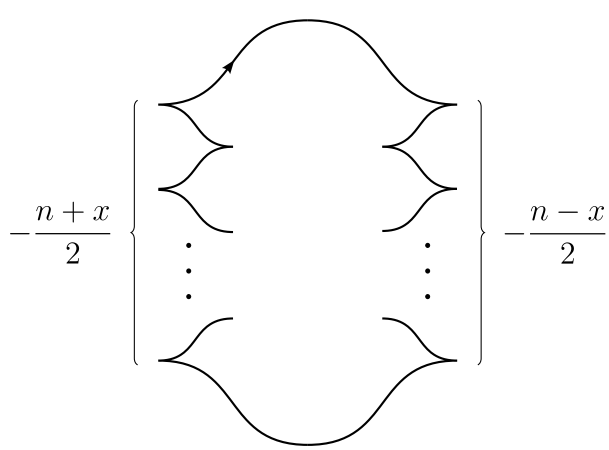

Figure 2.4: A Legendrian unknot with Thurston-Bennequin framing $n + 1 < 0$ and rotation number x.

necessarily has rotation number zero for any Stein handle attachment. Therefore, if we let K_i denote the attaching circle of the 2-handle in $\mathcal{C}_{p,q}$ whose belt-sphere is the meridian given by μ_i as labeled in Figure 2.3, we see that specifying rotation numbers only for K_i fixes a Stein structure on $\mathcal{C}_{p,q}$. With this in mind, for each $x = (x_0, \ldots, x_{\ell+1})$ chosen so that

$$
\begin{aligned}
x_0 &\in \{1 - s_0, 3 - s_0, \ldots, s_0 - 1\}, \\
x_i &\in \{-s_i, 2 - s_i, \ldots, s_i\},\ i \in \{1, \ldots, \ell\} \\
x_{\ell+1} &\in \{-1 - r_\ell, 1 - r_\ell, \ldots, r_\ell + 1\},
\end{aligned}
$$

we get a unique Stein structure on $\mathcal{C}_{p,q}$ inducing a distinct (up to isotopy) tight contact structure on $L(p^2, pq - 1)$. In an abuse of notation, we ignore the obvious dependence on p and q and choose to call this structure J_x. As constructed, Theorem 1.1.13 gives that

$$PDc_1(\mathcal{C}_{p,q}, J_x) = \sum_{i=0}^{\ell+1} x_i[K_i].$$

As each J_x has distinct first Chern class, no two can specify the same Stein structure. The uniqueness of the isotopy classes of the induced contact structure, ξ_{J_x}, follows from a result of Lisca and Matić [36]. It is a much more subtle fact, due to Honda and Giroux, that these Stein structures induce *all* the isotopy classes of tight contact structures on $L(p^2, pq - 1)$:

Theorem 2.1.5 ([23], Theorem 2.1; [20], Theorem 1.1)**.** *The number of distinct isotopy classes of tight contact structure on $L(p^2, pq - 1)$ is equal to*

$$s_0 \left(\prod_{i=1}^{\ell} (s_i + 1) \right) (r_\ell + 2).$$

It is known that $J_{x_{\min}}$ and $J_{x^{\max}}$ induce the two universally tight contact structures on $L(p^2, pq-1)$, where $x^{\max}$ fixes the largest allowable rotation number on each K^i and $x_{\min} = -x^{\max}$. Let ξ_x, $\xi_{\min}$ and $\xi^{\max}$ be the contact structures induced by J_x, $J_{\min}$ and $J^{\max}$ respectively; similarly define the spin$^{\mathbb{C}}$-structures $\mathfrak{t}_x$, $\mathfrak{t}_{\min}$ and $\mathfrak{t}^{\max}$. $\xi_{\min}$ and $\xi^{\max}$ are also induced by the Stein structures $(B_{p,q}, J_{p,q})$ and $(B_{p,p-q}, J_{p,p-q})$ specified in Example 1.1.12. Therefore, the spin$^{\mathbb{C}}$-structures $\mathfrak{t}_{\min}$ and $\mathfrak{t}^{\max}$ both extend over $B_{p,q}$ to spin$^{\mathbb{C}}$-structures $\mathfrak{s}_{\min}, \mathfrak{s}^{\max} \in \mathrm{Spin}^{\mathbb{C}}(B_{p,q})$. No other $\mathfrak{t}_x$ has this property:

Proposition 2.1.6. *Let $\Xi_{p,q}$ denote the set of homotopy classes of 2-plane fields induced by tight contact structures on $L(p^2, pq-1)$ and let $\mathcal{S} = \left\{\mathfrak{t}_\xi \in \mathrm{Spin}^{\mathbb{C}}(L(p^2, pq-1)) : \xi \in \Xi_{p,q}\right\}$, then $\mathcal{S}$ contains exactly two* spin$^{\mathbb{C}}$*-structures that extend across the ball $B_{p,q}$; both of which arise from contact structures contactomorphic to $\bar{\xi}_{st}$.*

Before we prove Proposition 2.1.6 we recall the obstruction to extending a given spin$^{\mathbb{C}}$-structure $\mathfrak{t} \in \mathrm{Spin}^{\mathbb{C}}(L(p^2, pq-1))$ across a rational ball bounding $L(p^2, pq-1)$. We can measure this obstruction against any fixed spin$^{\mathbb{C}}$-structure which is known to extend. As every 4-manifold admits a spin$^{\mathbb{C}}$-structure (which extends its restriction to the boundary), we always have such an element to measure against.

Lemma 2.1.7. *Suppose that $\mathcal{B}$ is a rational ball bounding $L(p^2, pq-1)$. For each pair $\mathfrak{t}_0, \mathfrak{t}_1 \in \mathrm{Spin}^{\mathbb{C}}(\partial\mathcal{B})$ such that $\mathfrak{t}_0$ extends across $\mathcal{B}$ to some $\mathfrak{s}_0 \in \mathrm{Spin}^{\mathbb{C}}(\mathcal{B})$, $\mathfrak{t}_1$ extends across $\mathcal{B}$ if and only if p divides the difference $\mathfrak{t}_0 - \mathfrak{t}_1 \in H^2(\partial\mathcal{B})$.*

Proof. From the standard fibration $S^1 \to \mathrm{Spin}^{\mathbb{C}}(4) \to SO(4)$, we find that extending $\mathfrak{t}_1$

amounts to the following lifting problem:

$$\begin{tikzcd}
 & B_{S^1} \arrow[d] \\
\partial\mathcal{B} \arrow[r, "\mathfrak{t}_1"] \arrow[d, hook] & B_{\mathrm{Spin}^{\mathbb{C}}(4)} \arrow[d] \\
\mathcal{B} \arrow[r, "\tau"'] \arrow[ur, dashed, "\varphi"] & B_{SO(4)}
\end{tikzcd}$$

where τ is the classifying map for the tangent bundle $T\mathcal{B}$. The obstructions to extending $\mathfrak{t}_1$ to such a map φ are in the cohomology groups $H^{i+1}\left(\mathcal{B}, \partial\mathcal{B}; \pi_i\left(B_{S^1}\right)\right)$. Since $B_{S^1} \simeq K(\mathbb{Z}, 2)$, the only obstruction occurs at $H^3(\mathcal{B}, \partial\mathcal{B}) \cong H_1(\mathcal{B}) \cong \mathbb{Z}_p$. By assumption $\mathfrak{t}_0$ extends, therefore $\mathfrak{t}_1$ extends if and only if the image of the difference $\mathfrak{t}_0 - \mathfrak{t}_1$ is trivial under the map

$$H^2(\partial\mathcal{B}) \cong \mathbb{Z}_{p^2} \xrightarrow{[p]} \mathbb{Z}_p \cong H^3(\mathcal{B}, \partial\mathcal{B})$$

sending $\mathfrak{t}_0 - \mathfrak{t}_1$ to its mod p reduction - giving the result. $\square$

We can use Lemma 2.1.7 to determine which other spin$^{\mathbb{C}}$-structures induced by some J_x extend over $B_{p,q}$. Note that for any spin-structure $s \in \mathrm{Spin}(L(p^2, pq-1))$ the difference

$$PD\left(\Gamma(\xi_y, s)\right) - PD\left(\Gamma(\xi_x, s)\right) = (\mathfrak{t}_y - s) - (\mathfrak{t}_x - s) = \mathfrak{t}_y - \mathfrak{t}_x$$

doesn't depend on the choice of spin-structure. Using Proposition 1.1.19, we calculate

$$PD(\mathfrak{t}_y - \mathfrak{t}_x) = \sum_{i=0}^{\ell+1} \frac{y_i - x_i}{2}\mu_i = \sum_{i=0}^{\ell+1}(-1)^i \frac{y_i - x_i}{2}\rho_{\ell-i+1}\mu_0$$

where the last equality[1] follows by applying Lemma 1.1.2 as well as Lemma 4.0.15 (which works out the determinants involved in Lemma 1.1.2) to write each μ_i as an appropriate multiple of μ_0. As an aside, since $c_1(\mathfrak{t}) = \mathfrak{t} - \bar{\mathfrak{t}}$ where $\bar{\mathfrak{t}}$ is the conjugate spin$^{\mathbb{C}}$-structure and since $\bar{\mathfrak{t}}_x = \mathfrak{t}_{-x}$ we can write down the Poincaré dual of the first Chern class for each tight contact structure on $L(p^2, pq-1)$ in terms of the standard generator for $H_1(L(p^2, pq-1))$:

$$PDc_1(\xi_x) = PD(\mathfrak{t}_x - \mathfrak{t}_{-x}) = \sum_{i=0}^{\ell+1} (-1)^i x_i \rho_{\ell-i+1}\mu_0.$$

Of course, this also follows by simply restricting $PDc_1(\mathcal{C}_{p,q}, J_x)$ to the boundary and applying Lemma 1.1.2.

Proof of Proposition 2.1.6. Suppose that $\mathfrak{t} \in \mathcal{S}$ extends across $B_{p,q}$. We can assume that $\mathfrak{t} = \mathfrak{t}_x$ for some Stein structure $(\mathcal{C}_{p,q}, J_x)$ on $\mathcal{C}_{p,q}$. Lemma 2.1.7 gives that $\mathfrak{t}_x$ extends if and only if p divides the difference $PD(\mathfrak{t}^{\max} - \mathfrak{t}_x)$ in $H_1(L(p^2, pq-1)$. Write $x = x^{\max} - 2c$ where $c = (c_0, c_1, \dots, c_{\ell+1})$ necessarily satisfies $c_0 \in \{0, 1, \dots, s_0 - 1\}$, $c_i \in \{0, 1, \dots, s_i\}$ for each $i \in \{1, 2, \dots, \ell\}$ and $c_{\ell+1} = \{0, 1, \dots, r_\ell + 1\}$. Then we find

$$PD(\mathfrak{t}^{\max} - \mathfrak{t}_x) = \sum_{i=0}^{\ell+1} (-1)^i \frac{x_i^{\max} - x_i}{2} \rho_{\ell-i+1}\mu_0 = \sum_{i=0}^{\ell+1} (-1)^i c_i \rho_{\ell-i+1}\mu_0.$$

Therefore, we investigate solutions to $\sum_{i=0}^{\ell+1} (-1)^i c_i \rho_{\ell-i+1} \equiv 0 \mod p$. We will prove in Corollary 4.0.17 that there are exactly two solutions - namely $c = 0$ and $2c = x^{\max}$ - giving that the only spin$^{\mathbb{C}}$-structures which extend correspond to $x^{\max}$ and $x_{\min} = -x^{\max}$ - which are known to induce the universally tight contact structures on $L(p^2, pq-1)$. $\square$

[1] Recall that $\{\rho_i\}_{i=-1}^{\ell+1}$ is the collection of remainders when applying the Euclidean algorithm to n and m as in Definition 1.2.4.

According to Theorem 1.1.17, two 2-plane fields (with torsion c_1) are homotopic if and only if they have the same Γ *and* d_3 invariants. Lisca proves that in the case of tight contact structures on a lens space, the Γ invariant alone is enough [33] - that is if $\Gamma(\xi_x, \mathbf{s}) = \Gamma(\xi_y, \mathbf{s})$, then ξ_x is homotopic to ξ_y (and their d_3 invariants necessarily coincide). Of course, one cannot expect the same result to hold with d_3 in place of Γ. However, the d_3-invariant does detect the universally tight structures on $L(p^2, pq-1)$. In fact by combining Proposition 2.1.6 with the "correction terms" from Heegaard Floer homology we arrive at the following Proposition known to experts:

Proposition 2.1.8. *Every tight contact structure ξ on $L(p^2, pq-1)$ with $d_3(\xi) = -1/2$ is universally tight.*

Ozsváth and Szabó define relatively $\mathbb{Z}$-graded homology groups $HF^{\pm}, HF^{\infty}$ associated to each 3-manifold endowed with a spin$^{\mathbb{C}}$-structure [44,45]. If the spin$^{\mathbb{C}}$-structure is torsion, one obtains absolute $\mathbb{Q}$-gradings [46]. Using this grading, Ozsváth and Szabó define the correction term $d(Y, \mathfrak{t})$ of any rational homology spin$^{\mathbb{C}}$ 3-sphere $(Y, \mathfrak{t})$ as the minimal degree of the image of a non-torsion element of $HF^{\infty}(Y, \mathfrak{t})$ in $HF^{+}(Y, \mathfrak{t})$ [43]. Of interest to the present problem, is the following result of Ozsváth, Stipsicz and Szabó.

Proposition 2.1.9 ([42], Corollary 1.7)**.** *Suppose (Y, ξ) is a rational homology 3-sphere equipped with a symplectically fillable contact structure ξ supported by a planar open book, then*

$$d_3(\xi) + \frac{1}{2} = -d\left(Y, \mathfrak{t}_{\xi}\right).$$

As every tight contact structure on a lens space is supported by a planar open book [53], we gain knowledge about the three-dimensionsal invariant d_3 from the correction term and vice versâ. In particular, compare Lemma 2.1.7 with the following result of Jabuka, Robins and Wang:

Proposition 2.1.10 ([24]). *If* $\mathfrak{t}_0$ *and* $\mathfrak{t}_1$ *are spin-c structures on* $L(p^2, pq-1)$ *satisfying that their respective correction terms vanish, then* p *divides* $\mathfrak{t}_0 - \mathfrak{t}_1 \in H^2(L(p^2, pq-1))$.

Proof of Proposition 2.1.8. As ξ is symplectically fillable and supported by a planar open book, Proposition 2.1.9 gives that

$$d\left(L(p^2, pq-1), \mathfrak{t}_\xi\right) = -d_3(\xi) - \frac{1}{2} = 0.$$

Proposition 2.1.10 then gives that p divides $\mathfrak{t}_{\bar{\xi}_{st}} - \mathfrak{t}_\xi$; and thus $\mathfrak{t}_\xi$ extends across $B_{p,q}$ as $\mathfrak{t}_{\bar{\xi}_{st}}$ does. Clearly $\xi \in \Xi_{p,q}$, so by Proposition 2.1.6, ξ is contactomorphic to $\bar{\xi}_{st}$. $\square$

2.2 Stein Structures on $A_{m,n}$

In this section, we show that $A_{m,n}$ admits a Stein structure. In light of the results of the previous section, the existence of such a structure immediately proves Theorem 1.2.1. To accomplish this, we use Eliashberg and Gompf's handle characterization of Stein surfaces stated in Theorem 1.1.11. This is done constructively; that is, we isotope the attaching circle of the 2-handle in $A_{m,n}$ so that it becomes Legendrian with respect to the tight contact structure on $S^1 \times S^2$ and so that the 2-handle is then being attached with framing one less than the resulting contact framing. For clarity, a worked example of Proposition 2.2.1 is contained in the Appendix (Figure A.1) for the rational ball $A_{3,5}$.

Proposition 2.2.1. *Each $A_{m,n}$ admits a Stein structure, $\widetilde{J}_{m,n}$, specified by the Stein handle decomposition of either Figure 2.5 or 2.6 depending upon the parity of ℓ where we assume $\{\rho_i\}_{i=-1}^{\ell+1}$ and $\{\sigma_i\}_{i=0}^{\ell}$ are as in Definition 1.2.4.*

Proposition 2.2.1, will be proved inductively. To motivate the proof as well as set up the base cases for induction we note that by sliding the 2-handle of $A_{m,n}$ once under the 1-handle (upper left of Figure 2.7) we find a route toward realizing the 2-handle in $A_{m,n}$ as a Stein handle attachment by an appropriate isotopy of the attaching circle K. Indeed if we refer to the portion of K passing behind the central plane of the two attaching balls of the 1-handle as the "bad" strand. We see that we can pair off negative crossings in the bad strand with positive crossings in K by "unraveling" the 2-handle. To accomplish this, begin by dragging the bad strand once over the 1-handle (bottom of Figure 2.7). By dragging the bad strand another $\sigma_0 - 1$ times over the 1-handle we find the bad strand now involves $\rho_1 - 1$ strands rather than the original $\rho_{-1} - 1$ strands (upper right of Figure 2.7). In fact, if $\rho_1 = 1$, then we immediately have the Stein structure $(A_{m,n}, \widetilde{J}_{m,n})$ of Proposition 2.2.1.

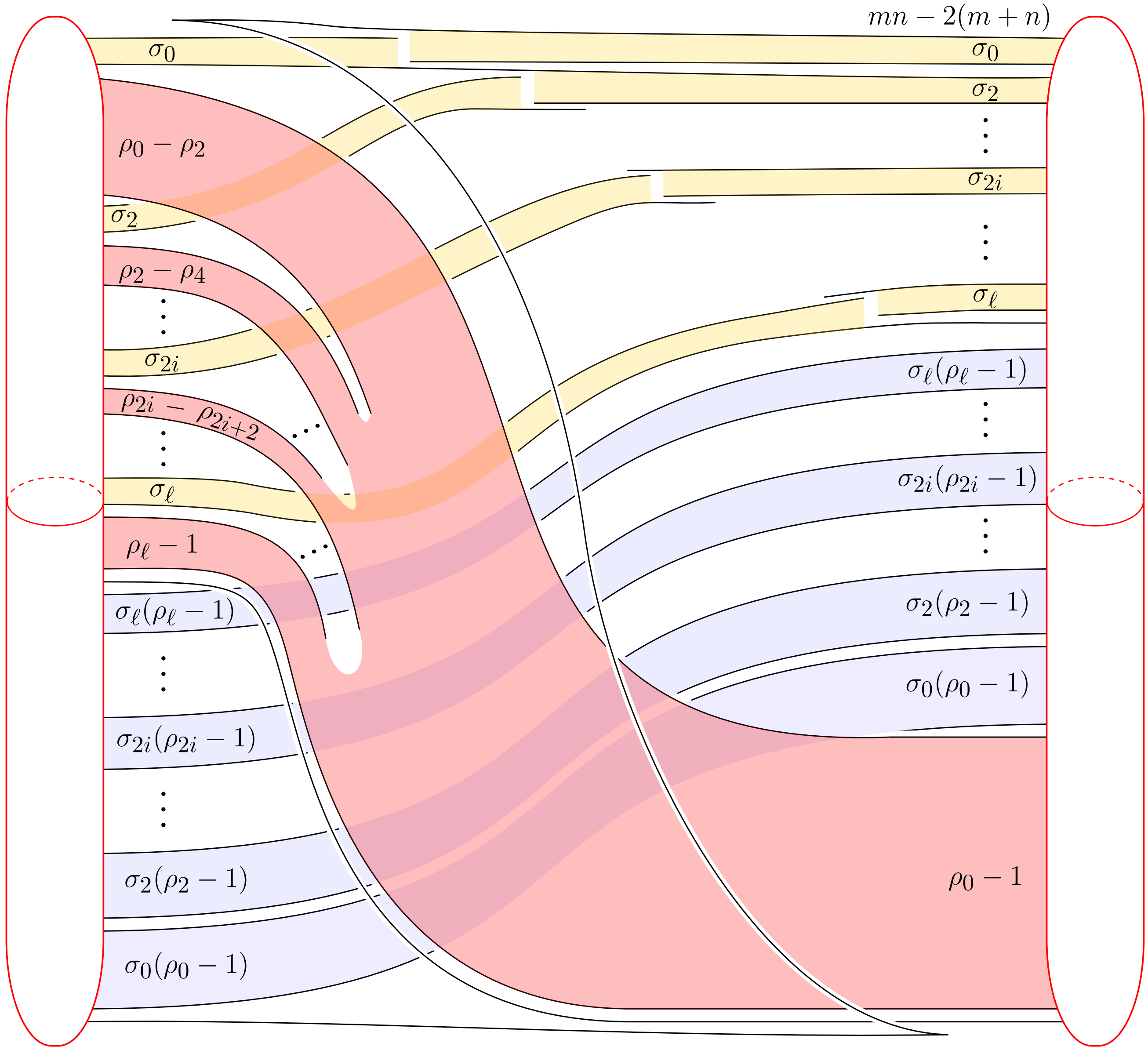

Figure 2.5: $(A_{m,n}, \widetilde{J}_{m,n})$ when $\ell \in 2\mathbb{Z}$. *Warning: The vertical scaling differs between the left and right foot of the 1-handle.*

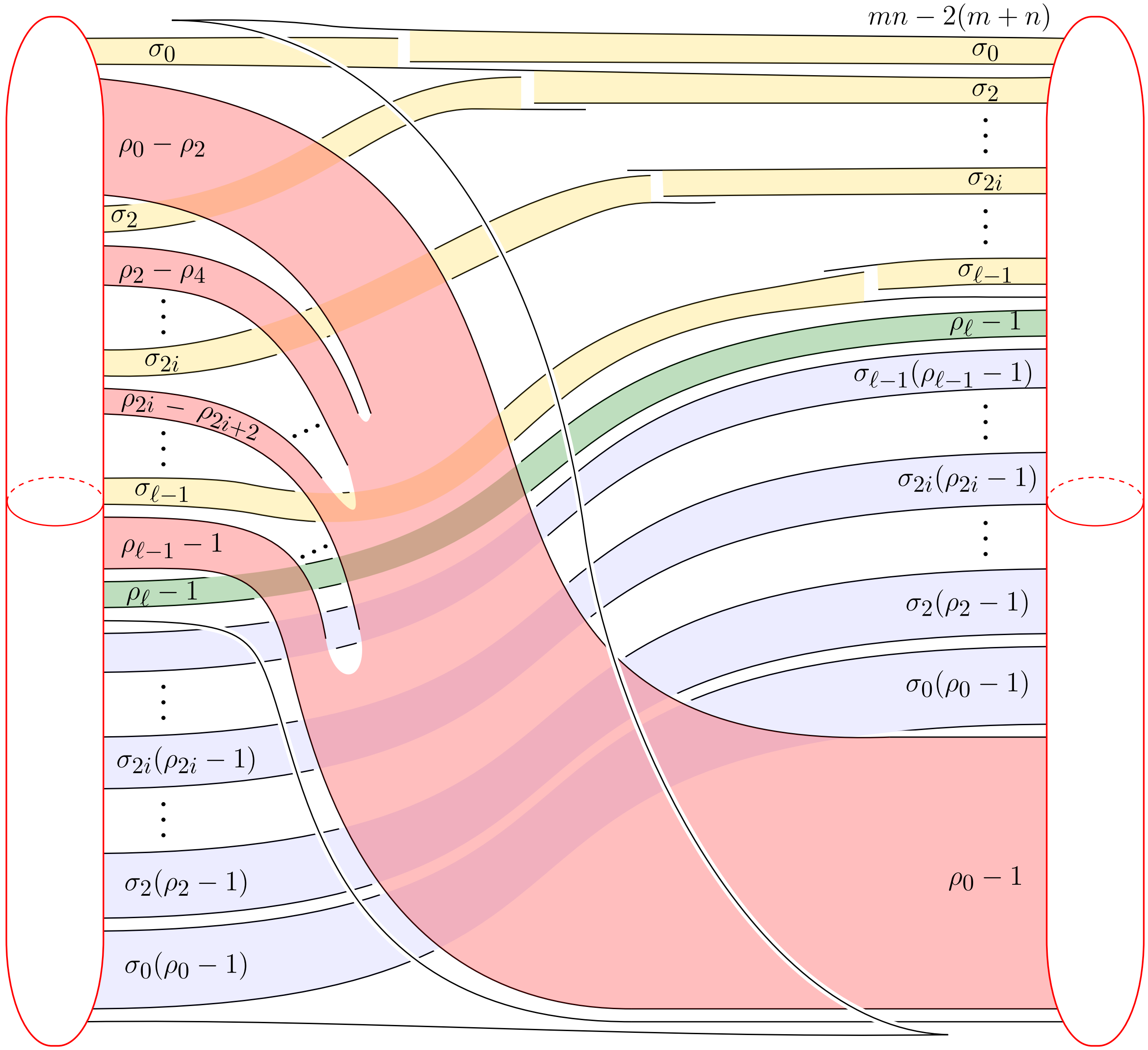

Figure 2.6: $(A_{m,n}, \widetilde{J}_{m,n})$ when $\ell \in 2\mathbb{Z}+1$. *Warning: The vertical scaling differs between the left and right foot of the 1-handle.*

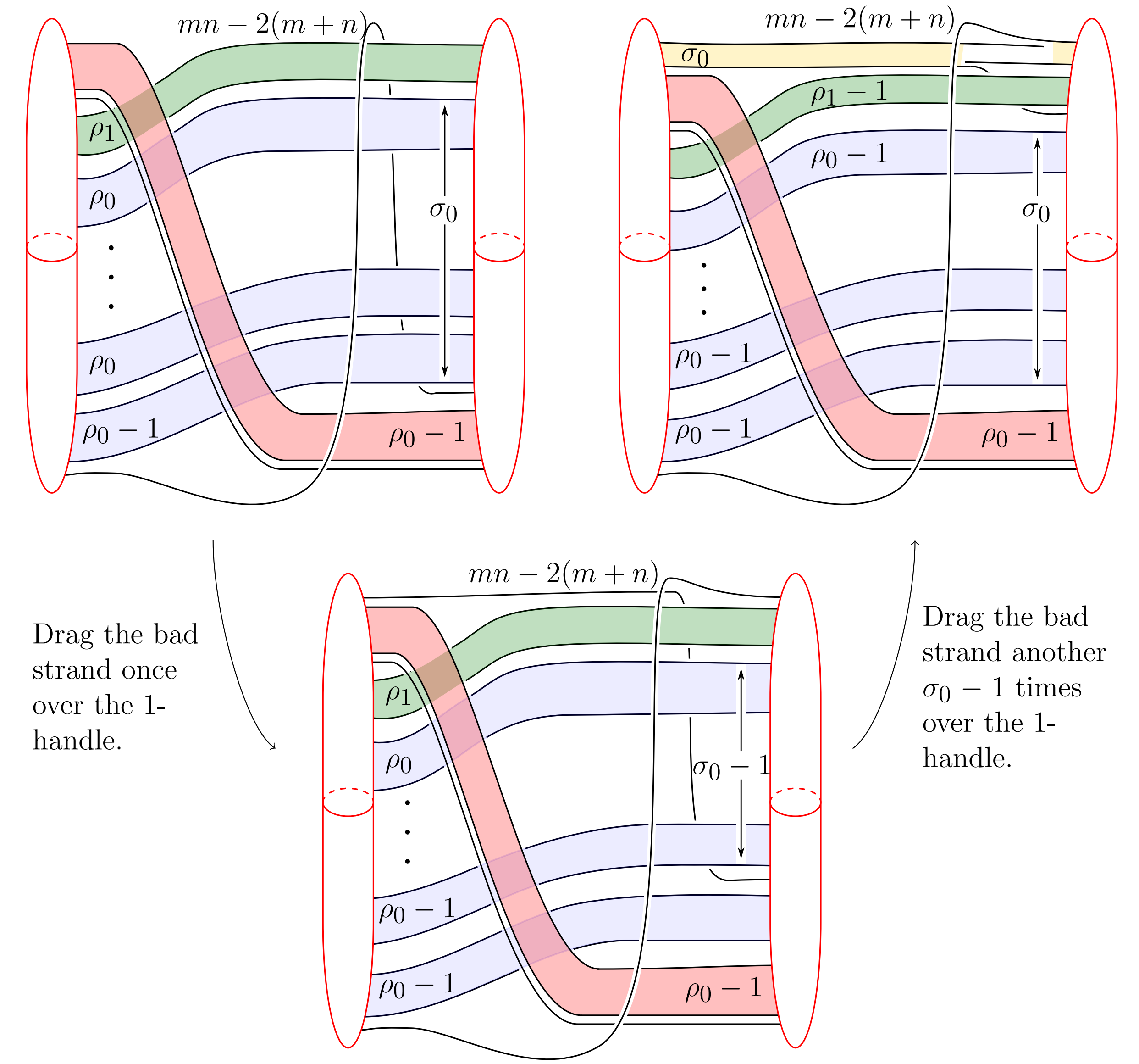

Figure 2.7: The result of sliding the attaching circle K once under the 1-handle, followed by isotopies of K as described.

Remark 2.2.2. We cannot assume $\rho_1 = 1$, that said, the same principle holds far more generally; that is, there exist isotopies of K taking the bad strand from involving $\rho_{2i-1} - 1$ strands to involving $\rho_{2i+1} - 1$ strands. This is the content of Proposition 2.2.3. Notice, for any tangle in the red band of K (upper right of Figure 2.7), can be shifted down ρ_1 strands by dragging it over the 1-handle $\sigma_0 + 1$ times. Similarly, any tangle in the blue bands can be shifted up $\rho_0 - 1$ strands by dragging it once over the 1-handle. Such isotopies will prove:

Proposition 2.2.3. *For each integer k such that $0 \le 2k \le \ell$, $A_{m,n}$ is specified by attaching a 2-handle with framing $mn + 2(m+n)$ along (the closure across the 1-handle of) the braid B_k defined in Figure 2.8.*

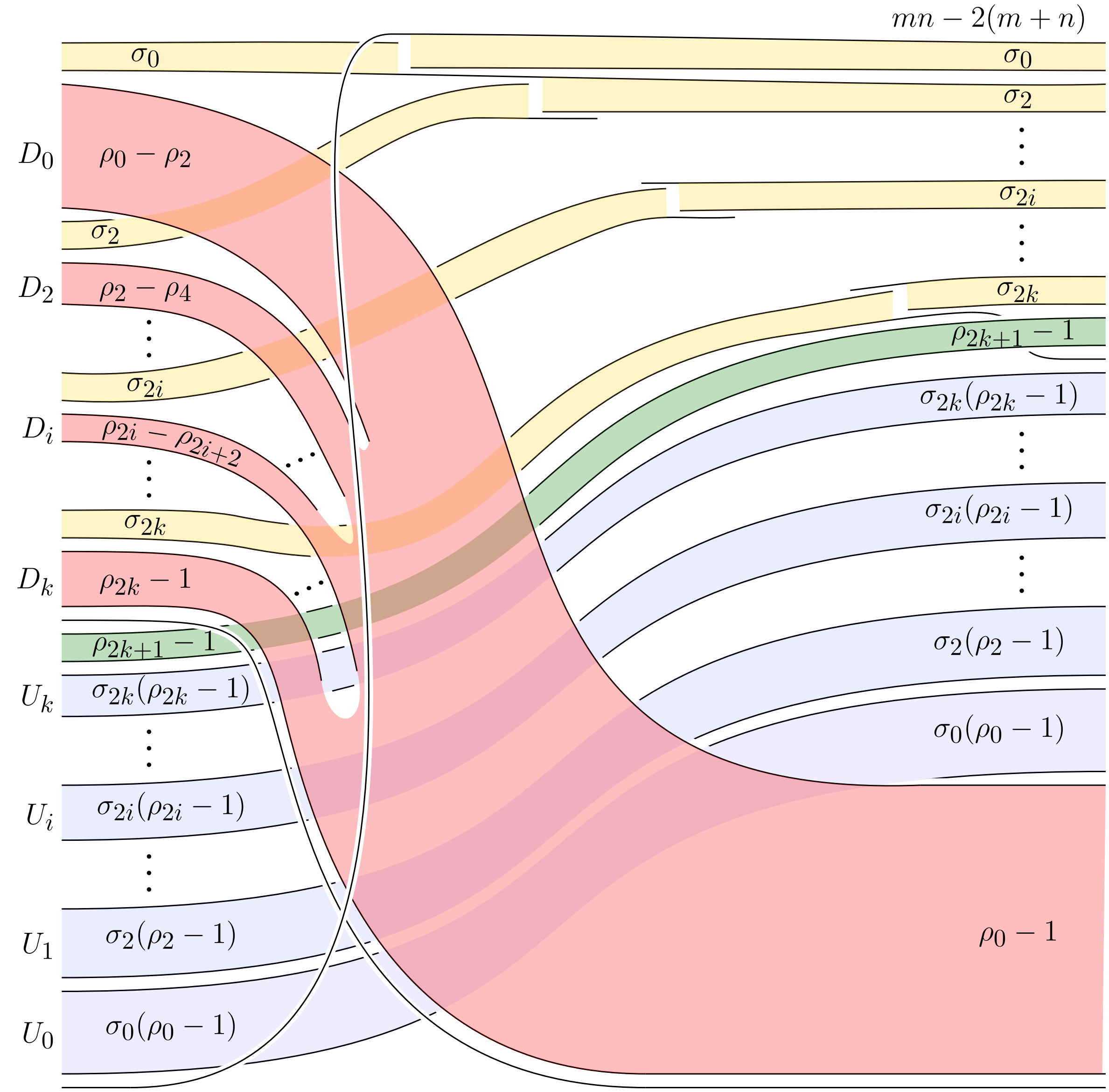

Figure 2.8: The braid B_k: Isotoping away the "bad strand" of the attaching circle for the 2-handle in $A_{m,n}$. The bands labeled D_i and U_i are those described in Lemma 2.2.4. *Warning: the 1-handle of $A_{m,n}$ has been suppressed and the braid does not preserve vertical scale from left to right.*

Proposition 2.2.3 immediately gives Proposition 2.2.1 in the case $\ell \in 2\mathbb{Z}$. This follows since $\rho_{\ell+1} - 1 = 0$ and the central band vanishes at the ℓ^{th} stage. To prove Proposition

2.2.3, we note that isotopies similar to those mentioned in Remark 2.2.2 hold in B_k as well. Denote the bands moving downward in B_k by D_i and those moving upward by U_i (as in Figure 2.8).

Lemma 2.2.4. *For each pair of integers $0 \leq i \leq k$ so that $0 \leq 2k \leq \ell$, the braid B_k admits an isotopy shifting any tangle T in the D_i-band down exactly ρ_{2i+1} strands. Similarly, B_k admits an isotopy shifting any tangle T' in the U_i-band up exactly $\rho_{2i} - 1$ strands.*

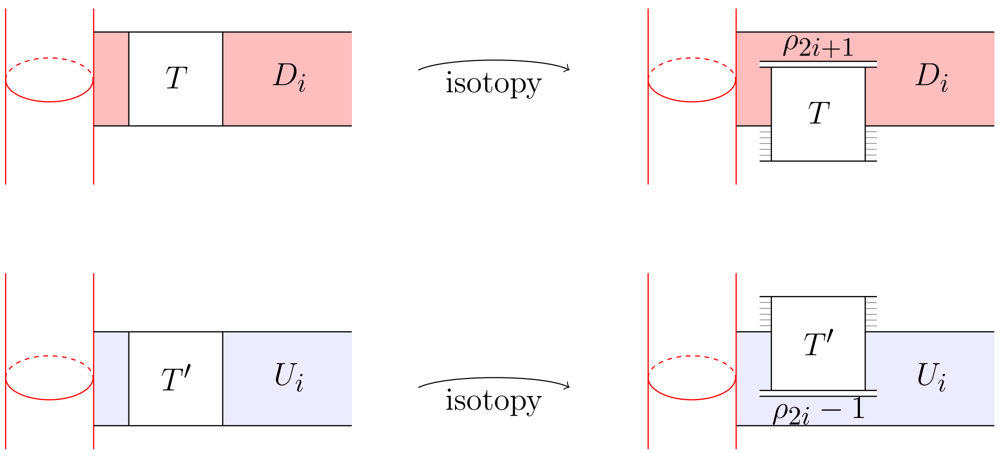

Figure 2.9: Moving tangles in D_i- and U_i- bands of the braid B_k.

Proof. We proceed by induction on k. The case when $i = k = 0$ is covered by Remark 2.2.2. Suppose the result holds for each $0 \leq i \leq k-1$ in B_{k-1}. It is immediate that the same isotopies persist in B_k. Therefore, we only need to show that tangles in the U_k and D_k bands can be moved up and down respectively. We prove that the isotopy on U_k holds first. Suppose we have moved the tangle T' into the D_i band for some $i < k$ as in Figure 2.10. Here, we use that $\rho_j - \rho_{j+2} = \sigma_{j+1}\rho_{j+1}$ and

$$\begin{aligned}
\rho_{2i+1} &= \sum_{j=i+1}^{k} \rho_{2j-1} - \rho_{2j+1} + \rho_{2k+1} \\
&= \sum_{j=i+1}^{k} \rho_{2j}\sigma_{2j} + \rho_{2k+1} = \sum_{j=i+1}^{k} (\sigma_{2j} + \rho_{2j}(\sigma_{2j} - 1)) + \rho_{2k+1}.
\end{aligned}$$

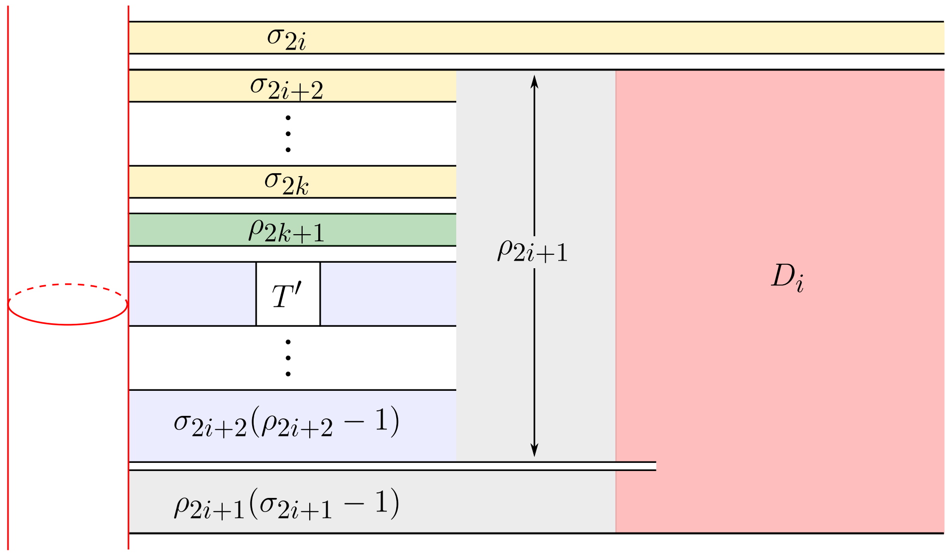

Figure 2.10: Moving T' through the D_i-band.

Clearly, by initially pushing the tangle T' once over the 1-handle, we can view T' in the D_0-band. By induction, there exists an isotopy of B_k taking any tangle in the D_i-band down exactly ρ_{2i+1} strands. Applying this isotopy a total of σ_{2i+1}-times moves T' into D_{i+1} in the same position as Figure 2.10 (with $i+1$ replacing i). Repeating this process for each $i < k$, moves T' as in Figure 2.11 giving the claimed isotopy of T' in U_k. With the isotopy

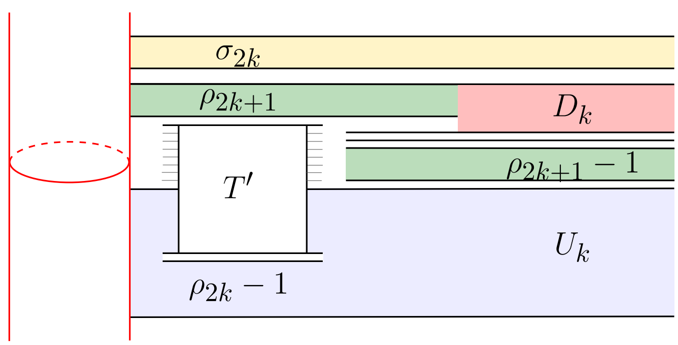

Figure 2.11: The desired isotopy on U_k.

for U_k in place, we show that the desired isotopy of a tangle T in D_k also exists. By first dragging the tangle T once over the 1-handle, we can view T in the U_0-band; suppose we have moved T into the U_i band for some $0 < i < k$ as in the left side of Figure 2.12. By induction, there exists an isotopy of B_k moving any tangle in the U_i-band up $\rho_{2i} - 1$-strands.

Applying this isotopy σ_{2i}-times places T in the U_{i+1} band. Repeating this process for each $i \le k$, moves T as in the right side of Figure 2.12 giving the claimed isotopy of T in D_k. $\square$

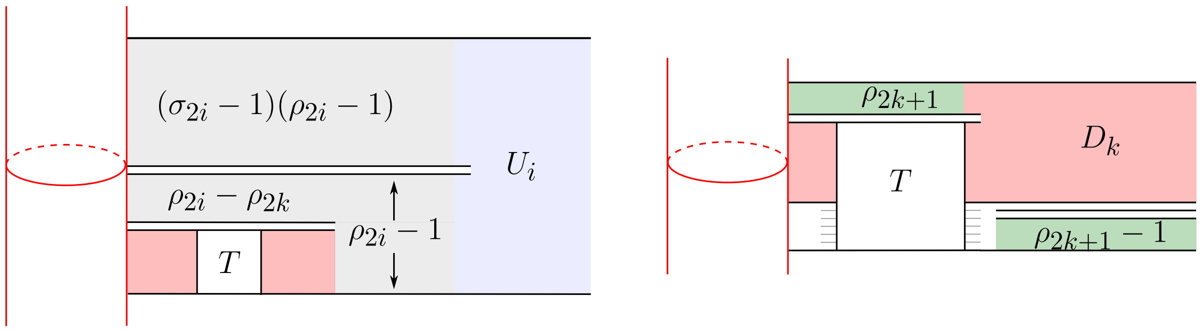

Figure 2.12: Left: Moving T through the U_i-band. Right: Moving T above the U_k-band giving the desired isotopy on D_k. .

Proof of Proposition 2.2.3. We proceed by induction on k. Figure 2.7 gives the case when $k = 0$. Suppose K has been isotoped to B_k for some k with $2k < \ell - 2$. We view the "bad" strand as a tangle on ρ_{2k+1} strands. By examining the proof of Lemma 2.2.4, we see that this tangle can be viewed in each D_i as a tangle directly above T' in Figure 2.10 (when $i < k$) and ultimately above T' in Figure 2.11. Lemma 2.2.4 allows us to move this tangle down ρ_{2k+1} strands as long as the tangle remains in D_k. As D_k consists of $\rho_{2k} - 1 = \rho_{2k+1}\sigma_{2k+1} + \rho_{2k+2} - 1$ strands, we can move the bad tangle down a total of σ_{2k+1} times before it begins to leave D_k. At this point, we find that $\rho_{2k+2} - 1$ strands of D_k as well as the strand directly below D_k can be pulled passed a single strand of the bad tangle (top of Figure 2.13). Repeating this process j times gives the bottom of Figure 2.13 Taking $j = \sigma_{2k+2}$ then gives B_{k+1}. $\square$

Proof of Proposition 2.2.1. As each isotopy from B_k to B_{k+1} is clearly writhe preserving. The writhe of B_k is that of B_0 which equals $mn - 2(m + n) + 2$. Therefore, the handle attachments of Figures 2.5 and 2.6 are Stein since their contact framings are easily seen to

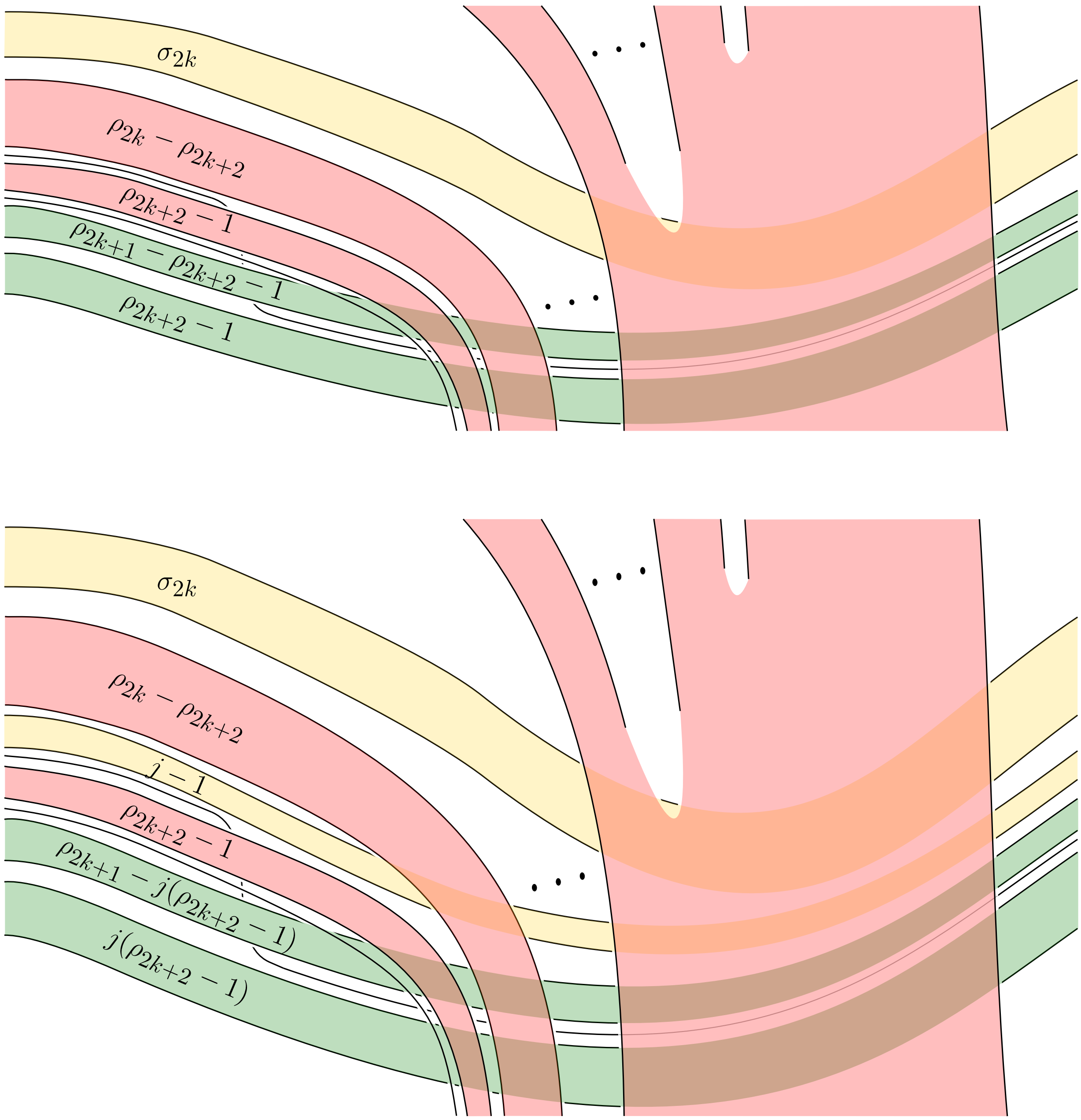

Figure 2.13: Isotoping B_k to B_{k+1}.

be one less than the writhe of B_k. That is $\mathrm{tb}(B_k) = mn - 2(m+n) + 1$. Then Proposition 2.2.3 immediately gives the result when $\ell \in 2\mathbb{Z}$ by isotoping to B_ℓ. On the other hand, if $\ell \in 2\mathbb{Z}+1$, applying the induction step of Proposition 2.2.3 one final time is easily seen to give the result in this case as well. $\square$

Proof of Theorem 1.2.1. The fact that $(\partial A_{m,n}, \xi_{\widetilde{J}_{m,n}})$ is contactomorphic to the universally tight lens space $(L(p^2, pq-1), \bar{\xi}_{\mathrm{st}})$ follows by noting that any almost complex structure on the rational ball $A_{m,n}$ (indeed any rational ball) satisfies that

$$\frac{c_1^2(A_{m,n}, J) - 2\chi(A_{m,n}) - 3\sigma(A_{m,n})}{4} = -\frac{1}{2},$$

thus $d_3(\xi_{\widetilde{J}_{m,n}}) = -1/2$. By Proposition 2.1.8, $\xi_{\widetilde{J}_{m,n}}$ is universally tight. As $(A_{m,n}, \widetilde{J}_{m,n})$ gives a symplectic filling of $(L(p^2, pq-1), \bar{\xi}_{\mathrm{st}})$, Theorem 1.1.15 gives that $A_{m,n} \approx B_{p,q}$. $\square$

Remark 2.2.5. Although Lisca's result allows us to conclude that $A_{m,n} \approx B_{p,q}$ whenever their boundaries coincide, it does not tell us anything about the Stein structures $\widetilde{J}_{m,n}$ versus $J_{p,q}$. It is worth noting that $B_{p,1} = A_{1,p-1}$ (they are specified by the *same* handle decomposition) and $\widetilde{J}_{1,p-1}$ coincides with $J_{p,1}$. Lekili and Maydanskiy note that it is unknown whether or not $B_{p,q}$ admits more than one Stein structure [32]. Clearly, Theorem 1.2.1 fails to answer this question; although, it does provide another candidate for study.

It appears that the Legerdrian isotopy class of the attaching circle for the 2-handle in Figure 2.5 or 2.6 is "maximal" in an appropriate sense: Consider the rational ball $\mathcal{A}_{m,n}^{mn+1}$ given by attaching the 2-handle in $A_{m,n}$ with framing $mn+1$ rather than mn (e.g. $\mathcal{A}_{2,3}^{7}$ is shown in Figure 2.14). Here we take "maximal" to mean that the following question should be settled in the negative.

Question 2.2.6. Is there any choice of $n > m > 1$ so that $\mathcal{A}_{m,n}^{mn+1}$ admits a Stein structure?

If the answer is no, then in particular there cannot be a smooth isotopy of the attaching circle of the 2-handle in $A_{m,n}$ to a Legendrian knot in the tight $S^1 \times S^2$ where the difference between the resulting contact and Seifert framings is more than one. This is clearly true in some cases.

For instance, it is easy to verify that $\partial\mathcal{A}^p_{1,p-1} \approx L(p,1)\# - L(p,1)$ - surger $L(p,1)\times I$ so that the boundary is connected. By a Theorem of Eliashberg's, any Stein structure on $\mathcal{A}^p_{1,p-1}$ would necessarily decompose as a boundary sum of Stein fillings of $L(p,1)$ and $-L(p,1)$ respectively [7]. However, it is known that no rational ball symplectically fills $-L(p,1)$ equipped with any tight contact structure [50]. Therefore, $\mathcal{A}^p_{1,p-1}$ fails to be Stein.

An arguably more interesting case: Kadokami and Yamada prove that $L(25,7)$ bounds $\mathcal{A}^7_{2,3}$ [25]. It is worth noting that Lawrence Roberts has this same result in 2008 [51]. Direct calculation shows that the tight contact structures on $L(25,7)$ have three-dimensional

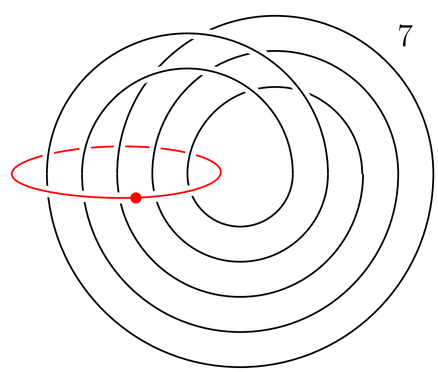

Figure 2.14: $\mathcal{A}^7_{2,3}$ - A rational ball bounding $L(25,7)$ which cannot symplectically fill any tight contact structure on its boundary.

invariants lying in $\{-1/50, 11/50, 19/50\}$. Therefore, this rational ball cannot syplectically fill $L(25,7)$ equipped with any tight contact structure; in particular $\mathcal{A}^7_{2,3}$ fails to admit a Stein structure.

$\mathcal{A}^7_{2,3}$ is the only member of $\mathcal{A}^{mn+1}_{m,n}$ bounding a lens space [25]. Using methods of Chapter 3, we can show that $\partial\mathcal{A}^{mn+1}_{m,n}$ is always Seifert fibered (for instance, one can verify $\partial\mathcal{A}^{4s+3}_{2,2s+1} \approx M(-3; -3-s, -\frac{s}{s-1}, -\frac{3}{2}))$. Unfortunately, classification results for fillings of general Seifert fibered spaces are less developed than those of lens spaces. That said, the answer to Question 2.2.6 is likely no in these cases as well.

Chapter 3

Identifying Rational Balls By Carving

Chapter 2 provided a complete answer to Question 1.1.9. However, this answer is a bit unsatisfying. Both the spaces $B_{p,q}$ and $A_{m,n}$ can be defined by their respective handle decompositions alone, yet the diffeomorphisms ensured by Theorem 1.2.1 provide little insight into how these two decompositions are related within the handle theory. This chapter aims to rectify this. Herein, we define the boundary diffeomorphisms of Theorem 1.2.2 and prove Corollary 1.2.3 that these maps extend across the interiors of $B_{p,q}$ and $A_{m,n}$. The latter is done through the method of carving.

3.1 Extending Maps through Carving

Carving, introduced by Akbulut [1], is a powerful tool for understanding handle decompositions (see also [2]). The method can be described as follows: suppose we have two 4-manifolds X and X' and a diffeomorphism $f : \partial X \to \partial X'$. Suppose that X admits a handle decomposition consisting of a single 0-handle, k 1-handles, and N 2-handles, where the ith 2-handle h_i is attached along a knot K_i in $\#k(S^1 \times S^2)$. Let μ_i denote the belt-sphere of h_i (i.e. a meridian of K_i). We attempt to extend f to a diffeomorphism between X and X'.

If f does extend, then in particular it extends across a neighborhood of the collection of cocores of the 2-handles in X. Thus, a necessary condition for f to extend is that the image of the belt-spheres $f(\mu_1) \cup \ldots \cup f(\mu_N)$ must be a slice link in $\partial X'$. That is, there exists a collection of properly embedded disks $D_i \subset X'$ such that $D_i \cap D_j = \emptyset$ and $\partial D_i = f(\mu_i)$. Assuming this, if f carries the 0-framing of each μ_i (induced by the cocore) to the framing of $f(\mu_i)$ induced by the slice disk, then f extends across the neighborhoods of the cocores of the 2-handles in X. In order to extend f across the rest of X, we are left needing to extend a map $f_0 : \#k(S^1 \times S^2) \to \#k(S^1 \times S^2)$. Laudenbach and Poenaru prove that every self diffeomorphism of $\partial(\natural k(S^1 \times B^3))$ extends [31]. Therefore, f_0 extends provided that

$$X' - \nu(D_1 \cup \ldots \cup D_N) \approx \natural k(S^1 \times B^3).$$

In practice, one defines f via framed link surgery, then traces each belt-sphere μ_i under f paying careful attention to how the map effects the 0-framing of μ_i. If each $f(\mu_i)$ bounds a disjoint disk in X', inducing the same framings as those traced, then f extends provided that surgering these disks gives $\natural k(S^1 \times B^3)$ - this last question is usually verified directly using 4-dimensional handle moves. We apply this process to $B_{p,q}$ and $A_{m,n}$.

3.1.1 Boundary Diffeomorphisms: $\partial B_{p,q}$

In this section, we exhibit explicit diffeomorphisms from $\partial B_{p,q}$ to $L(p^2, pq-1)$. To accomplish this, we find boundary diffeomorphisms to particular linear plumbings associated to p and q. Bearing in mind the carving procedure, outlined in the previous section, we trace the belt-sphere of the single 2-handle of $B_{p,q}$.

It's worth noting that such diffeomorphisms have been known previously. Yamada produces similar diffeomorphisms from $\partial A_{m,n}$ to $L(p^2, pq-1)$ expressed as the boundary of the unique linear plumbing of D^2-bundles over S^2 with Euler classes each ≤ -2 [59]. To accomplish this, one must carefully keep track of every stage of the Euclidean algorithm applied to $(p-q,q)=1$ - that is every time a_i is subtracted from b_i or b_i from a_i in Yamada's definition of $A(p-q,q)$ (see Lemma 4.0.11). We perform a courser bookkeeping of the Euclidean algorithm via Definition 1.2.4, which allows for arguably clearer definitions - however, we don't arrive at a linear plumbing with Euler classes ≤ -2. Yet, as shown in Corollary 2.1.3, through a sequence of blow-ups and blow-downs, one can easily get to that plumbing if so desired.

We first employ this method to $\partial B_{p,q}$. Again, for clarity a worked example of the diffeomorphisms defined in Proposition 3.1.1 as well as Corollary 3.1.3 is provided in the Appendix (Figure A.2) for the rational ball $B_{8,3}$.

Proposition 3.1.1. *Let* $\{r_i\}_{i=-1}^{\ell+2}$ *and* $\{s_i\}_{i=0}^{\ell+1}$ *be as defined in Definition 1.2.4. Then for each* $i \in \{0,\ldots,\ell+1\}$*,* $B_{p,q} \overset{\partial}{\approx} B^i_{p,q}$ *where* $B^i_{p,q}$ *is the 4-manifold given by Figure 3.1.*

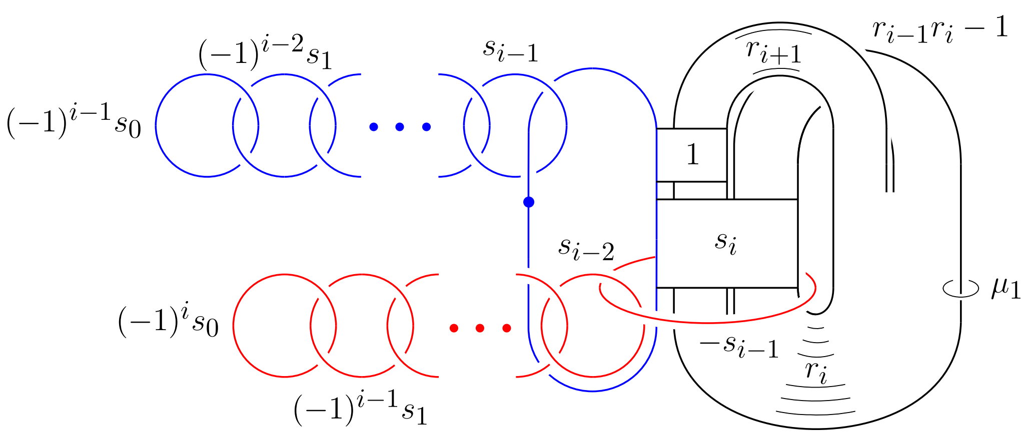

Figure 3.1: The 4-manifold $B^i_{p,q}$

Proof. We induct on i. When $i=0$, the result is immediate since $B^0_{p,q} \approx B_{p,q}$. Therefore, the proposition holds provided that $\partial B^i_{p,q} \approx \partial B^{i+1}_{p,q}$. Let K^i_1 be the attaching circle of the $r_{i-1}r_i-1$-framed 2-handle in $B^i_{p,q}$. Suppose the result holds for some $i \leq \ell$. For $i+1$, first, surger the single 1-handle and introduce a canceling pair of 1- and 2-handles to remove the s_i-full twists between K^i_1 and the, now surgered, 1-handle (Figure 3.2). Since K^i_1 links the

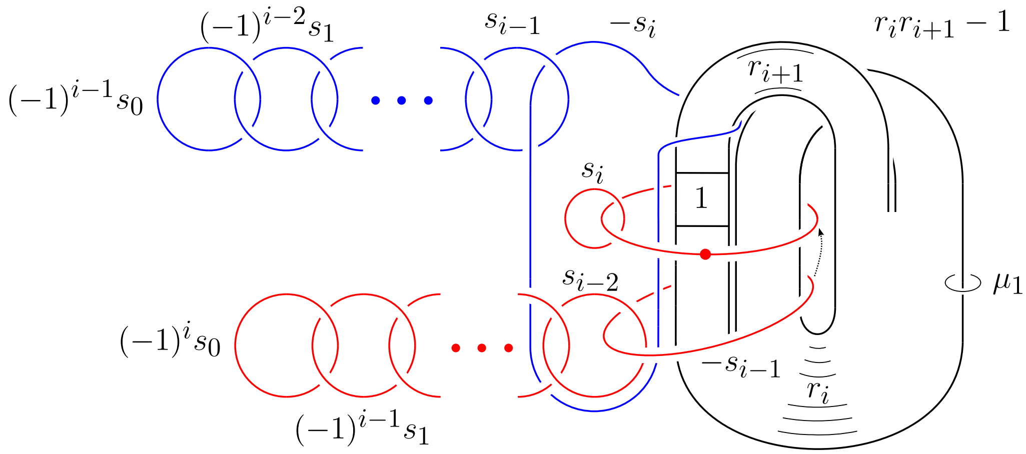

Figure 3.2: Introducing a canceling pair after surgery.

new 1-handle r_i times, the framing on K_1^i decreases by $s_i r_i^2$ and the new framing on K_1^i is

$$r_{i-1}r_i - 1 - s_i r_i^2 = r_i(r_{i-1} - s_i r_i) - 1 = r_i r_{i+1} - 1.$$

Sliding the $-s_{i-1}$-framed 2-handle under the new 1-handle as indicated in Figure 3.2, and isotoping the r_{i+1}-stranded band (see Figure 3.3) we find that the r_{i+1}-stranded band tra-

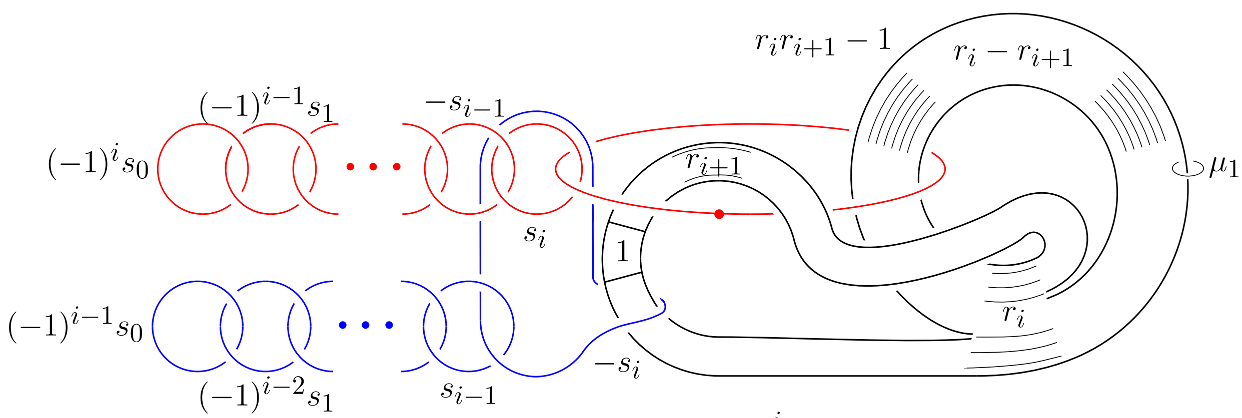

Figure 3.3: Isotoping K_1^i.

verses the 1-handle (positively) s_{i+1}-times as a complete band, while r_{i+2}-strands traverse an additional one time to make up the complete $s_{i+1}r_{i+1} + r_{i+2} = r_i$ linking. With this view in mind, we isotope K_1^i into a closed braid on r_{i+1} strands appropriately linking the carving disk of the 1-handle - Figure 3.4. The result holds by induction. □

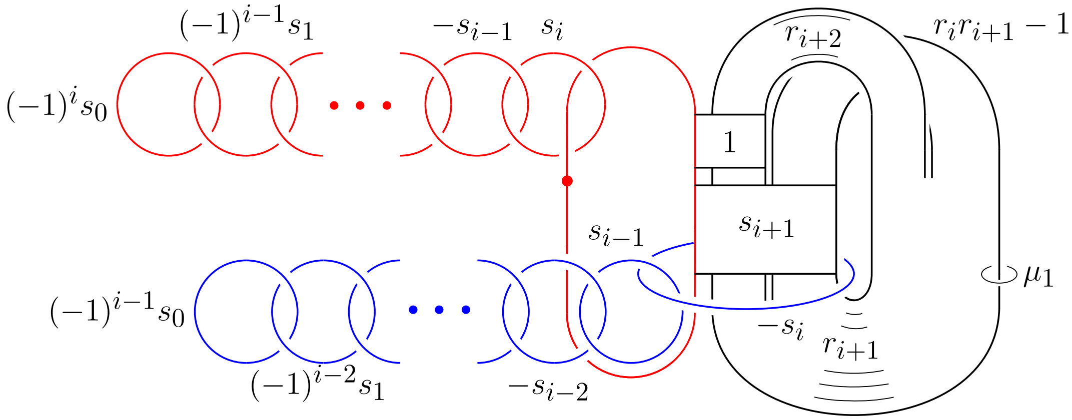

Figure 3.4: Further isotopy of K_1^i to K_1^{i+1}

Remark 3.1.2. At no point does μ_1, the meridian of K_1^i, get damaged under the boundary diffeomorphisms defined in Proposition 3.1.1. In particular, for each i, μ_1 bounds a disk in $B^i_{p,q}$ and the image of a collar neighborhood of μ_1 arising from such a disk persists under the boundary diffeomorphisms defined above - that is that each diffeomorphism preserves the 0-framing on μ_1.

Since $r_{\ell+1} = 1$ and $r_{\ell+2} = 0$, by definition, $s_{\ell+1} = s_{\ell+1}r_{\ell+1} + r_{\ell+2} = r_\ell$. So, by looking at $B^{\ell+1}_{p,q}$ we arrive at the following result of Casson and Harer [6].

Corollary 3.1.3. $\partial B_{p,q} \approx L(p^2, pq-1)$.

Proof. By Proposition 3.1.1, we have that $\partial B_{p,q} \approx \partial B^{\ell+1}_{p,q}$ (Figure 3.5). We show that $\partial B^{\ell+1}_{p,q}$

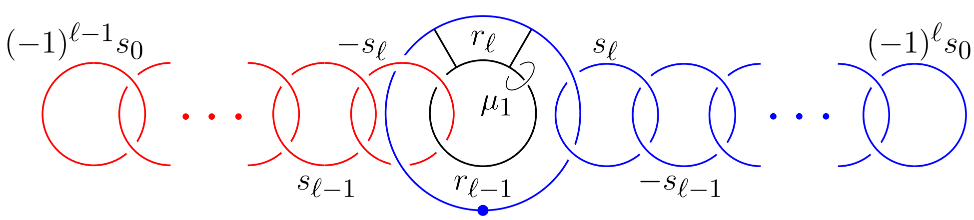

Figure 3.5: The space $B^{\ell+1}_{p,q}$.

is diffeomorphic to a linear plumbing of circle-bundles over S^2 as follows. Surger the 1-handle and introduce a canceling 1- and 2-handle, as in the induction step of Proposition 3.1.1, (top of Figure 3.6). Next, slide the $-s_\ell$-framed 2-handle as well as μ_1 under the 1-handle as indicated in the top of Figure 3.6 (middle of Figure 3.6). Surgering the new 1-handle and blowing down gives the linear plumbing (bottom of Figure 3.6). □

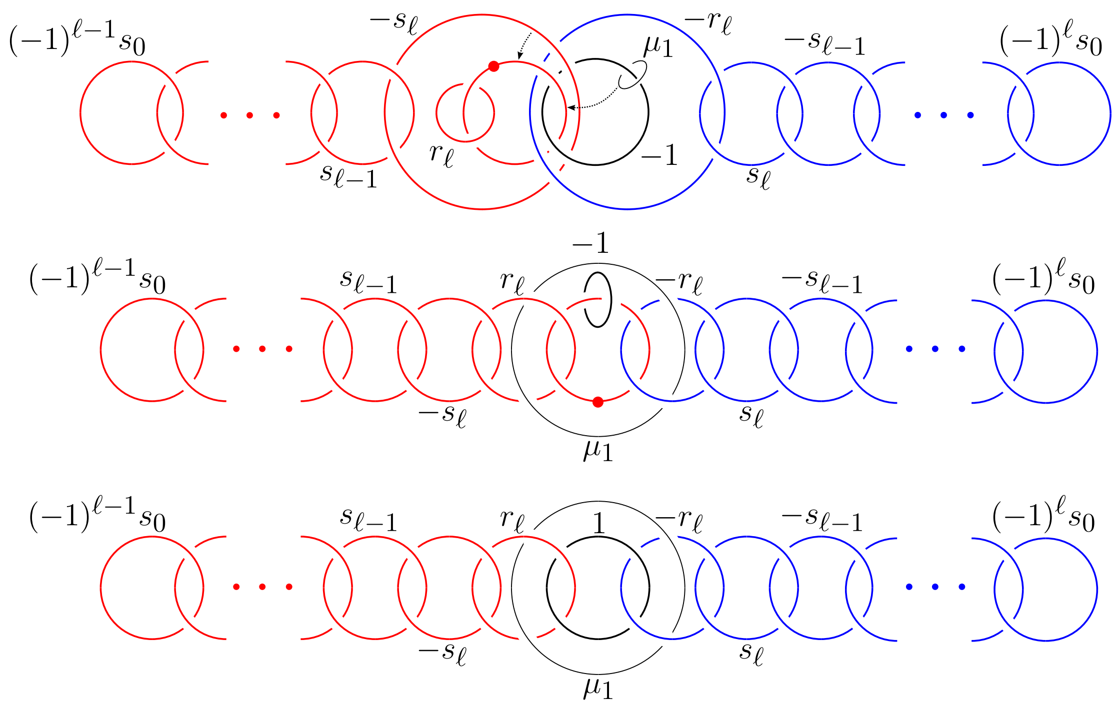

Figure 3.6: From top to bottom: The introduction of a canceling pair to $B_{p,q}^{\ell+1}$ after surgery; the result of the indicated slides; a linear plumbing associated to $\partial B_{p,q}$.

Remark 3.1.4. From Lemma 4.0.14, we see that the linear plumbing in Figure 3.6 bounds $L(p^2, pq-1)$. Indeed, we find that

$$[-s_0, s_1, \ldots, \pm r_\ell, 1, \mp r_\ell, \ldots, -s_1, s_0] = -\frac{p^2}{pq-1}.$$

3.1.2 Boundary Diffeomorphisms: $\partial A_{m,n}$

As in the previous section, we exhibit explicit diffeomorphisms, this time from $\partial A_{m,n}$ to $L(p^2, pq-1)$. As the image of μ_1 is given as the 0-framed push-off of the attaching circle of the central 1-framed unknot at the bottom of Figure 3.6. We'll trace where the curve, γ

in Figure 1.5, goes as well - finding that it too goes to the 0-framed push-off of the central 1-framed unknot via an appropriately defined diffeomorphism. We want to define these diffeomorphisms in a structurally similar manner to those of Proposition 3.1.1. To that end,

Lemma 3.1.5. *$A_{m,n}$ is given by Figure 3.7.*

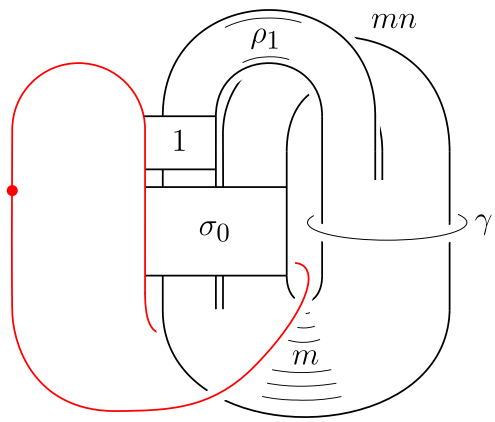

Figure 3.7: An alternative description of $A_{m,n}$.

Proof. As in Section 2.2, we are taking $n = m\sigma_0 + \rho_1$. The result follows from an isotopy of the 2-handle given in Figure 3.8. □

As with previous sections, we have provided a worked example in the case of $A_{3,5}$ in Figure A.3 of the Appendix. With Lemma 3.1.5 in place we prove:

Proposition 3.1.6. *Let $\{\rho_i\}_{i=-1}^{\ell+2}$ and $\{\sigma_i\}_{i=0}^{\ell+1}$ be as defined in Definition 1.2.4 (associated to $n > m \geq 1$). Then for each $i \in \{0, \ldots, \ell+1\}$, $A_{m,n} \overset{\partial}{\approx} A^i_{m,n}$ where $A^i_{m,n}$ is the 4-manifold given by Figure 3.9.*

Proof. We induct on i, treating the base case and the induction step simultaneously. For the base case, start with the handle decomposition from Lemma 3.1.5. For the induction step, suppose that the result holds for some $i \leq \ell$. Let K^i_1 be the attaching circle of the

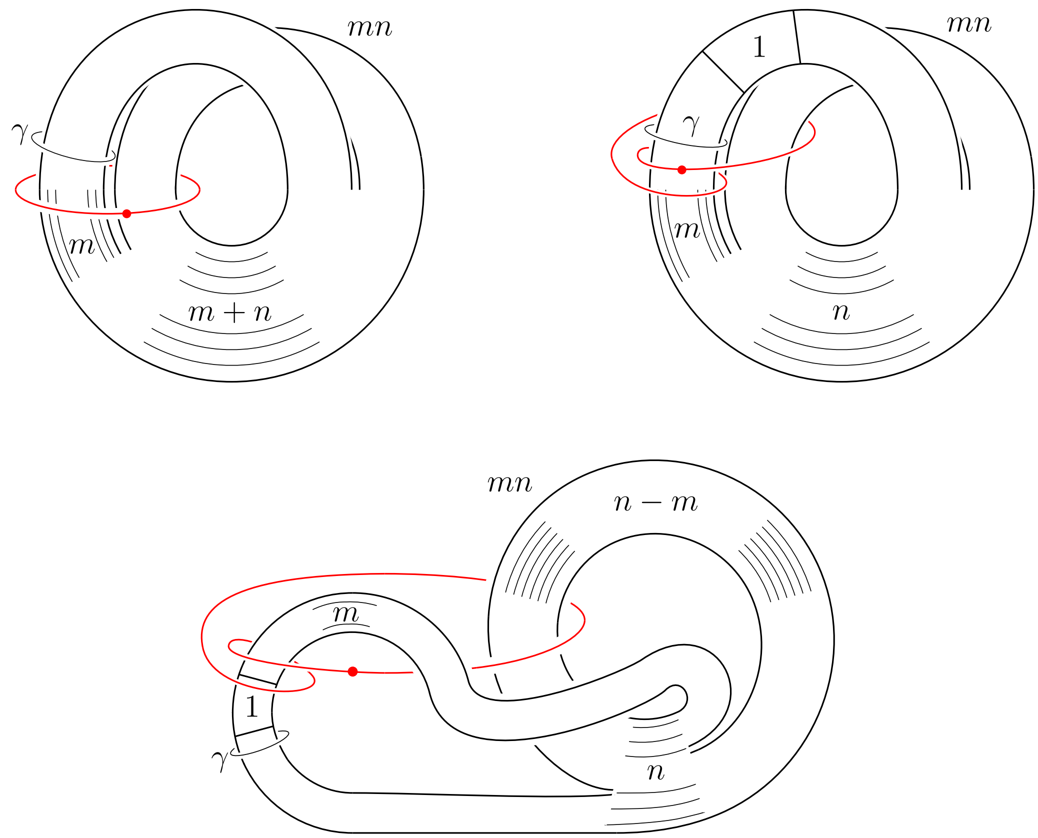

Figure 3.8: The isotopy of the 2-handle in $A_{m,n}$ used in the proof of Lemma 3.1.5.

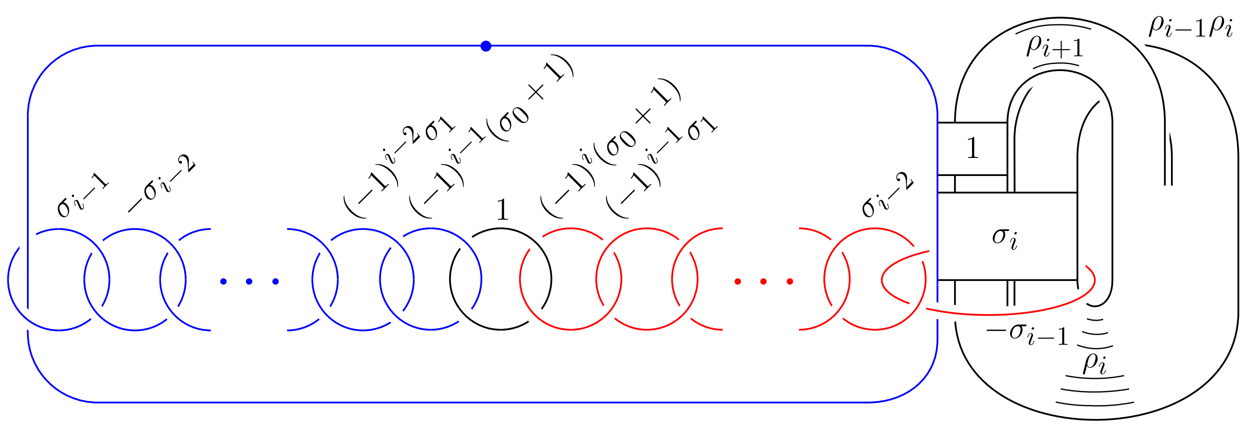

Figure 3.9: The 4-manifold $A^i_{m,n}$

$\rho_{i-1}\rho_i$-framed 2-handle in $A^i_{m,n}$. Surger the 1-handle and introduce a canceling 1- and 2-handle (for the base case see the left side of Figure 3.10, for the induction step see Figure 3.12). Notice, similar to Proposition 3.1.1 the framing of K^i_1 changes from $\rho_{i-1}\rho_i$ to $\rho_i\rho_{i+1}$. Slide the now surgered 1-handle as indicated in the respective figures and, for the base case,

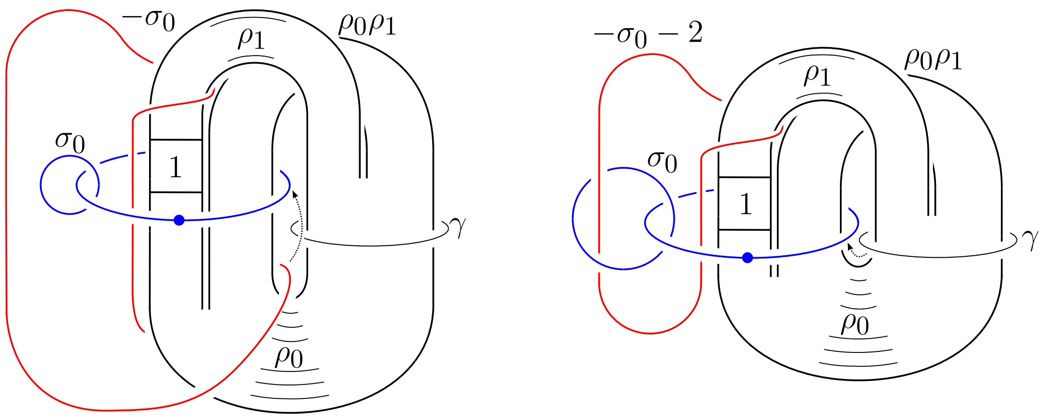

Figure 3.10: The base case of Proposition 3.1.6

blow-up once (right side of Figure 3.10). From here the base case follows similarly to the

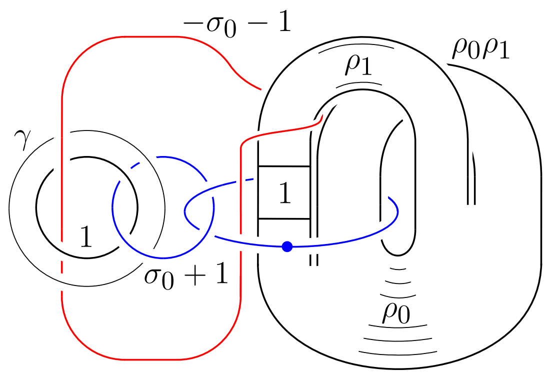

Figure 3.11: Finishing the base case of Proposition 3.1.6

induction step; both of which are structurally similar to Proposition 3.1.1. Indeed, isotope K_1^i to view a band with ρ_{i+1} stands traversing the 1-handle σ_{i+1}-times along with ρ_{i+2} of those strands traversing an extra time as in Figure 3.13.

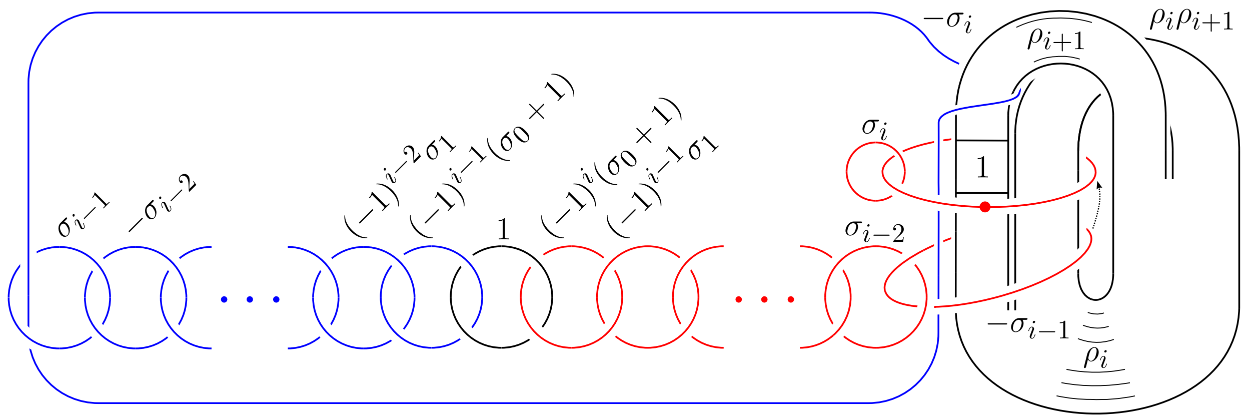

Figure 3.12: Introducing a canceling pair.

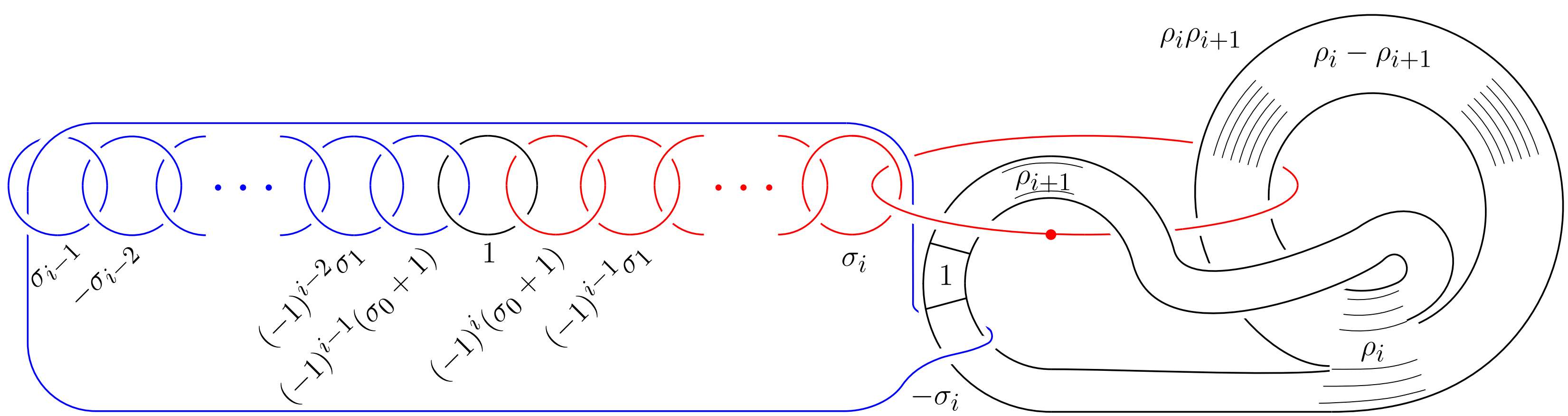

Figure 3.13: Isotoping K_1^i in $A_{m,n}^i$.

A further isotopy of K_1^i gives a closed braid on ρ_{i+1}-strands geometrically linking the carving disk of the new 1-handle ρ_i-times. Finally, notice that to get the appropriate linking on the chain of unknots, we have to wind the chain (as indicated in Figure 3.14) to add a total of i positive half-twists to the left of the euler-class 1 disk-bundle along with i negative half-twists to the right. The result follows by induction. $\square$

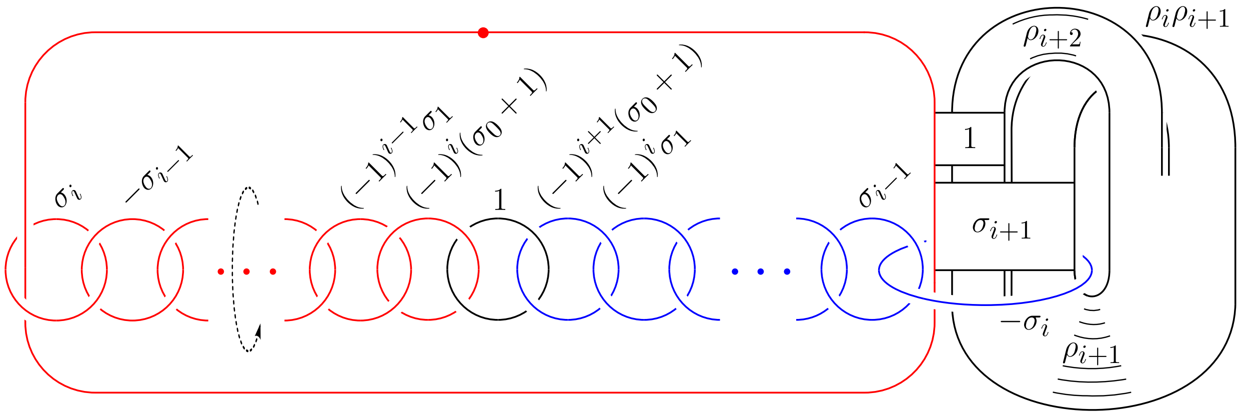

Figure 3.14: Further isotopy of K_1^i to K_1^{i+1} in $A_{m,n}^{i+1}$.

Corollary 3.1.7 ([59], Theroem 1.1)**.** *$\partial A_{m,n} \approx L(p^2, pq-1)$ for $(p-q, q) = A(m,n)$.*

Proof. By Proposition 3.1.6, $\partial A_{m,n} \approx \partial A_{m,n}^{\ell+1}$ (figure 3.15). We proceed as in Corollary 3.1.3.

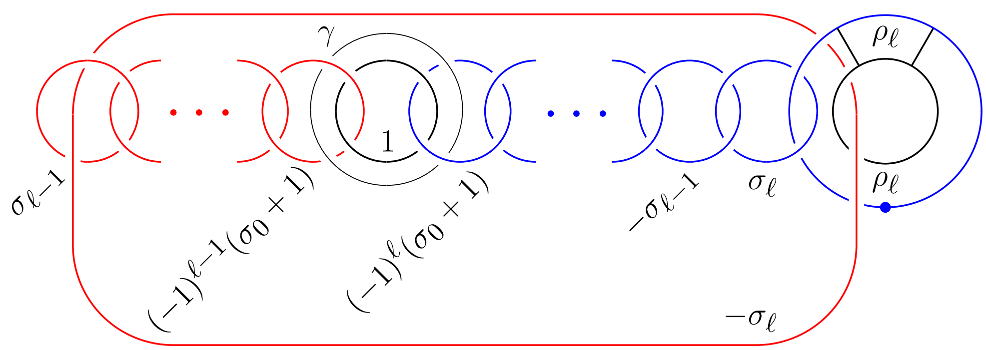

Figure 3.15: The space $A_{m,n}^{\ell+1}$

After surgering the 1-handle and introducing a canceling 1- and 2-handle (top of Figure 3.16), slide the $-\sigma_\ell$-framed 2-handle under the 1-handle and the $-\rho_\ell$-framed 2-handle over the 0-

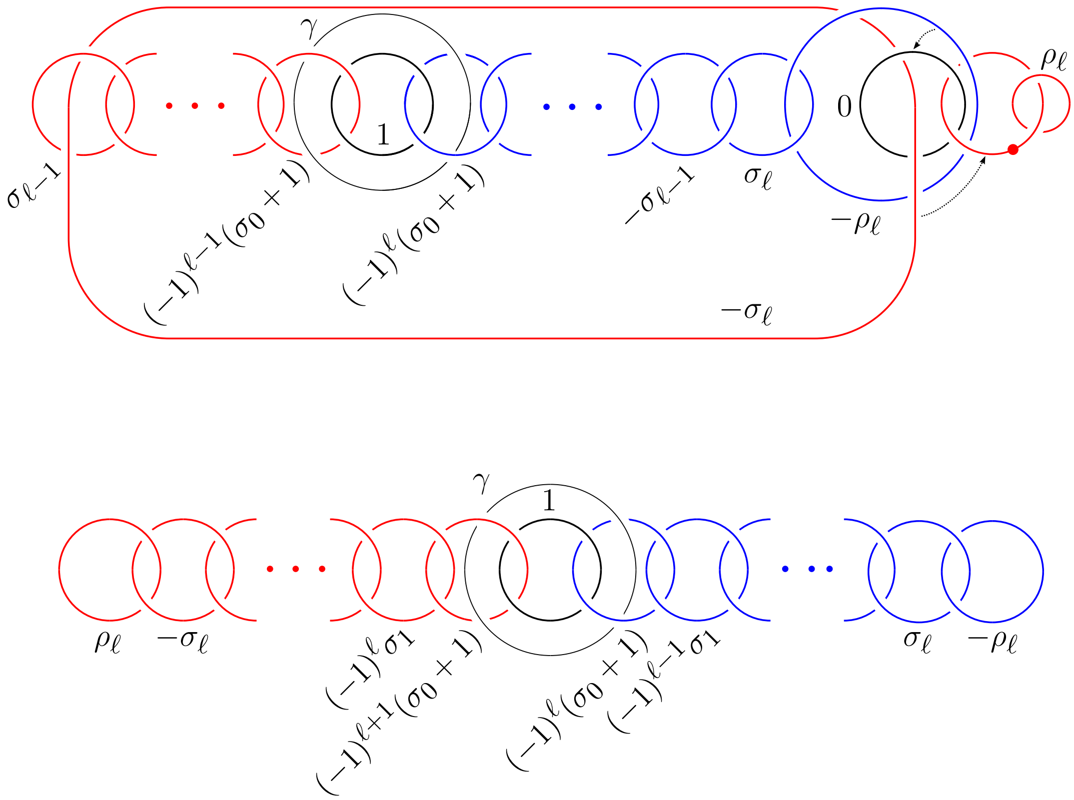

Figure 3.16: The result of surgering $A^{\ell+1}_{m,n}$ and introducing a canceling pair; a linear plumbing associated to $\partial A_{m,n}$

framed 2-handle as indicated in the top of Figure 3.16. Canceling the 1-handle with the 0-framed 2-handle gives the linear plumbing (bottom of Figure 3.16). $\square$

Remark 3.1.8. The fact that $\partial A_{m,n}$ is $L(p^2, pq-1)$ for $A(m,n) = (p-q, q)$ follows by noting that given p and q, or equivalently m and n, we can define the other pair by an appropriate identification of the linear plumbings in Corollaries 3.1.3 and 3.1.7 - provided that $s_0 > 1$ (that is, provided that $p - q > q$ - which we have assumed all along). In fact, this could be taken as the definition of the function A defined by Yamada [59]. The latter claim is the content of Lemma 4.0.11. Notice also that γ bounds a disk in each $\partial A^i_{m,n}$ as well as in the linear plumbing of Figure 3.16. Furthermore, each boundary diffeomorphism defined in Proposition 3.1.6 and those of Corollary 3.1.7 preserve the 0-framing of γ specified by those disks. Therefore, we can employ the carving method of Section 3.1 provided that carving along γ gives $S^1 \times B^3$ - which it does:

Proposition 3.1.9. *Carving $A_{m,n}$ along γ gives $S^1 \times B^3$.*

Proof. Carving $A_{m,n}$ along the curve γ means removing a neighborhood of the disk γ bounds inside $A_{m,n}$. The resulting handlebody decomposition is given by that of $A_{m,n}$ along with an extra 1-handle whose carving disk is γ. If we let γ_i be the analogous curve in A_{ρ_{i-1},ρ_i}, then the result of carving A_{ρ_{i-1},ρ_i} along γ_i is given in Figure 3.17. Notice that $A_{m,n} = A_{\rho_0,\rho_{-1}}$

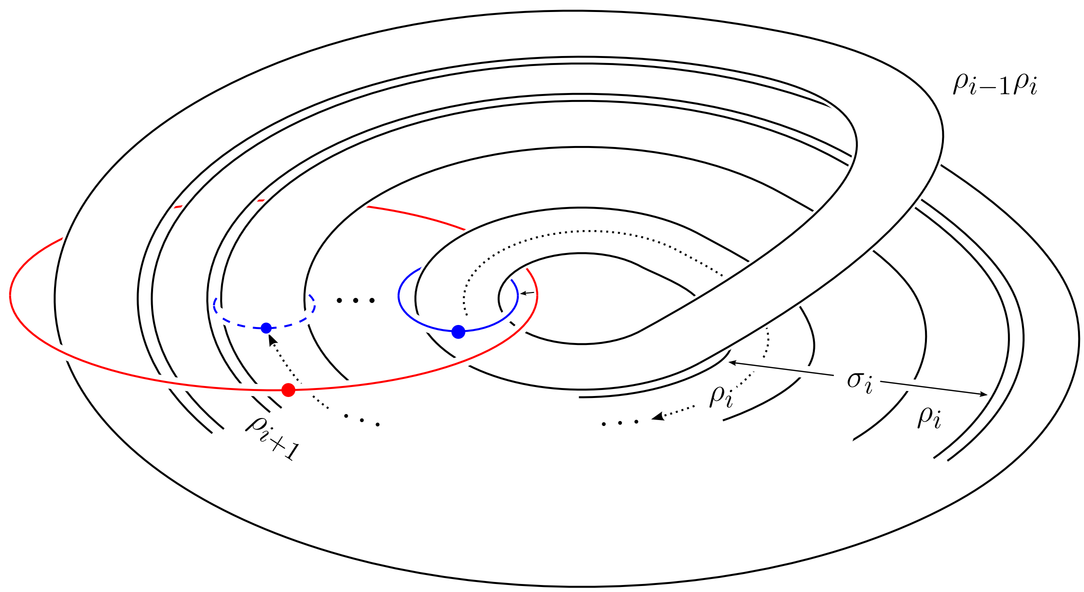

Figure 3.17: A_{ρ_{i-1},ρ_i} carved along γ_i.

and $\gamma = \gamma_0$. By sliding the original 1-handle across the newly carved 1-handle σ_i times, twisting the 1-handle σ_i-times (negatively) and finally sliding as indicated in the left side of Figure 3.18 we arrive at $A_{\rho_i,\rho_{i+1}}$ carved along γ_{i+1} (right side of Figure 3.18). Therefore, the result of carving along γ_i in A_{ρ_{i-1},ρ_i} is diffeomorphic to carving along γ_{i+1} in $A_{\rho_i,\rho_{i+1}}$. As carving A_{1,ρ_ℓ} along γ_ℓ gives $S^1 \times B^3$ we have the result. $\square$

Proof of Theorem 1.2.2. As $A(p-q,q) = (m,n)$, we can identify the plumbings of Figures 3.6 and 3.16. Then, by first, applying the diffeomorphisms of Proposition 3.1.1 we get a

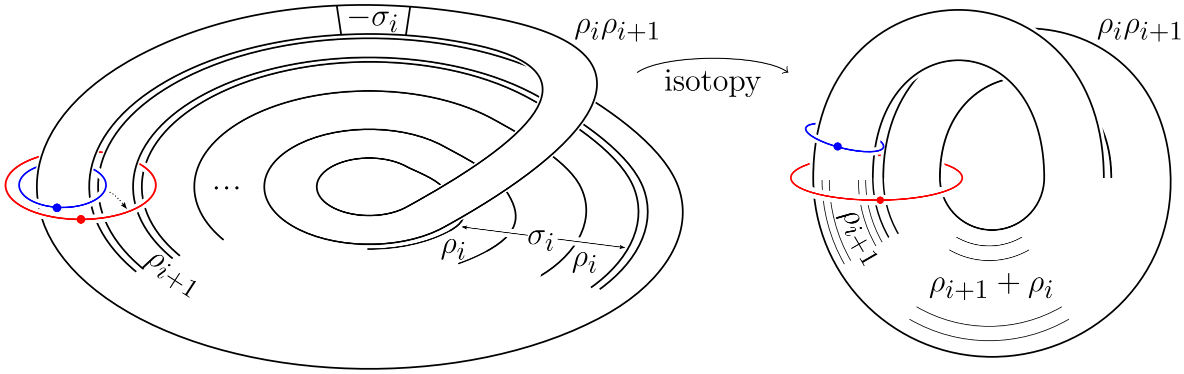

Figure 3.18: A_{ρ_{i-1},ρ_i} carved along γ_i after sliding and twisting σ_i-times.

diffeomorphism from $\partial B_{p,q}$ to the boundary of the linear plumbing of the bottom of Figure 3.6 carrying μ_1 as indicated. Then applying the diffeomorphisms of Proposition 3.1.6 in reverse from the boundary of the linear plumbing of Figure 3.16 to $A_{m,n}$ gives the required diffeomorphism $f : \partial B_{p,q} \to \partial A_{m,n}$. $\square$

3.2 Spin Structures and Orientations

In the interest of fully understanding the map f, we determine how it behaves with respect to elements of $H_1(\partial B_{p,q})$ as well as how f treats spin structures.

Remark 3.2.1. Lemma 1.1.2 allows us to determine $f_*^{-1}\gamma_0 \in H_1(\partial B_{p,q})$ where γ_0 is the meridian defined in Figure 1.5. From Proposition 3.1.6, we have that a meridian of $-(\sigma_0+1)$-framed unknot of figure 3.16 is carried to γ_0 in $\partial A_{m,n}$. Similarly, μ_0 is carried to a meridian of $-s_0$-framed unknot of Figure 3.6. Furthermore, by Corollary 4.0.18, we have that $\gamma_0 = \pm n\mu_0$ if $\ell \in 2\mathbb{Z}$ and $\gamma_0 = \pm m\mu_0$ if $\ell \in 2\mathbb{Z}+1$ where we view γ_0 and μ_0 as their respective images in the aforementioned linear plumbings. Now, by an appropriate choice of identification of the plumbings of Figures 3.16 and 3.6 we can always assume that

$$f_*^{-1}\gamma_0 = \begin{cases} +n\mu_0 & \text{if } \ell \in 2\mathbb{Z}, \\ +m\mu_0 & \text{if } \ell \in 2\mathbb{Z}+1. \end{cases}$$

Indeed, if as defined, $f_*^{-1}\gamma_0$ was $-m\mu_0$ or $-n\mu_0$, we can simply flip one pluming over before making the identification and redefine f accordingly!

Recall that $L(p^2, pq-1)$ admits a unique spin structure if p is odd and two spin structures if p is even. In the former case, f clearly maps the unique spin structure to itself. In the later case, we investigate how f behaves on spin structures by looking at characteristic sublinks due to Kaplan [26]:

Definition 3.2.2 ([26], Definition 1.10)**.** For a framed link $L \subset S^3$, a sublink $L' \subset L$ is characteristic if for each $K \subset L$,

$$\ell k(K, L') = \ell k(K, K) \mod 2.$$

When M^3 is given as (integral) surgery on L, spin structures on M are in bijection with characteristic sublinks of L. Furthermore, fixing a spin structure and thus a characteristic sublink of M, one can trace where that structure goes under a diffeomorphism specified via handle moves / blow-ups by tracing how the sublink evolves under those moves (see §5.7 of [22]). To accomplish this, we adopt the following notation to specify $(M, \mathbf{s})$ for $\mathbf{s} \in \text{Spin}(M)$ - the set of spin structures on M:

Notation 3.2.3. If M^3 is given by integral surgery on a framed link $L = K_1^{f_1} \cup \ldots \cup K_N^{f_N}$ with framings $f_i \in \mathbb{Z}$ and $\mathbf{s} \in \text{Spin}(M)$ is a spin structure with associated characteristic sublink $L' \subset L$, then we denote

$$(M, \mathbf{s}) = K_1^{(f_1;t_1)} \cup \ldots \cup K_N^{(f_N;t_N)}$$

where each $t_i \in \mathbb{Z}/2\mathbb{Z} = \{1, -1\}$ satisfies $t_i = -1$ if and only if $K_i \in L'$. (e.g. see Figure 3.19.)

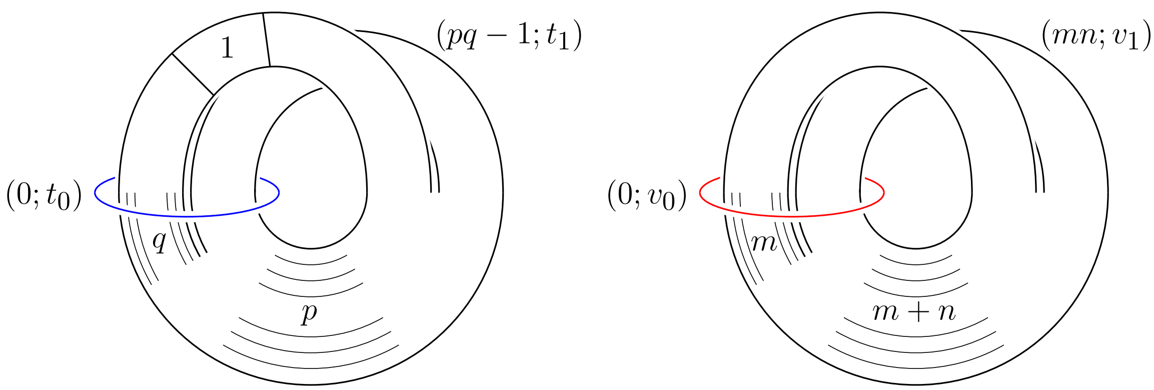

Figure 3.19: A choice of spin-structure on $\partial B_{p,q}$, respectively on $\partial A_{m,n}$.

When sliding K_i over K_j, $(f_i;t_i) \mapsto (f_i+f_j \pm 2\ell k(K_i,K_j);t_i)$ and $(f_j;t_j) \mapsto (f_j;t_it_j)$ [22]. Furthermore, blowing-up corresponds to the addition of $(\pm1;-1)$-decorated unknot. From these two observations, we immediately conclude the following lemma.

Lemma 3.2.4. *Suppose that a band of k strands has r strands contained in the characteristic sublink of a spin structure $\mathbf{s}$ on M and the remaining $k-r$ strands not in the characteristic sublink, then adding $-s_i$-full twists to the band, through the introduction of a canceling pair, effects the characteristic sublink as in Figure 3.20 with no change to the original characteristic sublink and with framings within the band changing in the obvious way.*

Thus, we can refine Proposition 3.1.1 to carry a fixed spin structure on $\partial B_{p,q}$ to each $\partial B^i_{p,q}$.

Lemma 3.2.5. *Let $\mathbf{s} \in \mathrm{Spin}(\partial B_{p,q})$ be specified by the pair $(t_0, t_1) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, then $\mathbf{s}$ corresponds to the spin structure on $\partial B^i_{p,q}$ in Figure 3.21 where $T_0 = t_0$ and for $1 \leq i \leq \ell+1$,*

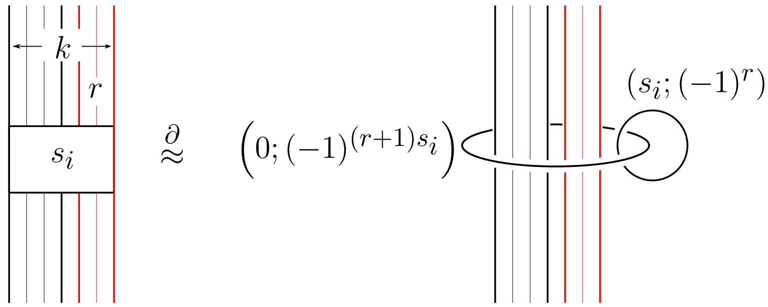

Figure 3.20: Tracing characteristic sublinks when introducing a canceling pair.

$T_i = (-1)^{1+\det A_{i-1}}(-t_0)^{\rho_{\ell+1-i}}(t_1)^{p\det A_{i-1}+ir_i}$ *such that* A_i *and* $\rho_{\ell+1-i}$ *are as defined in Lemma 4.0.11.*

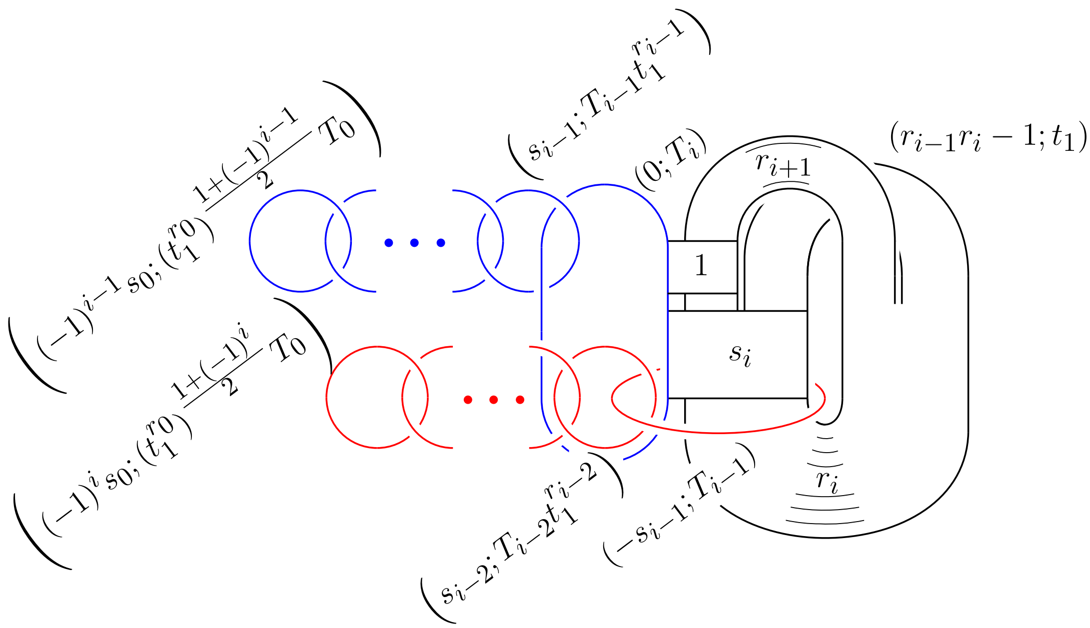

Figure 3.21: A fixed spin structure on $\partial B_{p,q}$ and $\partial B^i_{p,q}$.

Proof. Starting with (t_0, t_1) on $\partial B_{p,q}$ as in Figure 3.19, Lemma 3.2.4 combined with Proposition 3.1.1 gives that the T_j's in Figure 3.21 are defined recursively by $T_{-1} \doteq 0$, $T_0 \doteq t_0$, and $T_j = \left(-T_{j-1}t_1^{r_j-1}\right)^{s_j-1} T_{j-2}$. To see that the closed form for T_j is as claimed, note that we can assume $T_j = (-1)^{a_j}(t_0)^{b_j}(t_1)^{c_j}$ for sequences $\{a_j\}, \{b_j\}, \{c_j\} \subset \mathbb{Z}$ which only need to be determined to their respective parities. Then, the recursion on T_j descends to

$$\begin{array}{ccc} a_{-1} \doteq 0 & b_{-1} \doteq 0 & c_{-1} \doteq 0 \\ a_0 \doteq 0 & b_0 \doteq 1 & c_0 \doteq 0 \\ a_j = s_{j-1}(a_{j-1}+1) + a_{j-2}. & b_j = b_{j-1}b_{j-1} + b_{j-2}. & c_j = s_{j-1}(c_{j-1} + r_{j-1}) + c_{j-2}. \end{array}$$

By noting that $\rho_{\ell+1} = 1$, $\rho_\ell = s_0$ and $\rho_{\ell+1-j} = \rho_{\ell+1-(j-1)}s_{j-1} + \rho_{\ell+1-(j-2)}$ the result follows by induction on j. □

Remark 3.2.6. By Lemma 4.0.11, we have that $\det A_\ell = \pm d$ for d defined therein. Thus,

$$T_{\ell+1} = (-1)^{1+d}(-t_0)^m(t_1)^{pd+\ell+1}.$$

If $p \in 2\mathbb{Z}$, then $t_1 = -1$ for both spin structures on $\partial B_{p,q}$ and we can further reduce $T_{\ell+1}$ to $(-1)^{c+\ell}t_0$ (as m is necessarily odd and the parities of c and d always oppose each other in this case). Therefore, when $p \in 2\mathbb{Z}$, we can measure which spin structure $\mathbf{s}$ gives on $\partial B_{p,q}$ in the linear plumbing of Figure 3.6 by noting that the $-r_\ell$-framed unknot will be in the characteristic sublink associated to $\mathbf{s}$ if and only if $(-1)^{c+\ell}t_0 = -1$. Of course, we can also measure this by looking at the $-s_0$-framed unlink. However, to see which spin structure is induced on $\partial A_{m,n}$, it is convenient to look at $-r_\ell$. To that end, we have

Proposition 3.2.7. *Let $\mathbf{s}$ be the spin structure on $\partial B_{p,q}$ specified by (t_0, t_1) in Figure 3.19, then $f_*(\mathbf{s})$ is the spin structure on $\partial A_{m,n}$ specified by*

$$(v_0, v_1) = \left(\frac{(-1)^{c+\ell}t_0 + t_1 + (-1)^{c+\ell+1}t_0t_1 + 1}{2}, t_1 \right)$$

where the pair $(v_0, v_1) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is defined for $\partial A_{m,n}$ as in Figure 3.19.

3.2.1 Homotopy Invariants Revisited

In Chapter 2, we proved $(\partial B_{p,q}, \xi_{J_{p,q}})$ and $(\partial A_{m,n}, \xi_{\widetilde{J}_{m,n}})$ are necessarily contactomorphic. As an application of Proposition 3.2.7, we can compute the induced spin$^{\mathbb{C}}$-structures coming from $(\partial B_{p,q}, \xi_{J_{p,q}})$ and $(\partial A_{m,n}, \xi_{\widetilde{J}_{m,n}})$ directly from the definition of the Γ invariant of Proposition 1.1.19. This shows, unsurprisingly, that f can be arranged to give the contactomorphism.

Proposition 3.2.8. *For $p > q \geq 1$ relatively prime, the contact structure induced by the Stein structure, $J_{p,q}$, on $B_{p,q}$ given by Figure 1.4 has $\Gamma(\xi_{J_{p,q}}, \mathbf{s}) = \frac{pq}{2} \cdot \mu_0$ in an appropriate basis of $H_1(L(p^2, pq-1); \mathbb{Z})$ and for a fixed choice of $\mathbf{s}$ when $p \in 2\mathbb{Z}$.*

Proof. Let K_0 be the boundary of the carving disk of the 1-handle in Figure 1.4 let K_1 be the attaching circle of the single 2-handle, and let X_0 be the 4-manifold obtained from Figure 1.4 by surgering the 1-handle (exchanging the “dot” on K_0 for a 0-framed 2-handle). Then, let $\mathbf{s} \in \mathrm{Spin}(\partial B_{p,q})$ be the spin structure on $\partial B_{p,q}$ specified by (t_0, t_1) in Figure 3.21. As we have to slide the 2-handle under 1-handle q-times to arrive at Figure 1.4, we see that $\mathbf{s}$ corresponds to the characteristic sublink

$$L' = \frac{1 - t_0 t_1^q}{2} K_0 + \frac{1 - t_1}{2} K_1$$

in X_0. Orient the 2-handles so that $\mathrm{rot}(K_1) = q$ and so that $\ell k(K_0, K_1) = p$. In this orientation, let $\tilde{\mu}_i$ be a right handed meridian for K_i in X_0 and let μ_i be a right handed meridian for the corresponding (oriented) knots in $\partial B_{p,q}$ of Figure 3.21 so that

$$H_1(\partial X_0;\mathbb{Z}) = \langle \tilde{\mu}_0, \tilde{\mu}_1 : p\tilde{\mu}_1 = 0, p\tilde{\mu}_0 = (pq+1)\tilde{\mu}_1 \rangle \,,$$

$$H_1(\partial B_{p,q};\mathbb{Z}) = \langle \mu_0, \mu_1 : p\mu_1 = 0, p\mu_0 = (1-pq)\mu_1 \rangle \,,$$

where $\tilde{\mu}_0 = \mu_0 + q\mu_1$ and $\tilde{\mu}_1 = \mu_1$. Then, for $j = 0, 1$, by Proposition 1.1.19, we have

$$\rho([K_j]) = \frac{1}{2}\left(\frac{1-t_1}{2}p\right)(1-j) + \frac{1}{2}\left(q + \frac{3-t_0t_1^q}{2}p - \frac{1-t_1}{2}(pq+1)\right)j.$$

Noting that $\mu_1 = p\mu_0$, we find that

$$\begin{aligned}\Gamma(\xi_{J_{p,q}}, \mathbf{s}) &= \frac{1}{2}\left(\frac{1-t_1}{2}p\right)\tilde{\mu}_0 + \frac{1}{2}\left(q + \frac{3-t_0t_1^q}{2}p - \frac{1-t_1}{2}(pq+1)\right)\tilde{\mu}_1 \\ &= \left(\frac{pq}{2} + \left(\frac{3-t_0t_1^q}{2}\right)\frac{p^2}{2}\right)\cdot \mu_0.\end{aligned}$$

Since there is no 2-torsion in $\mathbb{Z}/p^2\mathbb{Z}$ if $p \in 2\mathbb{Z}+1$, $p^2/2 = 0$ in that case. If $p \in 2\mathbb{Z}$, then we can take $\mathbf{s}$ corresponding to $(t_0, t_1) = (1, -1)$. In either case, (fixing the spin structure) we have $\Gamma(\xi_{J_{p,q}}, \mathbf{s}) = \frac{pq}{2}\cdot \mu_0$. □

Proposition 3.2.9. *For $n > m \geq 1$ relatively prime, the contact structure induced by the Stein structure $(A_{m,n}, \widetilde{J}_{m,n})$ given by Figure 2.5 or 2.6 has*

$$\begin{aligned}&\Gamma(\xi_{\tilde{J}_{m,n}}, f_*(\mathbf{s})) \\ &\quad= \frac{m+n}{2}\left((d-c)^2 + \frac{1-t_1}{2}\left(1 + (d-c)^2\left(mn + \frac{1+(-1)^{c+\ell}t_0}{2}(m+n)\right)\right)\right)\gamma_0\end{aligned}$$

in an appropriate basis of $H_1(\partial A_{m,n};\mathbb{Z})$ where $cm + dn = 1$.

Proof. Let $\tilde{X}_0$ be the 4-manifold obtained from $A_{m,n}$ by surgering the 1-handle. Let $f_*(\mathbf{s}) \in \mathrm{Spin}(\partial A_{m,n})$ be the spin structure corresponding to the characteristic sublink (t_0, t_1) in $\partial B_{p,q}$. From Proposition 3.2.7, we have that $f_*(\mathbf{s}) = \left(\frac{(-1)^{c+\ell}t_0+t_1+(-1)^{c+\ell+1}t_0t_1+1}{2}, t_1\right)$. Then, since we slide the 2-handle once under the 1-handle to get to Figure 2.5 or 2.6, we consider the characteristic sublink

$$L' = \frac{1-t_1}{2}\left(\left(\frac{1+(-1)^{c+\ell}t_0}{2}\right)K_0 + K_1\right)$$

where K_0 is the 0-framed unkot arising from the surgery and K_1 is the Legendrian attaching circle of the single 2-handle. Orient K_0 and K_1 so that $\mathrm{rot}(K_1) = 1$ and so that $\ell k(K_0, K_1) = m+n$. With respect to this orientation, let γ_i be a right-handed meridian for K_i (viewed in $\partial A_{m,n}$ prior to the single handle slide). Then, by Proposition 1.1.19,

$$\Gamma(\xi_{\widetilde{J}_{m,n}}, f_*(\mathbf{s})) = \frac{1-t_1}{2}\frac{m+n}{2}\gamma_0 + \frac{1}{2}\left(1+\frac{1-t_1}{2}\left(mn+\frac{1+(-1)^{c+\ell}t_0}{2}(m+n)\right)\right)\gamma_1$$

To see that $\Gamma(\partial A_{m,n}, \mathbf{s})$ is as claimed, note that

$$H_1(\partial A_{m,n};\mathbb{Z}) = \langle \gamma_0, \gamma_1 : (m+n)\gamma_1 = 0, mn\gamma_1 = -(m+n)\gamma_0\rangle\,.$$

Combining this with the following observation; for c and d with $cm+dn=1$, we necessarily have $c(m+n)+(d-c)n = 1$ and $d(m+n)-(d-c)m = 1$. Multiplying these two equations gives that $-(d-c)^2 \cdot mn + (cd(m+n) - c(d-c)m + d(d-c)n) \cdot (n+m) = 1$. Thus,

$$\begin{aligned}
\gamma_1 &= \gamma_1 - (cd(m+n) - c(d-c)m + d(d-c)n) \cdot (n+m)\gamma_1 \\
&= (1 - (cd(m+n) - c(d-c)m + d(d-c)n) \cdot (n+m))\, \gamma_1 \\
&= -(d-c)^2 \cdot mn \cdot \gamma_1 \\
&= (d-c)^2 \cdot (m+n) \cdot \gamma_0.
\end{aligned}$$

Exchanging γ_1 for $(d-c)^2(m+n)\gamma_0$ in $\Gamma(\xi_{\tilde{J}_{m,n}}, f_*(\mathbf{s}))$ gives the result. $\square$

Remark 3.2.10. By applying $f^{-1} : \partial A_{m,n} \to \partial B_{p,q}$ of Theorem 1.2.2, we see that $\Gamma(f_*^{-1}\xi_{\tilde{J}_{m,n}}, \mathbf{s}) = \Gamma(\xi_{J_{p,q}}, \mathbf{s})$ for some spin structure $\mathbf{s} \in \mathcal{S}(\partial B_{p,q})$. Indeed, by Proposition 3.2.9 along with Remark 3.2.1 and Lemma 4.0.13 we have

$$\begin{aligned}
\Gamma(f_*^{-1}\xi_{\tilde{J}_{m,n}}, \mathbf{s}) &= f_*^{-1}\Gamma(\xi_{\tilde{J}_{m,n}}, f_*(\mathbf{s})) \\
&= \frac{p}{2}\left((d-c)^2 + \frac{1-t_1}{2}\left(1 + (d-c)^2\left(mn + \frac{1+(-1)^{c+\ell}t_0}{2}p\right)\right)\right) f_*^{-1}(\gamma_0) \\
&= \begin{cases} \frac{p}{2}\left((d-c)^2 + \frac{1-t_1}{2}\left(1 + (d-c)^2\left(mn + \frac{1+(-1)^{c}t_0}{2}p\right)\right)\right) n\mu_0 & \text{if } \ell \in 2\mathbb{Z}, \\ \frac{p}{2}\left((d-c)^2 + \frac{1-t_1}{2}\left(1 + (d-c)^2\left(mn + \frac{1+(-1)^{d}t_0}{2}p\right)\right)\right) m\mu_0 & \text{if } \ell \in 2\mathbb{Z}+1 \end{cases} \\
&= \frac{pq}{2}\mu_0 = \Gamma(\xi_{J_{p,q}}, \mathbf{s})
\end{aligned}$$

where the case when $\ell \in 2\mathbb{Z}+1$ follows from Lemma 4.0.13 by symmetry. It follows from Theorem 1.1.17 that $\xi_{J_{p,q}}$ and $f_*^{-1}\xi_{\tilde{J}_{m,n}}$ are in the same homotopy class and thus, by Theorem 1.1.16, isotopic. Therefore f^{-1} gives a contactomorphism from $(\partial A_{m,n}, \xi_{\tilde{J}_{m,n}})$ to $(\partial B_{p,q}, \xi_{J_{p,q}})$.

Chapter 4

The Algebraic Details

We have withheld some of the algebraic details used in the previous two chapters. In this chapter we state and prove these results. We start by giving a definition of the function A, defined by Yamada, which associates the relatively prime pair (m, n) to a given relatively prime pair $(p-q, q)$ [59]. Rather than relying on Yamada's original definition, we provide a description of A which dovetails with the boundary diffeomorphisms of Chapter 3. The following lemma gives that definition and proves that it is equivalent to Yamada's original definition.

Lemma 4.0.11. *Let $p-q > q \geq 1$ be relatively prime, and let $\{r_i\}_{i=-1}^{\ell+1}$ and $\{s_i\}_{i=0}^{\ell}$ be defined as in Definition 1.2.4. Define sequences $\{\sigma_i\}_{i=0}^{\ell}$ and $\{\rho_i\}_{i=-1}^{\ell+1}$ by $\sigma_0 \doteq r_\ell - 1$, $\sigma_i \doteq s_{\ell-i+1}$ for $i \in \{1, \ldots, \ell\}$. Recursively define ρ_i by setting $\rho_{\ell+1} \doteq 1$, $\rho_\ell \doteq s_0$, and setting*

$$\rho_i = \rho_{i+1}\sigma_{i+1} + \rho_{i+2}.$$

Let $m \doteq \rho_0$ and $n \doteq \rho_{-1}$. Then for m and n as defined, we have

$$A(p-q, q) = \begin{cases} (m, n) & \text{if } \ell \in 2\mathbb{Z}, \\ (n, m) & \text{if } \ell \in 2\mathbb{Z}+1. \end{cases}$$

$$(-1)^\ell(-c, d) = (|\det A_{\ell-1}| + (r_\ell - 1)|\det A_\ell|, |\det A_\ell|)$$

where c and d are the unique integers, with $0 < (-1)^{\ell+1}c, (-1)^{\ell}d < p$, satisfying $cm+dn = 1$, and where

$$A_i = \begin{pmatrix} s_1 & 1 & & \\ 1 & -s_2 & 1 & \\ & 1 & \ddots & 1 \\ & & 1 & (-1)^{i+1}s_i \end{pmatrix}.$$

Proof. Recall the definition of $A(p-q,q)$, as well as the pair (c,d) in [59]: Set $(a_0,b_0) \doteq (p-q,q)$, $(m_0,n_0) \doteq (1,1)$, $(c_0,d_0) = (0,1)$. If $a_i > b_i$,

$$(a_{i+1},b_{i+1}) \doteq (a_i - b_i, b_i), \qquad (m_{i+1},n_{i+1}) \doteq (m_i+n_i, n_i), \qquad (c_{i+1},d_{i+1}) \doteq (c_i, d_i+c_i)$$

and if $a_i < b_i$,

$$(a_{i+1},b_{i+1}) \doteq (a_i, b_i - a_i), \qquad (m_{i+1},n_{i+1}) \doteq (m_i, n_i+m_i), \qquad (c_{i+1},d_{i+1}) \doteq (c_i+d_i, d_i).$$

Then $A(p-q,q) \doteq (m_N,n_N)$ and $-c_N m_N + d_N n_N = 1$ for N such that $a_N = b_N = 1$ - which exists since $(p-q,q) = 1$. Since $p-q > q$, there is a subsequence $\{(a_{i_j},b_{i_j})\}_{j=1}^{\ell+2} \subset \{(a_i,b_i)\}_{i=0}^{N}$ satisfying

$$(a_{i_j},b_{i_j}) = \begin{cases} (r_j, r_{j-1}), & \text{if } j \in 2\mathbb{Z}+1, \\ (r_{j-1}, r_j), & \text{if } j \in 2\mathbb{Z} \end{cases}$$

for $j \in \{1,\ldots,\ell+1\}$, and $i_{\ell+2} = N$. Furthermore, for these indicies, we have

$$(m_{i_j},n_{i_j}) = \begin{cases} (\rho_{\ell-j+1}, \rho_{\ell-j+2}), & \text{if } j \in 2\mathbb{Z}+1, \\ (\rho_{\ell-j+2}, \rho_{\ell-j+1}), & \text{if } j \in 2\mathbb{Z}, \end{cases}$$

Thus for $j = \ell + 2$ we find that

$$A(p-q,q) = (m_N, n_N) = \begin{cases} (\rho_{-1}, \rho_0), & \text{if } \ell \in 2\mathbb{Z}+1, \\ (\rho_0, \rho_{-1}), & \text{if } \ell \in 2\mathbb{Z}. \end{cases}$$

To see that this gives the claim for (c,d) as well, we note for $j \leq \ell + 1$, we have

$$(c_{i_j}, d_{i_j}) = \begin{cases} \left(|\det A_{j-2}|, |\det A_{j-1}|\right), & \text{if } j \in 2\mathbb{Z}+1, \\ \left(|\det A_{j-1}|, |\det A_{j-2}|\right), & \text{if } j \in 2\mathbb{Z}. \end{cases}$$

where $A_{-1} \doteq 0$ and $A_0 \doteq 1$. Now, to produce such a subsequence, take $i_1 = s_0 - 1 > 1$ (so that $a_i > q$ for each $i < i_1$) similarly, take $i_{k+1} = s_k + i_k$ for $k \leq \ell$ and take $i_{\ell+2} = i_{\ell+1} + r_\ell - 1$. By definition,

$$(a_{i_1}, b_{i_1}) = (p - q - (s_0 - 1)q, q) = (r_1, r_0).$$

On the other hand

$$(m_{i_1}, n_{i_1}) = (1 + (s_0 - 1), 1) = (\rho_\ell, \rho_{\ell+1}), \qquad (c_{i_1}, d_{i_1}) = (0, 1+0) = (0,1).$$

For i_{k+1} we have (for $k < \ell + 1$),

$$(a_{i_{k+1}}, b_{i_{k+1}}) = \begin{cases} (r_k, r_{k-1} - s_k r_k), & \text{if } k \in 2\mathbb{Z}+1 \\ (r_{k-1} - s_k r_k, r_k), & \text{if } k \in 2\mathbb{Z} \end{cases} = \begin{cases} (r_k, r_{k+1}), & \text{if } k+1 \in 2\mathbb{Z} \\ (r_{k+1}, r_k), & \text{if } k+1 \in 2\mathbb{Z}+1. \end{cases}$$

and $(a_{i_{\ell+2}}, b_{i_{\ell+2}}) = (1,1)$. For $k \leq \ell+1$,

$$(m_{i_{k+1}}, n_{i_{k+1}}) = \begin{cases} (\rho_{\ell-k+1}, \rho_{\ell-k+2} + s_k \rho_{\ell-k+1}), & \text{if } k \in 2\mathbb{Z}+1 \\ (\rho_{\ell-k+2} + s_k \rho_{\ell-k+1}, \rho_{\ell-k+1}), & \text{if } k \in 2\mathbb{Z} \end{cases}$$
$$= \begin{cases} (\rho_{\ell-k+1}, \rho_{\ell-k+2} + \sigma_{\ell-k+1} \rho_{\ell-k+1}), & \text{if } k \in 2\mathbb{Z}+1 \\ (\rho_{\ell-k+2} + \sigma_{\ell-k+1} \rho_{\ell-k+1}, \rho_{\ell-k+1}), & \text{if } k \in 2\mathbb{Z} \end{cases}$$
$$= \begin{cases} (\rho_{\ell-k+1}, \rho_{\ell-k}), & \text{if } k+1 \in 2\mathbb{Z} \\ (\rho_{\ell-k}, \rho_{\ell-k+1}), & \text{if } k+1 \in 2\mathbb{Z}+1. \end{cases}$$

Finally notice that

$$\det A_i = (-1)^{i+1} s_i \det A_{i-1} - \det A_{i-2}$$

and that the sign of A_i coincides with the sign of $\sin(\pi i/2) + \cos(\pi i/2)$ giving that $|\det A_i| = s_i |A_{i-1}| + |A_{i-2}|$. Therefore,

$$(c_{i_{k+1}}, d_{i_{k+1}}) = \begin{cases} (|\det A_{k-2}| + s_k |\det A_{k-1}|, |\det A_{k-1}|), & \text{if } k \in 2\mathbb{Z}+1 \\ (|\det A_{k-1}|, |\det A_{k-2}| + s_k |\det A_{k-1}|), & \text{if } k \in 2\mathbb{Z} \end{cases}$$
$$= \begin{cases} (|\det A_k|, |\det A_{k-1}|), & \text{if } k+1 \in 2\mathbb{Z} \\ (|\det A_{k-1}|, |\det A_k|), & \text{if } k+1 \in 2\mathbb{Z}+1. \end{cases}$$

When passing to $k = \ell+2$, we have

$$(c_{i_{\ell+2}}, d_{i_{\ell+2}}) = \begin{cases} (|\det A_\ell|, |\det A_{\ell-1}| + (r_\ell - 1)|\det A_\ell|), & \text{if } \ell \in 2\mathbb{Z}+1, \\ (|\det A_{\ell-1}| + (r_\ell - 1)|\det A_\ell|, |\det A_{\ell-1}|), & \text{if } j \in 2\mathbb{Z}. \end{cases}$$

Giving that $(-1)^{\ell+1} (|\det A_{\ell-1}| + (r_\ell - 1)|)\, m + (-1)^\ell |\det A_\ell| n = 1$. $\square$

In general, c and d satisfying $cm+dn=1$ are far from unique. However, specifying them as in Lemma 4.0.11, (which are equivalent to the coefficients s and t that Yamada defines originally [59]) is crucial, since, as constructed:

Lemma 4.0.12 ([59], Lemma 2.5)**.** *Suppose that $A(p-q,q)=(m,n)$. If c and d are defined as in Lemma 4.0.11, giving that $cm+dn=1$, then $d-c=q$.*

Notice that if $A(p-q,q)=(n,m)$, then we clearly have $c-d=q$ instead. Lemma 4.0.12 allows us to simplify the quantity $f_*^{-1}\Gamma(\xi_{\tilde{J}_{m,n}}, f_*(\mathbf{s}))$ of Proposition 3.2.9. We only consider the case when $\ell \in 2\mathbb{Z}$ (giving that $A(p-q,q)=(m,n)$) since the case when $\ell \in 2\mathbb{Z}+1$ is symmetric by exchanging $m \leftrightarrow n$ and $c \leftrightarrow d$.

Lemma 4.0.13. *Suppose that $A(p-q,q)=(m,n)$, and that $cm+dn=1$ so that $d-c=q$, then for $(t_0,t_1) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, we have*

$$\frac{p}{2}\left(q^2+\frac{1-t_1}{2}\left(1+q^2\left(mn+\frac{1+(-1)^c t_0}{2}p\right)\right)\right)n=\frac{pq}{2}$$

in $\mathbb{Z}/p^2\mathbb{Z}$ whenever $p \in 2\mathbb{Z}+1$ or when $p \in 2\mathbb{Z}$ and $(t_0,t_1)=(1,-1)$.

Proof. Recall that $m+n=p$ and that $qn=1-cp$. Thus, in $\mathbb{Z}/p^2\mathbb{Z}$

$$\begin{aligned}
&\frac{p}{2}\left(q(1-cp)+\frac{1-t_1}{2}\left(n+m(1-cp)^2+\frac{1+(-1)^c t_0}{2}q(1-cp)p\right)\right)\\
&\quad=\frac{pq}{2}+\frac{p^2}{2}\left(-cq+\frac{1-t_1}{2}\left(1-2c+pc^2+\frac{1+(-1)^c t_0}{2}q\right)\right)\\
&\quad=\frac{pq}{2}+\frac{p^2}{2}\left(-cq+\frac{1-t_1}{2}\left(1+\frac{1+(-1)^c t_0}{2}q\right)\right).
\end{aligned}$$

If $p \in 2\mathbb{Z}+1$, then $\mathbb{Z}/p^2\mathbb{Z}$ lacks 2-torsion so that $p^2/2=0$. Suppose that $p \in 2\mathbb{Z}$ and that

$(t_0, t_1) = (1, -1)$, then the above reduces to

$$\frac{pq}{2} + \frac{p^2}{2}\left(-cq + 1 + \frac{1 + (-1)^c}{2} q\right) = \frac{pq}{2}$$

since in this case, $q \in 2\mathbb{Z} + 1$ and the quantity $-cq + 1 + \frac{1+(-1)^c}{2} q$ is necessarily even. $\square$

The following result is used to independently verify that $\partial B_{p,q} = L(p^2, pq - 1)$. To that end, we inductively build the linear plumbing of Figure 3.6 from the middle out. Furthermore, we choose signs on the weights so that $-s_0$ ends up on the left. Since, a posteriori, we have

$$[-s_0, s_1, \ldots, \pm r_\ell, 1, \mp r_\ell, \ldots, -s_1, s_0] = \frac{\det Q_{S_{\ell+1}}}{\det Q_{S_\ell^-}} = \frac{(-1)^\ell r_{-1}^2}{(-1)^\ell (1 - r_{-1} r_0)} = \frac{-p^2}{pq - 1}$$

where we use that if $[c_1, \ldots, c_n] = -p/q$ then $-p/q = \det C_n / \det C_{n-1}$ for the matrices C_i defined in Lemma 1.1.2.

Lemma 4.0.14. *Define $\{r_i\}_{i=-1}^{\ell+2}$ and $\{s_i\}_{i=0}^{\ell+1}$ as in Definition 1.2.4, let S_i be the 4-manifold given by plumbing D^2-bundles over S^2 according to the weighted graph in Figure 4.1. Let*

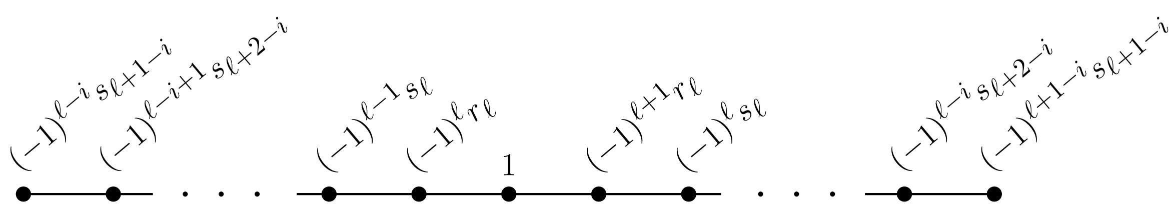

Figure 4.1: The 4-manifold S_i.

S_i^+ be the 4-manifold obtained by plumbing an Euler class $(-1)^{\ell-i-1} s_{\ell-i}$ disk bundle to the Euler class $(-1)^{\ell-i} s_{\ell+1-i}$ disk bundle in S_i. Let S_i^- be the 4-manifold obtained by plumbing an Euler class $(-1)^{\ell-i} s_{\ell-i}$ disk bundle to the Euler class $(-1)^{\ell+1-i} s_{\ell+1-i}$ disk bundle in

S_i. Then the intersection forms of S_i and $S_i^{\pm}$ satisfy

$$\det Q_{S_i} = (-1)^{i+1} r_{\ell-i}^2, \qquad \det Q_{S_i^+} = (-1)^{\ell}\left(r_{\ell-i-1}r_{\ell-i} + (-1)^{\ell+i}\right),$$
$$\det Q_{S_i^-} = (-1)^{\ell}\left((-1)^{\ell+i} - r_{\ell-i-1}r_{\ell-i}\right).$$

Proof. Induct on i by noting that

$$\det Q_{S_i^{\pm}} = (-1)^{\ell-i-(1\pm1)/2} s_{\ell-i} \det Q_{S_i} - \det Q_{S_{i-1}^{\mp}},$$

$$\det Q_{S_{i+1}} = (-1)^{\ell-i-1} s_{\ell-i} \det Q_{S_i^-} + (-1)^{\ell-i+1} s_{\ell-i} \det Q_{S_{i-1}^-} + \det Q_{S_{i-1}},$$

as well as the fact that, by definition, $r_k = r_{k+1}s_{k+1} + r_{k+2}$. $\square$

Lemma 1.1.2 requires that we understand certain determinants arising from the intersection form of a given linear plumbing. We calculate those determinants here - they are used to to measure the obstruction to a certain spin$^{\mathbb{C}}$-structures extending across $B_{p,q}$ as well as to express the generator, γ_0, of $H_1(\partial A_{m,n})$ in terms of $\mu_0 \in H_1(\partial B_{p,q})$.

Lemma 4.0.15. *Let $\{\rho_i\}_{i=-1}^{\ell+2}$ and $\{\sigma_i\}_{i=0}^{\ell+1}$ be as defined in Definition 1.2.4, (associated to n and m) then for each $i \leq \ell+1$ we have*

$$\det \begin{pmatrix} -\rho_\ell & 1 & & \\ 1 & \sigma_\ell & 1 & \\ & 1 & \ddots & 1 \\ & & 1 & (-1)^{\ell+1-i}\sigma_{\ell+1-i} \end{pmatrix} = -\left(\sin\left(\frac{\pi}{2}i\right) + \cos\left(\frac{\pi}{2}i\right)\right)\rho_{\ell-i}.$$

Proof. Induct on i, using that $\rho_{\ell+1} = 1$ and that $\rho_{\ell-i} = \rho_{\ell-i+1}\sigma_{\ell-i+1} + \rho_{\ell-i+2}$. $\square$

Lemma 4.0.16. *Fix integers $c_0 \in [0, s_0 - 1]$, $c_i \in [0, s_i]$ for $1 \le i \le \ell$ and $c_{\ell+1} \in [0, r_\ell - 1]$. Then for each $k < \ell + 1$ the following inequalities hold*

$$1 - \rho_{\ell - 2\left\lfloor \frac{k+1}{2} \right\rfloor + 1} \le \sum_{i=0}^{k} (-1)^i c_i \rho_{\ell-i+1} \le -1 + \rho_{\ell - \left\lfloor \frac{k}{2} \right\rfloor},$$

$$-p < 1 - \rho_0 \le (-1)^{\ell+1} \sum_{i=0}^{\ell+1} (-1)^i c_i \rho_{\ell-i+1} \le \rho_{-1} + 2\rho_0 - 1 < 2p.$$

Consequently, $\sum_{i=0}^{\ell+1} (-1)^i c_i \rho_{\ell-i+1} = 0$ if and only if each $c_i = 0$.

Proof. First, assume the inequalities; note $c_0 \rho_{\ell+1} = 0$ if and only if $c_0 = 0$. By way of induction, suppose the only solution to $\sum_{i=0}^{k} (-1)^i c_i \rho_{\ell-i+1} = 0$ is the trivial solution. Any purported nontrivial solution to $\sum_{i=0}^{k+1} (-1)^i c_i \rho_{\ell-i+1} = 0$, has $c_{k+1} > 0$ by induction; however,

$$c_{k+1} \rho_{\ell-k} > \rho_{\ell-k} - 1 \ge (-1)^k \sum_{i=0}^{k} (-1)^i c_i \rho_{\ell-i+1},$$

contradicting $\sum_{i=0}^{k+1} (-1)^i c_i \rho_{\ell-i+1} = 0$. The lower bounds follow by noting that the sum minimizes by taking c_i's maximal for odd indicies and zero otherwise: when $k < \ell + 1$,

$$\begin{aligned}
\sum_{i=0}^{k} (-1)^i c_i \rho_{\ell-i+1} &\ge \sum_{i=1}^{\left\lfloor \frac{k+1}{2} \right\rfloor} -c_{2i-1} \rho_{\ell-2i+2} \\
&\ge \sum_{i=1}^{\left\lfloor \frac{k+1}{2} \right\rfloor} -\sigma_{\ell-2i+2} \rho_{\ell-2i+2} \\
&= \sum_{i=1}^{\left\lfloor \frac{k+1}{2} \right\rfloor} (\rho_{\ell-2i+3} - \rho_{\ell-2i+1}) = \rho_{\ell+1} - \rho_{\ell - 2\left\lfloor \frac{k+1}{2} \right\rfloor + 1}
\end{aligned}$$

here we use that $s_i = \sigma_{\ell-i+1}$ and that $\rho_{i+1}\sigma_{i+1} = \rho_i - \rho_{i+2}$. The arguments are similar for the upper bounds and those when $k = \ell + 1$. $\square$

Corollary 4.0.17. *For c_i's as in Lemma 4.0.16, there are exactly two solutions to*

$$\sum_{i=0}^{\ell+1}(-1)^i c_i \rho_{\ell-i+1} \equiv 0 \mod p.$$

Proof. By Lemma 4.0.16, $\left|\sum_{i=0}^{\ell+1}(-1)^i c_i \rho_{\ell-i+1}\right| < 2p$, therefore, we only need to consider solutions with $\sum_{i=0}^{\ell+1}(-1)^i c_i \rho_{\ell-i+1} \in \{0, \pm p\}$. Notice, the last inequality in Lemma 4.0.16 implies that if there is a solution summing to $\pm p$ then there isn't one summing to $\mp p$. Lemma 4.0.16 also gives that there is exactly one solution summing to zero. Note that choosing the c_i's maximal gives

$$\sum_{i=0}^{\ell+1}(-1)^i c_i^{\max} \rho_{\ell-i+1} = s_0 - 1 + \sum_{i=1}^{\ell}(-1)^i s_i \rho_{\ell-i+1} + (-1)^{\ell+1}(r_\ell - 1)\rho_0 = (-1)^{\ell+1}p.$$

This solution is necessarily unique; whenever $\sum_{i=0}^{\ell+1}(-1)^i c_i \rho_{\ell-i+1} = (-1)^{\ell+1}p$, we have that

$$\sum_{i=1}^{\ell+1}(-1)^i (c_i^{\max} - c_i)\rho_{\ell-i+1} = 0,$$

forcing each $c_i = c_i^{\max}$. Therefore, there are exactly two solutions: $c_{\min} \equiv 0$ and $c^{\max}$. □

Corollary 4.0.18. *Let γ_0, $\eta_{\pm 1}$ each be meridians indicated in Figure 4.2. Then, fixing*

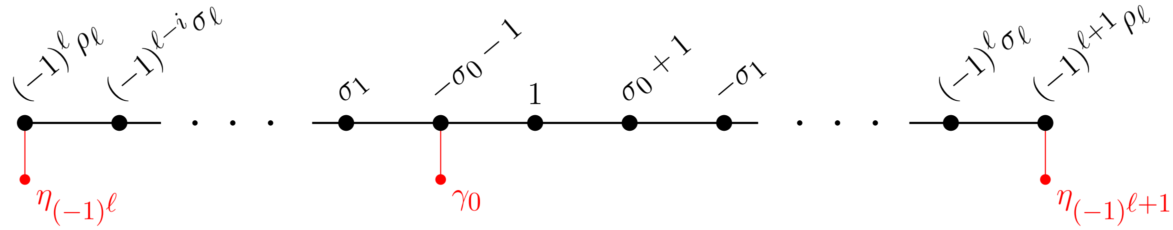

Figure 4.2: Expressing γ_0 in terms of a "preferred" generator, η_{-1}, for the lens space $\partial A_{m,n}$.

orientations so all linking is non-negative, we have

$$-\left(\sin\left(\frac{\pi}{2}\ell\right)+\cos\left(\frac{\pi}{2}\ell\right)\right)m\cdot\eta_{(-1)^{\ell}}=\gamma_0=-\left(\sin\left(\frac{\pi}{2}\ell\right)+\cos\left(\frac{\pi}{2}\ell\right)\right)n\cdot\eta_{(-1)^{\ell+1}}.$$

Proof. This follows immediately from Lemma 1.1.2 and Lemma 4.0.15. $\square$

APPENDIX

Appendix

An Example

For the benefit of the reader, we work out the major arguments of Propositions 2.2.1, 3.1.1 and 3.1.6 on the rational balls $B_{8,3}$ and $A_{3,5}$. To begin, with note that for $p = 8$ and $q = 3$ we find sequences $\{r_i\}_{i=-1}^{\ell+1}$ and $\{s_i\}_{i=0}^{\ell}$ as in Definition 1.2.4:

$$r_{-1} = 8, r_0 = 3, r_1 = 2, r_2 = 1, s_0 = 2, s_1 = 1.$$

Therefore, $\ell = 1$ in this example. According to Lemma 4.0.11, we can find $A(p-q, q) = A(5, 3) = (n, m)$ by constructing sequences $\{\rho_i\}_{i=-1}^{2}$ and $\{\sigma_0, \sigma_1\}$ where $\sigma_0 = r_\ell - 1 = 1$, $\sigma_1 = s_1 = 1$, $\rho_2 = 1$, $\rho_1 = s_0 = 2$ so that:

$$m = \rho_0 = \rho_1\sigma_1 + \rho_2 = 3,$$

$$n = \rho_{-1} = \rho_0\sigma_0 + \rho_1 = 5.$$

Theorem 1.2.1, as well as Corollary 1.2.3, shows that $B_{8,3} \approx A_{3,5}$. Figure A.1 illustrates the necessary isotopies, defined in the proof of Proposition 2.2.1, to realize $A_{3,5}$ as a Stein domain. Figure A.2 illustrates the boundary diffeomorphism from $\partial B_{8,3}$ to a linear plumbing. Figure A.3 illustrates the boundary diffeomorphism from that linear plumbing to $\partial A_{3,5}$.

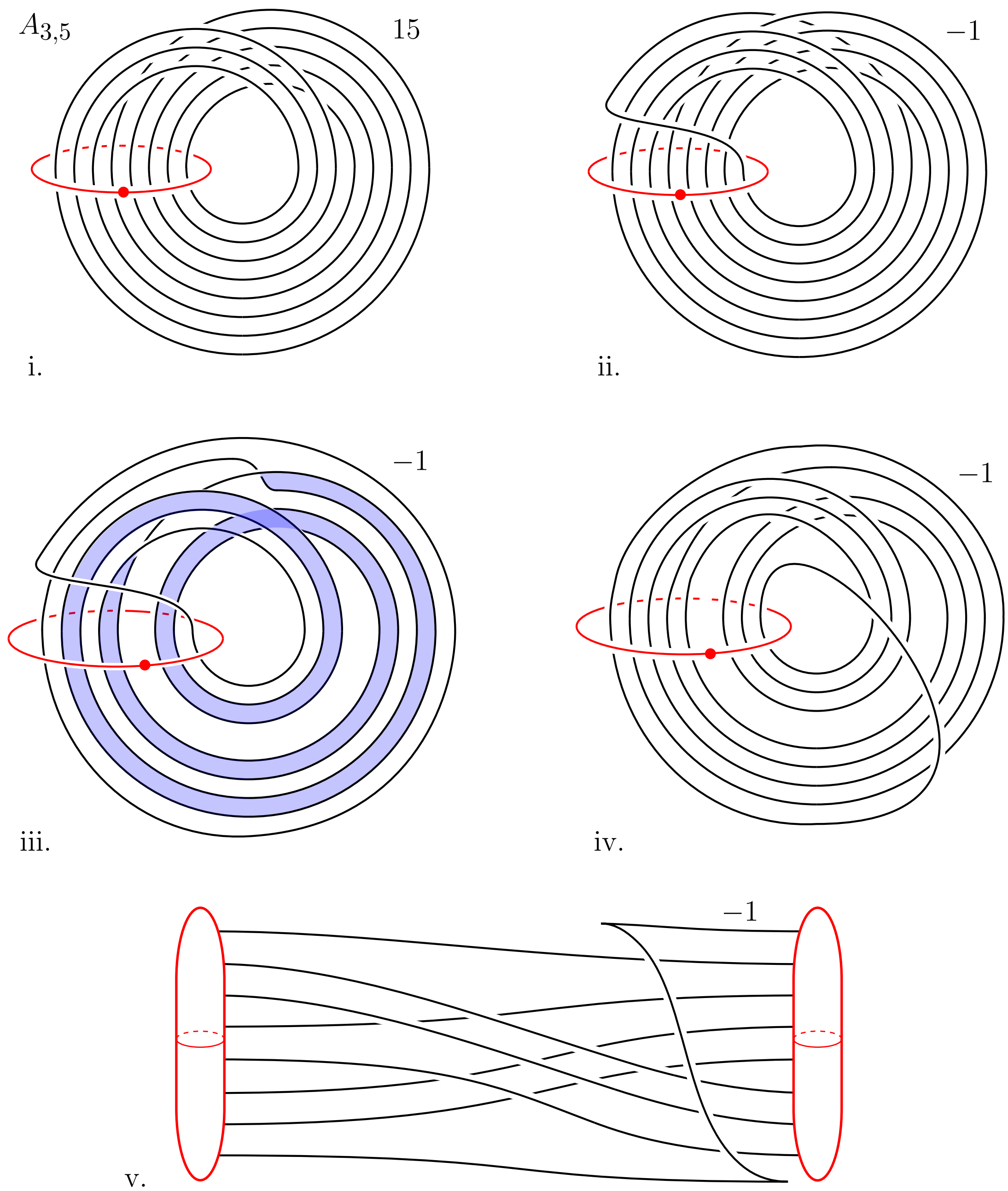

Figure A.1: The Isotpies of Proposition 2.2.1: i. $A_{3,5}$; ii. Slide the attaching circle K of the 2-handle once under the 1-handle; iii. Drag K over the 1-handle once. The shaded ribbon now represents the track of the isotopy needed to drag K over the 1-handle $\sigma_0+2 = 3$ more times; iv. Cancel the negative twist with positive twist at the ends of the shaded band; v. Pass to two ball notation and put K in Legendrian position. Notice that $\mathrm{tb}(K) = 8 - 7 - 1 = 0$. This is the Stein structure $(A_{3,5}, \widetilde{J}_{3,5})$.

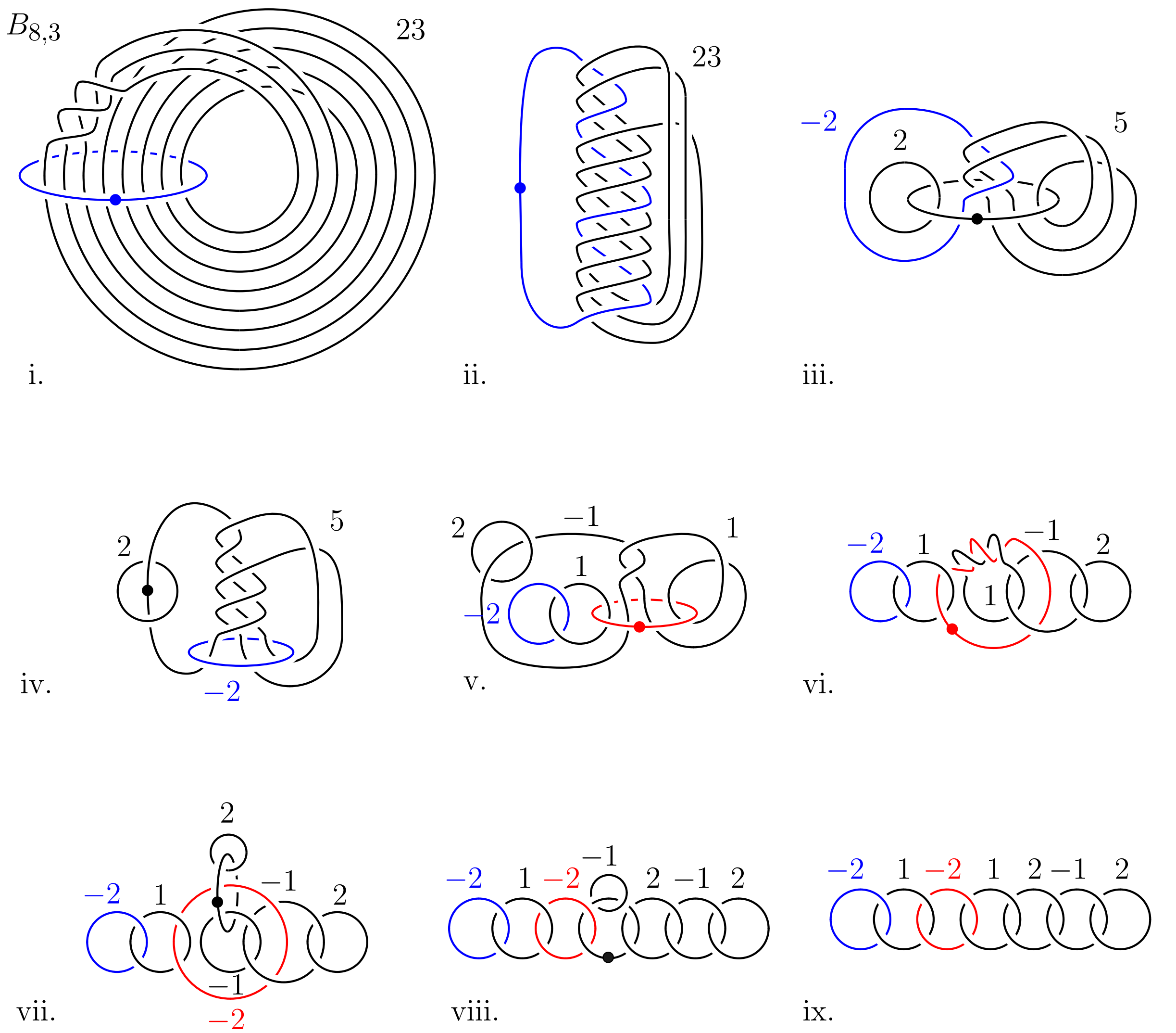

Figure A.2: The boundary diffeomorphisms of Proposition 3.1.1: i. $B_{8,3}$; ii. Isotope the attaching circle K by viewing K as a band of three strands traversing the 1-handle twice (with two strands traversing a third time); iii. Surger the 1-handle and unwind the two full twists by introducing a canceling pair. iv. Isotope the attaching circle of the 5-framed knot K by viewing K as a band of two strands traversing the 1-handle once (with one strands traversing an additional time); v. Again, surger the 1-handle and unwind the full twist by introducing a canceling pair. Slide the (blue) -2 framed 2-handle under the 1-handle. vi. Isotope the attaching circle of the rightmost 1-framed knot K; vii. Again, surger the 1-handle and unwind the two full twists by introducing a canceling pair. viii. Slide the -1 framed 2-handle under the 1-handle. ix. Surger the 1-handle and blow-down. This is the linear plumbing of Corollary 3.1.3 - showing directly that $\partial B_{8,3} \approx L(64, 23)$.

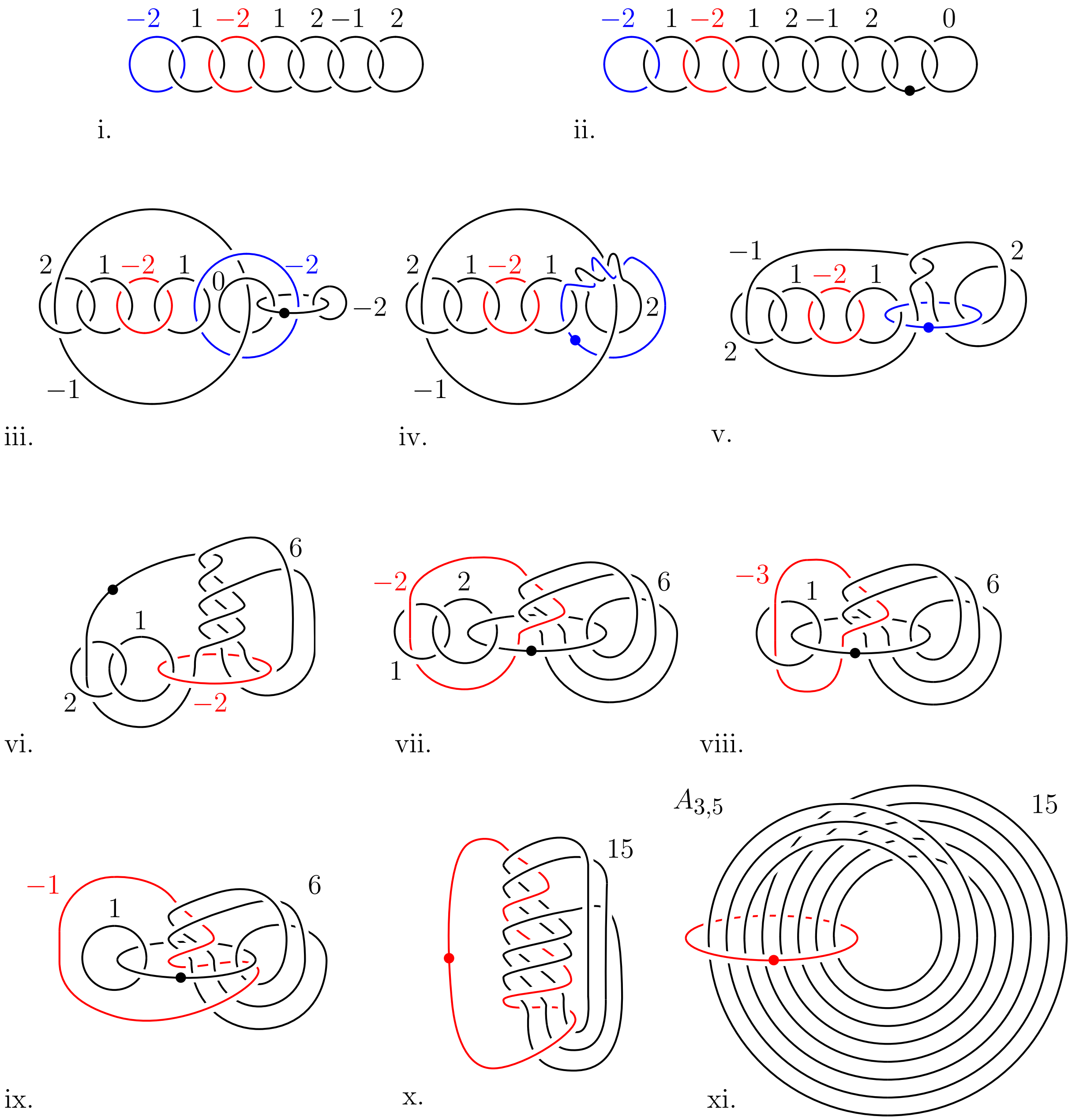

Figure A.3: The (inverse) boundary diffeomorphisms of Proposition 3.1.6: Working from xi. - i.: xi. $A_{3,5}$; x. Isotope the attaching circle K by first viewing the leftmost three strands as winding around the 1-handle twice with two strands winding a third time; ix. Surger the 1-handle and introduce a canceling pair of 1- and 2-handles to unwind the full twist; viii. Slide the (red) 2-handle under the 1-handle; vii. Blow-up once; vi. Isotope the 6-framed 2-handle by viewing it as two strands passing over the 1-handle once (with one strand passing over an additional time); v. Surger the 1-handle and unwind the full twist through the introduction of a canceling pair of 1- and 2-handles; iv. Isotope the rightmost 2-framed 2-handle; iii. Surger and unwind the two full twists through the introduction of a canceling pair; ii. Slide the (blue) -2 framed 2-handle over the 0-framed 2handle and slide the large -1 framed 2-handle under the 1-handle; i. Canceling the 1-handle gives the linear plumbing of Figure A.2.

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