This is to certify that the
dissertation entitled

TOPOLOGICAL TRANSITIVITY OF BOUNDED LINEAR
OPERATORS

presented by

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has been accepted towards fulfillment
of the requirements for the

Ph. D degree in Mathematics

Joel Shapiro
Major Professor's Signature

9 September 2003
Date

Topological Transitivity Of Bounded Linear Operators

By

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A DISSERTATION

Submitted to

Michigan State University

in partial fulfillment of the requirements

for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

2003

ABSTRACT

Topological Transitivity Of Bounded Linear Operators

By

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A bounded linear operator on a separable Banach space is said to satisfy the three-neighborhood condition if for every pair U, V of non-empty open subsets of X, and each open neighborhood W of zero in X there exists a positive integer n such that both $T^n U \cap W$ and $T^n W \cap V$ are non-empty. The operator is called syndetically hypercyclic if for any strictly increasing syndetic sequence of positive integers $\{n_k\}_k$, $\{T^{n_k}\}_k$ is a hypercyclic sequence of operators. We prove that these two conditions are equivalent to the Hypercyclicity Criterion. Then we prove the existence of topologically transitive multipliers on Banach algebras and study some necessary conditions for a multiplier to be topologically transitive on Banach algebras.

To my wife Maria Alejandra and my brother Cesar,
silent but close companions throughout this journey.

ACKNOWLEDGEMENTS

I would like to thank my advisor, Professor Joel H. Shapiro for his teaching, guidance, encouragement and patience through the time it took me to complete this thesis. I will always by deeply grateful to him for introducing me to the fascinating world of the dynamics of operators. His dedication to his work in mathematics is truly invaluable.

I would like to thank my friend, Professor Alfredo Peris, for the work we did together, which is the content of Chapter 3 of this thesis, and for all the personal communications and advice. Muchas gracias Chamo!

I would like to thank my friend, Professor Ruben Martínez, for all the small seminars we organize together and for all personal communications and encouragement. Gracias Chamaco!

I would like to thank Professors Michael Frazier, Shlomo Levental, William Sledd and Clifford Weil for all the seminars and classes they gave through the years and for serving on my committee.

Contents

Introduction

Let X be a separable Banach space and T a bounded linear operator on X; i.e., $T : X \to X$ is bounded and linear (referred to simply as operator). A sequence of operators $\{T_n\}_{n\geq 1}$ is said to be a *hypercyclic sequence* on X if there exists some $x \in X$ such that its orbit

$$Orb(\{T_n\}_n, x) := \{x, T_1x, T_2x, \dots\}$$

is dense in X. In this case the vector x is called *hypercyclic* for the sequence $\{T_n\}_n$. An operator T is *hypercyclic* on X if $\{T^n\}_n$ is a hypercyclic sequence of operators. Note that if $\{T_n\}_n$ is a hypercyclic sequence of operators on X, then X is necessarily separable.

A sufficient condition for hypercyclicity, the well known Hypercyclicity Criterion, independently discovered by Kitai [21] and Gethner and Shapiro [16], has been the fundamental tool for proving hypercyclicity. Throughout this thesis we will be using the following version of this Criterion.

Definition. (The Hypercyclicity Criterion) An operator T on a separable Banach space X is said to *satisfy the Hypercyclicity Criterion* provided there exists a strictly increasing sequence of positive integers $\{n_k\}_k$ for which there are

1. a dense subset $X_0 \subset X$ such that $T^{n_k}x \to 0$ for every $x \in X_0$, and
2. a dense subset $Y_0 \subset X$ and a sequence of mappings $\{S_k : Y_0 \to X\}_k$

such that

a) $S_k y \to 0$ for every $y \in Y_0$,

b) $T^{n_k} S_k y \to y$ for every $y \in Y_0$.

In the first three chapters of this thesis we study several new characterizations of the Hypercyclicity Criterion.

In Chapter 1 we explain the importance of hypercyclic operators and of the Hypercyclicity Criterion. We also mention some important known results about such operators related to this work. At the end of Chapter 1 we give our first characterization of the Hypercyclicity Criterion. (For terminology and notation see Chapter 1.)

Theorem. 1.4 *(The Hypercyclicity Criterion II) Let T be an operator on a separable Banach space X. Suppose that there exists a strictly increasing sequence of positive integers $\{n_k\}_{k\geq 1}$ for which there are:*

1. a dense subset $X_0 \subset X$ and $r_1 > 0$ such that $T^{n_k}x \to B(0, r_1)$ for every $x \in X_0$.

2. a well distributed $Y_0 \subset X$, a sequence of mappings $S_k : Y_0 \to X$ and $r_2 > 0$ such that:

a) $S_k y \to 0$ for every $y \in Y_0$

b) $T^{n_k} S_k y - y \to B(0, r_2)$ for every $y \in Y_0$.

Then T hypercyclic on X.

We also prove that the Hypercyclicity Criterion II is equivalent to the Hypercyclicity Criterion. (Theorem 1.7.)

In Chapter 2 we use the following condition given By Godefroy and Shapiro [16, Corollary 1.2], to get our second characterization of the Hypercyclicity Criterion.

Definition. 2.1 An operator T on a separable Banach space X is said to *satisfy the three-neighborhoods condition* if for every pair U, V of non-empty open subsets of X, and each open neighborhood W of zero in X there exists a positive integer n such that both $T^n U \cap W$ and $T^n W \cap V$ are non-empty.

Theorem. 2.3 *Let X be a separable Banach space and T an operator in X. Then T satisfies the Hypercyclicity Criterion if and only if T satisfies the three-neighborhoods condition.*

As a corollary of the proof of Theorem 2.3 we get the following.

Corollary. 2.4 *An operator T on a Banach space X satisfies the Hypercyclicity Criterion if and only if:*

(1) T is hypercyclic, and

(2) For any non-empty open subset $U \subset X$ and any open neighborhood of zero W there is an $n \in \mathbb{N}$ such that $T^n U \cap W \neq \emptyset$ and $T^n W \cap U \neq \emptyset$.

Being hypercyclic is equivalent to a property called *topological transitivity.* A sequence of continuous maps $\{T_n\}_n$ on a topological space X is *topologically transitive* if for any pair U, V of non empty open subsets of X there is a positive integer n_0 such that $T_{n_0}(U) \cap V \neq \emptyset$.

A single map $T : X \to X$ is *topologically transitive* if the sequence $\{T^n\}_n$ is topologically transitive.

The three-neighborhoods condition seems to be stronger than topological transitivity, but by Theorem 2.3, it turn out that proving that it is actually stronger is equivalent to giving an negative answer to the still open problem that every hypercyclic operator satisfies the Hypercyclicity Criterion.

Then we apply Theorem 2.3 to give a new proof of the following result due to León and Montes:

Corollary. 2.5 *Every hypercyclic bilateral weighted shift satisfies the Hypercyclicity Criterion.*

In [2], Ansari showed that if an operator T on a separable Banach space X is hypercyclic, then T^n is also hypercyclic on X, for every positive integer n. Moreover, Ansari also showed that T and T^n share the same set of hypercyclic vectors. Motivated by this result, Bés posed the following question: *suppose that T is hypercyclcic on a separable space X and $\{n_k\}_{k=1}^{\infty}$ is such that* $\sup_k\{n_{k+1}-n_k\} < \infty$ *(syndetic sequence). Is $\{T^{n_k}\}$ a hypercyclic sequence of operators on X?*

The topic of Chapter 3 is related to this question. We start with two definitions:

Definition. 3.1 An operator T on X is called *syndetically hypercyclic* if for any strictly increasing syndetic sequence of positive integers $\{n_k\}_k$, the sequence $\{T^{n_k} : X \to X\}_k$ is hypercyclic.

Definition. 3.2 Let X be a topological space and $T : X \longrightarrow X$ be a continuous map. T is called *weakly mixing* if $T \times T : X \times X \longrightarrow X \times X$ is topologically transitive.

Then we prove the main result of Chapter 3.

Theorem. 3.4 *Let $T : X \longrightarrow X$ be an operator on a separable Banach space X. Then the following are equivalent:*

(i) T satisfies the Hypercyclicity Criterion.

(ii) T is syndetically hypercyclic.

Theorem 3.4 is a consequence of the following result, which is also interesting in its own right.

Proposition. 3.3 *Let $T : X \longrightarrow X$ be a continuous map on a topological space X. Then the following are equivalent:*

(i) T is weakly mixing.

(ii) For any pair of non-empty open subsets $U, V \subseteq X$, and for any strictly increasing sequence $\{n_k\}_k$ with $\sup_k\{n_{k+1} - n_k\} < \infty$, there exists k_0 such that $T^{n_{k_0}}U \cap V \neq \emptyset$.

We then prove that if the sequence $\{n_k\}_k$ is such that $\sup_k\{n_{k+1} - n_k\} = \infty$, then there are hypercyclic operators (satisfying the Hypercyclicity Criterion) such that $\{T^{n_k}\}_k$ is not a hypercyclic sequence of operators.

Proposition. 3.5 *Suppose $\{n_k\}_k$ is such that $\sup_k\{n_{k+1} - n_k\} = \infty$. Then there exists a bounded sequence $\{w_n\}_n$ of positive scalars such that the unilateral weighted backward shift $T : l^2 \longrightarrow l^2$, given by*

$$Te_n = \begin{cases} w_n e_{n-1} & for \quad n \geq 2 \\ 0 & for \quad n = 1, \end{cases}$$

is hypercyclic but $\{T^{n_k}\}_k$ is not a hypercyclic sequence of operators in X.

We also show that even in the case that T satisfies the Hypercyclicity Criterion and $\{n_k\}_k$ is syndetic, T and T^{n_k} do not have to share the same set of hypercyclic vectors.

Theorem. 3.7 *Let T be a hypercyclic operator on a locally convex space X and let $x \in X$ be a hypercyclic vector for T. Then there exists a sequence of positive integers $\{n_k\}_k$ with $\sup_k\{n_{k+1} - n_k\} = 2$ such that $\{T^{n_k}x\}_k$ is somewhere dense but not everywhere dense.*

This establishes a difference not only with Ansari's result, but also between the full orbit and the sub-orbit associated with a sequence $\{n_k\}_k$, for an operator T. *If the* *orbit under T of any vector is somewhere dense, then the orbit is everywhere dense!* This is a result of Bourdon and Feldman [10].

The equivalence between hypercyclicity and topological transitivity (on separable Banach spaces) is used to extend the definition of hypercyclicity to spaces not necessarily separable via topological transitivity. This is the subject of the last two chapters of this thesis.

In Chapter 4 we extend some well known properties for hypercyclic operator (separable case) to topologically transitive ones (non-separable case). In particular we prove the following.

Theorem. 4.3 *Let X be a Banach space (separable or not) and $T \in L(X)$. If T is topologically transitive on X, $\sigma(T) \cap \{\lambda \in \mathbb{C} : |\lambda| = 1\} \neq \emptyset$.*

Generalizing the idea of Rolewicz, who showed that if $\lambda \in \mathbb{C}$ is such that $|\lambda| > 1$ and B is the backward shift in l^2 then λB is hypercyclic, J. H. Shapiro [4, pg. 1452] showed the existence of topologically transitive operators on any Hilbert space. In **Example 4.4** we present a more natural example of topologically transitive operators on a non-separable Banach space.

In Chapter 5 we study the topological transitivity of a special class of operators on Banach algebras.

Let A be a Banach algebra. Let T in $L(A)$. T is called a *left multiplier* if $T(uv) = T(u)v \;\forall\; u, v \in A$, and T is called a *right multiplier* if $T(uv) = uT(v) \;\forall u, v \in A$.

First we show the existence of topologically transitive multipliers.

Theorem. 5.1 *There exists a Banach algebra A and $\tilde{a} \in \tilde{A}$ (the unitization of A), such that the left multiplier $L_{\tilde{a}} \in L(A)$, given by $L_{\tilde{a}}(b) = \tilde{a}b \;\forall\; a \in A$ is topologically transitive on A.*

The Banach algebra of Theorem 5.1 is non-unital and this is not casual.

Theorem. 5.3 *No multiplier on a Banach algebra with unit element is topologically transitive.*

Then we consider some special cases when A is a non-unital Banach algebra. We start with commutative Banach algebras.

Theorem. 5.5 *No multiplier on a commutative Banach algebra with a non-zero, bounded, multiplicative linear functional is topologically transitive.*

Using the following Comparison Principle we show that a more general result holds with multiplication operators by elements in the algebra (multipliers of the form $L_a(b) = ab$ or $R_a(b) = ba$ for some $a \in A$ and for all $b \in A$.)

Lemma. 5.7*(A Comparison Principle) Let A_1, A_2 be Banach algebras and $T_i \in L(A_i)$, $i = 1, 2$. Let $\Psi : A_1 \to A_2$ be continuous with dense range such that $T_2 \circ \Psi = \Psi \circ T_1$. If T_1 is topologically transitive, then so is T_2.*

Proposition. 5.8 *Let A be a Banach algebra with a non-zero, bounded, multiplicative linear functional and let $a \in A$. The operator multiplication by a on A (from the left or from the right) is not topologically transitive.*

We also show that this result holds for general multipliers if the Banach algebra contains a bounded left approximate identity.

Theorem. 5.9 *No left (right) multiplier on a Banach algebra with non-zero, bounded, multiplicative linear functional and with a bounded left (right) approximate identity is topologically transitive.*

In Chapter 6 we give some final remarks and questions.

Chapter 1

Hypercyclic Fundamentals

Let X be a Banach space. Throughout this thesis $L(X)$ will denote the algebra of bounded linear operators, $T : X \to X$. The elements of $L(X)$ will be referred to simply as operators.

We will also be using the following standard notation. For each $x \in X$ and $\epsilon > 0$, let

$$B(x, \epsilon) = \{y \in X : ||y - x|| < \epsilon\}$$

Even though most of the results remain true for metrizable and complete topological vector space (F-spaces), we will keep the underlying space X as a Banach space.

Definition 1.1. A sequence of operators $\{T_n\}_n$ is said to be a *hypercyclic sequence* on X if there exists some $x \in X$ such that its orbit

$$\mathrm{Orb}(\{T_n\}_n, x) := \{x, T_1 x, T_2 x, ...\}$$

is dense in X. In this case the vector x is called *hypercyclic* for the sequence $\{T_n\}_n$. An operator T is *hypercyclic* on X if $\{T^n\}_n$ is a hypercyclic sequence of operators.

Note that if $\{T_n\}_n$ is a hypercyclic sequence of operators on X, then X is necessarily separable.

The importance of hypercyclic operators derives from several sources:

(1) *The invariant subset problem.* Note that $\overline{Orb(T,x)}$ is the smallest closed set, invariant under T containing the vector x. Thus, an operator lacks invariant closed subsets if and only if each non-zero vector is hypercyclic.

(2) *Density of hypercyclic vectors.* Suppose x is a hypercyclic vector for T. Then every element in $Orb(T,x)$ is hypercyclic for T. Moreover, if we let $HC(T)$ denote the set of hypercyclic vectors for T, and S is any countable dense subset of X, then $HC(T)$ can be written as

$$HC(T) = \bigcap_{s\in S}\bigcap_{k=1}^{\infty}\bigcup_{n=0}^{\infty}\{x \in X : ||T^n x - s|| < \frac{1}{k}\}.$$

For fixed $s \in S$, $k \geq 1$ and $n \geq 0$, the set in braces is $T^{-n}B(s, \frac{1}{k})$, so it is open by the continuity of T^n. Then if T is hypercyclic the set of hypercyclic vectors is a dense G_δ subset of X.

(3) *The study of chaotic operators.* In the sense of Devaney [13], an operator T on a separable Banach space X is *chaotic* if:

a) T is hypercyclic on X.

b) X has a dense subset of periodic points for T.

c) T has sensitive dependence on initial conditions; i.e., there exists a positive number δ such that for every $\epsilon > 0$ and every $x \in X$, there is a point $y \in B(x,\epsilon)$ such that $d(T^n x, T^n y) > \delta$ for the same positive integer n.

Godefroy and Shapiro showed that hypercyclic operators have a dramatic form of this property.

Theorem. *Suppose that X is a a Banach space and T is a hypercyclic operator on X. Then for every $x \in X$ there is a dense G_δ set of points $S(x) \subset X$, such that the* *set of orbit-differences* $\{T^n x - T^n y : n \geq 0\}$ *is dense in* X *for every* $y \in S(x)$.

Thus to prove that an operator is chaotic on a Banach space it is only necessary to check properties (a) and (b) in Devaney's definition.

(4) *There exist many hypercyclic operators.* It is surprising that an operator can actually be hypercyclic. In fact, Rolewicz [27, pg. 17] showed that no linear operator on a finite-dimensional space is hypercyclic. This result can easily be seen by using the following result due to Kitai.

Given a Banach space X, the dual space of X, denoted by X^*, is the space of continuous linear functionals on X. For $T \in L(X)$ the *adjoint* $T^* : X^* \to X^*$ is defined by $T^*\Lambda = \Lambda \circ T$, for $\Lambda \in X^*$.

Theorem. [21, Collorary 2.4] *If an operator on a Banach space is hypercyclic, then its adjoint has no eigenvalues.*

Since any operator on a finite-dimensional space has eigenvalues, and since the adjoint of a linear operator on a finite dimensinal space is again an operator on a finite-dimensional space, we get Rolewicz's result.

Moreover, Kitai's result can also be used to show that no compact operator on a Banach space can be hypercyclic. This provides another proof of Rolewicz's result since any operator on a finite-dimensional space is compact.

Kitai also showed that no normal (more generally hyponormal) operator can be hypercyclic on a Banach space.

Nonetheless, hypercyclic operators are more common than one might expect. In fact the only restrictions are that the underlying space be infinite-dimensional and separable.

Theorem. (Ansari [2], Bernal [5]) *Every infinite dimensional separable Banach space carries a hypercyclic operator.*

Moreover, Bés and Chan [6] recently showed that the set of hypercyclic operators on a separable Banach space X is dense in the strong operator topology (S.O.T) of $L(X)$. (The *strong operator topology* on $L(X)$ is the topology defined by the basic neighborhoods:

$$N(T; A, \epsilon) = \{R : R \in L(X), |(T - R)x| < \epsilon, x \in A\}$$

where A is an arbitrary finite subset of X. Thus in the strong operator topology a net $\{T_\alpha\}$ converges to T if and only if $\{T_\alpha x\}$ converges to Tx for every $x \in X$.)

(5) *Dense invariant hypercyclic vector manifolds.* The fact that the adjoint T^* of a hypercyclic operator has no eigenvalues was used by Bourdon [9] to show that if T is hypercyclic and $P(z)$ is any nonzero polynomial, then the operator $P(T)$ has dense range. Then he showed that every hypercyclic operator on a Banach space X has a dense invariant hypercyclic vector manifold. To see this, suppose that $x \in X$ is hypercyclic for T. The invariant manifold

$$M = \{P(T)x : P \text{ is a polynomial}\}$$

consists entirely, except for zero, of hypercyclic vectors. Indeed, if $0 \neq P(T)x$, then $Orb(T, P(T)x) = P(T)(Orb(T, x))$. But the image of a dense set under an operator with dense range is again dense. Thus $P(T)x$ is hypercyclic for T.

Note that since $Orb(T, x) \subset M$, then the hypercyclic vector manifold M is also dense.

(6) *We can derive new hypercyclic operators from old ones.* First note that hypercyclicity is invariant under similarity. Indeed, suppose T is a hypercyclic operator

on X and $S, T_1 \in L(X)$ are such that S is invertible and $ST = T_1 S$. If $x \in X$ is hypercyclic for T, then Sx is hypercyclic for T_1.

Another way of deriving new hypercyclic operators is using the next result due to J. H. Shapiro [29, pg. 111], which can be very useful in establishing hypercyclicity

Proposition. *(The Hypercyclicity Comparison Principle) Let T be a continuous linear operator on a topological vector space X. Let Y be a topological space such that Y is dense in X and the identity map $I_Y : Y \to X$ is continuous. If $T|_Y : Y \to Y$ is a well defined continuous and hypercyclic operator, then T is hypercyclic on X. In particular T has a hypercyclic vector in Y.*

(7) *Topological transitivity.* Being hypercyclic, for a single operator as well as for a sequence of commuting operators with dense range, is equivalent (see, for instance [18, Theorem 1 and Proposition 1]) to a property called *topological transitivity.* A sequence of continuous maps $\{T_n\}_n$ on a topological space X is *topologically transitive* if for any pair U, V of non empty open subsets of X there is a positive integer n_0 such that $T_{n_0}(U) \cap V \neq \emptyset$.

A single map $T : X \to X$ is *topologically transitive* if the sequence $\{T^n\}_n$ is topologically transitive.

Note that transitivity means that the orbit under T of any non-empty open set U is dense in X.

By elementary set theory $T^n(U) \cap V \neq \emptyset$ is equivalent to $U \cap T^{-n}V \neq \emptyset$. Then we get at once the following result due to Kitai.

Corollary. [21, Corollary 2.2] *Suppose T is an invertible operator on a Banach space X. Then T is hypercyclic if and only if T^{-1} is hypercyclic.*

In applications it is sometimes useful to use the following sequential version of topological transitivity given by Godefroy and Shapiro [16, pg. 233]. *For every pair of vectors $x, y \in X$ there exists a sequence $\{x_k\}$ of vectors convergent to x, and a subsequence $\{n_k\}$ of positive integers, such that $T_{n_k} x_k \to y$.*

But a sufficient condition for hypercyclicity, the well known Hypercyclicity Criterion, independently discovered by Kitai [21] and Gethner and Shapiro [16], has been the fundamental tool for proving hypercyclicity. The following version of the Hypercyclicity Criterion was given by Bés and Peris (see [7]).

Theorem. *(The Hypercyclicity Criterion) Let T be an operator on a separable Banach space X. Suppose that there exists a strictly increasing sequence of positive integers $\{n_k\}_k$ for which there are:*

1. *a dense subset $X_0 \subset X$ such that $T^{n_k} x \to 0$ for every $x \in X_0$, and*
2. *a dense subset $Y_0 \subset X$ and a sequence of mappings $\{S_k : Y_0 \to X\}_k$ such that*
   a) $S_k y \to 0$ for every $y \in Y_0$,
   b) $T^{n_k} S_k y \to y$ for every $y \in Y_0$.

Then T is hypercyclic on X.

T is said to satisfy the *Hypercyclicity Criterion* if it satisfies the hypothesis of last theorem.

In spite of the complicated statement of the Hypercyclicity Criterion, the result is often easy to use. To illustrate this, let's apply the Hypercyclicity criterion to the first examples of hypercyclic operators on Banach spaces (see Corollary 1.6 in [21]).

For $1 \leq p < \infty$, let B be the *backward shift* on l^p defined by $B((x_0, x_1, x_2, \ldots)) = (x_1, x_2, \ldots)$

Theorem. *(Rolewicz [27, pg. 17]) For every $\lambda \in \mathbb{C}$ with $|\lambda| > 1$ the operator λB is hypercyclic on l^p $(1 \leq p < \infty)$.*

Proof. Fix $1 \leq p < \infty$ and $\lambda \in \mathbb{C}$ with $|\lambda| > 1$. To apply the Hypercyclicity Criterion we need to find dense subsets X_0, Y_0, a sequence of positive integers $\{n_k\}_{k=1}^{\infty}$, and a sequence of mappings $S_k : Y_0 \longrightarrow l^p$ satisfying the hypothesis of this Criterion.

Let X_0 be the collection of finite sequences in l^p, and $\{n_k\} = \{k\}$ for $k = 1, 2, \ldots$. Then X_0 is dense in l^p and for each $x \in X_0$ $(\lambda B)^k x$ is eventually zero. Then trivially $(\lambda B)^k x \longrightarrow 0$ on X_0. Let Y_0 be l^p itself, and let U denote the forward shift on l^p:

$$U((x_0, x_1, \ldots)) = (0, x_0, x_1, \ldots).$$

Let $S_k = (\lambda^{-1}U)^k$. Then $(\lambda B)^k S_k = I$ on l^p (where I is the identity operator on l^p). Therefore, again trivially, $(\lambda B)^k S_k y \longrightarrow y$ in $Y_0 = l^p$. Finally since U is an isometry and $|\lambda| > 1$, then $S_k y \longrightarrow 0$ on $Y_0 = l^p$. This completes the proof. $\square$

In fact, every example of hypercyclic operators in the literature so far seems to satisfy the Hypercyclicity Criterion, but it is still an open question if every hypercyclic operator satisfies it.

The Hypercyclicity Criterion Problem: *Does every hypercyclic operator satisfy the Hypercyclicity Criterion?*

This problem has been open for over 15 years. An affirmative answer for this question would simplify the proof of known results and would answer some still open problems.

León and Müller [24] recently showed, using a very clever argument for semigroups of operators, that if T is a hypercyclic operator and $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, then λT is hypercyclic. If T satisfied the Hypercyclicity Criterion the proof of this theorem would be totally trivial; the same dense subsets, sequence of positive integers and sequence of mappings that satisfy the condition of the Hypercyclicity Criterion for T would do the job for λT!

In 1992 Herrero [20, Problem 1] proposed the following (still open) problem. *Let T be a hypercyclic operator on a Hilbert space H. Does it follows that the operator $T \oplus T$ on $H \oplus H$ is hypercyclic?* The answer is clearly (and trivially) positive if T satisfies the Hypercyclicity Criterion; if $X_0, Y_0, \{n_k\}$ and S_k do the job for T, then, $X_0 \oplus X_0$, $Y_0 \oplus Y_0$, $\{n_k\}$ and $S_k \oplus S_k$ do the job for $T \oplus T$ on $H \oplus H$.

In order to attack the Hypercyclicity Criteron problem, several characterization of this Criterion have been given.

Related to the Herrerro's problem, Bés and Peris ([7], 1999) gave the following characterization of the Hypercyclicity Criterion.

Theorem. [7, Theorem 2.3] *An operator T on a separable Banach space X satisfies the Hypercyclicity Criterion if and only if $T \oplus T$ is hypercyclic on $X \oplus X$.*

They also obtained the following result. An operator T on a separable Banach space X is called *hereditarily hypercyclic* provided there is a sequence of positive integers $\{n_k\}_{k=1}^{\infty}$ such that for any subsequence $\{n_{k_j}\}_{j=1}^{\infty}$ of $\{n_k\}_{k=1}^{\infty}$, $\{T^{n_{k_j}}\}$ is a hypercyclic sequence of operators on X.

Theorem. [7, Theorem 2.3] *An operator on a separable Banach space satisfies the Hypercyclicity Criterion if and only if it is hereditarily hypercyclic.*

Next we will provide an apparently weaker formulation of the Hypercyclicity Criterion. We will also show that this new formulation is equivalent to the usual Hypercyclicity Criterion.

Definition 1.2. A subset $A \subseteq X$ is called *well distributed* if there is an $r > 0$ such that $A \cap B(x,r) \neq \emptyset$, for every $x \in X$.

We will make use of the following recent result due to Feldman [14].

Theorem. [14, Theorem 2.1]. *Let X be a Banach space and $T \in L(X)$. Suppose that there is a vector $x \in X$ such that $Orb(T,x) := \{x, Tx, T^2x, ...\}$ is well distributed. Then T is hypercyclic.*

This result is far from obvious since a dense subset is well distributed, but the converse is false.

Definition 1.3. Let T be an operator on X, $A \subseteq X$, $r > 0$ and $\{n_k\}_{k=1}^{\infty}$ be a sequence of positive integers. We will say that T^{n_k} *converges pointwise to* $B(0,r)$ on A ($T^{n_k}y \to B(0,r)$ for every $y \in A$) if for each $y \in A$ there exists a positive integer k_0 $(= k_0(y))$ such that $T^{n_k}(y) \in B(0,r)$ $\forall$ $k \geq k_0$

Theorem 1.4. *(The Hypercyclicity Criterion II) Let T be an operator on a separable Banach space X. Suppose there exists a strictly increasing sequence of positive integers $\{n_k\}_{k \geq 1}$ for which there are:*

1. *a dense subset $X_0 \subset X$ and $r_1 > 0$ such that $T^{n_k}x \to B(0,r_1)$ for every $x \in X_0$.*

2. *a well distributed $Y_0 \subset X$, a sequence of mappings $S_k : Y_0 \to X$ and $r_2 > 0$ such that:*

   a) $S_k y \to 0$ for every $y \in Y_0$

   b) $T^{n_k}S_k y - y \to B(0,r_2)$ for every $y \in Y_0$.

Then T hypercyclic on X.

Proof. It is enough to prove that under the hypothesis of the theorem there is a positive number $r > 0$ and a vector $x \in X$ whose orbit under T comes within a distance r of every point in X. The result will follow using Feldman's theorem.

By assumption, there is a $d > 0$ such that $Y_0 \cap B(y, d) \neq \emptyset$, for every $y \in X$. Let $r_0 = r_1 + r_2 + d$ and $Z = \{z_i\}_{i \geq 1}$ be a countable dense subset of X. Fix $z_i \in Z$ and let

$$G(z_i) = \bigcup_{k=0}^{\infty} \{x \in X : ||T^{n_k}x - z_i|| < r_0\}.$$

Note that $G(z_i)$ is open. Moreover,

Claim: $G(z_i)$ is dense in X.

Proof of Claim: let $x_0 \in X$ and $\delta > 0$. We want to show that $B(x_0, \delta)$ contains points of $G(z_i)$. Since $T^{n_k}x \to B(0, r_1)$ on the dense subset X_0, $S_k \to 0$ pointwise on the well distributed set Y_0 (which intersect any ball of radius d), and $T^{n_k}S_k - I_{Y_0} \to B(0, r_2)$ on Y_0, for sufficiently large $k \in \mathbb{N}$, there exist:

1) $x_1 \in X$ such that $||x_1 - x_0|| < \frac{\delta}{2}$ and $||T^{n_k}x_1|| < r_1$,

2) $y_1 \in X$ such that $||y_1 - z_i|| < d$, $||S_k y_1|| < \frac{\delta}{2}$ and $||T^{n_k}S_k y_1 - y_1|| < r_2$.

For such a k, let $x = x_1 + S_k y_1$. Then

$$||x - x_0|| \leq ||x_1 - x_0|| + ||S_k y_1|| < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Therefore $x \in B(x_0, \delta)$. On the other hand, by linearity of T^{n_k} we have that $||T^{n_k}x|| = ||T^{n_k}x_1 + T^{n_k}S_k y_1||$. Then

$$\begin{aligned} ||T^{n_k}x - z_i|| &\leq ||T^{n_k}x_1|| + ||T^{n_k}S_k y_1 - z_1|| < r_1 + ||T^{n_k}S_k y_1 - y_1 - (z_i - y_1)|| \\ &< r_1 + ||T^{n_k}S_k y_1 - y_1|| + ||z_i - y_1|| < r_1 + r_2 + d = r_0. \end{aligned}$$

Therefore, $x \in G(z_i)$. Then $G(z_i)$ is dense in X. This concludes the proof of the claim.

But since z_i was an arbitrary element of Z, $G(z_i)$ is dense (and open) for all $i = 1, 2, \ldots$. By the Baire Category Theorem

$$\bigcap_{z_i \in Z} G(z_i) \neq \emptyset.$$

We have proved that the set D of vectors in X whose orbit comes within a distance r_0 of a countable dense subset of X is non-empty. Therefore for $y \in D$ there is an $r > 0$ such that the orbit of y comes within a distance r of any vector of X. Then $Orb(T, x)$ is well distributed, which implies that T is hypercyclic. $\square$

Remarks 1.5. By the last proof, under the hypothesis of Theorem 1.4 the set D of vectors whose orbit comes within a distance r of every vector of X is not only non-empty but dense (by Baire Category Theorem). Feldman [14] also showed if the orbit of a vector x comes within a distance r of every point of X, then for every $\epsilon > 0$, there exists a vector $x_\epsilon = \frac{\epsilon}{r}x$ whose orbit comes within a distance ϵ of every point in X. Therefore, the proof of Theorem 1.4 actually shows that the set of vectors whose orbit comes within a distance ϵ of every point is dense for every $\epsilon > 0$.

Definition 1.6. Let $T \in L(X)$. We say that T *satisfies the Hypercyclicity Criterion II* if T satisfies the conditions of Theorem 1.4 for a dense subset X_0, a well distributed subset Y_0, a strictly increasing sequence of positive integers (n_k), a sequence of mappings $S_k : Y_0 \to X$ and two positive numbers r_1 and r_2.

Theorem 1.7. *Let T be an operator on a separable Banach space. The following are equivalent:*

1) T satisfies the Hypercyclicity Criterion.

2) T satisfies the Hypercyclicity Criterion II.

Proof. 1) implies 2) Follows immediately from the definitions.

On the other hand if T satisfies the Hypercyclicity Criterion II for X_0, Y_0, the sequence of positive integers $(n_k)_{k=1}^{\infty}$, the sequence of mappings S_k, and $r_1, r_2 > 0$, then $T \oplus T$ satisfies the Hypercyclicity Criterion II for $X_0 \oplus X_0$, $Y_0 \oplus Y_0$, the sequence of positive integers $(n_k)_{k=1}^{\infty}$ and the sequence of mappings $S_k \oplus S_k$ and $r_1, r_2 > 0$. Therefore, by the Hypercyclicity Criterion II, $T \oplus T$ is hypercyclic, and then by Bés and Peris [7, Theorem 2.3], T satisfies the Hypercyclicity Criterion. □

Chapter 2

The Three-Neighborhoods Condition

Our second characterization of the Hypercyclicity Criterion is related to the following sufficient condition for hypercyclicity, given by Godefroy and Shapiro in 1991.

Theorem. [17, Corollary 1.3] *An operator T on a separable Banach space X is hypercyclic if for every pair U, V of non-empty open subsets of X, and each open neighborhood W of zero in X there exists a positive integer n such that both $T^nU \cap W$ and $T^nW \cap V$ are non-empty.*

Proof. We will show that the hypothesis imply the sequential version of topological transitivity. Let x and y be vectors in X. The hypothesis of this theorem imply that there are sequences (x'_k) converging to x, and (x''_k) converging to 0, and a subsequence $\{n_k\}$ of positive integers such that

$$T^{n_k}x'_k \to 0 \text{ and } T^{n_k}x''_k \to y.$$

Let $x_k = x'_k + x''_k$. By linearity of the operators T^n,

$$T^{n_k}x_k = T^{n_k}x'_k + T^{n_k}x''_k \to 0 + y = y.$$

□

Definition 2.1. An operator on a separable Banach space is said to *satisfy the three-neighborhoods condition* if it satisfies the hypothesis of last theorem.

Remark 2.2. The three-neighborhoods condition is equivalent to the apparently stronger requirement that there are infinitely many positive integers n such that both $T^nU \cap W$ and $T^n(W) \cap V$ are non-empty. Indeed, let U_1 and V_1 be non-empty open subsets of X and W_1 an open neighborhood of zero in X. If n is any fixed positive integer, by continuity of T^n and the three neighborhoods condition applied to the sets $U = U_1$, $V = T^{-n}V_1$ and $W = T^{-n}W_1 \cap W_1$, there is an n_0 such that

$$T^{n_0}U_1 \cap (T^{-n}W_1 \cap W_1) \neq \emptyset \text{ and } T^{n_0}(T^{-n}W_1 \cap W_1) \cap T^{-n}V_1 \neq \emptyset.$$

In particular,

$$T^{n_0}U_1 \cap T^{-n}W_1 \neq \emptyset \text{ and } T^{n_0}W_1 \cap T^{-n}V_1 \neq \emptyset.$$

By elementary set theory we get

$$T^{n_0+n}U_1 \cap W_1 \neq \emptyset \text{ and } T^{n_0+n}W_1 \cap V_1 \neq \emptyset.$$

It is natural to ask whether any hypercyclic operator on a separable Banach space satisfies the three-neighborhoods condition. By the next theorem an affirmative answer to this question will give an affirmative answer to the Hypercyclicity Criterion Problem.

Theorem 2.3. *Let X be a separable Banach space and T an operator on X. Then T satisfies the Hypercyclicity Criterion if and only if T satisfies the three-neighborhoods condition.*

Proof. Suppose that T satisfies the Hypercyclicity Criterion. Let U and V be non-empty open subsets of X and W an open neighborhood of zero in X. Let X_0 and Y_0 be the dense subsets of X, $\{n_k\}_{k=1}^{\infty}$ the sequence of positive integers, and $S_k : Y_0 \to X$ the sequence of mappings in the hypothesis of the Hypercyclicity Criterion. Since $T^{n_k}x \to 0$ pointwise in X_0, $S_k y \to 0$ pointwise in Y_0 and $T^{n_k}S_k y \to y$ pointwise in Y_0, we can choose $x_u \in U \cap X_0$, $y_v \in V \cap Y_0$ and n_k large enough such that

$$T^{n_k}x_u \in W, \;\; S_k y_v \in W, \;\; \text{and } T^{n_k}S_k y_v \in V.$$

Therefore for such n_k,

$$T^{n_k}U \cap W \neq \emptyset \text{ and } T^{n_k}W \cap V \neq \emptyset.$$

Then T satisfies the three-neighborhoods condition.

Conversely suppose that T satisfies the three-neighborhoods condition. Then by Godefroy and Shapiro's theorem, T is hypercyclic on X. Moreover, the set of hypercyclic vectors for T ($HC(T)$) is a dense G_δ set.

Claim: we can choose $z \in X$ hypercyclic for T and $\{n_k\}_{k=1}^{\infty}$ a sequence of positive integers such that $T^{n_k}z \to 0$ and $T^{n_k}(B(0, \frac{1}{k})) \cap B(z, \frac{1}{k}) \neq \emptyset$, for all $k \in \mathbb{N}$.

To see how the claim implies the Hypercyclicity Criterion proceed as follows. Pick such a hypercyclic vector $z \in X$ and such sequence of positive integers $\{n_k\}_{k=1}^{\infty}$. Recall that to show that T satisfies the Hypercyclicity Criterion we need to find X_0, Y_0 dense subsets of X, a sequence of positive integers $\{n_k\}_{k=1}^{\infty}$ and a sequence of mappings $S_k : Y_0 \to X$ satisfying the conditions of the Hypercyclicity Criterion. The sequence of positive integers has already been chosen. Let $X_0 = Y_0 = \{T^n z : n = 0, 1, \ldots\}$. Since z is a hypercyclic vector for T, X_0 and Y_0 are dense subsets of X. Also since z satisfies the claim, for every $k \in \mathbb{N}$, $T^{n_k}(B(0, \frac{1}{k})) \cap B(z, \frac{1}{k}) \neq \emptyset$, then, $T^{-n_k}B(z, \frac{1}{k}) \cap B(0, \frac{1}{k}) \neq \emptyset$. For each $k \in \mathbb{N}$, pick $w_k \in T^{-n_k}B(z, \frac{1}{k}) \cap B(0, \frac{1}{k})$ and define $S_k : Y_0 \to X$ by $S_k T^n z = T^n w_k$. Since $Orb(T, z)$ is dense, $T^{n_0}z \neq T^{n_1}z$ if $n_0 \neq n_1$. (In fact, it can be shown that $Orb(T,z)$ a a linearly independent subset of X.) Therefore S_k is well defined.

By the choice of z and $\{n_k\}$, using the claim we know that $T^{n_k}z \to 0$. Therefore for any fixed $n \in \mathbb{N}$, by continuity of T^n

$$T^{n_k}T^n z = T^n T^{n_k} z \to 0 \ \ (as\ k \to \infty).$$

Then $T^{n_k} \to 0$ pointwise on X_0. On the other hand, since $w_k \to 0$ and $S_k T^n z = T^n w_k$, again by continuity of T^n, $S_k \to 0$ pointwise on Y_0.

Finally, since $T^{n_k} S_k z = T^{n_k} w_k \in B(z, \frac{1}{k})$ for any $k \in \mathbb{N}$,

$$T^{n_k} S_k z \to z \ \ (as\ k \to \infty).$$

Moreover, for a fixed $n \in \mathbb{N}$, by continuity of T^n, and since

$$T^{n_k} S_k T^n z = T^{n_k} T^n w_k = T^n T^{n_k} w_k \to T^n z,$$

we have that $T^{n_k} S_k$ converges pointwise to the identity on Y_0. Therefore T satisfies the Hypercyclicity Criterion.

Proof of Claim: Let

$$P = \bigcap_{k=1}^{\infty} (\bigcup_{n=1}^{\infty} T^{-n}(B(0, \frac{1}{k})) \cap \{x \in X : T^n(B(0, \frac{1}{k})) \cap B(x, \frac{1}{k}) \neq \emptyset\})$$

We are going to show that P is a dense G_δ set. Then, by the Baire Category Theorem, $HC(T) \cap P$ will also be a dense G_δ set and the proof is complete.

Fix $k \in \mathbb{N}$. Since T is continuous, the set in parenthesis (G_k) is open. We want to show that it is also dense. Let $x \in X$ and $\epsilon > 0$. We can assume, without loss of generality, that $\epsilon < \frac{1}{2k}$. We want to show that $B(x,\epsilon) \cap G_k \neq \emptyset$. By the three-neighborhoods condition applied to the sets $U = V = B(x,\epsilon)$ and $W = B(0, \frac{1}{k})$, there exists an $n \in \mathbb{N}$ such that

$$T^n(B(x,\epsilon)) \cap B(0,\frac{1}{k}) \neq \emptyset \text{ and } T^n(B(0,\frac{1}{k})) \cap B(x,\epsilon) \neq \emptyset.$$

Let $y_1 \in B(x,\epsilon)$ with $T^n y_1 \in B(0,\frac{1}{k})$ and $y_2 \in B(x,\epsilon)$ with $y_2 \in T^n(B(0,\frac{1}{k}))$.

Subclaim: $y_1 \in G_k$.

Proof of Subclaim: Note that $y_2 \in B(x,\epsilon)$. Therefore, since $2\epsilon < \frac{1}{k}$,

$$y_2 \in T^n B(0,\frac{1}{k}) \cap B(y_1,\frac{1}{k})$$

(and $y_1 \in T^{-n}(B(0,\frac{1}{k}))$). Then by definition of G_k, $y_1 \in G_k$. Therefore

$$y_1 \in B(x,\epsilon) \cap G_k$$

which implies that G_k is dense. This completes the proof of the Claim. $\square$

From the proof of Theorem 2.3 we get the following corollary.

Corollary 2.4. *An operator T on a Banach space X satisfies the Hypercyclicity Criterion if and only if:*

(1) T is hypercyclic, and

(2) For any non-empty open subset $U \subset X$ and any open neighborhood of zero W there is an $n \in \mathbb{N}$ such that $T^n U \cap W \neq \emptyset$ and $T^n W \cap U \neq \emptyset$.

The result of Theorem 2.3 can be used to show that a very important class of hypercyclic operators satisfy the Hypercyclicity Criterion. Namely, let $H = l_2(\mathbb{Z})$. The operator T is a *bilateral (forward) weighted shift with respect to the canonical basis* $\{e_n : n \in \mathbb{Z}\}$ if $Te_n = a_n e_{n+1}$ where the sequence of weights $\{a_n : n \in \mathbb{Z}\}$ is a bounded subset of $\mathbb{C} \setminus \{0\}$. Since hypercyclicity is invariant under similarity, and since any bilateral weighted shift is similar to a bilateral weighted shift with positive weights, we can assume, without loss of generality, that each a_n is positive.

In 1995, Salas [28] gave necessary and sufficient conditions (on the sequence of weights) for a bilateral weighted shift to be hypercyclic on H.

Theorem. [28, Theorem 2.1] *Let T be a bilateral weighted shift with positive weight sequence $\{a_n\}$. Then T is hypercyclic if and only if given $\epsilon > 0$ and $q \in \mathbb{N}$, there exists $n \in \mathbb{N}$ arbitrarily large such that for all $|j| \leq q$*

$$\prod_{s=0}^{n-1} a_{s+j} < \epsilon \ \ \textit{and} \ \ \prod_{s=1}^{n} a_{j-s} > \frac{1}{\epsilon}$$

In the proof of this theorem, Salas showed that hypercyclic bilateral weighted shifts satisfy the following condition *(the S-condition.) If $\epsilon > 0$, and the vectors $g, h \in H$ are in the span of $\{e_j : |j| \leq q\}$, then there exists an arbitrarily large n and a vector u in the span of $\{e_j : -q - n \leq j \leq q - n\}$ such that*

$$||u|| < \epsilon, \ \ ||T^n u - g|| < \epsilon \text{ and } ||T^n h|| < \epsilon.$$

Leon and Montes[23, pg. 251] showed that every bilateral weighted shift satisfies the Hypercyclicity Criterion. Their proof is based on a direct application of the Hypercyclicity Criterion. Here we provide a proof using the three-neighborhoods condition.

Corollary 2.5. *Every hypercyclic bilateral weighted shift satisfies the Hypercyclicity Criterion.*

Proof. It is enough to show that the S-condition implies the three-neighborhood condition on $H = l_2(\mathbb{Z})$. Let U, V be open subsets of H and W an open neighborhood of zero in H. since the set of finite sequences is dense in H, we can pick $h \in U$,

$q_1 \in \mathbb{N}$ and $\epsilon_1 > 0$ such that $h \in \text{span}\{e_j : |j| \leq q_1\}$ and $B(h, \epsilon_1) \subset U$. Similarly, pick $g \in V$, $q_2 \in \mathbb{N}$ and $\epsilon_2 > 0$ such that $g \in \text{span}\{e_j : |j| \leq q_2\}$ and $B(g, \epsilon_2) \subset V$. Let $q = \max\{q_1, q_2\}$ and $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. Without loss of generality we may assume that $B(0, \epsilon) \subset W$. Therefore, $h, g \in \text{span}\{e_j : |j| \leq q\}$. By the S-condition, there exists an $n \in \mathbb{N}$ (arbitrarily large) and $u \in \text{span}\{e_j : -q - n \leq j \leq q - n\}$ such that:

$$||u|| < \epsilon, \quad (then \;\; u \in W)$$

$$|T^n u - g|| < \epsilon, \quad (then \;\; T^n u \in V) \;\; \text{and}$$

$$||T^n h|| < \epsilon, \quad (then \;\; T^n h \in W).$$

Therefore,

$$T^n U \cap W \neq \emptyset \;\; and \;\; T^n W \cap V \neq \emptyset$$

which implies that T satisfies the three-neighborhoods condition, and then, by Theorem 2.3, T satisfies the Hypercyclicity Criterion. $\square$

Chapter 3

Syndetically Hypercyclic Operators

Let X be a separable Banach space, and $T \in L(X)$. Recall that a sequence of operators $\{T_n\}_n$ is hypercyclic on X if there exists some $x \in X$ such that its orbit

$$Orb(\{T_n\}_n, x) := \{x, T_1x, T_2x, ...\}$$

is dense in X

Definition 3.1. A strictly increasing sequence of positive integers $\{n_k\}_k$ is said to be *syndetic* if $\sup_k\{n_{k+1} - n_k\} < \infty$. An operator T on X is called *syndetically hypercyclic* if for any strictly increasing syndetic sequence of positive integers $\{n_k\}_k$, $\{T^{n_k}\}_k$ is a hypercyclic sequence of operators.

We will show that $T \in L(X)$ is syndetically hypercyclic if and only if T satisfies the Hypercyclicity Criterion. This partially settles a question posed by Bés, who asked if every hypercyclic operator is syndetically hypercyclic. Bés' problem was motivated by a result of Ansari [1, Theorem 2.1], which asserts that the sequence $\{T^{pn}\}_n$ is hypercyclic for each $p \in \mathbb{N}$ whenever T is hypercyclic (see also [3, Theorem 2.5]). Again, by our equivalence, an affirmative answer to Bés' question would give an affirmative answer to the Hypercyclicity Criterion problem.

We will also show that if $\{n_k\}$ is not syndetic; i.e., if $\sup_k\{n_{k+1} - n_k\} = \infty$, there are examples of operators T satisfying the Hypercyclicity Criterion such that $\{T^{n_k}\}_k$ is not a hypercyclic sequence of operators

In the final part of this chapter we will show that, *for any hypercyclic operator $T \in L(X)$ on a general locally convex space X, and for any vector x hypercyclic for T, there exists a strictly increasing sequence of positive integers such that* $\sup_k\{n_{k+1} - n_k\} = 2$ *and $\{T^{n_k}x\}_k$ is not dense in X.* However, the sequence $\{T^{n_k}x\}_k$ turns out to be somewhere dense. This establishes a difference between sub-orbits and orbits of vectors under T. Bourdon and Feldman [10] recently proved that if a full orbit is somewhere dense, then it is everywhere dense.

We start with some results from topological dynamics related to the topic of this chapter.

Definition 3.2. Let X be a topological space and $T : X \longrightarrow X$ be a continuous map. T is called *weakly mixing* if $T \times T : X \times X \longrightarrow X \times X$ is topologically transitive.

Furstenberg [15, Prop. II.3] showed that if T is weakly mixing, then for any $m \in \mathbb{N}$

$$\underbrace{T \times T \times \cdots \times T}_{m-times} : X \times X \times \cdots \times X \longrightarrow X \times X \times \cdots \times X,$$

is topologically transitive.

Banks [3] gave several interesting conditions equivalent to weak mixing.

Theorem. *Let X be a topological space and $f : X \to X$ be continuous. The following are equivalent:*

(a) f is weakly mixing.

(b) For any $n \in \mathbb{N}$ and $U_1, U_2, \dots U_n, V_1, V_2 \dots V_n$ non-empty open subsets of X, there exists $k \in \mathbb{N}$ such that $f^k(U_i) \cap V_i \neq \emptyset$ for all $i = 1, \dots n$.

(c) Given U, V non-empty open subsets of X there exists $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$ and $f^n(V) \cap V \neq \emptyset$.

(d) Given U, V non-empty open subsets of X there exists $n \in \mathbb{N}$ such that $f^n(U) \cap U \neq \emptyset$ and $f^n(U) \cap V \neq \emptyset$.

(e) Given U, V, W non-empty open subsets of X there exists $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$ and $f^n(W) \cap W \neq \emptyset$.

(f) Given U, V_1, V_2 non-empty open subsets of X there exists $n \in \mathbb{N}$ such that $f^n(U) \cap V_1 \neq \emptyset$ and $f^n(U) \cap V_2 \neq \emptyset$.

(g) Given U, V, W non-empty open subsets of X there exists $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$ and $f^n(V) \cap W \neq \emptyset$.

(h) Given U_1, U_2, V non-empty open subsets of X there exists $n \in \mathbb{N}$ such that $f^n(U_1) \cap V \neq \emptyset$ and $f^n(U_2) \cap V \neq \emptyset$.

Banks' proof is given in this order:

(a) $\Rightarrow$ (b) is Furstemberg's Theorem.

(b) $\Rightarrow$(c), (b) $\Rightarrow$ (g), (b) $\Rightarrow$ (h) follows trivially.

(c) $\Rightarrow$ (d) $\Rightarrow$ (e) [3, Lemma 2 and Lemma 3]

(e) $\Rightarrow$ (f) $\Rightarrow$ (a) [3, Lemma 4 and lemma 4]

(g) $\Rightarrow$ (e) and (h) $\Rightarrow$ (e) [3, pg. 85].

If f is an operator on a separable Banach space, then by Bés and Peris [7, Theorem 2.3], f is weakly mixing if and only if f satisfies the Hypercyclicity Criterion. Thus each of the conditions (b) through (h) is also equivalent to f satisfies the Hypercyclicity Criterion.

Note that condition (g) in Banks' theorem is a formally stronger version of the three-neighborhoods condition of Chapter 2. (Recall that the three-neighborhoods condition defined in Chapter 2 required that the open set in the middle (in this case V) be a neighborhood of zero.)

Banks also showed [3, Lemma 8] that if a continuous function f on a topological space X is:

(1) *flip topologically transitive*; i.e., for any pair of non-empty open subsets U, V of X there is a positive integer n such that $f^n(U) \cap V \neq \emptyset$ and $f^n(V) \cap U \neq \emptyset$, and

(2) f^2 is transitive,

then f is weakly mixing.

In the case that f is an operator on a separable Banach space, if f is flip topologically transitive (which implies that f topologically transitive), then f^n is topologically transitive for all positive integer n (by Ansari [2]). Thus, if f is a flip topologically transitive operator, then f is weakly mixing. But then since weakly mixing is equivalent to the Hypercyclicity Criterion we get that *an operator on a separable Banach space is flip topologically transitive if and only if it satisfies the Hypercyclicity Criterion.*

In Corollary 2.4 we proved a similar result. The difference is that in the corollary we only required that the open set V, in the definition of flip topological transitivity, be a neighborhood of zero. But we also required that the operator be topologically transitive. (A condition that is part of the definition of flip topological transitivity.)

The first result of this chapter, which is fundamental for the desired equivalence between syndetically hypercyclic and the Hypercyclicity Criterion, remains valid for continuous maps on topological spaces. Thus we work initially within this general context.

Proposition 3.3. *Let $T : X \longrightarrow X$ be a continuous map on a topological space X.*

Then the following are equivalent:

(i) T is weakly mixing.

(ii) For any pair of non-empty open subsets $U, V \subseteq X$, and for any strictly increasing syndetic sequence $\{n_k\}_k$, there exists k_0 such that $T^{n_{k_0}}U \cap V \neq \emptyset$.

Implication (f) $\Rightarrow$ (a) in Banks' Theorem is necessary for the proof. Also (i) $\Rightarrow$ (ii) in the proposition is due to Furstenberg [15, Prop. II.11]. We include those proofs for the sake of completeness.

Lemma. [3, Lemma 5] *Let X be a topological space and $f : X \to X$ a continuous map. If for any open non-empty subsets $U, V_1, V_2 \subset X$ there is an $n \in \mathbb{N}$ such that $f^n(U) \cap V_i \neq \emptyset$, for $i = 1, 2$, then f is weakly mixing.*

Proof. Given U_1, U_2, W_1 and W_2 non-empty open sunsets of X, we need to find $k \in \mathbb{N}$ such that $f^k(U_i) \cap W_i \neq \emptyset$ for $i = 1, 2$. Apply the hypothesis to the sets U_1, U_2 and W_2. Then there exists $m \geq 1$ such that

$$S = f^m(U_1) \cap U_2 \neq \text{ and } T = f^m(U_1) \cap W_2 \neq \emptyset.$$

Then

$$S = U_1 \cap f^{-m}U_2 \neq \text{ and } T = U_1 \cap f^{-m}W_2 \neq \emptyset.$$

Now apply the hypothesis to the sets S, T and W_1. Then there exists $k \in \mathbb{N}$ such that

$$f^k(S) \cap T \neq \emptyset \text{ and } f^k(S) \cap W_1 \neq \emptyset.$$

Since $S \subseteq U_1$, we have $f^k(U_1) \cap W_1 \neq \emptyset$. Also $Y = S \cap f^{-k}(T)$ is non-empty and open. For $x \in Y \subseteq S \subseteq f^{-m}(U_2)$ we have $f^m(x) \in U_2$. But $x \in Y \subseteq f^{-k}(T)$; so $f^k(x) \in T \subseteq f^{-m}(W_2)$ and hence $f^m(f^k(x)) \in W_2$ which gives $f^k(U_2) \cap W_2 \neq \emptyset$. $\square$

Proof of Proposition 3.3: (i) implies (ii) [Furstenberg]: Suppose $\{n_k\}_k$ and U, V satisfy the hypothesis of (ii). Set $m := \sup_k\{n_{k+1} - n_k\}$. Since T is weakly mixing, the m-product map

$$\underbrace{T \times T \times \cdots \times T}_{m-times} : X \times X \times \cdots \times X \longrightarrow X \times X \times \cdots \times X,$$

is transitive. Then there is an $n \in \mathbb{N}$ such that

$$T^n U \cap T^{-i} V \neq \emptyset$$

for all $i = 1, \ldots, m$. This implies that $T^{n+i}U \cap V \neq \emptyset$ for all $i = 1, \ldots, m$. By the assumption on $\{n_k\}_k$, we have that $\{n_k : k \in \mathbb{N}\} \cap \{n+1, \ldots, n+m\} \neq \emptyset$. If we select n_{k_0} in this intersection, we get $T^{n_{k_0}}U \cap V \neq \emptyset$.

(ii) implies (i): We will show that, given non-empty open subsets $U, V_1, V_2 \subset X$, there is an $n \in \mathbb{N}$, such that $T^n U \cap V_i \neq \emptyset$, for $i = 1, 2$. This will imply that T is weakly mixing by Banks' result.

Fix $m \in \mathbb{N}$ such that $T^m V_1 \cap V_2 \neq \emptyset$. (Such m exists because (ii) is satisfied.) By continuity, we can find $\tilde{V}_1 \subset V_1$ open and non-empty such that $T^m \tilde{V}_1 \subset V_2$. Assumption (ii) implies the existence of some $l \in \mathbb{N}$ such that $T^{l+i}U \cap \tilde{V}_1 \neq \emptyset$, for all $i = 0, 1 \ldots, m$. (Otherwise we would find a strictly increasing sequence of positive integers $\{n_k\}_k$ such that $n_{k+1} - n_k \leq m + 1$, and $T^{n_k}U \cap \tilde{V}_1 = \emptyset$ for all $k \in \mathbb{N}$). In particular we have

$$T^{l+m}U \cap \tilde{V}_1 \neq \emptyset, \quad \text{and} \quad T^{l+m}U \cap T^m \tilde{V}_1 \quad \supset \quad T^m(T^l U \cap \tilde{V}_1) \neq \emptyset.$$

If we fix $n := l + m$, we conclude

$$T^n U \cap V_1 \neq \emptyset, \quad \text{and} \quad T^n U \cap V_2 \neq \emptyset,$$

which completes the proof. $\square$

We notice that condition (ii) can be equivalently formulated as follows. *For any pair of non-empty open subsets $U, V \subset X$, and for any $m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $T^{n+i}U \cap V \neq \emptyset$, $i = 0, \ldots, m$.*

Recall that Bés and Peris showed that an operator on a Banach space satisfies the Hypercyclicity Criterion if and only if it is weakly mixing. Combining this result with the previous proposition we obtain the next theorem.

Theorem 3.4. *Let $T : X \longrightarrow X$ be an operator on a separable Banach space X. Then the following are equivalent:*

(i) T satisfies the Hypercyclicity Criterion.

(ii) T is syndetically hypercyclic.

Proof. (i) implies (ii): If T satisfies the Hypercyclicity Criterion, then by [7, Theorem 2.3], T is weakly mixing. By Proposition 3.3, if $\{n_k\}_k$ is a syndetic sequence, then $\{T^{n_k}\}_k$ is topologically transitive and, therefore, hypercyclic.

(ii) implies (i): Condition (ii) means that, for any syndetic sequence $\{n_k\}_k$, we have that $\{T^{n_k}\}_k$ is hypercyclic; then topologically transitive, and thus, by Proposition 3.3, T is weakly mixing. Theorem 2.3 of [7] concludes that T satisfies the Hypercyclicity Criterion. $\square$

The observation after Proposition 3.3 now yields the following equivalence with the Hypercyclicity Criterion. Let X be a separable Banach space, $\mathcal{U}$ the family of all non-empty open subsets of X, and $T \in L(X)$. Then T satisfies the Hypercyclicity Criterion if and only if

$$\forall\, U, V \in \mathcal{U},\ \forall\, m \in \mathbb{N},\ \exists\, n \in \mathbb{N}: \quad T^i U \cap V \neq \emptyset, \quad i = n, \ldots, n+m.$$

Non-Syndetic Sequences.

We are going to show that if $\{n_k\}_k$ is such that $\sup_k\{n_{k+1}-n_k\}=\infty$, then there are hypercyclic unilateral weighted backward shift operators T on the Hilbert space l^2 of square-summable sequences such that $\{T^{n_k}\}_k$ is not a hypercyclic sequence of operators. Since hypercyclic weighted backward shifts operators are known to satisfy the Hypercyclicity Criterion (in fact any hypercyclic operator with dense generalized kernel satisfies the Hypercyclicity Criterion [7]), then even in the case that T satisfies the Hypercyclicity Criterion, it is not true that for any sequence $\{n_k\}$, $\{T^{n_k}\}$ is a hypercyclic sequence of operators.

Let $\{w_n\}_n$ be a bounded sequence of nonzero numbers, and $\{e_n\}_n$ be the standard basis of l^2. We will be using the following two results, the first due to Salas and the second to Bés and Peris.

Proposition. [28, Theorem 2.8] *Let T be a unilateral weighted backward shift on l^2 with positive weight sequence $\{w_n : n \in \mathbb{N}\}$. Then T is hypercyclic if and only if* $\sup_n \prod_{k=1}^n w_k = \infty$.

Proposition. [7, Proposition 3.1] *Let $\{n_k\}_k \subset \mathbb{N}$, and T be a unilateral weighted backward shift in l^2 with positive weight sequence $\{w_n : n \in \mathbb{N}\}$. Then $\{T^{n_k}\}_k$ is hypercyclic if and only if for all $\epsilon > 0$ and all $M, q \in \mathbb{N}$, there exists $m = m(\epsilon, q) \in \{n_k\}_k$, such that $m > M$ and*

$$w_{i+1}\cdots w_{i+m} > \frac{1}{\epsilon} \quad (1 \le i \le q).$$

Proposition 3.5. *Suppose $\{n_k\}_k$ is such that $\sup_k\{n_{k+1}-n_k\} = \infty$. Then there exists a bounded sequence $\{w_n\}_n$ of positive scalars such that the unilateral weighted* *backward shift $T : l^2 \longrightarrow l^2$, given by*

$$Te_n = \begin{cases} w_n e_{n-1} & for \;\; n \geq 2 \\ 0 & for \;\; n = 1, \end{cases}$$

is hypercyclic but $\{T^{n_k}\}_k$ is not a hypercyclic sequence of operators in X.

Proof. Take an increasing subsequence $\{n_{k_j}\}_j \subset \{n_k\}_k$ such that $n_{k_j+1} - n_{k_j} > 2j$ for all $j \geq 1$. Define the following sequence of weights. For $j = 1, 2, \ldots,$ let

$$w_n = \begin{cases} 2 & if \;\; n_{k_j} + 1 \leq n \leq n_{k_j} + j \\ \frac{1}{2} & if \;\; n_{k_j} + j + 1 \leq n \leq n_{k_j} + 2j \\ 1 & otherwise. \end{cases}$$

Let T be the unilateral weighted backward shift associated with $\{w_n\}_n$.

1. T is hypercyclic:

By the definition of the sequence of weights we have that $w_1 \cdots w_m = 2^j$ whenever $m = n_{k_j} + j$ for some $j \in \mathbb{N}$. Since j is arbitrarily large, we get $\sup_m w_1 \cdots w_m = \infty$ which implies the hypercyclicity of T (using Salas' result).

2. $\{T^{n_k}\}_k$ is not a hypercyclic sequence of operators in X:

Note that there are no $m \in \{n_k\}_k$ in between any sequence of $2's$ and $\frac{1}{2}'s$. Then for any $m \in \{n_k\}_k$ we have $w_2 \cdots w_m = 1$. Thus $w_2 \cdots w_{m+1} \leq 2$. Set $q := 1$, $\epsilon := 1/2$. The condition for hypercyclicity of $\{T^{n_k}\}_k$ given in Bés and Peris' proposition is not satisfied. □

Finally we will show that even in the case when an operator T satisfies the Hypercyclicity Criterion and $\{n_k\}_k$ is a syndetic sequence (then $\{T^{n_k}\}$ is hypercyclic) , the set of hypercyclic vectors for $\{T^{n_k}\}_k$ can be strictly contained in the set of hypercyclic vectors for T. This contrasts with Ansari's result [2] which shows that if for every $n \in \mathbb{N}$ T and T^n share the same set of hypercyclic vectors. (Of course that this set can be empty.)

More precisely, we will prove that if T is a hypercyclic operator and $x \in X$ is any hypercyclic vector for T, there exists a syndetic sequence $\{n_k\}_k$, such that the orbit

$$Orb(\{T^{n_k}\}_k, x) := \{x, T^{n_1}x, \dots\}$$

is somewhere dense but not everywhere dense. As we mentioned at the beginning of the chapter, this establishes a difference between the full orbit and the sub-orbit associated to a sequence $\{n_k\}_k$, for a single operator T, which should be compared with the result of Bourdon and Feldman [10].

We begin with a result for standard dynamical systems that came out of a conversation with L. Frerick.

Lemma 3.6. *Let X be a topological space without isolated points, $T : X \to X$ a continuous map, and $x \in X$ such that $Orb(T,x)$ is dense in X. Then, for any syndetic sequence $\{n_k\}_k$ of positive integers, the associated orbit $Orb(\{T^{n_k}\}_k, x)$ is somewhere dense.*

Proof. If $\{n_k\}_k$ is syndetic, we set $m := \sup_k\{n_{k+1} - n_k\}$. Without loss of generality $n_1 > m$. Since X has no isolated points and $Orb(T,x)$ is dense in X, we have

$$X = \overline{\{T^n x \ : \ n \geq n_1\}} = \bigcup_{i=0}^{m} \overline{\{T^{n_k - i}x \ : \ k \in \mathbb{N}\}}.$$

We define $M_i := \overline{\{T^{n_k - i}x \ : \ k \in \mathbb{N}\}}$, $i = 0, \dots, m$. If $int(M_0) \neq \emptyset$, then we are done. If not $X = \bigcup_{i=1}^{m} M_i$, and this would imply

$$X = \overline{T(X)} = \bigcup_{i=1}^{m} \overline{T(M_i)} = \bigcup_{i=0}^{m-1} M_i = \bigcup_{i=1}^{m-1} M_i.$$

By iterating this process we arrive at $X = M_1$. Thus $X = \overline{T(M_1)} = M_0$, which is a contradiction. $\square$

The next result holds for general locally convex spaces X.

Theorem 3.7. *Let T be a hypercyclic operator on a locally convex space X and let $x \in X$ be a hypercyclic vector for T. Then there exists a sequence of positive integers $\{n_k\}_k$ with $\sup_k\{n_{k+1} - n_k\} = 2$ such that $\{T^{n_k}x\}_k$ is somewhere dense but not everywhere dense.*

Proof. Let $x \in X$ be a hypercyclic vector for T. Then $Orb(T,x)$ is linearly independent. Therefore, $T^2(x) \notin \text{span}\{x, Tx, T^3x, T^4x\}$. Then there exists an element x^* in the dual X' of X such that $<x^*, T^2x>= 1$ and $<x^*, T^ix>= 0$ for $i = 0, 1, 3, 4$, where $<x^*, y>$ denotes $x^*(y)$, for any $y \in X$. Let $P : X \longrightarrow \mathbb{K}^3$ be given by

$$P(y) = (<x^*, y>, <x^*, Ty>, <x^*, T^2y>)$$

for all $y \in X$. P is linear, continuous and, since by definition $P(x) = (0, 0, 1)$, $P(Tx) = (0, 1, 0)$ and $P(T^2x) = (1, 0, 0)$, we have that P is surjective. We define

$$n_k = \begin{cases} k & \textit{if } \; |<x^*, T^{k+1}x>| > |<x^*, T^{k+2}x>| \\ k+1 & \textit{otherwise.} \end{cases}$$

(i) If $|<x^*, T^{k+1}x>| > |<x^*, T^{k+2}x>|$, then

$$P(T^{n_k}x) = P(T^kx) = (<x^*, T^kx>, <x^*, T^{k+1}x>, <x^*, T^{k+2}x>).$$

(ii) If $|<x^*, T^{k+1}x>| \leq |<x^*, T^{k+2}x>|$, then

$$P(T^{n_k}x) = P(T^{k+1}x) = (<x^*, T^{k+1}x>, <x^*, T^{k+2}x>, <x^*, T^{k+3}x>).$$

Consequently, for any $k \in \mathbb{N}$, the second coordinate of $P(T^{n_k}x)$ has magnitude greater than or equal to the first or the third coordinate of $P(T^{n_k}x)$. By continuity these inequalities pass on to anything in the closure of the set $\{P(T^{n_k}x)\}_k$. In particular

$(1,0,1) \notin \overline{\{P(T^{n_k}x) : k \in \mathbb{N}\}}$. The surjectivity of P implies that $\{T^{n_k}x\}_k$ can not be dense.

To complete the proof note that, by Lemma 3.6, this set is somewhere dense. $\square$

Chapter 4

Topologically Transitive Operators

Let X be a Banach space (separable or not). The equivalence between hypercyclic operators and topologically transitive ones (Chapter 1) suggests an extension of the notion of a hypercyclic operator to Banach spaces which are not necessarily separable.

Recall that an operator T is said to be *topologically transitive* on X if for any pair U, V of non-empty open subsets of X there exists a positive integer n such that

$$T^n(U) \cap V \neq \emptyset.$$

It is natural to ask what properties of hypercyclic operators (separable case) can be extended to topologically transitive ones (non-separable case). For instance the following two properties, well known for hypercyclic operators, are also enjoyed by topologically transitive ones:

1. Kitai [21, Corollary 2.4] showed that if T is hypercyclic, then T^* has no eigenvalues. Bermudez and Kalton [4, Proposition 3.3], extended this result to topologically transitive operators.

2. Kitai [21, Corollary 2.8] showed that if T is hypercyclic, then

$$\sigma(T) \cap \{\lambda \in \mathbb{C} : |\lambda| = 1\} \neq \emptyset.$$

An argument similar to the one used by Kitai can be used to show that the same result extends to topologically transitive operators. To prove this we start with two lemmas. Let $r(T)$ denote the *spectral radius* of T; i.e., $r(T) = \{|\lambda| : \lambda \in \sigma(T)\}$.

Lemma 4.1. *Let X be a Banach space (separable or not) and $T \in L(X)$.*

(a) If $r(T) < 1$, then T is not topologically transitive on X.

(b) If $\sigma(T) \subset \{\lambda \in \mathbb{C} : |\lambda| > 1\}$, then T can not be topologically transitive.

Proof. (a) Since $r(T) = \lim_{n\to+\infty} ||T^n||^{\frac{1}{n}}$, if $r(T) < 1$, then T is power bounded; i.e., there exists a positive constant C such that $||T^n|| < C$ for all $n \in \mathbb{N}$. But note that, as in the separable case, if T is power bounded, then T can not be topologically transitive on X.

(b) If $\sigma(T) \subset \{\lambda \in \mathbb{C} : |\lambda| > 1\}$, then T is invertible and $r(T^{-1}) < 1$. By (a), T^{-1} is not topologically transitive. But, as we mentioned in Chapter 1, if an invertible operator is topologically transitive, then the inverse is topologically transitive. Therefore T is not topologically transitive. $\square$

Lemma 4.2. *Suppose that $X_1, \ldots, X_n$ are Banach spaces and that $T_i \in L(X_i)$ for $i = 1, \ldots, n$. If $T_1 \oplus T_2 \ldots \oplus T_n$ is topologically transitive on $X_1 \oplus X_2 \ldots \oplus X_n$, then T_i is topologically transitive on X_i, for all $i = 1, \ldots, n$.*

Proof. Fix $i \in \{1, \ldots, n\}$. Let U, V be non-empty open subsets of X_i. Take

$$\tilde{U} = X_1 \oplus X_2 \ldots \oplus U_i \oplus X_{i+1} \ldots \oplus X_n,$$

and

$$\tilde{V} = X_1 \oplus X_2 \ldots \oplus V_i \oplus X_{i+1} \ldots \oplus X_n.$$

Since $\tilde{U}, \tilde{V}$ are non-empty open subsets of $X_1 \oplus X_2 \ldots \oplus X_n$, by assumption, there exists a positive integer n_0 such that

$$(T_1 \oplus T_2 \ldots \oplus T_n)^{n_0} \tilde{U} \cap \tilde{V} \neq \emptyset.$$

Therefore, $T_i^{n_0} U_i \cap V_i \neq \emptyset$. Then T_i is topologically transitive on X_i. □

Theorem 4.3. *Let X be a Banach space (separable or not) and $T \in L(X)$. If T is topologically transitive on X, $\sigma(T) \cap \{\lambda \in \mathbb{C} : |\lambda| = 1\} \neq \emptyset$.*

Proof. If $\sigma(T) \cap \{\lambda \in \mathbb{C} : |\lambda| = 1\} = \emptyset$, then $\sigma(T) = \sigma_1 \cup \sigma_2$, where $\sigma_1 = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$ and $\sigma_2 = \{\lambda \in \mathbb{C} : |\lambda| > 1\}$. We can assume that neither σ_1 nor σ_2 is empty. (If either one is empty, by Lemma 4.1, T is not topologically transitive.) Then σ_1 and σ_2 are non-empty, disjoint closed subsets whose union is $\sigma(T)$. By the Riesz decomposition theorem [26, pg. 131], there exists subspaces M_1 and M_2, invariant under T such that $X = M_1 \oplus M_2$, and $\sigma(T|M_i) = \sigma_i$ for $i = 1, 2$. If we let $T_i = T|M_i$ for $i = 1, 2$, then by Lemma 4.1, T_i is not topologically transitive. By Lemma 4.2, $T = T_1 \oplus T_2$ can not be topologically transitive on $X = M_1 \oplus M_2$. □

An argument similar to the one used by Bés and Peris ([7]) in their version of the Hypercyclicity Criterion provides a sufficient condition for topological transitivity.

Proposition. *(Topological Transitivity Criterion) Let T be a bounded linear operator on a complex Banach space X (not necessarily separable). Suppose that there exists a strictly increasing sequence of positive integers $\{n_k\}_{k=1}^{\infty}$ for which there are:*

(1) A dense subset $X_0 \subset X$ such that $T^{n_k} x \to 0$ for every $x \in X_0$.

(2) A dense subset $Y_0 \subset X$ and a sequence of mappings $S_k : Y_0 \to X$ such that:

(a) $S_k y \to 0$ for every $y \in Y_0$.

(b) $T^{n_k} S_k y \to y$ *for every* $y \in Y_0$.

Then T *is topologically transitive.*

This proposition can be used to show that there is a topologically transitive operator in any Hilbert space H. The idea (provided by J. H. Shapiro, see [4, Example on pg. 1452]) is as follows. If H is a non-separable Hilbert space, write $H = l_2(X)$ where X is a Hilbert space of the same density character and define T as twice the backward shift on $l_2(X)$; that is,

$$T(x_1, x_2, x_3, \dots) := 2(x_2, x_3, \dots).$$

An argument totally analogous to the one we used in Chapter 1 to show that λB is hypercyclic in $l_2(\mathbb{N})$ whenever $|\lambda| > 1$, shows that this operator T is topologically transitive in H.

But there are some differences. As we mentioned in Chapter 1, Ansari and Bernal showed that any infinite dimensional separable Banach space support hypercyclic operators. This property does not hold when considering non-separable cases. In [4] Bermudez and Kalton showed that if X is a non-reflexive quotient of a von Nuemann algebra (in particular $X = l_\infty$), then X does not support topologically transitive operators.

To finish this chapter we present a "more natural" example of a topologically transitive operator in a non-separable Banach space.

Example 4.4. Let

$$A = L_0^\infty(\mathbb{R}^+) = \{f \in L^\infty[0, +\infty] \ / \ \forall \ \epsilon > 0 \ \exists \ n \in \mathbb{R} : \text{ess sup}_{x \geq n} \ |f(x)| < \epsilon\}.$$

A is a subspace of $L^\infty(\mathbb{R}^+)$ and it is easy to check that under the same norm,

$$||f||_\infty = \text{ ess sup}_{R^+} \ |f| \ \ (\forall \ f \in A),$$

A is closed in $L^\infty(\mathbb{R}^+)$, and therefore it is a Banach subspace of $L^\infty(\mathbb{R}^+)$.

Also note that any function $g \in L^\infty[0,1]$ can be naturally embedded into A by setting $\epsilon : g \longrightarrow \tilde{g}$ where

$$\tilde{g}(x) = \begin{cases} g(x) & 0 \le x \le 1 \\ 0 & 1 < x \le \infty \end{cases}$$

Therefore A is a non separable Banach space.

Now, fix $a > 0$ and define $T : A \longmapsto A$ by

$$f(x) \longrightarrow 2f(x+a) \ \ (\forall\, x \in \ [0, +\infty]).$$

Then T is obviously in $L(A)$.

Claim: T is topologically transitive in A.

Proof of Claim:, let $f,\ g \in A$, $\epsilon > 0$, $U = \{h \in A : ||h - f||_\infty < \epsilon\}$, and $V = \{j \in A : ||j - g||_\infty < \epsilon\}$. We want to show that $\exists\ m \in \mathbb{N}$ such that $T^m(U) \cap V \neq \emptyset$. Since $f \in A$, there exists $n_0 \in \mathbb{R}$ such that ess $\sup_{x \ge n_0} |f(x)| < \epsilon/2$. Let m such that $ma > n_0$ and $2^{-m}||g||_\infty < \epsilon/2$, and

$$h(x) = \begin{cases} f(x) & 0 \le x \le n_0 \\ \frac{1}{2^m} g(x - ma) & x \ge ma \\ 0 & \text{otherwise} \end{cases}$$

Then $h \in A$ and, moreover,

$$\begin{aligned} ||h - f||_\infty &= \text{ess sup}_{x \ge n_0} |h(x) - f(x)| \\ &\le \text{ess sup}_{x \ge ma} |h(x)| + \text{ess sup}_{x \ge n_0} |f(x)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Therefore $h \in U$.

On the other hand,

$$T^m h = 2^m h(x + ma) = g(x), \ \ \forall\ \ x \ge 0.$$

Then $T^m h = g$ ($\in V$), and therefore T is topologically transitive on A.

Remark 4.5. It is also easy to check that this operator satisfies the Topological Transitivity Criterion. Indeed, take X_0 to be the dense subset of functions in A with finite support, $Y_0 = A$, $S : A \to A$ given by $S(f(x)) = \frac{1}{2}f(x-a)$, $n_k = k$ and $S_k = S^k$. But given U, V non-empty open subsets of A, the construction in the previous example gives an explicit vector, $h \in U$ and $m \in \mathbb{N}$ such that $T^m h \in V$.

Chapter 5

Topologically Transitive Multipliers

Let A be a Banach algebra and $T \in L(A)$. T is called a *left multiplier* if $T(uv) = T(u)v \ \forall \ u, v \in A$, and T is called a *right multiplier* if $T(uv) = uT(v) \ \forall u, v \in A$.

Note that if A is a commutative Banach algebra, a left multiplier is also a right multiplier, and *vice versa.*

The most natural examples of multipliers are the multiplication operators. A *left multiplication operator* on a Banach algebra A is an operator of the form $L_a(x) = ax$ for all $x \in A$. A *right multiplication operator* on A is an operator of the form $R_a(x) = ax$ for all $x \in A$.

A Banach space X has *the approximation property* if for every compact subset K of X and every $\epsilon > 0$ there exists a finite rank operator $T \in L(X)$ such that $||Tx - x|| < \epsilon$ whenever $x \in K$. Bonet, Peris and Martínez [8, Corollaries 1.3, 1.4] recently showed that if X is a Banach space with the approximation property such that X' (X' denotes the space of bounded linear functionals on X) is separable (then X is necessarily separable), $T \in L(X)$ satisfies the Hypercyclicity Criterion, and K is the Banach subalgebra of $L(X)$ of compact operators on X, then the left multiplication operator $L_T \in L(K)$ is topologically transitive on K.

Moreover, since any separable Banach space admits compact perturbations of the identity (operators of the form compact + identity), satisfying the Hypercyclicity Criterion (see [2] and [5]), and since the unitization $\tilde{K}$ of K is $\tilde{K} = K \oplus \mathbb{C}I$, we get the following.

Theorem 5.1. *There exists a Banach algebra A and $\tilde{a} \in \tilde{A}$ (the unitization of A), such that the left multiplication operator $L_{\tilde{a}} \in L(A)$ is topologically transitive on A.*

Remarks 5.2. 1) If T in $L(X)$ satisfies the Hypercyclicity Criterion, the same result holds if we consider the right multiplication operator $R_T : K \to K$.

2) As we mention in the introduction, the Banach algebra on Theorem 5.1 is necessarily non-unital, as we shall see next.

Suppose now that A is a Banach algebra with unit element e. Any left (right) multiplier T satisfies

$$T(v) = T(ev) = T(e)v \ \ (T(v) = T(ve) = vT(e)).$$

So if $a = T(e)$, we get $T(v) = av$ $(T(v) = va)$. So any left (right) multiplier T on A has the form L_a (R_a) for some $a \in A$.

Theorem 5.3. *No multiplier on a Banach algebra with unit element is topologically transitive.*

Proof. We only show the proof for T a left multiplier. The proof for a right multiplier is totally analogous.

Let A be a Banach algebra with unit element e. Suppose, in order to get a contradiction, that $T \in L(A)$ is a left multiplier and topologically transitive on A.

Let $U = \{x \in A : ||x - e|| < \frac{1}{2}\}$ and $V = \{x \in A : ||x|| < \frac{1}{2}\}$. By assumption $\exists\ m \in \mathbb{N}$ such that $T^m(U) \cap V \neq \emptyset$. Note that if $x \in U$, then x is invertible and

$$x^{-1} = \sum_{i=0}^{\infty} (e - x)^i.$$

Then

$$||x^{-1}|| \leq \sum_{i=0}^{\infty} ||e - x||^i \leq \sum_{i=0}^{\infty} (\frac{1}{2})^i = 2.$$

Since T is a left multiplier on A (unital), $\exists\ a \in A$ such that $T = L_a$. Suppose that $u \in U$ is such that $(L_a)^m(u) \in V$. Then $||a^m u|| < \frac{1}{2}$ and since u is invertible with $||u^{-1}|| \leq 2$, we get

$$||a^m|| = ||a^m u u^{-1}|| \leq ||a^m u|| ||u^{-1}|| < \frac{1}{2} 2 = 1.$$

For any positive integer n write $n = mk + r$ for some $k \in \{0, 1, 2, \dots\}$ and some $r \in \{0, 1, \dots, m-1\}$. Then

$$||a^n|| = ||a^{mk+r}|| \leq ||a^{mk}|| ||a^r|| \leq ||a^m||^k ||a^r|| < ||a^r||.$$

Therefore, $\forall\ n \in \mathbb{N}$,

$$||a^n|| \leq \max\{||a^r|| : 0 \leq r \leq m-1\} = M.$$

But $||T^n|| = ||L_a^n|| \leq ||a^n|| \leq M$, $\forall\ n \in \mathbb{N}$. Then T is power bounded and therefore T can not be topologically transitive. □

Remark 5.4. Chan [11] showed that if H is a separable Hilbert space and $T \in L(H)$ satisfies the Hypercyclicity Criterion, then $L_T : L(H) \to L(H)$ is topologically transitive in the strong operator topology. (For the definition of this topology see page 11 of this thesis.) But since $L(H)$ is a unital Banach algebra, L_T is not topologically transitive under the norm topology, for any $T \in L(H)$.

Next we will consider some special cases when A is a non-unital Banach algebra. Recall that we have just proved (Theorem 5.3) that no unital Banach algebra supports topologically transitive multipliers.

Theorem 5.5. *No multiplier on a commutative Banach algebra with a non-zero, bounded, multiplicative linear functional is topologically transitive.*

Proof. Let T be a multiplier on a commutative Banach algebra A. Assume that such a non-zero, bounded, multiplicative linear functional Φ exists. Then there exists $b \in A$ such that $\Phi(b) = 1$. For any $a \in A$

$$\Phi(T(ab)) = \Phi(aT(b)) = \Phi(a)\Phi(T(b)).$$

Similarly, by commutativity of A,

$$\Phi(T(ab)) = \Phi(b)\Phi(T(a)).$$

Therefore

$$\frac{\Phi(a)}{\Phi(b)}\Phi(T(b)) = \Phi(T(a)).$$

But $\Phi(b) = 1$; so

$$\Phi(a)\Phi(T(b)) = \Phi(T(a)) \ \forall \ a \in A.$$

Let $\alpha = \Phi(T(b))$. Then

$$\Phi(T(a)) = \alpha\Phi(a) \ \textit{and} \ T^*\Phi = \alpha\Phi.$$

Therefore Φ is an eigenvector for T^* associated to the eigenvalue α. Then T is not topologically transitive on A. $\square$

We previously showed that there exists a Banach algebra A and an element $\tilde{a}$ of its unitization $\tilde{A}$ such that $L_{\tilde{a}}$ is topologically transitive (Theorem 5.1). Now we are going to show an example of a Banach algebra A such that the set of multipliers on A' can also be identified with the unitization $\tilde{A}$ of A, but A does not support topologically transitive multipliers.

Example 5.6. Let bv be the space of all complex sequences $U = (u_n)_{n \in \mathbb{N}}$, for which

$$||U||_{b_v} := \sup_{n \in \mathbb{N}} |u_n| + \sum_{n=1}^{\infty} |u_{n+1} - u_n| < \infty.$$

This is *the space of sequences of bounded variation.* With componentwise operations bv is a commutative unital Banach algebra.

Note that the elements in bv are convergent sequences; so

$$\Phi(U) := \lim_{n \to \infty} u_n, \quad \forall\ U = (u_n)_{n \in \mathbb{N}} \in bv$$

is a bounded, multiplicative, non-zero linear functional on bv. Let $A = \ker \Phi$. Then A is a commutative, non-unital Banach subalgebra of codimension one in bv.

The evaluation functionals Φ_k, given by $\Phi_k(U) = u_k \ \forall\ U \in A$, are non-zero, bounded, multiplicative, linear functionals on A. Therefore no multiplier on A is topologically transitive. However, the set of multipliers on A can be identified with bv and $bv \approx A \oplus \mathbb{C}$, the unitization of A (See [22], pg. 306).

Suppose now A is a non-unital, non-commutative Banach algebra. In Theorem 5.5 we showed that if a Banach algebra is commutative and there are non-zero, bounded, multiplicative linear functionals, then no multiplier can be topologically transitive. The same is true in non-commutative Banach algebras when we consider left or right multiplications by elements in the algebra. (Note that any algebra can be identified with a subset of the set of multipliers in the algebra via: $a \longrightarrow L_a \quad \forall \quad a \in A$.)

For the proof we need the following version of the Comparison Principle (see Chapter 1).

Lemma 5.7. *(A Comparison Principle for Topologically Transitive Operators): Let A_1, A_2 be Banach algebras and $T_i \in L(A_i)$, $i = 1, 2$. Let $\Psi : A_1 \to A_2$ be continuous with dense range such that $T_2 \circ \Psi = \Psi \circ T_1$. If T_1 is topologically transitive, then so is T_2.*

Proof. Let U, V be non-empty open subsets of A_2. By the density of the range of Ψ, $\Psi^{-1}(U)$ and $\Psi^{-1}(V)$ are non-empty open subsets of A_1. By topologically transitivity of T_1, $\exists\ n_0 \in \mathbb{N}$ such that

$$T_1^{n_0}(\Psi^{-1}(U)) \cap \Psi^{-1}(V) \neq \emptyset.$$

Therefore

$$\Psi(T_1^{n_0}(\Psi^{-1}(U)) \cap \Psi^{-1}(V)) = \Psi T_1^{n_0} \Psi^{-1}(U) \cap \Psi\Psi^{-1}(V) \neq \emptyset.$$

But note that since $\Psi T_1 = T_2 \Psi$, we get that $\Psi T_1^n = T_2^n \Psi\ \ \forall\ n = 1, 2, \ldots$. Therefore

$$T_2^{n_0} \Psi\Psi^{-1}(U) \cap \Psi\Psi^{-1}(V) \neq \emptyset.$$

But $\Psi\Psi^{-1}(U) \subseteq U$ and $\Psi\Psi^{-1}(V) \subseteq V$. Therefore $T_2^{n_0}(U) \cap V \neq \emptyset$. Then T_2 is topologically transitive on A_2. □

Proposition 5.8. *Let A be a Banach algebra with a non-zero, bounded, multiplicative linear functional and $a \in A$. The operator multiplication by a on A (from the left or from the right) is not topologically transitive.*

Proof. Fix $a \in A$. We will give the proof for multiplication by a from the left. The proof for right multiplication by a is totally analogous.

Let Ψ be a non-zero, bounded, multiplicative linear functional. We can assume without loss of generality that $\Psi(a) \neq 0$. (If $\Psi(a) = 0$ then $\Psi(L_a(X)) = 0$, by multiplicativity of Ψ. But Since $\Psi \neq 0$ then Ψ is surjective, and since the image of a dense set under a functional with dense range is dense in $\mathbb{C}$, we get that $L_a(X)$ can not dense in X and then L_a is not topologically transitive on A.)

Let $g : \mathbb{C} \to \mathbb{C}$, given by $g(\alpha) = \Psi(a)\alpha$. It is easy to see that g is linear and continuous. Moreover,

$$g(\Psi(x)) = \Psi(a)\Psi(x) = \Psi(ax) = \Psi(L_a x).$$

Therefore, $g \circ \Psi = \Psi L_a$. So, by Lemma 5.7, if L_a were topologically transitive, g would be hypercyclic. But by Rolewicz (see Chapter 1) there are no hypercyclic operators on finite dimensional Banach spaces. $\square$

Since a Banach algebra is not only always contained (by the identification mentioned before) but usually properly contained in the set of multipliers on that algebra, Proposition 5.8 does not consider every multiplier on the Banach algebra. But we can extend Proposition 5.8 to general multipliers if the Banach algebra contains a bounded left approximate identity.

Theorem 5.9. *Let A be a Banach algebra with a non-zero, bounded, multiplicative linear functional and with bounded left (right) approximate identity. Then no left (right) multiplier on A is topologically transitive.*

Proof. Assume that T is a left multiplier on a Banach algebra A. The proof for right multipliers is totally analogous. Suppose $\{e_\alpha\}_{\alpha \in \Lambda} \subset A$ is such that $||e_\alpha|| \leq M$ $\forall\ \alpha \in \Lambda$ and $e_\alpha a \to a$, as $\alpha \to \infty$, $\forall\ a \in A$. Then

$$\Psi(T(e_\alpha a)) = \Psi(T(e_\alpha)a) = \Psi(T(e_\alpha))\Psi(a).$$

By continuity of Ψ

$$\Psi(T(e_\alpha a)) \to \Psi(T(a)), \ \ as \ \alpha \to \infty.$$

Therefore

$$\Psi(T(e_\alpha))\Psi(a) \longrightarrow \Psi(T(a)), \ \ as \ \alpha \to \infty.$$

But

$$|\Psi(T(e_\alpha))| \leq ||\Psi|| \, ||T||M.$$

So there exists a sub-net $\{e_{\alpha_j}\}$ of $\{e_\alpha\}_{\alpha\in\Lambda}$ such that $\Psi(T(e_{\alpha_j}))$ is convergent (independently of $a \in A$). Say, $\Psi(T(e_{\alpha_j})) \to \lambda \in \mathbb{C}$, as $j \to \infty$. Then $\Psi(T(a)) = \lambda\Psi(a)$ $\forall \ a \in A$. Therefore, Ψ is an eigenvector of T^* associated with the eigenvalue λ, which implies that T can not be topologically transitive on A. □

Example 5.10. Let G be a locally compact abelian group, with group operation $+$ and Haar measure m. Let $L^1(G)$ consist of all (equivalent classes of) Borel measurable complex valued functions on G which are integrable with respect to m. Endowed with the usual L^1-norm, and with pointwise vector space operations, $L^1(G)$ becomes a Banach space. For $f, g \in L^1(G)$, the *convolution product* $f * g \in L^1(G)$ is defined by

$$(f * g)(s) := \int_G f(s-t)g(t)\, dm(t),$$

for all $s \in G$. With convolution as a multiplication , $L^1(G)$ becomes a commutative Banach algebra, called the *group algebra* of G. If the group G is discrete, then $L^1(G)$ has identity in which case no multiplier can be topologically transitive in $L^1(G)$. If the group G is not discrete, $L^1(G)$ has no identity, but still, since there are non-zero multiplicative linear functionals on $L^1(G)$, we can see in two ways that no multiplier can be topologically transitive on $L^1(G)$.

(1) With convolution as multiplication $L^1(G)$ is a commutative Banach algebra. Then by Theorem 5.5, no multiplier can be topologically transitive on $L^1(G)$.

(2) $L^1(G)$ always has a bounded approximate identity. Then by Theorem 5.9, no multiplier can be topologically transitive on $L^1(G)$.

A celebrated result by Wendel ([30]) and Helson ([19]), asserts that, the set of multipliers on $L^1(G)$ can be identified with the Banach algebra $M(G)$ of all regular complex Borel measures on G. In this identification, $L^1(G)$ is identified with those measures in $M(G)$ which are absolutely continuous with respect to Haar measure on G.

Remark 5.11. If H is a Hilbert space and K is the ideal of compact operators in $L(H)$, then K has a bounded left approximate identity. Since, by Theorem 5.1, K admits topologically transitive left multipliers, Theorem 5.9 provides another way of seen that K does not admit non zero multiplicative linear functionals.

Chapter 6

Final Remarks and Questions

Feldman [14] showed that if X is a separable Banach space and $y \in X$ is such that that $Orb(T, y)$ is well distributed, then T is hypercyclic. His proof seems to work only for the full orbit.

Question 1: *Suppose that $Orb(T^{n_k}, y)$ is well distributed for some sequence $(n_k)_k$ and some $y \in X$. Is $\{T^{n_k}\}_k$ a hypercyclic sequence of operators on X?*

The equivalences to the Hypercyclicity Criterion showed in Chapters 2 and 3 can be used to reformulate the Hypercyclicity Criterion problem.

Question 2: *If T is hypercyclic, does T satisfy the three-neighborhood condition?*

Question 3: *If T is hypercyclic, is T syndetically hypercyclic?*

Condition 2 in Corollary 2.4 seems to be very strong. In fact it might happen that this condition itself is sufficient for hypercyclicity in which case condition 1 of this corollary would be redundant.

Question 4: *If T satisfies condition (2) of Corollary 2.4, is T hypercyclic?*

Using topological arguments and the fact that for every hypercyclic operator T there is a manifold consisting entirely, except for zero, of hypercyclic vectors for T, Ansari [1] showed that *if T is hypercyclic, then T^n is hypercyclic for any $n \in \mathbb{N}$*. This gives rise to the following question for the non-separable case.

Question 5: *If T is topologically transitive and $n \in \mathbb{N}$, is T^n topologically transitive?*

Since an operator is topologically transitive if and only if any non-empty open subset has a dense orbit, an affirmative answer to this question can be obtained as a corollary to the following:

Question 6: *Suppose that a non-empty open subset U of a Banach space X is such that $Orb(T, U)$ is dense in X. Is $Orb(T^n, U)$ dense in X for any $n \in \mathbb{N}$?*

Ansari also showed that $Orb(T, x)$ is dense if and only if $Orb(T^n, x)$ is dense for every $n \in \mathbb{N}$. So a positive answer to Question 6 would generalize, to some extent, Ansari's result.

Recently Chan and Sanders [12] began the study of weakly hypercyclic operators. An operator T on a Banach space X is called *weakly hypercyclic* if there is a vector $x \in X$ such that $Orb(T, x)$ is dense in X with respect to the weak topology. Obviously every hypercyclic operator is weakly hypercyclic, but Chan and Sanders showed that the converse does not hold.

Chan and Sanders also showed that an invertible weakly hypercyclic operator need not have weakly hypercyclic inverse [12, Corollary 5.5]. Thus the equivalence between hypercyclic operators and topologically transitive ones is no longer true when considering the weak topology. This might not seem surprising given that the main tool for proving such equivalence for the norm topology is the Baire Category Theorem which is not available in the weak topology. However the proof of Chan and Sanders is surprisingly deep.

Nonetheless we can define weak topological transitivity as follows. An operator

T on a Banach space X is called *weakly topologically transitive* if for any non-empty open subset $U \subset X$, $Orb(T, U)$ is dense in X with respect to the weak topology. It would be interesting to study weakly topologically transitive operators.

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