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LEARNING TO USE FRACTIONS AFTER LEARNING ABOUT FRACTIONS: A STUDY OF MIDDLE SCHOOL STUDENTS DEVELOPING FRACTION LITERACY

By

Debra I. Johanning

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ABSTRACT

LEARNING TO USE FRACTIONS AFTER LEARNING ABOUT FRACTIONS: A STUDY OF MIDDLE SCHOOL STUDENTS DEVELOPING FRACTION LITERACY

By

Debra I. Johanning

There is a large body of literature, both empirical and theoretical, that focuses on what is involved in learning fractions when fractions are the focus or goal of instruction. However, there is very little research that explores how students learn to use what they have learned about fractions outside instruction on fractions. The specific goal of this research was to explore how middle school students learned to use fraction knowledge, the fraction concepts and skills studied in formal curriculum units, in mathematical instructional settings where fractions were not the main focus of study, but rather supported the development of other mathematical content.

This study is sociocultural in nature. It is guided by a practice account of literacy (Scribner and Cole, 1981) and Barton's (1994) ecological approach to literacy. Studying literacy involves studying the practices that people engage in as they *use* knowledge for specific purposes in specific contexts of use. This research describes the practices that grade six and seven students engaged in when they had to use what they learned about fractions to make sense of mathematical contexts such as area and perimeter, decimal operations, probability, similarity, and ratio. In order to understand how the practices students engaged in when learning to use fractions differed from the practices students engaged in when learning about fractions, data collection and analysis focused on identifying and then comparing these two types of practices.

Data collection for this dissertation spanned approximately one and one-half school years. In the fall of 2002 and winter of 2003 I collected data during the two units where one class of sixth-grade students learned about fractions. In the spring of 2003 I began to collect data during three units where these sixth-grade students were using fractions as part of learning about area and perimeter, decimal operations, and probability. Data collection continued into seventh grade as I followed a subset of these sixth-grade students into their seventh-grade year. Data was collected during two seventh grade units were fractions were used in the context of similarity and ratio. Data collection ended in December of 2003. The data collected included fieldnotes, video recordings of whole class discussions, video-recording the small-group interactions of one group of four focus students, interviews with the four focus students, and copies of their written work.

The study's results revealed that students did not simply take the concepts and skills learned in the fractions units and use them. Understanding how to use fractions was tied to understanding situations in which they can be used. Students had to take into account both mathematical and situational contexts when making choices about how to use fractions. This led students to raise questions regarding what was appropriate when using fraction in these new contexts and how fractions and the new context were related. It was clear that the conversations these students had regarding the use of fractions were not only different from the conversations they had when learning about fractions, but potentially may not have occurred when learning about fractions directly. It is argued that providing students the opportunity to use fraction knowledge is critical to the development of fraction literacy.

To Nick

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TABLE OF CONTENTS

LIST OF TABLES viii

LISTS OF TABLES ix

CHAPTER 1
INTRODUCTION 1
  Personal Context and Merging Ideas 2
  Linking Literacy, Mathematical Literacy, and Fraction Literacy 4
  Fractions and Rational Number 8
    Fractions: Complexity of the Field 8
    Unanswered Questions Regarding Rational Number and Fraction Learning 14
  Upcoming Chapters 16

CHAPTER 2
LEARNING TO USE FRACTIONS: MAPPING LITERACY ONTO MATHEMATICS 17
  Learning to Use 17
  The Interaction Between Learning About and Learning to Use 22
  From Literacy to Fraction Literacy 25
  Research Questions 30
  Chapter Summary 31

CHAPTER 3
METHODS OF DATA COLLECTION AND ANALYSIS 32
  Timeline and Preliminary Fieldwork 32
  Data Collection 34
    Identifying Instructional Tasks 34
    Participants and Setting 45
    Sources of Data 48
  Data Analysis 51
    Analysis Related to Learning to Use Fractions Data 52
    Analysis Related to Learning About Fractions Data 57
  Chapter Summary 59

CHAPTER 4
LEARNING ABOUT FRACTIONS: CONVERSATIONS, QUESTIONS, AND PRACTICES 60
  The Teachers and Their Classrooms 61
    Developing Responsible Learners 62
    Letting Students Go Down the Wrong Path 62
    The Role of Language and Conceptually–Oriented Explanations 64
  Learning About Fractions: Fraction Learning Practices 65

Goals and Fraction Learning Practices for the First Fraction Unit 66
Goals and Fraction Learning Practices for the Second Fraction Unit 81
Summary 97

CHAPTER 5
LEARNING TO USE FRACTIONS: CONVERSATIONS, QUESTIONS, AND PRACTICES 99
The Practice of Determining Appropriateness 100
Subpractice One: Representing Repeating Decimals as Fractions 101
Subpractice Two: Moving From Fractions to Whole Numbers and Back 104
Subpractice Three: Drawing from Multiple Approaches and Algorithms 109
Subpractice Four: Choosing to Use Fractions When Fractions Are Not Explicit 113
Subpractice Five: Does Equivalence Matter? 116
Subpractice Six: How are Fractions and Ratios Related? 118
Summary of the Practice of Determining Appropriateness 124
The Practice of Connecting Fractions to Multiplication and Division Concepts 126
Prelude: An Overview of Fraction Multiplication and Division 126
Returning to the Findings: Connecting Fractions to Multiplication and Division 130
Summary: Connecting Fractions to Multiplication and Division 142
Chapter Summary 143

CHAPTER 6
LEARNING ABOUT FRACTIONS AND LEARNING TO USE FRACTIONS: IS THIS DISTINCTION NOTEWORTHY? 145
Comparing Fraction Learning Practices with Fraction Literacy Practices 145
The Role of Connections When Learning to Use Fractions 152
Summary: Does it Matter? 154

CHAPTER 7
POTENTIAL AND POSSIBILITY 155
Additional Considerations and Limitations 158
A Closing Thought 162

APPENDICES
Appendix A 164
Appendix B 166
Appendix C 197
Appendix D 200
Appendix E 202

REFERENCES 204

LIST OF TABLES

1. Barton's Characterization of Literacy, Literacy Events and Literacy Practices ... 26

2. Data Collection Timeline ... 35

3. Instructional Tasks Identified for Data Collection When Using Fractions in Other Contexts ... 38

4. Strategies for Placing Fractions in Intervals ... 76

5. Questions Supporting the Fundraiser Task ... 80

6. Questions Supporting the Land Problem ... 96

7. Frequency of Subpractices Associated with the Practice of *Determining Appropriateness* by Mathematical Content in the Unit ... 101

8. Questions Associated With the Practices of Learning About and Learning to Use Fractions ... 146

LIST OF FIGURES

1. Storm Shelters Task 36
2. Park Problem 39
3. Mixing Juice Task 42
4. Triangle Homework Task 43
5. Unanticipated Fraction Division Homework 44
6. Comparison Statement Task 45
7. Levels of Data Reduction for Storm Shelters Task 53
8. Mathematics Classroom Characteristics That Empower Students 61
9. Practice of Understanding Fractions as Quantities 68
10. Subpractices of the Practice of Understanding Fractions as Quantities 69
11. Fundraiser Task 70
12. Comparing Fractions to Benchmarks Task 75
13. Fraction and Decimal Equivalents Task 78
14. Questions associated with the Practice of Understanding Fractions as Quantities 81
15. Practice of Learning to Operate with Fractions 82
16. Problem B of Estimating Sums Task 85
17. Land Problem 86
18. TJ's Board Work for Part C of the Land Problem 88
19. Brownie Pan Problem 92
20. Modeling $\frac{1}{3} \times 2\frac{1}{2} = \frac{5}{6}$ 94
21. Questions Associated with the Practice of Learning to Operate with Fractions ... 97

22. Storm Shelters Data Table 105

23. Bryan's Partitioned Rectangle 117

24. Labeled Parallelograms 121

25 Questions Underlying Classroom Discussions 125

26. Two Representations for Division 128

27 Board Work 137

28. Similar Rectangles 138

Chapter One

INTRODUCTION

The National Council of Teachers of Mathematics (NCTM) 1989 Curriculum and Evaluation Standards center mathematics reform around notions of mathematical literacy and mathematical power, and in doing so, provided vision and direction for the mathematics education community. The intention of the laid out goals was to develop mathematically literate students with mathematical literacy being described in the following way:

> This term [mathematical literacy] denotes an individual's ability to explore, to conjecture, and to reason logically, as well as to use a variety of mathematical methods effectively to solve problems. By becoming literate, their mathematical power should develop. (p. 6)

This statement suggests that students have experiences where they engage in the processes of mathematics rather than focusing solely on learning how to create the products of mathematics. The notion of mathematical power, as well as the introduction of process standards by NCTM, highlights the need to recognize mathematics as more than a collection of skills and concepts to be mastered.

The NCTM Standards (1989, 2000) stress that knowing math is doing math. Not only should mathematics instruction focus on developing concepts and skills, it should be done in such a way that students see mathematics as useable. The value of knowledge "lies in the extent to which it is useful in the course of some purposeful activity" and instruction should "persistently emphasize 'doing' rather than 'knowing' that" (NCTM, 1989, p. 7). Both the 1989 and the 2000 Standards documents point out the importance of building on students' previous experiences rather than using a repetitive curriculum and the need to apply prior knowledge to increasingly more difficult situations. Students should be "responsible for what they have learned and for using that knowledge to understand and make sense of new ideas" (NCTM, 2000, p. 64). It is this distinction between learning about something and learning to use that knowledge in new settings that this dissertation explores.

As the title suggests, this theme is studied in the context of fractions. The first part of this chapter will discuss how literacy, fractions, and the curricular context came together for me as a researcher. The second part of the chapter will review the research literature on fractions–what these mean and why they are challenging to teach and learn. In this chapter I aim to provide a rationale for this dissertation, one whose specific purpose is to explore how students learn to use fraction knowledge, the fraction concepts and skills studied in formal curriculum units, in mathematical instructional settings where fractions are not the main focus of study, but rather support the development of other mathematical content.

Personal Context and Merging Ideas

This dissertation results from merging of several areas of work and professional interest. My Master's thesis focused on how writing in mathematics class influenced student learning. From this research it became apparent to me that it was not only the writing my students had done, but how reading, writing and speaking came together in my classroom, which greatly influenced their learning. This led me to think harder about what the 1989 NCTM Standards meant when they associated mathematical literacy with the ability to read, write, and speak mathematically. As I reviewed the research literature on reading, writing, and speaking in mathematics, I noted that the three were often separated in the mathematics literature. Studies often focus on a topic such as the role of writing, or classroom talk in mathematics learning. Reading in mathematics has been explored in the context of children's literature and through the lens of methods of teaching reading. I found mathematical literacy was often used but not defined in detail.

I began to explore the use of the term "literacy" in the field of language literacy[1]. Literacy, according to the New Literacy Studies, is viewed as more than the study of learning to read and write. Literacy has shifted from literacy as the study of reading and writing to one that is psychological and social in origin (Barton, 1994; Gee, 1999). While literacy is concerned with how people formally learn to read, write and speak, this shift attends to the importance of the social environment in which learning and use of literacy are embedded. It attends to the ways in which literacy is used and develops out of the everyday activities in which people engage. Because literacy is situated in social practice, studying literacy must push beyond a psychological view of literacy as mental state and a symbol system to be learned to include how this is linked with social situations that give meaning to the different forms and uses of literacy. Rather than examine orality and written literacy as separate (Street, 2001), the New Literacy Studies look at literacy as the integration of oral and written practices (Gee, 1999; Wilkinson & Silliman, 2000) with ways of acting, interacting, feeling, and knowing, as well as nonverbal symbols, sites, tools, objects, and technologies (Gee, 1999).

Barton (1994), a leading voice in the New Literacy Studies, argues that this new view must involve more than adding a social dimension to the study of literacy. While

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[1] I use the term language literacy to distinguish between literacy as used by those in the field of language arts, from the terms mathematical literacy and fraction literacy.

psychological approaches to studying literacy have played an important role, the role of social interaction in the development of thinking is also important.

> Instead of studying the separate skills which underlie reading and writing, it involves a shift to studying literacy, a set of social practices associated with particular symbol systems and their related technologies. To be literate is to be active; it is to be confident within these practices. (p. 32)

Literacy here, like mathematical literacy in the NCTM standards, is something that people use. It is a practice they engage in. There are social purposes for reading and writing. People do not read and write without a reason for doing so.

Barton's work, like mine, is influenced by the highly regarded work of Scribner and Cole (1981). Scribner and Cole propose what is called a "practice account" of literacy where literacy can only be understood in the context of the social practices where it is acquired and used. Of literacy Scribner and Cole write:

> ...we approach literacy as a set of socially organized practices which make use of a symbol system and a technology for producing and disseminating it. Literacy is not simply knowing how to read and write a particular script but applying this knowledge for specific purposes in specific contexts of use. (p. 236)

Scribner and Cole's definition of literacy captures NCTM's extension of mathematical knowledge from being informational to include being useful. This view of language literacy captures what I think of when I think of mathematical literacy or being able to use mathematical knowledge as a means rather than an end.

Linking Literacy, Mathematical Literacy, and Fraction Literacy

While trying to sort out what mathematical literacy was, I was working as a graduate student researcher for the Connected Mathematics Project II (CMP II). This project involved updating a set of published middle school mathematics curriculum materials. I was involved in revising two sixth-grade curriculum units on fractions. I spent part of my time observing in a classroom where draft versions of the new units were being field-tested.[2]

The CMP[3] curriculum is designed to emphasize connections across mathematical topics. One goal of the curriculum is to continually build and connect the ideas of one unit or topic to others so that students have to use previously explored concepts and procedures in new settings. This approach is intended to promote fluency and proficiency of one topic while exploring another (Lappan & Phillips, 1998). A second goal, one regarding skill development, suggests that skill is more than symbol manipulation. "Skill means that students can use the mathematical tools, resources, procedures, knowledge, and ways of thinking developed over time to make sense of new situations that they encounter" (Lappan & Phillips, 1998, p. 83). In the CMP curriculum, fraction concepts and procedures are formally taught in two sixth grade units: *Bits and Pieces I* (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2002/2003a) and *Bits and Pieces II* (Lappan et al., 2002/2003b). CMP does not have units that formally teach (or reteach) fractions in seventh and eighth grade. Rather, students revisit fraction concepts and procedures in other mathematical contexts across sixth, seventh and eighth grade.

As I observed the teaching of the two sixth-grade fraction units and interacted with students in the classroom where I was observing, I noted that students had learned a lot *about* fractions—both conceptually and procedurally. I also noted that this knowledge seemed fragile. The students had a strong knowledge base but their ability to *use* this knowledge did not seem well developed. I knew students would not study fractions

[2] The philosophy behind the development of the mathematics is the same in both the original and revised materials.
[3] CMP refers to the Connected Mathematics Project curriculum in a general sense. CMP II is a specific reference to the revised curriculum materials.

formally in seventh and eighth grade—as is typical in conventional curricula. However, I did know that students would have to use fraction concepts and procedures in other units in sixth, seventh, and eighth grade.

If we consider what it means to learn mathematics it is common to think about concepts and procedures for various topic strands such as number, algebra, etc. While topic areas are central to knowing mathematics, knowing mathematics does not automatically equip someone to effectively use mathematics. Using mathematics involves performing a series of activities that are dependent upon the problem being addressed (RAND, 2003). If we refocus these ideas around learning rather than knowing, it can help distinguish between learning to use and learning about.

Consider learning about fractions. In formal fraction units students typically learn about fractions by studying a set of concepts and procedures that are central to fractions. Documents such as *Adding it Up* (Kilpatrick, Swafford, & Findell, 2001) discuss the development of fractions in a chapter on developing proficiency with rational number and the NCTM Principles and Standards (2000) number strand includes statements that describe the body of knowledge that fraction instruction should cover. This includes but is not limited to comparing and ordering fractions, developing meaning for and connections among different forms of representations (fractions, decimals, and percents), equivalence, the role of the numerator and denominator, benchmarks and estimation, and operations with fractions[4]. Regarding mathematics as a body of knowledge, "simply

[4] Chapter Four will provide an extended discussion of what students learn about in the CMP fraction units *Bits and Pieces I* and *Bits and Pieces II*. This list is a generic list of what is commonly discussed regarding learning about fractions. For a more complete list, Appendix A provides a list of items directly linked with fractions by examining the list of expectations for each number standard (pre-K–2, 3–5, 6–8, and 9–12) in Principles and Standards for School Mathematics (NCTM, 2000).

knowing concepts does not equip one to use mathematics effectively" (RAND, 2003, p. 30). Learning about fractions and learning to use them are different ideas. When I use the term "learning about" I am referring to what students typically study in formal curriculum about the mathematical topic fractions. This involves learning about fraction topics like those previously listed. A person uses their knowledge of fractions for specific purposes when they come upon problems or situations where they have to use the fractions knowledge they learned about to achieve a purpose.

In this study the focus is on "learning" to use fractions. In this study "learning about fractions" occurs when students formally study fractions in fraction curriculum units. "Learning to use" fractions takes place in instructional tasks in curriculum units that follow the formal study of fractions, and support the development of other mathematical content. In these settings fractions are used to achieve a purpose and extends fraction knowledge from being a body of knowledge to a tool.

The development of concepts and procedures in the CMP curriculum provided a setting where I could explore how students learn to use mathematical knowledge, in this case fraction knowledge. This development was distinct from when they learned about fractions. For example, the sixth-grade area and perimeter unit followed the teaching of the formal fraction units. In the area and perimeter unit, I identified three problems where students had to use their knowledge about fractions in the context of area and perimeter.

The linking of literacy, fractions, and a particular mathematics curriculum serve to address the larger issue of "learning to use" mathematical ideas for specific purposes. Because this study centers around the context of fractions, I am framing literacy—both language literacy and mathematical literacy—around fraction literacy, a specific case of mathematical literacy. I will speak at length about literacy, fraction literacy, and this notion of learning to use knowledge in Chapter Two. The remainder of this chapter will review research literature on fractions and fraction learning in order to explicate fractions as a case of how students *learn to use* knowledge.

Fractions and Rational Number

Fractions: Complexity of the Field

Within the vast amount of literature on rational number there is both theoretical and empirical work that focuses on the teaching and learning of fractions—one form of rational number. Part of this literature focuses on the complexity of fractions. Fractions are one of student's first experiences where a symbol not only represents a number (location) on a number line but can also represent the relationship between two quantities or the operation of division. One-fourth can refer to $1 \div 4$, 1 of 4 discrete but equal-size objects, or 1 part of a single object partitioned into 4 equal parts. In addition, fractional quantities can be represented in multiple forms. For example, the fraction $\frac{1}{4}$ can also be represented as $\frac{2}{8}$ and 0.25. And to add even more complexity, consider the role the whole plays when one tries to decide how much one-fourth represents. For example, is it possible for you and I to each have one-fourth of a candy bar and not have the same amount of candy?

The work of Kieren (1976), as well as others (Behr, Harel, Post & Lesh, 1992; Behr, Lesh, Post, and Silver, 1983; Freudenthal, 1983; Ohlsson, 1987; Vergnaud, 1983), illuminates the various constructs or interpretations that one must understand and be able to use if he or she is to have a full understanding of rational numbers. These five constructs (Behr, Harel, Post & Lesh, 1992; Kieren, 1980), part-whole, quotient, operator, measure, and ratio number, are described as follows:

- *Part-whole* refers to the partitioning of quantities, continuous or discrete, into equal sized parts. Often the form $\frac{a}{b}$ is used to indicate the number of equally divided parts (b) and the number of those parts that are counted (a). For example, $\frac{3}{4}$ means that a whole is partitioned into 4 equal-sized parts and 3 are counted or used.
- *Quotient* refers to fractions as indicated division problem. Indicated means that the actual division is not carried out but $\frac{a}{b}$ is recognized as also representing $a \div b$. With $\frac{3}{4}$, this would mean recognizing that $\frac{3}{4}$ also represents $3 \div 4$.
- *Operator* involves using $\frac{a}{b}$ as an action upon a set. Here, $\frac{a}{b}$ does not represent quantity but refers to the action or operation of breaking some quantity into b parts and then using a of them. This will create a new set or quantity that is "$\frac{a}{b}$ of" the original. This interpretation, one that involves a size transformation, is used with multiplication of fractions where $\frac{3}{4} \times 12$ can be thought of as $\frac{3}{4}$ of 12.
- *Measure* indicates that a whole is partitioned into b part where $\frac{a}{b}$ is used to measure a unit of size $\frac{1}{b}$. For example, $\frac{3}{4}$ would mean 3 parts that are one-fourth in size or three one-fourth quantities.
- *Ratio* expresses a comparative relationship between two sets a and b. It can express an ordered pair or part-part relationship such as 3 boys to 4 girls. It can also express a ratio relationship between two different quantities as a rate. For example, 3 notebooks for $4.00.

Kieren (1976) argues that in order to understand rational number one must have adequate experience with and understand each construct as well as understand how they are interrelated. One important but complex relation is how these five constructs are related to fractions, as opposed to rational number. Fractions are one interpretation of rational number, numbers that can be expressed in $\frac{a}{b}$ form where $b \neq 0$. The first four constructs listed above indicate actions such as partitioning, operating, measuring, and scaling, and are tied to fraction operations such as comparing, ordering, adding, subtracting, multiplying and dividing.

The various constructs help make clear that different fractional situations lead to different ways of reasoning. These fractional ways of understanding are drawn upon when one tries to make sense of the many situations where fractions are used. In general, when one speaks of fractions he or she is speaking of the underlying rational number or the number the fraction represents (Lamon, 1999). But fractions are more than numbers to be operated upon. They can be operators too. For example, taking $\frac{3}{4}$ of something is a different interpretation than $\frac{3}{4}$ the number.

This fraction reasoning outlined by the first four constructs is different from the reasoning implied by the ratio construct. This construct refers to situations that involve the relationship between two quantities. Ratios can be expressions of part-part or part-whole relationships. Since fractions are representations of part-whole relationships, all fractions are ratios. Not all ratios are fractions though[5]. For example, part-part addition,

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[5] Clark, Berenson, & Cavey (2003) propose that within the literature there are five different models that relate fractions and ratios. The model drawn upon here is that fractions are a subset of ratios. In the curriculum unit where students work on fractions, fractions and ratios are presented as distinct sets. An extended discussion of fractions and ratios will be presented in Chapter 5.

relationships and relationships where zero is used in the denominator are not fractions. In the way one operates with a ratio can be different from the way one operates with fractions. Despite the differences, both can be expressed in $\frac{a}{b}$ form.

Part of understanding fractions involves knowing when something expressed in $\frac{a}{b}$ form is and is not being used as a fraction. The ability to make such a determination is also compounded by the need to understand not only how fractions are situated within the domain of rational number, but to also understand their place within the larger multiplicative conceptual field (Vergnaud, 1983, 1988). The multiplicative conceptual field is comprised of all the situations that call for or involve the use of multiplication, division, fractions, ratio, rational numbers, linear functions, dimensional analysis, and vector spaces. Vergnaud (1988) argues that you cannot think about the learning and teaching of fractions (or any of the aforementioned situations) independently of multiplicative structures because concepts develop in relationship with other concepts rather than in isolation. He also points out that empirical results indicate that the learning and understanding of the multiplicative field extends over a long period of time, one that extends from elementary through secondary school, using a wide variety of problems in a wide variety of settings.

While I focus on fractions in this study, I assume that part of understanding how and when to use fractions also involves understanding their place in the larger domain of rational number and within the larger multiplicative conceptual field. This implies that understanding fractions and being literate in their use is still being developed when fractions are not the topic of instruction. Referring to the work of Greeno (1991) on acquisition of number sense, Sowder, Philipp, Armstrong and Schappelle (1998) point out that part of understanding a domain means knowing one's way around in it. "Number and operations are resources, and navigating the environment calls for reasoning that is at times multiplicative, at times quantitative, at times proportional, and at times some combination of the three" (p. 6). This would also include knowing when and how to use fractions in situations that involve measurement or geometry. In the present study, the focus is on how students learn to use their fraction knowledge as a resource when trying to navigate other mathematical environments and how this contributes to their understanding of fractions both conceptually and procedurally.

Thompson (1995) suggests that making sense of number (number used in the broad sense) involves understanding the various contexts in which number is used by distinguishing between quantitative and numerical operations. Numerical operations include addition, subtraction, multiplication and division. "Quantitative operations are the conceptual operations one uses to *imagine* a situation and to *reason* about a situation—often independent of any numerical calculations" (p. 207). Quantitative reasoning is about making sense of the situations that numbers and calculations are applied to. It is reasoning about objects and their measurements rather than reasoning about numbers. For example, what does each $\frac{3}{5}$ mean when you need $\frac{3}{5}$ of a ribbon that is $\frac{3}{5}$ of a yard long? Thompson argues that before students can make sense of numbers and operations, in relationally complex or conceptually sophisticated settings, students must first make sense of the settings themselves.

This argument points to the need to help students not only understand fractions as a topic of study, but to understand how they are used and applied in contexts. In other words, part of knowing about fractions and fraction operations is understanding how they are used in various settings. For example, how is the area and perimeter of a rectangular area or figure affected when a scale factor of $\frac{1}{2}$ is applied? Answering this question as well as carrying out any numerical operations needed to apply the scale factor requires an understanding of how fractions are used in area and perimeter situations and scaling situations.

The Rational Number Project (RNP) has been researching the development of children's learning of rational number concepts since 1979. Project investigators put forth the need for students to reconceptualize their rational number understanding into progressively more complex settings (Behr, Lesh, Post, & Silver, 1983). They report it common to observe regression in concept understanding over several weeks and suggest that concepts not only need to be remembered, they need to be reconceptualized when they are extended to new domains. This is because mathematical ideas do not simply go from not understood to mastered. Ideas need to be embedded into progressively more complex systems that enrich their original interpretation. RNP researchers do not elaborate on what these more complex systems are, but the implication is similar to that of Thompson (1995) and Vergnaud (1988). Students need to make sense of rational numbers in new settings in order to develop a robust understanding of them. Rather than simply remembering what has been learned, understanding evolves as one makes use of ideas.

Others have suggested that fraction research needs to be linked to research in other content areas students study in order to understand developmental interrelationships across content areas (Behr, Harel, Post & Lesh, 1992). Similarly, NCTM (2000) has indicated that students' facility with rational numbers can be developed in concert with the study of other topics in the middle school curriculum. When comparing the percentage correct on fraction items that have the same item description, for eighth and twelfth-grade students, the 1990–2000 National Assessment of Educational Progress (NAEP) data indicates that students continue to improve in this area (Kloosterman & Lester, 2004). Other researchers have made similar observations (Smith, 1995). Since fractions are not a topic with strong curricular focus high school mathematics courses, this data suggests that students do develop strategies and ideas about fractions outside of instruction on fractions.

Unanswered Questions Regarding Rational Number and Fraction Learning

Empirical research on fractions has explored themes such as misconceptions (Hart, 1981), intuitive understanding (Mack, 1990; Smith, 1995), experimental curriculums or teaching experiments (Behr, Wachsmuth, Post, & Lesh, 1984; Cramer, Post & delMas, 2002; Lachance & Confrey, 2002; Moss & Case, 1999; Steencken & Maher, 2003), how students understand a certain aspect of fractions such as unitizing (Lamon, 1996), or a particular operation such as multiplication (Mack, 2001). Most of these studies focused on fraction learning during instruction about fractions[6], and are of great value when developing instructional models and curricula for teaching fractions. While this research is important, understanding how students acquire fraction knowledge in fraction instructional settings, and what knowledge they bring to such settings, is

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[6] Some of the cited studies involved data collected in classroom settings while others were teaching experiments done with small groups of students outside of regular classroom mathematics instruction. For example, Hart (1981) and Smith (1995) did not look at learning as part of instruction. Mack (1995, 2001) took on the role of teacher-researcher using a case-study approach with one-on-one instruction. LaChance and Confrey (2002), implemented their own curriculum in a classroom. My research draws from regular classroom interaction using the school's adopted curriculum as taught by the regular mathematics teacher.

different in nature from understanding how students learn to use this knowledge in settings where fractions support the development of other mathematical content.

In 1988 a group of researchers met at the Research Agenda Project Conference on Number Concepts in the Middle Grades (Hiebert & Behr). It was agreed that number, as presented in conventional middle school mathematics classrooms, did not help students understand the complexity of middle school number concepts. "Competence with middle school number concepts requires a break with simpler concepts of the past, and a reconceptualization of number itself" (Hiebert & Behr, 1988, p. 9). From this conference came a research agenda for moving beyond theorizing to empirical research and then returning to theory to reevaluate and move forward. Three areas were addressed: (1) growth of student competence, (2) the relationship between intuitive and formal competence, and (3) the role of instruction in growing competence.

An important contribution that the current study can make is addressed in the following conference agenda item put forth by Hiebert & Behr (1988):

> We need to build and test models of students' thinking at different points along the growth continuum and to trace children's thinking as it changes from intuitive, informal processes to more powerful and general formal processes. (p. 11)

The RAND Mathematics Study Panel (2003) also points to the need to better understand the practices involved in learning, doing, and using mathematics. This study proposes to articulate the practices students engage in as they learn to use mathematical knowledge, in this case fractions, by examining the practices students engage in as they learn to use the fraction knowledge learned in two fraction units in five subsequent units across sixth and seventh grade: (1) area and perimeter, (2) percent and decimal operations, (3) probability, (4) similarity, and (5) ratio, proportions and percents.

Upcoming Chapters

In Chapter Two I share a theoretical framework for conceptualizing and studying learning to use, learning about, and fraction literacy. I also state my research questions in Chapter Two. Chapter Three provides a description of the design of my study and my process of data analysis. Chapter Four includes a detailed look at the sixth and seventh-grade classroom learning environments as well as a summary of what students studied and learned about in the two curriculum units on fractions. The practices students engaged in when learning about fractions is also described in Chapter Four.

Chapter Five describes two fraction literacy practices students engaged in when learning to use their fraction knowledge. These two practices capture the ways students reason about, make connections among, and further deepen their understanding of fractions so that they can use fractions as tools in new situations. In Chapter Six I answer my research questions by discussing each practice and how it differs from the practices students engaged in when learning to use fractions. Finally, in Chapter Seven I summarize my findings and discuss the implications and limitations of this research.

Chapter 2

LEARNING TO USE FRACTIONS: MAPPING LITERACY ONTO MATHEMATICS

In the theoretical and empirical literature there is a sizeable collection of work that examines what it means to know fractions and how to shape instruction in order to broaden students' conception of rational number. My research questions are embedded in understanding how students *learn to use* what they know or have learned about the mathematical topic of fractions. In this chapter I provide a perspective for examining how students learn to use the knowledge they learned when they formally studied fractions and present the specific research questions this study investigates.

Learning to Use

In Chapter One I introduced Barton's (1994) ecological metaphor for literacy. This metaphor allows one to look at the interrelationship between the environment and human activity. By using an ecological approach to literacy one attempts to understand how literacy is embedded in other human activity. In this study I am not just interested in cognitive development but how it is shaped within the social context where it occurs.

Barton's ecological metaphor is influenced by Scribner and Cole's (1981) practice account of literacy where literacy can only be understood in the context of the social practice in which it is used and acquired. This view of literacy means that literacy is part of everyday events[1]. While I am limiting their model by only examining school-based

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[1] This study differs in focus from others, that have looked at how people use mathematics in real life or out of school contexts (Lave, 1988; Nunes, Schliemann, & Carrahar, 1993; Barton, 1994; Lee & Smagorinsky, 2000). I do not ignore or discount such work and its contributions, but rather feel it important to also consider how schooling can help students become literate users of mathematics. Teachers and schools too often address knowing-that or knowing-how (Mason & Spence, 1999) as the focus of their work.

events, this view of literacy allows me to think about how work with fractions, work that follows the formal teaching of fractions, is situated in using fractions in a context that is not about learning fractions in the formal sense. When students learn *about* fractions in formal fractions units they develop strategies to makes sense of what fractions are and strategies to read, write, compare, and operate with them. In such situations it is clear to students that they are studying fractions directly. In a practice approach or an ecological approach to literacy, fractions are more than a set of concepts and procedures to learn. Fraction literacy also includes the set of social practices that "make use of a symbol system" (Scribner & Cole, 1981). Meaning resides in how fractions are used.

These views of literacy are influenced by Vygotsky's social view of learning. Elaborating on Vygotsky, Wertsch (1998) examines how the social and the mental come together so that internalization occurs. Wertch (as discussed by Wilkinson & Silliman, 2000) proposed two levels of knowing. The first is skill or knowing how to apply a cultural tool for particular cognitive and social purposes. "When teachers support [student] learning, students learn how to participate in the dialogue and ultimately, learn the "skill" of reading. Skill development, then, emerges from mastery of particular cultural tools suggesting that performance supports competence" (Wilkinson & Silliman, 2000, p. 342). Wertsch goes on to suggest a richer view of internalization or appropriation where students make tools their own and spontaneously employ them as part of learning how to learn.

Moschkovich (2004), makes a similar statement regarding appropriation as related to the development of mathematical practices when a student and a tutor work together to explore algebraic and graphical representation of functions on a computer. Appropriation develops from joint productive activity but goes beyond imitation of ideas as learned from a more knowledgeable other. "Appropriation also involves taking what someone else produces during joint activity for one's own use in subsequent productive activity while using new meanings for words, new perspectives, and new goals and actions" (p. 51). Both Moschkovich's and Wertsch's view of appropriation support the role of knowledge transformation which leads to the development of tools that can be used to make sense of and explore other activities.

These perspectives on appropriation are of interest in this study. The definition of skill that grounds the goals and design of the CMP curriculum suggests a similar view where students are to use the procedures and concepts learned as tool to make sense of new situations. The fraction concepts and procedures learned about in the formal fraction units have to be used in different ways in units where fractions are not the focus. Students are asked to decide when to use fractions and how with the intent of deepening and strengthening their knowledge of fractions. Students are still provided with support of peers and teachers, but I suspect that the interaction and scaffolding that occurs in this setting is different from the conversations that take place when fractions are the object of instruction. Students have to master the use of fractions and be able to draw upon them as tools when they are needed to make sense of something.

Others have made distinctions between different forms of knowing and learning that are similar to the distinction between learning about and learning to use. In the mathematics education literature, Mason and Spence (1999) point out that formal mathematics education most often focuses on developing three types of knowledge that constitutes knowing about a subject: (1) knowing-that or factual knowledge, (2) knowing-how or *technique* and skill and (3) knowing-why or having a story that reconstructs actions. They argue that the central problem of education is that "knowing-about does not guarantee knowing-to" and that knowing-to act in the moment "depends on the structure of attention in the moment" (p. 135). It involves educating an awareness of knowing to act.

In the field of literacy, Gee (1992) focuses on learning versus acquisition arguing that you cannot formally teach that which is acquired. He defines the two as follows:

> Acquisition is a process of acquiring something subconsciously by exposure to models, a process of trial and error, and practice within social groups, without formal teaching. It happens in natural settings that are meaningful and functional in the sense that acquirers know that they need to acquire the thing they are exposed to in order to function and that they in fact want to so function. Learning is a process that involves conscious knowledge gained through teaching or through certain life experiences that trigger conscious reflection. This teaching or reflection involves explanation and analysis, that is, breaking down the thing that is to be learned into its analytic parts. It inherently involves attaining, along with the matter being taught, some degree of metaknowledge about the matter. (p. 113)

Gee is clear to point out that acquisition can occur in classrooms. It does not happen because of teaching but because of a process of apprenticeship and social practice. Gee would argue that you can teach someone about fractions but you cannot overtly teach them to use fractions—you can however let them practice using them with others who are capable. To Gee acquisition is a matter of being socialized into the use of knowledge or what he would call a "Discourse"[2].

Wells (2000) finds the view of knowledge as a "thing" problematic claiming that there is a relationship between knowing and acting. He suggests that of the various types of knowledge (instrumental, procedural, substantive, aesthetic, theoretical) that

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[2] Gee purposefully capitalizes this term. People engage in a variety of Discourses. Each Discourse involves talking, acting, interacting, valuing, etc. in such a way that they display membership in a particular social group with a common set of interests.

theoretical knowing is the most powerful[3]. However, theoretical knowledge is gained by trying out a theory, or as I would say, by putting knowledge to use. While you can know of something, you cannot really grasp its significance without using it. Wells argues that the development of theoretical knowing should be given high priority in schooling beginning in the middle school years, once basic literacy and numeracy are established.

> There should, whenever possible, be opportunities for gaining firsthand, practical experience of tackling problems in the relevant domain so that there will be a perceived need for the theoretical constructs that provide a principled basis for understanding those problems and making solutions to them. By the same token, because theoretical knowing should not be treated as an end to itself, there should also be opportunities to put the knowledge constructed to use in some situation of significance to the students so that, through bringing it to bear on some further problem, they may deepen their understanding. (p. 70)

Wells does not exclude other types of knowledge or find them less important, rather he argues for the importance of putting these other types of knowledge to use in situations of significance so students build deeper understanding. This notion of building deeper understanding by using previously developed concepts and procedures is a key tenet behind the design of the CMP curriculum. This study has the potential to provide empirical data on how deeper understanding may possibly develop in this setting.

Unlike notions of learning and transfer, which do not take into account social activity (Lave, 1988), the notion of *learning to use* explored in this study assumes social activity and language are inherent in the learning process. If we want to understand how students learn to use knowledge we cannot discount the role of others in this process. In addition we have to account for what students bring to this experience. Learning to use

[3] These modes of knowing represent stages of thinking that societies pass through as they become more literate and technologically oriented. Theoretical knowing is the most powerful but it is not the most superior. Modes of knowing that are distinguished prior to theoretical knowing are not displaced because theoretical knowing is built upon them.

knowledge has no meaning if we do not consider the school environments in which students participate—both social and mathematical—to give knowledge its purpose for use[4].

The mathematical environment in this study includes other mathematical topics, and their related contexts, which give the use of fractions meaning and provide the context for studying the practices that students engage in while learning to use fractions to achieve outcomes or goals. Helping students learn to engage in the various ways fractions are used involves looking at other strands of mathematics. For example, fractions are commonly used in measurement situations. This is tied to the goal of helping students understand that fractions are more than part-whole relationships. Fractions are also numbers, location, on a number line. When we discuss how tall some one is we can use fractions to create finer-grained measures rather than only being able to use whole number measures. In the real world weights of objects, distances traveled, the square footage of a building or a park involve non-whole number measures. Situations such as these, where fractions are used, provide a setting where students can begin to realize that there is the possibility of acting (Mason & Spence, 1999) and eventually of knowing to use.

The Interaction Between Learning About and Learning to Use

In this section Hiebert and Lefevre's (1986) work on conceptual and procedural understanding is reviewed. Prior to this, the reviewed research literature has focused on differentiating between learning about and learning to use. However, the development of conceptual and procedural knowledge is important when students learn about and learn to

[4] Describing the social and mathematical environments is the focus of Chapter Four.

use fraction knowledge. The fraction units where students learn about fractions do not develop one type of knowledge and the units that use fractions develop another. Rather, both are developed. Both are needed.

Conceptual knowledge is knowledge that is rich in relationships. It is constructed when information is networked with other information. This can happen when a relationship between two previously unlinked ideas develops or when new information is linked with existing knowledge. There are two levels at which relationships between knowledge can be established: primary and reflective. The information connected is at the same level of abstractness and is tied to a specific context. Hiebert and Lefevre point out that knowledge becomes more abstract when it is freed from specific contexts. Relationships at the reflective level involve recognizing similar core features in pieces of information that are superficially different.

The example that Hiebert and Lefevre use to describe each level of conceptual knowledge involves decimals but it can easily be restated to model how conceptual knowledge is developed when students are learning about fractions and learning to use fractions. When students learn about decimal addition there are two ideas that need to be related. First, each position to the right of the decimal represents a different place value and second, you need to line up place values when adding and subtracting decimals. "...a noteworthy characteristic of this primary relationship is that it connects two pieces of information about decimal numbers and nothing more. It is tied to the decimal context" (p. 5). This level of conceptual knowledge is reflective of the connections that may develop when students learn *about* decimals.

In contrast, the conceptual knowledge developed at the reflective level represents the kind of connections that can potentially develop when students are *using* fractions to make sense of decimals. In this case, the learner links lining up decimal places with adding fractions using common denominators.

> Now the connection between the position value and lining up decimal points to add decimal numbers is recognized as a special case of the general idea that you always add things that are alike in some crucial way, things that have been measured with a common unit...This kind of connection is at a reflective level because building it requires a process of stepping back and reflecting on the information being connected. (Hiebert & Lefevre, 1986, p. 5)

This connection is more abstract than the primary one established. It steps outside of the context of decimals and fractions but at the same time connects them in a new way leading to a deeper understanding of both.

Procedural knowledge includes two kinds of information. One involves familiarity with symbols and syntactic conventions for acceptable configurations of symbols. The second includes the rules or procedures used when solving problems. This knowledge often consists of sequences of procedures and is linear in nature. It differs from conceptual knowledge that is networked and connected.

An important aspect of conceptual and procedural knowledge involves how the two are related. Ideally, the two are closely linked with procedures being learned meaningfully. However, it is possible that procedures can be learned by rote. Hiebert and Lefevre (1986) argue that there are many benefits when conceptual and procedural knowledge is linked. Some of these include:

- More efficient memory storage and retrieval;
- Symbols have meaning;
- Better recall of procedures;

- Effective use of procedures;
- Ability to develop procedures needed to solve novel problems; and
- Procedural knowledge has the potential to promote concept development.

When procedures are learned conceptually they are linked to other ideas in the network. In a novel situation conceptual knowledge had the potential to extend a procedures range of applicability (Hiebert and Carpenter, 1992). In the case of students linking procedures for fraction addition and subtraction with decimal addition and subtraction it is possible to see how conceptual understanding of fractions and decimal numbers can lead to a deeper understanding of each as well as extend what one knows to a higher level.

From Literacy to Fraction Literacy

This section lays out the terminology and ideas behind what I am calling fraction literacy. In this study, when I refer to fraction literacy, I draw upon concepts such as Mason & Spence's (1999) concept of knowing-to and Scribner and Cole's (1981) notion of being able to use knowledge. I will also use Barton's (1994) ecological approach to literacy to look at fraction literacy as a set of social practices associated with a particular symbol system. When using an ecological approach to literacy one attempts to understand how literacy is embedded in human activity. In this study I explore how students learn to use fractions within the social context of the classroom where this learning takes place.

Table 1 characterizes Barton's description of literacy, literacy events, and literacy practices. Barton explains that literacy lacks a clear definition and there are many domains of literacy such as home literacies and school literacies. Within a domain, or social situation, people take on roles that require particular literacies. In general, literacy is best understood as a set of social practices that people use in certain situations or events. When studying literacy, literacy events and literacy practices are the basic unit of analysis with the starting point of the analysis being the interaction between individuals and the environment.

Table 1
Barton's Characterization of Literacy, Literacy Events and Literacy Practices

Literacy	▪ Social activity best understood in terms of the literacy practices which people draw upon in literacy events ▪ A symbolic system used for communication—it is a way of representing the world to ourselves and to others ▪ People have different literacies they make use of in different domains of life—the way we talk and interact at work versus the way we would act in a restaurant or the way we interact with children—people would act differently and use language differently in each of these places or situations ▪ A set of social practices associated with a particular symbol system and their related technologies-this includes being confident within these practices
Literacy Events	▪ Occasions where reading and writing are used ▪ Occasions when a person attempts to comprehend or produce graphic signs either alone or with others ▪ When talk revolves around a piece of writing
Literacy Practices	▪ Common patterns for using reading and writing in a particular situation—how a person would act at work or go about reading a menu ▪ Ways of using literacy that are carried from one situation to another

Studying associated ways of talking and writing can give insight into how participants make use of literacy practices. Methodologically it involves ethnographic approaches that examine the specific before generalities are suggested. "It starts out from a belief that it is necessary first to understand something within a particular situation before looking to generalities" (Barton, 1994, p. 37). In order to understand the intent of the research questions and the data collection methods associated with these questions, there are four terms that need to be defined: fraction literacy, fraction literacy events, fraction learning practices, and fraction literacy practices.

Fraction literacy is the way people talk and interact when taking part in situations where fractions are used. It involves understanding fractions both procedurally and conceptually. In Appendix A I listed part of the National Council of Teachers of Mathematics (NCTM, 2000) Number and Operations Standard that described what is to be expected when learning about fractions. This list represents what is involved in understanding fractions as a body of knowledge. However, fractions literacy is more than knowing about fraction concepts and skills. It also involves being able to use fractions to achieve a specific purpose and to engage in situations where fractions are used. This included being able to reason and operate with fractions, knowing when it is appropriate to do so, and being able to explain why one's reasoning and operational choices makes sense. Fraction literacy also involves being able to communicate ideas to others, both verbally and in written form. It involves knowing about fractions and being able to use knowledge about fractions to achieve goals for oneself, and to communicate with others confidently.

Fraction literacy events are situations that involve fractions. This involves fraction situations when fractions are being learned about and used. For this study I have identified mathematics instructional tasks in the school's adopted mathematics curriculum, CMP II, where fraction literacy events will occur. For example, one group of fraction tasks is the problems in the two sixth-grade fraction units. In addition, there are fraction tasks in units where fractions are used in other mathematical contexts. When students work on these tasks they have to engage with fractions and partake in fraction literacy events. Within the classroom, fraction literacy events can involve various types of student engagement. Students may work individually or with others when trying to solve a problem involving fractions. In addition, students may participate in whole-class discussions about their work when trying to solve a problem involving fractions.

Fraction literacy involves both knowing about and knowing to use fraction knowledge. In the design of this study I have identified task that will lead to fraction literacy events where one or the other is at the forefront and I will discuss the practices associated with each. For example, a majority of the work students do in the fractions units involves learning about fractions. However, students carry knowledge about equivalence and quantity from the first fraction unit to the second. In this respect there is opportunity for using fraction knowledge. However, it is explicit in the fraction units that the focus of instruction is on learning about fractions. This is not the case in units that follow the fraction units. In these units the main focus of instruction is on a non-fraction topic. When students encounter fractions in these settings they have to make decisions about if and how to use their fraction knowledge.

In order to distinguish between these two types of fraction practices I have chosen two terms: *fraction learning practices* and *fractions literacy practices*. In general, fraction practices are stable identifiable patterns of behavior that are associated with fractions and fraction events. The term *fraction practices* does not help distinguish between learning about and learning to use. In addition, fraction literacy involves "using" fraction knowledge. It is not assumed that because one has learned about fractions that one will be able to use them for specific purposes. It is not assumed that having learned about fractions automatically leads to fraction literacy.

Fraction learning practices are stable identifiable patterns of behavior that people engage in when learning about fractions. These practices are influenced by the curriculum, the teacher, and the students. In addition, the goals, focus, and development of ideas when learning about fractions is influenced by the field of mathematics and mathematics education. When these two perspectives come together the result is a set of practices that are both intended or established by the field and emergent in a particular classroom at the same time. In this study, these occur when students are studying fractions in the fraction units.

Fraction literacy practices are stable, identifiable patterns of behavior that people engage in when using fractions. The focus is on "using" fractions. These practices involve having fraction skills but go beyond knowing about fractions to knowing how to use them. I draw upon Scribner and Cole (1981) who argue that practice "always" refers to socially developed and patterned ways of using technology and knowledge to accomplish tasks. It assumes people are engaging with fractions for a reason—to accomplish some goal. This practice is tightly linked with Barton's definition of literacy where he says "Literacy is not simply knowing how to read and write a particular script but applying this knowledge for specific purposes" (p. 236).

In this study learning is associated with both fraction learning practices and fraction literacy practices. I assume the students in this study are learning in both settings—they are learning about fractions and learning to use them. They are somewhere in the continuum from novice to expert. I am not exploring the practices of proficient users of mathematics (see RAND, 2003). The practices of generalization, proof, or representation that the RAND Mathematics Study Panel focuses on are broad and apply to any mathematical context. I am looking at the practices that are particular to fractions that occur when *learning* about and *learning* to use fractions. In addition, I am not looking at general pedagogical practices or classroom practices.

The main focus of my research questions is on learning to use fractions but this involves also understanding what is involved when learning about fractions. By studying fraction literacy events that occur in instructional settings that follow explicit instruction with fractions in formal fraction curriculum units, and comparing and contrasting them to the fraction learning practices associated with learning about fractions, this study aims to provide insight into what is associated with developing fraction literacy.

Research Questions

The underlying basis of this study is that knowledge evolves as you "make use" of it. There is more involved in being able to use fraction knowledge than the concepts and ideas typically associated with teaching fractions. Guided by sociocultural theory and the notion that how and what one knows is influenced by what happens in the learning environment as a whole, the primary unit of analysis is the classroom learning environment. This study attempts to address the following general question: How do middle school students, within a classroom community, learn to use their knowledge of fraction concepts and procedures to achieve specific purposes during fraction literacy events that occur in units that follow the formal teaching of fractions? My specific research questions are:

1. What are the fraction literacy practices students engage in when learning to use fractions?
2. How are these practices similar and/or different from the practices students engaged in when learning about fractions?

By examining the classroom environment for practices associated with learning to use fractions, and understanding how these practices differ from practices associated with learning about fractions, this research aims to better understand what is involved when middle school students are in a position to learn to use fractions.

Chapter Summary

This chapter provided a framework for studying how students learn to use fractions. This involved developing constructs to situate mathematical literacy and using knowledge within the context of fractions. A set of research questions was also developed. Chapter Three will present the design of this study and methods for collecting and analyzing data.

Chapter Three

METHODS OF DATA COLLECTION AND ANALYSIS

In order to better understand how students learn to use fractions as a means for achieving other goals, I designed a study that investigated the mathematical practices students engaged in when they needed to use fractions in other mathematical content areas. My goal was to explore the research questions presented in Chapter Two by examining the fraction literacy practices that emerged in the content units that followed the teaching of the fraction units and to compare the practices students engaged in when learning to use fractions to the practices students engaged in when learning about fractions. In this chapter I describe the fieldwork in which I engaged in, the data I collected, the site where it was collected, and my methods of data analysis.

Timeline and Preliminary Fieldwork

Preliminary fieldwork and data collection took place across multiple school years and involved several stages. Initially, during the 2001–2002 school year I observed the teaching of the CMP II fraction units in sixth grade as part of my work for the NSF-funded project to revise the CMP curriculum. Here I met the sixth-grade teacher whose classroom was the eventual site for one phase of data collection. During this time I began to think about the notion of learning to use as opposed to learning about with regard to fractions. I began to examine the sixth-, seventh-, and eighth-grade curriculum units that followed the sixth-grade fraction units and identify potential tasks where students would have to use knowledge of fractions. While I would ideally have followed students through grades sixth, seventh, and eighth, I eventually decided to collect data in sixth grade during the 2002–2003 school year and seventh-grade data during the 2003–2004 year through the end of December 2003.

In Fall 2002 and Winter 2003, during the teaching of the fraction units *Bits and Pieces I* (Lappan, et al. 2002/2003a) and *Bits and Pieces II* (Lappan, et al. 2002/2003b), I began data collection. I took field notes and engaged in participant observation in the classroom three to four times per week. Once the teacher launched a task and students began to work on a problem in small groups, I moved from group to group listening to students' conversations, taking notes, asking them questions about their work, and at times, if appropriate, answering questions they would ask me. When the teacher and students engaged in whole-class conversations about the task I sat in the back of the room and took field notes. During this phase of data collection I identified four students who would be the focus of the small group work data I planned to collect as well as the students I would interview. Finally, during this time I finalized selection of instructional tasks where data would be collected to study the fraction literacy practices students engaged in when using fractions.

I point out that the original purpose for this data collection was to record what happened when students were learning about fractions. The data was suppose to be preliminary in nature and used to provide background information for the data collected when students were using fractions in other mathematical contexts. However, as data analysis and writing about the fraction literacy practices took place it became apparent that understanding what happened when students were learning to use fractions was tightly linked with what happened when students learned about fractions. Rather than using this data to provide a backdrop for the data on learning to use fractions, it became apparent that the learning about data and the learning to use data needed to be compared and contrasted.

The research questions presented at the end of Chapter Two were not the original questions. The original questions focused on learning to use fractions and did not address how students learned about fractions. Since this shift in focus was unanticipated, the type of data I planned to collect when students were learning about fractions did not include video recordings of classroom conversations. Only field notes were taken. However, as part of the work I was doing for the CMP revision project I did videotape six class sessions for another project. I used those videos when analyzing the data for the practices students engaged in when learning about fractions.

In February 2003 I began data collection in the first of five units where students would be using fractions as part of learning about other mathematical contents. Table 2 provides a timeline for data collection that focused on learning about fractions and learning to use fractions. The data collected to study the practices students engaged in when learning to use fractions including audio and video recordings of whole-class discussion, small group work, student work, and interviews. I continued to interact with students when doing small group work by asking them questions about their work.

Data Collection

Identifying Instructional Tasks

I surveyed the CMP II curriculum units that followed the formal fraction units and identified tasks in units where fraction literacy events that involved using fraction knowledge were likely to occur. The tasks were contextualized problems that asked students to draw upon the mathematics they knew and at the same time make sense of a

new idea(s). The tasks, as well as the curriculum they are taken from, use problems combined with an instructional model that encourages higher-level thinking and problem solving. Making sense of mathematics is at the core of the work which students' and teachers engage in as they work on a CMP task (Lappan, et al., 1996).

Table 2
Data Collection Timeline

Grade Unit Start Date	Instructional Unit: Mathematical Context	Focus of Data Collection
6th Grade October 22, 2002	Bits and Pieces I: Understanding Fractions, Decimals, and Percents (Lappan, et al., 2002/3a)	Learning About Fractions
6th Grade January 14, 2003	Bits and Pieces II: Using Rational Numbers (Lappan, et al., 2002/3b)	Learning About Fractions
6th Grade March 10, 2003	Covering and Surrounding: Two Dimensional Measurement (Lappan, et al., 2002/3c)	Learning to Use Fractions
6th Grade April 16, 2003	Data, Decimals, and Percents: Percents and Decimal Operations (Lappan, et al., 2002/3d)	Learning to Use Fractions
6th Grade May 15, 2003	How Likely Is It?: Probability (Lappan, et al., 2003)	Learning to Use Fractions
7th Grade October 29, 2003	Stretching and Shrinking: Similarity (Lappan, et al., 2004a)	Learning to Use Fractions
7th Grade December 12, 2003	Comparing and Scaling: Ratio, Proportion, and Percent (Lappan, et al., 2004b)	Learning to Use Fractions

An Example of a Task. Figure 1 shows the first instructional task where data on the use of fractions in other contexts was collected. This problem, Storm Shelters, was the fourth problem in a unit on area and perimeter. At this point in the unit there were two goals guiding the work: (1) understanding what it means to measure area and perimeter by relating area to covering and perimeter to surrounding and (2) there can be more than one area for a fixed perimeter and multiple perimeters for a fixed area. The Storm Shelters task is one of four tasks in the unit that explored maximum and minimum perimeters for fixed areas and maximum and minimum areas for fixed perimeters.

Problem 2.1 Storm Shelters

The rangers in Great Smoky Mountains National Park want to build several inexpensive storm shelters. The shelters must have rectangular shaped floors with 24 square meters of floor space. Suppose that the walls are made of flat rectangular sections that are 1 meter wide and cost $125.

A. Experiment with different *rectangular* shapes. Sketch each possible floor plan on grid paper. Record your data in a table with columns for length, width, perimeter, area and cost of the walls.

Length	Width	Perimeter	Area	Cost of Walls

B. **1.** What determines how many wall sections are needed, area or perimeter? Explain.
2. Based on the cost of the wall sections, which design would cost the least? expensive to build? Explain why this arrangement costs the least to build.
3. Which shelter plan has the most expensive set of wall sections? Explain why this arrangement costs the most to build.

C. Liz notices that as the length and width become closer in size, the perimeter decreases and the cost is less. The 6 × 4 storm shelter has side lengths that are closest in size, the smallest perimeter, and the lowest cost. Liz wants to see if using the fractional length $5\frac{1}{3}$ meters will give a smaller perimeter and reduce the cost even more.

1. If the area of the storm shelter is 24 square meters and the length is $5\frac{1}{3}$, what will the width of the storm shelter be?
2. What is the perimeter and the cost for wall sections? Add this data to your table in part A.
3. How does this rectangular floor plan compare in cost to the others in the table?

D. Suppose you want to consider a rectangular floor space of 36 square meters. Which design would have the least perimeter? The most perimeter? Explain your reasoning.

Figure 1 Storm Shelters Task

Previous tasks focused on area as a measure of square units and perimeter as the number of unit lengths that fit along the outside edge of an area. The tasks involved rectangular and non-rectangular shapes that could be neatly covered with whole square units. In these tasks the students could rely on counting whole square units or multiplication of length and width to generate dimensions for a given area of a rectangular space. The numbers representing the dimensions of these shapes were typically whole numbers associated with basic multiplication facts.

The intention of part C in the Storm Shelters task is to push students thinking beyond whole numbers to consider a situation with one known dimension and a known area, or one factor and the product. The fractional dimension pushed students to think beyond whole units of measure and leads them to drawing upon the inverse relationship between multiplication and division.

Identified Tasks for Data Collection. Nine sixth-grade instructional tasks and nine seventh-grade instructional tasks were identified as places where students might engage in using fractions. These tasks are listed in Table 3. In addition, there were eight unanticipated tasks. These will be discussed in the next section. The complete set of 26 tasks, planned and unanticipated, are included in Appendix B.

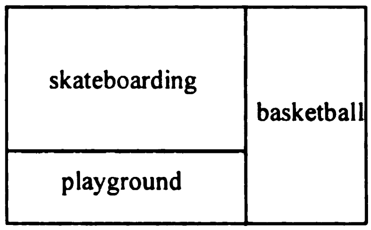
Two of the identified tasks in the sixth grade area and perimeter unit, *Covering and Surrounding* (Lappan, et al., 2002/2003c) involved fractional dimensions. The Storm Shelters problem had a missing length while the Area and Perimeter of Parallelograms problem provides a $3\frac{1}{2}$ by $4\frac{1}{2}$ rectangle on a centimeter grid. The dimensions are not given leaving students to measure the length and width they need to find the area and perimeter. The Park Problem, in Figure 2, uses fractions in a different way. Rather than fractions used as a unit of measure, fractions are used as an operator to think about a fraction *of* a measure (in this case length, width, and area).

Table 3
Instructional Tasks Identified for Data Collection When Using Fractions in Other Contexts

Unit Title: Mathematical Content Focus (Grade Level)	**Instructional Tasks**
Covering and Surrounding: Two-Dimensional Measurement (6^{th})	Investigation 2, Problem 1: Storm Shelters Investigation 4, Problem 1: Area/Perimeter of Parallelograms Investigation 4, Problem 3: Park Problem
Data, Decimals, and Percents: Percent and Decimal Operations (6^{th})	Investigation 1, Problem 3: Adding/Subtracting Decimals Investigation 2, Problem 1: Multiplying Decimals
How Likely Is It?: Probability (6^{th})	Investigation 1, Problem 1: Flipping for Breakfast Investigation 1, Problem 2: Tossing Paper Cups Investigation 1, Problem 3: Match/No Match Investigation 2, Problem 4: Exploring Probabilities
Stretching and Shrinking: Similarity (7^{th})	Investigation 3, Problem 1: Rep-Tile Quadrilateral Investigation 3, Problem 3: Scale Factors and Similar Shapes Investigation 4, Problem 1: Ratios Within Similar Parallelograms Investigation 4, Problem 2: Ratios Within Similar Triangles
Comparing and Scaling: Ratio, Proportion, and Percent (7^{th})	Investigation 1, Problem 1: Ads that Sell Investigation 1, Problem 2: Targeting an Audience Investigation 1, Problem 3: American Records Investigation 2, Problem 1: Mixing Juice Investigation 2, Problem 2: Sharing Pizza

Problem 4.3A The Park Problem

A. The Rochelle Park District set aside a rectangular section of land to create a park. After talking with students in the community the park district decided to create an area for skateboarding, an area with playground equipment, and an area with a basketball court.



1. A large rectangular area was set aside for skateboarding. A short fence was put up around the skateboarding area that took up $\frac{2}{3}$ of the length and $\frac{2}{3}$ of the width of the rectangular section of land. What fraction of the area of the entire rectangular section of land does the skateboarding park take up?

2. The basketball court is 35 feet by 60 feet. Use this information, and what you know about the skateboarding portion of the park to find the area and the perimeter of the playground area.

Figure 2. The Park Problem

In *Data, Decimals, and Percents* (Lappan et al., 2003/2004d), the unit on decimal and percent operations students explored decimal operations of addition, subtraction, multiplication and division from the perspective of place value and fractions. For example, two and three tenths plus one and twenty-two hundredths can be thought of as $2.3 + 1.22$ or as $2\frac{3}{10} + 1\frac{22}{100}$. The tasks observed, one on decimal addition and subtraction and another on decimal multiplication asked students to consider these operations while using the algorithms for fraction operations they learned in *Bits and Pieces II* (Lappan et al., 2002/2003b), the unit on fraction operations. For addition and subtraction, a task using a place value approach to decimal addition and subtraction, is followed by a task that asks students to rethink the same situational contexts with fraction addition and subtraction. Multiplication with decimals is first considered from the viewpoint of fraction multiplication and then place value. A task on decimal division using fraction division was planned for data collection but due to time constraints the students did not work on decimal division.

The probability unit, *How Likely Is It?* (Lappan et al., 2003), includes problems where students use fraction notation to express fractions as parts of whole and use fraction notation to express ratios or part-to-part or part-to-whole comparisons. Ratios are not the focus of the unit and students do not explicitly compare or contrast these fraction and ratio approaches to expressing an idea using $\frac{a}{b}$ notation. Nonetheless, I was curious to see if students would notice the different usages and how they would engage in the situations where fractions were explicit. This was the case in the Flipping for Breakfast task where students are asked to conduct an experiment to determine the probability of flipping a head or tail. Students are asked to express the number of heads as a fraction of the total coin tosses. As part of this task, the Tossing Paper Cup task and the Match/No Match task, students collect individual data and then combine it to create class data. In the process of combining students would all record their experimental probabilities and then add them together. Unlike when one adds fractions (and what is typically an error in addition of fractions), the students added up the total number of favorable trials (value in numerator) and the total number of trials (value in denominator). This treats fractions representing probabilities as ratios or relative frequencies. In the Match/No Match task and Exploring Probabilities task, students are asked to analyze specific probabilities such as the probability of drawing a red marble is $\frac{1}{6}$.

In the seventh grade unit on similarity, *Stretching and Shrinking* (Lappan, et al., 2004a) students use scale factors and ratios to describe the relationship between similar figures. In doing so, problems are designed to use whole numbers, fractions, and percents as scale factors. Ratios are introduced and explicitly related to fractions. Since ratios are often written using fraction notation and ideas about equivalent fractions are extended to equivalent ratios. The Reptile Quadrilateral task and the Scale Factors and Similar Shapes tasks involve students in making sense of fractional scale factors. The Ratios Within Similar Parallelograms task and Ratios Within Similar Triangles task explore ratios. Here students began to sort out when a fraction notation represents a fraction and when it represents a ratio.

Students continue to explore the differences between fractions as fractions (as defined for this study) and fractions as ratios in the ratio, proportion, and percent unit *Comparing and Scaling* (Lappan, et al., 2004b). In this unit students explore different ways to compare quantities using fractions, decimals, percents, differences, and rates. The different types of comparisons are introduced in the Ads That Sell task and carry across all five of the identified instructional tasks. In this unit students have to sort out whether a ratio expressed using fraction notation represents a part-whole relationship or a part-part relationship. The differentiation of ratio and fraction is formally introduced in the Targeting an Audience task. Students identify which of the various types of comparisons are being made and use data to write different types of comparisons in the American Records task. The last question in this task captures the intent of these problems. It asks: "When a problem requires comparison of counts or measurements, how do you decide whether to use difference, ratio, fractions, or percent ideas?" (Lappan et al., 2004b, p. 9). After exploring the various types of comparisons and when each is useful, students use the various comparisons to make sense of situations that involve comparisons. These situations lend themselves toward ratios and scaling in order to find equivalent ratios. Of these two tasks, Mixing Juice and Sharing Pizza, the Mixing Juice task is provided in Figure 3.

Problem 2.1 Mixing Juice

Every year, the seventh-grade students at Langston Hughes School go on an outdoor-education camping trip. During the week-long trip, the students study nature and participate in recreational activities. Everyone pitches in to help with the cooking and cleanup.

One year, Arvin and Mariah were in charge of making orange juice for all the campers. They planned to make the juice by mixing water and frozen orange juice concentrate. To find the mix that would taste best, Arvin and Mariah decided to test some recipes on a few of their friends.

Mix A		Mix B	
2 cups concentrate	3 cups cold water	5 cups concentrate	9 cups cold water

Mix C		Mix D	
1 cup concentrate	2 cups cold water	3 cups concentrate	5 cups cold water

A. Which recipe will make juice that is the most "orangey"? Explain your reasoning.

B. Which recipe will make juice that is the least "orangey"? Explain your reasoning.

C. Here are two comparison statements using fractions:

$\frac{5}{9}$ of mix B is concentrate. **$\frac{5}{14}$ of mix B is concentrate.**

Which do you think is correct? Why?

D. Assume that each camper will get $\frac{1}{2}$ cup of juice. For each recipe, how much concentrate and how much water are needed to make juice for 240 campers.

E. For each recipe, how much concentrate and how much water are needed to make 1 cup of juice?

Figure 3. Mixing Juice Task

Unanticipated Tasks Leading to Data Collection. In addition, there were eight fraction tasks that were unanticipated, occurring as teachable moments. While I was at the site waiting for a planned event to begin these additional events occurred. Sometimes I was waiting for a task to begin and had the video recorder available to capture what happened. At other times I only took field notes. In sixth grade there were three unanticipated events and in eighth grade there were five.

The first two sixth-grade events occurred during the area and perimeter unit Covering and Surrounding. The first event involved a conversation about the homework problem in Figure 4. The second event took place while going over a quiz the students had taken. The quiz had diagrams of two triangles and a rectangle. The shapes were not pre-measured. The students had to measure needed dimensions for each the shapes and find the area and perimeter. In both the homework problem and the quiz, the measures included non-whole numbers, leading students to have to operate with fractional lengths.

Calculate the area and perimeter of each polygon and briefly explain your reasoning for figures 1, 4, and 6. [Only the figure discussed in class is provided.]

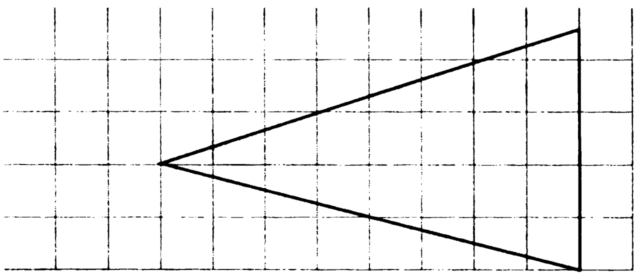


Figure 4. Triangle Homework Task

The third unanticipated sixth-grade event occurred during the decimal operation unit *Data, Decimals, and Percents* (Lappan, et al., 2002/2003d). While I was waiting to collect data on anticipated task 1.3 Adding/Subtracting Decimals, students were summarizing the preceding task that explored decimal addition and subtraction from the viewpoint of place value. When students were developing an algorithm for adding and subtracting decimals by lining up digits with like place value, a student suggested that if you wrote the quantities using fraction notation rather than decimal notation, that you could use the fraction addition and subtraction algorithm developed in the fraction unit on fraction operations.

The five remaining unanticipated events happened with the seventh-grade class. Four took place during the unit on similarity and one took place in the unit on ratio, proportion, and percent. The first three events were part of an opener or warm-up problem given at the start class. The first opener asked students to figure out what would happen if a 2 by 3 picture was put into a copier and enlarged 150%. The second opener asked students to find 20% of 120. This led to a long conversation where fractions were eventually used to make sense of the problem. The third task was $\frac{1}{2} \times 48$. This task was the focal point of a 30-minute discussion where students were trying to make sense of multiplication and division by one-half. This led to the fourth unanticipated seventh-grade task (See Figure 5). The next day students spent about half of the class period discussing their solutions to this problem.

Solve each of the following. Draw a picture to support your work.

1. $8 \div 2 =$ 2. $8 \div \frac{1}{2} =$ 3. $10 \div 2 =$ 4. $10 \div \frac{1}{2} =$

Figure 5. Unanticipated Fraction Division Homework.

The fifth and final unanticipated task took place during the *Comparing and Scaling* (Lappan, et al., 2004a) unit. The students had been writing different types of comparison statements. This task about participation in sporting activities, provided in

Figure 6, used large numbers and unlike the other problems students had been working on, the numbers did not lead to simple comparisons easily related through basic facts.

Participation in Sports Activities (1995)

Activity	Males	Females	Ages 12–17	Ages 55–64
Bicycle Riding	24,562,000	23,357,000	8,794,000	2,030,000
Camping	23,165,000	19,533,000	5,336,000	2,355,000
Exercise Walking	21,054,000	43,373,000	2,816,000	7,782,000
Fishing	30,449,000	14,885,000	4,945,000	3,156,000
Swimming	27,713,000	33,640,000	10,874,00	2,756,000
Total in Group	**111,851,000**	**118,555,000**	**21,304,000**	**20,922,000**

1. Write two percent statements from the 1995 data.
2. Write two fraction statements from the 1995 data.
3. Write two ratio statements from the 1995 data.

Figure 6. Comparison Statement Task

Participants and Setting

The setting for this study was a middle school of a small mid-western community. This school and its teachers had been involved with piloting and using CMP materials since its inception in the early 1990s. In the late 1990s CMP began piloting a set of updated materials and this district participated in the field-testing of these materials. From my work with the field-testing of the new CMP units, I had the opportunity to interact with the two teachers whose classrooms became the site for data collection.

These two teachers could be characterized as effective mathematics teachers (NCTM 1991, 2000). They had a strong understanding of the mathematics they taught and their students as learners of that mathematics. The data I collected needed to arise out of classroom discourse—both small group and whole class— and these teachers classroom organization and pedagogy would allow for this. Students would be engaging in conversation when working to solve problems and would be offering, as well as challenging, ideas that were put forth in conversation. From a research perspective, this type of learning environment would provide a window into how students, individually and as a whole class, were reasoning about the mathematics they engaged with.

During the 2002-2003 school year, a sixth-grade class with 23 students was the focus of data collection. In 2003-2004, a seventh-grade class of 23 students in the same school was observed. Of the 23 original sixth-graders, eight were in the seventh-grade class. The other students in the seventh-grade class were in sixth-grade classes in the same school with other sixth-grade teachers. Despite the seventh-grade students having had different sixth-grade teachers, all were exposed to the same mathematics curriculum in sixth grade.

Four target students were identified in sixth grade. The four target students, two girls and two boys, were video-taped during small group work and audio-taped during interviews. These four students, TJ, Ali, Katie and Bryan[1] were chosen for a number of reasons. Since Kent Middle School is a community with little transience, the teachers knew many of the families. The sixth-grade teacher, Mrs. Kay, and the school guidance counselor helped me determine which parents were most likely to request that their child be placed with the identified seventh-grade teacher, Mrs. Dew. Through these conversations I was able to identify a group of students who were likely to be placed in Mrs. Dew's class. When I identified the four focus students I was aware that there was the possibility that they could still be requested into a different seventh-grade teacher's class. Luckily this was not the case.

Three of the four focus students came from the local public elementary school that used a reform-based elementary mathematics curriculum. One student came from the

[1] Pseudonyms are used in place of all students' names, the teachers' names, and the name of the middle school.

local private religious school that used a traditional mathematics curriculum. The four focus students, Bryan, Ali, Katie, and TJ were considered average students. Their teacher shared her impression of numerous students being considered including their strengths and weaknesses. Based on these conversations, I learned that their grades ranged from Bs to Ds. The student who received the D was a bright and capable student who often did not turn in all of the homework assigned.

Ali worked hard and often came before or after school to get help when she was unclear about something. Mrs. Kay pointed out that Ali had shared that she greatly benefited from the whole class conversation that helped her pull ideas together. Katie, who came from the private school often commented in interviews that she really like to work out computations by hand but struggled to explain why the algorithms made sense. When someone shared an idea and Katie did not understand it, she would raise her hand and ask for more clarification. Both Katie and Bryan spoke up in class discussion when they did not think something someone offered was sensible. They would not agree if they were not convinced of something's reasonableness. A few times this left them on opposite sides of the table when they did not agree with each others' approach to a problem they were working on in small groups. Bryan was mathematically the strongest of the four students and the most outspoken of the four. TJ wanted to make sense of things for himself and if the others went too fast during small-group time he would tell them to wait. While TJ and Ali often offered ideas their group developed during class discussion, they were less likely than Bryan or Katie to object to someone's reasoning or offer a new idea.

An attribute that each of these students shared that drew me to them was that they interacted in small-group conversations and contributed to whole-class discussion. While interviewing the focus students Bryan asked why I chose him and not another student (he named her by name) who was smarter. My response was that she rarely contributed to class discussions and when she did she was usually called upon by the teacher. In addition, the focus students typically offered their ideas without being prompted. In addition, I explained that I did not want the "smartest" student. I wanted students who had to put forth effort to make sense of the problems so I could better understand what made sense to them and what caused them to struggle.

Sources of Data

Whole-Class and Small-Group Video Data. In order to capture what happened during fraction literacy events, video data was collected during each class session where an identified instructional task was being used. The class was taped during whole-class discussion. At the same time I took field notes to support as well as provide back up for video-data. When students were working on problems in small-groups, the four focus students formed a group and their interactions were videotaped. I also took field notes of their interactions. When an unexpected fraction literacy event took place, I videotaped the whole class conversations and the small-group work of the focus students if the camera was available and set up. If I did not have the camera with me I took detailed field notes.

Interview Audio Data. The interview process provided a setting in which to follow up and test out hunches that were emerging from the data. The interview questions were designed to provide confirming or disconfirming evidence for answering the first research question (Erickson, 1986). For example, when students reflected on videos from their work in class, I was able to determine if their interpretations did or did not match mine. By only observing and listening to students' conversations there was the possibility that my hunches about what they did or did not understand was wrong or incomplete. Students may have struggled with an instructional task because of the mathematical context that was being studied, for example, area and perimeter, rather than because the task involved the use of fractions.

Four audio-recorded interviews were conducted with the target students. Field notes were taken and any written work that students did as part of the interview was collected. Three interviews took place in sixth grade and one in seventh grade. The interviews focused on students' perception of what happened in class. The interview protocol (see Appendix C) is generic in form and was adapted to fit the different mathematical contexts that were investigated and the events that took place during class. For example, the interview process may have involved reflecting on a piece of the student's own work or reacting to a video clip from the class discussion, as well as students reflecting on the instructional task in general. While the generic protocol was followed, for each interview it was adjusted to be specific to what happened in class. For example, the sixth interview question "What do you think about what (insert name) brought up in the discussion?" was customized to address an event that took place in class. In the interview for the area and perimeter unit the question asked: "What did you think when Mrs. Kay asked why no one used their fraction multiplication algorithm that they learned in the fraction unit when they were finding the area of the rectangle in problem 4.1?" Prior to asking this question I played a video clip with the conversation that led to Mrs. Kay's proposed question.

During the group interview process the goal was to provide opportunities for each focus student to react to the ideas of the other focus students. During individual interviews the goal was to understand the individual students' thought process without having the other focus students influence what they offered. In order to achieve both goals the interviews used a variety of student groupings. In order for students to become comfortable with the interview process, the first interview was a group interview with all four students. This interview set the tone for other interviews by pushing students to reflect on their learning experiences during class. At the same time students were able to sit together, hear what others had to say, and react to others. The second interview was an individual interview. During the third interview session, students were interviewed in pairs. In the final interview, the seventh-grade interview, students were again interviewed individually.

Student Work. For each instructional task identified for data collection the work of the focus students was photocopied. Student responses to the end of unit reflection questions were collected for the *Covering and Surrounding* (Lappan, et al., 2002/2003c) unit and the *Data, Decimals and Percents* (Lappan, et al., 2002/2003d) unit. The seventh-grade teacher did reflections as part of the whole class conversation rather than individually. When these conversations occurred, I video-recorded them. During interviews I sometimes offered a problem for students to look at or work out. Any written work they did around such a problem was also collected. For example, based on the path of a small group and whole class conversation, when interviewed, I asked the focus students to solve and talk about the difference between two division problems $10 \div 4$ and $4 \div 10$.

Data Analysis

Data analysis was guided by Erickson's (1986) interpretive methods and participant observational fieldwork. This framework addresses the need to understand the social actions that take place in a particular setting. This is aligned with Barton's (1994) ecological approach to the study of literacy where one attempts to understand how literacy is embedded in human activity. In both Erickson's and Barton's approaches the focus is on understanding social interaction.

During the data collection associated with learning about fractions the primary data sources were fieldnotes of whole class discussions. The primary source of collected data that was associated with learning to use fractions was video recordings and fieldnotes from whole class discussions. In order to triangulate the data I also had video recordings and field notes of small group work of the focus students, the written work of the focus students and the focus student interviews.

Once I transcribed the audio and video data I was confronted with a huge amount of data. I read through the data for each instructional task several different times. This included reading the task itself, the small group data for the task, the whole class data, corresponding interview data that existed, and examining any student work that was collected. I began to identify major themes in the whole class data that were centered around the way the students and their teacher interacted with, made sense of, and used fractions. Due to the nature of the work when learning about fractions and when learning to use fractions the data analysis for each was slightly different.

Analysis Related to Learning to Use Fractions Data

I begin with discussion of the analysis of the learning to use fractions data since it preceded analysis of learning about fractions data. After transcribing the whole class data the transcript for each task was sectioned it into a set of events, each one capturing or focusing on a specific topic, thread of conversation or idea. I was trying to capture the major exchanges that occurred around fractions and their role in understanding and solving the task. This was done through a series of refinements. First I divided the actual transcript into sections that captured each key idea or focal point in the conversation. Each section was then summarized on a note card with a reduced photocopy of that part of the transcript (Erickson, 1986). I was able to create an outline that captured the major themes in that lesson. Figure 7 provides an example of each level of refinement using the first instructional task where the learning to use fractions data was collected, the Storm Shelter problem from the unit on area and perimeter.

As I created these notecard summaries I also revisited my other data sources—interview, student work, and small group work conversations of the focus students—looking for supporting evidence of the patterns that were emerging. For example, when I interviewed the focus students about this task two of them indicated that they had not considered Katie's idea to use division. They thought area was about multiplying. This limited understanding of area may not have surfaced if the problem did not involve non-whole number measures. I also saw in Katie's work how she struggled to actually carry out the fraction division. While she thought her idea for why to use division to find the missing value was accurate she was not sure until the whole class conversation actually occurred.

First Level Refinement: Excerpts from summary of major sections.

Lines 46–75: How do I operate with the fraction $5\frac{1}{3}$?

What decimal is equivalent to $5\frac{1}{3}$?

If $5\frac{1}{3} = 5.33333....$ Why doesn't $5.3333..... \times 4.5 = 24$?

* Note: not being able to rectify $5\frac{1}{3}$ interferes with finding a width in $5\frac{1}{3} \times W = 24$.

Lines 76–90: Katie asks: How did you get the width 4.5? Did you divide?

Is $4.6 = 4\frac{2}{3}$ or $4\frac{1}{6}$? Figure this out using fraction sense-part-whole reasoning.

Second Level Refinement: Complete lesson summary.

1. $5\frac{1}{3} \times 4\frac{2}{3} \rightarrow 5.3 \times 4.6 = 24.3$	Rejected
2. $5\frac{1}{3} \times 4\frac{1}{2} \rightarrow 5.3 \times 4.5$	Rejected
3. $5\frac{1}{3} \times 4\frac{1}{2} \rightarrow 5.33333 \times 4.5$	Rejected
4. $24 \div 5\frac{1}{3} \rightarrow 24 \div 5.333 = 4.528$	Not rejected at first.
5. $5\frac{1}{3} \times 4\frac{1}{2} \rightarrow \frac{16}{3} \times \frac{9}{2} = \frac{144}{6} = 24$	$4\frac{1}{2}$ is okay, but how did you get it? Did you divide? So, multiplication as justification is rejected initially.
6. $24 \div 5\frac{1}{3} \rightarrow 24 \div 5.333 = 4.528$	4.528 is rejected this time since 4.528 is not 4.5. How do you explain that your answer is not exactly 4.5?
7. $5\frac{1}{3} \times 4\frac{1}{2} \rightarrow \frac{16}{3} \times \frac{9}{2} = \frac{144}{6} = 24$	Explain how they got $4\frac{1}{2}$ using a combination of repeated addition and guess and check with multiplying. $4\frac{1}{2}$ is accepted as width.

Figure 7. Levels of data reduction for Storm Shelters task.

Erickson (1986) points out that materials collected from the field are not data themselves. Fieldnotes, transcripts, and videos are the data resources from which data is created. The process of creating summaries required numerous readings of transcripts and naturally led to the revisiting of the documentary sources in order to convert them into data. These summaries helped me focus on the big ideas that were taking place while not get caught up with other aspects of the data such as the teachers' general classroom practices. At the same time the summaries provide the linkages and records needed to find the associated full text record.

Through this process of refined data reduction and coding I began to look for data that helped me answer my research questions. I constructed a large grid and again revisited data resources such as the notecard summaries, transcripts, and audio and video recordings. Down the left side of the grid (See Appendix D) I listed potential themes I saw. Across the top of the grid I listed each instructional task-planned and unplanned. These themes were tied to my research questions. Initially I sorted the events on the notecards into two broad themes or categories. These themes, and the events that led to their development, are reflective of Erickson's (1986) methodology where one task of data analysis is to generate assertions largely through searching the full data corpus. One also needs to make evidentiary warrant for those assertions by reviewing and testing the assertions seeking confirming and disconfirming evidence. The chart was the first attempt to code data and was revised several times. As new potential codes or themes emerged I searched the data corpus to decide if I had enough evidence to warrant the assertion. The grid, which contained columns for each instructional task, provided a cross-sectional view of themes and where they occurred and didn't occur.

I used the lesson summaries for each task to begin the chart. The first two rows were used to sort the data into two categories: (1) places where fractions were used because they were part of the task and (2) places where students used fractions to make a point even though they were not explicit in the task. The parts of the lesson corresponding to each were entered into the chart. For example, when a student suggested using fraction notation to add 2.3 and 3.66 rather than decimal notation, this student made the choice to use fractions. The task did not require him to do so. This was entered into the cell for "fractions used when not explicit in problem" under the appropriate task heading, *Data, Decimals and Percents* (Lappan, et al., 2002/2003d) Problem 1.2. Adding Decimals/Place Value.

When I was sorting the lesson summaries into the two broad categories of explicit and implicit use of fractions, I noticed episodes that seem to revolve around equivalence (equivalent forms of representation, equivalent approaches, and equivalent outcomes). Initially I thought that this was a practice that students engaged in when using fraction knowledge. I could find an episode where issues of equivalence surfaced in the data for each of the first nine instructional tasks. I also found supporting data in the small group conversations and interviews. As I tried to write up vignettes from the data, and continued analysis of the seventh grade data, I began to realize that students were engaging in a practice but equivalence did not always or even completely capture what was happening. I subsequently determined that the questions and conversations that arose out of this data really centered around what was appropriate in a particular setting when using fractions. While equivalence was in many cases part of determining what was appropriate, the issues that surfaced were really about determining what an appropriate way to represent quantities was or what an appropriate way to operate with a set of quantities in a given situation was. Eventually the practice of determining equivalence was renamed to the practice of determining appropriateness.

Within this practice are several categories that capture how students engage in it. These categories led to the generation of a set of questions that capture the underlying focus of conversations regarding fraction use. For example, the episode where a student posed the use of fraction notation when adding decimals was categorized as Choosing to Use Fractions When Fractions Are Not Explicit and the underlying question that drives the conversation is: *Can we use fractions to make sense of or think about other mathematical ideas when fractions are not explicit in the problem?*

As I moved through the data I engaged in this process that began with transcribing and first level and second level refinement or data reduction. I then created categories to capture prevalent themes. From these themes I identified two practices that students and their teacher engaged in: the practice of determining appropriateness and the practice of connecting fractions to multiplication and division concepts. The episodes that were associated with each practice where then analyzed. In the case of the practice of determining appropriateness the episodes were broken into further categories. The episodes in each category were analyzed in order to create a question(s) that captured the underlying focus of these conversations. The practice of connecting fractions to multiplication and division concepts had fewer occurances and did not warrant the creation of additional categories. However a set of underlying questions was developed. These question then represented a set of questions that students were trying to sort out as they learned to use fractions in the new contexts.

Analysis Related to Learning About Fractions Data

Data analysis of the learning about fractions data occurred after I had analyzed the learning to use fractions data. While the data I drew upon was slightly different from the learning to use data, the end result was a set of practices and underlying questions. This analysis examined field notes from whole class and small group conversations, six videos of whole class discussion, and the two fraction unit texts that were used. The fraction texts were included because they influenced and supported the practices students engaged in. While I will further discuss the link between the fraction texts and the practices in Chapter 4, these teachers were teaching the intended curriculum making the tasks and the teacher's guide influential in the practices that were observed in the sixth grade classroom. A final influential factor in determining what practices students engaged in when learning about fractions was my role in helping to create the fraction units and the intended curriculum. I draw upon my understanding and input that lead to the mathematics developed in the unit, as well as the written materials a teacher would read in the teacher's edition, and the way students engaged with the mathematics. Finally, the way the teacher structures her classroom for learning and the choices teacher make regarding their students' interaction with the mathematical tasks in the unit greatly influences the practices that are observed in this sixth-grade classroom.

The analysis of learning about fractions data began by revisiting the goals for the unit and the task in the unit. Here I was making explicit to myself the reasons for writing task in the unit and what they were intended to do. I drew upon conversations I had when working on the development of the units, conversations with the authors regarding their rationale for including and designing particular problems, the literature the authors indicated influenced the development of the fraction units[2]. Finally I began to read the transcripts created from the six videos and my field notes. From these I began to identify practices and subpractices associated with learning about fractions.

I choose classroom conversations that highlighted the way these practices developed across the unit. In contrast to the set of more or less isolated or individual tasks where fractions were used, the fraction practices developed as the work of many problems interacted. From these conversations I created a list of questions that were asked with the purpose of illustrating the ideas that were driving conversations. I also looked at the questions in each task that students' work on and the mathematical goals for each task create a set of questions that representing the underlying mathematical focus of the task. These allowed me to account for the potential role of the tasks in the practices. Drawing upon these two sets of questions, I then created a third set of questions that captured the broad mathematical ideas underlying each fraction learning practice. This analysis will be made more explicit when I present the fraction learning practices in Chapter 4.

Chapter Summary

Within any set of data there are countless stories to be told and questions that could be answered. For myself I found all these possibilities fascinating as well as distracting. The data summaries on notecards and the grids created to reduce data and visually record themes in one large picture were important in helping me focus on answering my research questions. In the next two chapters I present my findings. In Chapter Four, I draw from the learning about fractions data and the curriculum units used

[2] A major influential work was Leen Streefland's *Fractions in Realistic Mathematics Education: A Paradigm of Developmental Research* (1991).

when students were learning about fractions. The first section in Chapter Four provides a sense of the classroom learning environment and the types of interactions that were commonplace in each teacher's classroom. In the latter sections I describe two fraction learning practices students engaged as they learned about fractions. Chapter Five presents the fraction literacy practices students engaged in when they were learning to use fractions. Chapter Six and Seven will include a discussion of these findings and how they help me answer my research questions as well as implications and limitations of this work.

Chapter Four

LEARNING ABOUT FRACTIONS: CONVERSATIONS, QUESTIONS, AND PRACTICES

Barton (1994) argues that literacy practices are situated within broader social practices and at the same time practices are created out of the past. What happens when students learn mathematics is not only based in what happens in the immediate environment, it is constructed historically within the conventions of classrooms, shaped in part by the teachers, in part by the students, and in part by the curriculum.[1] Cobb, Yackel, and Wood (1993) similarly note that mathematical learning is influenced by both the mathematical practices and the social norms students and their teachers engage in. One purpose of this chapter is to describe the two teachers in the study and their classroom learning environments. The social norms instituted greatly influence the data available to collect. The same curricular material implemented in another room may lead to a different set of social norms, mathematical practices, and fraction practices.

The second purpose of this chapter is to describe the fraction units and the practices students engaged in when learning about fractions. These practices will then be characterized by a set of underlying questions that capture the underlying basis of the conversations the students and their teacher engaged as they participated in each practice. The design of the curriculum is directly related to the practices that emerged in the sixth-grade classroom when students learned about fractions. In addition, the scope and sequence of lessons in the two fraction units and subsequent units play a role in determining the conversations students can legitimately have when learning about fractions and later when learning to use fractions in other units in this curriculum. I do not attempt to describe every fraction learning practice students and their teacher engaged in. Instead I offer ones that will help to show how the fraction literacy practices that students engaged in when learning to use fractions are similar to or different from the fraction learning practices students engaged in when learning about fractions.

The Teachers and their Classrooms

Previously, I characterized the two teachers as effective mathematics teachers (NCTM 1991, 2000) who had a strong understanding of the mathematics they taught and their students as learners of that mathematics. The Professional Standards for Teaching Mathematics (NCTM, 1991) suggest five characteristics that define the classroom learning environment and empower students as learners and doers of mathematics. These characteristics, presented in Figure 8, are representative of the learning environment that these two teachers work to develop in their classrooms.

- Classrooms should be mathematical communities
- Students should use mathematical evidence as verification
- Students should engage in mathematical reasoning
- Students should engage in conjecturing, inventing and problem solving
- Mathematics should be presented as a body of connected concepts, procedures, and ideas

Figure 8. Mathematics classroom characteristics that empower students.

[1] It is also important to note that the present is shaped by the past experiences in other classrooms and interaction outside of school. These are beyond the scope of this study.

Both teachers are strong proponents of the CMP curriculum and have been involved in providing professional development for others who use CMP. While I could say a great deal about each teacher and their perceived role as such, I am going to focus instead on two ideas that illuminate the experience of being a student in their classrooms.

Developing Responsible Learners

Both teachers send messages to their students regarding their role as learners of mathematics and of how they should interact in their classrooms. In sixth-grade Mrs. Kay is explicit in her role of preparing students to engage in an inquiry-oriented mathematics curriculum. She actively engages in an ongoing attempt to establish certain norms for interacting with others. These include:

- listening and responding to others
- offering your ideas
- talking to the class and not to the teacher when presenting an idea
- asking the speaker, not the teacher, a question about their work
- stating that you do not understand what is happening or being offered and possibly asking the speaker to say it again
- asking someone to help you when you are stuck
- coming to the board to explain without asking permission

In a conversation with Mrs. Kay she offered, "It is about them, the students, taking responsibility for their learning … If I take over, then they depend on me and do not listen and learn from each other. The goal is to have a classroom conversation, not a conversation between each student and me." In both classrooms it is important that students engage with one another's ideas as they work to make sense of the mathematics they are exploring.

Letting Students Go Down the Wrong Path and Regrouping.

Both teachers' planning is driven as much by their students as it is by their curricular goals. When students struggle both teachers are willing to take extra time, which may be in the form of days, to explore an idea. In addition, they do not try to immediately fix all the problems that arise when students are struggling. I have seen both teachers let students go down the wrong path so they can engage in deciding that they are stuck. Only when it is clear that the students have tried to make sense of an idea and are truly stuck will they step in. This most often leads to posing another question or task that will help redirect students.

For example, in seventh grade, while working on a warm-up problem, students began to debate whether the product of $\frac{1}{2} \times 48$ was 24 or 96. Unresolved, this led to another conversation about what $6 \div 2$ and $6 \div \frac{1}{2}$ meant. After almost 30 minutes without a resolution, Mrs. Dew gave students four division problems ($8 \div 2$, $8 \div \frac{1}{2}$, $10 \div 2$, $10 \div \frac{1}{2}$) to do for homework and asked them to include a diagram to support their solution. The next day students shared their diagrams and reasoning. After about 20 minutes of conversation, Mrs. Dew used the diagrams to talk about the two interpretations of division (sharing and grouping) before switching to work on the next task in the unit. Toward the end of class she used the similarity task students were working on as a case where students needed to understand the difference between multiplying and dividing by a fraction.

Wells (2000) writes: "[T]he goal for students is making, not learning… learning is an outcome that occurs because the making requires the student to extend his or her understanding in action" (p. 64). In these classrooms, both teachers make decisions about when to let students struggle, when to redirect, and when and what to tell. Both teachers are willing to step out of the conversation when it is sensible so students have the opportunity to make sense for themselves. In these teachers' classrooms, the students learn mathematics by engaging in making sense of it. This involves working on problems, presenting solutions and arguments for why solutions make sense, and evaluating others' solutions.

The Role of Language and Conceptually-Oriented Explanations

The role of language is critical not only to the learning process, but also to the data that is then available to collect. Because the students in both classrooms were encouraged to engage in dialogue while they were working on problems their thinking processes were made public. Both the classroom settings and the curriculum are inquiry-based. Because wondering and puzzling are allowed and encouraged, the data provides insight into the ideas and questions students wonder about and engage with as they try to make sense of the tasks. In another classroom these struggles may not have become objects of conversations because teachers are not always comfortable with letting students go down the wrong path or develop a classroom community where students are comfortable enough to speak out.

In the dialogue provided in this and the next chapter you will see that the curriculum as well as the teachers pose questions that require students to explain their reasoning or enforce the norm of explaining why something makes sense. You will also see the students say they are confused, raise important mathematical issues for the class to consider, and argue for what they think is mathematically reasonable. The conversational focus is often conceptual rather than calculational (Cobb, Stephan, McClain & Gravemeijer, 2001).

While a curriculum can provide the possibility for such conversations to arise, the conversations that do take place are unique to a classroom learning environment. I find these two classroom environments, as well as the curricular setting, well suited for this study because conversations illuminate what the students are wrestling with as they try to make sense of the mathematical ideas the curriculum aims to develop. The thoughts and voices of the students come to the forefront as they use language to communicate their ideas and make sense of mathematics.

Learning About Fractions: Fraction Learning Practices

This section will provide an overview of each of the two fraction units and describe associated fraction learning practices. For each unit, *Bits and Pieces I* (Lappan, et al., 2002/3a) and *Bits and Pieces II* (Lappan, et al., 2002/3b), there is a general fraction learning practice that shapes the development of the mathematics in that unit and therefore the teaching of the problems in the unit. The intended curriculum, one that is problem-based, is provided for a teacher in the teachers' edition where detailed descriptions for each lesson is provided. The fraction learning practices that emerged were shaped by the design or intent of the curriculum as well as by classroom activity. The teachers each made choices regarding how to support their students within the intent and spirit of the curriculum. The conversations that took place were both intended and emergent at the same time (Wells, 2000).[2]

[2] Wells (2000) argues that Vygotsky's concept of artifact-mediated joint activity, which involves change and transformation over time, is supported by several assumptions about learning and teaching. Outcomes that are aimed for and emerge go beyond covering a curriculum. While one may agree on the basic goals to be aimed for, the route depends upon the way human and material resources interact during problem solving.

In addition, the curriculum is structured to engage students in activities that move them to understand fraction concepts and skills as defined by the mathematics education community at large. In Chapter One I characterized learning about fractions by referring to documents such as *Adding it Up* (Kilpatrick, Swafford, & Findell, 2001) and *NCTM Principals and Standards* (2000). In Appendix A I provided a list of the *NCTM Principles and Standards* expectations for the development of number and operation that are related to fractions. The mathematical goals for learning about fractions in the CMP units are aligned with the expectations for NCTM's sixth- through eighth-grade grade-band expectations. Therefore, the fraction learning practices students engage in when learning about fractions also emerge from established views regarding what it means to understand fractions.[3]

Goals and Fraction Learning Practices for the First Fraction Unit

Goals for *Bits and Pieces I*. The first unit, *Bits and Pieces I* (Lappan, et al., 2002/2003a), asks students to make sense of fractions, decimals, and percents. The unit develops meaning for each of these three forms of representation and the relationships that exist between them. The unit emphasizes understanding various fraction models and interpretations of fractions. Interpretations include part-whole, operator[4], quotient, and measure. The problem contexts provide work with fraction-strip (bar), number line, grid-area, and partition models. The unit goals are as follows:

[3] Moschkovich (2004) makes a similar argument that while mathematical practices are emergent in classroom activity that is specific to particular mathematical ideas (also see Cobb, Stephan, McClain & Gravemeijer, 2001), they can simultaneously emerge as normative or established ways of working with a mathematical idea.

[4] The operator construct is used informally in situations that involve 2/3 of 300. It will be formally linked with fraction multiplication in *Bits and Pieces II* (Lappan, et al., 2002/2003b).

- Build an understanding of fractions, decimals, and percents and the relationships between and among these concepts and their representations;
- Develop ways to model situations involving fractions, decimals, and percents;
- Understand and use equivalent fractions to reason about situations;
- Compare and order fractions;
- Move flexibly between fraction, decimal, and percent representations;
- Use 0, $\frac{1}{2}$, 1, 1 $\frac{1}{2}$, and 2 as benchmarks to help estimate the size of a number or sum;
- Develop and use benchmarks that relate different forms of representations of rational numbers (for example, 50% is the same as $\frac{1}{2}$ and 0.5);
- Use physical models and drawings to help reason about a situation;
- Look for patterns and describe how to continue the pattern;
- Use context to help reason about a situation; and
- Use estimation to understand a situation.

The Practice of Understanding Fractions as Quantities. Figure 9 illustrates the fraction learning practice that students engage in during this unit. The practice of *understanding fractions as quantities* involves moving students among numerous modes of representation. Using a similar model, Post, Behr, and Lesh (1986) argue that the opportunity to move between and within different modes of representation makes ideas meaningful. Streefland (1991) uses a similar model that stresses the role of the problem context in concept formation. Both models are general models and could emerge in other mathematical content areas, but Streefland as well as Lesh, Post, and Behr use this notion of modes of representation to guide the development of problems in their work with fractions.[5]

[5] The use of multiple modes of representation is a foundational idea used to develop the CMP curriculum. Most of the problems in the curriculum are designed to move students between contexts, models, and symbolism. The inclusion of benchmarks and estimates is particular to the fraction units and perhaps other number units. The movement between the four modes does not occur with every problem in the fraction units but the places where it does happen indicates a regular reoccurring pattern of behavior.

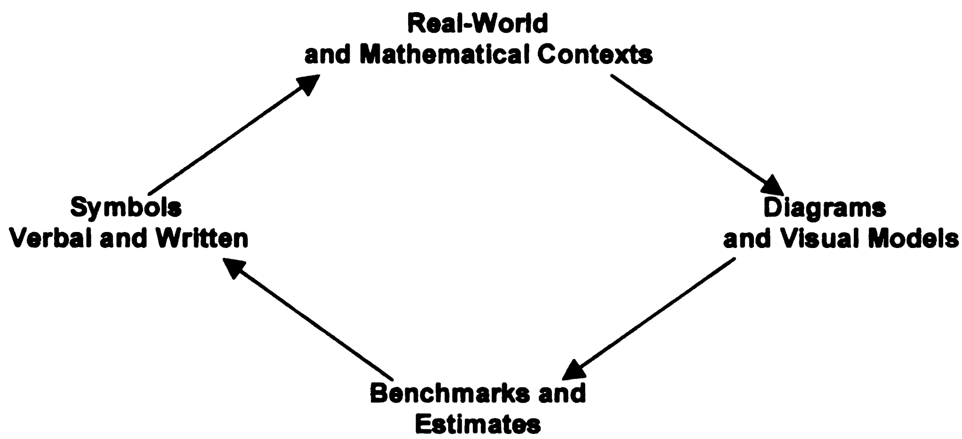


Figure 9. Practice of Understanding Fractions as Quantities.

The practice modeled in Figure 9 represents a recurrent pattern of activity throughout the first fraction unit. As students engaged with the problems in the unit, ones that were contextually as well as mathematically driven, they drew upon diagrams, models, symbols, benchmarks and estimates to make sense of problems[6]. As students participated with others in small and large group discussions to explain and justify their reasoning about a task, they moved toward understanding fractions as quantities that can be ordered and compared and that there can be multiple representations for the same quantity.[7]

Figure 10 illustrates three sub-practices that emerged as students participated in the practice of understanding fractions as quantities. The sub-practices involve understanding that (1) fractions are quantities, (2) fractions are quantities that can be ordered and compared, and (3) fractions are quantities with multiple representations. When students engaged in the practice of understanding fractions as quantities some or

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[6]Nine of 16 tasks in the unit and numerous homework problems resulted in movement between the four modes of representation presented in the Figure 9. All the tasks in the unit engage students in a practice where they move between contexts, models and symbolism. I have chosen the tasks where benchmarks are developed and used as they are significant in the development of understanding fractions as quantities. Understanding fractions as quantities is crucial if students are going to work meaningfully with fractions.

all of these sub-practices emerged. Students' ability to fully engage in the sub-practices developed across the problems in the unit. These sub-practices represent established practices that are important to understanding fractions as a mathematical idea.

There were sixteen instructional tasks in the *Bits and Pieces I* (Lappan, et al., 2002/2003a) unit with the practice of understanding fractions as quantities emerging in nine of them. The second task (see Figure 11) in the unit asks students to determine progress toward a goal at various stages in a fundraiser. The problem provides a context for finding different strategies for dividing up a length and then relating the parts to the whole in order to estimate progress toward a fundraising goal. Note how the fundraising thermometers are partitioned and shaded. For example, the first thermometer shows progress toward $\frac{1}{4}$ of the goal. However, the thermometer is not partitioned to show each of the four fourths. Identifying the fraction represented involved determining how to equally partition the thermometer and determine the part-whole relationship. The students will have to show the partitions to prove that the shaded portion is $\frac{1}{4}$ of the goal.

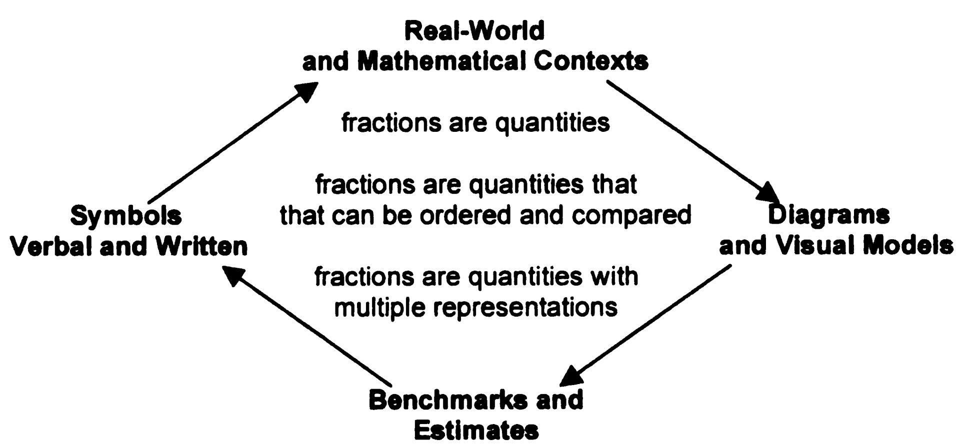
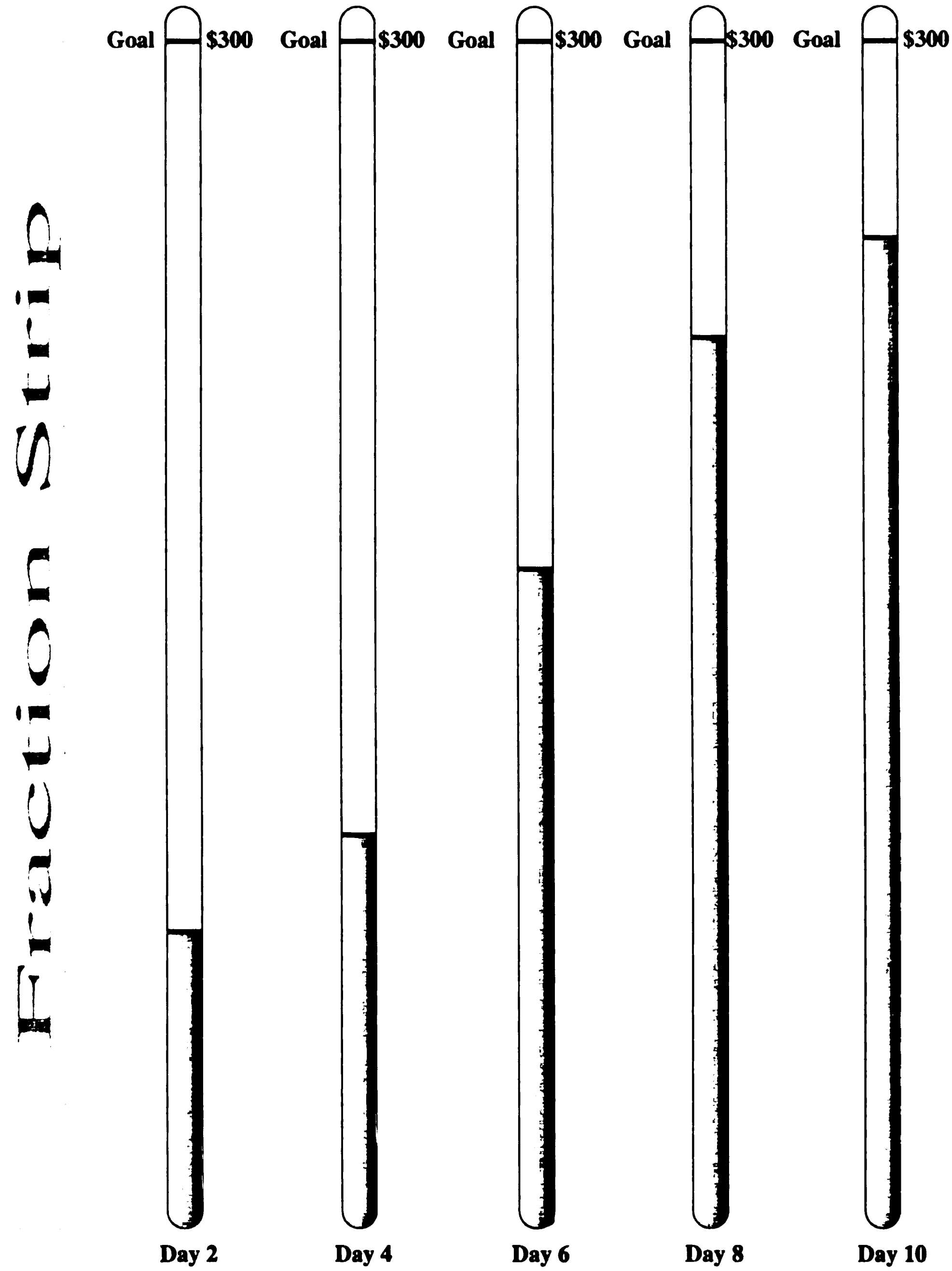


Figure 10. Subpractices of the Practice of Understanding Fractions as Quantities.



Problem 1.2 Exploring Strategies for Naming Parts of a Strip

The thermometers show the progress of the sixth-grade poster sale after two, four, six, eight, and ten days. The principal needs to understand what fraction of the goal the sixth-grade class had achieved after each of days 2, 4, 6, 8 and 10.

A. About what fraction of their goal did the sixth-graders achieve after each day?
B. What do the numerator and denominator of each fraction tell you about each thermometer?
C. What strategies did you use to estimate the fraction of the goal reached for each day?
D. About how much money had the sixth-graders raised at the end of each of these days?
E. After Day 9 the sixth graders had raised $240. What fraction of their goal have they reached? Show how the thermometer should be shaded at the end of Day 9?

Figure 11. Fundraiser Task.

The fundraiser context and the thermometer model, the symbols that will be used to represent the fraction of the goal that is met, as well as students use of benchmarks such as $\frac{1}{2}$ to talk about and justify the fraction of the goal that has been met all work together to help students understand that fractions are quantities with multiple names. The conversations about the task, taken from field notes written during data collection, illustrate the interaction among these four modes of representation.

This task occurs early in the unit. As you read the dialogue for Part A, note that in parts of the whole class conversation students struggled to understand the sub-practice fractions are quantities and to understand equivalence or the sub-practice fractions are quantities with multiple representations. By struggle, I mean that students were not able to explain why a solution made sense with regard to these two ideas. This is to be expected since the sub-practices emerged across the *Bits and Pieces I* (Lappan, et al., 2002/2003a) unit.

Mrs. Kay	Talk to us about Day Two. What fraction did you get and how did you get it?
Erica	[showed how she measured out $\frac{1}{4}$ sections with her fingers.] I knew this was $\frac{1}{4}$, so $\frac{2}{4}$, $\frac{3}{4}$, $\frac{4}{4}$.
Bryan	I used a ruler. I remembered from the other problem it was $\frac{1}{4}$ and \$75. This was the same thermometer.
Mrs. Kay	Why?
Bryan	If you do 75 four times it is 300.
Mrs. Kay	Do you agree?
Student A	75 + 75 = 150 and that is half of 300.
Mrs. Kay	Day Four?
Student B	I used Erica's strategy with her fingers. [shows how shaded length fills thermometer three times.] About $\frac{1}{3}$, $\frac{1}{3}$, $\frac{1}{3}$.
Student C	I got $\frac{3}{8}$. First I got $\frac{2}{8}$ for Day 2 and …

Mrs. Kay	How do you know $\frac{1}{4}$ is the same as $\frac{2}{8}$?
Student C	Two goes into 8 four times.
Student D	I disagree. If you mark $\frac{1}{4}$ I don't think the little left over is $\frac{1}{2}$ of $\frac{1}{4}$ and $\frac{3}{8}$ is $\frac{1}{4}$ + $\frac{1}{8}$ more.
Mrs. Kay	Which is more accurate?
Class	$\frac{1}{3}$

In this dialogue students used the thermometer model and repeated iteration to demonstrate the part-whole relationship of the shaded part to the whole thermometer. The students realized $\frac{1}{3}$ was a greater quantity than $\frac{1}{4}$. Student D displayed sophisticated reasoning for why $\frac{3}{8}$ cannot represent the fraction of the goal met for Day 4. He used partitioning to argue that $\frac{3}{8}$ was larger than $\frac{1}{3}$ indicating that he conceived of the *fractions as quantities that can be ordered and compared.* However, as the conversation moved to Day 6 the class as a whole struggled with the notion that *fractions are quantities* and that *fractions are quantities that can be ordered and compared.* In this conversation note how the benchmark $\frac{1}{2}$ was used to make sense of the goal on the thermometer but students could not offer a reason why the fractions $\frac{5}{8}$ and $\frac{5}{9}$ were greater than one half.

Student F	I got $\frac{1}{2}$ for Day 6. [several others agree]
Mrs. Kay	So if I fold over the line I will get 2 halves?
Student G	It [the wording in the task] says "about".
Mrs. Kay	I disagree. I think you can get more accurate.
Student G	I got $\frac{5}{8}$.
Mrs. Kay	Is that more than $\frac{1}{2}$?
Student G	Yes.
Student H	I was thinking $\frac{5}{9}$.

Mrs. Kay We will come back to this since it is giving us trouble.

[Conversation switched to Day 8.]

Student I I used $\frac{1}{4}$ from day 2 and got $\frac{3}{4}$.

Student J I used the top [unshaded part of thermometer] and marked down.

Troy I got 8ths, $\frac{6}{8}$.

Mrs. Kay How did you get 8ths? We have 4ths?

Student K Cut each [fourth] in half.

Mrs. Kay What do $\frac{3}{4}$ and $\frac{6}{8}$ have to do with each other?

Student K They are the same.

Mrs. Kay But fourths and eighths are not the same.

Student L Eighths are smaller.

During the Day 4 discussion students indicate that the goal is more than half but they do not use the benchmark $\frac{1}{2}$ to show how they know that the fractions $\frac{5}{8}$ or $\frac{5}{9}$ are greater than half. The teacher ended the Day 4 conversations saying that they would come back to this idea since it was giving them trouble. For Day 6 we see the teacher pushing students to explain why eighths and fourths can both be used to represent the fraction of the goal however the students are not offering a convincing explanation. For example, they do not use the context or the thermometer model to show what they mean when they say the fourths are cut in half. During Day 2 a student offered that $\frac{1}{4}$ was the same as $\frac{2}{8}$ using a computation rule. Like Day 8, the teacher let it ride but later said to me that she pushes students to think about equivalence in order to see if students will use the models to show why two fractions are equivalent but she does not push too hard because equivalence is the focus of a later set of instructional tasks.

With the Day 10 thermometer students had to determine that the thermometer could be partitioned into six-equal parts with five shaded. The problem context and the thermometer model leads students to argue for what fraction (symbol) is represented and in doing so provide the basis for a conversation about the role of the numerator and denominator. For example, with the Day 10 thermometer the student will need to defend why they used 5 and 6 and what each represents in the fraction $\frac{5}{6}$. This also provides an opportunity to talk about $\frac{5}{6}$ as being about one whole or between the benchmark values of $\frac{3}{4}$ (Day 8) and 1 therefore establishing that $\frac{3}{4}$ and $\frac{5}{6}$ are quantities that can be ordered and compared. However, students struggle with this indicating that the sub-practice fractions are quantities is an emerging one.

Student M $\frac{2}{3}$. It is almost the same as $\frac{3}{4}$.

Mrs. Kay Is $\frac{2}{3}$ more than $\frac{3}{4}$?

[Long Pause.]

Student N $\frac{7}{8}$

Mrs. Kay How?

The students could not answer this last question. They do not have a way to show which fraction is larger indicating that students do not yet have ways to make sense of these *fractions as quantities*. Since class was almost over the teacher let the conversation drop.

They returned to the problem the next day. As they discussed the other questions in the task they did make progress toward the subpractice of understanding that fractions are quantities. For example, part D of the task asks students to determine how much money was raised on each day. When discussing Day 10 a student offered $\frac{5}{6}$ and showed how the thermometer could be partitioned into six equal-size sections and the goal filled five of those sections. When talking about the fundraising money that was raised a student used one-half as a benchmark saying that half of \$300 is \$150. He then showed that the remaining $\frac{2}{6}$ was equivalent or the same size as the $\frac{1}{3}$ goal on the Day 4 thermometer, or \$100, making the total money raised \$250.

Figure 12 provides an example of a task where students determine what interval a fraction falls between. The task, seventh in the unit, is designed so students have to use $\frac{1}{2}$ as a benchmark for estimating a fractional quantity and ordering it. The data I am going to share next shows that as the students progress through the task other benchmarks (fourths, eighths and unit fractions) emerge as students have to make more accurate placements of the fractions in intervals and on a number line. Further, the excerpt of whole-class dialogue shows that students are more comfortable engaging in the practice of understanding fractions as quantities.[8] As a whole class, students' engagement with the sub-practices is much more developed and efficient than in earlier problems.

A. Decide whether each fraction below is in the interval between 0 and $\frac{1}{2}$, the interval between $\frac{1}{2}$ and 1, or between 1 and $1\frac{1}{2}$. Record your information in a table to show which fractions are in each interval.

$\frac{1}{5}$ $\frac{4}{5}$ $\frac{1}{3}$ $\frac{2}{3}$ $\frac{1}{10}$ $\frac{6}{10}$ $\frac{7}{10}$ $\frac{8}{10}$ $\frac{3}{8}$

$\frac{3}{9}$ $\frac{7}{8}$ $\frac{9}{8}$ $\frac{7}{9}$ $\frac{3}{4}$ $\frac{3}{12}$ $\frac{5}{6}$ $\frac{3}{7}$ $\frac{4}{7}$

B. Decide for each fraction in part A whether it is closest to 0, $\frac{1}{2}$, or 1.

C. Use benchmarks and other strategies to help you write the fractions from part A in order from smallest to largest.

Figure 12. Comparing Fractions to Benchmarks Task.

[8] What it means for two fractions to be equivalent was introduced, established, and defined in a previous task. By partitioning and labeling number lines and other linear models students first defined equivalent fractions as fractions that use different size pieces to show the same amount. They then developed strategies for finding or identifying equivalent fractions.

After the students worked on part A of the task, Mrs. Kay drew a number line and a chart with the appropriate intervals. As students were called upon to tell what interval a fraction belonged in, some used the number line to explain their reasoning. For the first few fractions ($\frac{3}{12}$, $\frac{3}{9}$, and $\frac{4}{5}$) students partitioned the number line using the denominator to determine how many equal parts to make. For the fraction $\frac{4}{5}$, the student partitioned the number line into five parts, identified $\frac{4}{5}$, and showed that it fell between $\frac{1}{2}$ and 1. Table 4 contains excerpt from my field notes of the strategies students offered for the next several fractions.

Table 4
Strategies for Placing Fractions in Intervals.

Fraction	Strategy
$\frac{3}{12}$	$\frac{3}{12}$ is the same as $\frac{1}{4}$ and $\frac{1}{4}$ would be between 0 and $\frac{1}{2}$. Pointed to places where fourths were marked on number line.
$\frac{3}{9}$	Student 1: $\frac{3}{9}$ is $\frac{1}{3}$ and $\frac{1}{3}$ is between 0 and $\frac{1}{2}$. Student 2: [Using the one-fourth and one-third partitioning marks on the number line] $\frac{1}{3}$ is a little bigger than $\frac{1}{4}$. If you go $\frac{1}{4}$, $\frac{2}{4}$, and $\frac{2}{4}$ is the same as $\frac{1}{2}$ on number line, then $\frac{1}{3}$ is a little bigger than $\frac{1}{4}$.
$\frac{4}{5}$	Marked number line into fifths, saying $\frac{5}{5}$ is 1 whole and $\frac{4}{5}$ would be right here (pointed to location of $\frac{4}{5}$ on number line) which is between $\frac{1}{2}$ and 1.
$\frac{4}{7}$	Student said more than $\frac{1}{2}$ but had trouble showing why. Mrs. Kay drew a fraction strip and asked what half of 7 would be. This brought out that half of 7 is $3\frac{1}{2}$ so 4 out of 7 is more than $\frac{1}{2}$ of 7.

After the fractions were placed on the number line, the strategies for ordering and comparing fractions were summarized. The following closing conversation shows students using the part-whole relationship or numerator and denominator and benchmarks to help them determine how large or small a quantity is represented by a fraction.

Mrs. Kay What strategies did you hear people say to decide where to place the fraction?

Katie	They cut the denominator in half. With $\frac{8}{10}$ they cut 10 in half to see what half was.
Mrs. Kay	So compare to $\frac{1}{2}$. Another?
Student A	Find equivalent fractions.
Student B	I cut up (partition) a line to find where mine is.
Student C	For $\frac{9}{8}$ I compared to 1 whole.
Mrs. Kay	Do you notice anything about the fractions that are less than half?
Student A	They are all 1 and 3.
Mrs. Kay	Do they have to be 1 and 3?
Student D	It could be 2 like in $\frac{2}{8}$.
Student E	It could be $\frac{14}{30}$.
Student F	All the numerators are less than half of the denominator.
Mrs. Kay	What about the $\frac{1}{2}$ to 1 interval?
Student G	The numerator is over half of the denominator.
Mrs. Kay	What about fractions greater than 1?
Student H	The numerator is greater than the denominator.
Mrs. Kay	Why is that true? Why does that make sense?
Student I	Because with 7ths seven sevenths is a whole and 8 is more than one whole.

Across the task students used various approaches including benchmarks, partitioning, and finding equivalent fractions. They had to engage with fractions as quantities, ones with multiple representations, and as numbers that can be ordered and compared. Earlier in the unit students usually drew on $\frac{1}{2}$ as a benchmark. Here fractions such as $\frac{1}{4}$ and $\frac{1}{3}$ are also used as benchmarks. The strategies offered show that students are engaging in a mathematical practice of understanding fractions as quantities where they realize that *fractions are quantities*, that *fractions can be ordered and compared*, and that *fractions are quantities with multiple representations*.

Later in the unit students explored how decimals are related to fractions. Figure 13 shows parts B and C of a task where students found decimal equivalents for fractions. This is the tenth task in the unit. As students engaged in the practice of understanding fractions as quantities the three sub-practices emerged. With decimals included in the work, students to expanded and further developed the practice of understanding fractions as quantities with multiple representations.

B. Using what you know about fractions and decimal, find decimal equivalents or approximations for the following fractions:

$\frac{1}{2}$ $\frac{1}{3}$ $\frac{1}{4}$ $\frac{1}{5}$ $\frac{1}{6}$ $\frac{1}{8}$ $\frac{1}{10}$ $\frac{2}{3}$ $\frac{3}{4}$

C. Use what you found in part B to find decimal equivalents for the following fractions:

$\frac{2}{5}$ $\frac{3}{5}$ $\frac{4}{5}$ $\frac{6}{5}$ $\frac{1}{20}$ $\frac{2}{8}$ $\frac{3}{8}$ $\frac{1}{16}$

$\frac{1}{12}$ $\frac{2}{6}$ $\frac{3}{6}$ $\frac{3}{12}$ $\frac{1}{9}$ $\frac{1}{18}$ $\frac{10}{8}$

Figure 13. Fraction and Decimal Equivalents Task.

This task used the mathematical context of equivalence. A number line was drawn on the board for students to use when explaining. The problem intent was for students to expand their repertoire of benchmarks to include the fractions in part B. Students used the number line and previously established work regarding fraction and decimal equivalents to name the decimal equivalents for the fractions in part B. As decimal equivalents were identified they were placed on the number line with their fraction equivalents. For example, in previous work students had established that $\frac{1}{8}$ was half of $\frac{1}{4}$ or 0.25 so $\frac{1}{8}$ was equivalent to half of 0.25 or 0.125. In part C a student used $\frac{1}{8} = 0.125$ as a benchmark for naming the decimal equivalent for $\frac{2}{8}$, $\frac{3}{8}$, and $\frac{10}{8}$. "Since $\frac{1}{8}$ is 0.125, two $\frac{1}{8}$s or $\frac{2}{8}$ is twice 0.125 or 0.25. Three $\frac{1}{8}$s or three 0.125s is 0.375".

Once again, as students engaged in a practice where they moved between the problem context, models, benchmarks and estimates, and symbolism they were engaged in sub-practices where they treated fractions as quantities that can be ordered and compared and have multiple names. Students' engagement with this practice is becoming more proficient. In the dialogue fewer questions and problem arose as solutions were presented.

Underlying Questions Associated with the Practice of Understanding Fractions as Quantities. Looking across the tasks in *Bits and Pieces I* (Lappan et al., 2002/2003a) where the practice of understanding fractions as quantities emerged, the classroom dialogue, the task itself, the enactment of the task, and the goals of the task suggest a set of questions that capture the underlying basis of the conversations students engaged in when learning about fractions. Using the Fundraiser Task as an example, Table 5 includes two sets of questions asked during the whole class conversation of the task and a set of questions. The first are questions from the student and teacher dialogue that capture the mathematical focus of the task as well as the underlying current of the actual discussion. These emerge from the enacted curriculum. The second set of questions, ones that represent the underlying mathematical focus of the task, are linked with the intended curriculum.

Table 5
<u>Questions Supporting Fundraiser Task</u>

Questions from Dialogue	What fraction did you get and how did you get it? Why? How do you know $\frac{1}{4}$ is the same as $\frac{2}{8}$? Which is the same as $\frac{1}{2}$? Is that more than half? How did you get eighths [when] we have fourths? What do $\frac{3}{4}$ and $\frac{6}{8}$ have to do with each other? Is $\frac{2}{3}$ more than $\frac{3}{4}$? How?
Questions representing the underlying mathematical focus of this task.	What fraction represents this fundraising situation? How do you know? How would you use the fraction of the goal to determine the amount of money raised? What role does the numerator and denominator in finding the fraction of the goal met?

Looking across these two sets of questions a third set of questions can be developed that captures the broad[9] mathematical ideas underlying basis of the conversations that took place with each of the nine tasks where the practice of understanding fractions as quantities appeared.[10] The underlying basis of the conversation that took place around the Fundraiser Task is captured in the following questions:

> *How can we represent concepts such as parts of a whole?*
> *Why can there be different names representing the same fractional quantity?*

Looking across all of the instructional tasks where the practice of understanding fractions as quantities emerged led to the complete set of questions in Figure 14. These questions characterize the conversations students and their teacher engaged in when participating in the practice of understanding fractions as quantities.

[9] The two sets of questions in the table are very specific to the fundraising task itself. The third set of questions are broad in that they capture mathematical ideas inherent in the task without making specific reference to the problem context.
[10] In Chapter 6 these questions will be compared to the set of questions generated for the faction literacy practices students engage in when learning to use fractions in the units following the two fraction units.

1. How can we represent concepts such as parts of a whole?
2. Why can there be different names representing the same fractional quantity?
3. How can we determine when two names refer to the same quantity?
4. How can we determine which of two fractions is greater or smaller?
5. How can equivalence and benchmarks help determine what a fraction, decimal or percent represents?
6. How is a decimal like a fraction?
7. How is a percent like a fraction?
8. What techniques can be used to find fraction, decimal and percent names for the same quantity?

Figure 14. Questions associated with the Practice of Understanding Fractions as Quantities.

As one might expect, these questions align with the body of knowledge that NCTM (2000) or Adding It Up (2001) indicate fraction instruction should address. The work of the unit and conversations lead to the development of practices that focus on developing understanding of what fraction notation can represent, what equivalent representations are, development of "how" to find equivalent representations, and strategies for ordering fractions. In addition, understanding why these ideas can exist or the conceptual basis for their existence and how they are related is also important.

Goals and Fraction Learning Practices for Second Fraction Unit

Goals for *Bits and Pieces II*. In the second fraction unit, *Bits and Pieces II* (Lappan et al., 2002/2003b) students develop an understanding of and algorithms for the fraction operations of addition, subtraction, multiplication, and division. The goals for this unit are to:

- Use benchmarks, and other strategies to estimate sums, differences, products, and quotients;
- Develop ways to model sums, differences, products, and quotients, including areas, strips and number lines;
- Understand when addition, subtraction, multiplication, or division is the appropriate operation to solve a problem;
- Develop strategies and algorithms for adding, subtracting, multiplying, and dividing fractions;
- Look for and generalize patterns in numbers; and
- Solve problems involving fractions.

The Practice of Learning to Operate With Fractions. In this unit strategies for operating with fractions arise as students make sense of situations where fraction operations are embedded. There are five elements of activity (see Figure 15) that come together or interact as one engages in the fraction learning practice of *learning to operate with fractions*. Each fraction operation arises through problem solving experiences that lead students to estimate, model, and write number sentences for particular situations. These situations ask students to combine and take quantities apart, partition, share, and group quantities. Eventually, students are asked to pull these ideas together and articulate an algorithm for each fraction operation. Through movement among the five representations shown in Figure 15 students engaged in two sub-practices: understanding how to operate with fractions and making sense of when each operation is appropriate.

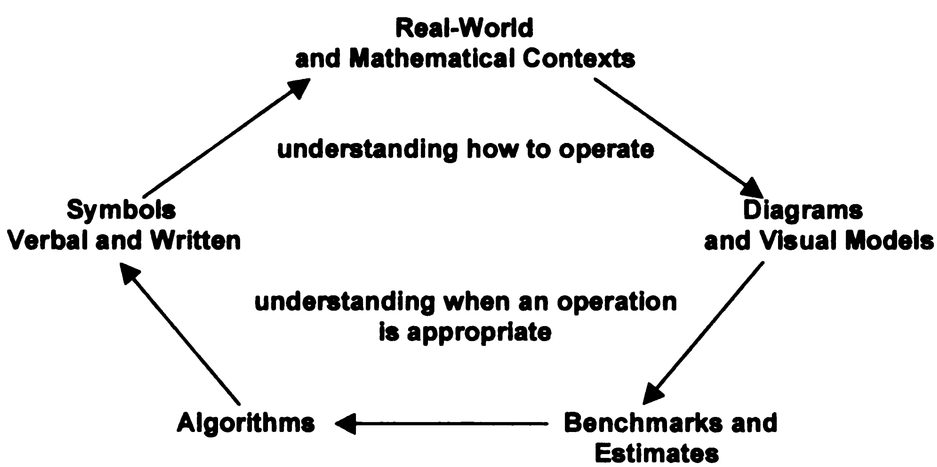


Figure 15. Practice of Learning to Operate with Fractions.

The fraction learning practice of learning to operate with fractions occurred in 6 of the 13 instructional tasks that were covered. There were other practices students engaged in when learning about fraction operations.[11] This one was chosen because benchmarks and estimation stood out as an important element in determining if an algorithm that was offered was indeed a legitimate algorithm. It was used by students to argue when someone's algorithmic approach did not make sense, by the teacher to help students decide on the validity of an algorithm that was proposed, and in the curriculum to help students develop operational number sense.[12]

The unit begins with two instructional tasks that focus on estimation with addition and subtraction. In the first task students play a game where they draw cards with various fractions and decimals on them. Students draw two cards and determine if the sum is closer to 0, 1, or 2. The teacher launched the instructional task by drawing a fraction or decimal card and asking students to explain where it would go on a number line marked with 0, 1, and 2. After spending a few minutes revisiting the location of fractions on the number line which includes understanding them as a quantity as well as a part-whole relationship, Mrs. Kay gave students two cards and asked them to work with their partner to estimate what the sum was. After a few minutes of working with their partners, Mrs.

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[11] A practice that was more predominant than the one displayed in Figure 4.8 involved interaction among the problem context, symbolism, diagrams and models and algorithms. The practice I am presenting is an extension in that the interactional element of benchmarks and estimation is involved. The description of this practice will provide a fair sense of how conversations looked when there was interaction between only four elements.

[12] The data for this study was collected during the teaching of the second of three revisions of Bits and Pieces II. During this revision the research and writing team noted that estimation and benchmarking was important not just for developing operational sense, as it was currently being used, but also as a tool that could be used to determine if an algorithm, even if derived from a model or diagram, was leading to a reasonable answer. Estimation was also important in this respect because there are cases where diagrams and models do not neatly lead to an algorithm (for example, fraction division and multiplication with mixed numbers) but the diagrams and models are important in making sense of situation where a particular operation is appropriate. The role of benchmarks and estimation in this data is clearly evidence but is more coherently developed in future versions of this unit.

Kay brought the class back together to debrief. They began discussing $\frac{3}{4}$ and 1.23 and then moved to $\frac{6}{10}$ and $\frac{6}{7}$. The conversation for these two pairs is as follows.

Student A	I want to see about how much $\frac{3}{4}$ is. It is 75 cents.
Mrs. Kay:	If you add about how much is it?
Student B	About \$1.90
Mrs. Kay	So, where is that on the number line?

[Student B came up to the number line indicating that 1.90 was just before 2.]

Mrs. Kay	How about $\frac{6}{10} + \frac{6}{7}$?
TJ:	Wouldn't they be $\frac{12}{17}$ if added?
Mrs. Kay	I don't know? Does it make sense? How much, about, is $\frac{12}{17}$?
Class	It is about $\frac{3}{4}$.
Mrs. Kay	About how much is $\frac{6}{10}$ and $\frac{6}{7}$?

[Students talk with their partners.]

Student C	$\frac{6}{10}$ is a little more than $\frac{1}{2}$ and $\frac{6}{7}$ is almost one whole.
Student D	$\frac{6}{7}$ is about $\frac{3}{4}$.
Mrs. Kay	Will the sum be more than $\frac{3}{4}$?
Class	Yes

In this conversation students used estimation to find a reasonable sum and disprove TJ's question about whether you can add the denominators and add numerators to find the sum of two fractions. They engaged in the practice of learning to operate with fractions by drawing upon the mathematical context of the problem, the diagram (and model) of a number line, symbolism, benchmarks and estimation, and a potential algorithm. It was interesting to see this issue raised by a student since the notion of adding denominators and adding numerators is purposefully raised in the second instructional task of the unit where a student named Billie is decorating a model of a house she designed for a school project (See Figure 16).

B. Billie needs to make curtains for the windows in the two model rooms. The pattern for one room calls for $\frac{7}{12}$ of a yard of material and for the other $\frac{5}{8}$ of a yard of material.

1. Is this a situation where you need an underestimate or an overestimate? Why?
2. She wrote on her paper the following computation: $\frac{7}{12} + \frac{5}{8} = \frac{23}{24}$

   Use estimation to check whether her computation makes sense. Explain your thinking.
3. Her friend Carl said that he represented the two quantities using the same denominator. He wrote $\frac{14}{24} + \frac{15}{24}$ and said, "Now the answer is easy."
   a. What do you think Carl would give as the sum? Does his thinking make sense?
   b. Is this an exact answer or an estimate?

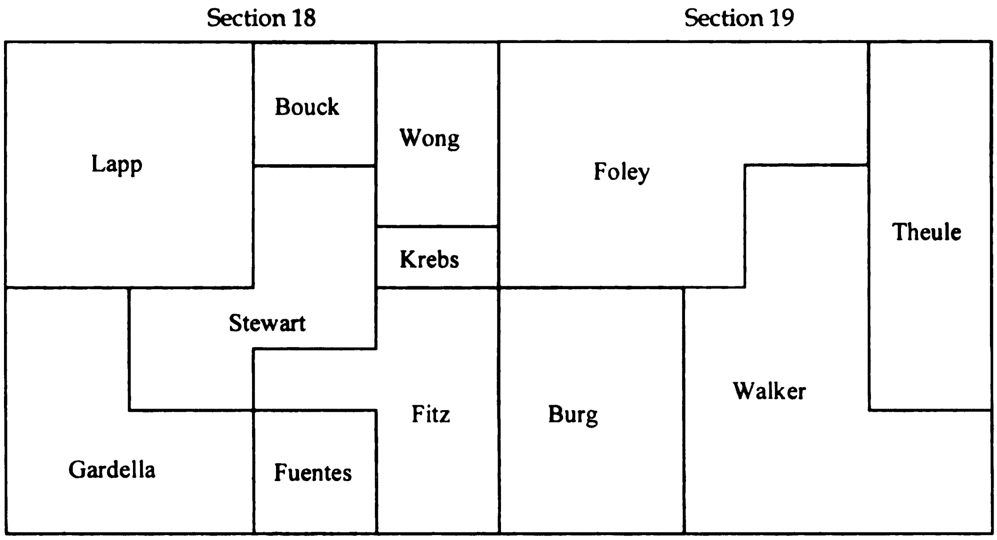
Figure 16. Problem B of Estimating Sums Task.

In Problem B of the Estimating Sums Task students had a second conversation about the reasonableness of adding numerators and denominators. In this conversation, like the one about $\frac{6}{10}$ and $\frac{6}{7}$, students use estimation to determine the reasonableness of such an approach. There is a focus on using number sense to determine what is reasonable. This is an important type of reasoning to establish so when students are working on tasks that lead to developing an algorithm for operating with fractions they use estimation to consider whether their approach leads to a reasonable solution.

The third task in the fraction operation unit uses an area model to investigate fraction addition and subtraction. In the Land Problem (See Figure 17), students begin by determining what fraction of a section of land a person owns. As people buy and sell their land, students have to put together (add) or compare (subtract) sections of land. The number sentences students write lead naturally to using equivalent fractions with common denominators. In support of their number sentences, students can also use the diagram to show how they arrive at their sums or differences. By having students explain their reasoning, they begin to understand why it is necessary to rename fractions.

When Tupelo Township was founded, the land was divided into sections that could be farmed. Each section is a square that is 1 mile long on each edge—that is, each section is 1 square mile of land. There are 640 acres of land in a 1-square-mile section.

The diagram below shows two side-by-side (adjacent) sections of land. Each section is divided among several owners. The diagram shows the part of a section that each person owns.



A. Determine what fraction of a section of land each person owns.

B. If Fuentes buys Theule's land, what fraction of a section would Fuentes own? Write a number sentence to show your solution.

C. Find two different combinations of owners whose combined land is equal to $1\frac{1}{2}$ sections of land. Write number sentences to show your solutions.

D. If Bouck and Lapp combined their land, they would have an amount equal to Foley's land. Find two other combinations of two or more people's land that add up to another person's land. Write number sentences to show each of your answers.

E. Determine how many acres of land each person owns. Explain your reasoning.

F. Lapp and Wong went on a land-buying spree and bought up all of section 18. Lapp bought all of Gardella's, Fuentes's and Fitz's land. Wong bought the rest.

1. When the buying was completed, what fraction of section 18 did Lapp own?
2. What fraction of section 18 did Wong own?
3. Who owned more and by how much?

Figure 17. Land Problem.

The problem structure of the Land Problem leads to opportunities for students to move between representations in four of the five areas of the practice of learning to operate with fractions: real-world and mathematical contexts, diagrams and visual models, symbolism, and an algorithm. In their written work and during class discussion, students naturally draw upon common denominators and their knowledge of equivalence, learned in Bits and Pieces I, to combine quantities. This use of equivalence is supported pictorially as students show how for example, Lapps $\frac{1}{4}$ section of land can be represented as four $\frac{1}{6}$-sized sections of land and showing that Bouck's land is $\frac{1}{16}$ of the entire section and four of Bouck's pieces of land fit into Lapp's.

The use of equivalence is also supported by asking students to write number sentences to represent their solutions. If students combine Bouck and Lapp's land they could write the number sentence $\frac{1}{4} + \frac{1}{16} = \frac{5}{16}$. In order to explain how they arrived at $\frac{5}{16}$, they often show how to repartition Lapp's land into $\frac{4}{16}$ allowing the use of equivalence to emerge naturally. In the descriptions that follow students are offering solutions to Part B through F of the Land Problem. Some students rename fractions to have common denominators in order to show how they are combining different pieces of land. In this respect a strategy is starting to emerge but a formal discussion about using equivalence as an algorithmic approach has not yet occurred.

An interesting development in the class discussion occurred when TJ was at the board talking about the number sentences he wrote to support his work for Part C which asked for combinations of land equal to $1\frac{1}{2}$ sections. (See Figure 18 for the work TJ put on the board before discussion began.) In the column on the left TJ lists all the landowners in Section 18 of the Land Problem. He does not use common denominators because all the landowners combined will equal one whole section. He writes the total as $\frac{16}{16}$. He also lists two landowners from Section 19 whose total equals $\frac{8}{16}$. While he does not say why he used sixteen as the denominator for the Section 18 total. It may be related to how he found and decided what fraction of a section each individual person owned in Part A. You can see that he uses common denominators, $\frac{5}{16} + \frac{3}{16}$, to combine Foley and Theule's land however, he may have been working visually.

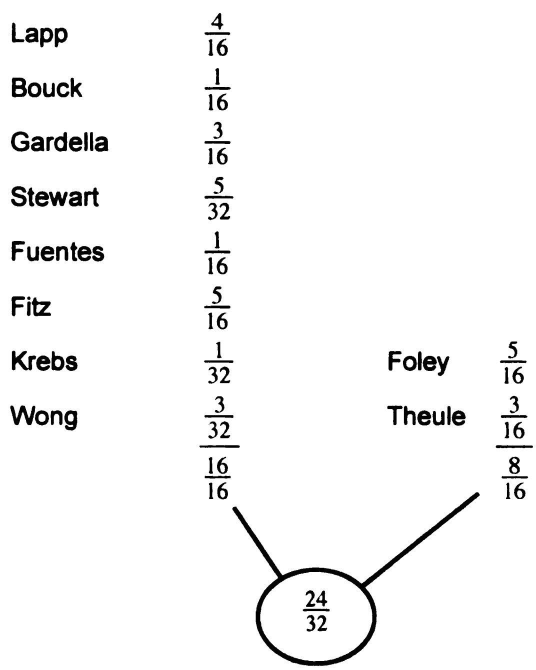


Figure 18. TJ's board work for Part C of the Land Problem.

TJ — I knew that this was all one section. That would equal $\frac{16}{16}$. This was half of the other section so I knew $\frac{8}{16}$ plus $\frac{16}{16}$ is $\frac{24}{32}$.

[Several students raise their hands.]

Mrs. Kay — Kristie do you have a comment for TJ?

Kristie — I did it a different way.

Mrs. Kay	Just a second. Are there people who disagree with TJ. Go ahead and talk to TJ.
Trevor	$\frac{24}{32}$ isn't even one whole. They are asking for $1\frac{1}{2}$.
Bryan	You have the right answer but not the...um...16 plus the 8...you don't have to add the denominators but you have to add the numerators.
Trevor	It has to be over one.
Bryan	16 plus 8. Its $\frac{24}{16}$, so you had it right but you added the denominators.
TJ	Oh, I see what you are saying Bryan.
Bryan	You don't add the denominators. I only had to add the numerators.
TJ	Oh I added the numerators but you only need to add the denominators.
Bryan	So you can change your answer.

[TJ revises and writes $\frac{24}{16}$.]

Mrs. Kay	Is $\frac{24}{16}$ a whole section and a half?
TJ	Yeah.

In these three examples we can see that benchmarks and estimation provide an argument for why adding both numerator and denominators are not reasonable. In itself though, estimation is not a justification for why you need common denominators and why you only add the numerators. As students continue to offer solutions to the Land Problem the conversation moves to why some are finding common denominators. In problems that follow this conversation Mrs. Kay makes this issue more explicit asking students why they need same size pieces.

Corey	I did almost like TJ but I did all of section 19 and a couple of people in section 18. I did Lapp, Wong, Krebs, Bouck, and Fuentes.

[Corey represented this on the board as $\frac{1}{4} + \frac{3}{32} + \frac{1}{32} + \frac{1}{16} + \frac{1}{16}$.]

Mrs. Kay	How do I know these all add up to one-half of a section?

Corey	Because Lapp is already $\frac{1}{4}$ and all the rest equal another fourth?
Mrs. Kay	How do you know these all the rest of these equal a fourth?
Corey	Because they are equal to 0.25. Well I am not sure.
Mrs. Kay	So you switched them to decimals?
Corey	I tried finding things that made one-fourth. Like Bouck and Fuentes that equals half of the one-fourth.
Mrs. Kay	Bouck and Fuentes equal half of a fourth, a sixteenth plus a sixteenth.
Corey	That is two sixteenths and I know you have to try to get four sixteenths for $\frac{1}{4}$. Wong and Krebs is two sixteenths. So two sixtheenths and two sixteenths is four sixteenths.
Mrs. Kay	For Part D you had to find two or more combinations where it would add up to another person's amount. Who did Part D?
Bryan	I just took one and found ones with same denominators, or I made same denominators, so I could see what all the fractions were. So $\frac{3}{16}$ plus $\frac{1}{16}$ plus $\frac{1}{16} = \frac{5}{16}$.
Mrs. Kay	So if they all are split into sixteenths they are pretty darn easy to add. Do you see how all of Bryans' combinations have the same denominator … so we can use equivalence to get them to have the same denominator.

Following the Land Problem, students explore addition and subtraction of mixed numbers in the context of mixing spices. This task begins with the statement: "In the following problems remember to think about the quantities before you start. Estimate what the answer will be before you figure out the exact answer" (Lappan et al., 2002/2003b, p. 21). Here estimation provides a way to determine if a strategy leads to a reasonable answer and is worth further exploration of why it works.

If you take into account the problem context, the number sentences students were writing, conversations about why you need to use common denominators, carry or borrow, and conversations like TJ's that justify whether or not an approach leads to a reasonable answer, you can begin to see how the five components of the practice of learning to operate with fractions come together to move students toward the sub-practice understanding how to operate. The problem context, maps and other diagrams or models, symbolism, benchmarks and estimates, and students beginning to articulate strategies or algorithms are all part of making sense of how to operate with fractions. In the broader context of student's work, estimation promotes number and operational sense.

This notion of number and operational sense is apparent when learning to multiply fractions. The tasks that focus on fraction multiplication are introduced as follows:

> Before you do any kind of computation, it is helpful to take a moment to estimate the size of your answer. Sometimes an estimate is all that you need to solve your problem or make a decision. At other times you need an exact answer, but the estimate helps you to know whether you have done the computation correctly. As you work on problems in this investigation, try to estimate the size of your answers before you compute. (Lappan, et al., 2002/3b, p. 33)

The first instructional task (See Figure 19) that addresses fraction multiplication is about selling brownies. Students engage with fraction of fraction situations in the context of selling part of pans of brownies that are only partly full. They develop an understanding of what happens when you take a fraction of a fraction or multiply fractions. This brownie pan model also supports estimation discussions such as whether $\frac{3}{4} \times \frac{2}{3}$ is greater than or less than 1, if $\frac{3}{4} \times \frac{2}{3}$ is greater than or less than $\frac{2}{3}$, and if $\frac{3}{4} \times \frac{2}{3}$ is greater than or less than $\frac{3}{4}$.

Estimation is also drawn upon when trying to decide if an algorithm is reasonable. As students move between the problem context, models, estimation, written and verbal symbolism they can decide if emerging strategies or algorithms are reasonable. For example, when students were trying to decide how to multiply a fraction by whole numbers and mixed number there was a suggestion that the algorithm they developed from the brownie pan problem for multiplying simple fractions times simple fractions would work when multiplying simple fractions by whole numbers and simple fractions by mixed numbers.

Paulo and Paula worked the brownie booth at the school fair. They ran into some interesting situations during the fair. Sometimes they had to find a fractional part of another fraction, like $\frac{1}{2}$ of $\frac{1}{3}$.

All of their pans of brownies are square. A pan of brownies costs \$24. You can buy any fractional part of a pan of brownies. You pay that fraction of \$24. For example, $\frac{1}{2}$ of a pan costs $\frac{1}{2}$ of \$24.

Below are two situations that Paulo and Paula had to solve. In each case, do the following:

- Draw a picture to show how each brownie pan looked before a customer bought part of what remained.
- Then draw a picture that shows how much of each pan the customer bought and how much was left.
- Use your drawings to check your computations for the brownie pan and the price for each customer.

A. Mr. Sims asked to buy half a pan that was two-thirds full. What fraction of a whole pan did Mr. Sims buy and what did he pay?

B. Aunt Serena bought $\frac{3}{4}$ of another pan that was half full. What fraction of a whole pan did she buy and how much did she pay?

Figure 19. Brownie Pan Problem.

Mrs. Kay	What if I told you that I had $2\frac{1}{2}$ pizzas and I ate a third of that. So I ate $\frac{1}{3}$ of $2\frac{1}{2}$ pizzas. How would you think about solving that kind of problem? Why don't you think about it on your paper.

[Gave students a few minutes to work on the problem.]

Mrs. Kay	Teresa how would you think about doing this problem?
Teresa	I could just ignore the 2 and then I can multiply the $\frac{1}{2}$ and the $\frac{1}{3}$.
Mrs. Kay	So totally ignore it. So you did $\frac{1}{3}$ times $\frac{1}{2}$. What did you get?
Teresa	$\frac{1}{6}$.
Mrs. Kay	So pretend the 2 is not there. So if I ate $\frac{1}{3}$ of $2\frac{1}{2}$ I ate $\frac{1}{6}$ of a pizza?

Teresa	And then I added the two to the $\frac{1}{6}$.
Mrs. Kay	So I would eat $2\frac{1}{6}$ pizzas? Okay, now I want to ask you if that makes sense. If I have $2\frac{1}{2}$ pizzas and I eat $\frac{1}{3}$ of it would I eat $2\frac{1}{3}$ pizzas? Would I eat both of those pizzas and $\frac{1}{3}$ of that one? [Pointing to diagram on board of $2\frac{1}{2}$ pizzas.] Does that seem reasonable? Ali what do you think?
Ali	I don't think that is reasonable because if it is one person, well, 2 is already over $\frac{1}{3}$ of it. Two wholes is already over $\frac{1}{3}$ of $2\frac{1}{2}$. I got a different answer.
Mrs. Kay	Talk about how you did it.
Ali	I did $\frac{1}{3}$ like when we did the 16 wholes.[13] I did $2\frac{1}{2}$ ones or $\frac{1}{3}$ of $2\frac{1}{2}$ ones [She is referring to writing $2\frac{1}{2}$ over 1.] and $\frac{1}{3}$ as it is and I did $2\frac{1}{2}$ times 1 and 3 times 1.
Mrs. Kay	Sara what do you think about that? She is trying to change this [$2\frac{1}{2}$] to a fraction without any mixed numbers
Sara	That is equivalent to my fraction.
Mrs. Kay	What did you do?
Sara	I changed the 2 into halves so I had 5 halves.
Mrs. Kay	Well this [the 2 wholes] is 4 halves and this is one more half so 5 halves. So [speaks and writes] $\frac{1}{3}$ of $\frac{5}{2}$. So you changed yours to fractions too but Ali changed hers to this fraction [points to $2\frac{1}{2}$] and you changed this to this fraction [points to $\frac{5}{2}$]. So what did you end up with for your answer?
Sara	$\frac{5}{6}$.
Mrs. Kay	$\frac{5}{6}$. so if you guys have two fractions with no whole parts you know how to multiply. The problem comes when I have some wholes in there. I heard Ali say that she took those and put them over one. Sometimes it gets a bit tricky because now we have to multiply by those again. I heard Sara say that she just switched this to an improper fraction so she could use her algorithm. But we have to do something. Our answer is almost a whole pizza. Does that make sense?
Student X	Yes.

[13] In the previous problem 3/4 × 16, one approach was to divide 16 by 4 and multiply by 3. Others used 3/4 × 16/1 and then applied the algorithm that emerged from the brownie problem of multiplying numerators and multiplying denominators.

Mrs. Kay	Why?
Bryce	Because if you are trying to get a third of $2\frac{1}{2}$, there is not even 3 whole pizzas so you can't get a whole pizza.
Mrs. Kay	Who can tell me what Bryce just said?
Corey	You have, since $\frac{1}{3}$ and 2, that you can't get one whole pizza since that is 3, $2\frac{1}{2}$. That is not a whole pizza so you can't get it.
Mrs. Kay	So if this was a whole pizza [pointing to the $\frac{1}{2}$ pizza] I'd get a third, then I'd get one whole pizza.
Corey	Since it is $2\frac{1}{2}$ can you split the two pizzas into halves and get 5 halves?
Mrs. Kay	That is what Sara was saying. Instead of leaving these pizzas whole let's cut them in half [See Figure 20] and then I would have 5 halves and I would have a regular fraction to deal with? Okay, if I want $\frac{1}{3}$ of these 5 halves, can I take $\frac{1}{3}$ of this one and $\frac{1}{3}$ of this half and $\frac{1}{3}$ of this half...So for each of the halves I split them into thirds and now I have got sixths. For each of those I took one piece which is almost a whole".

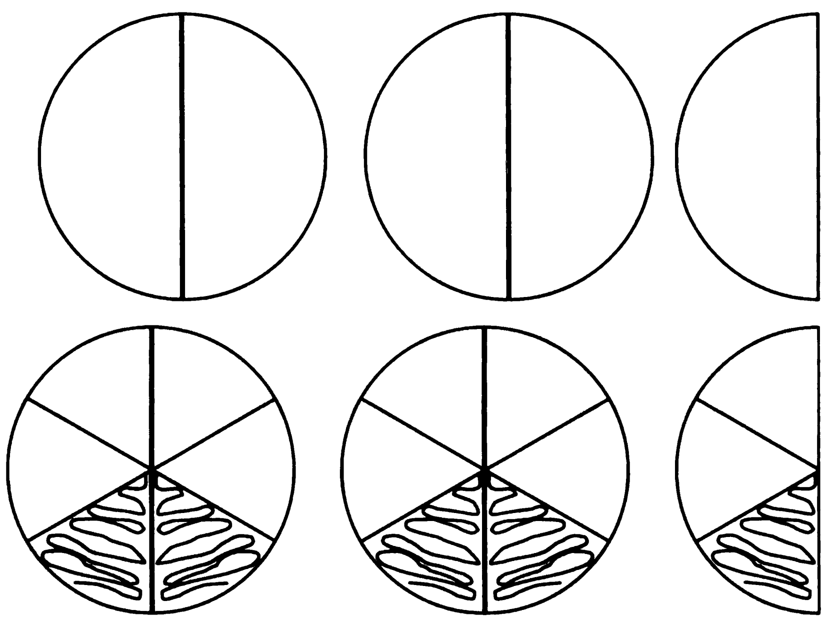


Figure 20. Modeling $\frac{1}{3} \times 2\frac{1}{2} = \frac{5}{6}$.

With regard to multiplication and division of fractions, understanding how to operate and when it is appropriate is more complex. Models don't make algorithms as obvious, especially when mixed and whole numbers are included. We saw the teacher ask students to use estimation to decide if algorithmic approaches led to reasonable products. The algorithm of multiplying numerators and multiplying denominators developed when was not assumed to work in situations that involved mixed numbers and whole numbers. The context, symbolism, estimation, diagrams and the algorithm were all used to make sense of multiplication of fractions.

This interaction of ideas in the practice of learning to operate with fractions also provides the basis for understanding when multiplication is an appropriate operation. In the dialogue we see how students move toward developing an algorithm for multiplying, but the models, diagrams, and problem contexts also help students understand what types of situations lead to multiplication. For example, in a unit that followed the fraction units a student explained that a situation was multiplication because it involved taking a part of a part. However, there were other times when students struggled to identify whether a situation involved addition, subtraction, multiplication, or division indicating that further learning, with regard to the sub-practice understanding when an operation is appropriate, is being developed.

Underlying Questions Associated with the Practice of Learning to Operate With Fractions. Like the questions developed to capture the underlying basis of the conversations in the first fraction unit, *Bits and Pieces I* (Lappan, et al., 2002/2003a), this unit also has a set of underlying questions. Table 6 lists the classroom conversation questions generated during the Land Problem and the questions that capture the mathematical focus of the task.

Table 6
Questions Supporting the Land Problem

Dialogue Questions	What fraction of the section does each person own and how do you know? You have $\frac{1}{4}$ for Lapp and she has $\frac{4}{16}$. Can both be right? How did you know the answer would be $\frac{4}{16}$? How did you know to add them? Can you tell us why if they have the same denominator why can we just add them? If they have the same denominator, what does that mean? Is $\frac{24}{16}$ a whole section and a half? How do I know all these add up to one-half of a section? How much are those together? What is $\frac{5}{16} + \frac{3}{16}$? How do you know that all the rest of these equal one-fourth? Do you see how all of Bryan's combinations have the same denominators? How did you decide what to divide it by? Why did you times it by 4? What does the 40 tell me? Who owns more? How can we figure out how much more this person owns than that person?
Questions representing the underlying mathematical focus of the task.	What number sentence represents this situation? How do you know? How many acres of land is a fraction of a section worth? How can I represent (including renaming) a person's land in such a way that I can easily combine it or compare it with another person's land? Why does renaming fractions to have common denominators help when trying to add and subtract fractions? How do I know my strategy leads to a reasonable answer?

By considering the questions that underlie the mathematical focus of the Land Problem task and the questions that were part of the classroom conversation, the underlying basis of the conversation is captured with the following questions:

What strategies can be used to add fractions? Subtract fractions?
Do these strategies lead to a reasonable result?
Why do the strategies work?

A similar set of questions can be asked for each task where students engaged in the practice of learning to operate with fractions. A complete set of questions representing the underlying nature of the work students engage in as part of the practice learning to operate with fractions is provided in Figure 21.

1. What strategies can be used to add fractions? Subtract fractions? Multiply fractions? Divide fractions?
2. Do these strategies lead to a reasonable result?
3. Why do the strategies work?
4. What makes this an addition situation? Subtraction situation? Multiplication situation? Division situation?

Figure 21. Questions associated with the Practice of Learning to Operate with Fractions.

Like the questions associated with the practice of understanding fractions as quantities, these questions align with the body of knowledge that documents such as NCTM (2000) and *Adding it Up* (Kilpatrick, Swafford, & Findell, 2001) specify. The work of the unit and the conversations lead students to understand "how" to operate with attention to both procedural and conceptual aspects of understanding how to operate. There is focus on developing number and operational sense so students can use these ideas as they develop strategies for operating and to monitor if their algorithmic work is leading to a reasonable solution. Through these experiences students also are exposed to situations that call for each operation leading them to develop meaning for each operation.

Summary

Across the two fraction units the described practices led students to focus on what fractions are and how to operate with them. As the chapter title suggested, student are learning "about" fractions. Their classroom work in the fraction units focused on development of both procedural and conceptual understanding of fractions. In general, the underlying focus of conversations was to understand the what, the how, and the why of fractions: What can we represent? How can we operate? and Why does that make sense?

The next chapter presents findings regarding how the students and the teachers in this study engaged in learning to use fractions to make sense of area and perimeter, decimal operations, scale factor, ratio, and making mathematical comparisons. There is a shift from fraction learning practices to fraction literacy practices. The examination of data regarding how students learn to use fraction ideas revealed two fraction literacy practices: the practice of "connecting fractions to multiplication and division concepts" and the practice of "determining appropriateness". Like the practices associated with learning about fractions, fraction literacy practices will be characterized as a set of questions characterizing the issues that are central to conversations students engage in when learning to use fractions in new mathematical contexts. Chapter Six will then look across the practices discussed in Chapter Four and Chapter Five in order to answer the posed research questions.

Chapter Five

LEARNING TO USE FRACTIONS: CONVERSATIONS, QUESTIONS, AND PRACTICES

The last chapter presented the fraction learning practices students engaged in when learning to use fractions. This chapter describes the fraction literacy practices students engaged in when learning to use fractions. Data analysis revealed two fraction literacy practices that these sixth- and seventh-grade students, and their respective teachers, engaged in while learning to use fractions. The practice of *determining appropriateness* and the practice of *connecting fractions to multiplication and division concepts* both revealed ways in which students engaged in conversations where they made connections between fractions and the mathematical contexts where fractions were being used. The conversational episodes indicated that the students did not simply pick up and use the knowledge they learned about fractions in these new settings of area and perimeter, decimal operations, similarity and ratio. Understanding how to use fractions is tied to understanding situations in which they can be used and the various ways fractions and other mathematical content merge together.

Before presenting my findings, I want to return to the notion of practice and how it was used with regard to fraction literacy practices. The construct fraction literacy practice is based on the notions of literacy (Barton, 1994; Scribner & Cole, 1981) and literacy practices (Barton, 1994). Literacy is a set of practices people use during literacy events and so in parallel fashion fraction literacy is then a set of practices that I call fraction literacy practices, which people draw upon during fraction literacy events. In this study it is assumed that students were learning to use fractions or moving toward fraction literacy rather than being fraction literate or proficient in the use of fractions in settings where fractions are used.

The Practice of Determining Appropriateness

Conversations that centered around the fraction literacy practice of determining appropriateness were found in 18 of 26 different instructional tasks across four of the five units[1]. The conversations indicated that students were not concerned with how to carry out an algorithm or if two forms of representation were equivalent. Instead, conversations point to students' need to make sense of the appropriateness of using particular fraction skills or concepts in a particular situation.

Like the fraction learning practices in Chapter Four, these episodes led to the development of a set of questions that captured the underlying basis of the conversations students engaged in as part of the practice of determining appropriateness. Since data collection occurred across several content units, there were numerous episodes where determining appropriateness occurred. Several episodes had common elements and were grouped into subpractices (See Table 7 for subpractices and Appendix E for chart with occurrences of both practices across all tasks.). For example, there were three different episodes regarding whether or not the standard multiplication algorithm would also work in situations where students drew upon a distribution approach to multiplication or an approach where they counted the square units that covered a shape. These three episodes were grouped into the subpractice "Drawing from Multiple Approaches and Algorithms".

[1] The probability unit did not have any instances where fractions led to a conversation about using them. When discussing that probability is between 0 and 1, students did not react or talk about this idea where fractions were concerned other than to say that the probabilities when expressed in fraction form had to total one. The idea appeared to be common sense and taken-as-shared (Cobb, P., Dillon, D. R., Simon, M., Wheatley, G., Wood, W., & Yackel. E., 1993). Perhaps the use of fractions is apparent here.

The question that captured the underlying basis of these conversations was *Is it appropriate to use the standard multiplication algorithm here*?

Table 7
Frequency of Subpractices Associated with the Practice of Determining Appropriateness by Mathematical Content in the Unit.

Subpractices	**Area and Perimeter**	**Decimal Operations**	**Similarity**	**Ratio**
Representing Repeating Decimals as Fractions	1		1	
Moving from Fractions to Whole Numbers and Back	1	1	1	
Drawing from Multiple Approaches and Algorithms	3			
Choosing to Use Fractions When Fractions Are Not Explicit		1	2	1
Does Equivalence Matter?	1	1		1
How are Fractions and Ratios Related?			2	4

As classroom episodes illustrating the practice of determining appropriateness are presented note what students know about fractions as they engage in situations where they are learning to use fractions. Consider what the focus of the conversations are, the actual questions students ask, and the knowledge that must be present for the students and their teacher to engage in the conversations presented. As you read, notice how the actual conversations and the questions that capture the underlying basis of conversations during the practice of determining appropriateness compare to those that the students and teachers engaged in when learning about fractions.

Subpractice One: Representing Repeating Decimals as Fractions

There were two episodes where conversations centered around representing repeating decimals as fractions. As we will see in the forthcoming classroom conversation, students knew that the fraction $\frac{1}{3}$ was a repeating decimal. Their conversation was about whether it was appropriate to use a decimal approximation for $\frac{1}{3}$

to solve a particular problem. The sixth-grade task Storm Shelters involved finding the width of a rectangular storm shelter with a floor area of 24 square meters and a length of $5\frac{1}{3}$ meters. During the whole class discussion of the task various decimals were offered and rejected to represent $5\frac{1}{3}$ when operating to find the missing width.

Cathy — I did 5.3 × 4.6 and got 24.3.

Bryan — I did 5.3 × 4.5 and got an even 24.

Trevor — Well $5\frac{1}{3}$ is not equal to 5.3. It is 5.3 with a line over it. [The teacher writes 5.33333 on board.] Yeah. So let's say that times 4.5 which equals 23.99999. [This is checked in a calculator.] It is pretty close to 24.

Here Trevor questions his own approach that did not produce exactly 24. Katie asked how people came up with 4.5 as a solution. This led the conversation to change over to a discussion of division as an approach to find the missing value.

Katie — You divide 24 by $5\frac{1}{3}$.

Mrs. Kay — Corey is that what you did?

Corey — Yeah.

Mrs. Kay — What did you get when you did that?

Katie — $4\frac{1}{3}$.

Corey — Four point 5 2 8.

As Katie tried to describe how she got $4\frac{1}{3}$, she realized she made an error, and rejected or withdrew her solution. Corey explained that he used a calculator and divided 5.333 by 4.5 to get 4.528. This was followed by Sara who showed that if you multiply the fractions $5\frac{1}{3} \times 4\frac{1}{2}$ the result was exactly 24. Again, Katie raised the issue of how Sara came up with the solution 4.5.

Mrs. Kay — So that works. $5\frac{1}{3}$ rows of $4\frac{1}{2}$. But how did you get that?

Corey — Maybe like I did. [Earlier Corey offered that he used 24 ÷ 5.333]

Mrs. Kay	Okay but you got 4.528. What would explain that?
Corey	I rounded off.
Mrs. Kay	That is the problem with sometimes switching to a decimal. If I don't I can get the exact answer.

In the end five approaches were rejected because using approximations for $5\frac{1}{3}$ did not lead to an exact area of 24. A similar situation took place in the seventh grade unit on similarity during the task Scale Factors and Similar Shapes. The class was given two rectangles, a 9 by 15 and a 12 by 20, and asked to determine whether they were similar. A scale factor of $1\frac{1}{3}$ had been proposed and students were checking to see if it worked. Like the sixth grade episode, using 1.3333 to represent $1\frac{1}{3}$ led to a discussion about whether or not it was appropriate to operate with 1.3333 for $1\frac{1}{3}$.

Katie	It is the point 3.
Bryan	But it is not point 3, it is point 33333.
Katie	Same thing.
Bryan	No it is not.
Mrs. Dew	So I guess the debate is if I have a fraction like $\frac{1}{3}$ and I use decimals I run into the problem of rounding, or estimating, and $\frac{1}{3}$ is not going to give me an exact decimal. So Janine is saying that instead of using the decimal when it is not exact, what if I left it as a fraction. If I leave it as a fraction, is Janine's multiplication and addition right?
Student A[2]	Yes.
Mrs. Dew	Is 15 times $1\frac{1}{3}$ [equal to] 20?
Student B	Yes.
Janine	$\frac{1}{3}$ has 3 equal parts in it and for .3333, if you add all that together you are gonna get .9999 and not a whole. So it is not really $\frac{1}{3}$ it is close but not exactly $\frac{1}{3}$. So using the fractions, $\frac{1}{3}$ plus $\frac{1}{3}$ plus

[2] References such as "Student A" or "Student B" indicate a student responded but I could not identify the name of the student.

	$\frac{1}{3}$ is one whole and it is exact so you know 20 is right. The point 3 will not add up to one whole so it is not $\frac{1}{3}$.
Mrs. Dew	So we have to be very careful when we decide if a fraction or decimal is what we want to use. If we change it to a decimal we want to be very careful. Do we have an exact decimal like point five for $\frac{1}{2}$ or if we have an estimated one like Janine was saying. As we get to the end we have to keep in mind that we will only have an estimate. If we don't want an estimate, and we want an exact answer then we have to use it as a fraction.

In both conversations the situational context is the launching point for the conversation. When making choices about using fractions and decimals to operate one needs to consider the situation and determine whether an exact or estimated value is appropriate. In both the sixth- and seventh-grade tasks the situation called for an exact solution. But there are situations, such as estimation, where using a decimal approximation such as 0.3 or 0.33 is appropriate for $\frac{1}{3}$.

It is clear that the students understand that some fractions have equivalent decimal representations while others only have approximations. In the Storm Shelters task students moved from using $4\frac{1}{2}$ to 4.5 without comment or question. With both $5\frac{1}{3}$ and $1\frac{1}{3}$ students know what the various decimal approximations are. However, this knowledge, which was learned in the fraction units, needs to be further refined so the students can determine the appropriate way to proceed and use their fraction knowledge when the situational context calls for an exact area of 24 meters.

Subpractice Two: Moving From Fractions to Whole Numbers and Back

A second form of the practice of determining appropriateness took place during two conversations where students were trying to extend a concept they understood in whole number situations to fraction situations. The question these conversations address are as follows:

Do concepts that work in whole number contexts also work when fractions are used in the same setting?

As part of the Storm Shelter conversation in the sixth grade classroom, Katie introduced the potential role of division in finding the missing width. She raised this issue after several people used multiplication to show that the length ($5\frac{1}{3}$) times some width resulted in an area of 24. When Katie shared that she divided 24 by $5\frac{1}{3}$ Mrs. Kay asked why she divided. In the conversation that follows Mrs. Kay returned to whole numbers as she pushed the class to consider whether or not division was appropriate. The following chart from the Storm Shelters task (See Figure 22) and dialogue shows how Mrs. Kay built her case.

Length	Width	Perimeter	Area	Cost of Walls
1	24	50	24	$6250
2	12	28	24	$3500
3	8	22	24	$2750
4	6	20	24	$2500
$5\frac{1}{3}$			24	

Figure 22. Storm Shelters Data Table

Mrs. Kay Katie, how do we even begin to figure out what goes here? [Points to the width column of the row in the chart with $5\frac{1}{3}$.]

Katie You divide 24 by $5\frac{1}{3}$.

Mrs. Kay Why?

Katie Because when you do you will find—because there are 24 blocks in the area—how many squares are in the area. And you are dividing how many there is on one side to find out the other.

Mrs. Kay Okay, I am going to stop you for one second. What you are saying is that if I knew this [points to known length 2 on chart], but didn't know width [points to the width 12], I could take 24, divide it into

	2 rows and see how many there were in there?
Katie	Yeah.
Mrs. Kay	So I could take 24 and divide by 3 rows and see how many are in each row?
Student A	8
Mrs. Kay	So I can take 24 and divide it by 4 rows to see how many are in each row? So does it make sense that I could do it here [points to $5\frac{1}{3}$] just because there is a fraction there?
	[pause]
Mrs. Kay	Let us say 24 and split it into $5\frac{1}{3}$ rows, and lets see how many are in each row. So that is what you did Katie?
Katie	Yeah.

In the seventh grade classroom during the unit on similarity students had been exploring the relationship between scale factor and perimeter and scale factor and area. The seventh-grade task Ratios Within Similar Parallelograms asked students to apply a scale factor of 2.5 to a four by eight rectangle and then determine both the new area and new perimeter[3]. While students were comfortable that each side length was multiplied by $2\frac{1}{2}$ becoming $2\frac{1}{2}$ times larger, they were struggling to figure out what was happening to the area. At one point someone said that the rectangle was $2\frac{1}{2}$ times larger. The teacher was pushing for an explanation of what that meant. Was the area or the perimeter becoming $2\frac{1}{2}$ times larger? This table, developed when working on the previous problem, was written on the board:

Scale Factor	Perimeter	Area
2	2	4
3	3	9
4	4	16

[3] The original problem stated that the scale factor as 2.5 but throughout the problem students interchange between 2.5 and $2\frac{1}{2}$.

The teacher had the students look at previous work for scaling figures using whole number scale factors. A student offered that you could multiply $2\frac{1}{2}$ by $2\frac{1}{2}$ to find the change in area another issue was raised and the conversation went down another path. A few minutes later in the discussion it was offered that you could square the scale factor.

Mrs. Dew	How much bigger is the [large] rectangle compared to the [small] one?
Bryan	6.25
Mrs. Dew	Why six and one-fourth?
Bryan	I did the 200 [area of large rectangle] divided by the 32 [area of small rectangle] and I got 6.25.
Mrs. Dew	What does that have to do with the scale factor? Bryan started us out by saying that he could figure out how the area is changing by knowing the scale factor is 2 point 5. And I have been lost ever since.
Bryan	If the rectangle was a square then the scale factor would have something to do with it.
Mrs. Dew	So the scale factor will not help me figure out how the area changed?
Ali	If you divide 200 by 32 you get the 6.25 and if you times it by 6.25 that would be what you would get so that would be your scale factor.
Mrs. Dew	If I take the area of 200 and I divide by 32, which is where Carl started us a few minutes ago, and I say I get 6 and $\frac{1}{4}$ what does that mean?
Student A	You get 6.25.
Mrs. Dew	What does that mean in terms of this picture?
Ali	That is how many times you multiplied the smaller one.
Student B	The area to get how many you need to make the bigger one.
Mrs. Dew	So $6\frac{1}{4}$ of these makes this?
Class	Yes.
Mrs. Dew	Is there any way I could have found that from where Bryan started this 10 minutes ago? Or do I have to know the area of this and the area of this and divide them to find out how many of these fit inside of this.

	[pause]
Janine	For problem 3.1 when you double it is 4, and when it is 3 it goes to 9 and 4 is 16 so the scale factor is 2.5 so wouldn't you square 2 point 5 or times it by itself to see the area change?
Mrs. Dew	What is 2.5 times 2.5?
Class	6.25.
Mrs. Dew	So, Janine is the same thing holding true?
Janine	Yes
Mrs. Dew	Just because I went to something that wasn't quite as nice and pretty as a 2, a 3, and a 4, the rule we had for a week now didn't go away. The way you prove that like Carl said, [dividing 200 by 32] just helped me solidify that. Even with an ugly scale factor I still have the same relationship. The scale factor times itself will tell me how that area will change.
Amy	So we spent almost a half an hour talking about that when we could have just done that?
Mrs. Dew	Yeah.

The last student's comment is humorous yet enlightening. It points to her realization that ideas carry from whole numbers to other forms of number, in this case fractions. We see the teacher making connections between what students are doing with fractions back to what they knew about whole numbers. In the sixth-grade conversation the teacher was asking students to determine if it was appropriate to use an idea that they could see worked with whole numbers in situations with fractions. In both the sixth- and seventh- grade excerpts students were pushed to use fractions in new ways with the conversations centering around whether or not it was appropriate to do so.

In settings such as area or scale factor using fractions helps students see how fractions and other types of numbers are related. Because fractions are a distinct topic in the curriculum, a topic with a unique set of associated concepts and skills, students may see whole numbers and fractions, even decimals, as unrelated sets of numbers. By extending situations that could be and often are explored with whole numbers, to include more complex numbers such as fractions, students see that the simplicity or complexity of the number or quantity is irrelevant. The actions and operations that can be performed with numbers sit above the situation.

The relationship that is established is at the reflective level because it requires stepping back form the situation and noting how two ideas are related and in case not completely different. For example, the concept of squaring the side length to find the change in area when a rectangle is scaled up is an idea in itself and it does not change when using fractional lengths rather than whole numbers. If students only explore mathematical topics such as those described with whole numbers they will not have the opportunity to learn that they can use and are capable of using fractions. Students need opportunities to develop the disposition and facility to engage with all kinds of numbers. This engagement is an important factor in the development of students' conception of number and in developing their fraction literacy.

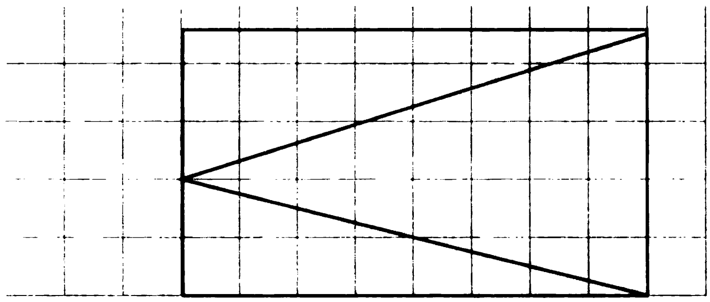
Subpractice Three: Drawing from Multiple Approaches and Algorithms

This subpractice involves situations where students are drawing upon one possible algorithm and the teacher is trying to get them to expand their thinking to consider the possibility of using another. The teacher is not disregarding the students' initial approach. Rather, the underlying tone of the conversations centers around the following question:

Is it appropriate to use the standard multiplication algorithm here?

There were three situations where this question was a theme during a whole class conversation.

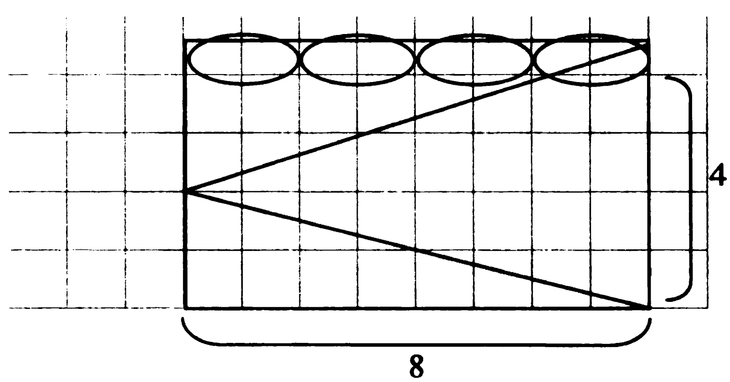
The first situation took place while discussing a homework problem from the sixth-grade area and perimeter unit. The students were given the following triangle on a grid and asked to find the area. TJ comes up to the overhead and draws the surrounding rectangle around the triangle.



He then wrote the following number sentences and explained how he found the area. The following conversation occurs:

TJ	So, $4\frac{1}{2} \times 8$ is	$4 \times 8 = 32$ $\frac{1}{2} \times 8 = \underline{4}$ 36
Ali	I don't get it.	

TJ returned to the overhead to show how the number sentences he wrote modeled the situation. He marked the 4 by 8 rectangle that comprises the 32 whole squares and the 8 half squares that made "4 more" for a total of 36.



Mrs. Kay	What if I just multiplied $4\frac{1}{2} \times 8$, the numbers?
Trevor	4.5 times 8 on a calculator
Mrs. Kay	What if I didn't want to go get the calculator. We spent all that time in the *Bits and Pieces II* unit learning to multiply fractions. How do we do that?

At this point the class talked about how to multiply using the general algorithm they had developed in the fraction operation unit. The general algorithm they developed was the standard algorithm commonly used for fraction multiplication. In this conversation the teacher raised the question of whether both approaches were appropriate for multiplying fractions and mixed numbers. Along with seeing the appropriateness of the algorithm she also wanted students to see it as efficient.

This issue surfaced again when working on a task for finding area and perimeter of parallelograms. Students were given several parallelograms on a grid including a $3\frac{1}{2}$ cm by $4\frac{1}{2}$ cm rectangle. Students were given rulers and asked to measure or in some way determine the needed lengths to find the perimeter and perimeter. When determining area, all of the non-right parallelograms had whole number bases and heights. For the rectangular parallelogram, most students visually determined the base to be $4\frac{1}{2}$ and the height to be $3\frac{1}{2}$[4].

During the summary of the rectangular parallelogram Trevor showed the measure of the base and height as $3\frac{1}{2}$ cm and $4\frac{1}{2}$ cm. He tried unsuccessfully to use a partial product approach to multiply the base and height. He determined the area to be $14\frac{3}{2}$

[4] Based on my classroom observations, Katie did use the standard algorithm. Initially Katie used a ruler to arrive at base and height measures of $3\frac{6}{10}$ cm and $4\frac{7}{10}$ cm. When she struggled to use the standard multiplication algorithm and find a reasonable product she switched to using $3\frac{1}{2}$ cm by $4\frac{1}{2}$ cm and a counting approach.

square centimeters. Two other students offered approaches that led to a correct solution. These approaches involved either counting or a combination of multiplication and counting of 15. For example, one student multiplied 3 by 4 to find the area of the rectangular region formed with whole square centimeter units, and then counted the half and quarter centimeter units to arrive at $15\frac{3}{4}$.

Noting that no one had offered an approach using the standard multiplication algorithm, Mrs. Kay pointed out that no one used the multiplication algorithm they had developed. After using the algorithm to determine the area, she asked "I am curious why no one uses the algorithm. Is it that you don't like the algorithm or that it doesn't make sense or what?" One student offered that he "forgets" about it sometimes. During the focus-group interview I asked the four focus students what they thought about when Mrs. Kay asked this. TJ indicated that it did not make sense to use the algorithm because the rectangle was small and on a grid. It was easy to count. TJ points out, "Why would you use the algorithm when the square is so small and you have all the squares right there around the outside." I drew a picture of a non-gridded rectangle with dimensions $3\frac{1}{2}$ by $4\frac{1}{2}$ labeled and asked how they would find the area in this case. All the focus students offered that the algorithm made more sense in this case.

When Mrs. Kay asked students to consider the plausibility of using the standard algorithm she was not disregarding their approaches. For Mrs. Kay, it was important to raise the possibility of using the standard multiplication algorithm because as the class continued to work on finding area and perimeter they were going to move away from figures drawn on a grid. While the students' partial product approach may work in some cases, the standard algorithm the students developed in the fraction unit is not only appropriate, but very efficient. At the end of the $3\frac{1}{2}$ by $4\frac{1}{2}$ rectangle conversation she asked the class how many counted square units. As students raised their hands she said, "It is okay, I was just curious." By asking them to consider whether or not the standard algorithm would work, she was asking them to decide if it was an appropriate approach for the problem.

Multiple strategies arise when we aim to solicit invented algorithms from students. When these students studied fraction operations they explored and developed several approaches for finding area and perimeter. Having multiple approaches provide the user with a variety of options to draw upon. However, making choices about which strategy or strategies are most efficient for a situation is a skill that is developed by having the opportunity to decide which approach to use and why it would be reasonable and appropriate for that instance. For the focus students it was not a matter of knowing how to use the standard fraction multiplication algorithm but one of appropriateness. While it was mathematically appropriate, the students did not think it was situationally appropriate.

Subpractice Four: Choosing to Use Fractions When Fractions Are Not Explicit

When I identified instructional tasks where fractions would be used I was unable to anticipate certain tasks. In most, but not all cases, I chose a task because fractions were an explicit part of the problem. The tasks presented in this section were unanticipated because the problems did not have fractions in them. Rather, students or the teacher made a choice to use fractions in order to make sense of another idea. This practice emerged in four different conversations (refer back to Table 7). The underlying question for these cases is:

Can we use fractions to make sense of or think about other mathematical ideas when fractions are not explicit in the problem?

When learning to add and subtract decimals the sixth graders began by exploring the role of place value. After working on decimal addition and subtraction problems, sharing solutions, and talking about the role of place value, the following problem was proposed:

> I ran 5.2 miles on Monday, 6.08 miles on Tuesday, and 2.455 miles this morning. How many total miles have I gone so far?

A student presented a solution that involved lining up the place values and decimal point. The class talked about why you line up decimal points, the role of place value and the meaning of the carrying that was used when adding the decimals. They wrote out a general procedure for adding and subtracting decimals using place value. At this point a student suggested that there was another approach, one that involved fractions.

Trevor	Like say 5 point 2, say like 5 and 2 tenths.
Mrs. Kay	Oh, 5 and 2 tenths. [Writes $5\frac{2}{10}$ on board.]
Trevor	And like 6 point zero eight as 6 and 8 hundredths and 2 point four four five as two and 4 hundred 55 thousandths.

Both the decimal and fraction from of the problem were written on the board.

$$\begin{array}{r} 5.2 \\ 6.08 \\ +\underline{2.455} \end{array} \qquad \begin{array}{r} 5\frac{2}{10} \\ 6\frac{8}{100} \\ \underline{+\ 2\frac{455}{1000}} \end{array}$$

The conversation turned toward determining if fractions were an appropriate form of representation for decimals when adding and subtracting.

Mrs. Kay	How is doing this [fraction representation] like doing this [decimal representation]?
Bryan	Add another zero onto the decimal then the decimal, then the denominator gets higher just like with fractions.

Mrs. Kay	Oh, so when I added these two zeros on [onto 5.2 to make 5.200], how is that like doing this [$5\frac{200}{1000}$]?
Bryan	Because you add two zeros onto the [point] 2 and two zeros onto the 10.

Trevor's idea raised several questions. Can fractions be used in place of decimals to operate? Are the forms equivalent? How are these two forms of representation related? The students went on to formally explore these ideas more in the next task as they reworked the contextual problems where they used a place value approach to decimals, this time using fraction addition and subtraction.

In the seventh grade classroom fractions were introduced to make sense of a percent problem offered in an opener. Initially, several solution approaches–some sensible and some not–were offered for finding 20% of 120. Some included:

$120 \div 20 = 6$ 0.20 of 120 = 6 $120 \div 20 = 12$

20% of 120 = 6   20% of 120 = 24

As various number sentences were offered and written on the board Mrs. Dew asked why the different approaches made sense. When students struggled to answer her question she pointed out to students that they needed to figure out if their responses were reasonable. In other words, a number sentence with its solution was not proof for finding 20% of 120.

At one point Mrs. Dew suggested the use of a simpler problem: 20% of 100. This led to more of the same response types. After roughly 15 minutes of discussion Mrs. Dew redirected the conversation having students get out paper. She wrote 20% of 100 and suggested using fractions to make sense of the problem.

Mrs. Dew:	I keep hearing people say that you want to multiply with the word *of* from last year. Tell me what 20% means.
Class:	[several responses including "point 2", "20 hundredths", and "2 tenths"]

Mrs. Dew	What if I take it to what you are more comfortable with? What if I make it $\frac{20}{100}$ and of means you want to multiply. What is 100 as a fraction? [Asking how to represent 100 with fraction notation.]

In both examples fractions were introduced into the conversation as a tool for making sense of a related but more complex concept. Students were pushed to think about how fractions were related to percents and decimals and how using fractions could help them better understand decimals and percents in these situations. The example that involved connecting fraction addition to decimal addition is the same example that Heibert and Lefevre (1986) used to describe making connections at the reflective level.

In the first fraction unit students learned about and discussed relationships among fractions, decimals, and percents. The examples shared in this section show how students were learning to use these relationships by being provided with situational contexts where it made sense to use fractions. Students needed support in understanding if they could use fractions in these situations and that it was appropriate to do so. While the field recognizes the importance of helping students learn how fractions, decimals and percents are related we also need to recognize that students need opportunities to draw upon and learn to use that knowledge.

Subpractice Five: Does Equivalence Matter?

The Park Problem in the area and perimeter unit raised the question of whether or not something is reasonable given the context of the problem. Rather than dimensions with fractional lengths, this problem uses fractions as operators to think about a fraction *of* a length. If a skateboard arena takes up $\frac{2}{3}$ of the length and $\frac{2}{3}$ of the width of a park, can we multiply $\frac{2}{3}$ by $\frac{2}{3}$ to find out what fraction of the park's area the skateboarding arena takes up? This conversation took place when the group of focus students were working together to find a solution to the task. The underlying basis of their conversation is:

Is it appropriate to use equivalent forms when solving problems?

This question surfaced across three different episodes in this study as shown in Table 7. Here I share two of these conversations. When working in small groups, Bryan used $\frac{2}{3}$ by $\frac{4}{6}$ to partition the park while the other three used $\frac{2}{3}$ by $\frac{2}{3}$. The other students argued that Bryan was wrong. They did not realize that his solution of $\frac{8}{18}$ was equivalent to their solution of $\frac{4}{9}$. Once the issue of equivalence was worked out, there remained the question of why you would want to do that? Bryan argued that for him the use of $\frac{4}{6}$ provided a partitioned rectangle where the units visually looked more like square units (See the $\frac{2}{3}$ by $\frac{4}{6}$ diagram in Figure 23). The teacher and the other three focus students were asking if this approach was situationally appropriate. In the end they were not concerned with whether the use of $\frac{2}{3}$ and $\frac{4}{6}$ was mathematically equivalent or appropriate, but whether or not it made sense or provided insight to the problem situation to use $\frac{2}{3}$ and $\frac{4}{6}$.

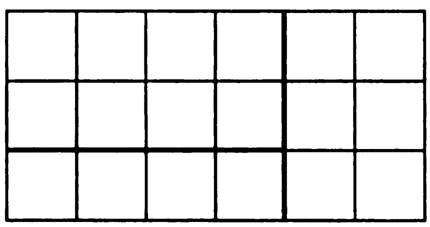


Figure 23. Bryan's Partitioned Rectangle

Another conversation regarding equivalence surfaced during the sixth-grade decimal and percent unit. Students were asked to rework addition and subtraction problems they had done this time using decimals with fractions. During small group work students engaged in conversations regarding appropriateness of the form of representation and quantity. One problem students worked read as follows:

> Emma signed up to clean 1.5 miles with the cross-country team. She stopped when it started to rain after 0.25 of a mile. How much did she have left to clean when the rain stopped?

As students worked in small groups two focus students represented the solution to 1.5 – 0.25 using different forms of representation. TJ wrote $\frac{150}{100} - \frac{25}{100} = \frac{125}{100}$ while Ali wrote $1\frac{50}{100} - \frac{25}{100} = 1\frac{25}{100}$. When TJ questioned Ali's solution she pointed out that her solution was equivalent to his. "You did improper fractions and [I] did a mixed number." TJ replied, "Can mine be a mixed number? Is that okay?" TJ's concern was not whether the two forms represent equivalent quantities. He did not ask if $\frac{125}{100}$ and $1\frac{25}{100}$ were the same or different. He was trying to determine if mixed numbers were appropriate to use in this situation. This was also evident when he then asked if he had to do it the same way every time. A similar conversation took place between two other students. One student asked, "Should we use $\frac{25}{100}$ or $\frac{1}{4}$?" The other student replied, "It doesn't matter because $\frac{25}{100}$ is $\frac{1}{4}$."

Subpractice Six: How are Fractions and Ratios Related?

Students first encounter with ratio was in sixth grade during the probability unit. Students were not explicitly introduced to ratio. In fact, they did not seem to notice that they were using fractions as ratios any differently than fractions as fractions. For example, students recorded the number of coin tosses out of 30 that resulted in heads on the board as the number of heads over the total number of coin tosses. Once each group's data was listed the class began to compare each groups' individual data and cumulative data. As the data, expressed as fractions, were combined students automatically added all the denominators and all of the numerators.

In seventh grade students were formally introduced to ratio in the unit on similarity. They had been using scale factor, same general shape, and congruent angles to determine if figures were similar. As part of launching the task Ratios Within Similar Parallelograms, Mrs. Dew drew upon the context of an earlier problem involving similarity transformations. In the earlier problem students used scale factor to determine if two rectangles were similar. Using a rectangle whose length to width ratio was 4 to 1, Mrs. Dew introduced the students to the ratio representations a to b and a:b. When she introduced the $\frac{a}{b}$ or fraction form of representation for ratio, students were unsure if this representation was a fraction or not. When Mrs. Dew asked the class if they were comfortable with $\frac{4}{1}$ several said no.

Bryan	Four oneths.
Mrs. Dew	Are you comfortable with that one?
Class	[Several say no]
Mrs. Dew	It looks like a fraction. Is it a fraction?
Class	[Several respond no followed by several responding yes]
Mrs. Dew	What does 4 over 1 say to you in the fraction world?
Class	[Several responses were called out.] Four whole. Or four ones. Or four to one
Mrs. Dew	So what do you think? Is this a fraction?
Bryan	No, it is four. Yes it is a fraction but it is a fraction of a higher number because that is four right there. Or you could say eight seconds or sixteen fourths.
Student A	It looks like a fraction.
Mrs. Dew	What is the definition of a fraction.
Chris	A piece of a piece.

Mrs. Dew	Is that what I am looking at when I wrote 4 to 1.
Class	[At once, some say no and some say yes.]
Mrs. Dew	Am I looking at 4 out of the 1?
Class	[At once, no, yes, possibly.]
Student B	No, you are not.
Student C	I don't know?
Mrs. Dew	Katie?
Katie	It is like 4 over 1 that length is 4 and the width is 1, so its 4 over 1.
Mrs. Dew	So is it the same as the fraction world?
Student D	Seems like it.
Mrs. Dew	Janine says no. Let's keep that one out there a while.

As the conversation continued with other ratios being added to the chart that was started with the 4 to 1 ratio, students began to compare the ratios and discuss whether the ratios indicated that the two rectangles were similar. When 8 to 2 and 12 to 3 were introduced students were looking for patterns pointing out that if you double or triple both numerator and denominator in $\frac{4}{1}$ you will get other similar ratios.

Katie	If you do four oneths times 3 that equals 12 thirds. It is like, its kind of like fractions but not really.
Mrs. Dew	Why is it like fractions?
Bryan	It is kind of a fraction way but it is really four wholes. Like if you had a pizza sliced into fourths. Like if you had a pizza sliced into one...yeah and then not sliced.

Each of these episodes show students engaging in the practice of determining appropriateness with the following questions serving as the underlying basis of the conversations:

Are ratios like fractions? Are fractions like ratios?

Can fraction equivalence be used with ratios?

Students are trying to decide if ratios and fractions behave in the same way. Since ratios can behave like fractions, students see qualities of fractions in ratios. These questions underlie six conversations in both of the seventh grade units where data was collected.

After this introduction to ratios the students were given a task with three parallelograms whose dimensions were labeled (See Figure 24). The students were asked to find the ratio of the long side to the short side and then use the ratio to decide if the parallelograms were similar. Most students eliminated Parallelogram E right away because of the difference in angle measure. A majority of the students determined that Parallelogram F and G were similar by showing that a scale factor existed between the corresponding side lengths. However, they seemed unclear about what the ratio was.

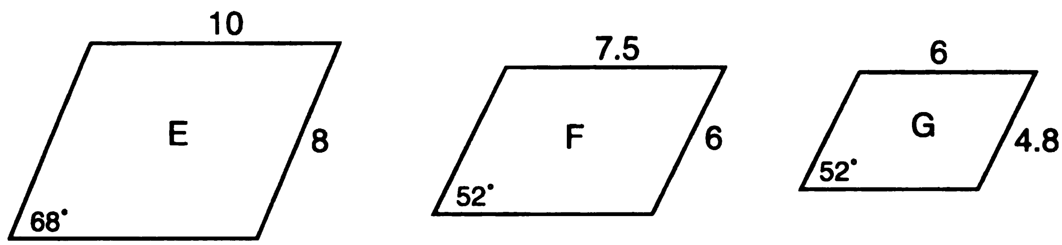


Figure 24. Labeled parallelograms.

During the whole class discussion Mrs. Dew wrote the long side to short side ratios for Parallelogram F and G on the board as follows:

$$\begin{array}{c} \times 1.25 \\ \dfrac{7.5 \longleftarrow 6}{6 \longleftarrow 4.8} \\ \times 1.25 \end{array}$$

Using 1.25 as the scale factor, the class decided that the two rectangles were similar.

At this point Janine asked if it was possible to use the relationship between each rectangle's length and width (internal ratio) to determine similarity rather than comparing the corresponding parts of two different rectangles with a scale factor.

Janine	I have a question about the 6. Since there is one on the bottom and one on the other side you want to times 1.25 by 6. Which one would you want to multiply it by. The other one or the one on the bottom?
Bryan	You lost me.
Mrs. Dew	The fact that there is a 6 here and a 6 there is what her question is about.
Janine	Whichever one you use you get the same answer. Like which one would you have if you had a different one?
Mrs. Dew	What if we sort of did what you are saying. What if we divided this way. What if we took 7.5 and divided by 6?
Student A	1.25
Mrs. Dew	What is 6 divided by 4.8?
Student B	1.25
Mrs. Dew	Now that is interesting.
Janine	What if they are not the same? Which one would you use? Would you multiply it by the one on the bottom or over?
Mrs. Dew	Well let's do one like that. If we had $\frac{3}{4}$ and $\frac{12}{16}$? We were going across. The question is that we have been always going across. Janine was noticing that with both of these 6s here it got confusing. What if I did it vertically? What if you did 3 divided by 4?
Student C	0.75
Mrs. Dew	What is 12 divided by 16?
Student C	0.75
Mrs. Dew	Why are they coming out equal?
Class	They are they same.
Bryan	Because the fractions are equivalent.
Mrs. Dew	If their fractions are equal, then their percents or decimals are equal. So Janine does it matter what way we look at it?
Janine	No.
Mrs. Dew	Great question.

Janine's question indicated that she was not comfortable with or clear about the appropriateness of applying ideas used with fractions in this situation with ratios. This conversation and the one about whether $\frac{4}{1}$ was a fraction and a ratio it focused on how fractions and ratios are alike. In addition, part of understanding when and how to use fractions and ratios is tied to understanding how they are different. Another possible interpretation of Janine's question is that she is not sure if part-part ratios can be compared in the same way that part-whole ratios or ratios that are also fractions can be compared.

As students move into the next unit, *Comparing and Scaling* (Lappan, et al., 2004a), students have to make choices about whether or not to use fractions and ratios. They also begin to name different types of comparisons[5]. For example, when determining which juice recipe was the strongest in the Mixing Juice task, most groups made part-to-whole or fraction comparisons. They may have then represented the fractions as decimals or percents. One group shared that they initially used ratios and then switched to fractions and percents. Their written work showed that they started writing part-part (water to juice concentrate) ratios and then changed to part-whole (juice concentrate to prepared juice)

Carl	We used ratios. It was hard.
Mrs. Dew	[She writes the ratios on the board: 3:2 9:5 2:1 5:3] What was annoying about trying to use them?
Amy	They are not out of the same.
Mrs. Dew	Can you make them out of the same but keep them as ratios?
Amy	You could but it would be hard.

[5] One goal of *Comparing and Scaling* (Lappan, et al., 2004a) is to engage students in making different types of comparisons such as difference, ratio, rates, percents, and fractions. Students make choices about which type of comparison to make in a situation and why it is useful.

Carl	That is why we didn't do it.
Amy	You could do them out of 100 and do a percent.
Mrs. Dew	So you are taking them to a whole and using a fraction. Is there any way I can keep them as ratios or parts and use them?
Janine	See what they are compared to each other. I would still get the same answer.

Mrs. Dew and the students explore ways to compare the ratios using equivalence. The conversation ended with Mrs. Dew asking, "Could I left them as ratios to compare?" and the class responding "yes". While students appear to be comfortable comparing part-whole ratios as fractions they were not comfortable using fraction ideas such as equivalence with part-part ratios. This idea needed continued development before students could see that comparing part-part ratios was appropriate in situations like the Mixing Juice task and that mathematically comparing part-part ratios was done in the same way as comparing part-whole ratios or fractions.

Summary of the Practice of Determining Appropriateness

I have presented six sub-practices within the fraction literacy practice I have named *"determining appropriateness"*. These sub-practices support a set of questions (See Figure 25) that underlie the conversations that these students and teachers engaged in when using fractions to make sense of other mathematical contexts. These questions capture what these students talked about and had to make sense of when trying to use fractions. They provide insight into the way students learn to use fraction knowledge.

1. What is the appropriate way to represent a fraction with a repeating decimal, or one that involves using a decimal approximation, when operating?
2. Is it appropriate to use a decimal representation?
3. Do concepts that work in whole number contexts also work when fractions are used in the same setting?
4. Is it appropriate to use the standard multiplication algorithm here?
5. Can we use fractions to make sense of or think about other mathematical ideas when fractions are not explicit in the problem?
6. Is it appropriate to use equivalent forms (mixed or improper) when solving problems?
7. Are ratios like fractions? Are fractions like ratios?
8. Can fraction equivalence be used with ratios?

Figure 25. Questions underlying classroom discussions.

The various classroom episodes show that the students brought knowledge about fractions to these situations. They recognized equivalent forms and could move from fraction to decimal notation when representing quantities. We have seen examples where the students could use fraction addition and multiplication algorithms. The multitude and variety of episodes involving determining appropriateness, ones that cut across a variety of mathematical content, indicate that students do have questions regarding how and when to use fractions. The conversations highlighted the need to make sense of situations and determine what was situationally appropriate.

The examples show students engaging in a level of conceptual knowing that is reflective (Hiebert & Lefevre, 1986). Students are developing rich relationships between fractions and the new mathematical contexts. The classroom episodes show students making connections between fractions and the contexts in which they are used. The classroom episodes also indicate that the students do not simply take the concepts and skills learned in the fraction units and use them. The students had questions about how to extend their fraction knowledge and when it was appropriate to do so.

The Practice of Connecting Fractions to Multiplication and Division Concepts

In this section I share the fraction literacy practice of connecting fractions to multiplication and division concepts. In contrast to the practice of determining appropriateness which involves a two-way connection between fractions and another mathematical context such as ratio, this practice involves creating a three-way connection between fractions, multiplication and division concepts, and the new context being studied. These conversational episodes, eight in all, occurred less frequently than the episodes associated with the practice of determining appropriateness. They lack the smooth entrance and departure that the determining appropriateness episodes had. The determining appropriateness conversations seem to rise naturally from the problem contexts. The practice of connecting fractions to multiplication and division concepts took place because students were struggling with a concept and/or the teacher wanted to make clear to the students that they needed to attend to aspects of multiplication and division when using fractions with a task. As you read the episodes related to this fraction literacy practice, note the role the teacher plays in initiating and carrying out these conversations.

Prelude: An Overview of Fraction Multiplication and Division

Before presenting findings I provide a brief discussion of the relationship of fractions to multiplication and division. Keep in mind that students have worked with multiplication and division concepts and operations in whole number settings throughout elementary school[6] as well as in the sixth grade fraction units. However, how much work they have done exploring how multiplication and division are related to each other is

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[6] While I cannot say what form elementary instruction took, the district uses and the students participated in Investigations, an NSF-funded inquiry-based curriculum where attention is devoted to development of both conceptual and procedural understanding of multiplication and division of whole numbers.

unclear. Like the previous chapter, because students are expected to and do openly contribute to the development and path of conversations in class, the issues that come to the forefront provide insight into the ways students think and behave as they try to make sense of fractions and the ways in which they are used.

Relating Multiplication and Division. The relationship between multiplication and division is much more complicated than the relationship between addition and subtraction. On the surface this may not appear to be the case. With addition there are two addends and a sum; with multiplication there are two factors and a product. However, what the addends, sums, factors, and products represent are very different. With addition each quantity in the problem has the same type of unit. For example, $2 + 3 = 5$ can be thought of as 2 apples plus 3 apples equals 5 apples. The referent for the addends and sum are each a number of apples.

This is not the case with multiplication and division. In the problem $3 \times 4 = 12$ each number, factors and product, have a different referent. For example, $3 \times 4 = 12$ may represent 3 boxes with 4 chocolates in each box equals 12 chocolates in all. Number of boxes, number of chocolates per box, and number of chocolates are the units represented. The same number sentence can also be represented as three chocolates given to four people. While these are just two of many interpretations for multiplication, division has two primary interpretations–partitive and measurement–sometimes referred to as sharing and grouping. These two types of division are related to the two types of multiplication problems:

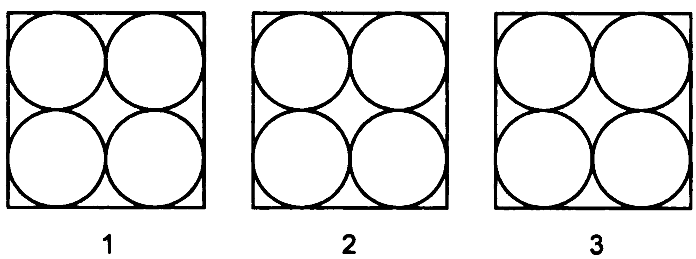
(1) Grouping

3	×	4	=	12	→	12	÷	4	=	3
Unit		Rate		Total		Total		Rate		Unit
Number of Boxes		Chocolates per Box		Number of Chocolates		Number of Chocolates		Chocolates per Box		Number of Boxes

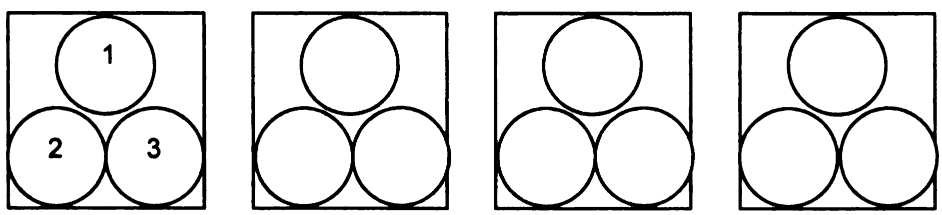
(2) Sharing

3	×	4	=	12	→	12	÷	4	=	3
Rate Chocolates per Box		Unit Number of Boxes		Total Number of Chocolates		Total Number of Chocolates		Unit Number of Boxes		Rate Chocolate per Box

Division is about forming groups. There are two types of groups that can be formed leading to two different meanings for division. You are either forming groups of a certain size or you are forming a certain number of groups. The diagram in Figure 26 shows how the two ideas differ. The grouping problem is asking how many groups of size 4 can be made from 12. The other problem is a sharing division problem. You have 4 boxes and the question being asked is how many will be in a group or box if there are 12 chocolates. You are finding the size of each group.



Grouping: You have 12 chocolates and you put four chocolates in a box. How many boxes can you fill?



Sharing: You have 12 chocolates and four boxes. How many chocolates will you put in each box.

Figure 26. Two representations for division.

Division with fractions is not conceptually different from division with whole numbers. However, the sharing model is much more difficult to represent than the grouping model. Consider the problem $6 \div \frac{1}{2}$ and the grouping situation above. Rather than find how many groups of four, or boxes of 4 are contained within 12, you are finding how many groups of $\frac{1}{2}$ are in 6. There are 12 one-halves in 6. In the sharing model you are forming a certain number of groups. With $6 \div \frac{1}{2}$ you are forming $\frac{1}{2}$ of a group. If $\frac{1}{2}$ represents the number of groups formed, the question is how many are in each whole group. If $\frac{1}{2}$ of a group has 6 in it, then a whole group has 12 in it.

Fractions as a Conceptual Scheme. Thompson and Saldanha (2003) propose that coherent fraction reasoning develops by interrelating several conceptual schemes not often associated with fractions where conceptual schemes are the reasoning or ways of thinking needed to understand situations in particular ways. These schemes are not abstract and involve imagining, connecting, inferring and understanding situations in particular ways. They develop out of understanding situations where fractions are used. They include division schemes, multiplications schemes, and measurement schemes as well as fraction schemes.

> We emphasize conceptualizations of measurement, multiplication, division, and fractions. This is not the same as measuring, multiplying, and dividing. The latter are activities. The former are images of what one makes through doing them. (p. 100)

Thompson and Saldanha contrast their work to Kieren's (1976, 1980) construct theory arguing that such work is unsatisfactory when designing instruction for an integrative understanding of fractions. Rather they argue for placing fraction reasoning within multiplicative reasoning because the motivations for developing the mathematical system of rational numbers did not emerge from meanings or subconstructs but rather from images that are part of understanding fractions relationally (Skemp, 1978) or both conceptually and procedurally (Hiebert and Lefevre, 1986).

Each conceptual schemes focuses on quantitative operations rather than numerical operations (Thompson, 1995). The two schemes that are most relevant to the results of this research study are division schemes and fractions schemes. Division schemes were described in the previous section and involve understanding division as sharing, as grouping, and how the two are related. An important point that Thompson and Saldanha (2003) make regarding how grouping and sharing are related involves understanding that any measure of a quantity induces a partition of it and any partition of a quantity induces a measure of it.

A conceptual scheme for fractions is based on conceiving two quantities as being in a reciprocal relationship of relative size. Thompson and Saldanha (2003) describe this relationship in the following way:

> [The] two amounts in comparison are each measured in units of the other. Saying amount B is seven times as large as amount A is saying that amount B is measured in units of A; saying amount A is $\frac{1}{7}$ is as large as amount B is saying that amount A is measured in units of B. (p. 107)

This conceptualization of fractions also involves conceptualizations of measure (measure as a number of segmented units and comparison where something is 3 times the size of some measure), multiplication (as the making of identical copies) and division (grouping and sharing).

Returning to the Findings: Connecting Fractions to Multiplication and Division

I began this chapter showing that the practice of determining appropriateness involved linking fractions with some aspect of the mathematical context in the unit being studied. For example, fraction and decimal equivalence were linked with fraction addition to make sense of decimal addition. The data presented in this section in turn will provide insight into how students' ability to use fractions is linked with their ability to connect what they learned about fractions, with conceptualizations of multiplication and division, and to connect this to mathematical contexts such as area or similarity. In the episodes presented here, understanding how multiplication and division interacts with using fractions, and the mathematical content being explored, is a critical part of making sense of the instructional task.

Conversations that centered around the fraction literacy practice of connecting fractions to multiplication and division concepts were found in 8 of 26 different instructional tasks across ? of the five units explored in this study. In addition, there will be instances where students engaged in learning to use fractions but in doing so also drew upon practices that were identified with learning about fractions. The underlying questions that characterize the conversations associated with this practice are as follows:

- *Which operation is appropriate here: multiplication or division?*
- *What makes this a multiplication situation? A division situation?*
- *What situations bring out and make use of the inverse nature of multiplication and division?*
- *What situations bring out and make use of the reciprocal nature of fractions?*

These underlying questions indicate that learning to use fractions and learning about fractions are not distinct here. The first question is a derivation of the sub-practice understanding when an operation is appropriate from the practice learning to operate with fractions. The second question is one of the underlying questions that characterized the conversations students had when engaging in the practice of learning to operate with fractions.

However, this is not all that surprising. Learning about some aspects of fractions is more meaningful when done in situations where fractions are used to make sense of other mathematical contexts. This is part of Thompson and Saldanha's (2001) argument that coherent fraction reasoning develops by interrelating several conceptual schemes and that these conceptualizations are images one makes through doing them. For example, consider the case of reciprocals. When students learn about fractions they can be introduced to reciprocals. However, how one would use reciprocals and when they would want to use them makes sense in contexts that give meaning to their use. The scale factor episode that was presented is one example where using fractions provides a powerful context to learn about this relationship *and* how it is used.

One sub-practice of the practice of determining appropriateness involved returning to whole numbers to make sense of relationships that also held in fraction situations. Recall the Storm Shelters problem where students were trying to find the width given a length of $5\frac{1}{3}$ meters and an area of 24 square meters. During the summary of that problem there was a conversation about the relationship between multiplication and division and how recognizing this relationship provided a strategy for finding the missing width in the storm shelter problem. Katie asked how people found the value of the unknown width and whether or not they divided. Mrs. Kay pushed the class to consider whether or not Katie's idea to use division would be appropriate and in doing so brought up the inverse nature of multiplication and division.

Mrs. Kay	I could take 24, divide it into 2 rows and see how many there were in there?
Katie	Yeah.
Mrs. Kay	So I could take 24 and divide by 3 rows and see how many are in each row?

Student A	8
Mrs. Kay	So I can take 24 and divide it by 4 rows to see how many are in each row? So does it make sense that I could do it here [points to $5\frac{1}{3}$] just because there is a fraction there?
	[pause]
Mrs. Kay	Let us say 24 and split it into $5\frac{1}{3}$ rows, and lets see how many are in each row. So that is what you did Katie?

An important aspect of this conversation, one not highlighted in the discussion of determining appropriateness, is that the teacher was linking fractions, area and perimeter, and the relationship between multiplication and division. The teacher, based on Katie's comments, prompted this conversation.

When asked about her approach in an interview Katie explained that she noticed "if you go 24 times 1 that is 24 and you just kind of multiplied them and then if you divided it is the same thing so if you didn't have the number you needed to multiply then that [division] makes sense". TJ shared that when originally solving the problem he did not think of division because to find area you multiply length times width. Ali also shared that she did not think division made sense because when finding area you multiply length times width[7]. This conversation was important in helping students to see that the inverse nature of multiplication and division can be useful in area situations, including ones that involve fractions.

The next set of conversations regarding the relationship between multiplication and division took place in the seventh-grade similarity unit *Stretching and Shrinking* (Lappan, et al., 2004b) and were related to scale factor. This series of related conversations are prompted by instances where students did not differentiate between

[7] Students have only explored and developed a strategy for finding area of rectangles and parallelograms at this point.

fraction multiplication and division. This is in part tied to not seeing how multiplication and division are related inversely as well as the reciprocal nature of each. The data also indicates that students are struggling with what type of situations lead to multiplication and which lead to division. While the fraction learning practice of learning to operate with fractions involved understanding when an operation was appropriate, we see students need support and further experience in this area. This work may seem tied to the practice of determining appropriateness but in these episodes what stood out was that students were struggling with the connection between fractions, the context of scale factor, and how multiplication and division are related to each other.

In this first conversation students were discussing the following warm-up problem:

> If you put a picture of a 2 by 3 rectangle into a copier and pushed 150%, what would happen?

While it never becomes part of the class conversation, in the background you hear a student say that 150% is the "same as multiplying by 1 and dividing by $\frac{1}{2}$." Multiplication by 1 would mean that when you take 100% of something it does not change the size of the object. A scale factor of 1 is equivalent to taking 100%. The fifty percent is related to increasing the size an additional 50%. This should require using a scale factor of $\frac{1}{2}$ or multiplying by $\frac{1}{2}$ rather than dividing by $\frac{1}{2}$.

The next day the teacher started class with the warm-up problem $\frac{1}{2} \times 48$. Ali offers 24 saying "I got that because I remember that multiplying by half is the same as dividing by $\frac{1}{2}$ … Because it is a fraction and you are taking a fraction of a number." Jen disagreed. "That is not right. Dividing by 2 and multiplying by $\frac{1}{2}$ is the same." When

Mrs. Dew asked the class if they were the same or different she got a mixture of responses. Mrs. Dew wrote the problem $48 \div 2$ on the board.

Mrs. Dew	Don't tell me the answer, tell me what that means?
Dan	It means that there are, like you are dividing the larger number in half.
Class	[several begin to talk at once]
Lori	You are taking 48 and sharing it in 2 groups.
Mrs. Dew	And when I get my answer what does the answer tell me?
Lori	How many are in each group of 2.
Mrs. Dew	So what does this one say? [Writes $48 \div \frac{1}{2}$ on board.]
Student A	How much you are dividing by.
Ali	What we are dividing by.
Mrs. Dew	Is that [$48 \div 2$] the same as that [$48 \div \frac{1}{2}$]?
Class	[Students begin to talk to each other with some saying yes and others saying no.]
Carl	You are confusing us. Just tell us.
Mrs. Dew	I don't think so. Is $48 \div \frac{1}{2}$ the same as $48 \div 2$?

One student talked about $48 \div \frac{1}{2}$ as $\frac{1}{2}$ of 48 and another suggested that they wanted to know how many halves were in 48. This led to a debate about whether or not the solution to $48 \div \frac{1}{2}$ was 24 or 96. Katie offered 24 saying that two halves make a whole and there are 24 twos in 48.

Mrs. Dew	How many halves are in 48? Let's try [drawing] that.
Janine	No it wouldn't. Two halves make a whole. It would have to be 96.
Dan	It would have to be 96.
TJ	Yeah, you are right. It would have to be 96
Mrs. Dew	Why can't it be 24?
TJ	Because 24 times 2 is 48 so that well 24 times 1 is 24 and 24 times $\frac{1}{2}$ is 12 but if you do 48 times $\frac{1}{2}$ its 24 so half times 96 is 48.
Mrs. Dew	Is 48 divided by $\frac{1}{2}$ the same as 48 divided by 2?

Again, some students said yes and others said no. At this point Mrs. Dew had students get paper and tried to redirect using smaller numbers ($6 \times \frac{1}{2}$ and $6 \div \frac{1}{2}$) so that it would be easier to draw diagrams. During the conversation $6 \div 2$ was characterized as "3 goes into 6 two times" and "How many 2s are in 6". Mrs. Dew asked, "If there are 3 twos in six, are there 3 halves in six?" Students said no there were 12. Numerous times throughout the conversation Mrs. Dew reposed the question of whether or not multiplying by $\frac{1}{2}$ and dividing by $\frac{1}{2}$ were the same. Some students continued to say they were.

Janine	I think it, $48 \div \frac{1}{2}$ is 24.
Mrs. Dew	Why?
Janine	Because I think that they are the same answers but they mean different things. $48 \div 2$ is [96] and 48 divided by $\frac{1}{2}$ that is like you are cutting it in half and you will have two parts and so 48 in two parts, since they are equal you will have 24. But for 6 divided by $\frac{1}{2}$ I don't think it is 12. I think it is 3. They mean different things. Like 6 divided by 2, that means 2 groups of 3 is 6 but 6 divided by $\frac{1}{2}$ with fractions, you wouldn't you put it that way because it is wrong. I am not sure.

Here we see that Janine is using one interpretation of division when dividing by 2 and another when dividing by $\frac{1}{2}$. She realizes that she is thinking about each case differently but is not sure why this is happening. This issue comes out when drawing diagrams (See Figure 27) on the board to represent $6 \div 2$ and $6 \div \frac{1}{2}$.

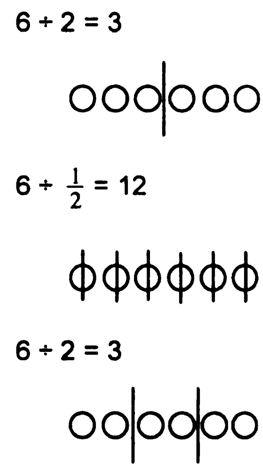


Figure 27. Board work.

Mrs. Dew	If I have 6 things and I want to divide it by 2 and I want to show it on my picture, how would I do that?
Ali	Make two groups out of the big group.
TJ	Two groups would be the same size.
Mrs. Dew	Make two groups.
Ali	But they have to be equal groups out of the set so make it three.
Student A	So three and three, cut it down the middle.
Mrs. Dew	So I am gonna divide this into two equal groups of three each. Like that?
Class	Yeah
Mrs. Dew	So this picture will help me understand this [$6 \div 2 = 3$]. What will help with this [$6 \div \frac{1}{2}$]? How can I divide this by $\frac{1}{2}$?
Tara	Draw a line in each circle which represents a half of each.

[There are lots of objections.]

Student B	It seems to not be the same as the top one. No, you want to get half of the whole thing.
Bryan	If you had six soccer players you would not cut them in half.
Mrs. Dew	What if we tried to make our pictures say the same thing. [Drew six more circles.]

Here we see the students struggling with the two interpretations of division, grouping and sharing, at play in this fraction setting. At this point they drew the third picture in Figure 27. One student said that it showed thirds. Another student suggested that they vote. Many objected to voting saying that it would not explain why. With about 25 minutes of conversation having past, and the students becoming frustrated, Mrs. Dew reluctantly had the students get a calculator and find the solutions[8]. The students were surprised to find

[8] Mrs. Dew shared her reluctance with me in a conversation saying that she did not want the students to think that the calculator justified the answer. She wanted them to be convinced of what the answer was and be able to explain why.

that $48 \div \frac{1}{2}$ is 96. They could not explain it. Mrs. Dew decided to stop the conversation. She gave the students four problems ($8 \div 2$, $8 \div \frac{1}{2}$, $10 \div 2$, and $10 \div \frac{1}{2}$) for homework and asked them to draw pictures to show their reasoning and solution.

The class returned to their work with scale factor. The students were discussing the problem in Figure 28. They were asked to find the missing lengths and angles knowing that the rectangles were similar. When the students moved from the small (side length of 3) to the large rectangle they said the scale factor was two. However when going from the large (side length of 9) to small rectangle they said divide by 2.

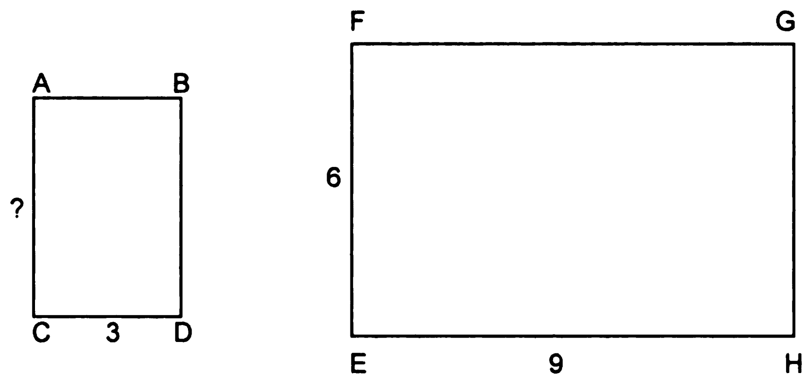


Figure 28. Similar rectangles.

Mrs. Dew	If my scale factor had to be something that I multiply by, if you are saying that I am going to divide by 2, but I wanted to say my scale factor is what I multiply my side lengths by. What would I be multiplying by?
Josh	2
Mrs. Dew	You would multiply 9 by 2 to get the other side?
Josh	No you would divide by 2.
Mrs. Dew	Okay I don't think anyone disagrees that if I divided by 2 I will get $4\frac{1}{2}$. If I divided by 2 that is the same as multiplying by what?

Several students responded with two. Mrs. Dew points out that $9 \div 2 = 4\frac{1}{2}$ and that scale factor is the number that dimensions are multiplied by.

Mrs. Dew	If I am dividing by 2 I can say, well, what am I actually

	multiplying by to get $4\frac{1}{2}$?
Nick	$\frac{1}{2}$
Mrs. Dew	Nick says it is the same as multiplying by $\frac{1}{2}$.
Class	[There are several who agree and several who do not.]
Amy	I am confused.
Mrs. Dew	That is why when Carl said are we spending another 40 minutes On this [dividing by 2 and dividing by $\frac{1}{2}$], it is part of this unit. I have to know what is going on with these dimensions and if I am multiplying or dividing.
Nick	Man, we have got issues.

At this point they laugh and leave this conversation for the next day when they share their diagrams and reasoning regarding the four homework problems. However, Mrs. Dew's point is important. Understanding when you are multiplying, when you are dividing, and how the two are related is part of similarity and scale factor. In this example, using fractions is linked with ideas such as multiplication and division as inverse operations, both interpretations of division, and reciprocals.

The next day when the students shared the diagrams they drew for homework they again had a sense that the way they were modeling whole number division was different from the way they were interpreting the fraction division situation. The students were using the sharing model when dividing by two and the grouping model when dividing by $\frac{1}{2}$. Janine commented that the diagrams were not representing the same model. Mrs. Dew talked to them about the two interpretations for division making each explicit. At the end of this conversation the class concluded that multiplying by $\frac{1}{2}$ and dividing by $\frac{1}{2}$ were different. They also saw that multiplying by $\frac{1}{2}$ had the same outcome as dividing by two.

There were three subsequent places in the data where scale factor was related to division and multiplication. In each case students were prompted by either the teacher or another student to attend to the reciprocal nature of fractions. One such example took place at the very end of data collection during the *Comparing and Scaling* (Lappan, et al., 2004a) unit on ratio, proportions and percents. Students were given a recipe for making orange juice by mixing 2 cups concentrate with 3 cups water. They were asked to consider how much of each was needed to make one cup of juice. At the beginning of the class conversation ideas were offered but none of them involved maintaining a 2 to 3 ratio of concentrate to water. Once it was clarified that the 2 to 3 ratio had to be maintained a student said, "If you have a whole measuring cup and it had fifths in it and if it is $\frac{2}{5}$ of concentrate and $\frac{3}{5}$ of water it is the same thing. You are using the same fractions but it is out of a cup instead". In the conversation that followed several interpretations of fractions came out including those described by Kieren (1976, 1988) and by Thompson and Saldanha (2003).

Carl	You have 60% of water and 40% of concentrate. Can't you do 60% of the water cup and 40% of the concentrate cup?
Janine	Just do $\frac{2}{5}$ and $\frac{3}{5}$.
Mrs. Dew	Why does $\frac{2}{5}$ and $\frac{3}{5}$ make sense to you, Janine?
Janine	$\frac{2}{5}$ is the same thing as 40% and $\frac{3}{5}$ is the 60%. It has the same denominator. If you make it out of one cup you are still going to get the same juice. Because 5 cups would be like a cup each and you could scale it down and do it out of one cup.
Mrs. Dew	All together it makes five cups. If I want to take five cups and scale it down to one, what would be my scale factor?
Dan	1
Student A	5
Student B	5
Mrs. Dew	Be careful if you say five I am thinking to multiply by five.

Mrs. Dew prompted students to pay attention to the role of multiplication and division, and their relationship to each other, when working with scale factor. In this conversation we also see that the students linked the part-part ratio of 2 to 3 with the part-whole relationship of $\frac{2}{5}$ and $\frac{3}{5}$. Students refer to the part-whole ratio written in fraction form as a fraction. Janine's point about using $\frac{2}{5}$ of a cup of water and $\frac{3}{5}$ of a cup of water is a different interpretation of $\frac{2}{5}$ and $\frac{3}{5}$. Here she is thinking of the fraction as a number or quantity rather than a relationship. As the conversation continues you will see that Janine is not sure of her conjecture but is trying to determine the appropriateness of her reasoning. In doing so she attended to the reciprocal relationship between multiplication and division when determining scale factor.

Janine	No. It is point five. Wait no.
Nick	That is a half that is point five. It is point 2.
Mrs. Dew	Or $\frac{1}{5}$?
Janine	Yeah.
Mrs. Dew	If I multiply by $\frac{1}{5}$ I will be down to one cup. If I multiply my total by $\frac{1}{5}$ what am I going to do to my parts?
Janine	Divide by 5.
Mrs. Dew	Divide by five or multiply by $\frac{1}{5}$. So if my whole thing is scaled down to 1 cup then each of these [points to 2 cups concentrate and 3 cups water in the recipe], have to be scaled down in the same way or it is not going to taste the same. So if I divide by 5, isn't that what these are saying? [Points to $\frac{2}{5}$ and $\frac{3}{5}$] Two divided by five and 3 divided by 5.
Janine	Is $\frac{2}{5}$ and $\frac{3}{5}$ in cups the answer?
Mrs. Dew	Will this make one whole cup? $\frac{2}{5}$ of one and $\frac{3}{5}$ of the other?

In this discussion we can see several key ideas coming together. In the fraction units where students learned about fractions they studied fraction–decimal and fraction–

percent relationships. They also worked with fractions as measures, as indicated divisions and as numbers[9]. Students also have to understand $\frac{2}{5}$ and $\frac{3}{5}$ can represent a relationship or ratio and a number. The connection between these two uses is supported by the use of scale factor and understanding the reciprocal nature of multiplication and division. This problem provided a window into the multitude of ideas that students must pull together to make sense of when using fractions including an understanding of the relationship between multiplication and division as it is related to fractions.

Summary: Connecting Fractions to Multiplication and Division

When looking across the data and considering what is involved in using fractions the important role of multiplication and division concepts was very clear. The data supports Thompson & Saldanha's (2003) theory that coherent fraction reasoning develops by interrelating several conceptual schemes not often associated with fractions and that this reasoning develops out of understanding situations where fractions are used. While some of these schemes and concepts, for example reciprocal, were developed when learning about fractions these settings provided a place in which to make sense of that knowledge and develop a much deeper understanding of it by exploring situations in which is it used. These findings also support Vergnaud's (1983, 1988) work that situates understanding rational number within understanding the multiplicative conceptual field.

In the episodes presented here, understanding how multiplication and division interacts with using fractions, and the mathematical content being explored, is a critical part of making sense of the instructional task. Initially one may think that students

[9] Students look at Janine's proposed idea in another problem with 4 cups concentrate and 7 cups water. A student wanted to rename the fractional part of a cup as a decimal the teacher is concerned that students do not see 4/11 as a number. She says to the class, "I guess what is bothering me is that when you see this you feel the need that you have to do something else. Isn't 4/11 a number?"

engagement with these tasks are a case of application the data associated with this practice indicates that students are learning new ideas about fractions and the way they are used. For example, students could study how fraction multiplication and division are related when studying about fraction operations, but the settings where this usage of these ideas occurs provides the most meaningful situations for understanding that use.

As with the practice of determining appropriateness, in these situations students were engaging in a level of conceptual knowing that is reflective (Heibert & Lefevre, 1986). Relationships were developed between fractions, the new context, and multiplication and division. Unlike the practice of determining appropriateness where students had questions about how to extend their fraction knowledge and when it was appropriate to do so, the data shows that this fraction literacy practice is much more teacher-driven. It highlights the importance of teacher's having a strong understanding of how fractions are used in other settings and the way this use of fractions involves making connections among a multitude of concepts—concepts that extend beyond what students learned in the fractions units.

Chapter Summary

This chapter addressed my second research question regarding what are the practices students engage in when learning to use fractions. I presented two fraction literacy practices—the practice of determining appropriateness and the practice of connecting fractions to multiplication and division concepts. The discussion of each fraction literacy practice involved highlighting questions that supported the underlying focus of conversations where fractions were used. In the next chapter I will reflect upon the fraction literacy practices presented in this chapter and the fraction learning practices that were presented in Chapter Four. I will discuss my second research question of how these two types of practices are similar and different. I will also discuss why these differences and similarities are important. This will lead to a concluding chapter where I discuss implications and an agenda for future work.

Chapter Six

LEARNING ABOUT FRACTIONS AND LEARNING TO USE FRACTIONS: IS THIS DISTINCTION NOTEWORTHY?

The broad purpose of this study was to understand how students learn to use fraction knowledge in mathematical instructional settings where fractions are not the main focus of study, but rather support the development of other mathematical content. In the last two chapters I described the fraction practices sixth-grade and seventh-grade middle school students engaged in when learning about fractions and when learning to use fractions. The analysis of this study's data indicated that these two fraction practices are indeed quite different. In this section I discuss the main differences in order to argue that providing students with the opportunity to learn to use fraction knowledge is crucial to the development of fraction literacy.

Comparing Fraction Learning Practices with Fraction Literacy Practices

To begin the discussion about the differences between the two kinds of learning practices explored in this study I draw attention to the questions that captured the nature of the conversations that took place in each of the two types of curricular units. The questions associated with the practice of learning to use fractions are different in nature from the questions underlying the practice of learning about fractions. The first two practices in Table 8 are fraction learning practices students engaged in when learning about fractions. The associated questions focus on issues such as "How can we …?", "Why can we ...?" , "What techniques can we use to …" and "Why do they work?" The last two practices in Table 6.1 are the fractions literacy practices students engaged in when learning to use fractions. The associated questions focus on issues such as "Is it appropriate to …?", "Can we use this idea to …?", and "What situations lead to the use of …?"

Table 8
Questions Associated With the Practices of Learning About and Learning to Use Fractions

Practice	Underlying Questions
Learning About Fractions	
Understanding Fractions as Quantities	How can we represent concepts such as parts of a whole? Why can there be different names representing the same fractional quantity? How can we determine when two names refer to the same quantity? How can we determine which of two fractions is greater or smaller? How can equivalence and benchmarks help determine what a fraction, decimal or percent represents? How is a decimal like a fraction? How is a percent like a fraction? What techniques can be used to find fraction, decimal and percent names for the same quantity?
Learning to Operate With Fractions	What strategies can be used to add fractions? Subtract fractions? Multiply fractions? Divide fractions? Do these strategies lead to a reasonable result? Why do the strategies work? What makes this an addition situation? Subtraction situation? Multiplication situation? Division situation?
Learning to Use Fractions	
Determining Appropriateness	What is the appropriate way to represent a fraction with a repeating decimal, or one that involves using a decimal approximation, when operating? Is it appropriate to use a decimal representation? Do concepts that work in whole number contexts also work when fractions are used in the same setting? Is it appropriate to use the standard multiplication algorithm here? Can we use fractions to make sense of or think about other mathematical ideas when fractions are not explicit in the problem? Is it appropriate to use equivalent forms (mixed or improper) when solving problems? Are ratios like fractions? Are fractions like ratios? Can fraction equivalence be used with ratios?
Connecting Fractions to Multiplication and Division Concepts	Which operation is appropriate here: multiplication or division? What makes this a multiplication situation? A division situation? What situations bring out and make use of the inverse nature of multiplication and division? What situations bring out and make use of the reciprocal nature of fractions?

Broadly speaking, the fraction learning practices often involved understanding how, why and what. For example, in the first fraction unit students explored how to determine if two fractions were equivalent, why there can be multiple names for a fractional quantity and what strategies could be used to find equivalent quantities. In the second fraction unit questions such as what strategies can be used to operate with fractions and why do they work were the focus. In contrast, appropriateness of use was a strong theme in the conversations where students engaged in fraction literacy practices. Students were trying to decide how to use what they knew about fractions in a particular situation. In turn, the conversations on learning to use fractions were less focused on how to carry out the fraction operations needed to find the answer. In most cases students did not have questions regarding concepts and procedures they learned in the fraction units.

Another way to conceptualize the difference between fraction learning practices and fraction literacy practices is by considering the nature of the conversations associated with each practice. The conversations students had as part of their engagement in learning to use fractions were more likely to focus on quantitative operations, or ways to reason and imagine a situation, than on numerical operations such as addition or multiplication (Thompson, 1995). While some conversations eventually led to the use of numerical operations, the conversations did not focus on how to add or multiply fractions. Rather they focused on whether the use of a particular fraction concept or procedure would be appropriate in a particular setting. For example, when students were trying to distinguish between $48 \div \frac{1}{2}$ and $48 \div 2$ they were more focused on understanding and creating a representation for the problem than carrying out a numerical procedure.

In contrast, when students were learning about fractions they were likely to focus on both quantitative and numerical operations. An important goal when learning about fractions was to understand operations quantitatively as well as numerically. An example of this can be found in the practice of understanding fractions as quantities. For example, there were conversations that focused on understanding equivalence quantitatively. These are captured in the underlying question: "Why can there be different names for the same fractional quantity?" There were also conversations that focused on strategies for finding equivalent fractions. Some of these strategies included the use of numerical operations. These conversations are represented by the question: "How can we determine when two names refer to the same quantity?"

One area of overlap between fraction literacy and fraction learning practices can be found in students' attempts to understand when an operation was appropriate. *Understanding when an operation is appropriate* was a sub-practice of the practice learning to operate with fractions. This is similar in nature to two of the underlying questions that supported the fraction literacy practice of *connecting fractions to multiplication and division concepts*: (1) Which operation is appropriate here: multiplication or division? and (2) What makes this a multiplication or division situation? For example, when students were discussing whether $48 \div \frac{1}{2}$ was the same as $48 \div 2$ and if $6 \times \frac{1}{2}$ and $6 \div \frac{1}{2}$ was the same they were working on ideas that had their origin in the fraction learning subpractice of *understanding when an operation is appropriate.* The difference between the two practices became evident when students then linked this work to their work with scale factor and the role of reciprocal relationships. This extension of ideas created an additional connection to a situation that would not have been possible in the fractions units. And as indicated by the association of these classroom episodes with the practice of *connecting fractions to multiplication and division concepts*, students were making connections between fractions, the scale factor context and multiplication and division concepts.

The difference between the fraction learning practices and the fraction literacy practices is also apparent when one considers the answers to the questions that underlie each practice. The fraction learning practices lead to skills, concepts and procedures commonly associated with the study of fractions in middle school curricula. In Chapter One I offered that learning about fractions includes but is not limited to comparing and ordering fractions, developing meaning for and connections among different forms of representations (fractions, decimals, and percents), equivalence, the role of the numerator and denominator, benchmarks and estimation, and operations with fractions. Therefore it is no great surprise that the underlying questions and their answers direct us back to what one would expect students to learn about fractions.

In looking back at the questions that characterized the practice of *determining appropriateness* and considering what can be learned by answering these questions one can see that this practice entails learning about the differences among number systems and how to move among them. To illustrate consider the example around the underlying question of whether concepts that work with whole numbers work when fractions are used in the same setting. Here students wrestled with the question of whether they could use fractions in situations where they were not explicit. While the connections between fractions, decimals, and percents were studied in the fraction chapters, once outside of the fraction units students were not sure if or even when this connection was appropriate to use fractions in place of other equivalent representations. For example, could they use fraction form to represent quantities in decimal form when adding and subtracting decimals. A second example occurred when students were discussing which situations required a particular representation such as a fraction rather than decimal as was the case with $\frac{1}{3}$ and 0.333... during the Storm Shelter task. These two tasks and the conversations they generated made it possible for students to learn that in some situations, such as adding decimals, one could be flexible and choose either the decimal or fraction form of representation but in other situations fraction form is preferable.

The *NCTM Principles and Standards* (2000) indicate that understanding the difference between fractions, decimals, and percents is at the heart of flexibility with rational numbers. But they go on to say that once student have learned to generate and recognize equivalent forms, as occurred in the fractions units in this study, they also need to build on and extend this experience so they can become fluent in using fractions, decimals and percents meaningfully. The results of this study reveal that students do indeed need opportunities beyond work in the fraction units to build meaning for fractions. For these students having to use fractions when studying other mathematical content led to questions regarding appropriateness and use that did not surface when fractions were the focus of instruction. As Vergnaud (1988) points out, "A single concept usually develops not in isolation, but in relationship with other concepts, through several kinds of problems and with the help of several wordings and symbolisms" (pp. 131-2). As the students in this study engaged with fractions in other mathematical contexts the analysis of their class conversations indicated that further learning was taking place.

Beyond the learning of content, the inquiries these students made also indicate they were developing the disposition to evaluate a situation and make choices for that situation. NCTM (2000) indicates that middle grade students need experiences with number that help them learn to make choices by considering the particular context, the question asked, and the numbers involved. Should we change the fraction to a decimal or do we need to operate with the given fractions? Can we use fractions in this situation? Which algorithmic approach would be most reasonable in this setting?

An important highlight of this work involves developing situational understanding in conjunction with mathematical understanding. Being able to situate the concepts and skills learned in the fraction units into new contextual and mathematical settings was at the center of many of the conversations presented. This supports Thompson's (1995) theory that making sense of number involves understanding the various contexts in which number is used. It is through making sense of the many situations and coming to realize the potential for using fractions that students develop ways to employ them and use them as a tool.

As Sowder, Philipp, Armstrong, and Schappelle (1998) suggest, number and operation are resources one uses to navigate an enviroment. In the fraction units students were learning about fractions and developing a set of tools they could use to make sense of settings where fractions were used. Through work in these other settings students were learning how to use the tools as resources. The data indicates that students do not simply take the ideas they learned about in the fractions units and use them. They have questions regarding their use. The opportunities these students had to learn to use fractions and the conversations they and their teachers engaged in were critical in helping students learn to navigate in settings where fractions were used. This finding supports a practice account of literacy (Scribner & Cole, 1981), in this case fraction literacy, where literate use of fractions develops out of understanding situations where fractions are used.

The Role of Connections When Learning to Use Fractions

The conversations associated with fraction learning practices indicated that this work is rich in connections. The fraction learning practice of *understanding fractions as quantities* and the fraction learning practice of *learning to operate with fractions* involved interrelating many ideas including symbolism, contexts or situations, and models. Questions such as what strategies can be used to operate with fractions and why they work are important questions to explore when learning about fractions. I would argue that much of the conceptual work that these students and their teacher engaged in when learning about fractions occurred at the primary level (Heibert & Lefevre, 1986) where information is connected at the same level of abstractness and tied to a specific context. The connections students made within each practice where they learned about fractions were primarily about fractions.

However, this research indicates that when students learn to use fraction in other contexts that they develop a much richer understanding of fractions that extends beyond fractions. The connections that these students developed were at Hiebert and Lefevre's reflective level where students step outside a single context and connect two different contexts in a new way that leads to a deeper understanding of both. For example, using fractions within the context of area and perimeter led to a deeper understanding of both. Not only did the many students who used guess and check with multiplication to find the missing width in the Storm Shelter task learn that they could use division to find the missing value in this problem, two of the focus students indicated in an interview that they did not consider division because in their mind finding the area of a rectangle involved multiplication. When a student suggested that fractions could be used to add decimals, the resulting conversation and work led to an understanding of how fractions, decimals, place value, addition and subtraction where all related. This research highlights that what students learned in situations where fractions were used was not only different but above and beyond what they learned about fractions. It indicates that there is much to be learned beyond the concepts and skills studied when learning about fractions.

Beyond content students are learning that they need to "be responsible for what they have learned and for using that knowledge to understand and make sense of new ideas" (NCTM, 2000, p. 64). NCTM goes on to point out that students need to develop the disposition to look for and use connections. When middle school students are given opportunities to learn to use fractions in new mathematical contexts it provides a foundation for work in grades 9-12 where students need to take advantage of and use connections. As students begin to look for and use connections they will come to view mathematics as a collection of connected useable ideas rather than a set of arbitrary rules.

This study indicates that as students try to take existing knowledge about fractions and use it they need support in doing so and they have questions regarding what is appropriate. Schoenfeld (1986) points out the following regarding the business of mathematics, "Much of the intrinsic power of mathematics comes from perception of the structure–from seeing connections and exploiting them" (p. 259). The questions that underlie the conversations associated with the fraction literacy practices I have reported here indicate that the students, with encouragement and guidance from their teacher, are trying to exploit what they know and are starting to see that the possibility of acting (Mason & Spence, 1999) exists.

Summary: Does it Matter?

The title of this chapter is posed as a question. Is the distinction between learning about and learning to use noteworthy? My response to this question is yes it does matter. This research indicates that providing students with opportunities to learn to use fractions leads to further learning that includes an awareness of the need to use the ideas they have learned in new settings. As students engaged in tasks where they had to use fractions it was evident that they brought a great deal of knowledge about fractions to the problem situations. They also developed a great deal of knowledge as they worked on the tasks. The experiences these students and their respective teacher had when learning to use fractions led to conversations and learning that went beyond what they learned about fractions in the fraction units.

Chapter Seven

POTENTIAL AND POSSIBILITY

This research adds to the empirical literature on fractions and fraction learning by studying learning from a different position than is typical. Rather than focus on what students know or understand about fractions this study focuses on what it involved in learning to use fractions. This study examines how students learn to use fractions by examining points in the curriculum where fractions are not the focus of instruction.

The NCTM Standards (1989, 2000) have suggested that students learn to apply prior knowledge in increasingly more difficult situations and that they should be responsible for using what they have learned to make sense of new ideas. This study revealed that students can learn substantial mathematics in the area of fractions, beyond what is typically found in units on fractions, when fractions are used in other mathematical contexts. Using fractions did not come naturally to the students featured in this study. However, as they engaged in fraction literacy events in their mathematics classrooms students were beginning to understand various ways in which fractions could be used.

As the students in this study tried to make connections between fractions and the contexts in which they were used, they needed support as well as the opportunity to engage in a thinking process where they had to make decisions. As students engaged in the two fraction literacy practices they were developing connections between what they knew about fractions and the new contexts, and in some cases between fractions, a new context and multiplication and division concepts. Being able to situate the concepts and skills learned in the fractions units into new mathematical settings was at the center of many conversations. It was through making sense of the many situations and coming to realize the potential for using fractions that students were developing ways to employ them and use them as a tool. In order to act upon knowledge about fractions students had to make sense of new situations, they had to draw upon and use previous knowledge, and they had to develop new knowledge about how old ideas, new ideas, and contexts are related. Determining appropriate uses for fractions in specific situations was a common theme in classroom conversations.

In Chapter Two I defined fraction literacy by describing what one should be able to do both from a mathematical perspective and a discourse perspective. From a mathematical perspective I included three elements by drawing upon the NCTM's (2000) Number and Operations Standard.

- Understand numbers, ways of representing numbers, relationships among numbers, and number systems;
- Understand meanings of operations and how they relate to one another;
- Compute fluently and make reasonable estimates.

The results of this research support the importance of providing students with opportunities in their mathematical experiences where they use fractions in order to allow for this type of development. As students engaged in fraction literacy events their conversations and questions revealed that the first two elements were crucial when trying to make sense of situations where fractions were used. The third element involving computation and estimation did not really surface explicitly in the conversations that focused on learning to use fractions. While students did raise questions about when to compute or estimate, or whether it was appropriate to do so, their conversations did not center around how to compute or estimate.

The second part of my definition of fraction literacy involved knowing about fractions and using knowledge about fractions to achieve goals for oneself and to communicate with others. Engaging with multiple contextual situations where fractions are used supported the enculturation of students into ways of thinking and acting that are important when using fractions. When studying scale factor and the role of reciprocals a student pointedly asked what scale factor had to do with fractions. As the role of fractions became apparent when the reciprocal nature of scale factor emerged, the students as a whole were experiencing what role fractions could play in another mathematical content area. Opportunities such as these were not apparent when students learned about fractions.

In this study the questions associated with the fraction literacy practices of determining appropriateness and connecting fractions to multiplication and division concepts differed from those associated with fraction learning practices. The fraction literacy questions represent ways of acting, thinking and being that are automatic for a person who has acquired and mastered the discourse of using fractions. While these students have not mastered or acquired this discourse, they were supported in this effort as they actively engage with situations where fractions are used. Gee (1992) argues that acquisition is not a product of overt teaching and conscious attempts to learn something. Rather, acquisition happens in natural settings where the person knows they must acquire it to be functional. This work argues for the important step of providing students with the opportunity to be in a position to acquire the ability to use fractions.

Additional Considerations and Limitations

A factor that contributed to the outcomes of this research was the type of tasks that students engaged in. Earlier I argued that these tasks were substantial problems for students to engage in. They were more than simple application problems. The problems identified for data collection were complex and asked students to use their fraction knowledge but at the same time to add to their knowledge base for fractions. For example, while you can teach students about reciprocals as part of learning about fractions, understanding how they are used and where reciprocal relationships occur is a topic that can be addressed outside of direct study of fractions. Streefland (1991) refers to this interaction of knowledge as intertwining lines of learning. Rather than singling fractions out as a topic of study, one might also consider how fractions are situated within other mathematical contexts. While I am not suggesting that students not directly study fractions, this work does suggests that as we design courses of study we carefully consider what aspects of fractions are appropriate as part of learning about fractions and what aspects of fractions are better served by introducing them in mathematical contexts where they are used.

Related to Streefland's notion of intertwined learning lines was an unanticipated but interesting observation made during data collection involving the interaction between fractions and the new mathematical context. Often the use of fractions in the new context provided opportunities to observe how well students were engaging with the mathematical context they were learning about. For example, the Storm Shelter task revealed that some students chose to use multiplication as part of a guess and check approach to find the missing dimension because finding area involved multiplication. They did not think division was possible.

Another example occurred when finding the area of a $3\frac{1}{2}$ by $4\frac{1}{2}$ rectangle represented on a grid. The dimensions were not labeled leaving students to determine what quantities were needed to find area and perimeter. Many students counted the total number of square units to find area. In many cases they counted the 12 whole square units, the 7 half-units and one quarter-sized unit to get a total area of $15\frac{3}{4}$. After finding area the conversation shifted to finding perimeter. When showing on a diagram of the rectangle on the grid what they were measuring and counting to find perimeter many students counted the partial $\frac{1}{2}$ by $\frac{1}{2}$ square unit with an area of $\frac{1}{4}$ as a square with dimensions of $\frac{1}{4}$ by $\frac{1}{4}$. Students were focusing on the numbers or quantities associated with measures they identified for finding area. They did not take into account what the quantities where a measure of. When the students switched to finding perimeter they did not attend to the change in unit being used. Compare this to a gridded rectangle with whole number dimensions of 3 cm by 4 cm. In whole number situations it may appear that students understand what type of measurement unit is being used because the numbers used to find the area and perimeter are the same. The type of unit and what is being measured is different for area and perimeter, but the numbers used to operate are the identical. If students only work with simple units of measure such as whole number units and they do not use more complex units the teacher may assume that students understand what they are measuring with area and with perimeter. However, in real life most area and perimeter situations are not simple whole number measures.

Hiebert and colleagues (1996) point out the importance of allowing the subject matter to be problematic. The types of tasks students engaged in when learning about fractions as well as when learning to use fractions led students to engage in sense-making. The role of an inquiry-oriented curriculum is important however I also argue that the situations students engaged in when learning to use fractions in this study pushed beyond the work commonly thought of when designing fraction instruction—even when inquiry oriented. It is not uncommon for teachers as well as curricula to avoid the use of messy numbers such as fractions when learning about area and perimeter or ratios. This data shows that students need support when they need to use fractions and that the rigor created by their use helped students develop a more complete conceptual understanding of the mathematics they were studying.

The role of fractions in the development of multiplicative and proportional thinking is one that needs further exploration. The data indicated that students were trying to make sense of the relationship between ratios and fractions. In particular, equivalence raised numerous issues. More analysis of this data as well as further data collection extending beyond what was done in this study would be beneficial to understanding the interaction between fractions and ratios and the role this interaction plays in the development of multiplicative and proportional reasoning.

An additional contributing factor to the results of this study is the pedagogical beliefs of the teachers and how this influenced their implementation of the curriculum. The willingness of these teachers to let students engage in the unknown and struggle to make sense of ideas as well as the very deliberate way they did this was an important contribution to the data available to collect. Both teachers allowed students to explore ideas they were uncomfortable with and to state when they were uncomfortable. They students talk through ideas and were quiet so that other students could react to other students. If data was collected in another classroom where teachers intervened earlier or student participation was less common, the practice of determining appropriateness may not have surfaced.

Other mathematics curricula have restructured the traditional scope and sequence used to teach fractions in sixth through eighth grade. The CMP II curriculum is structured by teaching a topic formally and then continually revisiting it in other mathematical contexts. It would also be beneficial to study a variety of curricular trajectories. For example, how students learn to use fractions in settings where students both use ideas in other contexts and learn about fractions in sixth, seventh, and eighth grade. Research on how students learn to use fractions can also be informed by studying fraction use in high school mathematics settings where fractions are most often not studied formally.

This research is limited in that it only looked at two classrooms where students learn to use fractions. It only looked at some of the units in the CMP curriculum where students use fractions. There are other seventh-grade units and eighth-grade units where fractions are used as well. This study and its data result from studying one curricular setting in one school and only drew from two classrooms within that one school. These results do not generalize to other classrooms. These results are particular to this setting. The identified fraction learning practices and fraction literacy practices may vary or not exist in other classrooms. However, this research does provide a case of what could happen. There is strong evidence that engaging students in situations where they learn to use mathematics is important and beneficial.

The focus of this research on studying how students "learn to use" knowledge is also limited in that it only examines fractions and the development of fraction literacy. Research that looks at how students learn to use other mathematical contents would be needed to develop a clearer picture of how students learn to become literate users of mathematics. In addition, studying how students "learn" to use mathematics does not completely answer the question of whether or not students actually do develop into literate users of mathematics.

A Closing Thought

In closing I would like to return to an idea that this study began with–mathematical power. The intention of the goals and vision laid out by NCTM (1989, 2000) is to develop mathematically literate students where becoming literate means that a student's mathematical power should develop. Not only should mathematics instruction focus on developing concepts and skills, it should be done in such a way that students see mathematics as useable. The results of this study provide an argument for how instruction could be structured to not only develop fractions ideas as useful, but as a set of concepts and procedures that students can engage with and use.

The conversations that the students in this study participated in when learning to use fractions were different in nature from conversations characteristic of learning about fractions. These conversations indicated that the students engaged in practices that involved actively trying to connect what they learned about fractions to new mathematical contexts. By providing opportunities for students to learn to use what they have learned about fractions, and recognizing the need to support students as they engage in practices where they learn to use fractions, there is the possibility that they can become literate users of fractions.

APPENDIX A

DEVELOPMENT OF FRACTIONS AS OUTLINED IN THE NCTM PRINCIPLES AND STANDARDS NUMBER STRAND

Grade Band	Expectations for NCTM (2000) Number & Operations Standard
Pre-K–2	• Understand and represent commonly used fractions, such as $\frac{1}{4}$, $\frac{1}{3}$, and $\frac{1}{2}$.
3–5	• Develop understanding of fractions as parts of unit wholes, as parts of a collection, as locations on number lines, and as divisions of whole numbers. • Use models, benchmarks, and equivalent forms to judge the size of fractions. • Recognize and generate equivalent forms of commonly used fractions, decimals, and percents. • Explore numbers less than 0 by extending the number line and through familiar applications. • Develop and use strategies to estimate computations involving fractions and decimals in situations relevant to students' experience. • Use visual models, benchmarks, and equivalent forms to add and subtract commonly used fractions and decimals.
6–8	• Work flexibly with fractions, decimals, and percents to solve problems. • Compare and order fractions, decimals, and percents efficiently and find their approximate locations on a number line. • Understand the meaning and effects of arithmetic operations with fractions, decimals, and integers. • Use the associative and commutative properties of addition and multiplication over addition to simplify computations with integers, fractions, and decimals. • Select appropriate methods and tools for computing with fractions and decimals from among mental computation, estimation, calculators or computers, and paper and pencil depending upon the situation and apply the selected methods. • Develop and analyze algorithms for computing with fractions, decimals and integers and develop fluency in their use. • Develop and use strategies to estimate the results of rational number computations and judge the reasonableness of the results.
9–12	(none directly related to fractions)

APPENDIX B

INSTRUCTIONAL TASKS

Covering and Surrounding 2.1: Storm Shelters

The rangers in Great Smoky Mountains National Park want to build several inexpensive storm shelters. The shelters must have rectangular shaped floors with 24 square meters of floor space. Suppose that the walls are made of flat rectangular sections that are 1 meter wide and cost $125.

A. Experiment with different *rectangular* shapes. Sketch each possible floor plan on grid paper. Record your data in a table with columns for length, width, perimeter, area and cost of the walls.

Length	Width	Perimeter	Area	Cost of Walls

B. **1.** What determines how many wall sections are needed, area or perimeter? Explain.

2. Based on the cost of the wall sections, which design would be the least expensive to build? Explain why this arrangement costs the least to build.

3. Which shelter plan has the most expensive set of wall sections? Explain why this arrangement costs the most to build.

C. Liz notices that as the length and width become closer in size, the perimeter decreases and the cost is less. The 6 × 4 storm shelter has side lengths that are closest in size, the smallest perimeter, and the lowest cost. Liz wants to see if using the fractional length $5\frac{1}{3}$ meters will give a smaller perimeter and reduce the cost even more.

1. If the area of the storm shelter is 24 square meters and the length is $5\frac{1}{3}$, what will the width of the storm shelter be?

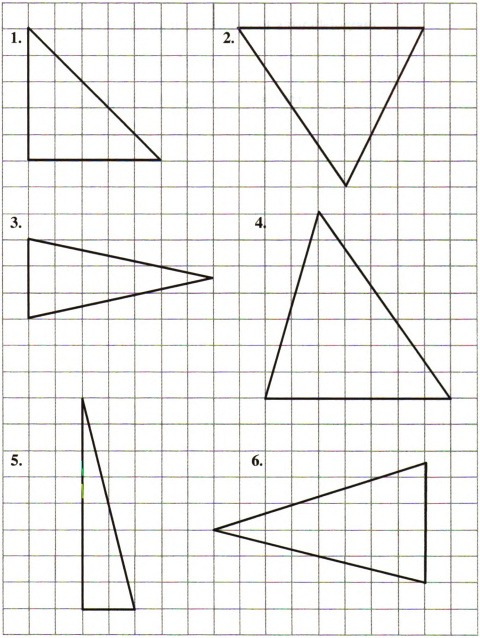
2. What is the perimeter and the cost for wall sections? Add this data to your table in part A.

3. How does this rectangular floor plan compare in cost to the others in the table?

D. Suppose you want to consider a rectangular floor space of 36 square meters. Which design would have the least perimeter? The most perimeter? Explain your reasoning.

Unanticipated Task: Covering and Surrounding Triangle Homework Task

Calculate the area and perimeter of each polygon and briefly explain your reasoning for figures 1, 4, and 6.



Covering and Surrounding 4.1: Area/Perimeter of Parallelograms (Page 1 of 2)

On the next page six parallelograms labeled A–F are drawn on a grid.

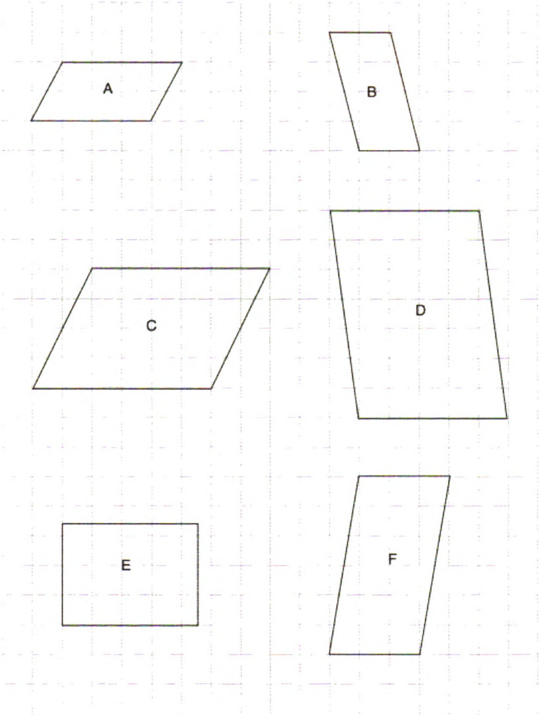
A. **1.** Find the perimeter of each parallelogram.

2. Describe a rule for finding the perimeter of a parallelogram.

B. **1.** Find the area of each parallelogram.

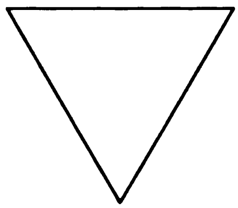
2. Describe the strategies you used to find the areas.

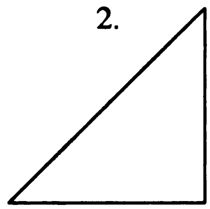
Covering and Surrounding 4.1: Area/Perimeter of Parallelograms (Page 2 of 2)

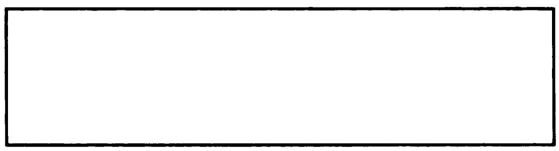


Unanticipated Task: Covering and Surrounding Area/Perimeter Pop Quiz

Find the area of each shape. To find needed measurements use a ruler and measure to the nearest whole centimeter.

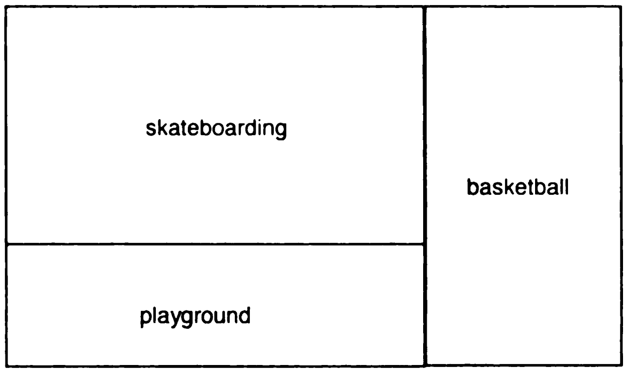
1. 

2. 

3. 

Covering and Surrounding 4.3: Park Problem

A. The Rochelle Park District set aside a rectangular section of land to create a park. After talking with students in the community the park district decided to create an area for skateboarding, an area with playground equipment, and an area with a basketball court.



1. A large rectangular area was set aside for the skateboarding. A short fence was put up around the skateboarding area that took up $\frac{2}{3}$ of the length and $\frac{2}{3}$ of the width of the rectangular section of land. What fraction of the area of the entire rectangular section of land does the skateboarding park take up?

2. The basketball court is 35 feet by 60 feet. Use this information, and what you know about the skateboarding portion of the park to find the area and the perimeter of the playground area.

Unanticipated Task: Data, Decimals, and Percents 1.2: Adding/Subtracting Decimals – Place Value

- Solve each problem.
- Write a mathematical sentence using decimal notation showing your computation.
- Make a table using the headings below. You will add to this table in Problem 1.3.

Person	Mathematical sentence (decimal notation)	*(Leave this column blank for Problem 1.3)*
Emma		

A. 1. Emma signed up to clean 1.5 miles with the cross-country team. She stopped when it started to rain after 0.25 of a mile. How much did she have left to clean when the rain stopped?

2. Pam, a member of the choir, cleaned 0.25 of a mile and cleaned another 0.375 of a mile with the math club. How much did she clean altogether?

3. Jim, a member of the Chess Club, cleaned 0.287 of a mile and then kept going to clean another 0.02 of a mile. How much of a mile did he clean altogether?

4. Teri didn't notice that she had finished her section of highway until she was 0.005 of a mile past the 0.85 of a mile she signed up to do. She claims she cleaned nine tenths of a mile. Is she correct? Explain.

B. 1. Explain what place value has to do with adding and subtracting decimals.

2. Test your ideas about place value and adding and subtracting decimals on the following problems:

a. 27.9 + 103.2	b. 0.45 + 1.2	c. 0.982 – 0.23
d. 34.023 – 1.76	e. 2.011 – 1.99	f. 2.011 + 1.99

Data, Decimals, and Percents 1.3: Adding/Subtracting Decimals – Fractions

Decimals are an extension of our place value system. When you think of decimals this way you have to be sure to only add or subtract digits that have the same place value. You can make sure that digits with like value are added by writing the addition or subtraction problem in column form and lining up the decimal points.
Another way to look at addition and subtraction of decimals is to remember that decimals are also fractions with 10, 100, 1000, 10000, etc in their denominator. Let's revisit the Quick Shop Clerk and think of the money amounts as fractions.

Think About This!

Remember that Sally Jane and her friend Zeke went to Quick Shop to buy snacks. They picked out a half-gallon of cider for $1.97 and a half-dozen donuts for $0.89.

$0.89 = \frac{89}{100}$

$1.97 = \frac{197}{100}$

So the total cost is $\frac{89}{100} + \frac{197}{100} = \frac{286}{100} = 2.86$

This is like thinking of the cost in pennies and then finally writing the sum in dollars.

For parts A, B, and C, work each problem by writing the decimal numbers in fraction form with denominators of 10, 100, 1000, etc. and adding or subtracting the fractions. Write a mathematical sentence using fraction notation that shows your computation and add it to your table from Problem 1.2.

A. Emma signed up to clean 1.5 miles with the cross-country team. She stopped when it started to rain after 0.25 of a mile. How much did she have left to clean when the rain stopped?

B. Pam, a member of the choir, cleaned 0.25 of a mile and cleaned another 0.375 of a mile with the math club. How much did she clean altogether?

C. Jim, a member of the Chess Club, cleaned 0.287 of a mile and then kept going to clean another 0.02 of a mile. How much of a mile did he clean altogether?

D. Use your table to compare your answers to those you got in Problem 1.2 A, B, and C. How are they the same and how are they different?

Data, Decimals, and Percents 2.1: Multiplying Decimals (Page 1 of 3)

In this investigation what you already know about whole number and fraction multiplication will be extended to make sense of multiplication with decimal numbers.

Think About This!

Examine each of the following situations and decide what operation is needed and then estimate the size of the answer:

- Billie bought 8 packages of blueberry muffins. Each package contains a dozen muffins. How many muffins did she buy?
- Betty has $12 dollars allowance for transportation. How many times can she ride the bus if it costs $0.75 a trip?
- The Spartan Marching Band lines up in 18 rows of 12 people at the football stadium. How many are in the band?
- Stacey needs $39.99 for a new pair of sports shoes. She has $22.53 in her pocket and a check from her babysitting job for $15. Can she buy the shoes?
- Mr. Smyth asks for $\frac{2}{3}$ of the red beads in a jar at *Bedazzle*. The beads in the jar weigh 1.2 pounds. How many pounds will Mr. Smyth get?
- Big Bob ate five 0.875-ounce bags of chips on Saturday afternoon at the picnic. How many ounces of chips did he eat?

What strategies did you use to decide what operation is needed?

What strategies did you use to estimate the answers?

Data, Decimals, and Percents 2.1: Multiplying Decimals (Page 2 of 3)

In *Bits and Pieces I* you learned to write fractions as decimals and decimals as fractions. This makes sense because decimals are just fractions written in a different notation. You know that $0.63 = \frac{63}{100}$ and that $0.0063 = \frac{63}{10,000}$. Let's see how fraction multiplication can help us with decimal multiplication.

Think About This!

If we need to find the product of 0.52 and 2.3, the first thing we ask ourselves is, "About how big is the product?"

About how big do you think the product of a number that is about $\frac{1}{2}$ and a number that is a little more than 2 will be? Will it be less than or greater than 2?

Now, we can look at the problem using equivalent fractions.

$$0.52 = \frac{52}{100} \text{ and } 2.3 = 2\frac{3}{10} \text{ or } \frac{23}{10}$$

so,

$$0.52 \times 2.3 = \frac{52}{100} \times \frac{23}{10}$$

What is the product equal to in fraction notation?

What is the product equal to, when written as a decimal?

How can knowing the product as a fraction help you to know how many places will be in the decimal form of the product?

Data, Decimals, and Percents 2.1: Multiplying Decimals (Page 3 of 3)

In each problem:

- estimate about how large the answer will be,
- write the decimals in fraction notation with denominators of 10, 100, 1000, etc.,
- use fraction multiplication to find the product, and then,
- change the answer from fraction form back to a decimal.

A. At the One-A-Day roadside fruit stand the following purchases took place on Saturday:

1. Kelly bought 0.4 of a pound of Granny Smith apples on special at $0.55 a pound. What was her bill?

2. There is a special on Red Delicious apples and Onur bought 1.7 pounds at $0.50 a pound. What was his bill?

3. Carl wanted his mother to make a pie so he bought Northern Spys. He bought 5.125 pounds at $1.10 a pound. What was his bill?

4. Look over your work in parts 1–3. Is there a relationship between the denominator of the fractions you multiplied and the number of decimal places in the product?

B. 1. The dimensions of a rectangular sheet of gold leaf are 7.1 cm by 10.1 cm. What is the area of the sheet of gold leaf?

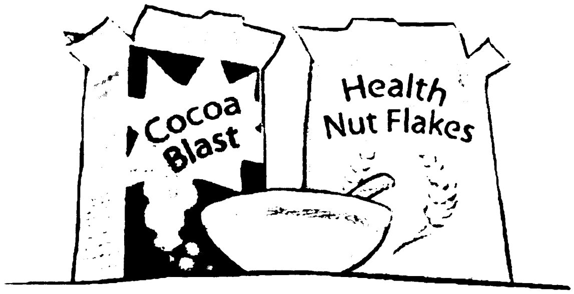
2. If a sheet of gold costs $3 a square centimeter, how much is the rectangle in part 1 worth?

C. Describe the steps in multiplying decimals by using their fraction form.

How Likely Is It? 1.1: Flipping for Breakfast

Kalvin always has cereal for breakfast. He likes Cocoa Blast cereal so much that he wants to eat it every morning. Kalvin's mother wants him to eat Health Nut Flakes at least some mornings because it is more nutritious than Cocoa Blast.

Kalvin and his mother have come up with a fun way to determine which cereal Kalvin will have for breakfast. Each morning, Kalvin flips a coin. If he flips a head, he will have Cocoa Blast. If he flips a tail, he will have Health Nut Flakes.



Problem 1.1

A. How many days in June do you think Kalvin will eat Cocoa Blast?

B. Conduct an experiment to test your prediction from part A. Flip a coin 30 times (one for each day in June), and record your results in a table like this:

Date	Result of Flip (H or T)	Number of Heads So Far	Fraction of Heads So Far	Percent of Heads So Far
1				
2				

C. As you added more and more data, what happened to the percent of flips that were heads?

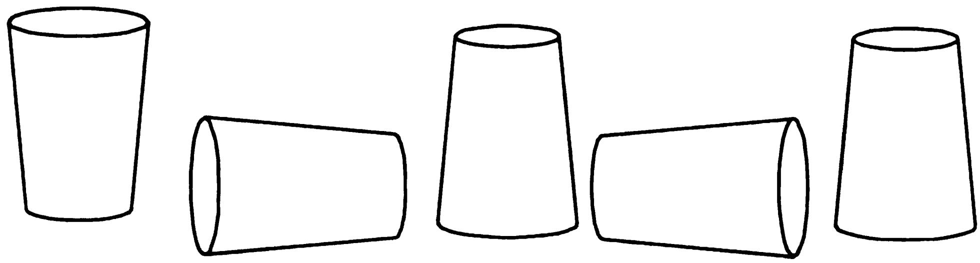
D. Work with your teacher and your classmates to combine the result from all the groups. Start with the data for one group, then add the data for a second group, then a third group, and so on. Each time you add a group's data, recalculate the percent of the total number of flips that were heads.

1. What percent of the total number of flips for your entire class were heads?
2. As your class added more and more data, what happened to the percent of flips that were heads?
3. Based on what you found for June, how many times would you expect Kalvin to eat Cocoa Blast cereal in July? Explain your reasoning.

E. Kalvin's mother told him that the chances of getting a head when he flips a coin are $\frac{1}{2}$. Does this mean that every time he flips a coin twice he will get one head and one tail? Explain your reasoning.

How Likely Is It? 1.2: Tossing Paper Cups

Kalvin loves Cocoa Blast cereal so much that he wants to find something else to flip that will give him a better chance of eating it each morning. Kalvin looked through the kitchen cupboard and found a package of paper cups. He wonders if a paper cup would be a good thing to flip. Because Kalvin wants to eat Cocoa Blast cereal most of the time, he needs to determine if the cup lands in one position—either on its side or on one of its ends—more often. If so, he will ask to use a paper cup instead of a coin for deciding his cereal each morning.



Problem 1.2

A. Which of the cup's landing positions—end or side—do you think Kalvin should use to represent Cocoa Blast? (Remember, he wants to eat Cocoa Blast as often as possible.)

B. Conduct an experiment to test your prediction from part A. Toss a paper cup 50 times. Create a table to keep track of your results.

C. Use the results of your experiment to answer the following questions:

1. For what fraction of your 50 tosses did the cup land on one of its ends? What percent is this?
2. For what fraction of your 50 tosses did the cup land on its side? What percent is this?
3. When you toss a cup, do the two landing positions—end and side—have the same chance of occurring? If not, which is more likely? Explain.
4. If Kalvin's mother agrees to let him use a cup instead of a coin to decide his breakfast, which of the cup's landing positions—end or side—should Kalvin use to represent Cocoa Blast? Explain your reasoning.

D. Work with your teacher and your classmates to combine results from all the groups. Compare the class results with your group's data. Based on the class data, would you change your answers to questions 3 or 4 of part C? Explain why or why not.

E. Kalvin's mom agrees to let him use a cup to decide his cereal each morning. On the first morning, the cup lands on its end. On the second morning, it lands on its side. Kalvin says, "This cup isn't any better than the penny—it lands on an end 50% of the time!" Do you agree or disagree with Kalvin? Explain.

How Likely Is It? 1.3: Match/No-Match

By now Kalvin has become intrigued with probability situations. He comes up with one more way to use probability to decide his breakfast cereal. This time, he tosses two coins.

- If the coins match, he gets to eat Cocoa Blast.
- If the coins do not match, he eats Health Nut Flakes.

His mother agrees to let him use this method for the month of June.

A. How many days in June do you think Kalvin will eat Cocoa Blast?

B. Conduct an experiment to test your prediction from part A. Toss two coins 30 times. Keep track of the number of times a match and the number of times a "no-match" occur. Based on your data, what is the experimental probability of getting a match? Of getting a no-match?

C. Combine your data with the data collected by your classmates.

1. Based on the class data, what is the experimental probability of getting a match? Of getting a no-match? Compare these probabilities with the probabilities you computed in part B.

2. Based on the class data, do you think the two results—match and no match—have the same chance of occurring? Explain.

D. Think about the possible results when you toss two coins.

1. In how many ways can a match occur?

2. In how many ways can a no-match occur?

3. Based on the number of ways each result can occur, do you think a match and a no-match have the same chance of occurring? Explain.

E. Kalvin's friend Lucy comes up with yet another way for Kalvin to determine his breakfast cereal. She suggests that he toss a thumbtack. If it lands on its side, Kalvin gets to eat Cocoa Blast. If it lands on its head, he eats Health Nut Flakes. Kalvin says they must first experiment to find the probabilities involved. Lucy does 11 tosses. Kalvin does 50 tosses. Here are the probabilities Lucy and Kalvin calculated based on their experiments:

Lucy: P(head) = $\frac{6}{11}$ Kalvin: P(head) = $\frac{13}{50}$

Which result do you think better predicts the chance that the tack will land on its head when tossed? Why?

How Likely Is It? 2.4: Exploring Probabilities

A. A bag contains two yellow marbles, four blue marbles, and six red marbles. One marble is drawn from the bag.

1. What is the probability the marble is yellow? The probability it is blue? The probability it is red?

2. What is the sum of the probabilities from part 1?

3. Which color is the marble most likely to be?

4. What is the probability the marble is *not* blue?

5. What is the probability the marble is either red or yellow?

6. What is the probability the marble is white?

7. Mary said the probability the marble is blue is $\frac{12}{4}$. Anne said this is impossible. Who is correct? Explain your reasoning.

B. Imagine the bag in part A has twice as many of each color marble. Do the probabilities change? Explain.

C. How many blue marbles must be added to the bag in part A to make the probability of drawing a blue marble equal to $\frac{1}{2}$?

D. A bag contains several marbles. Each marble is either red, white, or blue. The probability of drawing a red marble is $\frac{1}{3}$, and the probability of drawing a white marble is $\frac{1}{6}$.

1. What is the probability of drawing a blue marble? Explain.

2. What is the smallest number of marbles that could be in the bag? If the bag contains the smallest number of marbles, how many of each color does it contain?

3. Could the bag contain 48 marbles? Is so, how many of each color would it contain?

4. If the bag contains 8 red marbles and 4 white marbles, how many blue marbles does it contain?

Stretching and Shrinking 3.1: Rep-Tile Quadrilaterals

You've seen that a square is a rep-tile. In the next problem, you will look at rectangles and non-rectangular parallelograms to see if they might be rep-tiles as well.

Sketch and make several copies of each of the following shapes:

- a non-square rectangle
- a non-rectangular parallelogram
- a trapezoid

A. Which of these shapes can fit together to make a larger shape that is similar to the original? Make a sketch showing how the copies fit together.

B. Look at your sketches from part A.

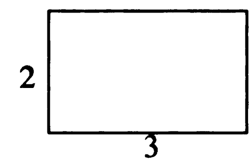
1. What is the scale factor from the original figure to the larger figure? Explain how you found this scale factor.
2. How is the perimeter of the new figure related to the perimeter of the original?
3. How is the area of the new figure related to the area of the original?

C. 1. Extend the rep-tile patterns you created in part A by adding copies of the original figure to create other, larger figures that are similar to the original.

2. Make sketches showing how the figures fit together.
3. Find the scale factor from each original figure to each new figure. Explain how you found the scale factor.
4. Explain what the scale factor indicates about the corresponding side lengths, perimeters, and areas.

Unanticipated Task: Stretching and Shrinking Copy Machine Warm-Up

Find 150% of a 2 by 3 picture put into a copier.



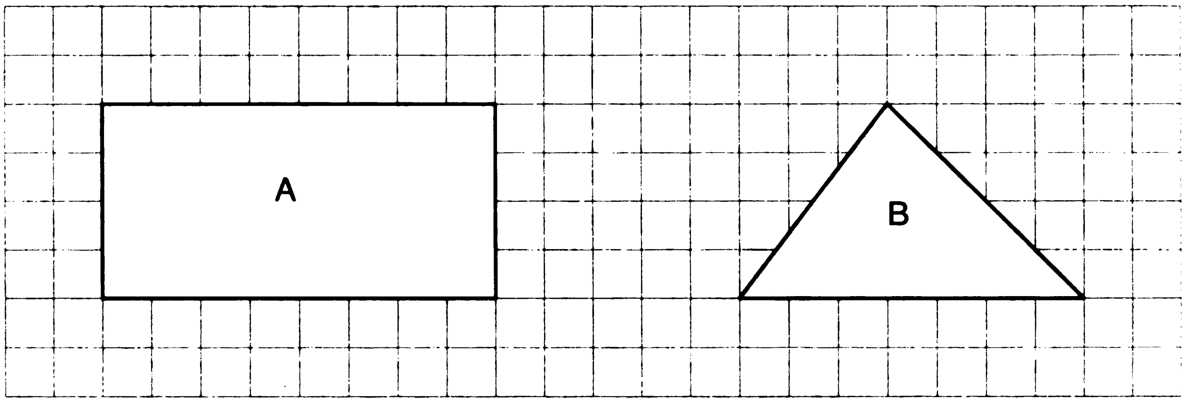
Unanticipated Task: Stretching and Shrinking 20% if 120 Warm-Up Task

Find 20% of 120.

Stretching and Shrinking 3.3: Scale Factors and Similar Shapes (Page 1 of 2)

You have seen that the scale factor from one figure to another gives you important information about how the side lengths, perimeters, and areas of the figures are related. In the next problem, you will use what you have learned.

Below are two figures drawn on a grid.



A. In parts 1–3, draw a rectangle similar to rectangle A that fits the given description. Label the dimensions of each new rectangle.

1. The scale factor from rectangle A to the new rectangle is 2.5.

2. The area of the new rectangle is $\frac{1}{4}$ the area of rectangle A.

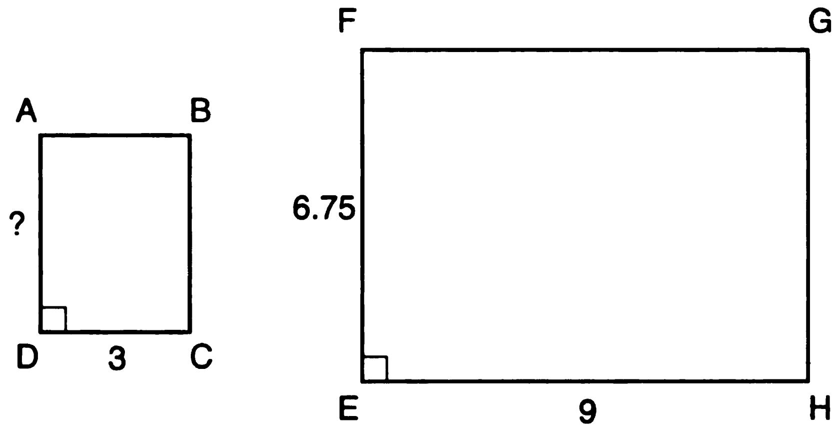
3. The perimeter of the new rectangle is three times the perimeter of rectangle A.

B. In parts 1–2, draw a triangle similar to triangle B that fits the given description. Label the base and height of each new triangle.

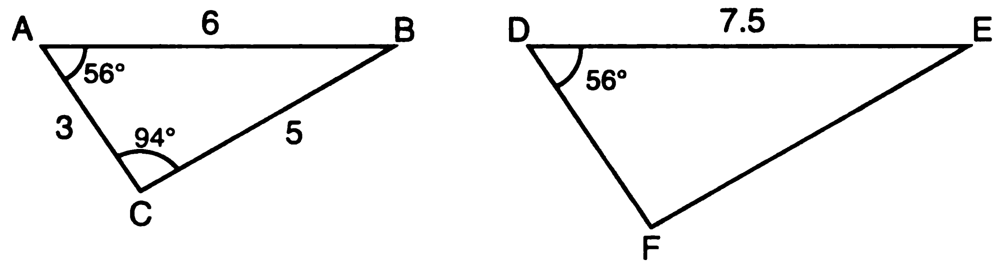
1. The area of the new triangle is nine times the area of triangle B.

2. The scale factor from triangle B to the new triangle is $\frac{1}{2}$.

C. 1. Rectangles ABCD and EFGH are similar. Find the missing side lengths. Explain your reasoning.



2. Triangles ABC and DEF are similar. Find the missing side lengths and angle measures. Explain.



Unanticipated Task: Stretching and Shrinking Warm-Up

What is $\frac{1}{2}$ of 48?

Unanticipated Task: Stretching and Shrinking Fraction Division Homework

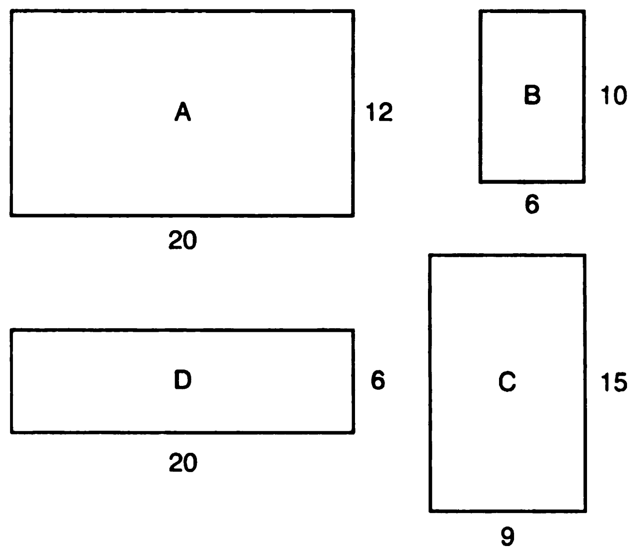
Solve each of the following. Draw a picture to support your work.

1. $8 \div 2 =$ 2. $8 \div \frac{1}{2} =$ 3. $10 \div 2 =$ 4. $10 \div \frac{1}{2} =$

Stretching and Shrinking 4.1: Ratios Within Similar Parallelograms (Page 1 of 2)

You know that if two figures are similar, then there is a scale factor that relates each length in one figure to the corresponding length in the other. In addition, there is a relationship between two lengths in a figure that also holds for the corresponding lengths in a similar figure. You will explore this relationship further in Problem 4.1.

A. The lengths in centimeters of two sides are given for each rectangle. Look closely at these rectangles.



1. For each rectangle, find the ratio of the length of a long side to the length of a short side.

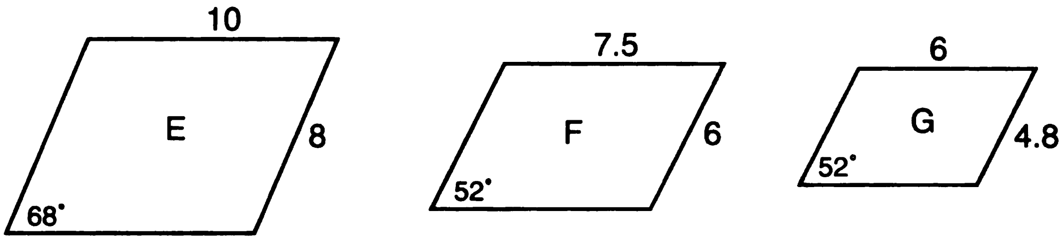
2. Identify the rectangles that are similar to each other.

3. Compare the ratios in part 1 for these rectangles.

4. For two similar rectangles, find the scale factor from the smaller rectangle to the larger rectangle. What information does the scale factor give about two similar figures?

5. Compare the information given by the scale factor to the information given by the ratios of side lengths.

B. Study the parallelograms below.



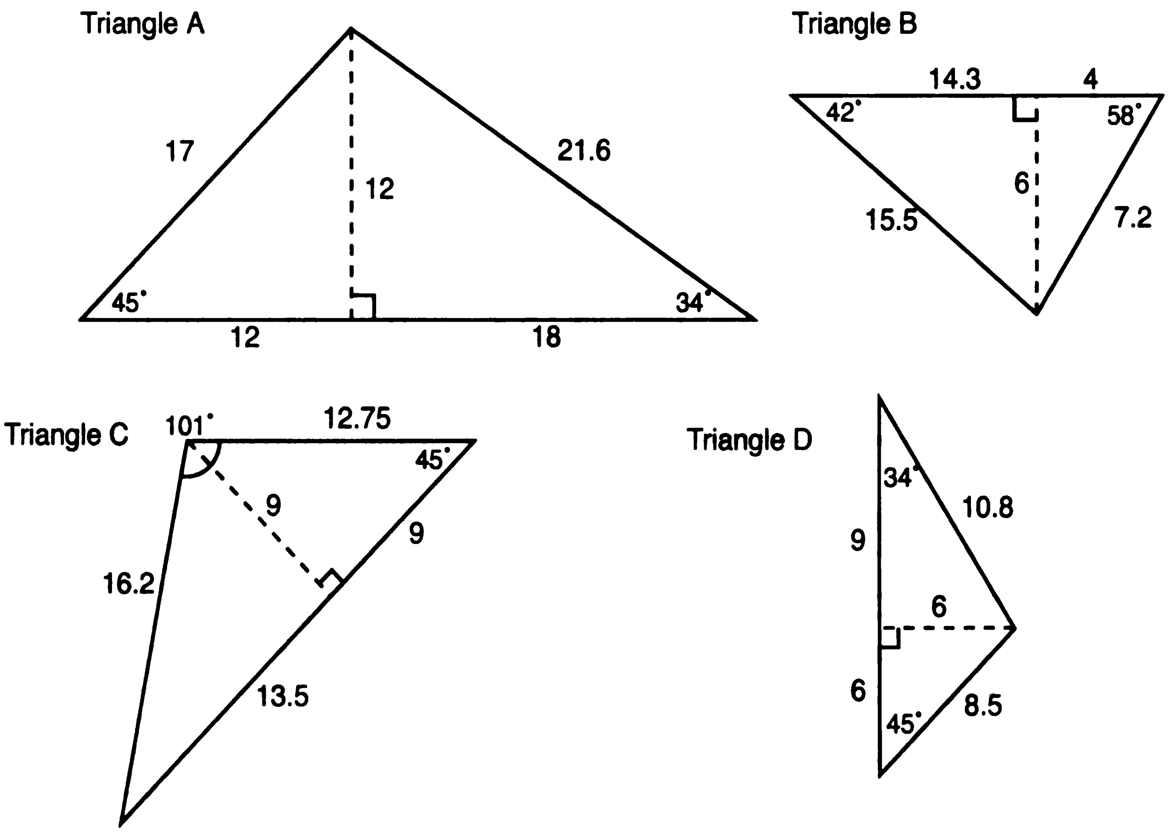
1. Which of the parallelograms are similar? Explain how you decided.

2. For each parallelogram, find the ratio of the length of a longer side to the length of a shorter side. How do the ratios compare?

C. If the ratio of adjacent side lengths in one parallelogram is equivalent to the ratio of the corresponding side lengths in another, can you conclude that the parallelograms are similar? Explain your reasoning.

Stretching and Shrinking 4.2: Ratios Within Similar Triangles

In this problem, you will use what you know about side-length ratios and angle measures to find pairs of similar triangles. Look closely at the four triangles below. The lengths are in centimeters.



A. Identify the triangles that are similar to each other. Explain how you decided.

B. **1.** For each triangle, find the ratio of the base to the height. How do these ratios compare for the similar triangles? For the non-similar triangles?

2. How do the ratios of side lengths compare for two similar triangles?

3. How do the ratios of side lengths compare for triangles that are not similar?

Comparing and Scaling 1.1: Ads That Sell

1.1 Ads That Sell

An ad for the soft drink *Bolda Cola* starts like this:

> **WHICH SOFT DRINK DO YOU LIKE BETTER?**
>
> ***Bolda Cola* or *Cola Nola***
>
> **Take the Cola Taste Test Yourself!**

To complete the ad, the Bolda Cola company plans to report the results of taste tests. A copywriter for the ad department has proposed four possible concluding statements. (Add a letter label for each ad so that students can refer to them by letter. Please make these look more like ads.)

> In a taste test, people who preferred ***Bolda Cola*** outnumbered those who preferred *Cola Nola* by a ratio of 17,139 to 11,426.

> In a taste test, 5,713 more people preferred ***Bolda Cola***.

> In a taste test, 60% of the people preferred ***Bolda Cola***.

> In a taste test, people who preferred ***Bolda Cola*** outnumbered those who preferred *Cola Nola* by a ratio of 3 to 2.

A. Describe what you think each statement means in the **Bolda Cola** ads.

B. Which of the proposed statements do you think would be most effective in advertising **Bolda Cola**? Why?

C. Is it possible that all four advertising claims are based on the same survey data? Explain your reasoning.

D. In what other ways could you express the claims in the four proposed advertising statements? Explain your reasoning.

E. If you were to survey 1,000 cola drinkers, what numbers of **Bolda Cola** and **Cola Nola** drinkers would you expect based on this survey?

Comparing and Scaling 1.2: Targeting an Audience

A survey of 150 students at Neilson Middle School found that 100 students prefer watching television and 50 prefer listening to the radio.

A. How would you compare student preferences for radio or television?

B. Which of the following statements accurately report results of the Neilson Middle School survey?

1. At Neilson Middle School $\frac{1}{3}$ of the students prefer radio to television.

2. **a.** Students prefer television to radio by a ratio of 2 to 1.

   b. The ratio of students who prefer radio to television is 1 to 2.

3. The number of students who prefer television is 50 more than the number of students who prefer radio.

4. The number of students who prefer television is two times the number who prefer radio.

5. 50% of the students prefer radio to television.

6. Compare the statements in parts 3 and 4. Which is more informative and why?

C. Which of the accurate statements about preferences of students for television or radio would best convince merchants to place ads on

1. radio? Give reasons for your choice.

2. television? Give reasons for your choice.

Comparing and Scaling 1.3: American Records

One way to describe the size of a tree is by comparing it to other trees or familiar things. In the United States, the *American Forests* organization keeps records that we can use. Some examples from the current list are given in the table on the following page.

Tree Type (State)	Circumference	Height	Spread/Diameter
Giant Sequoia (CA)	83.2 feet	275 feet	107 feet
Coast Redwood (CA)	79.2 feet	321 feet	80 feet
Swamp Chestnut Oak (TN)	23.0 feet	105 feet	216 feet
Florida Crossopetalum (FL)	0.4 feet	11 feet	3 feet
White Oak (MD)	31.8 feet	96 feet	119 feet

[Source: *Washington Post* 12/25/2000 "Tree Lovers on a Crusade to Clone 'Champions'" by Rick Weiss, page A19.]

A. Each of the following questions ask about how the largest trees relate to each other.

1. **a.** How many of the largest Coast Redwoods would it take to equal the spread of the largest white oak?

   b. Kenning says that the spread of the largest white oak is greater than that of the largest Coast Redwood by a *ratio* of about 3 to 2. Is he correct?

2. Mary says the *difference* between the heights of the largest Coast Redwood and the largest Giant Sequoia is 46 feet. Is she correct?

3. **a.** How many of the largest Giant Sequoias would it take to equal the spread of the largest Swamp Chestnut Oak?

   b. T.Y. says the spread of the largest Giant Sequoia is less than *50%* of the spread of the largest Swamp Chestnut Oak. Is he correct?

4. Betty says the circumference of the largest Swamp Chestnut Oak is about *three-quarters* the circumference of the largest White Oak. Is she correct?

B. The tallest person in history, according to the *Guinness Book of World Records*, was Robert Wadlow who was nearly 9 feet tall. Write two accurate statements comparing Wadlow to the world's tallest trees. In your statements, use ideas of fractions, ratios, percents, or differences.

C. Average waist, height, and arm span measurements for a small group of adult men are:

Waist = 32 inches Height = 72 inches Arm Span = 73 inches

Write two statements comparing the data on men to the world's tallest trees. Use ideas of fractions, ratios, percents, or differences.

D. When a problem requires comparison of counts or measurements, how do you decide whether to use *difference*, *ratio*, *fraction*, or *percent* ideas?

Comparing and Scaling 2.1: Mixing Juice

Every year, the seventh-grade students at Langston Hughes School go on an outdoor-education camping trip. During the week-long trip, the students study nature and participate in recreational activities. Everyone pitches in to help with the cooking and cleanup.

One year, Arvin and Mariah were in charge of making orange juice for all the campers. They planned to make the juice by mixing water and frozen orange juice concentrate. To find the mix that would taste best, Arvin and Mariah decided to test some recipes on a few of their friends.

Mix A	
2 cups concentrate	3 cups cold water

Mix B	
5 cups concentrate	9 cups cold water

Mix C	
1 cup concentrate	2 cups cold water

Mix D	
3 cups concentrate	5 cups cold water

Answer these questions that Arvin and Mariah faced.

A. Which recipe will make juice that is the most "orangey"? Explain your reasoning.

B. Which recipe will make juice that is the least "orangey"? Explain your reasoning.

C. Here are two comparison statements using fractions:

$\frac{5}{9}$ of mix B is concentrate.

$\frac{5}{14}$ of mix B is concentrate.

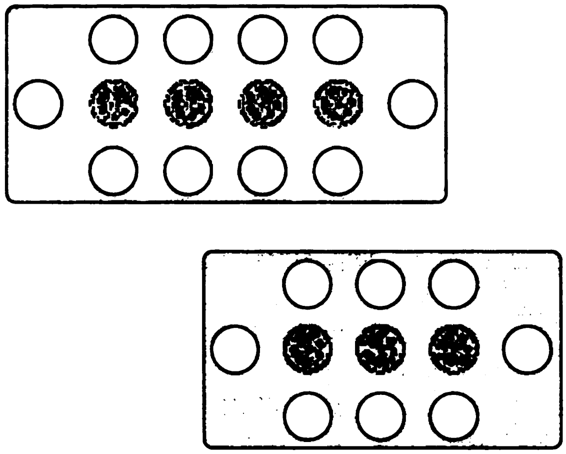
Which do you think is correct and why?

D. Assume that each camper will get $\frac{1}{2}$ cup of juice. For each recipe, how much concentrate and how much water are needed to make juice for 240 campers?

E. For each recipe, how much concentrate and how much water are needed to make 1 cup of juice?

Comparing and Scaling 2.2: Sharing Pizza

On the last day of camp, the cook served pizza. The camp dining room has two kinds of tables. A large table seats 10 people, and a small table seats 8 people. The students serving dinner put four pizzas on each large table and three pizzas on each small table.



A. If the pizzas are shared equally by everyone at the table, will a person sitting at a small table get the same amount as a person sitting at a large table? Explain your reasoning.

B. 1. Do you agree or disagree with Caroline's method of reasoning?

Caroline: $10 - 4 = 6$ and $8 - 3 = 5$ so the large table is better.

2. If you put 9 pizzas on the large table, what answer would Caroline's method give? Does this answer make sense?

3. Which table relates to $\frac{3}{8}$ and what do the 3 and the 8 mean? Is the $\frac{3}{8}$ a part to whole comparison or a part to part comparison?

D. 1. The ratio of large tables to small tables in the dining room is 8 to 5. There are exactly enough seats for the 240 campers. How many tables of each kind are there?

2. What fraction of the campers will sit at small tables?

3. What percent of the campers will sit at large tables?

Unanticipated Task: Comparing and Scaling Comparison Statement Task

Participation in Sports Activities (1995)

Activity	Males	Females	Ages 12–17	Ages 55–64
Bicycle Riding	24,562,000	23,357,000	8,794,000	2,030,000
Camping	23,165,000	19,533,000	5,336,000	2,355,000
Exercise/Walking	21,054,000	43,373,000	2,816,000	7,782,000
Fishing	30,449,000	14,885,000	4,945,000	3,156,000
Swimming	27,713,000	33,640,000	10,874,00	2,756,000
Total in Group	**111,851,000**	**118,555,000**	**21,304,000**	**20,922,000**

1. Write two percent statements from the 1995 data.

2. Write two fraction statements from the 1995 data.

3. Write two ratio statements from the 1995 data.

APPENDIX C

INTERVIEW PROTOCOL

Interview Protocol

1. Tell me about your initial thoughts when you read this problem?
    a. Did you instantly know what to do?
    b. Did you know the answer immediately?
    c. What was clear? Unclear?
    d. [If they are talking about the content focus of the class lesson, such as area, push them to talk about how they thought about fractions.] What were you thinking about when you saw the fractions in the problem? How did you interpret the fractions in the problem?

2. What did you think about as you were trying to make sense of the problem?
    a. What did you try to do?
    b. Why did that seem sensible?
       or
       Why did you change your mind about that approach?
    c. Are there other ways you could have tried to do the problem?

3. Did other people say or do something that helped you make sense of this problem?
    a. Did others (students or teacher) have thoughts or ideas that helped you? How were they helpful?
    b. Did others have thoughts or ideas that confused you? What was confusing?
    c. When XXX said XXX, what did you think about that?

4. When you did this problem, before the class discussion, how well do you think you understood this problem?
    a. Did you understand the problem in the same way as you do right now? Why?
    b. Were you making progress in solving the problem before the class discussion?
    c. Why do you think you were or were not making progress?

Questions that focus on what happened during the class discussion?

5. You said that you XXX when working on the problem, did that change during the class discussion?
    a. What changed? How are you thinking about XXX now?
    b. What happened in the discussion that caused that change in thinking to occur?
    c. Did the teacher or a student do or talk about something that you think is really and important part of understanding this problem? Why is this important?

6. What do you think about XXX that XXX brought up in the discussion? [Their ideas may not have changed, but something may have expanded what they thought.]
    a. Did this confuse you? How?
    b. Did this help you? How?
    c. Did anything happen that made you think differently about how you could do the problem, even thought your approach makes sense too?

7. In the discussion the class started to talk about XXX, but people did not seem to understand or want to try that.
    a. Why do you think that happened?
    b. How did you feel about that idea? Does it makes sense (not make sense)? Why?

8. When you worked on XXX in XXX, this idea about fractions was difficult for you.
    a. How do you feel about this now?
    b. Is it making more sense? How is it making more sense?
    c. Is there something that has helped make this idea easier for you?

9. What would you say to another student who is having a hard time with this problem, especially with the fractions (fraction ideas)?

APPENDIX D

STRUCTURE AND DESIGN OF INITIAL CODING CHART

Structure and Design of Initial Coding Chart

	Storm Shelters	Triangle Homework Task	...	Bonus Problem Task
Fraction Use Explicit in Task				
Fraction Use Implicit in Task				
Determining ~~Equivalence~~ Appropriateness				
×/÷ Discussions				
Linking relationships with other number systems and fractions (or extending whole numbers?)				
Katie Interview Notes				
Ali Interview Notes				
TJ Interview Notes				
Bryan Interview Notes				

1. $5\frac{1}{3} \times 4\frac{2}{3} \rightarrow 5.3 \times 4.6 = 24.3$ Rejected
2. $5\frac{1}{3} \times 4\frac{1}{2} \rightarrow 5.3 \times 4.5$ Rejected
3. $5\frac{1}{3} \times 4\frac{1}{2} \rightarrow 5.33333 \times 4.5$ Rejected
4. $24 \div 5\frac{1}{3} \rightarrow 24 \div 5.333 = 4.528$ Not rejected at first.
5. $5\frac{1}{3} \times 4\frac{1}{2} \rightarrow \frac{16}{3} \times \frac{9}{2} = \frac{144}{6} = 24$ $4\frac{1}{2}$ is okay, but how did you get it? Did you divide? So, multiplication as justification is rejected initially.
6. $24 \div 5\frac{1}{3} \rightarrow 24 \div 5.333 = 4.528$ 4.528 is rejected this time since 4.528 is not 4.5. How do you explain that your answer is not exactly 4.5?
7. $5\frac{1}{3} \times 4\frac{1}{2} \rightarrow \frac{16}{3} \times \frac{9}{2} = \frac{144}{6} = 24$ Explain how they got $4\frac{1}{2}$ using a combination of repeated addition and guess and check with multiplying. $4\frac{1}{2}$ is accepted as width.

APPENDIX E

OCCURRANCES OF FRACTION LITERACY PRACTICES ACROSS TASKS

Instructional Task	Unit	Practice of Determining Appropriateness	Practice of Connecting Fractions with Multiplication and Division Concepts
Storm Shelters	Area and Perimeter	✔	✔
Area/Perimeter Triangle Homework	Area and Perimeter	✔	
Area/Perimeter of Parallelograms	Area and Perimeter	✔	
Area/Perimeter Pop Quiz	Area and Perimeter	✔	
Park Problem	Area and Perimeter	✔	
Addition/Subtraction Decimals-Place Value	Decimal Computation	✔	
Addition/Subtraction Decimals-Fractions	Decimal Computation	✔	
Multiplying Decimals	Decimal Computation	✔	
Flipping For Breakfast	Probability	N/A	N/A
Tossing Paper Cups	Probability	N/A	N/A
Match/No Match	Probability	N/A	N/A
Exploring Probability	Probability	N/A	N/A
Rep-tile Quadrilaterals	Similarity		✔
Copy Machine Task	Similarity	✔	✔
20% of 120 Warm-Up	Similarity	✔	
Scale Factor and Similar Shapes	Similarity	✔	
1/2 × 48 Warm-Up	Similarity		✔
Fraction Division Homework Task	Similarity		✔
Ratios Within Similar Parallelograms	Similarity	✔	✔
Ratios Within Similar Triangles	Similarity	✔	
Bolda Cola	Ratio	✔	
Target Audience	Ratio	✔	
American Records	Ratio	✔	
Mixing Juice	Ratio	✔	✔
Sharing Pizza	Ratio		✔
Comparison Statement Task	Ratio	✔	

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