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ROBUST OUTPUT REGULATION OF MINIMUM PHASE NONLINEAR SYSTEMS USING CONDITIONAL SERVOCOMPENSATORS

By

Sridhar Seshagiri

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ABSTRACT

ROBUST OUTPUT REGULATION OF MINIMUM PHASE NONLINEAR SYSTEMS USING CONDITIONAL SERVOCOMPENSATORS

By

Sridhar Seshagiri

The design of robust output feedback controllers for output regulation of minimum-phase nonlinear systems with well-defined normal form is considered, with emphasis on their transient performance. Previous work has shown how to design such controllers by incorporating a linear servocompensator in a continuous sliding mode design, but the asymptotic error regulation is usually achieved at the expense of poor transient performance. In this work, we present an approach to improve the transient performance. The servocompensator is designed as a "conditional" one that provides servocompensation only inside the boundary layer, effectively eliminating the performance degradation. We give both regional as well as semi-global results for error convergence, and show that the output feedback continuous sliding mode controller with conditional servocompensator can be tuned to recover the performance of a state feedback ideal sliding mode control.

To my family

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गुरुस्त्वं शिवस्त्वं च शक्तिस्त्वमेव
त्वमेवासि माता पिताऽसि त्वमेव ।
त्वमेवासि विद्या त्वमेवासि बुद्धिः
गतिर्मे मतिर्देवि सर्वं त्वमेव ॥

श्रीकामाक्षीपरदेवतायाः पादारविन्दयोः भक्तिभरेण समर्पितम् ॥

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Chapter 1

Introduction

1.1 Background and Motivation

The output regulation problem is one of the most fundamental problems in control theory, dealing with the design of a controller to make the output of a fixed plant asymptotically track a reference signal, while asymptotically rejecting a disturbance signal, both of which are produced by an autonomous system called the *exosystem*. For multivariable, time-invariant, finite-dimensional, linear systems, an exhaustive account of the available theory can be found, for instance, in the works of Davison [19] and Francis and Wonham [27]. It was established in these papers that the solvability of the problem requires the solvability of a system of two linear matrix equations, called the *regulator equations*. This, in turn, is equivalent, as illustrated by Hautus [30], to a certain property of the transmission polynomials of a composite system which incorporates the plant and the exosystem. A striking result of the theory is the observation that any controller which solves the problem can always be viewed as the interconnection of two components, called the servocompensator and the stabilizing compensator. The former is a device that incorporates an *internal model* of the exosystem, i.e., a model capable of generating the reference and disturbance signals produced by the exosystem. The role of the latter is then to stabilize the augmented system formed of the plant and the servocompensator. The generic setup is shown in Fig 1.1. The above mentioned property is known as the *internal model principle*, and reduces in the special case of constant references and disturbances to the well known idea of integral control. In fact, one of the earliest formal acknowledgements of the internal model principle can be found in the work of Minorsky [62], where he observes that his "second class" of controllers, popularly known today as proportional-integral (PI) controllers, 'has the remarkable result that such a constant disturbance has no influence upon the device'.

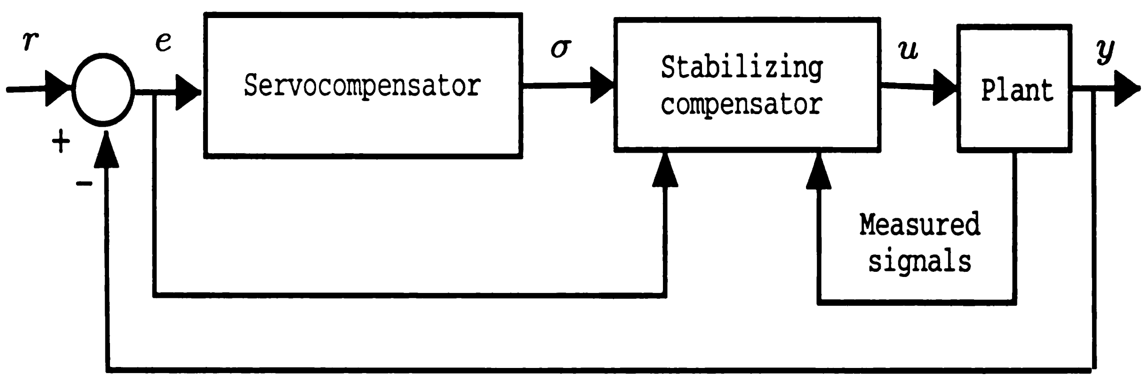

Figure 1.1: The general setup for the solution of the robust output regulation problem.

In this thesis, we concentrate on the design of controllers to solve the output regulation problem for a class of nonlinear systems, robustly with respect to parameter uncertainties, *with emphasis on their transient performance.* Specifically, the class of systems that we consider are mimimum-phase nonlinear systems transformable to the normal form, uniformly in a compact set of the unknown system parameters. For this class of systems, robust continuous feedback control techniques like min-max control, or sliding mode control (SMC) can be used to ensure convergence of the tracking error to a small ball, while rejecting bounded disturbances. However, making the error arbitrarily small requires the use of high-gain feedback near the origin; see, for example, [7, 18, 21].

The classical idea of servocompensator + stabilizing compensator design has been used by Khalil and co-workers in [43, 45, 46, 57, 58] to achieve asymptotic output regulation. For the case of constant references and disturbances [46, 57], the servocompensator is simply an integrator driven by the tracking error; and its inclusion creates an equilibrium point at which the tracking error is zero. For the more general case [43, 45, 58], a linear internal model is identified, which generates the trajectories of the exosystem and, along with them, a number of higher-order harmonics generated by the system nonlinearities. [1] This is then used to synthesize a servocompensator, the inclusion of which creates an invariant manifold on which the error is zero. In order to achieve nonlocal stabilization of the disturbance dependent equilibrium point or zero-error manifold, the stabilizing compensator is designed using the robust control techniques mentioned in the previous paragraph. Furthermore, the controller uses only error feedback, and is designed using the separation approach of Esfandiari and Khalil [24], where a state feedback controller is first designed and then a saturated high-gain observer is used to recover the performance of the state feedback design.

While the above mentioned designs achieve robust output regulation, they do not address the issue of transient performance. In fact, the steady-state performance achieved in these papers often happens at the expense of degradation of the transient performance. This is due in part to the increase in system order as a result of the servocompensator, and in part to the interaction of the servocompensator with the control saturation, which in the case of integral control leads to the well-known problem of *windup* [4].

The goal of this dissertation is to address the issue of transient performance degradation in the "conventional" integrator and servocompensator designs of [43,

[1] See Sections 1.2.3 and 4.2 for further discussion of this point.

45, 46, 57, 58]. To that end, the main contribution of the dissertation is a new approach to introducing integral and servo action, done within the continuous sliding mode control (CSMC) framework of [45, 46]. In the new approach, the integrator or servocompensator is "active" only inside the boundary layer, resulting in "conditional" integrators and servocompensators that effectively eliminate the degradation in transient performance. Analytical results for the stability of the closed-loop system and asymptotic output regulation are provided, and the improvement in transient performance is confirmed by showing that the output feedback CSMC with conditional integrator/servocompensator can be tuned to recover the performance of a state feedback ideal SMC.

The rest of this chapter is organized as follows. In the next section, we briefly review some of the main elements of this dissertation. These include

1. Input-output linearizable systems and the normal form
2. Integral control of nonlinear systems
3. Output regulation of nonlinear systems
4. Sliding mode control, and
5. Output feedback using high-gain observers.

The normal form can be considered the starting point of the controller designs in Chapters 2 to 4, which use the technique of sliding mode control to design the stabilizing compensator. Integral control is the topic of Chapters 2 and 3, and its extension to the output regulation problem is the topic of Chapter 4. Output feedback using high-gain observers is used in Chapters 2 and 4. We conclude this chapter with an overview of the thesis in Section 1.3.

1.2 Preliniminaries

1.2.1 Input-output Linearizable Systems and the Normal Form

The single-input single-output (SISO) nonlinear system

$$\dot{x} = f(x) + g(x)\, u, \;\; y = h(x) \tag{1.1}$$

with state $x \in R^n$, input $u \in R$, and output $y \in R$, where f and g are sufficiently smooth vector fields on a domain $D \in R^n$ and $h : D \to R$ is a sufficiently smooth function, is said to have relative degree ρ, $1 \leq \rho \leq n$, in a region $D_0 \in D$ if $L_g L_f^{i-1} h(x) = 0$, for $i = 1, 2, \cdots, \rho - 1$ and $L_g L_f^{\rho-1} h(x) \neq 0$ for all $x \in D_0$, where $L_f h(x) = \frac{\partial h}{\partial x} f(x)$ is the *Lie derivative* of h with respect to f, and $L_f^i h(x) = L_f(L_f^{i-1} h(x))$.

Remark 1.1 *It follows from the definition that the relative degree is the smallest integer ρ for which the control u appears in the expression for $y^{(\rho)}$.*

It can be shown (see, for example, [37]) that the relative degree condition guarantees the existence of a local change of variables $[\eta^T \; \xi^T]^T = T(x)$, $\xi \in R^\rho$, $\eta \in R^{n-\rho}$, that transforms (1.1) to the *normal form*

$$\left.\begin{array}{rcl} \dot{\eta} & = & \phi(\eta, \xi) \\ \dot{\xi} & = & A_c \xi + B_c \; \gamma(x) \; [u - \alpha(x)] \\ y & = & C_c x \end{array}\right\} \tag{1.2}$$

where the pair (A_c, B_c, C_c) is a canonical form representation of a chain of ρ integrators, $\gamma(x) = L_g L_f^{\rho-1} h(x)$, and $\alpha(x) = -L_f^\rho h(x) / L_g L_f^{\rho-1} h(x)$. The state feedback control $u = (\alpha(x)\gamma(x) + v)/\gamma(x)$ reduces the input-output map to $\dot{\xi}_\rho = y^{(\rho)} = v$, i.e., renders it linear, and consequently makes the component η unobservable from the output. The equation $\dot{\eta} = \phi(\eta, \xi)$ is called the *internal dynamics* of the system; when $\xi = 0$, it is called the *zero dynamics*.

For the controller designs in this thesis, we ensure the boundedness and convergence of ξ by a robust control design. Consequently, in order to show stability of the full system, some sort of a stability condition has to be imposed on the internal dynamics. The assumption that we make is that the internal dynamics $\dot{\eta} = \phi(\eta, \xi)$ is input-to-state stable (ISS) with ξ as the driving input, which guarantees that η is bounded whenever ξ is, and that η tends to zero as ξ tends to zero [47].

Extensions of the above concepts to multi-input multi-output (MIMO) systems can be found, for example, in [37, 65].

1.2.2 Integral Control of Nonlinear Systems

Integral control achieves robust output regulation for the case of constant or asymptotically constant exogenous signals. As mentioned in Section 1.1, the inclusion of an integrator creates an equilibrium point at which the error is zero. While external disturbances or uncertainties in the system model shift the equilibrium point, the integrator ensures that the tracking error is zero, as long as the controller stabilizes this equilibrium point.

Integral control of nonlinear systems has been studied by several researchers. It was shown in Hepburn and Wonham [31] that integral control suffices to guarantee *structurally stable regulation*, i.e., regulation in the presence of "small" parameter variations. Similar results were given by Desoer and Lin [22], for exponentially stable plants having a strictly increasing dc steady-state input-output map. Local results were also given in Huang and Rugh [35] using the method of extended linearization. Semi-global results for fully linearizable systems using state-feedback were reported in Freeman and Kokotovic [28]. Regional and semi-global results for input-output linearizable systems with asymptotically stable zero dynamics using output-feedback were given by Mahmoud and Khalil [57] and Khalil [46].

1.2.3 Output Regulation of Nonlinear Systems

The integral control idea of the the previous section can be generalized to the case of exogenous signals generated by a neutrally stable system under some additional assumptions. In particular, as mentioned in Section 1.1, once an internal model is identified, an appropriate servocompensator can be designed and augmented with the plant. The servocompensator is the generalization of the integrator, and its inclusion creates an invariant manifold on which the error is zero. As was the case with the equilibrium point, the zero-error manifold is disturbance dependent, but the servocompensator ensures that the tracking error is zero, as long as the controller stabilizes this zero-error manifold.

The output regulation problem for nonlinear systems has been studied by many researchers in the past two decades, among whom we specially mention Isidori and co-workers [12, 13, 14, 38, 39, 67, 68, 69], Huang and co-workers [16, 32, 33, 34, 35, 36] and Khalil and co-workers [43, 45, 58]. The work was initiated, to the best of our knowledge, by Hepburn and Wonham [31], who presented an extension of the notion of the internal model for nonlinear systems defined on differentiable manifolds. The pioneering work of Isidori and Byrnes [39] showed how the results of Francis and Wonham [27] could be extended to nonlinear plants and nonlinear neutrally stable exosystems with their formulation of the *nonlinear regulator equations.* They also showed that the transmission polynomial interpretation of Hautus [30] had its natural extension in terms of the zero dynamics, which is the nonlinear analog of the notion of transmission zeros. The results in [39] were local, and required smallness of both the exogenous signals and the initial states. A different "computational approach" to the problem, involving a power series expansion of the solution of the regulator equations, was pursued by Huang and Rugh [35, 36]. Their method allowed for large exogenous signals, but was still local in the initial states. Regional and semi-global results first appeared, for the case of fully linearizable systems, in the work of Khalil

[43]. An important contribution of [43] was the observation that in the nonlinear case the internal model must be able to generate not only the trajectories of the exosystem, but also a number of their higher-order harmonics. This idea was also independently elaborated by Huang and Lin [34, 32] and by Delli Priscoli [63]. Appealing to the concept of *immersion*, Isidori [37] further refined the idea of an internal model, and provided a complete set of necessary and sufficient conditions for the existence of a solution to the local output regulation problem. Extensions of the design of [43] to the case of nontrivial zero dynamics were given in Mahmoud and Khalil [58]. By combining the structurally stable output regulation approach of [37, Chapter 8] with the robust control approach of [43], an interesting generalization of the results of [43] was given in Isidori [38]. A succinct overview of the output regulation problem for nonlinear systems, summarizing most the results discussed above, can be found in [12, 13]. In the next paragraph, we summarize some of the more recent results.

A simplification of the robust servocompensator design of [58] is given in Khalil [45], where the only precise information that is required in the design of the controller is the relative degree of the plant, the sign of its high-frequency gain and the linear internal model. A semi-global controller that relaxes the assumption of input-to-state stability of the zero dynamics made in [45, 58] can be found in Serrani *et al* [68], and one that allows the frequencies of the exosystem to be unknown, making use of an adaptive internal model, can be found in [69]. A recent result that relaxes an assumption made in almost all previous works, including [45, 58, 68, 69], that the solution of the regulator equations be a polynomial in the exogenous signals, can be found in Chen and Huang [16].

1.2.4 Sliding Mode Control

Following [45, 46], the robust control technique that we use in this thesis to design the stabilizing compensator is a continuous version of sliding mode control.

Sliding mode control can guarantee asymptotic tracking with zero steady-state error for a wide class of nonlinear systems, and its design is accomplished through a two-step process. The first step is the design of a sliding surface function s, so designed that when the system trajectories are on the sliding surface $s = 0$, the system has the desired behavior. The second step is the design of the control to force the trajectories to reach the surface $s = 0$ in finite time and remain on it thereafter. The motion thus consists of a *reaching phase* during which trajectories starting off the surface $s = 0$ move towards it and reach it in finite time, and a *sliding phase* during which motion is confined to the surface $s = 0$. In its original form, often referred to as ideal SMC, finite-time convergence to and invariance of the surface $s = 0$ is accomplished by a discontinuous control that switches sign across the surface. As a result of switching non-idealities such as time-delays and parasitic sensor/actuator dynamics, the variable s does not remain identically at zero at the end of the reaching phase, but oscillates about it. Fig 1.2 shows a typical zig-zag motion about the surface due to a time-delay in implementing the control. The trajectory starts in the region $s > 0$ and heads towards the surface $s = 0$, first hitting it at the point P. In ideal SMC, the sliding phase should begin at this instant of time. However, in reality, there is a delay between the time the sign of s changes and the time the control switches. During this period, the trajectory crosses the surface into the region $s < 0$. When the control switches, the trajectory reverses its direction and the process repeats. The above phenomenon, known as *chattering*, is an important drawback of ideal SMC, and can excite unmodelled high-frequency dynamics, degrade system performance, and even result in instability. Various approaches have been proposed to reduce/eliminate chattering; see, for example, Bartolini *et al* [8], Young *et al* [75] and the references therein. The most common one is to replace the discontinuous control by a continuous approximation in a boundary layer of the sliding surface. This method can reduce chattering but often at the expense of a finite steady-state error.

Asymptotic error regulation can be recovered in a CSMC design by augmenting the system with a servocompensator and including the output of the servocompensator as part of the sliding variable s. Such an approach has been pursued for the case of constant exogenous signals by Chang [15], Khalil [46], and Baik *et al.* [6], and for the more general case by Khalil [45].

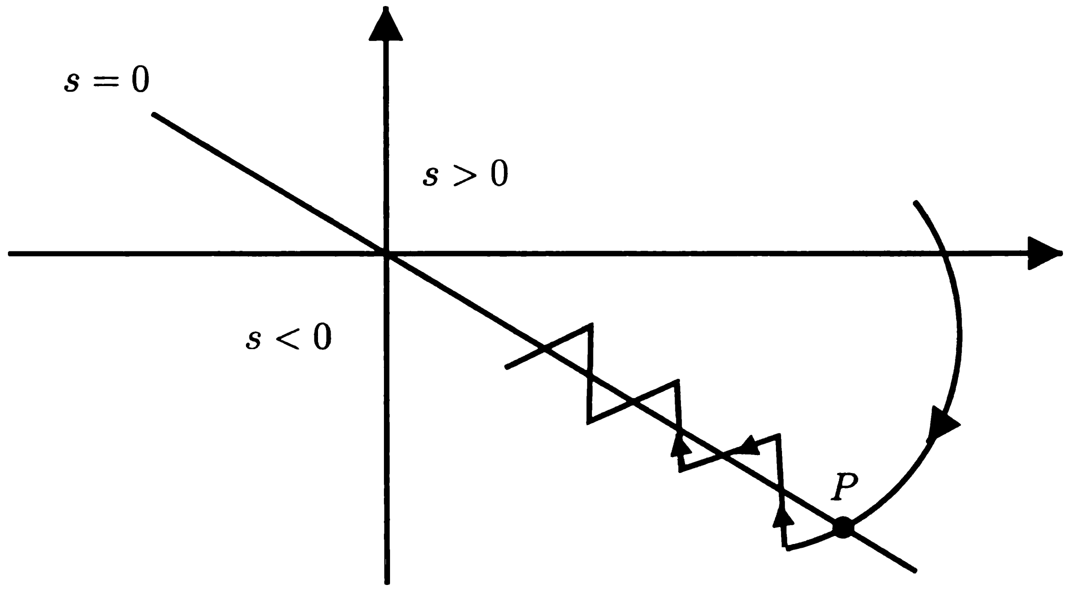

Figure 1.2: Chattering due to delay in control switching.

1.2.5 High-gain Observers

High-gain observers provide an important technique for the design of output feedback controllers for nonlinear systems. A high-gain observer is essentially an approximate differentiator that robustly estimates the derivatives of the output. Its gain depends on a small parameter ϵ, which can be adjusted to guarantee that the estimation error decays to an $\mathcal{O}(\epsilon)$ value within an arbitrarily small time interval. The use of high-gain observers in the design of output feedback control of input-output linearizable minimum phase nonlinear systems was studied by Esfandiari and Khalil in [24]. A key contribution of their study is the use of saturation nonlinearities to overcome the *peaking phenomenon* associated with high-gain observers. The observer is designed to assign the observer eigenvalues at $O(1/\epsilon)$ values in the open left-half complex plane. This results in exponential modes of the form $(1/\epsilon)^m \exp(-ta/\epsilon)$ for some positive constant a and positive integer m. While the decay of the exponential term $\exp(-ta/\epsilon)$ can be made faster by decreasing ϵ, the amplitude term $(1/\epsilon)^m$ grows larger with such decrease, resulting in an impulse-like peaking in the estimates. For nonlinear systems, this peaking can destabilize the system because of the possibility of finite escape times. Esfandiari and Khalil showed that saturating the feedback control law outside a compact region of interest protects the plant from the effects of peaking. Because the peaking transients decay rapidly, the saturation period is small. During this period, the state of the plant is close to initial value, and the estimation error decays to an $O(1/\epsilon)$ value. As a result, the output feedback controller recovers the performance under state feedback as ϵ tends to zero.

Since its introduction, the technique in [24] has received a lot of attention in output-feedback designs and has been included in a few textbooks [37, 47, 64]. One of the important consequences of this technique is the ability to separate the design of output feedback control for nonlinear systems into a state feedback design followed by the design of the high-gain oberver. Teel and Praly [72] developed a generic separation principle, which showed that (semi)global stabilizability via state feedback plus uniform observability imply semiglobal stabilizability via output feedback. Atassi and Khalil [5] provided a more comprehensive separation principle and showed that the output feedback controller recovers the performance of the state feedback controller in the sense of recovering asymptotic stability of an equilibrium point, its region of attraction, and its trajectories. An extensive survey of the use of high-gain observers in nonlinear control can be found in [44].

1.3 Overview of the Thesis

In Chapter 2, we consider the problem of robust output regulation for multi-input multi-output input-output linearizable minimum phase systems. For this case, we design a continuous sliding mode control with conditional integrators that achieve asymptotic error regulation with good transient performance. Analytical results are given for regional and semiglobal regulation, and we prove that the output feedback controller with conditional integrators recovers the performance of a state feedback ideal SMC that does not include integral action.

In Chapter 3, we consider a state-feedback version of the design considered in Chapter 2, with the specific goal of achieving global regulation. In addition to the global regulation result, we also prove a performance recovery result that is slightly sharper than the one of Chapter 2.

In Chapter 4, we consider the extension of the conditional integrator design of Chapter 2 to the more general output regulation problem, where the reference and disturbance signals are generated by a neutrally stable exosystem. For this case, we design a conditional servocompensator that provides servocompensation only inside the boundary layer, achieving asymptotic output regulation with good transient performance. As before, we give regional and semiglobal results for the regulation and also show the performance recovery of an ideal SMC design. A result on the effect of small internal model perturbations on the error is also provided.

In Chapter 5, we apply the designs of Chapters 2 to 4 to two applications : temperature control of a continuous stirred tank reactor (CSTR) and position control of a permanent magnet stepper motor (PMSM). For the CSTR, we apply the design of Chapter 2, while for the PMSM, we consider the designs of Chapters 2 to 4.

Finally we summarize our results and provide directions for future research in Chapter 6.

Chapter 2

Regulation Using Conditional Integrators

2.1 Introduction

We consider the problem of robust output regulation for multi-input multi-output (MIMO) minimum phase nonlinear systems transformable into the normal form, uniformly in a set of constant disturbances and uncertain parameters. For this class of systems, prior work has shown how to achieve zero steady-state error by introducing integral action in the controller [46, 57]. Integral control creates an equilibrium point at which the tracking error is zero. Robust control is designed to bring the trajectories to a small neighborhood of the equilibrium point, within which the control acts as a high-gain feedback that stabilizes the equilibrium point. While [57] accomplishes this through continuous min-max control, [46] uses continuous sliding mode control (CSMC) to design the controller as a universal one, where the only precise information about the plant that is used is its relative degree and the sign of its high-frequency gain.

The asymptotic regulation achieved by integral action happens at the expense of degrading the transient performance. Even in the absence of control saturation, integral action makes the response more oscillatory. When the control saturates, integrator buildup results, causing large overshoots and settling times. In this chapter, we present a new approach for introducing integral action to alleviate this transient performance degradation. This is done within the continuous SMC framework of [46]. The integrator is modified to provide integral action only inside the boundary layer, i.e., only "conditionally", effectively eliminating the transient performance deterioration. Both regional as well as semi-global results for asymptotic regulation are provided. The improvement in transient performance is shown analytically by proving that the output feedback continuous SMC with the conditional integrator recovers the performance of a state feedback ideal SMC without an integrator.

2.2 Motivating Example

Consider the second order system

$$\left.\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & ax_1^2 + bx_2 + cx_2^3 + u \\ y & = & x_1 \end{array}\right\} \tag{2.1}$$

The constants a, b and c are assumed to be unknown, but bounded with known bounds. The control objective is to regulate the output y to a constant value r. In ideal SMC design, the sliding surface can be chosen as $s = k_1 e_1 + e_2$, where $e_1 = y - r$, $e_2 = \dot{e}_1$, and $k_1 > 0$. This ensures that when motion is constrained to $s = 0$, the error e_1 converges asymptotically to zero. Differentiating, one obtains

$$\dot{s} = k_1 e_2 + a(e_1 + r)^2 + be_2 + ce_2^3 + u$$

Finite-time convergence to, and invariance of, $s = 0$ can be achieved by choosing $u = u_1 + u_2$, where the equivalent control u_1 is designed to cancel known or nominal terms in the expression for $\dot{s}$ and can be taken as

$$u_1 = -k_1 e_2 - \hat{a}(e_1 + r)^2 - \hat{b} e_2 - \hat{c} e_2^3$$

where $\hat{a}$, $\hat{b}$, and $\hat{c}$ are nominal values of a, b, and c respectively. The switching control u_2 is designed to handle the uncertain terms in the resulting expression for $\dot{s}$ and can be taken as

$$u_2 = -[\alpha(e_1 + r)^2 + \beta|e_2| + \gamma|e_2|^3 + \delta] \ \mathrm{sgn}(s)$$

where the positive constants α, β, and γ are upper bounds on $|a - \hat{a}|$, $|b - \hat{b}|$, and $|c - \hat{c}|$, respectively, and $\delta > 0$. This choice ensures that s converges to zero in finite time and stays there for all future time, which guarantees that e_1 and e_2 converge to zero asymptotically. However, as is well-known, this design suffers from chattering in the presence of switching nonidealities or unmodeled high-frequency dynamics. Various approaches have been proposed to reduce or eliminate chattering [75], the most common one being replacing the discontinuous term $\mathrm{sgn}(s)$ by its continuous approximation $\mathrm{sat}(s/\mu)$. This method can eliminate chattering but often at the cost of a non-zero steady-state error, that is proportional to μ. In order to obtain smaller errors, it is therefore necessary to make μ smaller, which in turn, leads to chattering again.

It is possible to recover the asymptotic regulation achieved by ideal SMC by using integral control within a continuous SMC setting. Integral action is conventionally introduced by augmenting the system with an integrator driven by the tracking error, i.e., $\dot{\sigma} = e_1$. In the case of the particular example, suppose we do this and also modify the sliding surface to $s = k_0\sigma + k_1 e_1 + e_2$, where now k_0 and k_1 are chosen to ensure that the polynomial $\lambda^2 + k_1\lambda + k_0$ is Hurwitz, which guarantees that when motion is restricted to $s = 0$, the error e_1 converges asymptotically to zero. The previous steps can then be repeated to design u. In particular, we take

$$\begin{aligned} u_1 &= -k_0 e_1 - k_1 e_2 - \hat{a}(e_1 + r)^2 - \hat{b} e_2 - \hat{c} e_2^3 \\ u_2 &= -[\alpha(e_1 + r)^2 + \beta|e_2| + \gamma|e_2|^3 + \delta] \operatorname{sat}(s/\mu) \end{aligned}$$

The presence of integral action guarantees that there is an equilibrium point, within $O(\mu)$ of the origin, at which $e_1 = 0$. Now, to achieve asymptotic regulation we do not need μ to be arbitrarily small; we only need it to be "small enough" to stabilize the equilibrium point. [1] However, while integral control, as designed above, can achieve asymptotic regulation, the transient response deteriorates when compared to that under ideal SMC.

To address the transient response degradation with conventional integral control, we modify the integrator design as follows. Let s be as before, i.e., $s = k_0\sigma + k_1 e_1 + e_2$, but now $k_0 > 0$ is arbitrary, $k_1 > 0$ is retained from the ideal SMC design, and σ is the output of

$$\dot{\sigma} = -k_0\sigma + \mu \operatorname{sat}(s/\mu) \tag{2.2}$$

To see the relation of (2.2) to integral control, observe that inside the boundary layer $\{|s| \le \mu\}$, (2.2) reduces to $\dot{\sigma} = k_1 e_1 + e_2 = k_1 e_1 + \dot{e}_1$, which implies that $e_1 = 0$ at equilibrium. Thus (2.2) represents a "conditional integrator" that provides integral action only inside the boundary layer. The control is taken as in the continuous approximation of ideal SMC, i.e., $u = u_1 + u_2$, where

$$\begin{aligned} u_1 &= -k_1 e_2 - \hat{a}(e_1 + r)^2 - \hat{b} e_2 - \hat{c} e_2^3 \\ u_2 &= -[\alpha(e_1 + r)^2 + \beta|e_2| + \gamma|e_2|^3 + \delta] \operatorname{sat}(s/\mu) \end{aligned}$$

The simulation results are shown in Figure 2.1. Numerical values used in the simu-

[1] We naturally expect this fact to be of consequence when there are switching nonidealities, and will show so through simulation later on.

lation are $a = 0.6$, $b = 2.5$, $c = 0.1$, $\hat{a} = 1$, $\hat{b} = 2$, $\hat{c} = 0$, $r = 1$, $\alpha = 0.5$, $\beta = 0.6$, $\gamma = 0.1$, $\delta = 1$, $\mu = 0.1$, and $x_1(0) = x_2(0) = \sigma(0) = 0$. The constant $k_1 = 5$ in the ideal SMC case, its continuous approximation, and the conditional integrator design, with $k_0 = 1$ in the conditional integrator design. The values of k_0 and k_1 in the conventional integrator design are taken as 25 and 10 respectively. The following observations can be made from Figure 2.1 : (i) the conventional integral control recovers the asymptotic regulation that is lost in the continuous approximation of ideal SMC (without integral control), but at the expense of degraded transient performance; in particular, the error convergence to zero is sluggish; (ii) the conditional integral design also achieves the task of asymptotic regulation, but without any degradation in transient performance; in fact, in the subplot for the transient behavior, the responses of the ideal SMC and the conditional integrator design are almost indistinguishable.

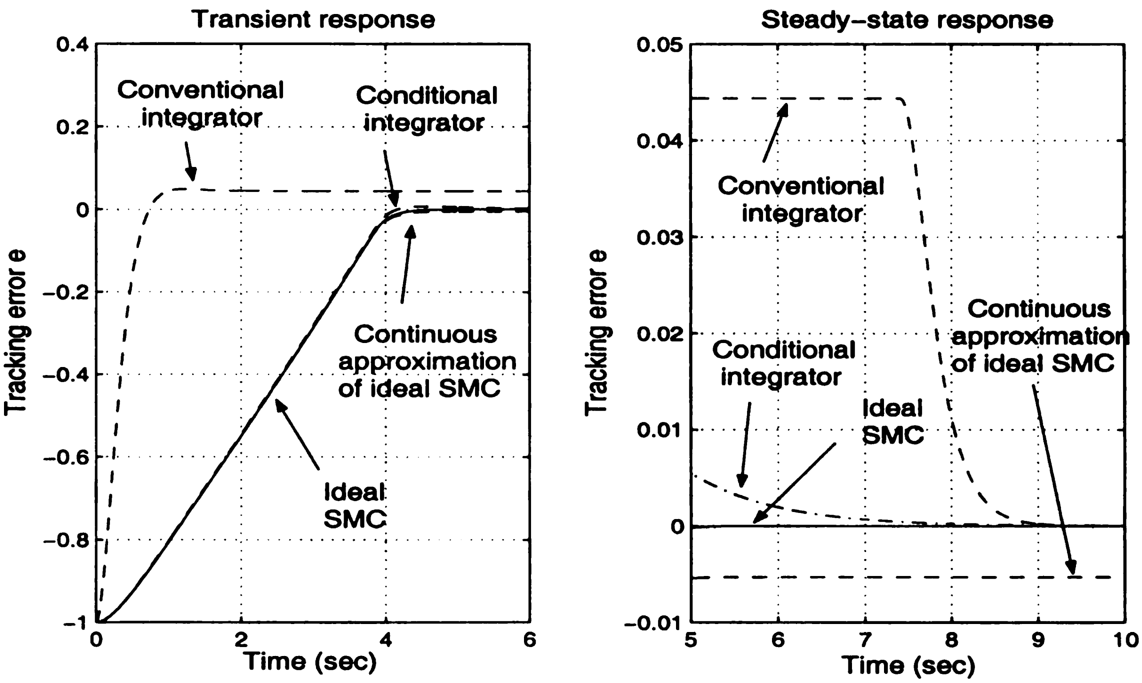

Figure 2.1: Asymptotic error regulation with improved transient performance using the "conditional integrator".

The transient performance recovery property of the conditional integrator design is also retained under output feedback, when the state e_2 is replaced by its estimate

$\hat{e}_2$ obtained from the high-gain observer (HGO)

$$
\begin{aligned}
\dot{\hat{e}}_1 &= \hat{e}_2 + \alpha_1(e_1 - \hat{e}_1)/\epsilon \\
\dot{\hat{e}}_2 &= \alpha_2(e_1 - \hat{e}_1)/\epsilon^2
\end{aligned}
$$

The positive constants α_1 and α_2 are chosen to assign the roots of the Hurwitz polynomial $\lambda^2 + \alpha_1\lambda + \alpha_2$, and ϵ is chosen sufficiently small. In order to take care of the peaking phenomenon associated with high-gain observers [24], the control is saturated outside a compact set of interest. Simulation results are shown in Figure 2.2, with $\alpha_1 = 15$, $\alpha_2 = 50$, $\hat{e}_1(0) = \hat{e}_2(0) = 0$, and a saturation level of 50 for the control.

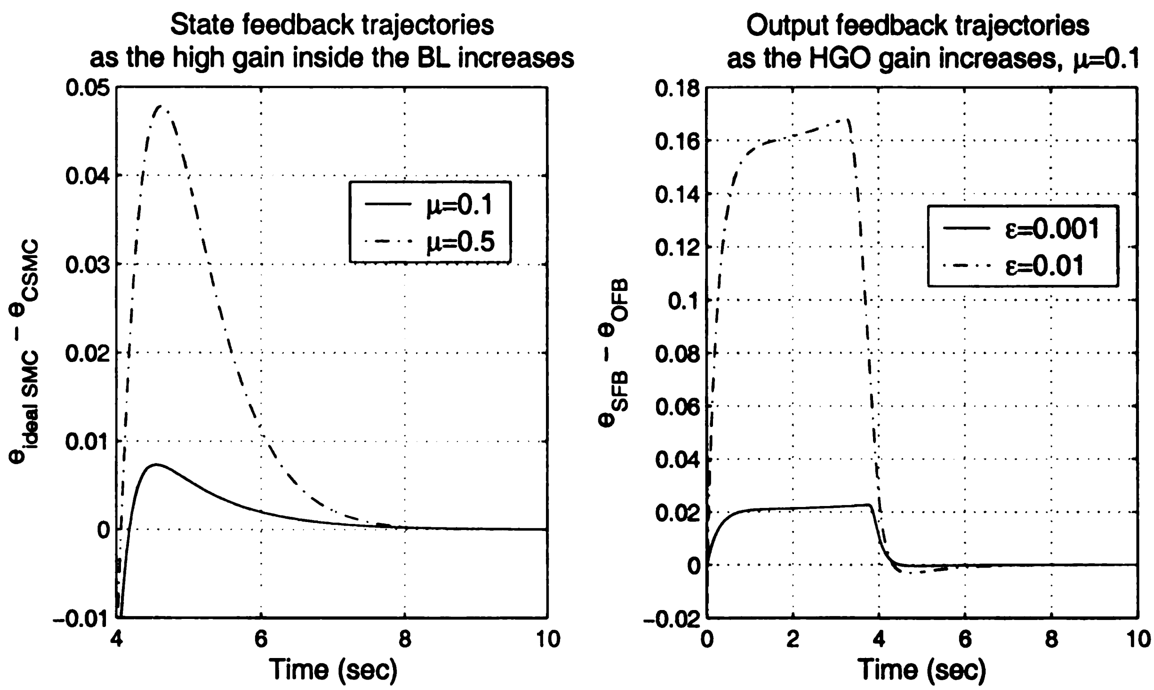

Figure 2.2: Performance recovery under the output feedback conditional integrator design.

Performance recovery in Figure 2.2 is shown in two steps : (i) as μ tends to zero, the response under state feedback continuous SMC approaches ideal SMC; (ii) for fixed μ, the response under the output feedback continuous SMC approaches that under state feedback continuous SMC as ϵ tends to zero. For the general case, we will prove in Section 2.4.2 that the closed-loop trajectories under output feedback continuous SMC with the conditional integrator approach those of the state feedback ideal SMC as μ, ϵ tend to zero.

The previous discussion showed how the design of the controller proceeds in general. Since our design requires that the control be bounded, a possible simplification of the controller is to choose the equivalent control to be zero and the coefficient of the switching component to be constant, i.e.,

$$u = -k \ \mathrm{sat}(\hat{s}/\mu)$$

Since, in practical applications, a constraint on the control magnitude appears naturally as an actuator limit, one might simply choose k as the maximum permissible control magnitude. Since the only precise information about the plant that such a controller uses is its relative degree and the sign of its high-frequency gain, it is referred to as a *universal integral regulator* [46]. We will discuss the universal integral regulator design further in Section 2.5 and show that the integrator modification (2.2) can be interpreted, in this case, as a special choice of a traditional anti-windup scheme.

To continue with this discussion, note that the control magnitude required to accommodate a step change in r increases as the step increases. One way to deal with this when the control is constrained with an apriori specified bound is through trajectory planning schemes. [2] In order to illustrate this, we consider the following modification to the previous simulations. The control is replaced by $u = -k \ \mathrm{sat}(\hat{s}/\mu)$, with $k = 50$. All other values are retained from the previous simulations, except r, which is increased to 1.5. One can verify, for example, by simulation, that the sliding condition is not satisfied. However, when the constant reference r is "smoothed"

[2] A more detailed discussion on this issue can be found in the next section.

by passing it through the filter $1/(\tau s+1)^2$, with $\tau = 0.5$, the control magnitude is now sufficient to overcome the uncertain terms and the sliding condition is satisfied. Furthermore, since $s(0) = 0$, the sliding condition ensures that s stays inside the boundary layer for all future time, which along with $e_1(0) = 0$ implies that the error e_1 itself is small for all time, under both the conventional as well as the conditional integrator designs. When the trajectory stays inside the boundary layer during the transient period, the conditional integrator acts as an integrator all the time; hence, we do not expect any significant difference between the transient responses of the two designs. The advantage of the conditional integrator design becomes clear when we consider an unexpected disturbance that causes an abrupt change in the state of the system. For example, consider an additive impulse-like disturbance $d(t)$ of magnitude 75 acting at the input of the system between $t = 5$ and $t = 5.1385$ seconds. The response of the two designs are shown in Figure 2.3.

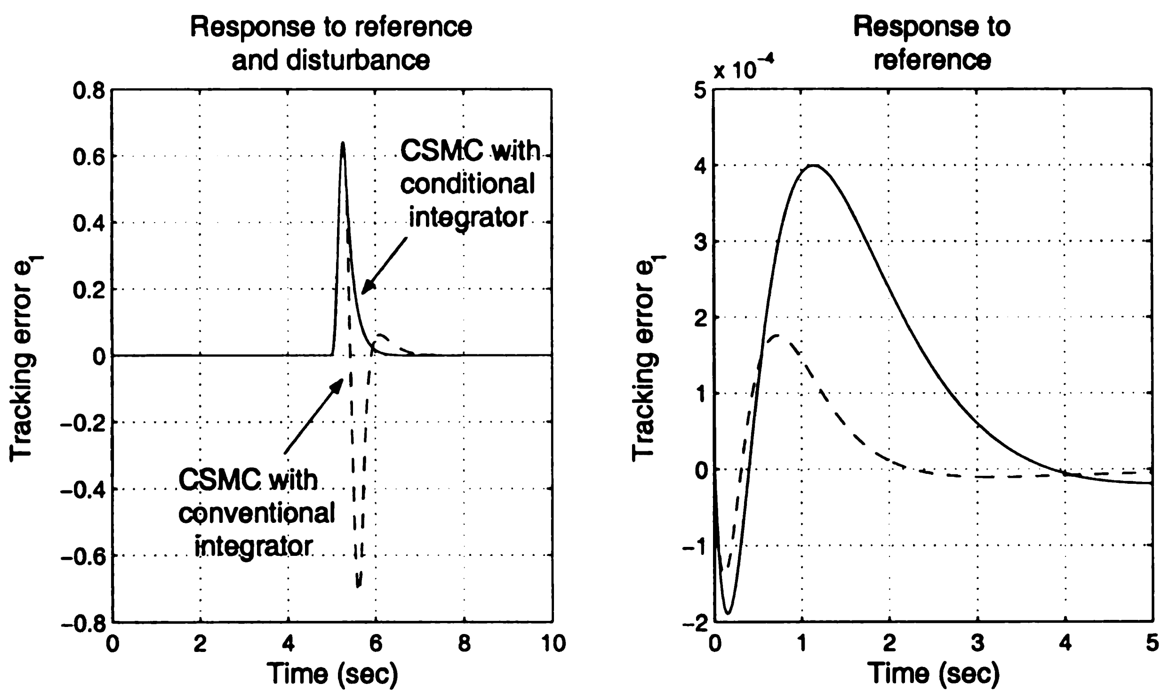

Figure 2.3: Effect of disturbance on the conventional and the conditional integrator designs.

We see from the plot on the left that while the system responses are almost identical (indistinguishable in that plot) before the onset of the disturbance, the response to the disturbance is significantly degraded with the conventional integrator design. The response before the disturbance is seen better in the plot on the right.

Lastly, before we present the system description and the problem statement for the general case, we make a small digression. In order to highlight the issue of chattering, we repeat the first two simulations under the assumption that a time delay of $T = 0.01$ seconds precedes the control input. The results are shown in Figure 2.4.

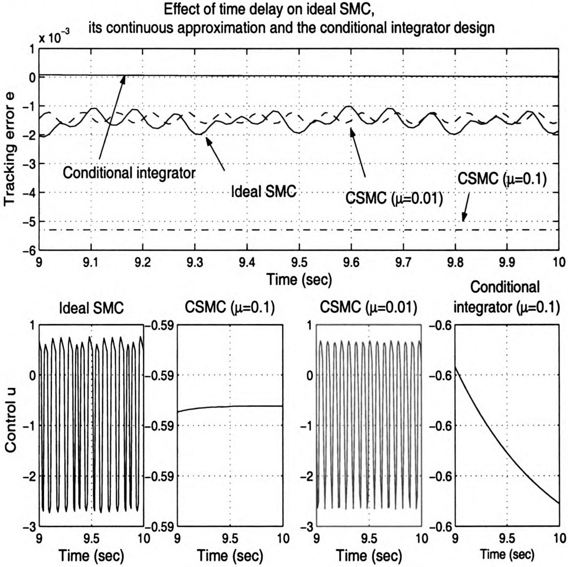

Figure 2.4: Effect of time delay on the ideal SMC and the conditional integrator design.

We see from the figure that there is chattering in the control and that the property of asymptotic regulation is lost with ideal SMC. Replacing the discontinuous control with its continuous approximation eliminates chattering when $\mu = 0.1$, but at the expense of a relatively large non-zero steady-state error. Reducing μ to 0.01 results in chattering again. The non-zero steady-state error can be handled by the conditional integrator design, where as mentioned earlier, the value of μ does not have to be made arbitrarily small and hence we can expect that this design will not suffer from chattering. This is validated by the simulation results of Figure 2.4. While we have included a simulation with a time delay to highlight a merit of this approach, we do not present any analysis for this case in the succeeding sections.

2.3 Problem Statement

Consider an MIMO nonlinear system, modeled by

$$\left.\begin{array}{rcl} \dot{x} & = & f(x,\theta) + \sum_{i=1}^{m} g_i(x,\theta)[u_i + \delta_i(x,\theta,w)], \\ y_i & = & h_i(x,\theta),\ 1 \le i \le m \end{array}\right\} \tag{2.3}$$

where $x \in R^n$ is the state, $u \in R^m$ is the control input, $y \in R^m$ is the output, θ is a vector of unknown constant parameters that belongs to a compact set $\Theta \subset R^p$, $w(t)$ is a piecewise continuous exogenous signal that belongs to a compact set $W \subset R^q$, $f(\cdot)$ and $g_i(\cdot)$ are smooth vector fields on $D \stackrel{\text{def}}{=} D_x \times \Theta$, where D_x is an open connected subset of R^n, $h_i(\cdot)$ are smooth functions on D, and the disturbances $\delta_i(\cdot)$ are continuous functions on $D \times W$. The formulation in (2.3) allows for matched disturbances that may depend on time-varying exogenous signals. We will specify a restriction on w shortly, when we are ready to state the control objective. Our first assumption is that the disturbance-free system (2.3) has a well-defined normal form, possibly with zero dynamics [37].

Assumption 2.1 *The system*

$$\dot{x} = f(x,\theta) + g(x,\theta)u, \;\; y = h(x,\theta)$$

has a strong vector relative degree $\{\rho_1, \rho_2, \ldots, \rho_m\}$ *in* D_x*, i.e.,*

$$L_{g_j} L_f^k h_i(x,\theta) = 0 \text{ for } 0 \le k \le \rho_i - 2, 1 \le i \le m, 1 \le j \le m$$

and $A(x,\theta) \stackrel{\text{def}}{=} \{L_{g_j} L_f^{\rho_i - 1} h_i\}$ *is nonsingular for all* $x \in D_x$ *and* $\theta \in \Theta$*. Furthermore, the distribution* $\text{span}\{g_1, \cdots, g_m\}$ *is involutive, uniformly in* θ*, and there is a change of variables*

$$\begin{bmatrix} \eta \\ \xi \end{bmatrix} = T(x,\theta) = \begin{bmatrix} T_1(x,\theta) \\ T_2(x,\theta) \end{bmatrix}, \;\; \eta \in R^{n-\rho}, \;\; \xi \in R^{\rho} \tag{2.4}$$

where $\xi = \{\xi^i\}$*, with* $\xi^i_j = L_f^{j-1} h_i$*,* $1 \le j \le \rho_i$*,* $1 \le i \le m$*, and* $\rho = \rho_1 + \rho_2 + \cdots + \rho_m$*, such that* $L_{g_j} \eta_i = 0 \;\forall\; 1 \le j \le m$*,* $1 \le i \le n - \rho$*, and* $T(x,\theta)$ *is a diffeomorphism of* D_x *onto its image.*

The vector relative degree and involutivity of the distribution $\text{span}\{g_1, \cdots, g_m\}$ guarantee the existence of the change of variables (2.4) locally [37]. Assumption 2.1 goes beyond that by requiring (2.4) to hold on a given region, uniformly in θ.

With the change of variables (2.4), we rewrite (2.3) in the normal form

$$\left.\begin{array}{rcl} \dot{\eta} & = & \phi(\eta, \xi, \theta) \\ \dot{\xi}^i & = & A_i \xi^i + B_i \left[b_i(\eta,\xi,\theta) + \sum_{j=1}^{m} a_{ij}(\eta,\xi,\theta)(u_j + \delta_j(\eta,\xi,\theta,w)) \right] \end{array}\right\} \tag{2.5}$$

where, for $1 \le i \le m$, the pair (A_i, B_i) is a controllable canonical form that represents a chain of ρ_i integrators, $b_i(\cdot) = L_f^{\rho_i} h_i$, and $\{a_{ij}(\cdot)\} = A(\cdot)$.

Our interest is in the regulation problem. To that end, we require that the exogenous signal $w(t)$ approaches a constant limit w_{ss}, i.e., $\lim_{t \to \infty} w(t) = w_{ss}$. In a similar vein, the reference $r_i(t)$ that the output y_i is required to asymptotically track has the following two properties:

- $r_i(t)$ and its derivatives up to the ρ_ith derivative are bounded, and $r_i^{(\rho_i)}(t)$ is piecewise continuous, for all $t \geq 0$
- $\lim_{t\to\infty} r_i(t) = r_{iss}$ and $\lim_{t\to\infty} r_i^{(j)}(t) = 0$ for $1 \leq j \leq \rho_i$.

This class of signals includes constant signals as a special case. Formulating the problem with time-varying references, which are asymptotically constant, accommodates a common practice in many applications; for example, in "trajectory planning" schemes employed to achieve point-to-point motion in the control of robotic manipulators [66, Chapter 5], or in pre-filter smoothing of a step command in the control of electric drives [55, Chapter 15]. The formulation also takes advantage of the robust control approach to designing the stabilizing controller, in the following sense. When the reference satisfies the first property, the robust control design ensures ultimate boundedness of the tracking error. When the second property is satisfied as well, the integral action gaurantees that the error asymptotically converges to zero.

Let $r_{ss} = \{r_{iss}\}$, $\tilde{w}(t) = w - w_{ss}$, $\nu^i(t) = [r_i - r_{iss}, r_i^{(1)}, \cdots, r_i^{(\rho_i - 1)}]^T$, $\varpi(t) = \{r_i^{(\rho_i)}\}$, and $\nu(t) = \{\nu^i\}$. By construction, $\tilde{w}(t)$, $\nu(t)$, and $\varpi(t)$ are bounded for all $t \geq 0$ and converge to zero as $t \to \infty$. Let $X \subset R^m$, $\Lambda \subset R^\rho$, and $\Lambda_0 \subset R^m$ be compact sets such that $r_{ss} \in X$, $\nu(t) \in \Lambda$, and $\varpi(t) \in \Lambda_0$ for all $t \geq 0$, and l_0 be a positive constant such that $\|\nu\| \leq l_0$ for all $\nu \in \Lambda$. Set $d = (r_{ss}, \theta, w_{ss})$ and $D_d = X \times \Theta \times W$. To solve the regulation problem, it is necessary that for every $d \in D_d$, there exist an equilibrium point at which $y = r_{ss}$ and a control input that can maintain equilibrium. This is guaranteed by our next assumption.

Assumption 2.2 *For each $d \in D_d$, there exist a unique equilibrium point $\bar{x} = \bar{x}(d) \in$*

D_x *and a unique control* $\bar{u} = \bar{u}(d)$ *such that*

$$\left.\begin{array}{rcl} 0 & = & f(\bar{x},\theta) + g(\bar{x},\theta)[\bar{u} + \delta(\bar{x},\theta,w_{ss})], \text{ and} \\ r_{ss} & = & h(\bar{x},\theta) \end{array}\right\}$$

With the change of variables (2.4), the equilibrium point $\bar{x}(d)$ maps into $(\bar{\eta}(d), \bar{\xi}(d))$, where $\bar{\xi}^i(d) = [r_{iss}, 0, \cdots, 0]^T$. Let $z = \eta - \bar{\eta}$, and $e^i = \xi^i - \bar{\xi}^i - \nu^i$, and rewrite (2.5) as

$$\left.\begin{array}{rcl} \dot{z} & = & \phi(z, e+\nu, d) \\ \dot{e}^i & = & A_i e^i + B_i \left[b_i(z, e+\nu, d) - r_i^{(\rho_i)} + \sum_{j=1}^m a_{ij}(z, e+\nu, d)(u_j + \delta_j(z, e+\nu, d, \tilde{w}))\right] \end{array}\right\} \tag{2.6}$$

where, for convenience, we write the functions ϕ, b_i, a_{ij}, and δ_j in terms of the new variables. Since we do not necessarily require our assumptions to hold globally, we need to restrict our analysis in the (z, e) variables to a region that maps back into the domain D_x. The following assumption states such a restriction.

Assumption 2.3 *There exist positive constants* l_1 *and* l_2*, independent of* d*, such that for all* $d \in D_d$*,* $w \in W$*,* $\nu \in \Lambda$ *and* $\varpi \in \Lambda_0$*,*

$$e \in \mathcal{E} \stackrel{\text{def}}{=} \{\|e\| < l_1\} \text{ and } z \in \mathcal{Z} \stackrel{\text{def}}{=} \{\|z\| < l_2\} \Rightarrow x \in D_x$$

In the output feedback case, the only components of the state (z, e) that are available for feedback are $e_1^i = y_i - r_i, 1 \leq i \leq m$. The unavailability of the partial-state e is dealt with by using a high-gain observer to estimate its unmeasured components. The unavailability of z is not an issue because we will design the control u to regulate the error e to zero and then rely on a minimum-phase-like assumption, stated below, to guarantee boundedness of z. The assumption has two parts. The first part states that with $(e + \nu)$ as the driving input, the system $\dot{z} = \phi(z, e + \nu, d)$ is input-to-state stable over a certain region [47], which implies that with $e + \nu = 0$, the origin of

$\dot{z} = \phi(z, 0, d)$ is asymptotically stable. This is strengthened in the second part of the assumption to local exponential stability of the origin.

Assumption 2.4 (i) *There exist a C^1 proper function $V_z : \mathcal{Z} \to R_+$, possibly dependent on d, and class $\mathcal{K}$ functions $\lambda_i : [0, l_2) \to R_+ (i = 1, 2, 3)$ and $\gamma : [0, l_0 + l_1) \to R_+$, independent of d, such that*

$$\lambda_1(\|z\|) \leq V_z(t, z, d) \leq \lambda_2(\|z\|)$$

$$\frac{\partial V_z}{\partial t} + \frac{\partial V_z}{\partial z}\phi(z, e + \nu, d) \leq -\lambda_3(\|z\|), \ \forall \ \|z\| \geq \gamma(\|e + \nu\|)$$

for all $e \in \mathcal{E}$, $z \in \mathcal{Z}$, $\nu \in \Lambda$, and $d \in D$. Furthermore, $\gamma(l_0) < \lambda_2^{-1}(\lambda_1(l_2))$.

(ii) *The equilibrium point $z = 0$ of $\dot{z} = \phi(z, 0, d)$ is exponentially stable, uniformly in d.*

2.4 Controller Design

Relying on the separation principle [5] that is common to the output feedback designs of [46] and [57], we pursue the same procedure for designing the controller used in those papers. First, a globally bounded partial state-feedback controller that meets the design objectives is designed under the assumption that e is available for feedback. Next, a high-gain observer is used to estimate the derivatives of the measured outputs e_1^i.

2.4.1 Partial State Feedback Design

The first step in the sliding mode design is to specify a sliding surface on which sliding motion occurs [75]. In the absence of integral action, we define the sliding surface $s_i = 0$ by

$$s_i = \sum_{j=1}^{\rho_i - 1} k_j^i e_j^i + e_{\rho_i}^i \tag{2.7}$$

where the positive constants $k_1^i, \cdots, k_{\rho_i-1}^i$ are chosen such that the polynomial

$$\lambda^{\rho_i - 1} + k_{\rho_i-1}^i \lambda^{\rho_i - 2} + \cdots + k_1^i$$

is Hurwitz, which guarantees that when motion is constrained to the surface $s_i = 0$, the tracking error e_1^i and its derivatives converge to zero. Differentiating (2.7) and using (2.6), we have

$$\dot{s}_i = F_i(z, e, \nu, d, r_i^{(\rho_i)}) + \sum_{j=1}^{m} a_{ij}(\cdot)[u_j + \delta_j(\cdot)] \tag{2.8}$$

where $F_i(\cdot) = b_i(\cdot) - r_i^{(\rho_i)} + \sum_{j=1}^{\rho_i-1} k_j^i e_{j+1}^i$. Let $F(z, e, \nu, d, \varpi) = \{F_i(z, e, \nu, d, r_i^{(\rho_i)})\}$. In the SISO case, (2.8) reduces to

$$\dot{s} = F(\cdot) + a(\cdot)[u + \delta(\cdot)]$$

and a standard assumption in this case is to require the sign of $a(\cdot)$ to be known and $a(\cdot)$ to be bounded away from zero. Our next assumption can be thought of as a straightforward extension to the MIMO case.

Assumption 2.5 *$A(z, e+\nu, d) = \Gamma(z, e+\nu, d)\hat{A}(e, \nu)$ where $\hat{A}$ is a known nonsingular matrix and $\Gamma = \mathrm{diag}[\gamma_1, \cdots, \gamma_m]$, with $\gamma_i(\cdot) \geq \gamma_0 > 0,\ 1 \leq i \leq m$, for all $e \in \mathcal{E}$, $z \in \mathcal{Z}$, $\nu \in \Lambda$, $d \in D_d$, and some positive constant γ_0.*

In the ideal SMC case, the control u can then be taken as

$$u = \hat{A}^{-1}(e, \nu)[-\hat{F}(e, \nu, \varpi) + v], \quad v_i = -\beta_i(e, \nu, \varpi)\ \mathrm{sgn}(s_i) \tag{2.9}$$

where $\hat{F}_i(\cdot)$ is a nominal value of $F_i(\cdot)$, which could be, but not restricted to

$$\hat{F}_i(\cdot) = \sum_{j=1}^{\rho_i - 1} k_i^j e_i^{j+1} - r_i^{(\rho_i)} + \hat{b}_i(\cdot)$$

$\hat{b}_i(\cdot)$ is a nominal value of $b_i(\cdot)$, and the component v_i is designed to handle uncertainties. Note that $\hat{F}_i(\cdot) = 0$ is possible. The choice of $\beta_i(\cdot)$ will be made clear shortly.

Guided by the motivating example in Section 2.1, we introduce integral action as follows. First, the ideal sliding surface function s_i of (2.7) is modified to

$$s_i = k_0^i \sigma_i + \sum_{j=1}^{\rho_i - 1} k_j^i e_j^i + e_{\rho_i}^i \tag{2.10}$$

where σ_i is the output of

$$\dot{\sigma}_i = -k_0^i \sigma_i + \mu_i \operatorname{sat}\left(\frac{s_i}{\mu_i}\right), \quad \sigma_i(0) \in [-\mu_i/k_0^i, \mu_i/k_0^i] \tag{2.11}$$

with $k_0^i > 0$, and μ_i a small positive parameter to be specified later. Furthermore, as was done in Section 2, the ideal SMC (2.9) is modified to the continuous control

$$u = \hat{A}^{-1}(e, \nu)[-\hat{F}(e, \nu, \varpi) + v], \quad v_i = -\beta_i(e, \nu, \varpi) \operatorname{sat}(s_i/\mu_i) \tag{2.12}$$

Inside the boundary layer $\{|s_i| \leq \mu_i\}$,

$$\dot{\sigma}_i = \sum_{j=1}^{\rho_i - 1} k_j^i e_j^i + e_{\rho_i}^i \stackrel{\text{def}}{=} e_a^i$$

where the "augmented error" e_a^i is a linear combination of the tracking error e_1^i and its derivatives up to order $(\rho_i - 1)$. At equilibrium, $e_a^i = 0$, which implies that $e_1^i = 0$. Hence, equation (2.11) represents a conditional integrator, which provides integral action only inside the boundary layer. In the present case, the resulting equation for $\dot{s}_i$ can be written as

$$\dot{s}_i = \Delta_i(z, e, \varpi, \sigma, d, \tilde{w}) - \gamma_i(z, e+\nu, d)\beta_i(e, \nu, \varpi)\ \mathrm{sat}(s_i/\mu_i) \tag{2.13}$$

where

$$\Delta(\cdot) = \{\Delta_i(\cdot)\} = F(\cdot) - \Gamma(\cdot)\hat{F}(\cdot) + A(\cdot)\ \delta(\cdot) + \{k_0^i(-k_0^i\sigma_i + \mu_i\ \mathrm{sat}(s_i/\mu_i)\}$$

In order to specify how $\beta_i(\cdot)$ is chosen, we make the following standard assumption.

Assumption 2.6 *Let*

$$\max\left|\frac{\Delta_i(\cdot)}{\gamma_i(\cdot)}\right| \leq \varrho_i(e, \nu, \varpi),\ \ 1 \leq i \leq m \tag{2.14}$$

for some known functions $\varrho_i(\cdot)$, where the maximization is taken over all $(z, e, \sigma) \in \Psi_c$, $d \in D_d$, $\nu \in \Lambda$, $\varpi \in \Lambda_0$, and $w \in W$.

The compact set Ψ_c will be defined in the next section using Lyapunov functions, and will serve as an estimate of the region of attraction. The functions β_i are chosen as $\beta_i(\cdot) = \varrho_i(\cdot) + q_i$, where $q_i > 0$. From (2.13) and (2.14), it follows that inside Ψ_c, $s_i\dot{s}_i \leq -\gamma_0\ q_i\ |s_i|$, whenever $|s_i| \geq \mu_i$. We note that the right-hand side of (2.14) is independent of z and σ, even though Δ_i may depend on z and σ. The former, while restrictive, is necessiated by the fact that z is unavailable for feedback, and is justified since (2.14) is only required to hold over a compact set. The latter is done purely for convenience, and is not restrictive, since, as we shall see later on, $\|\sigma\| = O(\max_i\ \mu_i)$, so that the contribution of σ is not significant, provided the constants μ_i are sufficiently small.

2.4.2 Output Feedback Design

The output feedback design uses the following high-gain observer to estimate e^i.

$$\left.\begin{array}{rcl} \dot{\hat{e}}^i_j & = & \hat{e}^i_{j+1} + \alpha^i_j(e^i_1 - \hat{e}^i_1)/(\epsilon_i)^j, \; 1 \le j \le \rho_i - 1 \\ \dot{\hat{e}}^i_{\rho_i} & = & \alpha^i_{\rho_i}(e^i_1 - \hat{e}^i_1)/(\epsilon_i)^{\rho_i} \end{array}\right\} \tag{2.15}$$

where $\epsilon_i > 0$ is a design parameter, and the positive constants α^i_j are chosen such that the roots of

$$\lambda^{\rho_i} + \alpha^i_1 \lambda^{\rho_i - 1} + \cdots + \alpha^i_{\rho_i - 1}\lambda + \alpha^i_{\rho_i} = 0$$

have negative real parts. In (2.15), $\hat{e}^i_j$ is an estimate of e^i_j, the $(j-1)^{th}$ derivative of e^i_1. Let

$$\hat{s}_i = k^i_0 \sigma_i + \sum_{j=1}^{\rho_i - 1} k^i_j \hat{e}^i_j + \hat{e}^i_{\rho_i} \tag{2.16}$$

be the corresponding estimate of s_i, [3] where σ_i is now the output of

$$\dot{\sigma}_i = -k^i_0 \sigma_i + \mu_i \text{ sat}\,(\hat{s}_i/\mu_i) \tag{2.17}$$

We replace e and s with their estimates $\hat{e}$ and $\hat{s}$ in the control (2.12), and saturate the control outside a compact set of interest. In particular, rewrite the control (2.12) as

$$u_i = \Upsilon_i(e, \nu, \varpi, \sigma), \text{ where } \Upsilon(\cdot) = \hat{A}^{-1}(\cdot)[-\hat{F}(\cdot) + v]$$

Inside Ψ_c, e belongs to Λ_e, a compact subset of R^ρ. Let S_i be the maximum value of $|\Upsilon_i(e, \nu, \varpi, \sigma)|$, where the maximization is taken over all $\nu \in \Lambda$, $\varpi \in \Lambda_0$, $|\sigma_i| \le \mu_i/k^i_0$ and $e \in \Lambda_{ee}$, where Λ_{ee} is a compact set that contains Λ_e in its interior. The control u is then taken as

$$u_i = S_i \text{ sat}(\Upsilon_i(\hat{e}, \nu, \varpi, \sigma)/S_i) \tag{2.18}$$

[3] We can take $\hat{e}^i_1$ as the estimate provided by (2.15) or the measured output e^i_1.

In summary, the output feedback controller is given by (2.15)-(2.18), where

$$\left.\begin{aligned} \Upsilon(\hat{e},\nu,\varpi,\sigma) &= \hat{A}^{-1}(\hat{e},\nu)[-\hat{F}(\hat{e},\nu,\varpi)+v] \\ v_i &= -\beta_i(\hat{e},\nu,\varpi)\,\mathrm{sat}(\hat{s}_i/\mu_i) \\ \beta_i(\hat{e},\nu,\varpi) &= \varrho_i(\hat{e},\nu,\varpi)+q_i \end{aligned}\right\} \tag{2.19}$$

To complete the controller design, we must specify how μ_i and ϵ_i are chosen. The parameters μ_i result from replacing an ideal SMC with its continuous approximation, and hence should be chosen "sufficiently small" to recover the performance of the ideal SMC. Similarly, in order for the output-feedback controller to recover the performance under state-feedback, the high-gain observer parameters ϵ_i should also be chosen "sufficiently small". Therefore, one might view μ_i and ϵ_i as tuning parameters and first reduce μ_i gradually until the transient response under partial state feedback is close enough to the ideal SMC, and then reduce ϵ_i gradually until the transient response under output feedback is close enough to that under state feedback. The asymptotic theory of the next section guarantees that this tuning procedure will work.

2.5 Closed-Loop Analysis

For $i = 1,\cdots,m$, define $\zeta^i \in R^{\rho_i - 1}$ by $(e^i)^T = [(\zeta^i)^T \; e^i_{\rho_i}]$ and write the closed-loop system in the standard singularly perturbed form

$$\left.\begin{aligned} \dot{\sigma}_i &= -k_0^i\sigma_i + \mu_i\,\mathrm{sat}((s_i - N_i(\epsilon_i)\varphi^i)/\mu_i) \\ \dot{\zeta}^i &= M_i\zeta^i + C_i(s_i - k_0^i\sigma_i) \\ \dot{s}_i &= \Delta_i(z,e,\varpi,\sigma,d,\tilde{w}) - \gamma_i(z,e+\nu,d)\;\beta_i(e,\nu,\varpi)\,\mathrm{sat}(s_i/\mu_i) + \Delta_i^*(\cdot) \\ \dot{z} &= \phi(z,e+\nu,d) \\ \epsilon_i\dot{\varphi}^i &= L_i\varphi^i + \epsilon_i B_i\,[b_i(\cdot) - r_i^{(\rho_i)} + \textstyle\sum_{j=1}^m a_{ij}(\cdot)(u_j+\delta_j(\cdot))] \end{aligned}\right\} \tag{2.20}$$

where

$$M_i = \begin{bmatrix} 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \\ -k_1^i & -k_2^i & \cdots & \cdots & -k_{\rho_i-1}^i \end{bmatrix}, \; L_i = \begin{bmatrix} -\alpha_1^i & 1 & \cdots & \cdots & 0 \\ \vdots & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ \vdots & \cdots & \cdots & 0 & 1 \\ -\alpha_{\rho_i}^i & 0 & \cdots & \cdots & 0 \end{bmatrix}, \; C_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

$$\Delta_i^*(\cdot) = k_0^i \mu_i \left[\mathrm{sat}(\hat{s}_i/\mu_i) - \mathrm{sat}(s_i/\mu_i)\right] + \sum_{j=1}^{m} a_{ij}(\cdot) \left[S_j \,\mathrm{sat}(\Upsilon_j(\hat{e}, \nu, \varpi, \sigma)/S_j) - \Upsilon_j(e, \nu, \varpi, \sigma)\right]$$

$N_i(\epsilon_i) = [0 \; k_i^2 {\epsilon_i}^{\rho_i-2} \; \cdots \; k_i^{\rho_i-1} \epsilon_i \; 1]$, and the scaled estimation error $\varphi^i = \{\varphi_j^i\}$ is defined by

$$\varphi_j^i = (e_j^i - \hat{e}_j^i)/(\epsilon_i)^{\rho_i - j}$$

The matrices M_i and L_i are Hurwitz by design. Let $\mu = \{\mu_i\}$ and $\epsilon = \{\epsilon_i\}$.

2.5.1 Boundedness and Convergence

The stability analysis shares many details with [46] and [57]. The main difference between the present analysis and its counterparts in [46] and [57] is treating σ_i and ζ^i separately, while in [46] and [57], they are lumped together in one vector. As in [46] and [57], it is both convenient as well as instructive to present the analysis in two parts. In the first part, we show that the controller parameters can be chosen to bring the trajectories to an arbitrarily small neighborhood of an equilibrium point, at which the tracking error is zero. In the second part, we show that the controller parameters can be further tuned to ensure asymptotic stabilization of this equilibrium point. To show the first part, we define appropriate Lyapunov functions for each of the five components of (2.20), i.e., s_i, σ_i, ζ^i, z, and φ^i, and use them to construct a compact set of interest $\Psi_c \times \Sigma_\epsilon$ that serves as an estimate of the region of attraction. We show that this set is positively invariant for a suitable choice of the controller parameters and that trajectories starting in the interior of this set will eventually reach a "small" set $\Psi_\mu \times \Sigma_\epsilon$ that shrinks to the origin as $\|\mu\|_\infty$ and $\|\epsilon\|_\infty$ tend to zero. To that end, let $Q_i = Q_i^T > 0$ and $P_i = P_i^T > 0$ be the solutions of the Lyapunov equations $Q_i M_i + M_i^T Q_i = -I$ and $P_i L_i + L_i^T P_i = -I$ respectively. For the components s, σ, ζ, and φ, we use the quadratic Lyapunov functions

$$V_i^s(s_i) \stackrel{\text{def}}{=} \frac{1}{2}s_i^2,\ V_i^\sigma(\sigma_i) \stackrel{\text{def}}{=} \frac{1}{2}\sigma_i^2,\ V_i^\zeta(\zeta^i) \stackrel{\text{def}}{=} \zeta^{i^T} Q_i \zeta^i, \text{ and } V_i^\varphi(\varphi^i) \stackrel{\text{def}}{=} \varphi^{i^T} P_i \varphi^i$$

respectively, and we use the Lyapunov function $V_z(t, z, d)$ for z. The sets Ψ_c and Σ_ϵ are defined by $\Psi_c \stackrel{\text{def}}{=} \Omega_c \times \Omega_{cz}$, $\Omega_c \stackrel{\text{def}}{=} (\prod_{i=1}^m \Omega_{c_i})$, $\Sigma_\epsilon \stackrel{\text{def}}{=} \prod_{i=1}^m \Sigma_{\epsilon_i}$, where

$$\left.\begin{array}{lcl} \Omega_{c_i} & \stackrel{\text{def}}{=} & \{V_i(\zeta^i) \le (c_i + \mu_i)^2 \chi_i, V_i^s(s_i) \le \frac{1}{2}c_i^2, V_i^\sigma(\sigma_i) \le \frac{1}{2}(\mu_i/k_0^i)^2\}, \\ \Omega_{cz} & \stackrel{\text{def}}{=} & \{V_z(t, z, d) \le \lambda_4(l_0 + l_3\|c\|)\}, \\ \Sigma_{\epsilon_i} & \stackrel{\text{def}}{=} & \{V_i^\varphi(\varphi^i) \le \epsilon_i^2 \vartheta_i\} \end{array}\right\} \tag{2.21}$$

$c_i > \mu_i$ is a positive constant, $c = \{c_i\}$, $\lambda_4 = \lambda_2 \circ \gamma$ is a class $\mathcal{K}$ function, and χ_i, l_3, and ϑ_i are positive constants independent of μ_i and ϵ_i to be specified shortly.

Before we show that $\Psi_c \times \Sigma_\epsilon$ serves as an estimate of the region of attraction, we need to ensure that $(z, e, \sigma) \in \Psi_c$ implies that $(z, e) \in \mathcal{Z} \times \mathcal{E}$. It can be verified that $\|e\| \le l_3\|c\|$ in Ω_c, where l_3 is a positive constant independent of c. Using this fact, along with Assumption 2.4(i), it follows that choosing c to ensure that

$$l_3\|c\| < \min\{l_1, \lambda_4^{-1}(\lambda_1(l_2)) - l_0\}$$

guarantees that $(z, e) \in \mathcal{Z} \times \mathcal{E}$ for all $(z, e, \sigma) \in \Psi_c$.

Since the boundaries of the set $\Psi_c \times \Sigma_\epsilon$ are formed of Lyapunov surfaces, to show that this set is positively invariant, it suffices to show that the derivatives of the corresponding Lyapunov functions are non-positive on the respective boundaries.

Using the fact that $|s_i| \le c_i$, $|\sigma_i| \le \mu_i/k_0^i$ in Ψ_c, and the inequality

$$\dot{V}_i^\zeta \le -\|\zeta^i\|^2 + 2\|\zeta^i\| \; \|Q_iC_i\| \; (|s_i| + k_0^i|\sigma_i|)$$

it is easy to show that $\dot{V}_i^\zeta \le 0$ on the boundary $V_i = (c_i + \mu_i)^2\chi_i$ for the choice $\chi_i = 4\|Q_iC_i\|^2\lambda_{max}(Q_i)$. Since

$$\sigma_i\dot{\sigma}_i \le -k_0^i|\sigma_i|^2 + \mu_i|\sigma_i|$$

it follows that $\dot{V}_i^\sigma \le 0$ on the boundary $V_i^\sigma = \frac{1}{2}(\mu_i/k_0^i)^2$. Next, we consider the $\dot{s}_i$ equation, which differs from (2.13) only in the term Δ_i^*. Inside Σ_{ϵ_i}, $\|\varphi^i\| = O(\epsilon_i)$, which can be used to show that, for sufficiently small ϵ_i, the control is not saturated inside $\Psi_c \times \Sigma_\epsilon$. Using this, along with $s_i = \hat{s}_i + N_i(\epsilon_i)\varphi^i$, it can be shown that Δ_i^* is $O(\|\epsilon\|_\infty)$ inside $\Psi_c \times \Sigma_\epsilon$. Let ϵ_i be small enough that $|\Delta_i^*(\cdot)| < \gamma_0 q_i$. On the boundary $V_i^s = \frac{1}{2}c_i^2$ we have $\text{sat}(s_i/\mu_i) = \text{sgn}(s_i)$, so that

$$\dot{V}_i^s \le -|s_i|[\gamma_i(\cdot)\beta_i(\cdot) - |\Delta_i(\cdot)| - |\Delta_i^*(\cdot)|]$$

Using (2.14), the definition of β_i, and the fact that $|\Delta_i^*(\cdot)| < \gamma_0 q_i$, it follows that $\dot{V}_i^s < 0$ on the boundary $V_i^s = \frac{1}{2}c_i^2$. Assumption 2.4(i) shows that $\dot{V}_z \le 0$ on the boundary $V_z = \lambda_4(l_0 + l_3\|c\|)$. Finally, using the inequality

$$\dot{V}_i^\varphi \le -\frac{\|\varphi^i\|^2}{\epsilon_i} + 2\;\|\varphi^i\|\;\|M_iB_i\|\;\Xi_i$$

where $\Xi_i = \max|b_i(\cdot) - r_i^{(\rho_i)} + \sum_{j=1}^m a_{ij}(\cdot)[u_j + \delta_j(\cdot)]|$, with the maximization taken over all $(z, e, \sigma) \in \Psi_c$, $d \in D_d$, $\nu \in \Lambda$, $\varpi \in \Lambda_0$, $w \in W$, and $\varphi \in \Sigma_\epsilon$, it follows that $\dot{V}_i^\varphi \le 0$ on the boundary $V_i^\varphi = \epsilon_i^2\vartheta_i$ for the choice $\vartheta_i > 4\;\|M_iB_i\|^2\;\lambda_{max}(M_i)\;\Xi_i{}^2$. [4]

[4]Though we only require equality, i.e., $\vartheta_i = 4\;\|M_iB_i\|^2\;\lambda_{max}(M_i)\;\Xi_i{}^2$ to show that $\dot{V}_i^\varphi \le 0$, we replace it with the strict inequality above, in order to arrive at the succeeding inequality in the next step of the analysis.

Hence, $\Psi_c \times \Sigma_\epsilon$ is positively invariant.

Our next step is to show that for any bounded $\hat{e}(0)$, and any $(z(0), e(0), \sigma(0)) \in \Omega_b$, where $0 < b_i < c_i$, it is possible to choose ϵ_i such that the trajectory enters the set $\Psi_c \times \Sigma_\epsilon$ in finite time. Since, for all $(z, e, \sigma) \in \Omega_c$, the right-hand side of the slow equation of (2.20) is bounded uniformly in ϵ, for all $(z(0), e(0), \sigma(0)) \in \Omega_b$ there is a finite time T_0, independent of ϵ, such that for all $0 \le t \le T_0$, $(z(t), e(t), \sigma(t)) \in \Omega_c$. During this interval, using the definition of ϑ_i, we have

$$\dot{V}_i^\varphi \le -\alpha_\varphi^i \|\varphi^i\|^2 \;\; \text{for } V_i^\varphi(\varphi^i) \ge \epsilon_i^2 \vartheta_i$$

for some positive constant α_φ^i. This inequality can be used to show that $\varphi^i(t)$ enters Σ_{ϵ_i} within the time interval $[0, T(\epsilon)]$, where $\lim_{\epsilon \to 0} T(\epsilon) = 0$ [5]. Therefore, by choosing ϵ_i small enough we can ensure that $T(\epsilon) < T_0$.

The argument that the set $\Psi_c \times \Sigma_\epsilon$ is positively invariant can be extended (see Appendix A) to show that trajectories starting inside it reach the set $\Psi_\mu \times \Sigma_\epsilon$ in finite time and stay there for all future time, where

$$\left.\begin{array}{lcl} \Psi_\mu & \stackrel{\text{def}}{=} & \Omega_\mu \times \{V_z(t, z, d) \le \lambda_4(\|\mu\|_\infty r^*)\} \\ \Omega_\mu & \stackrel{\text{def}}{=} & \prod_{i=1}^m \{(e^i, \sigma_i) : |s_i| \le \mu_i(1 - \varsigma_i),\ |\sigma_i| \le \frac{\mu_i}{k_0^i},\ V_i^\zeta(\zeta^i) \le 16\mu_i^2 \chi_i\} \\ \Sigma_\epsilon & \stackrel{\text{def}}{=} & \prod_{i=1}^m \{\varphi^i \in R^{\rho_i} : V_i^\varphi(\varphi^i) \le \|\epsilon\|_\infty^2 \vartheta_i\} \end{array}\right\} \quad (2.22)$$

$0 < \varsigma_i < 1$ is chosen such that $\max \varrho_i \le q_i/(2\varsigma_i) - q_i$, ϵ_i is small enough that $|N_i(\epsilon_i)\varphi^i| < \varsigma_i\ \mu_i$, and $r^* > \alpha^*$, where α^* is a positive constant such that $\|e\| \le \|\mu\|_\infty \alpha^*$ for all $e \in \Omega_\mu$. An argument similar to the one for $\Psi_c \times \Sigma_\epsilon$ can be used to show that $\Psi_\mu \times \Sigma_\epsilon$ is positively invariant. This completes the first part of the analysis.

To prove the second part, note that when $\tilde{w} = 0$, $\nu = 0$ and $\varpi = 0$, the system has a unique equilibrium point $(z = 0, e = 0, \sigma_i = \bar{\sigma}_i, \varphi = 0)$. Let $\bar{s}_i = k_0^i \bar{\sigma}_i$ be the corresponding equilibrium value of s_i, $\tilde{\sigma} = \sigma - \bar{\sigma}$, and $\tilde{s} = s - \bar{s}$. By the converse

Lyapunov theorem [47], Assumption 2.4(ii) implies that in some neighborhood of $z = 0$ there is a Lyapunov function $V_{zz}(z, d)$ that satisfies

$$\lambda_5\|z\|^2 \leq V_{zz} \leq \lambda_6\|z\|^2, \ (\partial V_{zz}/\partial z)\phi(z, 0, d) \leq -\lambda_7\|z\|^2, \text{ and } \|\partial V_{zz}/\partial z\| \leq \lambda_8\|z\| \tag{2.23}$$

for some positive constants λ_5 to λ_8, independent of d. Let $Q = \text{blockdiag}[Q_1, \cdots, Q_m]$, and $P = \text{blockdiag}[P_1, \cdots, P_m]$. Using

$$V = V_{zz}(z, d) + \lambda_9\zeta^T Q\zeta + \frac{1}{2}\lambda_{10}\|\tilde{\sigma}\|^2 + \frac{1}{2}\|\tilde{s}\|^2 + \varphi^T P\varphi \tag{2.24}$$

as a Lyapunov function candidate, where $\lambda_9, \lambda_{10} > 0$, it can be verified that (see Appendix B), by first taking λ_9 large enough, then λ_{10} large enough, then $\|\mu\|_\infty$ small enough, and lastly $\|\epsilon\|_\infty$ small enough, $\dot{V}$ satisfies an inequality of the form

$$\dot{V} \leq -\lambda_{11}V + \lambda_{12}\sqrt{V}(\|\nu(t)\| + \|\varpi(t)\| + \|\tilde{w}(t)\|) \tag{2.25}$$

for some positive constants λ_{11} and λ_{12}, uniformly in μ and ϵ. Since $\tilde{w}$, $\nu(t)$, $\varpi(t) \to 0$ as $t \to \infty$, the preceeding inequality can be used to show that all trajectories approach the equilibrium point $(z = 0, e = 0, \sigma = \bar{\sigma}, \varphi = 0)$ as t tends to infinity. If all assumptions hold globally, the controller can achieve semiglobal regulation. We summarize our conclusions in the following theorem.

Theorem 2.1 *Suppose Assumptions 2.1 through 2.6 are satisfied, the constants c_i, χ_i, ϑ_i, and l_3 are chosen as described before, $\hat{e}(0)$ is bounded, and the initial states $(z(0), e(0), \sigma(0))$ belong to the set Ψ_b, where $0 < b_i < c_i$. Then, there exists $\mu^* > 0$, and for each μ with $\|\mu\|_\infty \in (0, \mu^*]$, there exists $\epsilon^* = \epsilon^*(\mu) > 0$, such that, for $\mu_i \in (0, \mu^*]$ and $\epsilon_i \in (0, \epsilon^*]$, all state variables of the closed-loop system under the output feedback controller (2.15)-(2.19) are bounded, and $\lim_{t\to\infty} e(t) = 0$. If, in addition, all the assumptions hold globally, then, given compact sets $\mathcal{N} \subset R^n$ and*

$\mathcal{M} \subset R^{\rho}$, *the foregoing conclusion holds for all* $(z(0), e(0)) \in \mathcal{N}$ *and* $\hat{e}(0) \in \mathcal{M}$, *provided* Ψ_b *is chosen large enough to include* $\mathcal{N}$.

2.5.2 Performance

We saw in Section 2.1, via simulation, that the output feedback continuous SMC with a conditional integrator recovers the performance of the state feedback ideal SMC. The following theorem shows that the closed-loop trajectories under the two controllers can be made arbitrarily close.

Theorem 2.2 *Let* $X = (z, e)$ *be part of the state of the closed-loop system for (2.6) under the output feedback continuous SMC (2.15)-(2.19), and* $X^* = (z^*, e^*)$ *be the state of the closed-loop system under the state feedback ideal SMC control (2.7), (2.9), with* $X(0) = X^*(0)$. *Then, under the hypotheses of Theorem 2.1, for every* $\tau > 0$, *there exists* $\mu^* > 0$, *and for each* μ *with* $\|\mu\|_\infty \in (0, \mu^*]$, *there exists* $\epsilon^* = \epsilon^*(\mu) > 0$, *such that, for* $\mu_i \in (0, \mu^*]$ *and* $\epsilon_i \in (0, \epsilon^*]$, $\|X(t) - X^*(t)\| \leq \tau \ \forall \ t \geq 0$.

Proof. We prove the theorem in two parts. First, we look at the trajectories under state feedback continuous SMC with the conditional integrator. Let $X^\dagger = (z^\dagger, e^\dagger)$ be part of the state of the closed-loop system under the control (2.10)-(2.12), with $X^\dagger(0) = X^*(0)$. For this case, we show that, for sufficiently small μ_i, $X^\dagger(t) - X^*(t) = O(\|\mu\|_\infty) \ \forall \ t \geq 0$. Let $s^\dagger$ and s^* be the corresponding sliding surface functions of the two systems. Let $\mathcal{I}_\mathcal{M} = \{1, \cdots, m\}$ and $t_1 = \min\{t_1^\dagger, t_1^*\}$, where

$$t_1^\dagger = \min_{i \in \mathcal{I}_\mathcal{M}} \{t : |s_i^\dagger(t)| \leq \mu_i\} \text{ and } t_1^* = \min_{i \in \mathcal{I}_\mathcal{M}} \{t : |s_i^*(t)| = 0\}$$

If $t_1 > 0$, using $\text{sat}(s_i^\dagger(t)/\mu_i) = \text{sgn}(s_i^*(t)) \ \forall \ 0 \leq t < t_1$, it can be shown that $X^\dagger(t) = X^*(t) \ \forall \ 0 \leq t \leq t_1$. Next, we consider $X^\dagger(t)$ and $X^*(t)$ in the time interval $t \geq t_1$. Let

$$I_1 = \{i : |s_i^\dagger(t_1)| \leq \mu_i\} \cup \{i : s_i^*(t_1) = 0\}$$

Since $X^{\dagger}(t_1) = X^{*}(t_1)$, $|s_i^{\dagger}(t_1) - s_i^{*}(t_1)| = |k_0^i\ \sigma_i^{\dagger}(t_1)| \leq \mu_i$. Using this, along with the definition of I_1 and the fact that $|s_i^{\dagger}(t)|$ and $|s_i^{*}(t)|$ monotonically converge to the positively invariant sets $\{|s_i^{\dagger}| \leq \mu_i\}$ and $\{0\}$ respectively, it can be shown that for all $i \in I_1$, $|s_i^{\dagger}(t) - s_i^{*}(t)| \leq 3\mu_i$ for all $t \geq t_1$. It follows that for all $i \in I_1$, $s_i^{\dagger}(t) - s_i^{*}(t) = O(\mu_i)\ \forall\ t \geq 0$. Since the equations for $\zeta^{i\dagger}$ and ζ^{i*} are identical stable linear equations, driven by inputs $s_i^{\dagger} - k_0^i\ \sigma_i^{\dagger}$ and s_i^{*} respectively, where $|k_0^i\ \sigma_i^{\dagger}| \leq \mu_i$ and $s_i^{\dagger} - s_i^{*} = O(\mu_i)$, continuity of solutions on the infinite time interval [47, Theorem 9.1] can be used to show that for sufficiently small μ_i, $\zeta^{i\dagger}(t) - \zeta^{i*}(t) = O(\mu_i)$ and hence $e^{i\dagger}(t) - e^{i*}(t) = O(\mu_i)$ for all $i \in I_1$ and $t \geq t_1$. In particular, if $I_1 = \mathcal{I}_{\mathcal{M}}$, then $e^{\dagger}(t) - e^{*}(t) = O(\|\mu\|_{\infty})$ for all $t \geq t_1$, which can then be used to show that $z^{\dagger}(t) - z^{*}(t) = O(\|\mu\|_{\infty})$ for all $t \geq t_1$, so that the result follows.

If $I_1 \neq \mathcal{I}_{\mathcal{M}}$, let $t_2 \in (t_1, \infty) = \min\{t_2^{\dagger}, t_2^{*}\}$, where

$$t_2^{\dagger} = \min_{i \in \mathcal{I}_{\mathcal{M}} \backslash I_1} \{t : |s_i^{\dagger}(t)| \leq \mu_i\} \text{ and } t_2^{*} = \min_{i \in \mathcal{I}_{\mathcal{M}} \backslash I_1} \{t : |s_i^{*}(t)| = 0\}$$

Let $X_1^{\dagger}$ be the part of the state $X^{\dagger}$ with the components $e^i, i \in I_1$, deleted and X_1^{*} be the corresponding part of X^{*}. For $i \in \mathcal{I}_{\mathcal{M}} \backslash I_1$, we have $\mathrm{sat}(s_i^{\dagger}/\mu_i) = \mathrm{sgn}(s_i^{*}) = \mathrm{sign}(s_i^{\dagger})\ \forall\ t_1 \leq t < t_2$, so that, during this period, the right-hand side of the equations for $X_1^{\dagger}$ and X_1^{*} are Lipschitz functions of their arguments. Viewing $X_1^{\dagger}$ and X_1^{*} as states of systems driven by inputs $(\sigma_i^{\dagger}, e^{i\dagger})$ and e^{i*} respectively, $i \in I_1$, and using the fact that $|k_0^i\ \sigma_i^{\dagger}| \leq \mu_i$, and $e^{i\dagger} - e^{i*} = O(\mu_i)$, the results of [47, Theorem 3.4], dealing with continuity of solutions on compact time-intervals, can be used to show that, for sufficiently small μ_i, $X_1^{\dagger}(t) - X_1^{*}(t) = O(\|\mu\|_{\infty})\ \forall\ t_1 \leq t \leq t_2$. Using this, the previous arguments involving [47, Theorem 9.1] can then be repeated to show that, for sufficiently small μ_i, $e^{i\dagger}(t) - e^{i*}(t) = O(\mu_i)\ \forall\ t \geq t_2$ and all $i \in I_2$, where

$$I_2 = \{i \in \mathcal{I}_{\mathcal{M}} \backslash I_1 : s_i^{\dagger}(t_2)| \leq \mu_i\} \cup \{i \in \mathcal{I}_{\mathcal{M}} \backslash I_1 : s_i^{*}(t_2) = 0\}$$

In particular, if $I_1 \cup I_2 = \mathcal{I}_{\mathcal{M}}$, then $e^\dagger(t) - e^*(t) = O(\|\mu\|_\infty)\ \forall\ t \geq t_2$, which can then be used to show that $z^\dagger(t) - z^*(t) = O(\|\mu\|_\infty)\ \forall\ t \geq t_2$, so that the result follows. If $I_1 \cup I_2 \neq \mathcal{I}_{\mathcal{M}}$, the result follows by an inductive argument that uses [47, Theorem 3.4] and [47, Theorem 9.1] alternately. In particular, this completes the first part of the proof, which shows that there exists $\mu^* > 0$ such that $\|\mu\|_\infty \in (0, \mu^*] \Rightarrow \|X^\dagger(t) - X^*(t)\| \leq \tau/2\ \forall\ t \geq 0$.

In the second part of the proof, we use the idea in [5] to show that the trajectories X of the system under output feedback approach the trajectories $X^\dagger$ under state feedback as $\epsilon \to 0$. In particular, we show that there exists $\epsilon^* = \epsilon^*(\mu)$ such that for all $\epsilon_i \leq \epsilon^*$, $\|X(t) - X^\dagger(t)\| \leq \tau/2\ \forall\ t \geq 0$. This is done by dividing the time interval $[0, \infty)$ into three sub-intervals $[0, T(\epsilon)]$, $[T(\epsilon), T_3]$ and $[T_3, \infty)$ and showing that the inequality $\|X(t) - X^\dagger(t)\| \leq \tau/2$ holds over each of these sub-intervals. From asymptotic stability of the two systems, we know that there exists a finite time T_3, independent of ϵ, such that $\|X(t) - X^\dagger(t)\| \leq \tau/2\ \forall\ t \geq T_3$. Also, as mentioned in Section 5.1, there is a time interval $[0, T(\epsilon)]$, with $T(\epsilon) \to 0$ as $\epsilon \to 0$, during which the fast variable φ decays to an $O(\|\epsilon\|_\infty)$ value. It can be shown that global boundedness of the controls implies that over this interval, $\|X(t) - X^\dagger(t)\| \leq \lambda_0 T(\epsilon)$, for some positive constant λ_0 that is independent of ϵ. Since $T(\epsilon) \to 0$ as $\epsilon \to 0$, for small enough $\|\epsilon\|_\infty$, $\|X(t) - X^\dagger(t)\| \leq \tau/2\ \forall\ t \in [0, T(\epsilon)]$. Lastly, noting that $X(T(\epsilon)) - X^\dagger(T(\epsilon)) \to 0$ as $\epsilon \to 0$ and φ is $O(\|\epsilon\|_\infty)$, and using the continuous dependence of the solutions of differential equations on compact time intervals [47, Theorem 3.4], one can show that it is possible to choose ϵ to satisfy the inequality $\|X(t) - X^\dagger(t)\| \leq \tau/2$ over the time interval $[T(\epsilon), T_3]$. This shows that $\|X(t) - X^\dagger(t)\| \leq \tau/2\ \forall\ t \geq 0$. The conclusion of Theorem 2 then follows from the triangle inequality.

2.6 Universal Integral Regulator

For SISO systems, the flexibility that is available in the choice of the functions $\hat{F}$ and β can be exploited to simplify the controller to

$$u = -k \text{ sat}(\hat{s}/\mu) = -k \text{ sat}\left(\frac{k_0\sigma + k_1 e_1 + k_2\hat{e}_2 + \cdots + \hat{e}_\rho}{\mu}\right) \tag{2.26}$$

As mentioned in Section 2.1, this particular design, while having a simple structure, is also natural since the control is required to be bounded. It is clear from (2.26) that the only precise knowledge about the plant that is used is its relative degree and the sign of its high-frequency gain $L_g L_f^{\rho-1} h$. This "universal" design was first presented in [46], for the conventional integrator, where was shown that the structure of the universal integral regulator coincides with the classical PI and PID controllers, followed by saturation, for relative degree one and two plants, respectively.

In the present case, when $\rho = 1$, the integrator equation can be rewritten as

$$\dot{\sigma} = e_1 + (\mu/k)(\hat{u} - u)$$

where

$$u = -k \text{ sat}\left(\frac{k_0\sigma + e_1}{\mu}\right) \text{ and } \hat{u} = -k\left(\frac{k_0\sigma + e_1}{\mu}\right)$$

The term $\hat{u}$ is the "unsaturated version" of the control u, so that the controller (2.26) has the structure shown in Figure 2.5. It is a PI controller with "anti-windup" [25], followed by saturation.

In the relative degree ρ case, the integrator equation can be rewritten as

$$\dot{\sigma} = e_a + (\mu/k)(\hat{u} - u)$$

where $e_a = \sum_{j=1}^{\rho-1} k_j e_j + e_\rho$ is the augmented error that was defined in Section 2.3. It is

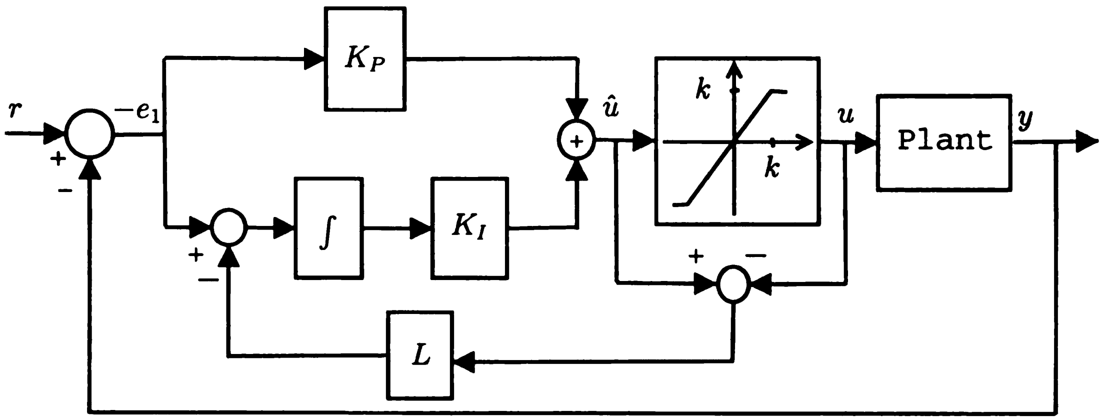

Figure 2.5: Universal regulator for relative degree one systems : PI controller with anti-windup, followed by saturation; $K_I = kk_0/\mu$, $K_P = k/\mu$, and $L = \mu/k$.

clear from the expression for $\dot{\sigma}$ that the anti-windup structure of Figure 2.5 is retained in this case as well. The control (2.26) now represents a "PID$^{\rho-1}$ controller", with a conditional (anti-windup) integrator, followed by saturation. This interpretation of the conditional integrator as a specially tuned version of an anti-windup scheme for the universal integral regulator design was presented in [70].

2.7 Conclusions

We have presented a new approach to introducing integral action in the control of nonlinear systems, which captures the regional and semi-global asymptotic regulation results of [46] and [57], while improving the transient response. In the new approach, the integrator is designed in such a way that it provides integral action only "conditionally", effectively eliminating the performance degradation. The improvement in performance is demonstrated analytically by showing that the output-feedback continuous sliding mode controller, with conditional integrator, recovers the performance of an ideal state-feedback sliding mode controller, without integral action, as the controller parameters μ_i and ϵ_i tend to zero. In view of this result, the control design can start with the ideal state-feedback sliding mode control, where the parameters of the sliding surface $s_i^* = 0$ are chosen to meet the transient response specifications. Then, integral action is introduced by modifying s_i^* to $s_i = k_i^0\sigma_i + s_i^*$, with $\dot{\sigma}_i = -k_i^0\sigma_i + \mu_i \operatorname{sat}(s_i/\mu_i)$. The discontinuous term $\operatorname{sgn}(s_i^*)$ in the ideal SMC is replaced by $\operatorname{sat}(s_i/\mu_i)$. The parameters μ_i are reduced gradually until the transient response is close enough to the ideal case. Finally, a high-gain observer is brought in to estimate the derivatives of the tracking error. The observer parameters ϵ_i are gradually reduced until the transient performance is close enough to the ideal case. Note that in the ideal SMC design, the inequality that corresponds to (2.14) will have Δ_i terms that do not account for the σ_i variables. However, since σ_i is $O(\mu_i)$, the β_i's of the ideal SMC design will still work in the presence of the conditional integrator, provided the μ_i's are sufficiently small.

While modifying the integral control designs of [46] and [57] from conventional to conditional integrators, we have also extended the problem statement to MIMO systems and allowed time-varying matched disturbances. Moreover, we proved that the trajectories under output feedback approach those under state feedback as $\epsilon \to 0$. This property also holds for [46] and [57], but was not proved there.

Appendix A Derivation of (2.22)

The arguments of Section 2.4.1 show that $\dot{V}_i^s < 0$ whenever $|s_i| \geq \mu_i$. Suppose $\mu_i(1 - \varsigma_i) \leq |s_i| \leq \mu_i$, so that

$$\dot{V}_i^s \leq -\frac{\gamma_i(\cdot)\beta_i(\cdot)|s_i|^2}{\mu_i} + |\Delta_i(\cdot)||s_i| + |\Delta_i^*(\cdot)||s_i|$$

Using Assumption 2.6 and the definition of $\beta_i(\cdot)$, it can be shown that

$$-\frac{\gamma_i(\cdot)\beta_i(\cdot)|s_i|^2}{\mu_i} + |\Delta_i(\cdot)||s_i| \leq -\frac{\gamma_0 q_i |s_i|}{2}$$

for $\mu_i(1 - \varsigma_i) \leq |s_i| \leq \mu_i$, provided ς_i is small enough that $\max \varrho_i \leq q_i/(2\varsigma_i) - q_i$. Choosing ϵ_i small enough that $|\Delta_i^*(\cdot)| < \gamma_0 q_i/2$, it follows that $\dot{V}_i^s < 0$ whenever $|s_i| \geq \mu_i(1 - \varsigma_i)$, which shows that $s_i(t)$ reaches the set $\{|s_i| \leq \mu_i(1 - \varsigma_i)\}$ in finite time and stays there for all future time. Thereafter,

$$|s_i| + k_0^i|\sigma_i| \leq 2\mu_i$$

which along with the inequality

$$\dot{V}_i^\zeta \leq -\|\zeta^i\|^2 + 2\|\zeta^i\| \; \|Q_i C_i\| \; (|s_i| + k_0^i|\sigma_i|)$$

and the definition of χ_i, can be used to show that $\dot{V}_i^\zeta \leq -\|\zeta^i\|^2/2$, whenever $V_i^\zeta \geq 16\mu_i^2\chi_i$. This shows that $\zeta^i(t)$ reaches the set $\{V_i^\zeta \leq 16\mu_i^2\chi_i\}$ in finite time and stays therein for all future time. Next, we note that $e \in \Omega_\mu$ implies that $\|e\| \leq \|\mu\|_\infty \alpha^*$, where α^* is a positive constant independent of μ. Since $\lim_{t\to\infty}\nu(t) = 0$, it follows that there is a finite time after which $\|e + \nu\| \leq r^*\|\mu\|_\infty$, where r^* is any positive constant that satisfies $r^* > \alpha^*$. From Assumption 2.4(i) and the definition of λ_4, it follows that $\dot{V}_z \leq -\lambda_3(\|z\|)$ for $V_z \geq \lambda_4(\|\mu\|_\infty r^*)$, which shows that $z(t)$ reaches the set $\{V_z \leq \lambda_4(\|\mu\|_\infty r^*)\}$ in finite time and stays therein. Lastly, the fact that $\varphi^i(t)$ reaches the set $\{V_i^\varphi(\varphi^i) \leq \|\epsilon\|_\infty^2 \vartheta_i\}$ in finite time and stays therein was already shown in Section 2.4.1.

This completes the proof of the statement that every trajectory starting inside the set $\Psi_c \times \Sigma_\epsilon$ enters the set $\Psi_\mu \times \Sigma_\epsilon$ in finite time and stays therein for all future time.

Appendix B Derivation of (2.25)

The derivative of V of (2.24) is given by

$$\begin{aligned}\dot{V} =& \frac{\partial V_{zz}}{\partial z}\phi(z, e+\nu, d) \\ &- \lambda_9\|\zeta\|^2 + 2\lambda_9\sum_{i=1}^{m}\zeta^{i^T}Q_iC_i(\tilde{s}_i - k_0^i\tilde{\sigma}_i) \\ &- \lambda_{10}\sum_{i=1}^{m}k_0^i\tilde{\sigma}_i^2 + \lambda_{10}\sum_{i=1}^{m}\tilde{\sigma}_i\ (\tilde{s}_i - N_i(\epsilon_i)\varphi^i) \\ &+ \sum_{i=1}^{m}\tilde{s}_i\left[-\frac{\gamma_i(\cdot)\beta_i(\cdot)s_i}{\mu_i} + \Delta_i(\cdot) + \Delta_i^*(\cdot)\right] \\ &\sum_{i=1}^{m} -\frac{\|\varphi^i\|^2}{\epsilon_i} + 2\sum_{i=1}^{m}(\varphi^i)^TP_iB_i\ [b_i(\cdot) - r_i^{(\rho_i)} + \sum_{j=1}^{m}a_{ij}(\cdot)(u_j + \delta_j(\cdot))]\end{aligned} \tag{2.27}$$

We arrange (2.27) in a quadratic form of $\Pi = [\|z\|\ \|\zeta\|\ \|\tilde{\sigma}\|\ \|\tilde{s}\|\ \|\varphi\|]^T$. From (2.23), we have

$$\begin{aligned}\frac{\partial V_{zz}}{\partial z}\phi(z, e+\nu, d) &= \frac{\partial V_{zz}}{\partial z}\phi(z, 0, d) + \frac{\partial V_{zz}}{\partial z}[\phi(z, e+\nu, d) - \phi(z, 0, d)] \\ &\leq -\lambda_7\|z\|^2 + \lambda_8L_\phi\|z\|(\|e\| + \|\nu\|)\end{aligned}$$

where L_ϕ is the Lipschitz constant of $\phi(z, \cdot, d)$. Using $e = \{e^i\}$, where $e^i = [(\zeta^i)^Te_{\rho_i}^i]^T$, $e_{\rho_i}^i = \tilde{s}_i - k_0^i\tilde{\sigma}_i - K^i\zeta^i$, and $K^i = [k_1^i, \cdots, k_{\rho_i-1}^i]$, it can be verified that

$$\|e\| \leq \lambda_{13}\|\zeta\| + \lambda_{14}\|\tilde{\sigma}\| + \lambda_{15}\|\tilde{s}\| \tag{2.28}$$

for some positive constants λ_{13} to λ_{15}, so that the first term of (2.27) satisfies

$$\frac{\partial V_{zz}}{\partial z}\phi(z, e+\nu, d) \leq -\lambda_7\|z\|^2 + \lambda_8L_\phi\|z\|(\lambda_{13}\|\zeta\| + \lambda_{14}\|\tilde{\sigma}\| + \lambda_{15}\|\tilde{s}\| + \|\nu\|) \tag{2.29}$$

It is easily verified that the second and third terms of (2.27) satisfy

$$-\lambda_9\|\zeta\|^2 + 2\lambda_9 \sum_{i=1}^{m} \zeta^{i^T} Q_i C_i(\tilde{s}_i - k_0^i \tilde{\sigma}_i) \leq -\lambda_9\|\zeta\|^2 + 2\lambda_9\|\zeta\|(\lambda_{16}\|\tilde{\sigma}\| + \lambda_{17}\|\tilde{s}\|) \quad (2.30)$$

and

$$-\lambda_{10}\sum_{i=1}^{m} k_0^i\tilde{\sigma}_i^2 + \lambda_{10}\sum_{i=1}^{m} \tilde{\sigma}_i\,(\tilde{s}_i - N_i(\epsilon_i)\varphi^i) \leq -\lambda_{10}\bar{k}_0\|\tilde{\sigma}\|^2 + \lambda_{10}\|\tilde{\sigma}\|(\|\tilde{s}\| + \lambda_{18}\|N(\epsilon)\|_\infty\|\varphi\|) \quad (2.31)$$

respectively, for some positive constants λ_{16} to λ_{18}, where $\bar{k}_0 = \min_i k_0^i$, and $N(\epsilon) = \{N_i(\epsilon_i)\}$. Note that $\lambda_{18}\|N(\epsilon)\|_\infty$ can be replaced by any constant $\tilde{\lambda}_{18}$ such that $\tilde{\lambda}_{18} > \lambda_{18}$ by making $\|\epsilon\|_\infty$ sufficiently small, so that by redefining λ_{18}, the right hand side of inequality (2.31) can be replaced simply by $-\lambda_{10}\bar{k}_0\|\tilde{\sigma}\|^2 + \lambda_{10}\|\tilde{\sigma}\|(\|\tilde{s}\| + \lambda_{18}\|\varphi\|)$, and we use this idea in the definition of some of the constants later on.

For the fourth term of (2.27), the idea is to rearrange expressions by adding and subtracting appropriate terms. We show this in detail for one expression, and a similar procedure applies to the others. For example

$$\begin{aligned}
\gamma_i(z, e+\nu, d)\beta_i(e, \nu, \varpi) = {} & \gamma_i(z, e+\nu, d)\beta_i(e, \nu, \varpi) - \gamma_i(0, e+\nu, d)\beta_i(e, \nu, \varpi) \\
& + \gamma_i(0, e+\nu, d)\beta_i(e, \nu, \varpi) - \gamma_i(0, \nu, d)\beta_i(0, \nu, \varpi) \\
& + \gamma_i(0, \nu, d)\beta_i(0, \nu, \varpi) - \gamma_i(0, 0, d)\beta_i(0, 0, \varpi) \\
& + \gamma_i(0, 0, d)\beta_i(0, 0, \varpi) - \gamma_i(0, 0, d)\beta_i(0, 0, 0) \\
& + \gamma_i(0, 0, d)\beta_i(0, 0, 0)
\end{aligned}$$

Consequently, using (2.28), it can be verified that $\widetilde{\gamma_i(\cdot)\beta_i}(\cdot) \overset{\text{def}}{=} \gamma_i(z, e+\nu, d)\beta_i(e, \nu, \varpi) - \gamma_i(0, 0, d)\beta_i(0, 0, 0)$ satisfies

$$|\widetilde{\gamma_i(\cdot)\beta_i}(\cdot)| \leq \lambda_{19}\|z\| + \lambda_{20}\|\zeta\| + \lambda_{21}\|\tilde{\sigma}\| + \lambda_{22}\|\tilde{s}\| + \lambda_{23}\|\nu\| + \lambda_{24}\|\varpi\|$$

for some positive constants λ_{19} through λ_{24}. Likewise, $\tilde{\Delta}_i(\cdot) \stackrel{\text{def}}{=} \Delta_i(z, e, \varpi, \sigma, d, \tilde{w}) - \Delta_i(0, 0, 0, \bar{\sigma}, d, 0)$ satisfies an inequality similar to the one above, except that $\|\nu\|$ is replaced by $\|\tilde{w}\|$. Rewriting

$$
\begin{aligned}
-\frac{\gamma_i(\cdot)\beta_i(\cdot)s_i}{\mu_i} + \Delta_i(\cdot) = &- \frac{\gamma_i(\cdot)\beta_i(\cdot)\tilde{s}_i}{\mu_i} - \frac{\widetilde{\gamma_i(\cdot)\beta_i}(\cdot)\bar{s}_i}{\mu_i} + \tilde{\Delta}_i(\cdot) \\
&- \frac{\gamma_i(0,0,d)\beta_i(0,0,0)\bar{s}_i}{\mu_i} + \Delta_i(0,0,0,\bar{\sigma},d,0)
\end{aligned}
$$

and using, from (2.13), the fact that $\gamma_i(0,0,d)\beta_i(0,0,0)(\bar{s}_i/\mu_i) = \Delta_i(0,0,0,\bar{\sigma},d,0)$, we have

$$
\tilde{s}_i\left[-\frac{\gamma_i(\cdot)\beta_i(\cdot)s_i}{\mu_i} + \Delta_i(\cdot) + \Delta_i^*(\cdot)\right] \le -\frac{\gamma_i(\cdot)\beta_i(\cdot)\tilde{s}_i^2}{\mu_i} - \frac{\widetilde{\gamma_i(\cdot)\beta_i}(\cdot)\bar{s}_i\tilde{s}_i}{\mu_i} + \tilde{\Delta}_i(\cdot)\tilde{s}_i + \Delta_i^*(\cdot)\tilde{s}_i
$$

Finally, using the facts that $|\bar{s}_i| \le \mu_i$, $\gamma_i(\cdot) \ge \gamma_0$, $\beta(\cdot) \ge q_0 \stackrel{\text{def}}{=} \min_i q_i$, and that inside the set $\Psi_\mu \times \Sigma_\epsilon$

$$
\Delta_i^*(\cdot) = -k_0^i N_i(\epsilon_i)\varphi^i + \sum_{j=1}^{m} a_{ij}(z, e+\nu, d)\ [\Upsilon_j(\hat{e}, \nu, \varpi, \sigma) - \Upsilon_j(e, \nu, \varpi, \sigma)]
$$

it can be verified that the fourth term of (2.27) satisfies

$$
\begin{aligned}
\sum_{i=1}^{m} \tilde{s}_i\left[-\frac{\gamma_i(\cdot)\beta_i(\cdot)s_i}{\mu_i} + \Delta_i(\cdot) + \Delta_i^*(\cdot)\right] &\le -\left(\frac{\gamma_0 q_0}{\|\mu\|_\infty} - \lambda_{25}\right)\|\tilde{s}\|^2 \\
+ \|\tilde{s}\|\ (\lambda_{26}\|z\| + \lambda_{27}\|\zeta\| &+ \lambda_{28}\|\tilde{\sigma}\| + \lambda_{29}\|\varphi\| + \lambda_{30}\|\nu\| + \lambda_{31}\|\varpi\| + \lambda_{32}\|\tilde{w}\|)
\end{aligned} \tag{2.32}
$$

for some positive constants λ_{25} through λ_{32}.

Noting that

$$
b_i(\cdot) - r_i^{(\rho_i)} + \sum_{j=1}^{m} a_{ij}(\cdot)(u_j + \delta_j(\cdot)) = \dot{\tilde{s}}_i - k_0^i\dot{\tilde{\sigma}}_i - k_{\rho_i-1}^i(\tilde{s}_i - k_0^i\tilde{\sigma}_i) - \tilde{K}^i\zeta^i
$$

where $\tilde{K}^i = k_{\rho_i-1}^i K^i - [0\ k_1^i\ \cdots\ k_{\rho_i-2}^i]$, and using the results of the previous discussion, it can be verified that the final term of (2.27) satisfies

$$\sum_{i=1}^{m} -\frac{\|\varphi^i\|^2}{\epsilon_i} + 2\sum_{i=1}^{m}(\varphi^i)^T P_i B_i \left[b_i - r_i^{(\rho_i)} + \sum_{j=1}^{m} a_{ij}(u_j+\delta_j)\right] \leq -\left(\frac{1}{\|\epsilon\|_\infty} - \lambda_{33}\right)\|\varphi\|^2$$
$$+ \|\varphi\|(\lambda_{34}\|z\| + \lambda_{35}\|\zeta\| + \lambda_{36}\|\tilde{\sigma}\| + \lambda_{37}\|\tilde{s}\| + \lambda_{38}\|\nu\| + \lambda_{39}\|\varpi\| + \lambda_{40}\|\tilde{w}\|) \tag{2.33}$$

for some positive constants λ_{33} through λ_{40}.

From (2.29) to (2.33), it follows that the derivative of V can be arranged in the form

$$\dot{V} \leq -\Pi^T \mathcal{P} \Pi + \lambda_{41}\|\Pi\|(\|\nu\| + \|\varpi\| + \|\tilde{w}\|) \tag{2.34}$$

where λ_{41} is a positive constant, and the symmetric matrix $\mathcal{P}$ has the form

$$\mathcal{P} = \begin{bmatrix} \lambda_7 & -\lambda_{1a} & -\lambda_{1b} & -\lambda_{1c} & -\lambda_{1d} \\ & \lambda_9 & -\lambda_{2b} & -\lambda_{2c} & -\lambda_{2d} \\ & & \bar{k}_0\lambda_{10} & -\lambda_{3c} & -\lambda_{3d} \\ & & & \frac{\gamma_0 q_0}{\|\mu\|_\infty} - \lambda_{25} & -\lambda_{4d} \\ & & & & \frac{1}{\|\epsilon\|_\infty} - \lambda_{33} \end{bmatrix}$$

where the positive constants λ_{33}, and λ_{1d} to λ_{4d} are independent of ϵ; λ_{25} and λ_{1c} to λ_{3c} are independent of μ; λ_{1b} and λ_{2b} are independent of λ_{10}; and λ_{1a} is independent of λ_9. Therefore, by choosing λ_9 large enough, then λ_{10} large enough, then $\|\mu\|_\infty$ small enough, and lastly $\|\epsilon\|_\infty$ small enough, we can make $\mathcal{P}$ positive definite. Given this fact, the equivalence between (2.34) and (2.25) follows easily from (2.23) and (2.24).

Chapter 3

Global Regulation under State Feedback

3.1 Introduction

In the previous chapter, we showed that the output feedback continuous sliding mode controller with a conditional integrator could be tuned to achieve semi-global regulation when all the conditions hold globally. However, it does not achieve global regulation. The semi-global result is a limitation of the high-gain observer based design, which requires that the control be globally bounded. In this chapter, under certain additional assumptions, we show that the semi-global result of the output feedback design can be extended to a global result under state feedback.

3.2 Problem Statement

Consider a MIMO nonlinear system, modeled by

$$\left.\begin{aligned} \dot{x} &= f_0(x) + \Delta f(x,\theta) + \textstyle\sum_{i=1}^{m} g_i(x,\theta)[u_i + \delta_i(x,\theta,w)], \\ y_i &= h_i(x),\ 1 \le i \le m \end{aligned}\right\} \tag{3.1}$$

where $x \in R^n$ is the state, $u \in R^m$ is the control input, $y \in R^m$ is the output, θ is a vector of unknown constant parameters that belongs to a compact set $\Theta \subset R^p$, $w(t)$ is a piecewise continuous exogenous signal that belongs to a compact set $W \subset R^q$, $f_0(\cdot)$ and $h_i(\cdot)$ are smooth functions on R^n, $\Delta f(\cdot)$ and $g_i(\cdot)$ are smooth functions on $R^n \times \Theta$, and the disturbances $\delta_i(\cdot)$ are continuous functions on $R^n \times \Theta \times W$. The function $f_0(x)$ is assumed to be precisely known, so that the uncertainty is lumped into the term $\Delta f(x, \theta)$. The output functions $h_i(x)$ are assumed to be precisely known. The uniform vector relative degree assumption of Assumption 2.1 is modified as follows.

Assumption 3.1 *The system*

$$\dot{x} = f_0(x) + g(x, \theta)u, \quad y = h(x)$$

has a strong vector relative degree $\{\rho_1, \rho_2, \ldots, \rho_m\}$ *in* D_x, *i.e.,* $L_{g_j} L_{f_0}^k h_i(x) = 0$ *for* $0 \le k \le \rho_i - 2, 1 \le i \le m, 1 \le j \le m$, *and* $A(x, \theta) \stackrel{\text{def}}{=} \{L_{g_j} L_{f_0}^{\rho_i - 1} h_i\}$ *is nonsingular for all* $x \in D_x$ *and* $\theta \in \Theta$. *Furthermore, the distribution* $\text{span}\{g_1, \cdots, g_m\}$ *is involutive, uniformly in* θ, *and there is a change of variables*

$$\begin{bmatrix} \eta \\ \xi \end{bmatrix} = T(x, \theta) = \begin{bmatrix} T_1(x, \theta) \\ T_2(x) \end{bmatrix}, \quad \eta \in R^{n-\rho}, \ \xi \in R^\rho \tag{3.2}$$

where $\xi = \{\xi^i\}$, *with* $\xi_j^i = L_{f_0}^{j-1} h$, $1 \le j \le \rho_i$, $1 \le i \le m$, *and* $\rho = \rho_1 + \rho_2 + \cdots + \rho_m$, *such that* $L_{g_j} \eta_i = 0 \ \forall \ 1 \le j \le m$, $1 \le i \le n - \rho$, *and* $T(x, \theta)$ *is a global diffeomorphism of* R^n *into* R^n.

Remark 3.1 *The transformation* T_2 *no longer depends on the unknown parameter* θ, *i.e., it is known. Consequently, since the state* x *is available for feedback, it follows that* ξ *is available for feedback.*

Assumption 3.2 *The uncertain term $\Delta f(x,\theta)$ satisfies*

$$\Delta f(x,\theta) \in Ker[dh_i, dL_{f_0}h_i, \cdots, dL_{f_0}^{\rho_i-2}h_i],\ 1 \le i \le m \tag{3.3}$$

Assumption 3.2 places a restriction on $\Delta f(x,\theta)$. It can be verified that the state model (3.1), along with Assumptions 3.1 and 3.2 allow one to work with a class of systems that includes those in which the uncertainty satisfies the generalized matching condition of [65, Chapter 9], as defined in Remark 3.2 below.

Remark 3.2 *Consider the MIMO system*

$$\left.\begin{array}{rcl} \dot{x} &=& f_0(x) + \Delta f(x,\theta) + \sum_{i=1}^{m}[g_i(x) + \Delta g_i(x,\theta)]u_i, \\ y_i &=& h_i(x),\ 1 \le i \le m \end{array}\right\}$$

where the nominal system $\dot{x} = f_0(x) + g(x)u,\ y = h(x)$ has a strong vector relative degree $\{\rho_1, \rho_2, \ldots, \rho_m\}$. The uncertain terms $\Delta f(x,\theta)$ and $\Delta g_i(x,\theta)$ are said to satisfy the generalized matching condition [65, Chapter 9], if, for $1 \le i \le m$,

$$\left.\begin{array}{rcl} L_{\Delta f}L_{f_0}^{j}h_i &=& 0 \text{ for } 0 \le j \le \rho_i - 2, \\ L_{\Delta g_k}L_{f_0}^{j}h_i &=& 0 \text{ for } 0 \le j \le \rho_i - 1,\ 1 \le k \le m. \end{array}\right\}$$

The generalized matching condition of Remark 3.2 is weaker than the matching condition, and therefore the class of sytems in Remark 3.2 includes those where the uncertain terms satisfy the matching condition. It is also clear that the class of systems described by (3.1), and Assumptions 3.1 and 3.2 include the ones described in Remark 3.2. A class of systems and accompanying assumptions, similar to the ones in our work, for the problem of robust output tracking of MIMO systems using sliding mode control, can be found in [23].

Assumptions 3.1 and 3.2 allow (3.1) to be rewritten in the normal form

$$\left.\begin{array}{rcl} \dot{\eta} & = & \phi(\eta, \xi, \theta) \\ \dot{\xi}^i & = & A_i \xi^i + B_i \left[b_i(x) + \Delta b_i(x, \theta) + \sum_{j=1}^{m} a_{ij}(x, \theta)(u_j + \delta_j(x, \theta, w)) \right] \end{array} \right\} \quad (3.4)$$

where, for $1 \leq i \leq m$, the pair (A_i, B_i) is a controllable canonical form that represents a chain of ρ_i integrators, $b_i(\cdot) = L_{f0}^{\rho_i} h_i$, $\Delta b_i(\cdot) = L_{\Delta f} L_{f0}^{\rho_i - 1} h_i$, and $\{a_{ij}(\cdot)\} = A(\cdot)$.

The output y is required to asymptotically track a reference signal $r(t)$. As before, we assume that the exogenous signal $w(t)$ and the reference signal $r(t)$ satisfy the following properties:

- $\lim_{t \to \infty} w(t) = w_{ss}$
- $r_i(t)$ and its derivatives up to the ρ_ith derivative are bounded, and $r_i^{(\rho_i)}(t)$ is piecewise continuous, for all $t \geq 0$
- $\lim_{t \to \infty} r_i(t) = r_{iss}$, and $\lim_{t \to \infty} r_i^{(j)}(t) = 0$ for $1 \leq j \leq \rho_i$.

Define $r_{ss} = \{r_{iss}\}$, $\tilde{w}(t) = w - w_{ss}$, $\nu^i(t) = [r_i - r_{iss}, r_i{}^{(1)}, \cdots, r_i{}^{(\rho_i - 1)}]^T$, $\varpi(t) = \{r_i{}^{(\rho_i)}\}$, $\nu(t) = \{\nu^i\}$, and let $X \subset R^m$, $\Lambda \subset R^\rho$, and $\Lambda_0 \subset R^m$ be compact sets such that $r_{ss} \in X$, $\nu(t) \in \Lambda$, and $\varpi(t) \in \Lambda_0$ for all $t \geq 0$. Set $d = (r_{ss}, \theta, w_{ss})$ and $D_d = X \times \Theta \times W$. Assumption 2.2 is modified as follows.

Assumption 3.3 *For each $d \in D_d$, there exist a unique equilibrium point $\bar{x} = \bar{x}(d)$ and a unique control $\bar{u} = \bar{u}(d)$ such that*

$$\left.\begin{array}{rcl} 0 & = & f_0(\bar{x}) + \Delta f(\bar{x}, \theta) + g(\bar{x}, \theta)[\bar{u} + \delta(\bar{x}, \theta, w_{ss})], \text{ and} \\ r_{ss} & = & h(\bar{x}) \end{array} \right\}$$

With the change of variables (3.2), the equilibrium point $\bar{x}(d)$ maps into $(\bar{\eta}(d), \bar{\xi}(d))$, where $\bar{\xi}^i(d) = [r_{iss}, 0, \cdots, 0]^T$. Let

$$z = \eta - \bar{\eta} \text{ and } e^i = \xi^i - \bar{\xi}^i - \nu^i$$

and rewrite (3.4) as

$$\left.\begin{aligned} \dot{z} &= \phi(z, e+\nu, d) \\ \dot{e}^i &= A_i e^i + B_i \left[b_i(x) + \Delta b_i(x,\theta) - r_i^{(\rho_i)} + \textstyle\sum_{j=1}^{m} a_{ij}(x,\theta)(u_j + \delta_j(x,\theta,\tilde{w}))\right] \end{aligned}\right\} \tag{3.5}$$

Remark 3.3 *Since $e^i = \xi^i - \bar{\xi}^i - \nu^i$, where ξ^i, $\bar{\xi}^i$, and ν^i are known, it follows that e is available for feedback.*

Assumption 2.3 is clearly irrelevant for the global problem, and Assumption 2.4 is modified to hold with class $\mathcal{K}_\infty$ functions $\lambda_i(\cdot)$, $(i = 1, 2, 3)$ and $\gamma(\cdot)$, i.e.,

Assumption 3.4 *There exist a C^1 proper function $V_z : \mathcal{Z} \to R_+$, possibly dependent on d, and class $\mathcal{K}_\infty$ functions $\lambda_i : [0, \infty) \to R_+ (i = 1, 2, 3)$ and $\gamma : [0, \infty) \to R_+$, independent of d, such that*

$$\lambda_1(\|z\|) \le V_z(t, z, d) \le \lambda_2(\|z\|)$$

$$\frac{\partial V_z}{\partial t} + \frac{\partial V_z}{\partial z}\phi(z, e+\nu, d) \le -\lambda_3(\|z\|), \ \forall \ \|z\| \ge \gamma(\|e+\nu\|)$$

for all $e \in \mathcal{E}$, $z \in \mathcal{Z}$, $\nu \in \Lambda$, and $d \in D$. Furthermore, the equilibrium point $z = 0$ of $\dot{z} = \phi(z, 0, d)$ is exponentially stable, uniformly in d.

Assumption 3.4 implies that the system $\dot{z} = \phi(z, e+\nu, d)$ is input-to-state stable (ISS) with $(e + \nu)$ as the driving input [47].

3.3 Controller Design

The control design proceeds exactly as before, i.e., let

$$s_i = k_0^i \sigma_i + \sum_{j=1}^{\rho_i - 1} k_j^i e_j^i + e_{\rho_i}^i \tag{3.6}$$

where the positive constants $k_1^i, \cdots, k_{\rho_i-1}^i$ are chosen such that the polynomial

$$\lambda^{\rho_i - 1} + k_{\rho_i-1}^i \lambda^{\rho_i - 2} + \cdots + k_1^i$$

is Hurwitz, $k_0^i > 0$, and σ_i is the output of

$$\dot{\sigma}_i = -k_0^i \sigma_i + \mu_i \ \mathrm{sat}(s_i/\mu_i), \ \ \sigma_i(0) \in [-\mu_i/k_0^i, \mu_i/k_0^i] \tag{3.7}$$

This results in

$$\dot{s}_i = k_0^i[-k_0^i \sigma_i + \mu_i \ \mathrm{sat}(s_i/\mu_i)] + F_i(x, e^i, r_i^{(\rho_i)}, \theta) + \sum_{j=1}^{m} a_{ij}(\cdot)[u_j + \delta_j(\cdot)]$$

where

$$F_i(\cdot) = b_i(x) + \Delta b_i(x, \theta) - r_i^{(\rho_i)} + \sum_{j=1}^{\rho_i - 1} k_j^i e_{j+1}^i$$

Assumption 3.5 *$A(x,\theta) = \Gamma(x,\theta)\hat{A}(x)$ where $\hat{A}(x)$ is a known nonsingular matrix and $\Gamma = \mathrm{diag}[\gamma_1, \cdots, \gamma_m]$, with $\gamma_i(\cdot) \geq \gamma_0 > 0, \ 1 \leq i \leq m$, for all $x \in R^n$, and $\theta \in \Theta$, for some positive constant γ_0.*

The control u is taken as

$$u = \hat{A}^{-1}(x)[-\hat{F}(x, e, \varpi) + v], \ \ v_i = -\beta_i(x, e, \varpi) \ \mathrm{sat}(s_i/\mu_i) \tag{3.8}$$

where, as before, $\hat{F}(\cdot) = \{\hat{F}_i(\cdot)\}$ is a nominal value of $F(\cdot) = \{F_i(\cdot)\}$, which could be, but not restricted to,

$$\hat{F}_i(\cdot) = b_i(x) - r_i^{(\rho_i)} + \sum_{j=1}^{\rho_i - 1} k_i^j e_i^{j+1}$$

and the component v_i is designed to handle uncertainties.

Remark 3.4 *The state feedback control (3.8) is more general than the one in Section 2.3.1, because it uses full state feedback rather than partial state feedback.*

With the control (3.8), the expression for $\dot{s}_i$ becomes

$$\dot{s}_i = \Delta_i(x, e, \sigma, \theta, \varpi, \tilde{w}) - \gamma_i(x, \theta)\ \beta_i(x, e, \varpi)\ \mathrm{sat}(s_i/\mu_i)$$

where

$$\Delta(\cdot) = F(\cdot) - \Gamma(\cdot)\hat{F}(\cdot) + A(\cdot)\ \delta(\cdot) + \{k_0^i(-k_0^i \sigma_i + \mu_i\ \mathrm{sat}(s_i/\mu_i)\}$$

Assumption 3.6 *Let*

$$\sup \left| \frac{\Delta_i(\cdot)}{\gamma_i(\cdot)} \right| \leq \varrho_i(x, e, \varpi),\ 1 \leq i \leq m$$

for some known functions $\varrho_i(\cdot)$, where the supremum is taken over all $x \in R^n$, $e \in R^\rho$, $|\sigma_i| \leq \mu_i/k_0^i$, $\theta \in \Theta$, $\varpi \in R^m$, and $w \in W$.

Note that the inequality in Assumption 3.6 holds for all $\varpi \in R^m$, i.e., while we require that $\varpi \in \Lambda_0$, where Λ_0 is compact, the inequality in Assumption 3.6 is independent of Λ_0. The functions β_i are chosen as

$$\beta_i(\cdot) = \varrho_i(\cdot) + q_i,\ q_i > 0$$

As before, the parameters μ_i should be chosen sufficiently small, in order to recover the performance of the ideal SMC. This completes the design of the controller.

3.4 Closed-Loop Analysis

For $i = 1, \cdots, m$, define $\zeta^i \in R^{\rho_i - 1}$ by $(e^i)^T = [(\zeta^i)^T \; e^i_{\rho_i}]$ and write the closed-loop system in the form

$$\left.\begin{array}{rcl} \dot{\sigma}_i &=& -k_0^i \sigma_i + \mu_i \, \mathrm{sat}(s_i/\mu_i) \\ \dot{\zeta}^i &=& M_i \zeta^i + C_i(s_i - k_0^i \sigma_i) \\ \dot{s}_i &=& \Delta_i(x, \sigma, \theta, \varpi, \tilde{w}) - \gamma_i(x, \theta) \; \beta_i(x, \varpi) \; \mathrm{sat}(s_i/\mu_i) \\ \dot{z} &=& \phi(z, e + \nu, d) \end{array}\right\} \tag{3.9}$$

where M_i and C_i are as defined in Section 2.5, and note that M_i is Hurwitz by design. Let $Q_i = Q_i^T > 0$ be the solution of the Lyapunov equation $Q_i M_i + M_i^T Q_i = -I$ and $V_i^\zeta(\zeta^i) \stackrel{\text{def}}{=} \zeta^{iT} Q_i \zeta^i$.

Our first step is to show that for any initial state $x(0)$, all trajectories of the closed-loop system are bounded. To that end, let c_i be a positive constant such that $c_i > \mu_i$, and define the set $\Psi_c \stackrel{\text{def}}{=} \Omega_c \times \Omega_{cz}$, $\Omega_c \stackrel{\text{def}}{=} (\prod_{i=1}^m \Omega_{c_i})$,

$$\left.\begin{array}{rcl} \Omega_{c_i} &\stackrel{\text{def}}{=}& \{V_i^\zeta(\zeta^i) \leq (c_i + \mu_i)^2 \chi_i, |s_i| \leq c_i, |\sigma_i| \leq \mu_i / k_0^i\}, \\ \Omega_{cz} &\stackrel{\text{def}}{=}& \{V_z(t, z, d) \leq \lambda_4(l_0 + l_3 \|c\|)\} \end{array}\right\}$$

$c = \{c_i\}$, $\lambda_4 = \lambda_2 \circ \gamma$ is a class $\mathcal{K}_\infty$ function, $\chi_i = 4\|Q_i C_i\|^2 \lambda_{max}(Q_i)$, and l_3 is a positive constant such that $\|e\| \leq l_3 \|c\|$ in Ω_c. It can be verified that $\sigma_i \dot{\sigma}_i \leq 0$ on the boundary $|\sigma_i| = \mu_i / k_0^i$, $\dot{V}_i^\zeta \leq 0$ on the boundary $V_i^\zeta = (c_i + \mu_i)^2 \chi_i$, $s_i \dot{s}_i \leq 0$ on the boundary $|s_i| = c_i$, and that $\dot{V}_z \leq 0$ on the boundary $V_z = \lambda_4(l_0 + l_3\|c\|)$. Since the set Ψ_c can be chosen large enough to include any state $x(0)$ and the control is independent of c, the preceeding argument shows that all states of the closed-loop system (3.9) are bounded. In fact, the preceeding argument can be extended to show that for any bounded $x(0)$, the trajectories of the system reach the positively invariant set Ψ_μ in finite time, where

$$\left.\begin{array}{lcl} \Psi_\mu & \stackrel{\text{def}}{=} & \Omega_\mu \times \{V_z(t,z,d) \leq \lambda_4(\|\mu\|_\infty r^*)\} \\ \Omega_\mu & \stackrel{\text{def}}{=} & \prod_{i=1}^m \{(e^i,\sigma_i) : |s_i| \leq \mu_i,\ |\sigma_i| \leq \frac{\mu_i}{k_0^i},\ V_i^\zeta(\zeta^i) \leq 16\mu_i^2\chi_i\} \end{array}\right\}$$

$\mu = \{\mu_i\}$, $r^* > \alpha^*$, and α^* is a positive constant such that $\|e\| \leq \|\mu\|_\infty \alpha^*$ for all $e \in \Omega_\mu$. Inside this set, the system has a unique equilibrium point $(z = 0, e = 0, \sigma_i = \bar{\sigma}_i)$ when $\tilde{w} = 0$, $\nu = 0$ and $\varpi = 0$. The equilibrium analysis of the preceeding chapter can be repeated almost verbatim to show that for sufficiently small μ_i, all trajectories starting inside Ψ_μ approach the equilibrium point $(z = 0, e = 0, \sigma = \bar{\sigma}, \varphi = 0)$ as t tends to infinity. In particular, we have the following result.

Theorem 3.1 *Suppose Assumptions 3.1 through 3.6 are satisfied, then, for any bounded initial state $x(0)$, the state $x(t)$ of the closed-loop system under the state feedback controller (3.6)-(3.8) is bounded for all $t \geq 0$. Moreover, there exists $\mu^* > 0$, such that, for $\mu_i \in (0, \mu^*]$, $\lim_{t\to\infty} e(t) = 0$.*

In order to state the analogous result of Theorem 2.2 of Section 2.4.2, dealing with the closeness of trajectories of the CSMC to ideal SMC, consider the ideal SMC control

$$\left.\begin{array}{lcl} s_i & = & \sum_{j=1}^{\rho_i-1} k_j^i e_j^i + e_{\rho_i}^i \\ u & = & \hat{A}^{-1}(x)[-\hat{F}(x,e,\varpi) + v],\ v_i = -\beta_i(x,e,\varpi)\,\mathrm{sgn}(s_i/\mu_i) \end{array}\right\} \qquad (3.10)$$

The arguments of Theorem 2.2 can be repeated to prove the following result.

Theorem 3.2 *Let x^* be the state of the closed-loop system under the ideal SMC control (3.10) and x be part of the state of the closed-loop system under the continuous SMC (3.6)-(3.8). Then, under the hypotheses of Theorem 3.1, given any compact subset $\mathcal{S}$ of R^n, with $x^*(0) = x(0) \in \mathcal{S}$, there exists $\mu^* > 0$, such that, $\mu_i \in (0, \mu^*] \Rightarrow x^*(t) - x(t) = O(\|\mu_\infty\|)\ \forall\ t \geq 0$.*

3.5 Conclusions

In this chapter, we considered the design of robust sliding mode control with conditional integrators for uncertain MIMO nonlinear systems, with the specific objective of achieving global regulation when all the assumptions hold globally. We showed that, under certain additional assumptions, the semi-global output feedback result of the previous chapter can be extended to a global result, by means of a combined of full state/error feedback design. The result is global with respect to the initial state of the plant and bounded reference signals, with the control being independent of the bound on the reference signal; however, it is dependent on the bound on the exogenous (disturbance) signal. Analytical results for the improvement in performance over conventional integral control were provided. Since we deal with state feedback, the performance recovery result, dealing with closeness of the closed-loop system trajectories to those under ideal sliding mode control, is sharper than the one obtainable under output feedback.

Chapter 4

Tracking Using Conditional Servocompensators

4.1 Introduction

We consider the design of a robust controller for the output regulation problem, where the exogenous signals are generated by a neutrally stable exosystem. Previous work [45] has shown how to do this by incorporating a servocompensator in a sliding mode design, but the transient performance is degraded when compared to ideal SMC. Extending the conditional integrator idea of Chapter 2, we design the servocompensator as a conditional one that is active only inside the boundary layer, achieving asymptotic output regulation, but with improved transient performance. Both regional as well as semi-global asymptotic results are provided, and we analytically show that the controller can be tuned to recover the performance of an ideal SMC. A result from [45] regarding ultimate boundedness of the tracking error under internal model perturbation is recalled, and a simulation example is included to demonstrate the improvement in transient performance over the conventional servocompensator design of [45], and the effect of the internal model perturbation.

4.2 Problem Statement

Consider a single-input single-output nonlinear system, modeled by

$$\left.\begin{aligned} \dot{x} &= f(x,\theta) + g(x,\theta)u + \beta(x,d,\theta), \\ y &= h(x,\theta) + \gamma(d,\theta) \end{aligned}\right\} \tag{4.1}$$

where $x \in R^n$ is the state, $u \in R$ is the control input, $y \in R$ is the measured output, $d \in R^p$ is a time-varying disturbance input. The functions f, g, β, h and γ depend continuously on θ, a vector of unknown constant parameters, which belongs to a compact set $\Theta \subset R^l$. We assume that, for all $\theta \in \Theta$, the functions are sufficiently smooth on U_θ, an open connected subset of R^n that could depend on θ, for all d in a compact set of interest. The functions β and γ vanish at $d = 0$, i.e., $\beta(x, 0, \theta) = 0$ and $\gamma(0, \theta) = 0$ for all $\theta \in \Theta$ and $x \in U_\theta$. Our first assumption is that the disturbance-free system has a well-defined normal form, possibly with zero dynamics.

Assumption 4.1 *The system (4.1), with $d = 0$, has a uniform relative degree $\rho \le n$ for all $x \in U_\theta$ and $\theta \in \Theta$; i.e., $L_g h(x,\theta) = L_g L_f h(x,\theta) = \cdots = L_g L_f^{\rho-2} h(x,\theta) = 0$ and $|L_g L_f^{\rho-1} h(x,\theta)| \ge g_0 > 0$ where g_0 is independent of θ. Moreover, there exists a diffeomorphism*

$$\begin{bmatrix} \eta \\ \xi \end{bmatrix} = T(x,\theta) \tag{4.2}$$

of U_θ onto its image that transforms (4.1), with $d = 0$, into the normal form [1]

$$\left.\begin{aligned} \dot{\eta} &= \phi(\eta,\xi,\theta) \\ \dot{\xi}_i &= \xi_{i+1}, 1 \le i \le \rho - 1 \\ \dot{\xi}_\rho &= b(\eta,\xi,\theta) + a(\eta,\xi,\theta)u \\ y &= \xi_i \end{aligned}\right\} \tag{4.3}$$

[1] For $\rho = n$, η and the $\dot{\eta}$-equation are dropped.

Assumption 4.2 *In the presence of disturbance, the change of variables (4.2) transforms the system into the form* [2]

$$\left.\begin{aligned} \dot{\eta} &= \phi_a(\eta, \xi_1, ..., \xi_m, d, \theta) \\ \dot{\xi}_i &= \xi_{i+1} + \Psi_i(\xi_1, ..., \xi_i, d, \theta), 1 \le i \le m-1 \\ \dot{\xi}_i &= \xi_{i+1} + \Psi_i(\eta, \xi_1, ..., \xi_i, d, \theta), m \le i \le \rho - 1 \\ \dot{\xi}_\rho &= b(\eta, \xi, \theta) + a(\eta, \xi, \theta)u + \Psi_\rho(\eta, \xi, d, \theta), \\ y &= \xi_i + \gamma(d, \theta) \end{aligned}\right\} \tag{4.4}$$

where $1 \le m \le \rho - 1$. The functions Ψ_i vanish at $d = 0$.

Examples of physical systems which are transformable into the normal form in Assumption 4.1, uniformly in a compact set of system parameters, can be found, for example in [37, Section 4.10]. Geometric conditions under which a system can be transformed into the form in Assumption 4.2 can be found, for example, in [60].

Assumption 4.3 *Let ρ_0 be the disturbance relative degree and $\tilde{\rho} = \rho - \rho_0$. The disturbance and reference signals $d(t)$ and $r(t)$ have the following properties for all $t \ge 0$:*

- *$d(t)$ and its derivatives up to the $\tilde{\rho}$th derivative are bounded, and $d^{(\tilde{\rho})}(t)$ is piecewise continuous*
- *$r(t)$ and its derivatives up to the ρth derivative are bounded, and $r^{(\rho)}(t)$ is piecewise continuous*
- $\lim_{t\to\infty}[\mathcal{D}(t) - \bar{\mathcal{D}}(t)] = 0$ *and* $\lim_{t\to\infty}[\mathcal{Y}(t) - \bar{\mathcal{Y}}(t)] = 0$, *where*

$$\mathcal{D}^T(t) = [d(t), \cdots, d^{(\tilde{\rho})}(t)], \ \mathcal{Y}^T(t) = [r(t), \cdots, r^{(\rho)}(t)]$$

[2] For $m = 1$, the first $\dot{\xi}_1$-equation is dropped.

and $\bar{\mathcal{D}}(t)$ and $\bar{\mathcal{Y}}(t)$ are generated by the known exosystem

$$\left.\begin{array}{rcl} \dot{w} & = & S_0 w, \\ \begin{bmatrix} \bar{\mathcal{D}} \\ \bar{\mathcal{Y}} \end{bmatrix} & = & \Gamma_0 w \end{array}\right\} \tag{4.5}$$

where S_0 has distinct eigenvalues on the imaginary axis and $w(t)$ belongs to a compact set W.

Let D and Y be compact subsets of $R^{(\bar{\rho}+1)p}$ and $R^{(\rho+1)}$ respectively, such that $\mathcal{D} \in D$ and $\mathcal{Y} \in Y$, and $\bar{d}(w)$ and $\bar{r}(w)$ denote the steady-state values of d and r as determined by the exosystem (4.5). Define $\pi_1(w,\theta)$ to $\pi_m(w,\theta)$ by

$$\begin{aligned} \pi_1 &= \bar{r} - \gamma(\bar{d},\theta), \\ \pi_{i+1} &= \frac{\partial \pi_i}{\partial w} S_0 w - \Psi_i(\pi_1, \cdots, \pi_i, \bar{d}, \theta), \ 1 \le i \le m-1. \end{aligned}$$

Assumption 4.4 *There exists a unique mapping $\lambda(w,\theta)$ that solves the partial differential equation*

$$\frac{\partial \lambda}{\partial w} S_0 w = \Phi_a(\lambda, \pi_1, \cdots, \pi_m, \pi_{m+1}, \bar{d}, \theta)$$

for all $w \in W$, where

$$\pi_{m+1} = \frac{\partial \pi_m}{\partial w} S_0 w - \Psi_m(\lambda, \pi_1, \cdots, \pi_m, \bar{d}, \theta)$$

Let

$$\pi_{i+1} = \frac{\partial \pi_i}{\partial w} S_0 w - \Psi_i(\lambda, \pi_1, \cdots, \pi_i, \bar{d}, \theta), \ m+1 \le i \le \rho - 1.$$

The steady-state value of the control u on the zero-error manifold $\{\eta = \lambda(w,\theta), \xi = \pi(w,\theta)\}$ is given by

$$\chi(w,\theta) = \frac{1}{a(\lambda,\pi,\theta)}\left[\frac{\partial \pi_\rho}{\partial w} S_0 w - b(\lambda,\pi,\theta) - \Psi_\rho(\lambda,\pi,\bar{d},\theta)\right]$$

Assumption 4.5 *There exists a set of real numbers $c_0, \cdots, c_{q-1}$, independent of θ, such that $\chi(w,\theta)$ satisfies the identity*

$$L_s^q \chi = c_0 \chi + c_1 L_s \chi + \cdots + c_{q-1} L_s^{q-1} \chi \tag{4.6}$$

for all $(w,\theta) \in W \times \Theta$, where $L_s\chi = (\frac{\partial \chi}{\partial w}) S_0 w$ and the characteristic polynomial

$$p^q - c_{q-1}\, p^{q-1} - \cdots - c_0$$

has distinct roots on the imaginary axis.

Motivation for Assumption 4.5 comes from the nonlinear counterpart of the internal model principle, which recognizes that in the nonlinear case, the controller must be able to reproduce not only the trajectories generated by the exosystem, but also a number of higher-order nonlinear deformations thereof, an idea that was elaborated independently in [34], [43], [63]. Assumption 4.5, along with the notion of *immersion* [37, Chapter 8], allow the construction of a finite dimensional linear internal model, as we will soon show. However, before we do so, a couple of remarks are in order.

Note that, among other things, the matrix S_0 in Assumption 4.3, and hence the frequencies of the exosystem need to be precisely known. For the case where the frequencies of the exosystem are unknown, an alternate design, that makes use of an adaptive internal model whose "natural frequencies" are automatically tuned to match those of the unknown exosystem, can be found in [69]. A recent result, which relaxes Assumption 4.5, thereby removing the restriction that the solution of the regulator equations be a polynomial in the exogenous signals, and allows for a nonlinear internal model, can be found in [16].

Defining

$$S = \begin{bmatrix} 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \\ c_0 & \cdots & \cdots & \cdots & c_{q-1} \end{bmatrix}, \ \tau = \begin{bmatrix} \chi \\ L_s\chi \\ \vdots \\ L_s^{q-2}\chi \\ L_s^{q-1}\chi \end{bmatrix}$$

and $\Gamma = [1\ 0 \cdots\ 0]_{1\times q}$, it can be shown that $\chi(w,\theta)$ is generated by the internal model

$$\begin{aligned} \frac{\partial\tau(w,\theta)}{\partial w}S_0 w &= S\tau(w,\theta), \\ \chi(w,\theta) &= \Gamma\tau(w,\theta) \end{aligned}$$

To tackle the tracking problem, we apply the change of variables

$$z = \eta - \lambda(w,\theta) \text{ and } e_i = y^{(i-1)} - r^{(i-1)},\ 1 \le i \le \rho$$

and

$$\nu^T(t) = [\mathcal{D}^T(t) - \bar{\mathcal{D}}^T(t), \mathcal{Y}^T(t) - \bar{\mathcal{Y}}^T(t)]$$

and note that $\nu(t)$ belongs to a compact set Λ and $\lim_{t\to\infty}\nu(t) = 0$. With this change of variables, system (4.4) can be rewritten as

$$\left.\begin{aligned} \dot{z} &= \phi_0(z,e,\nu,w,\theta) \\ \dot{e}_i &= e_{i+1}, 1 \le i \le \rho - 1 \\ \dot{e}_\rho &= b_0(z,e,\nu,w,\theta) + a_0(z,e,\nu,w,\theta)u, \\ y_m &= e_1 \end{aligned}\right\} \tag{4.7}$$

where y_m is the measured tracking error. The functions $\phi_0(\cdot)$, $a_0(\cdot)$ and $b_0(\cdot)$ satisfy

$$\begin{aligned}
\phi_0(0,0,0,w,\theta) &= 0 \\
a_0(0,0,0,w,\theta) &= a(\lambda(w,\theta),\pi(w,\theta),\theta) \\
b_0(0,0,0,w,\theta) &= -\chi(w,\theta)\, a(\lambda(w,\theta),\pi(w,\theta),\theta)
\end{aligned}$$

In the new variables, the zero-error manifold is given by $\{z = 0, e = 0\}$. The next two assumptions of similar to corresponding ones in Chapter 2.

Assumption 4.6 *There exist positive constants r_1 and r_2, independent of (ν, w, θ), such that for all $(\nu, w, \theta) \in \Lambda \times W \times \Theta$,*

$$\|e\| < r_1 \text{ and } \|z\| < r_2 \Rightarrow x \in U_\theta$$

Define the balls

$$\mathcal{E} = \{e \in R^\rho : \|e\| < r_1\} \text{ and } \mathcal{Z} = \{z \in R^{n-\rho} : \|z\| < r_2\}$$

Since Λ is compact, there exists $r_3 > 0$ such that $\|\nu\| < r_3$ for all $\nu \in \Lambda$. Therefore, $\|(e^T, \nu^T)\| < r_1 + r_3$ for all $e \in \mathcal{E}$ and $\nu \in \Lambda$.

Assumption 4.7 (i) *There exist a C^1 proper function $V_z : \mathcal{Z} \times W \to R_+$, possibly dependent on θ, and class $\mathcal{K}$ functions $\alpha_i : [0, r_2) \to R_+ (i = 1,2,3)$ and $\delta : [0, r_1 + r_3) \to R_+$, independent of (w, θ), such that*

$$\begin{gathered}
\alpha_1(\|z\|) \le V_z(z,w,\theta) \le \alpha_2(\|z\|), \\
\frac{\partial V_{0z}}{\partial z}\phi_0(z,e,\nu,w,\theta) + \frac{\partial V_z}{\partial w} S_0 w \le -\alpha_3(\|z\|)
\end{gathered}$$

for all $\|z\| \ge \delta(\|(e^T,\nu^T)\|)$, $\|(e^T,\nu^T)\| < r_1 + r_3$ and $(z,w,\theta) \in \mathcal{Z} \times W \times \Theta$. Furthermore, $\delta(r_3) < \alpha_2^{-1}(\alpha_1(r_2))$.

(ii) *There exists a Lyapunov function $V_{zz}(z, w, \theta)$, defined in some neighborhood of $z = 0$, and positive constants λ_1 to λ_4, independent of (w, θ), such that*

$$\lambda_1 \|z\|^2 \le V_{zz}(z, w, \theta) \le \lambda_2 \|z\|^2$$
$$\frac{\partial V_{zz}}{\partial z} \phi_0(z, 0, 0, w, \theta) + \frac{\partial V_{zz}}{\partial w} S_0 w \le -\lambda_3 \|z\|^2$$
$$\left\| \frac{\partial V_{zz}}{\partial z} \right\| \le \lambda_4 \|z\|$$

4.3 Controller Design

Our design of the conditional servocompensator follows very closely that of the conditional integrator in [70]. Basically, it involves modifying the servocompensator

$$\dot{\sigma} = S\sigma + Je_1, \quad J^T = [0, \cdots 0, 1]$$

in [45] to make it "active" only inside the boundary layer. Assume for the present that the state e is available for feedback. To simplify the notation in what is to come, we define

$$\zeta^T = [e_1 \; e_2 \; \cdots \; e_{\rho-1}] \text{ and } K_2 = [k_1 \; k_2 \; \cdots \; k_{\rho-1}]$$

In the absence of the servocompensator, one could take the sliding surface as

$$s = K_2 \zeta + e_\rho$$

with K_2 chosen such that the polynomial $\lambda^{\rho-1} + k_{\rho-1}\lambda^{\rho-2} + \cdots + k_2\lambda + k_1$ is Hurwitz. This guarantees that when motion is confined to the manifold $s = 0$, the error e_1 converges to zero asymptotically. Servocompensation is then introduced by modifying the sliding surface to

$$s = K_1\sigma + K_2\zeta + e_\rho \tag{4.8}$$

where σ is the output of the conditonal servocompensator

$$\dot{\sigma} = (S - JK_1)\sigma + \mu J\text{sat}(s/\mu) \tag{4.9}$$

with $\mu > 0$ being the width of the boundary layer and K_1 chosen such that $S - JK_1$ is Hurwitz, which is always possible since the pair (S, J) is controllable. Equation (4.9) represents a perturbation of the exponentially stable system $\dot{\sigma} = (S - JK_1)\sigma$, with the norm of the perturbation bounded by the small parameter μ. Inside the boundary layer, i.e., when $|s| \leq \mu$, equation (4.9) reduces to

$$\dot{\sigma} = S\sigma + Je_a \tag{4.10}$$

where the "augmented error" $e_a = K_2\zeta + e_\rho$ is a linear combination of the tracking error and its derivatives up to order $\rho - 1$. Equation (4.10) coincides with the servocompensator of [45] only in the case when $\rho = 1$.

Since the state e is unavailable for feedback, we use the following linear high-gain observer to robustly estimate the derivatives of e_1:

$$\left.\begin{aligned} \dot{\hat{e}}_i &= \hat{e}_{i+1} + g_i(e_1 - \hat{e}_1)/\epsilon^i, \ 1 \leq i \leq \rho - 1 \\ \dot{\hat{e}}_\rho &= g_\rho(e_1 - \hat{e}_1)/\epsilon^\rho \end{aligned}\right\} \tag{4.11}$$

where $\epsilon > 0$ is a design parameter to be specified, and the positive constants $g_1, \cdots, g_\rho$ are chosen such that the polynomial $\lambda^\rho + g_1\lambda^{\rho-1} + \cdots + g_{\rho-1}\lambda + g_\rho$ is Hurwitz. We replace s by its estimate $\hat{s}$, given by

$$\hat{s} = K_1\sigma + k_1e_1 + \sum_{i=2}^{\rho-1} k_i\hat{e}_i + \hat{e}_\rho \tag{4.12}$$

where σ is the the output of

$$\dot{\sigma} = (S - JK_1)\sigma + \mu J \mathrm{sat}(\hat{s}/\mu) \tag{4.13}$$

The control is taken as

$$u = -k \ \mathrm{sign}(L_g L_f^{\rho-1} h) \ \mathrm{sat}(\hat{s}/\mu) \tag{4.14}$$

To complete the design, we need to specify the design parameters k, μ and ϵ. The constant k is an upper bound on the control u. Since in typical applications the control has to satisfy magnitude constraints, we can simply choose k to be the maximum permissible control magnitude. The parameters μ and ϵ should be chosen sufficiently small. In particular, we show in the next section that there exists $\mu^* > 0$, and for each $\mu \in (0, \mu^*)$, there is an $\epsilon^* = \epsilon^*(\mu) > 0$ such that asymptotic tracking is achieved for each $0 < \mu < \mu^*$ and $0 < \epsilon < \epsilon^*$. The only precise knowledge about the plant that is required to calculate and implement the control (4.14) is its relative degree ρ, the sign of its high-frequency gain $L_g L_f^{\rho-1} h$, and the characteristic polynomial of Assumption 4.5.

4.4 Closed-Loop Analysis

The current analysis shares many points in common with the ones in [45], [70], and the previous chapter, so we only outline the idea, taking care to highlight any differences. Analogous to the case in Chapter 2, the main difference between the analysis in this section and that in [45] is treating σ and ζ separately in the current work, while in [45], they are lumped together in one vector.

We write the closed-loop system in the singularly perturbed form

$$\left.\begin{array}{rcl} \dot{w} &=& S_0 w \\ \dot{z} &=& \phi_0(z, e, \nu, w, \theta) \\ \dot{\sigma} &=& A_\sigma \sigma + \mu J \mathrm{sat}(s - N(\epsilon)\varphi)/\mu) \\ \dot{\zeta} &=& A_\zeta \zeta + B_1(s - K_1\sigma) \\ \dot{s} &=& \Delta(z, \sigma, e, \varphi, \nu, w, \theta) - k|a_0(\cdot)|\ \mathrm{sat}(s - N(\epsilon)\varphi)/\mu) \\ \epsilon\dot{\varphi} &=& A_\varphi \varphi + \epsilon B_2 \left[b_0(\cdot) - k|a_0(\cdot)|\ \mathrm{sat}(s - N(\epsilon)\varphi)/\mu)\right] \end{array}\right\} \qquad (4.15)$$

where

$$A_\zeta = \begin{bmatrix} 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \\ -k_1 & -k_2 & \cdots & \cdots & -k_{\rho-1} \end{bmatrix},\ A_\varphi = \begin{bmatrix} -g_1 & 1 & \cdots & \cdots & 0 \\ \vdots & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ \vdots & \cdots & \cdots & 0 & 1 \\ -g_\rho & 0 & \cdots & \cdots & 0 \end{bmatrix},\ B_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

$$\Delta(\cdot) = b_0(\cdot) + K_1 A_\sigma \sigma + \mu K_1 J \mathrm{sat}(s - N(\epsilon)\varphi)/\mu) + K_2\ [e_2\ e_3\ \cdots\ e_\rho]^T,$$

$$N(\epsilon) = [\ 0 \quad k_2\epsilon^{\rho-2} \quad k_3\epsilon^{\rho-3} \quad \cdots \quad k_{\rho-1}\epsilon \quad 1\ ],$$

$A_\sigma \stackrel{\text{def}}{=} S - JK_1$, and the scaled estimation φ is given by

$$\varphi_i = \frac{1}{\epsilon^{\rho-i}}(e_i - \hat{e}_i),\ 1 \le i \le \rho$$

Noting that A_ζ, A_φ, and A_σ are Hurwitz, we define the Lyapunov functions

$$V_\zeta(\zeta) \stackrel{\text{def}}{=} \zeta^T P_\zeta \zeta,\ V_\varphi(\varphi) \stackrel{\text{def}}{=} \varphi^T P_\varphi \varphi,\ \text{and } V_\sigma(\sigma) \stackrel{\text{def}}{=} \sigma^T P_\sigma \sigma$$

where the symmetric positve definite matrices P_ζ, P_φ, and P_σ are the solutions of

$$P_\zeta A_\zeta + A_\zeta^T P_\zeta = -I,\ P_\varphi A_\varphi + A_\varphi^T P_\varphi = -I,\ \text{and}\ P_\sigma A_\sigma + A_\sigma^T P_\sigma = -I$$

respectively. Given a positive constant $c > \mu$, define the compact set Ω_c by

$$\Omega_c \stackrel{\text{def}}{=} \{(z, e, \sigma) : |s| \le c,\ V_\sigma(\sigma) \le \mu^2 \rho_1,\ V_\zeta(\zeta) \le (c+\mu\rho_2)^2 \rho_3,\ V_z(t, z, d) \le \alpha_4(c\rho_4 + r_3)\}$$

where ρ_1, ρ_2, ρ_3, and ρ_4 are positive constants, independent of c, to be specified shortly and α_4 is a class $\mathcal{K}$ function defined in terms of the functions α_2 and δ of Assumption 4.7 by $\alpha_4 = \alpha_2 \circ \delta$. Our analysis will be restricted to trajectories starting inside a product set of which Ω_c is a component. Therefore, in view of Assumption 4.6, Ω_c will have to be chosen to ensure that $(z, e, \sigma) \in \Omega_c$ implies that $(z, e) \in \mathcal{Z} \times \mathcal{E}$. Using

$$e = K_3 \zeta + B_2(s - K_1 \sigma),\ K_3 = \begin{bmatrix} I \\ -K_2 \end{bmatrix}$$

it can be verified that inside Ω_c, $\|e\| \le \rho_4 c$, where $\rho_4 = (1+\rho_2)\|K_3\|\sqrt{(\rho_3/\lambda_{min}(P_\zeta)} + 1 + \|K_1\|\sqrt{(\rho_1/\lambda_{min}(P_\sigma)}$. Using this, along with Assumption 4.7, it can be shown that choosing c to satisfy

$$c\rho_4 < \min\{r_1, \alpha_4^{-1}(\alpha_1(r_2)) - r_3\} \tag{4.16}$$

guarantees that inside Ω_c, we have $(z, e) \in \mathcal{Z} \times \mathcal{E}$.

Define the compact set Σ_ϵ by

$$\Sigma_\epsilon \stackrel{\text{def}}{=} \{\varphi \in R^\rho : V_\varphi(\varphi) \le \epsilon^2 \rho_5\}$$

where ρ_5 is a positive constant to be specified shortly. We wish to show that for a suitable choice of the controller parameters, the set $\Omega_c \times \Sigma_\epsilon$ is a positively invariant set of the closed-loop system (4.15). Using the inequality

$$\dot{V}_\sigma \leq -\|\sigma\|^2 + 2\mu\|\sigma\|\|P_\sigma J\|$$

it is easy to show that $\dot{V}_\sigma \leq 0$ on the boundary $V_\sigma = \mu^2\rho_1$ for the choice $\rho_1 = 4\|P_\sigma J\|^2\lambda_{max}(P_\sigma)$. Inside the set Ω_c, $\|\sigma\| \leq \mu\sqrt{\rho_1/\lambda_{min}(P_\sigma)} \stackrel{\text{def}}{=} \mu\rho_2/\|K_1\|$. Using this, along with $|s| \leq c$, and the inequality

$$\dot{V}_\zeta \leq -\|\zeta\|^2 + 2\|\zeta\| \ \|P_\zeta B_1\| \ (|s| + \|K_1\|\|\sigma\|)$$

it is easy to show that $\dot{V}_\zeta \leq 0$ on the boundary $V_\zeta = (c + \mu\rho_2)^2\rho_3$, for the choice $\rho_3 = 4\|P_\zeta B_1\|^2\lambda_{max}(P_\zeta)$. Next we evaluate $s\dot{s}$ on the boundary $|s| = c$. To that end, let ϵ be small enough that $|N(\epsilon)\varphi| \leq c - \mu$, so that

$$\text{sat}\left(\frac{s - N(\epsilon)\varphi}{\mu}\right) = \text{sgn}\left(\frac{s - N(\epsilon)\varphi}{\mu}\right) = \text{sgn}(s)$$

and hence

$$s\dot{s} \leq |\Delta(\cdot)||s| - k|a_0(\cdot)||s|$$

Choosing k and c to satisfy [3]

$$k \geq \rho_6 + \gamma_1(c) \tag{4.17}$$

where $\rho_6 > 0$ and $\gamma_1(c) = \max|\Delta(\cdot)|/|a_0(\cdot)|$, with the maximization taken over all $(z, e, \sigma) \in \Omega_c$, $\nu \in \Lambda$, $w \in W$, and $\theta \in \Theta$, we have $s\dot{s} < 0$ on the boundary $|s| = c$. Assumption 4.7 shows that $\dot{V}_z \leq 0$ on the boundary $V_z = \alpha_4(c\rho_4 + r_3)$. Finally, using the inequality

$$\dot{V}_\varphi \leq -\frac{1}{\epsilon}\|\varphi\|^2 + 2\|\varphi\| \ \|P_\varphi B_2\|\gamma_2(c)$$

[3]Inequality (4.17) can be viewed in two ways. Given $c > 0$, it is a constraint on the minimum value k. Alternatively, given k, it is a constraint on the estimate of the region of attraction.

where $\gamma_2(c) = \max|b_0(\cdot) - k|a_0(\cdot)|\text{sat}((s - N(\epsilon)\varphi)/\mu)|$, with the maximization taken over the same set as that for $\gamma_1(c)$, it follows that $\dot{V}_\varphi \leq 0$ on the boundary $V_\varphi \leq \epsilon^2\rho_5$ for the choice $\rho_5 > 4\|P_\varphi B_2\|^2\gamma_2^2(c)\lambda_{max}(P_\varphi)$. It follows that the set $\Omega_c \times \Sigma_\epsilon$ is positively invariant.

Our next step is to show that for any bounded $\hat{e}(0)$, and any $(z(0), e(0), \sigma(0)) \in \Omega_b$, where $0 < b < c$, it is possible to choose ϵ such that the trajectory enters the set $\Omega_c \times \Sigma_\epsilon$ in finite time. Using the fact that for $(z, e, \sigma) \in \Omega_c$, the right-hand side of the slow equation of (4.15) is bounded uniformly in ϵ, it follows that for all $(z(0), e(0), \sigma(0)) \in \Omega_b$, there is a finite time T_0, independent of ϵ such that for all $0 \leq t \leq T_0$, $(z(t), e(t), \sigma(t)) \in \Omega_c$. During this interval, using the definition of ρ_4, we have

$$\dot{V}_\varphi \leq -\rho_7\|\varphi\|^2, \text{ for } V_\varphi(\varphi) \geq \epsilon^2\rho_5$$

for some $\rho_7 > 0$. This inequality can be used to show that $\varphi(t)$ enters Σ_ϵ within a time interval $[0, T(\epsilon)]$, where $\lim_{\epsilon\to 0} T(\epsilon) = 0$. Therefore, by choosing ϵ small enough we can ensure that $T(\epsilon) < T_0$.

The argument that $\Omega_c \times \Sigma_\epsilon$ is positively invariant can be extended (see Appendix A) to show that, for sufficiently small ϵ, all trajectories starting inside it reach the positively invariant set $\Psi_{\mu,\epsilon} \stackrel{\text{def}}{=} \Omega_\mu \times \Sigma_\epsilon$ in finite time, where

$$\Omega_\mu \stackrel{\text{def}}{=} \{(z, e, \sigma) : |s| \leq \mu(1 - \delta_0),\ V_\sigma(\sigma) \leq \mu^2\rho_1,\ V_\zeta(\zeta) \leq \mu^2\rho_8,\ V_z(t, z, d) \leq \alpha_4(\mu\rho_9)\} \tag{4.18}$$

where $0 < \delta_0 < \rho_6/(4k) < 1/4$, ϵ is small enough that $|N(\epsilon)\varphi| < \mu\delta_0$, and ρ_8, ρ_9 are positive constants independent of μ. The set $\Psi_{\mu,\epsilon}$ shrinks to the origin as $\mu, \epsilon \to 0$.

Lastly, we show that every trajectory in $\Psi_{\mu,\epsilon}$ approaches an invariant manifold on which the error is zero. To do this, we first note that inside $\Psi_{\mu,\epsilon}$, the closed-loop system is given by

$$
\left.\begin{array}{rcl}
\dot{w} & = & S_0 w \\
\dot{z} & = & \phi_0(z, e, \nu, w, \theta) \\
\dot{\sigma} & = & A_\sigma \sigma + J(s - N(\epsilon)\varphi) \\
\dot{\zeta} & = & A_\zeta \zeta + B_1(s - K_1\sigma) \\
\dot{s} & = & \Delta(\cdot) - k|a_0(\cdot)| \left(\frac{s-N(\epsilon)\varphi}{\mu}\right) \\
\epsilon\dot{\varphi} & = & A_\varphi \varphi + \epsilon B_2 \left[b_0(\cdot) - k|a_0(\cdot)| \left(\frac{s-N(\epsilon)\varphi}{\mu}\right)\right]
\end{array}\right\} \qquad (4.19)
$$

Next, we claim that there exists a unique matrix M such that

$$SM = MS \text{ and } -K_1 M = \Gamma$$

To see this, note that since A_σ is Hurwitz and S has eigenvalues on the imaginary axis, the Sylvester equation $A_\sigma X - XS = J\Gamma$ has a unique solution. That this solution satisfies $SX - XS = 0$ and $K_1 X + \Gamma = 0$ is shown in [49]. Thus $M = X$ is the required matrix. Defining

$$\mathcal{M}_\mu = \{z = 0, \sigma = \bar{\sigma}, e = 0, \varphi = 0\}$$

where

$$\bar{\sigma} = (\mu/k)\ \text{sign}(L_g L_f^{\rho-1} h) M\tau(w, \theta)$$

it is easy to verify by direct substitution that $\mathcal{M}_\mu$ is an invariant manifold of (4.19) when $\nu = 0$, for all $w \in W$. Let

$$\bar{s} = K_1\bar{\sigma},\ \tilde{\sigma} = \sigma - \bar{\sigma},\ \text{and } \tilde{s} = s - \bar{s}$$

Using

$$V = V_{zz}(z, w, \theta) + \lambda_5 V_\zeta(\zeta) + \lambda_6 V_\sigma(\tilde{\sigma}) + \frac{1}{2}\tilde{s}^2 + V_\varphi(\varphi) \tag{4.20}$$

as a Lyapunov function candidate, where λ_5, λ_6 are positive constants, it can be verified (see Appendix B) that, by first taking λ_5 large enough, then λ_6 large enough, then μ small enough, and lastly ϵ small enough, $\dot{V}$ satisfies an inequality of the form

$$\dot{V} \leq -\lambda_7 V + \lambda_8 \sqrt{V} \|\nu(t)\| \tag{4.21}$$

for some positive constants λ_7 and λ_8, uniformly in μ and ϵ. Since $\nu(t) \rightarrow 0$ as $t \rightarrow \infty$, the preceeding inequality can then be used to show that every trajectory inside $\Psi_{\mu,\epsilon}$ approaches $\mathcal{M}_\mu$ as $t \rightarrow \infty$. Our conclusions can be summarized in the following theorem.

Theorem 4.1 *Suppose Assumptions 4.1 through 4.7 are satisfied and consider the closed-loop system formed of the plant (4.7) and the output feedback controller (4.11)-(4.14). Suppose* $\hat{e}(0)$ *is bounded and the initial states* $(z(0), e(0), \sigma(0)) \in \Omega_b$, *where* $0 < b < c$, *and* c *satisfies (4.16) and (4.17). Then, there exists* $\mu^* > 0$ *and for each* $\mu \in (0, \mu^*]$, *there exists* $\epsilon^* = \epsilon^*(\mu)$, *such that for all* $\mu \in (0, \mu^*]$ *and* $\epsilon \in (0, \epsilon^*]$, *all the state variables of the closed-loop system are bounded and* $\lim_{t\rightarrow\infty} e(t) = 0$.

The estimate of the region of attraction Ω_b is limited only by two factors: the region of validity of our assumptions, and the control level k. If all the assumptions hold globally and k can be chosen arbitrarily large, the controller can achieve semi-global regulation.

Corollary 4.1 *Suppose Assumptions 4.1 through 4.7 are satisfied globally, i.e.,* $U_\theta = R^n$, *the functions* $\alpha_i(\cdot)$ $(i = 1, 2, 3)$ *and* $\delta(\cdot)$ *are class* $\mathcal{K}_\infty$ *functions, and* k *can be chosen arbitrarily large. Given compact sets* $\mathcal{N} \subset R^n$ *and* $\mathcal{L} \subset R^\rho$, *choose* $c > b > 0$ *such that* Ω_b *is large enough to contain* $\mathcal{N}$, *and choose* k *large enough to satisfy (4.17).*

Then, there exists $\mu^* > 0$ and for each $\mu \in (0, \mu^*]$, there exists $\epsilon^* = \epsilon^*(\mu)$, such that for all $\mu \in (0, \mu^*]$ and $\epsilon \in (0, \epsilon^*]$, and for all initial states $(z(0), e(0)) \in \mathcal{N}$ and $\hat{e}(0) \in \mathcal{L}$, all the state variables of the closed-loop system formed of the plant (4.7) and the output feedback controller (4.11)-(4.14) are bounded and $\lim_{t\to\infty} e(t) = 0$.

We conclude this section with the following theorem on the performance of the controller, which states that the controller recovers the performance of ideal state-feedback SMC, without servocompensation. Consider the ideal SMC

$$\left.\begin{array}{rcl} u &=& -k\ \mathrm{sign}(L_g L_f^{\rho-1} h)\ \mathrm{sgn}(s) \\ s &=& \sum_{i=1}^{\rho-1} k_i e_i + e_\rho \end{array}\right\} \tag{4.22}$$

Theorem 4.2 *Let $X = (z, e)$ be part of the state of the closed-loop system for the system (4.7) with the output feedback control (4.11)-(4.14) and $X^* = (z^*, e^*)$ be the state of the closed-loop system with the state feedback control (4.22), with $X(0) = X^*(0)$. Then, under the hypotheses of Theorem 4.1, for every $\varrho > 0$, there exists $\mu^* > 0$ and for each $\mu \in (0, \mu^*]$, there exists $\epsilon^* = \epsilon^*(\mu)$, such that for all $\mu \in (0, \mu^*]$ and $\epsilon \in (0, \epsilon^*]$, $\|X(t) - X^*(t)\| \le \varrho\ \forall\ t \ge 0$.*

Proof. We prove the theorem in two parts. First, we compare the trajectories under ideal SMC with those under state feedback continuous SMC with the conditional servocompensator. Let $X^\dagger = (z^\dagger, e^\dagger)$ be part of the state of the closed-loop system under the control (4.8), (4.9) and

$$u = -k\ \mathrm{sign}(L_g L_f^{\rho-1} h)\ \mathrm{sat}(s/\mu)$$

with $X^\dagger(0) = X^*(0)$. For this case, we show that, for sufficiently small μ, $X^\dagger(t) - X^*(t) = O(\mu)\ \forall\ t \ge 0$. Let $s^\dagger$ and s^* be the corresponding sliding surface functions of the two systems and

$$t_0 = \min\{t : |s^\dagger(t)| \leq \mu(1+\rho_2)\ \forall\, t \geq t_0\}$$

If $t_0 > 0$, then since $|K_1 \sigma^\dagger(t)| \leq \mu\rho_2\ \forall\, t$, it follows that

$$\mathrm{sat}(s^\dagger(t)/\mu) = \mathrm{sgn}(s^\dagger(t)) = \mathrm{sgn}(s^\dagger(t) - K_1\sigma^\dagger(t))\ \forall\, 0 \leq t \leq t_0$$

which can then be used to show that $X^\dagger(t) = X^*(t)\ \forall\, 0 \leq t < t_0$. The result holds trivially if $t_0 = 0$.

We now consider $X^\dagger(t)$ and $X^*(t)$ in the time interval $t \geq t_0$. Since $X^\dagger(t_0) = X^*(t_0)$, we have

$$|s^\dagger(t_0) - s^*(t_0)| = |K_1\ \sigma^\dagger(t_0)| \leq \mu\rho_2$$

Using this, along with the fact that $|s^\dagger(t)|$ and $|s^*(t)|$ monotonically converge to the positively invariant sets $\{|s^\dagger| \leq \mu\}$ and $\{0\}$, respectively, it can be shown that

$$|s^\dagger(t) - s^*(t)| \leq 2\mu(1+\rho_2)\ \forall\, t \geq t_0$$

It follows that $s^\dagger(t) - s^*(t) = O(\mu)\ \forall\, t \geq 0$. Since the equations for $\zeta^\dagger$ and ζ^* are identical stable linear equations, driven by inputs $s^\dagger - K_1\ \sigma^\dagger$ and s^* respectively, where $|K_1\ \sigma^\dagger| \leq \mu\rho_2$ and $s^\dagger - s^* = O(\mu)$, continuity of solutions on the infinite time interval [47, Theorem 9.1] can be used to show that for sufficiently small μ, $\zeta^\dagger(t) - \zeta^*(t) = O(\mu)$ and hence $e^\dagger(t) - e^*(t) = O(\mu)$ for all $t \geq t_0$, which can then be used to show that $z^\dagger(t) - z^*(t) = O(\mu)$ for all $t \geq t_0$, so that the first part of the proof follows. In particular, there exists $\mu^* > 0$ such that

$$\mu \in (0, \mu^*] \Rightarrow \|X^\dagger(t) - X^*(t)\| \leq \tau/2\ \forall\, t \geq 0$$

In the second part of the proof, the idea in [5], that was used in Theorem 2.2 in Chapter 2, can be repeated to show that the trajectories X of the system under output feedback approach the trajectories $X^{\dagger}$ under state feedback as $\epsilon \to 0$. In particular, there exists $\epsilon^* = \epsilon^*(\mu) > 0$ such that

$$\epsilon \in (0, \epsilon^*] \Rightarrow \|X(t) - X^{\dagger}(t)\| \leq \tau/2 \ \forall \ t \geq 0$$

The conclusion of Theorem 4.2 then follows from the triangle inequality.

4.5 Internal Model Perturbation

As mentioned in the concluding remarks in Section 4.3, our design requires that the constants c_0 to c_q in Assumption 4.5, and hence the frequencies of the exosystem, be precisely known. In addition, as noted in the remarks following it, Assumption 4.5 is equivalent to requiring that the control input, when restricted to the zero-error manifold, be a polynomial function of the exogenous signals [13]. A violation of either of these conditions results in a perturbation of the internal model. To make the idea precise, let

$$\chi(w, \theta) = -b_0(0, 0, 0, w, \theta)/a_0(0, 0, 0, w, \theta)$$

and suppose $\bar{\chi}(w, \theta)$ is a nominal value of $\chi(w, \theta)$ that satisfies (4.6). As mentioned above, there are two sources for this perturbation. First, the frequencies of the exosystem are unknown, so that χ satisfies (4.6) with unknown coefficients c_0 to c_q, while $\bar{\chi}$ does so with nominal coefficients $\bar{c}_0$ to $\bar{c}_q$, which are used to construct the internal model. Second, the assumption that the control input, when restricted to the zero-error manifold, is a polynomial function of the exogenous signals, does not hold, so that χ does not satisfy (4.6), but an approximation $\bar{\chi}$ of it does. In this case, we note that since any continuous function can be approximated to arbitrary accuracy on compact sets by polynomials, a linear internal model that generates $\bar{\chi}$ can be used to approximate χ arbitrarily closely. Regardless of the source of the perturbation, using the results of [45, Section 5], it can be shown that provided the perturbation is small, the controller of the previous sections achieves ultimate boundedness of the tracking error, with the bound being proportional to the size of the perturbation. In particular, let $\tilde{\chi}(w,\theta) = \bar{\chi}(w,\theta) - \chi(w,\theta)$, and suppose

$$|\tilde{\chi}(w,\theta)| \leq \delta_\chi \ \forall \ (w,\theta) \in W \times \Theta$$

The analysis proceeds exactly as in Section 4.4 upto the point of showing that the trajectories enter the set $\Psi_{\mu,\epsilon}$. Using the fact that

$$b_0(0,0,0,w,\theta) = -\chi(w,\theta)a(\lambda,\pi,\theta) = -\bar{\chi}(w,\theta)a(\lambda,\pi,\theta) + \tilde{\chi}(w,\theta)a(\lambda,\pi,\theta)$$

it can be shown that inside the set $\Psi_{\mu,\epsilon}$, when $\nu = 0$, the closed-loop equation (4.19) can be written as

$$\left.\begin{array}{rcl}
\dot{w} & = & S_0 w \\
\dot{z} & = & \phi_0(z,e,0,w,\theta) \\
\dot{\sigma} & = & A_\sigma \sigma + J(s - N(\epsilon)\varphi) \\
\dot{\zeta} & = & A_\zeta \zeta + B_1(s - K_1\sigma) \\
\dot{s} & = & \bar{\Delta}(z,\sigma,e,\varphi,0,w,\theta) - k|a_0(z,\sigma,e,0,w,\theta)|\left(\frac{s-N(\epsilon)\varphi}{\mu}\right) + \tilde{\chi}(w,\theta)a(\lambda,\pi,\theta) \\
\epsilon\dot{\varphi} & = & A_\varphi \varphi + \epsilon B_2\left[\bar{b}_0(z,\sigma,e,0,w,\theta) - k|a_0(\cdot,0,\cdot)|\left(\frac{s-N(\epsilon)\varphi}{\mu}\right) + \tilde{\chi}(w,\theta)a(\lambda,\pi,\theta)\right]
\end{array}\right\} \tag{4.23}$$

where

$$\bar{\Delta}(z,\sigma,e,\varphi,\nu,w,\theta) = \Delta(z,\sigma,e,\varphi,\nu,w,\theta) - b_0(0,0,0,0,w,\theta) - \bar{\chi}(w,\theta)a(\lambda,\pi,\theta)$$

$$\bar{b}_0(z,\sigma,e,\nu,w,\theta) = b_0(z,\sigma,e,\nu,w,\theta) - b_0(0,0,0,0,w,\theta) - \bar{\chi}(w,\theta)a(\lambda,\pi,\theta)$$

It can be verified that the system (4.23) has $\mathcal{M}_\mu$ as an invariant manifold when $\tilde{\chi} = 0$. Equation (4.23) takes the form of [45, Equation (A1)], and satisfies all the assumptions of [45, Lemma 2], so that the results of [45, Lemma 2] can be applied to show that (4.23) has an exponentially attractive manifold $\bar{\mathcal{M}}_\mu$ that is $O(\delta_\chi)$ close to $\mathcal{M}_\mu$, and on which $e = O(\delta_\chi)$. The Lyapunov analysis of the final part of the proof of Theorem 4.1 can be repeated to show that all trajectories inside $\Psi_{\mu,\epsilon}$ approach $\bar{\mathcal{M}}_\mu$ as t tends to infinity. Our results can be summarized in the following theorem.

Theorem 4.3 *Under the hypotheses of Theorem 4.1, there exists $\mu^* > 0$ and for each $\mu \in (0, \mu^*]$, there exists $\epsilon^* = \epsilon^*(\mu) > 0$ and $\delta_\chi^* = \delta_\chi^*(\mu) > 0$, such that for all $\delta_\chi \in [0, \delta_\chi^*]$, $\mu \in (0, \mu^*]$, and $\epsilon \in (0, \epsilon^*]$, all the state variables of the closed-loop system are bounded and converge to an invariant manifold where $e = O(\delta_\chi)$.*

4.6 Simulation Example

To show the performance improvement with the conditional servocompensator, we consider a second-order system modelled by the equations

$$\dot{x}_1 = x_2, \;\; \dot{x}_2 = -\theta_1(x_1 - x_1^3/3!) + \theta_2 u, \;\; y = x_1 \tag{4.24}$$

with the reference signal $r(t) = r_0 sin(\omega t)$, which is generated by the exosystem

$$\dot{w} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} w, \;\; w^T(0) = [0, \;\; r_0], \;\; r(t) = w_1$$

It can easily be verified that

$$\chi = \frac{1}{\theta_2}[-\omega^2 w_1 + \theta_1(w_1 - w_1^3/3!)]$$

and that χ satisfies the identity $L_s^q\chi = c_0\chi + c_1L_s\chi + \cdots + c_{q-1}L_s^{q-1}\chi$ with $q = 4$, $c_0 = -9\omega^4$, $c_1 = 0$, $c_2 = -10\omega^2$ and $c_3 = 0$. We show the performance of four designs: the first is an ideal SMC, the second is a continuous approximation that does not use a servocompensator, the third uses the fourth order conventional servocompensator $\dot{\sigma} = S\sigma + Je_1$, and the last design uses the conditional servocompensator (4.13). In the first two designs, $\hat{s} = k_1e_1 + \hat{e}_2$, while $\hat{s} = K_1\sigma + k_1e_1 + \hat{e}_2$ in the last two designs. For all designs except the conventional servocompensator, the scalar k_1 is chosen as any positive constant. In the conditional servocompensator design, K_1 is chosen to make $(S - JK_1)$ Hurwitz. For the conventional servocompensator, k_1 and K_1 are chosen to make the matrix

$$\mathcal{A}_h = \begin{bmatrix} S & J \\ -K_1 & -k_1 \end{bmatrix}$$

Hurwitz (see [45]). The estimate $\hat{e}_2$ is provided by the high-gain observer

$$\dot{\hat{e}}_1 = \hat{e}_2 + g_1(e_1 - \hat{e}_1)/\epsilon, \quad \dot{\hat{e}}_2 = g_2(e_1 - \hat{e}_1)/\epsilon^2$$

with g_1 and g_2 chosen such that the polynomial $\lambda^2 + g_1\lambda + g_2$ is Hurwitz. The control is taken as $u = -k\ \mathrm{sat}(\hat{s}/\mu)$.

We use the following numerical values in the simulation: $\theta_1 = 1$, $\theta_2 = 3$, $\omega = 0.5\ rad/s$, $r_0 = 1$, $k = 10$, $\mu = 0.1$, $k_1 = 5$ in the first, second and last designs, with K_1 chosen to assign the eigenvalues of $S - JK_1$ at -0.5, -1, -1.5 and -2. For the third design, we choose k_1 and K_1 to assign the eigenvalues of $\mathcal{A}_h$ at -0.5, -1, -1.5, -2 and -3. The observer parameters are chosen as $g_1 = 6$, $g_2 = 5$ and $\epsilon = 0.01$. The results of the simulation are shown in Fig 4.1, and the improvement in the transient performance with the conditional servocompensator is clear. In particular, the transient response of this design is close (almost indistinguishable in the figure) to that of the ideal SMC design. As expected, the transient response of the CSMC design without a servocompensator is also close to that of the ideal SMC, but does not result in asymptotic error convergence, while the conditional servocompensator design does. Zero steady-state error is also achieved with the third design, which employs a conventional servocompensator, but at the expense of degraded transient performance.

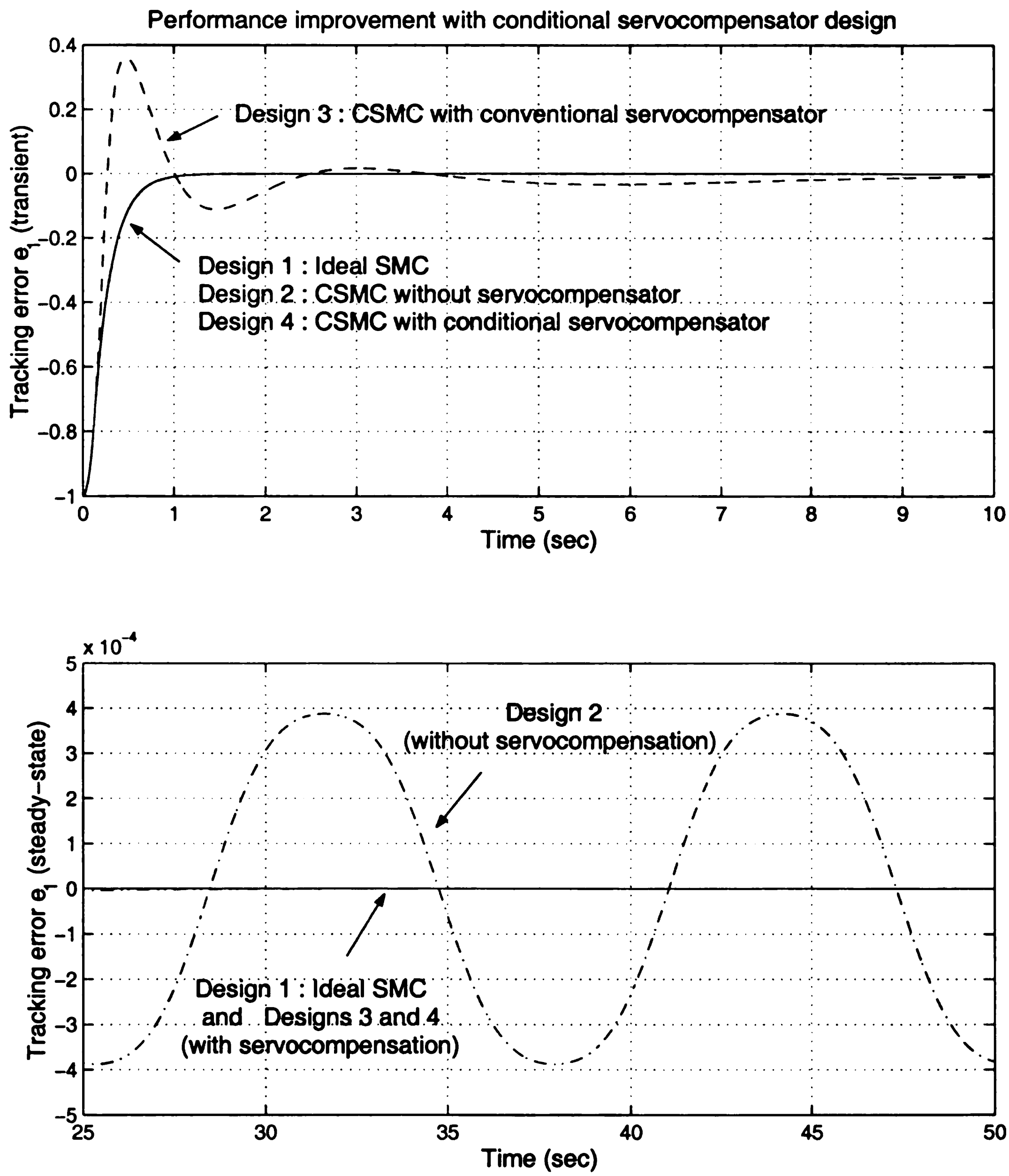

Figure 4.1: Performance improvement over the conventional servocompensator design using a conditional servocompensator.

Equation (4.24) represents an approximation of the pendulum equation. To show the effect of internal model perturbations, suppose that the system in question is the simple pendulum, described by the equation

$$\dot{x}_1 = x_2,\ \dot{x}_2 = -\theta_1 \sin(x_1) + \theta_2 u,\ y = x_1 \tag{4.25}$$

With the control objective the same as that in the previous simulation, it can be verified that, in the current case

$$\chi = \frac{1}{\theta_2}[-\omega^2 w_1 + \theta_1 \sin(w_1)]$$

so that Assumption 4.5 does not hold. Suppose that $\sin(w_1)$ is approximated by the successively higher order polynomials $p_1(w_1) = w_1$, $p_2(w_1) = w_1 - w_1^3/3!$, and $p_3(w_1) = w_1 - w_1^3/3! + w_1^5/5!$ respectively, to which correspond the perturbed nominal values of the steady-state control

$$\bar{\chi}_i = \frac{1}{\theta_2}[-\omega^2 w_1 + \theta_1 p_i(w_1)],\ i = 1, 2, 3$$

It can be verified that $\bar{\chi}_1$ satisfies (4.6) with $q = 2$, $c_0 = -\omega^2$, and $c_1 = 0$, while $\bar{\chi}_3$ does so with $q = 6$, $c_0 = -225\omega^6$, $c_1 = 0$, $c_2 = -259\omega^4$, $c_3 = 0$, $c_4 = -35\omega^2$, and $c_5 = 0$. The constants for $\bar{\chi}_2$ are as specified in the previous simulation. We compare the performance of three conditional servocompensator designs, of orders 2, 4, and 6, corresponding to the polynomial approximations $p_1(\cdot)$, $p_2(\cdot)$, and $p_3(\cdot)$ respectively. For the servocompensator of order 2, K_1 is chosen to assign the eigenvalues of $S - JK_1$ at -0.5 and -1, for that of order 4, at -0.5, -1, -1.5 and -2, and for that of order 6, at -0.5, -1, -1.5, -2, -2.5 and -5. All other values are retained from the previous simulation, except k, which is chosen as 20. The results are shown in Fig 4.2a. For comparison, we also show the performance of the conventional servocompensator design of [45], with the eigenvalues of $\mathcal{A}_h$ placed as in [45]. As expected from the results of Theorem 4.3, for both designs, there is a reduction in the steady state tracking error going from the lowest order approximation to the highest.

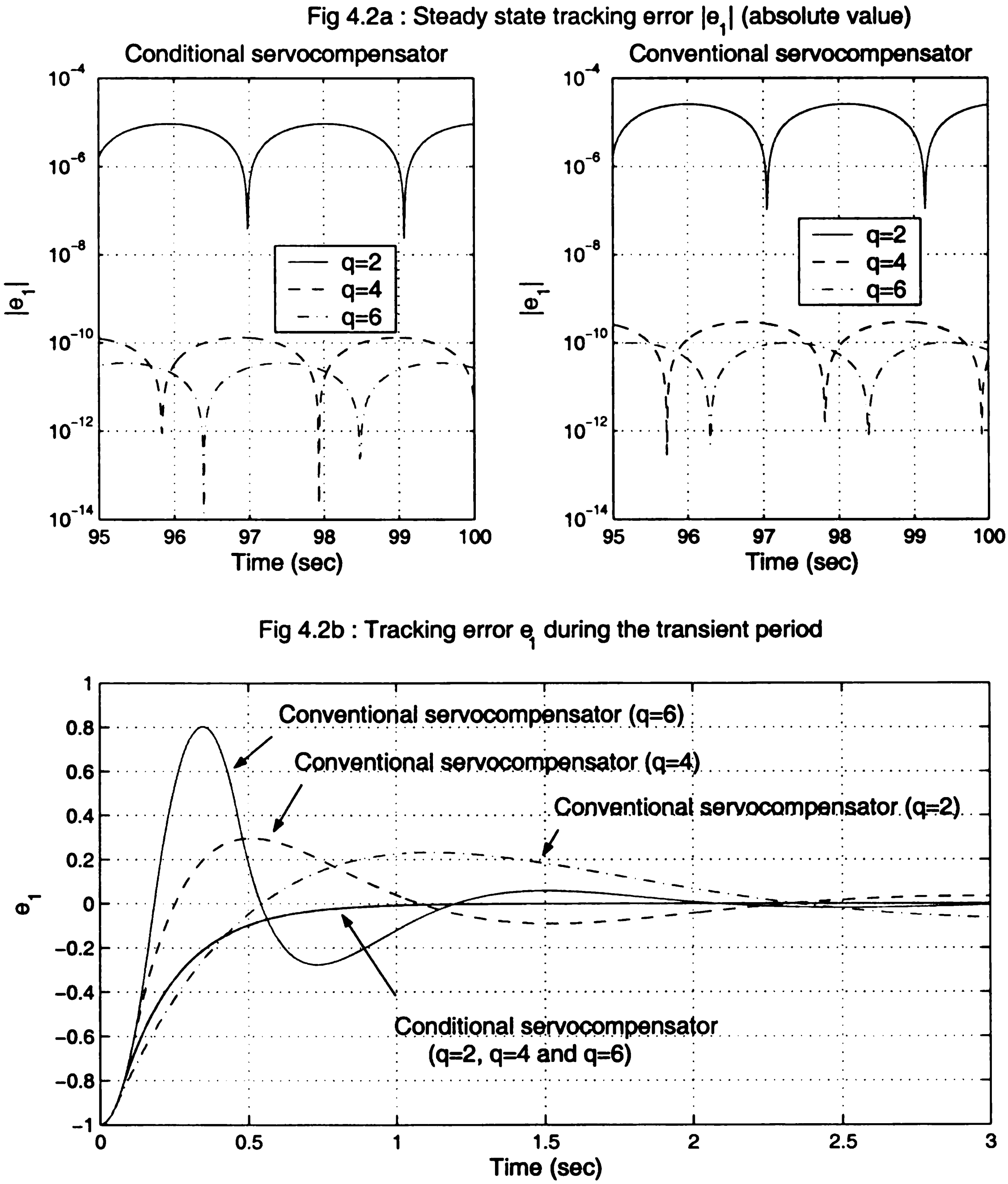

Figure 4.2: Effect of internal model perturbation on the tracking error.

Fig 4.2b shows the transient response of the controllers. We see that while the transient responses are almost identical for the three designs in the case of the conditional servocompensator (indistinguishable in the figure), they become progressively degraded as the order of the approximation increases in the case of the conventional servocompensator design.

4.7 Conclusions

In this chapter, we extended the conditional integrator design of Chapter 2 to that of a conditional servocompensator, also within a sliding mode control framework for minimum-phase nonlinear systems. As before, in the new approach, servocompensation is provided only "conditionally", i.e., inside the boundary layer, thus effectively eliminating the transient performance degradation brought about by the conventional servocompensator design. Analytical results are provided for regional and semi-global asymptotic tracking, and the improvement in performance is shown analytically by proving that the performance of the output feedback continuous sliding mode controller, with a conditional servocompensator, can be tuned to recover the performance of an ideal state feedback sliding mode controller, without a servocompensator.

We also studied the effect of internal model perturbations on the tracking error, and showed that in the presence of perturbation, the tracking error is ultimately bounded, with a bound that depends on the magnitude of the perturbation. In the case of such perturbations resulting from the approximation of a continuous function by polynomials, the magnitude of the perturbation can be made arbitrarily small, by increasing the order of the approximating polynomial. However, doing so increases the order of the internal model, and hence the system order. While in general, the transient response of a system becomes worse as its order increases, such is not the case with the conditional servocompensator. In particular, our result shows that the performance with the conditional servocompensator is always "close" to that with an ideal sliding mode controller of fixed order, regardless of the order of the conditional servocompensator. This shows as an advantage of the conditional servocompensator design over the conventional one, in which the transient reponse becomes progressively degraded as the order of the approximating internal model increases.

Extensions to relax the design to include an equivalent control component and allow the coefficient of the switching component to be error and/or time dependent should be straightforward, as should be extensions to the multi-input, multi-output case. Such extensions are carried out in the special case of integral control in [71].

Appendix A Derivation of (4.18)

The arguments of section 4.4 show that $s\dot{s} \le -\rho_6 g_0 |s|$ whenever $|\hat{s}| \ge \mu$, provided ϵ is small enough that $|N(\epsilon)\varphi| < \mu$. Next we consider s in the set

$$\{|s| \ge \mu(1-\delta_0)\} \cap \{|\hat{s}| \le \mu\}$$

where $\delta_0 < \rho_6/(4k)$ and ϵ is chosen small enough that $|N(\epsilon)\varphi| < \mu\delta_0$. Inside this set,

$$\mathrm{sat}(\hat{s}/\mu) = \hat{s}/\mu = (s - N(\epsilon)\varphi)/\mu$$

so that

$$\begin{aligned} s\dot{s} &\le |\Delta(\cdot)||s| - k|a_0(\cdot)||s|\frac{|s|}{\mu} + k|a_0(\cdot)||s|\frac{|N(\epsilon)\varphi|}{\mu} \\ &\le |\Delta(\cdot)||s| - k|a_0(\cdot)||s|(1-\delta_0) + k|a_0(\cdot)||s|\delta_0 \\ &\le -|a_0(\cdot)||s|(k - \gamma_1 - \rho_6/2) \end{aligned}$$

which along with (4.17) shows that $s\dot{s} \le -\frac{1}{2}\rho_6 g_0 |s|$ whenever $\{|s| \ge \mu(1-\delta_0)\}$, so that $s(t)$ reaches this set in finite time and stays there for all future time. Thereafter,

$$|s| + |K_1\sigma| \le \mu(1+\rho_2)$$

which along with the inequality

$$\dot{V}_\zeta \le -\|\zeta\|^2 + 2\|\zeta\|\ \|P_\zeta B_1\|\ (|s| + |K_1\sigma|)$$

can be used to show that $\dot{V}_\zeta \le -\|\zeta\|^2/2$ whenever $V_\zeta \ge \mu^2\rho_8$ for the choice $\rho_8 = 4\rho_3(1+\rho_2)^2$. This shows that $\zeta(t)$ reaches the set $\{V_\zeta \le \mu^2\rho_8\}$ in finite time and stays therein for all future time. It can be verified that inside Ω_μ, $\|e\| < 2\rho_4\mu$. Since

$\lim_{t\to\infty}\nu(t) = 0$, it follows that there is a finite time after which $\|(e^T, \nu^T)\| \leq 2\rho_4\mu$. From Assumption 4.7 and the definition of α_4, it follows that $\dot{V}_z \leq -\alpha_3(\|z\|)$ for $V_z \geq \alpha_4(\mu\rho_9)$, where $\rho_9 = 2\rho_4$. This shows that $z(t)$ reaches the set $\{V_z \leq \alpha_4(\mu\rho_9)\}$ in finite time and stays therein. Lastly, the fact that $\varphi(t)$ reaches the set $\{V_\varphi(\varphi) \leq \epsilon^2\rho_5\}$ in finite time and stays therein was already shown in Section 4.3.

This completes the proof of the statement that every trajectory starting inside the set $\Omega_c \times \Sigma_\epsilon$ enters the set $\Psi_{\mu,\epsilon}$ in finite time and stays therein for all future time.

Appendix B Derivation of (4.21)

We consider each term in the derivative of V of (4.20) separately and then arrange the derivative in a quadratic form of $\Pi = [\|z\| \; \|\zeta\| \; \|\tilde{\sigma}\| \; |\tilde{s}| \; \|\varphi\|]^T$.

Using Assumption 4.7 and the fact that $e = K_3\zeta + B_2(\tilde{s} - K_1\tilde{\sigma})$, an argument almost identical to the corresponding one in Appendix B of Chapter 2 can be used to show that the first term of the derivative of V satisfies

$$\frac{\partial V_{zz}}{\partial z}\phi_0(z, e, \nu, w, \theta) + \frac{\partial V_{zz}}{\partial w}S_0 w \le -\lambda_3\|z\|^2 + \|z\|(\lambda_9\|\zeta\| + \lambda_{10}\|\tilde{\sigma}\| + \lambda_{11}|\tilde{s}| + \lambda_{12}\|\nu\|)$$

for some positive constants λ_9 to λ_{12}.

Using $\dot{\zeta} = A_\zeta\zeta + B_1(\tilde{s} - K_1\tilde{\sigma})$, it can be verified that the second term in the derivative of V satisfies an inequality of the form

$$\lambda_5\dot{V}_\zeta \le -\lambda_5\|\zeta\|^2 + \|\zeta\|(\lambda_{13}\|\tilde{\sigma}\| + \lambda_{14}|\tilde{s}|)$$

for some positive constants λ_{13} and λ_{14}.

The expression for $\dot{\tilde{\sigma}}$ can be computed as

$$\begin{aligned}
\dot{\tilde{\sigma}} &= A_\sigma\sigma + J(s - N(\epsilon)\varphi) - (\mu/k)\ \mathrm{sign}(L_gL_f^{\rho-1}h)\ M(\partial\tau/\partial w)S_0 w \\
&= A_\sigma\sigma + J(s - N(\epsilon)\varphi) - (\mu/k)\ \mathrm{sign}(L_gL_f^{\rho-1}h)\ MS\tau(w,\theta) \\
&= A_\sigma\tilde{\sigma} + J(\tilde{s} - N(\epsilon)\varphi) + S\bar{\sigma} - (\mu/k)\ \mathrm{sign}(L_gL_f^{\rho-1}h)\ MS\tau(w,\theta) \\
&= A_\sigma\tilde{\sigma} + J(\tilde{s} - N(\epsilon)\varphi)
\end{aligned}$$

from which it can be shown that the third term satisfies an inequality of the form

$$\lambda_6\dot{V}_\sigma \le -\lambda_6\|\tilde{\sigma}\|^2 + \|\tilde{\sigma}\|(\lambda_{15}|\tilde{s}| + \lambda_{16}\|\varphi\|)$$

for some positive constants λ_{15} and λ_{16}.

The expression for $\dot{\tilde{s}}$ is given by

$$
\begin{aligned}
\dot{\tilde{s}} &= \dot{s} - K_1(\mu/k)\ \mathrm{sign}(L_gL_f^{\rho-1}h)\ MS\tau(w,\theta) \\
&= \Delta(\cdot) - k|a_0(\cdot)| \left(\frac{s - N(\epsilon)\varphi}{\mu}\right) - K_1(\mu/k)\ \mathrm{sign}(L_gL_f^{\rho-1}h)\ MS\tau(w,\theta) \\
&= b_0(\cdot) + K_1A_\sigma\sigma + K_1J(s - N(\epsilon)\varphi) + K_2\ [e_2\ e_3\ \cdots\ e_\rho]^T \\
&\quad - k|a_0(\cdot)| \left(\frac{s - N(\epsilon)\varphi}{\mu}\right) - K_1(\mu/k)\ \mathrm{sign}(L_gL_f^{\rho-1}h)\ MS\tau(w,\theta)
\end{aligned}
$$

It can be verified that $K_1A_\sigma\sigma + K_1Js = K_1A_\sigma\tilde{\sigma} + K_1J\tilde{s} + K_1S\bar{\sigma}$. Using this, along with the fact that $K_1S\bar{\sigma} = K_1S(\mu/k)\ \mathrm{sign}(L_gL_f^{\rho-1}h)\ M\tau(w,\theta)$, and $SM = MS$, we have

$$
\begin{aligned}
\dot{\tilde{s}} &= b_0(\cdot) + K_1A_\sigma\tilde{\sigma} + K_1J\tilde{s} - K_1JN(\epsilon)\varphi - k|a_0(\cdot)| \left(\frac{s - N(\epsilon)\varphi}{\mu}\right) + K_2\ [e_2\ e_3\ \cdots\ e_\rho]^T \\
&= K_4\zeta + K_5\tilde{\sigma} + k_\rho\tilde{s} - K_1JN(\epsilon)\varphi + b_0(\cdot) - k|a_0(\cdot)| \left(\frac{s - N(\epsilon)\varphi}{\mu}\right)
\end{aligned}
$$

In arriving at the last line, we have used the property that $K_2\ [e_2\ e_3\ \cdots\ e_\rho]^T$ can be rewritten as $K_4\zeta + k_{\rho-1}(\tilde{s} - K_1\tilde{\sigma})$ for a suitable K_4, and then appropriately defined K_5 and k_ρ.

Using the property that $b_0(0,0,0,w,\theta) + a_0(0,0,0,w,\theta)\ \bar{u} = 0$, where $\bar{u} = -k\ \mathrm{sign}(L_gL_f^{\rho-1}h)\ (K_1\bar{\sigma}/\mu)$, and rewriting

$$
\begin{aligned}
b_0(\cdot) - k|a_0(\cdot)| \left(\frac{s - N(\epsilon)\varphi}{\mu}\right) &= b_0(z,e,\nu,w,\theta) - b_0(0,0,0,w,\theta) \\
&\quad + b_0(0,0,0,w,\theta) + a_0(0,0,0,w,\theta)\ \bar{u} \\
&\quad + k\ |a_0(0,0,0,w,\theta)|\ (K_1\bar{\sigma}/\mu) - k\ |a_0(0,0,0,w,\theta)| \left(\frac{\tilde{s} - N(\epsilon)\varphi + K_1\bar{\sigma}}{\mu}\right) \\
&\quad + k\ |a_0(0,0,0,w,\theta)| \left(\frac{s - N(\epsilon)\varphi}{\mu}\right) - k\ |a_0(z,e,\nu,w,\theta)| \left(\frac{s - N(\epsilon)\varphi}{\mu}\right)
\end{aligned}
$$

it can be verified that the fourth term in the derivative of V satisfies an inequality of the form

$$\tilde{s}\dot{\tilde{s}} \le -\left(\frac{kg_0}{\mu} - \lambda_{17}\right)|\tilde{s}|^2 + |\tilde{s}|(\lambda_{18}\|z\| + \lambda_{19}\|\zeta\| + \lambda_{20}\|\tilde{\sigma}\| + \lambda_{21}\|\varphi\| + \lambda_{22}\|\nu\|)$$

for some positive constants λ_{17} to λ_{22}, where λ_{17} is independent of μ.

A similar argument can then be used to show that the final term satisfies an inequality of the form

$$\dot{V}_\varphi \le -\left(\frac{1}{\epsilon} - \lambda_{23}\right)\|\varphi\|^2 + \|\varphi\|(\lambda_{24}\|z\| + \lambda_{25}\|\zeta\| + \lambda_{26}\|\tilde{\sigma}\| + \lambda_{27}|\tilde{s}| + \lambda_{28}\|\nu\|)$$

for some positive constants λ_{23} to λ_{28}, where λ_{23} is independent of ϵ.

It follows that the derivative of V can be arranged in the form

$$\dot{V} \le -\Pi^T \mathcal{P} \Pi + \lambda_{29}\|\Pi\|\|\nu\| \tag{4.26}$$

where λ_{29} is a positive constant, and the symmetric matrix $\mathcal{P}$ has the form

$$\mathcal{P} = \begin{bmatrix} \lambda_3 & -\lambda_{1a} & -\lambda_{1b} & -\lambda_{1c} & -\lambda_{1d} \\ & \lambda_5 & -\lambda_{2b} & -\lambda_{2c} & -\lambda_{2d} \\ & & \lambda_6 & -\lambda_{3c} & -\lambda_{3d} \\ & & & \dfrac{kg_0}{\mu} - \lambda_{17} & -\lambda_{4d} \\ & & & & \dfrac{1}{\epsilon} - \lambda_{23} \end{bmatrix}$$

where the positive constants λ_{23}, and λ_{1d} to λ_{4d} are independent of ϵ; λ_{17} and λ_{1c} to λ_{3c} are independent of μ; λ_{1b} and λ_{2b} are independent of λ_6; and λ_{1a} is independent of λ_5. Therefore, by choosing λ_5 large enough, then λ_6 large enough, then μ small enough, and lastly ϵ small enough, we can make $\mathcal{P}$ positive definite. The equivalence between (4.26) and (4.21) follows from Assumption 4.7 and (4.20).

Chapter 5

Applications

5.1 Introduction

In this chapter, we apply the controller designs of Chapters 2 through 4 to two applications : temperature control of a continuous stirred tank reactor (CSTR) and position control of a permanent magnet stepper motor (PMSM).

For the CSTR, our interest is in the regulation problem where the references and disturbances are asymptotically constant and the control magnitude is constrained, so that the universal integral regulator of Section 2.6 is a natural choice for the controller. Since the particular class of CSTR systems that we study in this chapter is relative degree one, the universal integral regulator is simply a PI controller with anti-windup. Control of the CSTR problem under input constraints and varying assumptions on the class of CSTR, available measurements, and control objectives has been widely studied; see, for example, [1, 2, 40, 41, 73, 74] and the references therein. Some of these references deal specifically with PI controllers [1, 2, 40] and the issue of anti-windup.

For the PMSM, we examine both the integral control design of Chapters 2 and 3 for the case of asymptotically constant references and disturbances, as well as the servocompensator design of Chapter 4 for the case of references and disturbances generated by a neutrally stable exosystem. Whereas the design in Chapter 4 was done for SISO systems, as mentioned in the concluding remarks of the chapter, the results could have been extended to MIMO systems, and we do so by simulation for the PMSM. For the output feedback design of Chapter 2, we show that the controller reduces to a MIMO version of the universal integral regulator design of Section 2.6. The problem of position control of a PMSM that we study in this chapter has been widely studied, under varying assumptions, by several authors; see, for example, [9, 10, 11, 17, 20, 48, 51, 52, 53, 59, 61, 76, 77]. Also, while we specifically apply our designs to the PMSM, the results can be extended to other types of motors, such as permanent magnet synchronous motors [6, 54], and brushed and brushless dc motors, studied, for example, in [20].

5.2 Continuous Stirred Tank Reactor

The continuous stirred tank reactor is widely used in the chemical process industry. It consists of a well-stirred tank into which there is a continuous flow of reacting material and from which the reacted or partially reacted material passes continuously. It is generally assumed to be homogeneous and therefore modelled as having no spatial variations in concentration, temperature or reaction rate throughout the vessel. In fact, the main assumption concerning the CSTR dynamics is that perfect mixing occurs inside the reactor. A schematic of such a chemical reactor, with typical process symbolism [3], is shown in Fig 5.1.

While a wide variety of methodologies, including input-output feedback linearization [74] and adaptive versions of feedback linearization [26, 50] have been explored for the stabilization of chemical reactors, in virtually all present day industrial applications the problem is efficiently solved using PI controllers. Whereas traditional

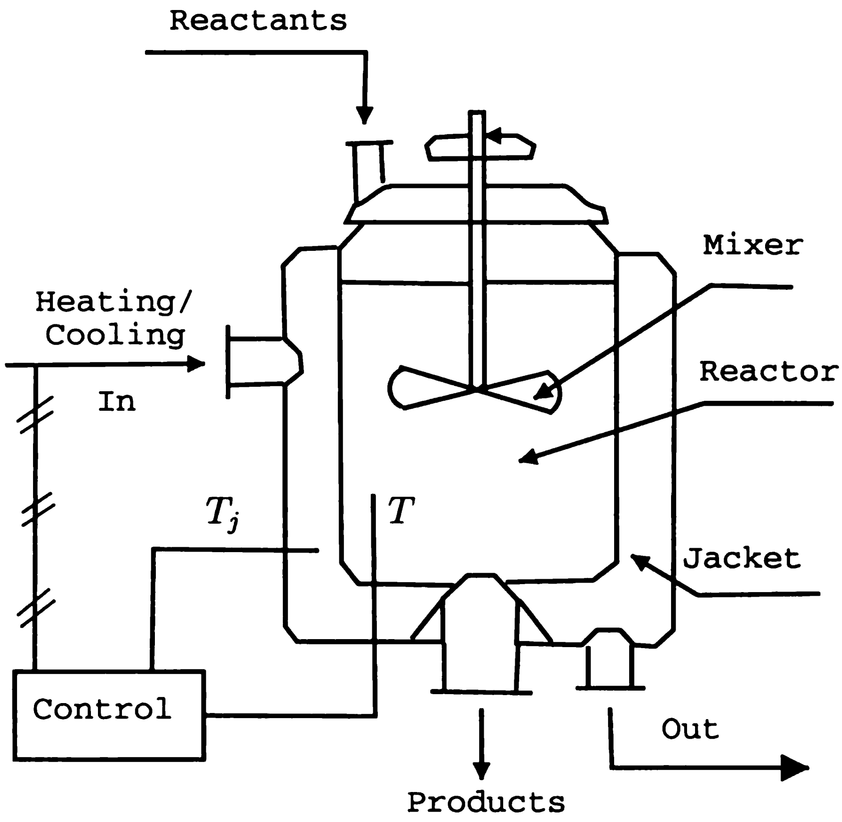

Figure 5.1: Block schematic of a CSTR

analysis of CSTRs controlled with PI control algorithms resorted to linear system analysis together with linearized models, recently, there have been few works which address the nonlinear problem directly. The stabilization of chemical reactors by output feedback with PI-type controllers has been reviewed and treated in detail in the Ph.D. thesis of Jadot [40]. A robust control scheme in the face of uncertain kinematics for a class of CSTRs has been proposed by Alvarez-Ramirez *et al* [2], where it has been shown that the proposed controller has the structure of PI control. A more recent result, which goes beyond just the analysis of closed-loop stability, and focuses on transient performance, has been discussed in Alvarez-Ramirez *et al* [1]. In this chapter, we focus our attention on the same class of CSTRs studied in [1, 2]. The design of Section 2.6, when applied to this class of CSTRs, yields a PI controller with an anti-windup structure.

5.2.1 System Model

We consider the first-order, irreversible, exothermal chemical reaction, which occurs in a constant volume continuous stirred tank reactor. The process dynamics are given by [1, 2]

$$\left.\begin{aligned} \dot{c} &= \theta(c_{in} - c) - \mathcal{R}(c, T), \\ \dot{T} &= \theta(T_{in} - T) + \Delta H_r \mathcal{R}(c, T) + \gamma(T_j - T) \end{aligned}\right\} \tag{5.1}$$

where c is the reactor concentration, c_{in} is the inlet concentration, T is the reactor temperature, T_{in} is the inlet temperature, T_j is the jacket temperature, ΔH_r is the reaction enthalpy, θ is the dilution rate, $\gamma > 0$ is the heat transfer coefficient, and $\mathcal{R}(c, T)$ is the reaction rate, given by

$$\mathcal{R}(c, T) = c\beta_0 \exp(-E_A/RT)$$

The control objective is to regulate the reactor temperature T via manipulation of the jacket temperature T_j. The system (5.1) has relative degree one uniformly in the system parameters, and satisfies Assumption 2.2 for all physically allowable values of the parameters, i.e., given any desired reactor temperature $T_{eq} > 0$, there is a unique set of equilibrium values c_{eq} and T_{jeq} of the reactor concentration c and the jacket temperature T_j respectively, such that

$$\left.\begin{aligned} 0 &= \theta(c_{in} - c_{eq}) - \mathcal{R}(c_{eq}, T_{eq}), \\ 0 &= \theta(T_{in} - T_{eq}) + \Delta H_r \mathcal{R}(c_{eq}, T_{eq}) + \gamma(T_{jeq} - T_{eq}) \end{aligned}\right\}$$

We assume the following values for the constants, taken from [2], $\beta_0 = \exp(25)$, $E_A/R = 10^4$, $\Delta H_r = 200$, $\theta = 1$, $\gamma = 1$, $c_{in} = 1$, $T_{in} = 350$ and $\bar{T}_j = 350$, where $\bar{T}_j$ is the nominal value of the jacket temperature. Due to limitations in cooling/heating equipment, the jacket temperature T_j is subject to saturation constraints. Similar to [2], we assume that $T_j \in [300, 400]$.

5.2.2 PI controller design

Defining the states $\eta = c$, $\xi = T$, input $u = \Delta T_j = T_j - \bar{T}_j$ and output $y = T$, we rewrite (5.1) in the form of (2.5), where the functions $\phi(\cdot)$, $b(\cdot)$ and $a(\cdot)$ are given by [1]

$$\begin{aligned} \phi(\eta, \xi) &= 1 - \eta - \eta \exp\left(25 - \tfrac{10^4}{\xi}\right), \\ b(\eta, \xi) &= 200\ \eta \exp\left(25 - \tfrac{10^4}{\xi}\right) - 2\xi + 700,\ \ a(\eta, \xi) = 1 \end{aligned} \tag{5.2}$$

Note that for the specified numerical values for the constants, the CSTR has an unstable open-loop equilibrium point $(c_{eq}, T_{eq}) = (0.5, 400)$ and that the control u is bounded in magnitude by 50.

For the purpose of simulation, we assume that the desired temperature at which the CSTR is to be stabilized corresponds to the unstable equilibrium point $(c, T) = (0.5, 400)$. Therefore, we take the reference T_{ref} as a step of magnitude 400. Since the system is relative degree one, the sliding surface function is taken as

$$s = k_0 \sigma + e, \ \ e = \xi - r \tag{5.3}$$

where $k_0 > 0$ and σ is the output of the conditional integrator

$$\dot{\sigma} = -k_0 \sigma + \mu \ \mathrm{sat}(s/\mu) \tag{5.4}$$

The control is taken as

$$u = -k \ \mathrm{sat}(s/\mu) \tag{5.5}$$

which, as mentioned in Section 2.6, is a PI controller with anti-windup.

[1]Even though we assume that all the constants in (5.1) are known, so that $\phi(\cdot)$, $b(\cdot)$ and $a(\cdot)$ do not depend on an unknown parameter, it is clear that it is possible to allow for such unknown terms.

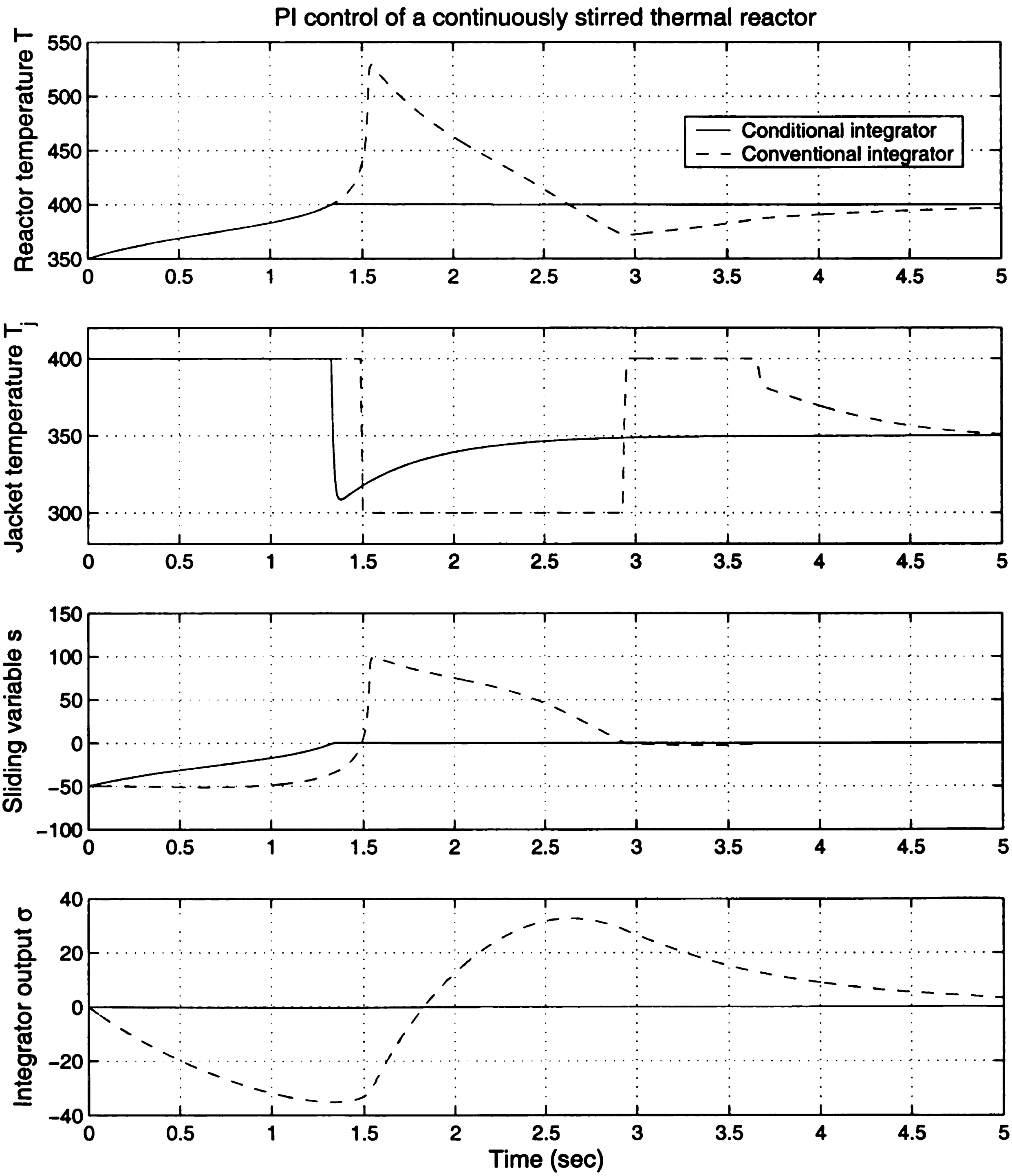

Figure 5.2: Temperature control of CSTR using a PI controller with conditional integrator

Numerical values for the controller parameters are chosen as $k = 50$, $\mu = 0.5$ and $k_0 = 1$. The initial conditions of the states are taken as $\eta(0) = 0.5$, $\xi(0) = 350$, and $\sigma(0) = 0$. Fig 5.2 shows the reactor temperature T, the jacket temperature T_j, the sliding variable s and the integrator output σ. For comparison, we also plot the variables for the corresponding conventional integrator design, which uses $\dot{\sigma} = e_1$, with expressions for s and u and other numerical values retained from the conditional design. The following observations can be made from the figure. The inclusion of the conventional integrator causes the sliding condition $s\dot{s} < 0$ to be violated, since the control needs to be large enough to overcome the term $k_0|\dot{\sigma}| = k_0|e|$, which can be large if $|e(0)|$ is large. The subplot for the sliding variable s shows that the sliding condition is not satisfied, and the integrator winds up, causing the control to be in saturation for a longer period of time. The large buildup in the integrator (windup) with the conventional integrator design can be observed in the last subplot, and the extended period of saturation in the controller as a consequence is clear from the subplot of the jacket temperature T_j. On the other hand, with the conditional integrator, $k_0|\dot{\sigma}| = k_0|-k_0\sigma + \mu \operatorname{sat}(s/\mu)|$, which is "small" even with large k_0, provided μ is small. From the subplot for s, we see that the sliding condition is satisfied for the conditional integrator design. It is clear from the figure that the tracking performance with the conventional integrator design is significantly degraded over the conditional integrator design.

Suppose that, in order to have the sliding condition satisfied, we choose k_0 small in the conventional integrator design. To that end, we let $k_0 = 0.01$ in the conventional integrator design, but retain the value of $k_0 = 1$ for the conditional integrator design. It can be verified that the sliding condition is satisfied for both designs for the specified values. The results are shown in Fig 5.3, and we see from the figure that the convergence of the error to zero is sluggish with the conventional integrator design when compared to the conditional integrator design.

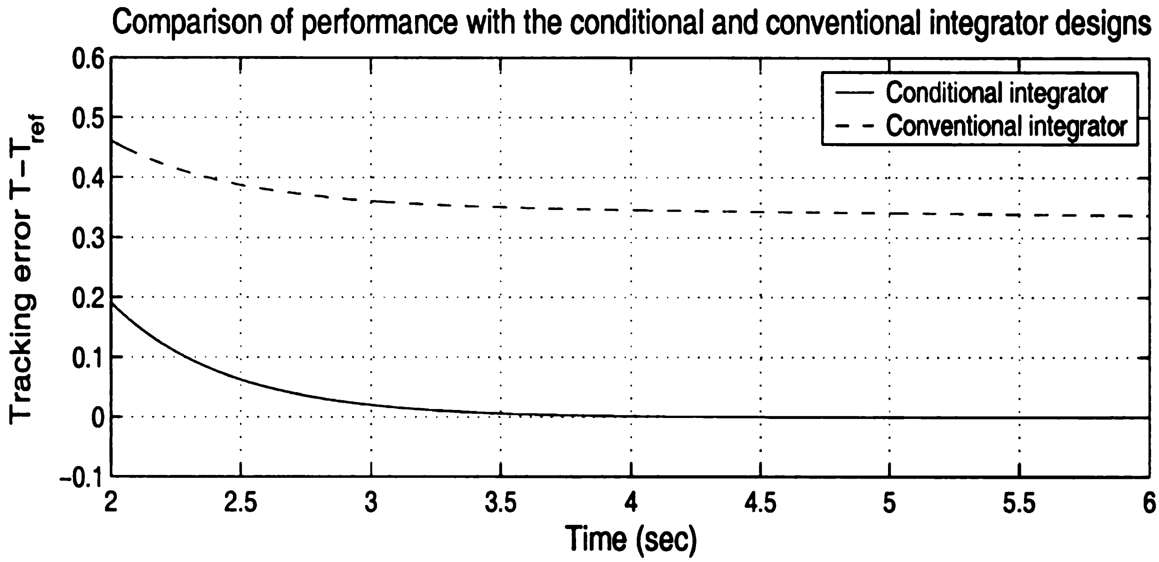

Figure 5.3: Comparison of performance with the conditional and the conventional integrator designs.

As mentioned in Section 2.6, the only precise information about the CSTR system that the control (5.5) uses is its relative degree and the sign of the high frequency gain. It does not use precise information about any system parameters. To show that the controller is robust to uncertainties, we consider a -10% step disturbance in T_{in} at $t = 6$ and a setpoint change in T_{ref} from 400 to 410 at $t = 8$, which are the same uncertainties considered in [2] to demonstrate the robustness of their PI controller. The results are shown in Fig 5.4, and we see that the response is not degraded in the face of such uncertainties. In fact, the effect of the step disturbance in T_{in} at $t = 6$ is almost negligible, as can be seen from the subplot of the reactor temperature T, and can be explained as follows. When the parameter T_{in} changes at $t = 6$, it changes the equilibrium point. Prior to this change, s is inside the boundary layer $\{|s| \leq \mu\}$. The magnitude of the change is not large enough to force s outside the boundary layer, as can be inferred from the fact that the control does not reach its saturation limits. Since $e = s - k_0\sigma$, from $|s| \leq \mu$, and $k_0|\sigma| \leq \mu$, it follows that $|e| \leq 2\mu$ in this case. For the step change in T_{ref} at $t = 8$, we have $e(8^+) = -10$, which forces s outside the boundary layer. The trajectory re-enters the boundary layer at $t \simeq 8.3$, after which the error e satisfies $|e| \leq 2\mu$ and asymptotically approaches zero. Our results are comparable to the ones reported in [2].

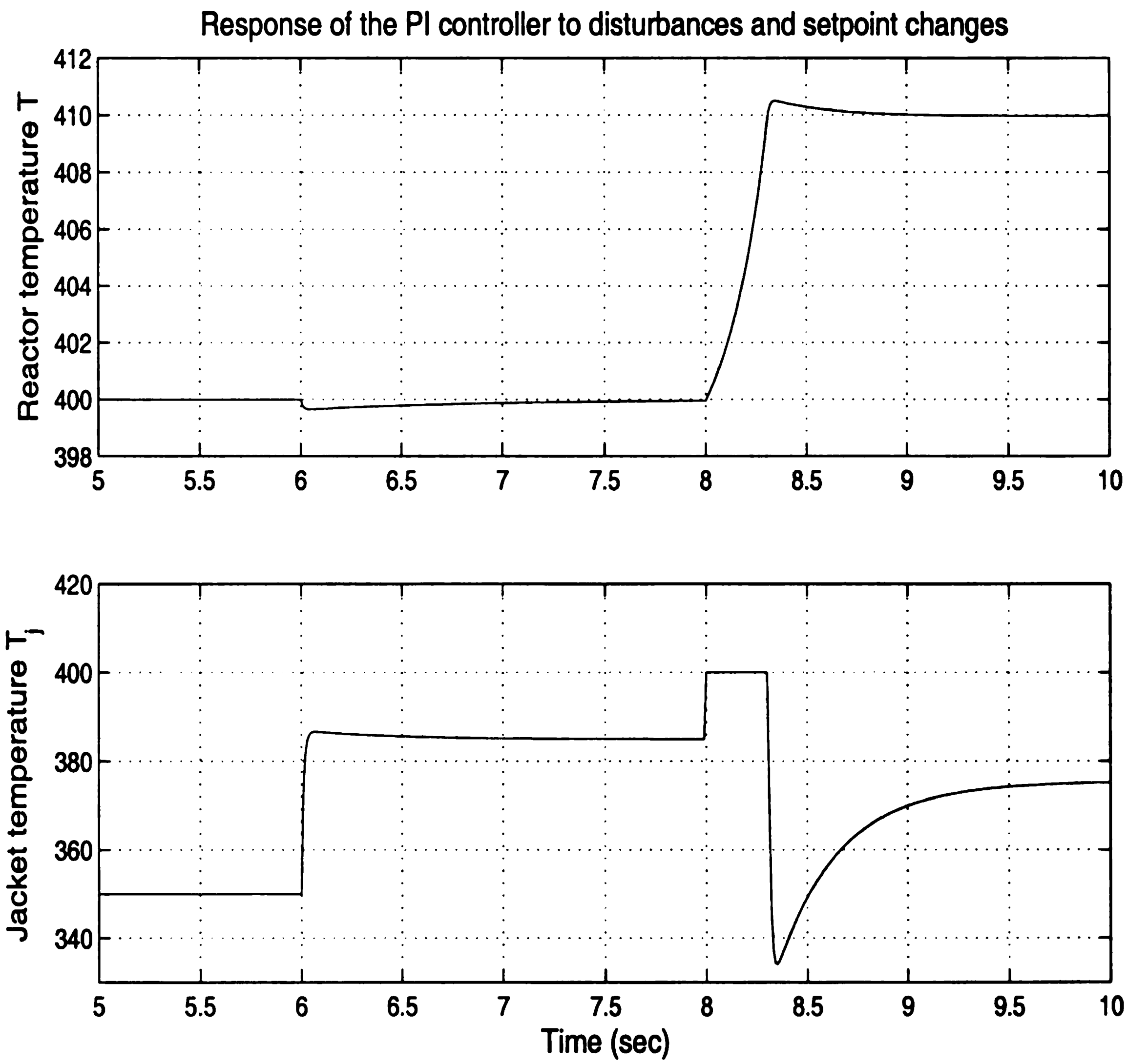

Figure 5.4: Effect of uncertainties on the response of the PI controller with conditional integrator

5.3 Permanent Magnet Stepper Motor

Permanent magnet stepper motors (PMSM) have become a popular alternative to the traditionally used brushed DC motors (BDCM) for many high performance motion control applications for several reasons : better reliability due to the elimination of mechanical brushes, better heat dissipation as there are no rotor windings, higher torque-to-inertia ratio due to a lighter rotor, lower price, and easy interfacing with digital systems [20, 77]. They are now widely used in numerous motion control applications such as robotics, printers, process control systems etc. Some of the drawbacks of PM machines when operated in open loop are the occurrence of large overshoots and settling times, especially when the load inertia is high, and the fact that microstepping is not possible in the open loop mode of operation. As a result, over the years, many control algorithms that can improve the performance of PMSMs in a closed loop operation have been examined.

Zribi and Chiasson [76] used the technique of exact feedback linearization using full state feedback, with extensions to the partial state feedback case in [10, 17], and experimental validation of the controller in [10]. Adaptive solutions to the problem, under varying assumptions on the measurable states and on what parameters in the system are partially or wholly known, have appeared, for example, in the works of Dawson and co-workers [11, 20], Khorrami and co-workers [48, 51, 52, 53, 61] and others [59]. A sliding mode controller along with implementation results was reported in Zribi *et al.* [77]. In order to avoid the chattering problem associated with the "static" discontinuous SMC, a "dynamic" or "second-order" SMC was proposed, where the discontinuitites were relegated to the derivatives of the control input. Integral action occurs in the second-order SMC as a result of designing with the derivative of the input, so that the controller in [77] is the closest in similarity to the SMC with (conditional) integral action that we design.

5.3.1 System Model

A schematic of a PMSM that has a slotted stator with two phases, and a PM rotor is shown in Fig 5.5.

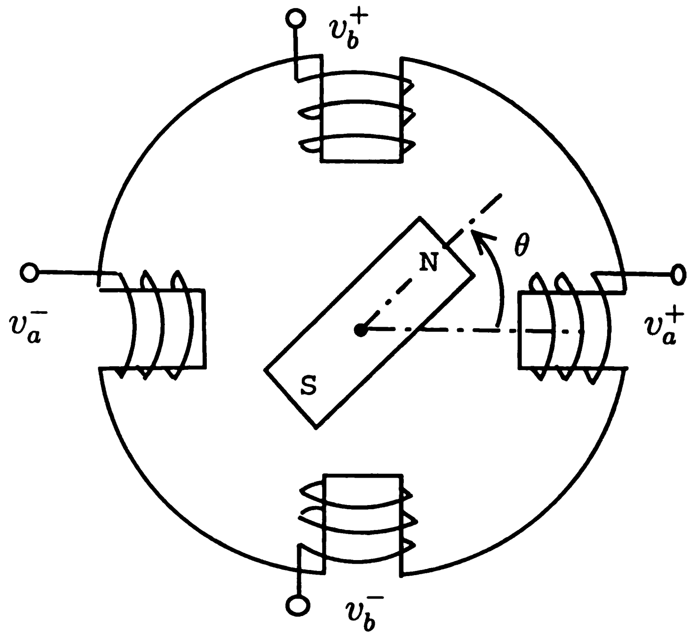

Figure 5.5: Schematic of a two phase PMSM.

The mathematical model of the PMSM is given below [10, 76, 77]

$$\left.\begin{array}{l} \frac{di_a}{dt} = \frac{1}{L}(v_a - Ri_a + K_m\omega \sin(N_r\phi)) \\ \frac{di_b}{dt} = \frac{1}{L}(v_b - Ri_b + K_m\omega \cos(N_r\phi)) \\ \frac{d\omega}{dt} = \frac{1}{J}(-K_m i_a \sin(N_r\phi) + K_m i_b \cos(N_r\phi) - B\omega - \tau_L) \\ \frac{d\phi}{dt} = \omega \end{array}\right\} \tag{5.6}$$

where i_a is the current in winding A, i_b is the current in winding B, ϕ is the angular displacement of the shaft of the motor, ω is the angular velocity of the shaft of the motor, v_a is the voltage across winding A, v_b is the voltage across winding B, N_r is the number of rotor teeth, J is the rotor and load inertia, B is the viscous friction coefficient, L and R are the inductance and resistance respectively of the phase windings, K_m is the motot torque (back-emf) constant, and τ_L is the load torque. The model neglects the slight magnetic coupling between the phases, the small change in inductance as a function of the rotor position, the *detent* torque [48], and the variation in inductance due to magnetic saturation. A nonlinear transformation, known as the direct-quadrature (DQ) transformation can be used to transform these equations into a form more suitable for designing nonlinear controllers, i.e., one to which feedback linearization techniques can be applied [10, 76, 77]. This transformation changes the frame of reference from the fixed phase axes to axes that are moving with the rotor. The DQ transformation from the fixed axes variables (x_a, x_b) to the dq axes variables (x_d, x_q) is defined by

$$\begin{bmatrix} x_d \\ x_q \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} \cos(N_r\phi) & \sin(N_r\phi) \\ -\sin(N_r\phi) & \cos(N_r\phi) \end{bmatrix} \begin{bmatrix} x_a \\ x_b \end{bmatrix}$$

The direct current i_d corresponds to the component of the stator magnetic field along the axis of the rotor magnetic field, while the quadrature current i_q corresponds to the orthogonal component. Defining the states, inputs, and outputs as $x_1 = i_d$, $x_2 = i_q$, $x_3 = \omega$, $x_4 = \phi$, $u_1 = v_d$, $u_2 = v_q$, $y_1 = i_d$, and $y_2 = \phi$, it can be verified that (5.6) can be rewritten in the form of the following state model for the PMSM

$$\left.\begin{aligned} \dot{x}_1 &= -k_1x_1 + k_5x_2x_3 + k_6u_1 \\ \dot{x}_2 &= -k_1x_2 - k_5x_1x_3 - k_2x_3 + k_6u_2 \\ \dot{x}_3 &= k_3x_2 - k_4x_3 - d_0 \\ \dot{x}_4 &= x_3 \\ y_1 &= x_1 \\ y_2 &= x_4 \end{aligned}\right\} \tag{5.7}$$

where the constants k_1 to k_6 and d_0 are related to N_r, J, B, L, R, K_m and τ_L by

$$k_1 = \frac{R}{L}, k_2 = \frac{K_m}{L}, k_3 = \frac{K_m}{J}, k_4 = \frac{B}{J}, k_5 = N_r, k_6 = \frac{1}{L} \text{ and } d_0 = \frac{\tau_L}{J}$$

5.3.2 Integral Control - State Feedback

It is desired that the rotor angular position and direct-axis current track given references $\phi_d(t)$ and $I_{dd}(t)$ that tends to constant values as $t \to \infty$. It can be verified that the system has full vector relative degree $\rho = \{1, 3\}$, globally in R^4. Consequently the sliding surface functions s_1 and s_2 are given by

$$\left.\begin{aligned} s_1 &= k_0^1\sigma_1 + e_1, \; e_1 \stackrel{\text{def}}{=} y_1 - I_{dd}, \\ s_2 &= k_0^2\sigma_2 + k_1^2 e_2 + k_2^2\dot{e}_2 + \ddot{e}_2, \quad e_2 \stackrel{\text{def}}{=} y_2 - \phi_d \end{aligned}\right\} \tag{5.8}$$

where σ_1 and σ_2 are the outputs of the conditional integrators

$$\left.\begin{aligned} \dot{\sigma_1} &= -k_0^1\sigma_1 + \mu_1 \text{sat}(s_1/\mu_1) \\ \dot{\sigma_2} &= -k_0^2\sigma_2 + \mu_2 \text{sat}(s_2/\mu_2) \end{aligned}\right\} \tag{5.9}$$

$k_0^1, k_0^2 > 0$, and k_1^2 and k_2^2 are chosen such that the polynomial $x^2 + k_2^2 x + k_1^2$ is Hurwitz.

Since $\ddot{e_2} = k_3x_2 - k_4x_3 - d_0 - \phi_d^{(2)}$ is required to construct s_2, the parameters k_3, k_4 and d_0 will need to be known. We assume that k_1, k_2 and k_6 are unknown, corresponding to uncertainties in the resistance R and inductance L of the phase windings, while k_5, being the number of rotor teeth, is precisely known. Consequently, the vector of unknown parameters is taken as $\theta = [k_1, k_2, k_6]^T$. It can be verified that the expressions for $\dot{s}_i$ of Section 3.3 take the form

$$\left.\begin{aligned} \dot{s}_1 &= k_0^1(-k_0^1\sigma_1 + \mu_1\text{sat}(s_1/\mu_1)) + F_1(x, e_1, I_{dd}^{(1)}, \theta) + a_{11}(x,\theta)u_1 \\ \dot{s}_2 &= k_0^2(-k_0^2\sigma_2 + \mu_2\text{sat}(s_2/\mu_2)) + F_2(x, e_2, \dot{e}_2, \ddot{e}_2, \phi_d^{(3)}, \theta) + a_{22}(x,\theta)u_2 \end{aligned}\right\} \tag{5.10}$$

with $F_1(\cdot) = -\theta_1x_1 + k_5x_2x_3 - I_{dd}^{(1)}$, $F_2(\cdot) = k_1^2\dot{e}_2 + k_2^2\ddot{e}_2 - \phi_d^{(3)} - k_4(k_3x_2 - k_4x_3 - d_0) - k_3\theta_1x_2 - k_3k_5x_1x_3 - k_3\theta_2x_3$, $a_{11}(\cdot) = \theta_3$, and $a_{22}(\cdot) = k_3\theta_3$. Specifically, the matrix $A(x,\theta)$ in Assumption 3.5 is given by $A(x,\theta) = \text{diag}[\theta_3, k_3\theta_3]$, so that the natural choices for $\Gamma(x,\theta)$ and $\hat{A}(x)$ become $\Gamma(x,\theta) = (\theta_3/\hat{\theta}_3)I_{2\times2}$ and $\hat{A}(x) = \text{diag}[\hat{\theta}_3, k_3\hat{\theta}_3]$, where $\hat{\theta}_3 > 0$ is a nominal value of θ_3. With this choice, the control u in (3.8) becomes

$$u_i = \frac{-\hat{F}_i(x, e, \varpi) - \beta_i(x, e, \varpi)\ \mathrm{sat}(s_i/\mu_i)}{\hat{a}_{ii}}, \quad i = 1, 2 \tag{5.11}$$

where $e^T = [e_1, e_2, \dot{e}_2, \ddot{e}_2]$, and $\varpi^T = [I_{dd}^{(1)}, \phi_d^{(3)}]$. We will choose the nominal control component $\hat{F}_i(\cdot)$ to cancel all known/nominal terms in $F_i(\cdot)$, i.e.,

$$\begin{aligned} \hat{F}_1(\cdot) &= -\hat{\theta}_1 x_1 + k_5 x_2 x_3 - I_{dd}^{(1)} \\ \hat{F}_2(\cdot) &= k_1^2 \dot{e}_2 + k_2^2 \ddot{e}_2 - \phi_d^{(3)} - k_4(k_3 x_2 - k_4 x_3 - d_0) - k_3\hat{\theta}_1 x_2 - k_3 k_5 x_1 x_3 - k_3 \hat{\theta}_2 x_3 \end{aligned}$$

where $\hat{\theta}_1 > 0$ and $\hat{\theta}_2 > 0$ are nominal values of θ_1 and θ_2 respectively. To make the choice of $\beta_i(\cdot)$ precise, suppose that $\theta_i \in [\theta_i^m, \theta_i^M]$, where $0 < \theta_i^m < \theta_i^M$ and θ_i^m and θ_i^M are known. Let

$$\Delta_i(\cdot) = F_i(\cdot) - (\theta_3/\hat{\theta}_3)\hat{F}_i(\cdot)$$

and $\rho_i(x, e, \varpi)$ be such that

$$\sup \left| \frac{\hat{\theta}_3 \Delta_i(\cdot)}{\theta_3} \right| \leq \varrho_i(x, e, \varpi)$$

where the supremum is taken over all $x \in R^4$, $e \in R^4$, $\theta_i \in [\theta_i^m, \theta_i^M]$ and $\varpi \in R^2$. The functions β_i are chosen as $\beta_i(\cdot) = \varrho_i(\cdot) + q_i$, where $q_i > 0$. Note that we have not included the term $\{k_0^i(-k_0^i\sigma_i + \mu_i\ \mathrm{sat}(s_i/\mu_i))\}$ in the definition of $\Delta_i(\cdot)$ as was done in Section 3.3. This is done purely as a matter of convenience. The contribution of this term to the left hand side of the inequality in Assumption 3.6 is $O(\mu_i)$ and can therefore be accounted for by the term q_i, provided μ_i is sufficiently small. A remark to this effect was also made in Section 2.7. The results of Theorem 3.1 allow us to conclude that the controller (5.11) achieves global regulation, provided μ_1, μ_2 are sufficiently small.

For the purpose of simulation, we use the following values for the system parameters, obtained from [77] : $K_m = 0.1349\ Nm/A$, $J = 4.1295 \times 10^{-4}\ kgm^2$, $B = 0.0013\ Nm/rad/s$, $N_r = 50$ and $\tau_L = 0.2\ kg$. The resistance R and inductance L are assumed to be unknown, with nominal values of $\hat{R} = 20\ \Omega$ and $\hat{L} = 35\ mH$ respectively. Also, $R \in [R^m, R^M]$, and $L \in [L^m, L^M]$, with $L^m = 30\ mH$, $L^M = 40\ mH$, $R^m = 19\ \ \Omega$, and $R^M = 21\ \ \Omega$. The actual values of the resistance and inductance are taken as $R = 19\ \Omega$ and $L = 40\ mH$. The current reference is taken as $I_{dd}(t) = 0A$ [54]. The values of the controller parameters are taken as $k_0^1 = 20$, $k_0^2 = 100$, $k_1^2 = 7.5 \times 10^4$, $k_2^2 = 550$, $\mu_1 = 0.1$, and $\mu_2 = 50$. Initial values for all the states are taken as zero. The switching terms $\beta_i(\cdot)$ are taken as

$$\begin{aligned}
\beta_1(\cdot) &= [(R^M - \hat{R})|x_1| + (L^M - \hat{L})N_r|x_2x_3| + q_1]/\hat{L},\ q_1 = 2.1k_0^1\mu_1L^M \\
\beta_2(\cdot) &= [(R^M - \hat{R})k_3|x_2| + (L^M - \hat{L})|k_1^2\dot{e}_2 + k_2^2\ddot{e}_2 - k_4(k_3x_2 - k_4x_3 - d_0) \\
&\quad -k_3k_5x_1x_3| + q_2]/\hat{L},\ q_2 = 2.1k_0^2\mu_2L^M
\end{aligned}$$

The desired angular position is taken as $\phi_d(t) = 0.03142\ [u(t) + u(t - 0.5)]$, where $u(t)$ is the unit step function. The results of the simulation are shown in Fig 5.6. We focus our attention on the error e_2, since it corresponds to the variable ϕ of physical interest and also because it is the harder of the two outputs to control. However, our observations regarding e_2 also hold for e_1. From the figure, it is clear that good tracking performance with very little overshoot is achieved, independent of the magnitude of $\phi_d(t)$.

5.3.3 Integral Control - Output Feedback

Suppose that the angular velocity ω of the motor shaft is unavailable for feedback. It is easy to verify that for the conditions of Assumption 2.1 to be satisfied, i.e., for the system to have a uniform vector relative degree and be transformable to normal form, none of the positive constants k_i and d_0 need to be exactly known. Accordingly, we will assume that in the present case, in addition to the resistance R and inductance

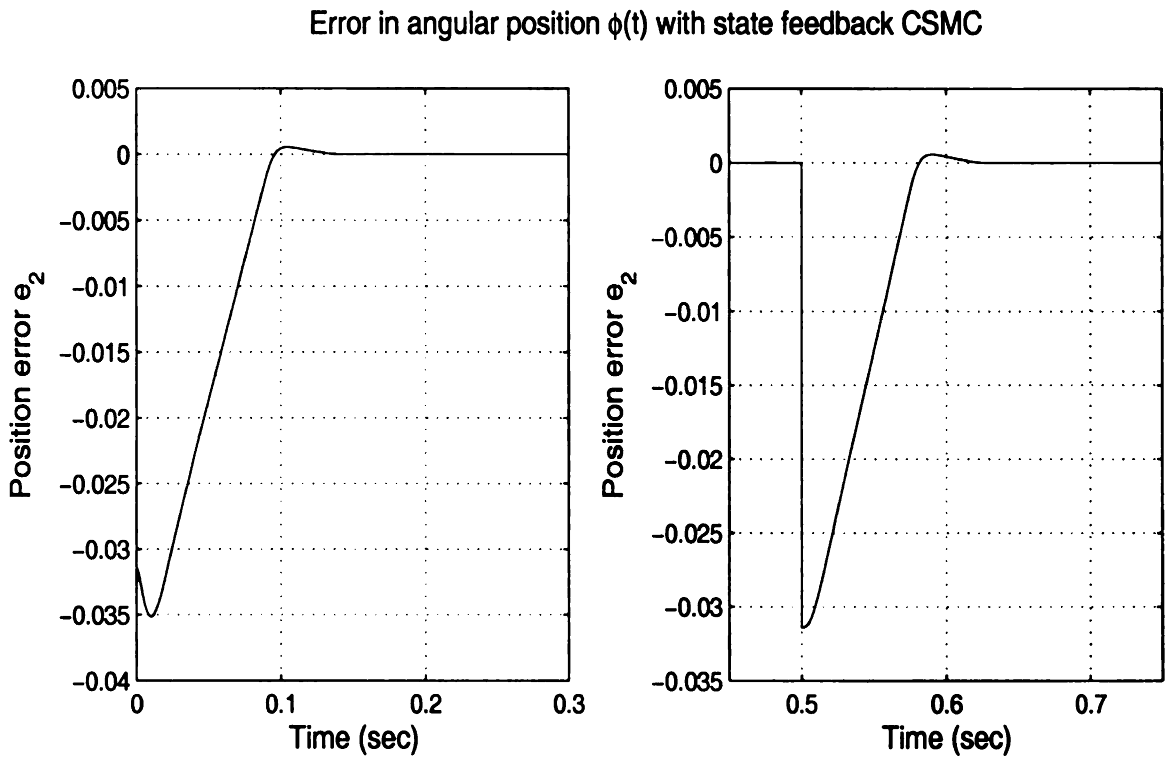

Figure 5.6: Tracking error performance under state-feedback integral control.

L, the parameters K_m, B, J and the load torque τ_L are all unknown, and take the vector θ of unknown parameters as $\theta = [k_1, k_2, k_3, k_4, k_6, d_0]^T$. Since ω is unavailable for feedback, so is $\dot{e}_2 = \omega - \dot{\phi}_d$. Furthermore, even if ω were available for feedback, since $\ddot{e}_2 = k_3 i_q - k_4\omega - d_0 - \phi_d^2$, and k_3, k_4 and d_0 are unknown, $\ddot{e}_2$ would be unavailable for feedback. Therefore, we estimate $\dot{e}_2$ and $\ddot{e}_2$ in (5.8) using the high-gain observer (HGO)

$$\left.\begin{array}{rcl} \dot{z}_1 & = & z_2 + \alpha_1(e_2 - z_1)/\epsilon \\ \dot{z}_2 & = & z_3 + \alpha_2(e_2 - z_1)/\epsilon^2 \\ \dot{z}_3 & = & \alpha_3(e_2 - z_1)/\epsilon^3 \end{array}\right\} \tag{5.12}$$

where the positive constants α_1, α_2, and α_3 are chosen such that the polynomial $\lambda^3 + \alpha_1\lambda^2 + \alpha_2\lambda + \alpha_3$ is Hurwitz. We replace $\dot{e}_2$, $\ddot{e}_2$, and also s_2 in (5.9) by their estimates z_2, z_3, and $\hat{s}_2 = k_0^2\sigma_2 + k_1^2 e_2 + k_2^2 z_2 + z_3$ respectively. Finally, motivated in part by the goal of simplifying the design, and in part by the need to work with saturated controls, we (i) make use of the flexibility in choosing $\hat{F}_i$, and let $\hat{F}_i = 0$, and (ii) choose $\beta_i(\cdot)$ as a constant M_i (say), which is equal to the maximum physically allowable value for the control component $|u_i|$. As we previously pointed out, the matrix corresponding to $A(\cdot)$ in Assumption 2.5 is diagonal, so that $\hat{A}$ can be taken as the identity. With this choice, the final expression for the control (2.18), (2.19) then becomes

$$\begin{aligned} u_1 &= -M_1 \operatorname{sat}(s_1/\mu_1) \\ u_2 &= -M_2 \operatorname{sat}(\hat{s}_2/\mu_2) \end{aligned} \tag{5.13}$$

which can be considered the MIMO version of the universal integral control design of Section 2.6.

For the purpose of simulation, we let K_m, J, B, N_r, τ_L, k_0^1, k_0^2, k_1^2, k_2^2, μ_1, μ_2 and I_{dd} retain their values from the previous simulation. Also we take $R = 19.5\ \Omega$, $L = 30\ mH$, and $\phi_d = 0.03142\ rads$. The HGO gains are taken as $\alpha_1 = 17$, $\alpha_2 = 80$ and $\alpha_3 = 100$, and the saturation levels for the controls as $M_1 = M_2 = 50$. We compare the performance of the output feedback controller with the partial state feedback design $u_i = -M_i \operatorname{sat}(s_i/\mu_i)$, which makes use of measurements of e_1, e_2, $\dot{e}_2$ and $\ddot{e}_2$. This could, for instance, be the case when the full state x is available for feedback and the parameters k_3, k_4 and d_0 are known. Fig 5.7(a) shows the results of the simulation for $\epsilon = 10^{-4}$, and we see that good tracking performance is achieved by the output feedback controller, which uses minimal information about the system. Fig 5.7(b) shows the effect of ϵ on the performance recovery of the state feedback design, and it is clear from the figure that the error e_2 under output feedback approaches the error e_2 under state feedback as ϵ tends to zero.

Lastly, to show the merit of the proposed scheme versus ideal SMC, consider a sampled-data implementation of the above controller, i.e., we assume that the inputs to the controller are sampled and held signals, with a zero-order hold, and likewise for the controller outputs. We redo the previous simulation (with the state feed-

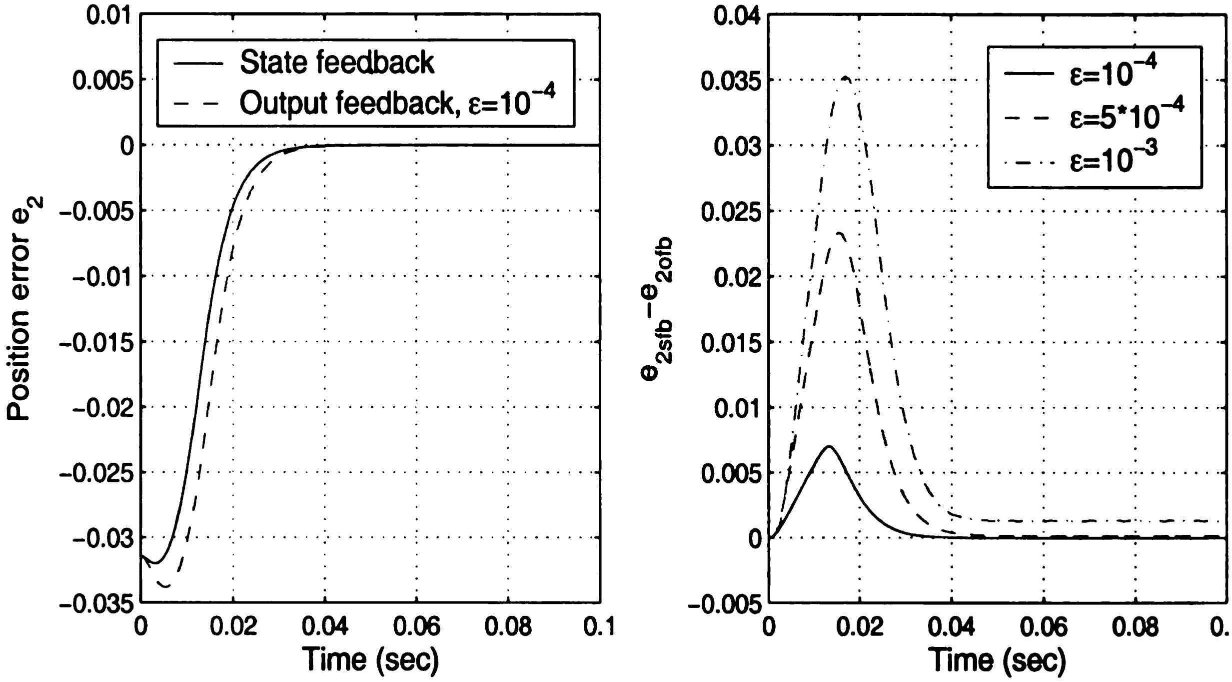

Fig 5.7(a) : MIMO universal integral controller

Fig 5.7(b) : Performance recovery of sfb as ε tends to zero

Figure 5.7: Tracking error performance under the output-feedback MIMO universal integral control.

back CSMC) and compare the results versus the ideal sliding mode control $u_i = -M_i \ \mathrm{sgn}(s_i)$. The sampling period is assumed to be $T = 0.1$ms. The results are shown in Fig 5.8, and we see that asymptotic regulation is lost with the ideal SMC, and there is considerable chattering in the control v_q. Asymptotic regulation is retained with the CSMC with conditional integrator, and there is no chattering in the control.

5.3.4 Servocompensator Design

The integral control designs of the previous subsections were done with the goal of point-to-point motion of the PMSM. In many positioning applications, the desired trajectory for the position is a sinusoid [20]. Specifically, suppose that the desired

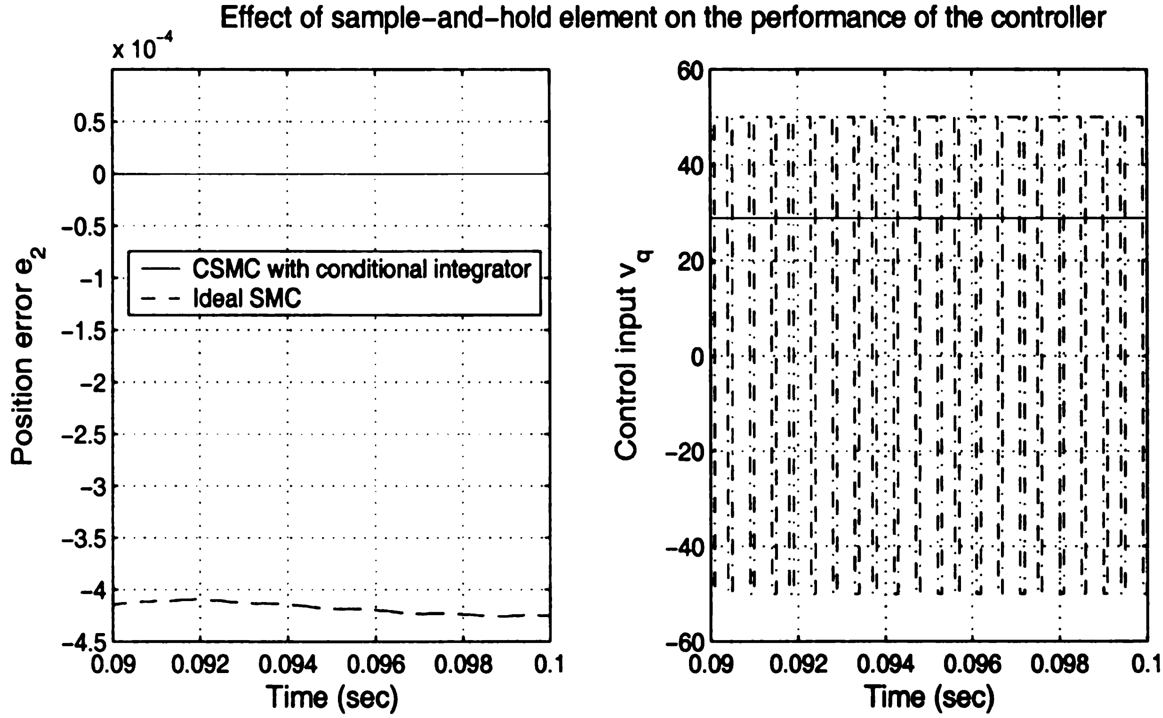

Figure 5.8: Effect of sampled-time implementation on ideal SMC and CSMC with conditional integrator.

trajectory asymptotically converges to $\phi_d(t) = r_0 \sin(\omega_0 t)$. [2] We show that the servocompensator design of Chapter 4 can be applied to this case. To that end, our first goal is to identify a suitable linear internal model that generates the steady-state values of the control inputs v_d and v_q. As before, the desired reference for the current i_d is a constant I_{dd}. It can be verified that with steady-state values $x_{1ss}(t) = I_{dd}$ and $x_{4ss}(t) = r_0 \sin(\omega_0 t)$ respectively for x_1 and x_4, the steady-state values of x_2 and x_3 are given by $x_{2ss} = [d_0 + k_4 r_0 \omega_0 \cos(\omega_0 t) - r_0 \omega_0^2 \sin(\omega_0 t)]/k_3$ and $x_{3ss} = r_0 \omega_0 \cos(\omega_0 t)$ respectively, and the steady state values of the control inputs v_d and v_q are given by

$$\begin{aligned} u_{1ss} &= \gamma_1 + \gamma_2 \cos(\omega_0 t) + \gamma_3 \sin(2\omega_0 t) + \gamma_4 \cos(2\omega_0 t) \\ u_{2ss} &= \gamma_5 + \gamma_6 \sin(\omega_0 t) + \gamma_7 \cos(\omega_0 t) \end{aligned} \tag{5.14}$$

[2]More general reference trajectories can be considered, as long as they satisfy the conditions of Assumption 4.3.

for some constants γ_1 to γ_7. The steady state value of the control u_{1ss} satisfies (4.6) of Assumption 4.5 with $q = 5$, $c_0 = 0$, $c_1 = -4\omega_0^4$, $c_2 = 0$, $c_3 = -5\omega_0^2$, and $c_4 = 0$, while u_{2ss} does so with $q = 3$, $c_0 = 0$, $c_1 = -\omega_0^2$ and $c_2 = 0$. Let

$$S_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -4\omega_0^4 & 0 & -5\omega_0^2 & 0 \end{bmatrix}, \; S_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\omega_0^2 & 0 \end{bmatrix}$$

be the internal model matrices corresponding to u_{1ss} and u_{2ss}, and

$$J_1 = [\begin{array}{ccccc} 0 & 0 & 0 & 0 & 1 \end{array}]^T, \; J_2 = [\begin{array}{ccc} 0 & 0 & 1 \end{array}]^T$$

be the corresponding J matrices, as defined in Section 4.3. We take σ_1 and σ_2 as outputs of the conditional servcompensators

$$\dot{\sigma}_i = (S_i - J_i K_0^i)\sigma_i + \mu_i J_i \text{sat}(\hat{s}_i/\mu_i) \tag{5.15}$$

where

$$\left. \begin{array}{rcl} s_1 & = & K_0^1 \sigma_1 + e_1, \; e_1 \stackrel{\text{def}}{=} y_1 - I_{dd}, \\ s_2 & = & K_0^2 \sigma_2 + k_1^2 e_2 + k_2^2 \dot{e}_2 + \ddot{e}_2, \; e_2 \stackrel{\text{def}}{=} y_2 - \phi_d \end{array} \right\} \tag{5.16}$$

The matrices K_0^i are chosen such that $S_i - J_i K_0^i$ are Hurwitz, the scalars k_1^2 and k_2^2 are chosen such that the polynomial $x^2 + k_2^2 x + k_1^2$ is Hurwitz, $\hat{s}_1 = s_1$, $\hat{s}_2 = K_0^2 \sigma_2 + k_1^2 e_2 + k_2^2 z_2 + z_3$, where z_2 and z_3 are estimates of $\dot{e}_2$ and $\ddot{e}_2$ respectively, provided by the high-gain observer (5.12). The control is taken as in (5.13), i.e.,

$$u_i = -M_i \; \text{sat}(\hat{s}_i/\mu_i) \tag{5.17}$$

This completes the design of the controller. For the purposes of simulation, all values are retained from the one in the previous subsection, except for the reference $\phi_d(t)$ and the matrices K_0^i. The former is chosen as $\phi_d(t) = r_0 \cos(\omega_0 t)$ [3], for which two sets of values of (r_0, ω_0) are used, $(r_0, \omega_0) = (\pi/2, 2)$ and $(r_0, \omega_0) = (\pi/10, 5)$. The latter are chosen to place the eigenvalues of $S_1 - J_1 K_0^1$ and $S_2 - J_2 K_0^2$ at $\{-1, -2 \pm j, -3 \pm j\}$ and $\{-1, -2 \pm j\}$ respectively. We compare the performance against a CSMC design that does not include a servocompensator, i.e., $\hat{s}_1 = e_1$, $\hat{s}_2 = k_1^2 e_2 + k_2^2 z_2 + z_3$, where z_2 and z_3 are as defined above, and $u_i = -M_i \text{ sat}(\hat{s}_i/\mu_i)$. The results of the simulation are shown in Fig 5.9. As before, we see that good tracking performance is achieved by the output feedback MIMO servocompensator design, which uses minimal information about the system. The transient performance of the CSMC without servocompensator is close to the one with a servocompensator (indistinguishable in the figure), but the steady-state error is non-zero.

Before we present our conclusions, we make the following observation. Throughout, we have assumed that the load torque τ_L is constant. It is not hard to see that this assumption can be relaxed. For the integral control design of sections 5.3.2 and 5.3.3, it can be verified that all that is required is that the load torque be asymptotically constant. Since in this case, the angular position $\phi(t)$ and the angular velocity $\omega(t)$ asymptotically approach a constant and zero respectively, we can allow position or velocity dependent load torques, i.e., $\tau_L = f_\tau(\phi(t), \omega(t))$, where $f_\tau(\cdot)$ is a suficiently smooth function of its arguments. For the servocompensator design of this section, the situation is slightly more restrictive on account of Assumption 4.5. It can be verified that in this case, for Assumption 4.5 to be satisfied, $f_\tau(\cdot)$ will have to be a

[3]This is conceptually not different from the $\phi_d(t) = r_0 \sin(\omega_0 t)$ that we designed for, just phase-shifted from it by $\pi/2$, and was chosen this way to have a non-zero initial error $e_2(0)$.

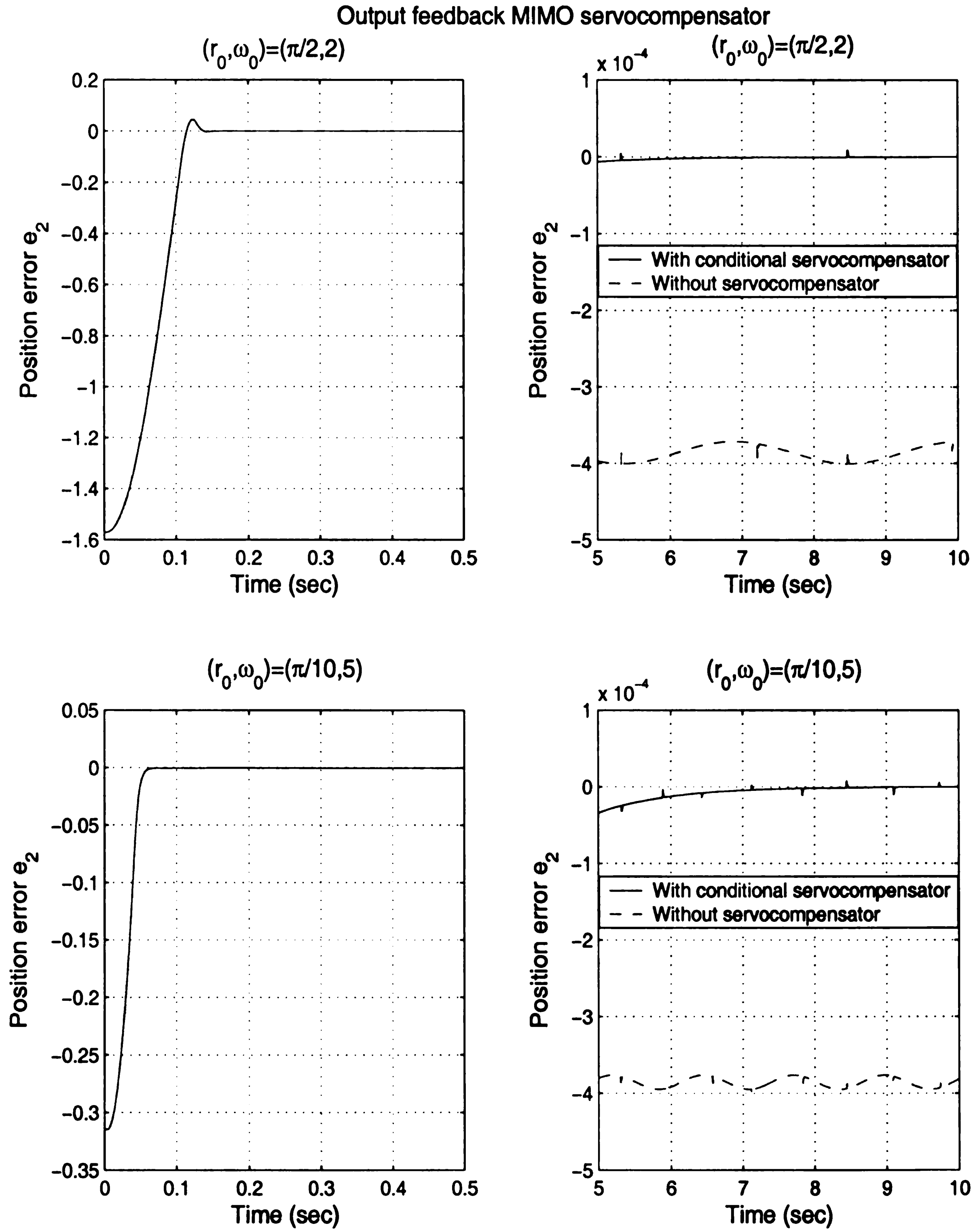

Figure 5.9: Sinusoidal reference tracking using MIMO servocompensators.

polynomial function of its inputs, and that its form must be known. For example, if

$$f_\tau(\phi,\omega) = \sum_{i\in I, j\in J} \alpha_{ij}\phi^i\omega^j$$

where $I, J \subset Z_{\geq 0}$, the sets I and J will have to be known, but not α_{ij}. When the polynomial condition is violated, for example, when the load torque is of the form $\tau_L = N\sin(\phi)$ say, then, as mentioned in Chapter 4, polynomial approximations may be used to achieve practical regulation of the error.

5.4 Conclusions

In this chapter, we considered the application of the controller designs presented in Chapters 2 through 4 to two applications : temperature control of a continuous stirred tank reactor (CSTR) and position control of a permanent magnet stepper motor (PMSM).

For the CSTR, we considered the problem of constant references and disturbances, to which the design of Section 2.6 could be applied. Since the class of CSTRs considered in this chapter is relative degree one, the resulting controller is a PI controller with anti-windup. Such a structure was also considered in [1, 2]. The design was initiated in [2], based on modelling error estimation ideas, and with the goal of handling uncertain parameters, and the resulting structure was shown to be a PI controller with anti-windup. Furthermore, it was shown in [1] that the controller could recover the performance of a saturated inverse feedback control. The corresponding result in our case is the recovery of the performance an ideal sliding mode control. Even though a direct analytic comparison of the two controllers is not possible, it was observed via simulations that the results in [2] and our work are comparable.

For the PMSM, we considered both constant as well as sinusoidal references. In the constant references case, we looked at both the state feedback design of Chapter 3 for the global regulation problem, and the output feedback design of Chapter 2 for the regional problem. The output feedback controller was designed as the MIMO version of the universal integral regulator of Section 2.6. Good tracking performance was achieved with both designs, in spite of partial knowledge of the machine parameters. The same was also seen to be true with sinusoidal references, in which case the controller is the MIMO version of the universal servocompensator of Chapter 4. While we specifically considered PMSMs in this chapter, our results can be extended to other types of motors, such as permanent magnet synchronous and dc motors.

Chapter 6

Conclusions

In this thesis, we have addressed the problem of robust output regulation by output feedback for input-output linearizable minimum-phase nonlinear systems, with emphasis on the transient performance. As mentioned in Chapter 1, the issue of transient performance needs to be addressed because conventional approaches to designing servocompensators often result in poor transient performance. To that end, we have presented a new approach to introducing servo action that results in improved transient performance over the conventional approach. Analytical results have been provided for regional and semi-global regulation and also for the performance recovery of a state feedback ideal SMC design. A summary of results by chapter is provided below.

6.1 Summary of Results

In Chapter 2, we considered the output regulation problem for the special case of (asymptotically) constant exogenous signals. For this case, we designed a continuous sliding mode controller with a conditional integrator that achieves asymptotic error regulation with good transient performance. The conditional integrator is so designed that it provides integral action only inside the boundary layer. Analytical results were given for regional and semiglobal regulation, and we proved that the output feedback controller with conditional integrators recovers the performance of a state feedback ideal SMC that does not include integral action. Advantages of the proposed method over the conventional approach as regards the transient performance and over ideal SMC as regards the problem of chattering were shown by simulation.

In Chapter 3, we considered the extension of the semi-global regulation result of Chapter 2 to a global one under full state feedback.

In Chapter 4, we considered an extension of the design of Chapter 2 for the more general output regulation problem. To that end, we designed a conditional servocompensator that provides servocompensation only inside the boundary layer. As before, regional and semiglobal results for regulation were given and also for performance recovery of a state feedback ideal SMC design. We also studied the effect of internal model perturbations on the tracking error and showed that, in the presence of such perturbation, the tracking error is ultimately bounded by a bound that depends on the magnitude of the perturbation.

In Chapter 5, we applied the designs of Chapters 2 through 4 to two application examples, temperature control of a continuous stirred tank reactor (CSTR) and position control of a permanent magnet stepper motor (PMSM). The controller for the CSTR problem was based on the design in Section 2.6, and for the particular class of CSTR systems that we considered was simply a PI controller with anti-windup. For the PMSM, we considered simulations involving all of the designs in Chapters 2 to 4. Whereas the results of Section 2.6 and Chapter 4 were presented for the single-input single-output case, their application to the PMSM shows how such results can be extended to a multi-input multi-output problem. For both the CSTR as well as the PMSM example, the simulation results show that the controller designs of this thesis result in good performance.

6.2 Future Work

The conditional servocompensator design of this thesis is based on the idea of providing servocompensation only conditionally, inside the boundary layer of a sliding mode design. However, it is clear that any controller that has the structure of a saturated high-gain feedback is a promising candidate for the application of the conditional servocompensator design. Identifying previous works that use servocompensators within such a controller structure and applying the conditional servocompensator idea to see if the performance of such designs can be improved is an interesting line of future work.

A second promising direction in which research can be pursued is understanding whether the servocompensator design can be modified to be used in conjunction with other controller designs that extend the class of systems or relax the assumptions made in this thesis. Examples of two such designs are the ones by Serrani *et al* (i) for the output regulation of nonminimum phase nonlinear systems [68] and (ii) using adaptive internal models when the frequencies of the exosystem are unknown [69].

It will also be useful to evaluate other techniques for the output regulation of systems with constraints on the control, and see how they compare to the results of this thesis. One such technique [1] was mentioned when we discussed the CSTR example in Chapter 5. Related to [1] is the work of Kapoor and Daoutidis [42], which extends the results of [1] along two directions : (i) it allows the system relative degree to be greater than one, and (ii) deals with the general output regulation problem, not just the case of constant exogenous signals. Another design for the output regulation of constrained linear systems, but with the goal of semi-global regulation when the control level is fixed apriori, can be found in Lin *et al* [56].

Finally, further work needs to be done on understanding how to tune the controller parameters. For example, it is not hard to see that when the level of the control is fixed apriori, there is a trade-off between the region of attraction and the speed of convergence, which is dictated by the choice of the sliding surface parameters. Identifying such trade-offs will offer insight into how the parameters can be tuned to achieve specific objectives, and also identify possible limitations on the achievable performance. A partial account of such details can be found in the Ph.D. thesis of Grognard [29].

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