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PSEUDO-HOLOMORPHIC MAPS IN FOLDED SYMPLECTIC MANIFOLDS

By

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ABSTRACT

PSEUDO-HOLOMORPHIC MAPS IN FOLDED SYMPLECTIC MANIFOLDS

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We define moduli spaces for rational pseudo-holomorphic maps into oriented, closed folded symplectic 4-manifolds with circle-invariant folds. Its elements are stable *folded holomorphic maps* that are discontinuous across the folding hypersurface. The boundary values on the fold are given by *tunneling maps* which are punctured generalized holomorphic maps into the folding hypersurface with prescribed asymptotics on closed characteristics.

We show that the linearized operator of this boundary value problem is Fredholm and thus obtain well behaved finite dimensional moduli spaces and give examples.

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1 Introduction

In the last two decades new techniques have been devised to study symplectic manifolds and Hamiltonian dynamics. In particular, M. Gromov showed in [Gro85] that the tools of complex geometry that exist on Kähler manifolds can be transferred to symplectic manifolds. In the last decade that has led to a vibrant new field based on the study of "pseudo-holomorphic curves" in symplectic manifolds.

Unfortunately, these methods do not apply to all manifolds: many smooth manifolds do not admit symplectic forms. There are different possibilities to extend the theory of pseudo-holomorphic curves to a broader class of manifolds. One approach, being pursued by C. Taubes, begins with the observation that every compact oriented 4-manifold with intersection form that is not negative definite admits a closed 2-form that degenerates along a disjoint union of circles. Taubes has made a detailed study of the behavior of pseudo-holomorphic curves approaching these circles ([Tau02]).

In [CGW00] A. Cannas, V. Guillemin and C. Woodward introduced the notion of folded symplectic structures, which we describe in Section 2.2. Every orientable 4-manifold admits a folded symplectic structure.

In this thesis I construct finite-dimensional moduli spaces of folded holomorphic maps into a certain subclass of folded symplectic manifolds (see Theorems 7.3 and 7.11). In physics pseudo-holomorphic maps come up as vacuum solutions of classical closed strings. Adopting this point of view, the theory of folded holomorphic maps that I develop describes strings that are scattered at the fold singularity and exit at a location which is in general different from where they enter. Thus pseudo-holomorphic maps in folded symplectic manifolds are discontinuous at the fold. This "scattering" or "tunneling" map, defined in Section 5, is the central object of this thesis.

Let (X, ω) be an oriented compact folded symplectic 4-manifold with folding hypersurface Z. As in Symplectic Field Theory [BEH$^+$03], we assume that the hypersurfaces Z are dynamically stable, i.e. the embedding of Z into X extends to a family foliating a neighborhood of Z with ω inducing the same dynamics on each leaf. In contrast to the symplectic case, one cannot always find a smooth almost complex structure on a folded symplectic manifold. Instead we equip folded symplectic manifolds with an almost complex structure J that is discontinuous across the fold in a controlled manner. We set this up in Section 2.4.

The almost complex structure J can be used to define "folded pseudo-holomorphic maps" as follows. Domains for folded holomorphic maps are oriented 2-dimensional surfaces Σ, separated into two parts called Σ_+ and Σ_- by a collection of disjoint embedded circles σ (see Section 4.1). A *folded holomorphic map* consists of a complex structure j on Σ and a pair of pseudo-holomorphic maps $u_\pm : \Sigma_\pm \to X_\pm$ satisfying the "matching condition"

$$(u_+, u_-)|_\sigma \in \Delta^Z$$

where the folded diagonal $\Delta^Z \subset \mathrm{Map}(\sigma, Z) \times \mathrm{Map}(\sigma, Z)$ is given by the scattering map. We give a precise meaning to this in Section 4.3.

The folded diagonal Δ^Z is the crucial component in this definition. Unfortunately, the obvious choice – requiring that the images of the maps $u_\pm$ match along Z – does not not work (it does not lead to a Fredholm problem as explained in Sections 3 and E). In fact, in order to obtain a well-behaved finite dimensional moduli space one needs to allow the maps to tunnel across the fold, exiting at a possibly different location from where they entered.

The folded diagonal is constructed in Section 5.2 by considering $\mathcal{H}$-holomorphic

maps called *tunneling maps.* These are maps $v : S \to Z$ from a punctured Riemann surface with boundary $\partial S = \sigma$ identified with the domain fold satisfying

$$J\pi_F\, dv = \pi_F\, dv\, j; \qquad d(v^*\alpha \circ j) = 0.$$

$\mathcal{H}$-holomorphic are essentially equivalent to the generalized holomorphic maps as introduced by Hofer for domains without boundary, in [ACH04]; for a discussion of this see Section D. We deviate from the setup of generalized holomorphic maps in that use different function spaces, defined in Section 4.2, that lead to a more natural Fredholm theory for our setup. This is done in Sections C and C.1, where show that, just as in the familiar case of holomorphic maps into cylinders, the punctures are asymptotic to closed characteristics.

Tunneling maps v_+ and v_- start at boundary conditions given by maps u_+ into X_+ and u_- into X_- and cap these off to closed characteristics. The folded diagonal is then defined as the boundary values of pairs (v_+, v_-) of tunneling maps that are "reflections" of one another through a special kind of tunneling map called "Abbas solution". This reflection process is defined in Section 5.2 and explained in a simple setting in Section 5.3.

The space of folded holomorphic maps breaks up into components labeled by relative homology classes

$$[u_\pm \sqcup_\partial v_\pm] \in H_2(X_\pm, \mathcal{R}; \mathbb{Z}),$$

where $\mathcal{R}$ is the space of closed characteristics. As a consequence, the energy of folded holomorphic maps is constant in families, as explained in Section 7.1.

Deformations of the folded diagonal are studied in Section 6. This leads to the result that the folded diagonal poses elliptic boundary conditions for the linearized

operator at a solution (Theorem 7.1). This leads to the main result, which I have proved for the case that the fold is an S^1-bundle over a Riemann surface with the vertical subspaces coinciding with the characteristic foliation.

Theorem 1.1 (Theorem 7.3 and 7.11). *For generic almost complex structure, the moduli space of folded holomorphic maps is a smooth finite dimensional manifold whose dimension is the index (7.42)of the linearized operator.*

As examples, I consider folded holomorphic maps into S^4. While S^4 does not admit either symplectic or almost complex structures, it does have a canonical folded symplectic structure. In this case, the spaces of relative homology data reduce to the degree with which the tunneling maps wrap the closed characteristic. In Section 9.2 I give an explicit example of a family of folded holomorphic maps of degree 1 into S^4. As a second example (see Section 9.1) I show how in certain special cases (e.g. if X is a folded elliptic fibration) folded holomorphic maps on a folded symplectic manifold can be equated with pseudo-holomorphic maps into a (different) symplectic manifold.

The results presented here are the beginnings of a general program whose ultimate goal is to construct Gromov-Witten type invariants for non-symplectic 4-manifolds and develop techniques for computing these invariants. That will involve in particular compactifying the moduli space constructed in my thesis and generalizing the admissible structures on the fold. First steps in this direction are carried out in Section 8.

While tunneling maps may seem a novel feature of the folded symplectic setting, they are already implicitly present in the gluing and degeneration arguments of Ionel and Parker, Hofer and Eliashberg. Tunneling maps give the difference between the smooth and the degenerate case and keep track of relative homology data.

As an application of this work, I hope to be able to use the theory of folded holomorphic maps to make progress toward the "recognition problem" for some folded symplectic manifolds, in particular for S^4. The space of rational folded holomorphic maps of degree 1 has the property that it "sweeps out" S^4_- in a similar fashion as lines in $\mathbb{P}^2$ sweep out $\mathbb{P}^2$ (see Figures 10 and 11). Arguments of Gromov show how to use this behavior of lines in $\mathbb{P}^2$ to recognize the complex projective plane, and these were generalized by McDuff and Lalonde to ruled surfaces. Similar arguments may work to recognize certain folded symplectic manifolds.

2 Background and Basic Definitions

2.1 Symplectic Manifolds

Definition 2.1 (Symplectic Manifold). *A symplectic manifold is a pair (X, ω) where X is a smooth manifold and ω is a closed non-degenerate 2-form on X.*

The existence of a symplectic form on a manifold gives us tools to study these manifolds. For example the study of symplectic submanifolds leads to Donaldson's description of symplectic manifolds via braids. The study of J-holomorphic curves leads to Gromov-Witten invariants for symplectic manifolds. Taubes showed that in dimension 4 certain invariants constructed in this way are actually independent of the choice of symplectic form and only depend on the underlying smooth structure.

Not every manifold is symplectic, but we can try to generalize the tools that we have on symplectic manifolds to a larger class of manifolds. Here we will weaken some of the assumptions on the symplectic form ω and try to extend symplectic results to this case. There are two different approaches, based on the following observations.

(1) Every four-manifold with intersection form that is not negative definite, there

exists a closed 2-form ω with $\omega^2 \geq 0$ and $(\omega^2)^{-1}(0) = Z$ is a disjoint union of embedded circles. Taubes used this approach to define J-holomorphic curves. (See [Tau98] and [Tau02].)

(2) Every oriented four-manifold carries a folded symplectic form. This was shown by Ana Cannas in [Can02].

We follow the second approach.

2.2 Folded Symplectic Manifolds

The content of this section is based on [CGW00].

Definition 2.2 (Folded Symplectic Structure). *Let X be a smooth $2n$-dimensional manifold. A* folded symplectic structure *ω is a closed 2-form such that ω^n is transverse to 0 (so $Z = (\omega^n)^{-1}(0)$ is a, possibly empty, smooth codimension 1 hypersurface) and $\omega^{n-1}|_Z$ is non vanishing. Z is called the fold.*

The last condition means that the kernel of ω, which by transversality is a real 2-plane bundle over Z, is transverse to TZ. This is equivalent to the requirement that

$$L := \ker(\omega)|_Z \subset TZ$$

is a 1-dimensional foliation. L is called the *characteristic foliation*. Thus intrinsically, the fold of a folded symplectic manifold is indistinguishable from an orientable hypersurface in a symplectic manifold.

Here are some examples:

1. $(\mathbb{R}^{2n}, \omega)$ where

$$\omega = x_1 dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + \ldots + dx_n \wedge dy_n \tag{2.1}$$

is folded symplectic. To see this, note that $\omega^n = x_1(dx_1 \wedge dy_1 \wedge \ldots \wedge dx_n \wedge dy_n)$ is transverse to zero, with fold defined by $Z = \{x_1 = 0\}$. Also, $\omega|_Z = dx_2 \wedge dy_2 + \ldots + dx_n \wedge dy_n$, so $\omega^{n-1}|_Z = dx_2 \wedge dy_2 \wedge \ldots \wedge dx_n \wedge dy_n \neq 0$.

2. S^{2n} has a folded symplectic form defined as follows. View $S^{2n} \subset \mathbb{R}^{2n+1}$ as the unit sphere, and let $\pi : S^{2n} \to \mathbb{R}^{2n}$ be the restriction of the coordinate projection $\mathbb{R}^{2n+1} = \mathbb{R}^{2n} \times \mathbb{R} \to \mathbb{R}^{2n}$ to S^{2n}. The standard symplectic form ω_0 on $\mathbb{R}^{2n}$ pulls back to a folded symplectic form $\omega = \pi^{-1}(\omega_0)$ on S^{2n}. Indeed, ω^2 degenerates on $Z = S^{2n} \cap \mathbb{R}^{2n} \times \{0\} \approx S^{2n-1}$ and it intersects zero transversely. Also $\omega|_Z \neq 0$.

3. The folded symplectic form in Example (2) is invariant under the antipodal map, so it descends to a folded symplectic form on the quotient $\mathbb{RP}^{2n}$.

4. On a Riemann surface Σ, any 2-form ω is closed. It is a folded symplectic structure provided it is transverse to the zero section. This condition is open and dense among all 2-forms.

5. On a 4-manifold, the conditions that a closed 2-from ω has a square that is transverse to the zero section is again open and dense in the space of closed 2-forms, while the condition that $\omega|_Z$ never vanishes is open but not generic in the space of closed 2-forms.

As with symplectic structures and contact structures, folded symplectic structures can locally be put in standard form.

Theorem 2.3 (Darboux). *For every folded symplectic form ω there exist local coordinates near the fold such that ω has the form (2.1).*

More generally, in [CGW00] it is proved that for any $\alpha \in \Omega^1(Z)$ that does not vanish on L we can extend the inclusion $i : Z \hookrightarrow X$ of the fold to an orientation preserving diffeomorphism

$$\phi : (-\varepsilon, \varepsilon) \times Z \to U \tag{2.2}$$

onto a tubular neighborhood U of the fold such that

$$\phi^*\omega \quad = \quad \pi^* i^* \omega + d(r^2 \pi^* \alpha), \tag{2.3}$$

where r is the coordinate function on $(-\varepsilon, \varepsilon)$ and π is the projection $\pi(r, z) \to z$.

Definition 2.4. *A* morphism $\psi : X_1 \to X_2$ *of folded symplectic manifolds* (X_1, ω_1) *and* (X_2, ω_2) *is a diffeomorphism satisfying*

$$\psi^*\omega_2 = \omega_1.$$

Such morphisms automatically take folds to folds.

2.3 Folded Connect Sums

The most important example of an operation on folded symplectic manifolds is the connect sum along symplectic submanifolds of arbitrary dimension. This procedure, described in Theorem 2.5 below generalized the "symplectic connect sum" operation that joins two symplectic submanifolds along a codimension 2 symplectic submanifold.

In the symplectic category connect sum can only be performed along codimension 2 symplectic submanifolds. The ordinary connect sum is performed by removing neighborhoods of points on each manifold and then gluing the boundaries. If this could be performed symplectically, then the manifold obtained by gluing up the two remaining pieces should also be symplectic. But this manifold is isomorphic to S^{2n}

and the only sphere that admits a symplectic form is S^2. Thus the connect sum operation cannot be performed symplectically in dimension ≥ 4.

On the other hand, S^{2n} does admit a folded symplectic form, so there is the possibility that connect sum can be performed in the folded symplectic category. Here is Ana Canna's construction ([CGW00]) for folded symplectic connect sum:

Let (M_1, ω_1) and (M_2, ω_2) be folded symplectic manifolds and let $p_i \in M_i$ be points not on the folds. Then we can obtain the manifold $(M = M_1 \sharp_{p_1,p_2} \overline{M_2}, \omega)$ in the following way. Near the points p_i the forms ω_i are non-degenerate, so we can pick symplectic Darboux charts near these points. Then pick annuli $A_i \approx S^{2n-1} \times [1,2]$ around p_i contained in such a chart. Let $\pi : S^{2n-1} \times [1,2] \to S^{2n-1}$ and α be the standard contact 1-form on S^{2n-1}, so $\omega|_{A_i}$ is diffeomorphic to $d(r_i \pi^*(\alpha))$, $r_i \in [1,2]$. Choose coordinates t_i such that $r_i = t_i^2$ for $t_i > \varepsilon$. Then we can extend ω across the symplectic sum by defining

$$\omega = d[(1+t^2) \wedge \pi^* \alpha], \qquad \text{where} \quad t = \begin{cases} -t_1 & \text{if } t < -\varepsilon \\ t_2 & \text{if } t > \varepsilon \end{cases}$$

In [CGW00] it is shown how to generalize this construction to gluing manifolds along almost contact manifolds with certain compatibility conditions. We modify this argument to show that we can perform connect sum along arbitrary codimension symplectic submanifolds to obtain a folded symplectic manifold.

Theorem 2.5 (Folded Connect Sum). *Let (M_1, ω_1) and (M_2, ω_2) be symplectic manifolds with symplectomorphic symplectic submanifolds V_1 and V_2, respectively, with symplectomorphic normal bundles.*

Then for any boundary Z of a small enough tubular neighborhood of V_1 in M_1 there exists a folded symplectic manifold (M, ω) with fold Z, obtained from M_1 and $\overline{M_2}$ by taking connect sum along V_1 and V_2, such that there exists tubular neighborhoods U_1

of V_1, U_2 *of* V_2 *and* U *of* Z *with*

$$(M \setminus U, \omega) = (M_1 \setminus U_1, \omega_1) \cup (\overline{M_2 \setminus U_2}, \omega_2).$$

Proof. By the Symplectic Neighborhood Theorem there exist neighborhoods U_1 of V_1 and U_2 of V_2 and a symplectomorphism $\phi : U_1 \to U_2$ extending the given symplectomorphism $V_1 \mapsto V_2$. Choose a tubular neighborhood $\tilde{U}_1$ of V_1 properly contained in U_1 and set $Z = \partial\tilde{U}_1$.

Define the manifolds

$$\begin{aligned} \hat{M}_1 &= M_1 \setminus \tilde{U}_1 \\ \hat{M}_2 &= M_2 \setminus \phi(\tilde{U}_1) \end{aligned}$$

with boundary Z identified via ϕ. Since ϕ is a symplectomorphism we have $\omega_1|_Z = \omega_2|_Z$ which we denote by ω_Z.

Let $\alpha \in \Omega^1(Z)$ with $\alpha > 0$ on the characteristic foliation. By the Coisotropic Embedding Theorem there exist collar neighborhoods $Z \times [0, \varepsilon) \subset U_i$ on which the symplectic forms ω_i, pull back to

$$\omega_i = \omega_Z + d(t_i\alpha), \qquad t_i \in [0, \varepsilon); \qquad i = 1, 2$$

Define the annulus $A_i \subset U_i$,

$$A_i = \{(z, t_i) \in Z \times [0, \varepsilon) | 0 < t_i < \varepsilon\}$$

and the orientation reversing diffeomorphism

$$\sigma : A_1 \to A_2 \qquad \sigma(z, t) = (z, \varepsilon - t).$$

Then define the manifold

$$M = \hat{M}_1 \cup_\sigma \overline{\hat{M}_2}$$

with open submanifold A the image of A_i.

Choose a monotone coordinate function $t : (-\varepsilon, \varepsilon) \to \mathbb{R}$ such that

$$t^2 = t_1 \quad \text{if} \quad t < -\sqrt{\varepsilon/2}$$

$$t^2 = t_2 \quad \text{if} \quad t > \sqrt{\varepsilon/2}$$

and define the folded symplectic 2-form ω on M by

$$\omega(x) = \begin{cases} \omega_Z + d(t^2\alpha) & x = (z,t) \in A \\ \omega_1 & x \in \hat{M_1} \setminus \{(z,t) | t \leq \frac{1}{2}\varepsilon\} \\ \omega_2 & x \in \hat{M_2} \setminus \{(z,t) | t \leq \frac{1}{2}\varepsilon\} \end{cases}$$

Then ω is a folded symplectic form on M with fold Z. □

2.4 CR Structures on the Fold

Given a folded symplectic 2n-dimensional manifold (X, ω) with fold Z, and an orientation on X, we obtain some additional structure. Since X is oriented, $X \setminus Z$ is the disjoint union of the open manifolds X_+ and X_-, the regions where the orientation agrees or disagrees with the one induced by ω, respectively. Therefore Z is also oriented.

Observe that we have a canonical 2-dimensional subbundle

$$E = \ker(\omega) \subset T_Z X$$

and the 1-dimensional subbundle

$$L = \ker(\omega_Z) \subset E.$$

Both L and E are oriented.

In [HZ94], Hofer and Zehnder made the definition of a stable hypersurfaces in a symplectic manifold. As observed earlier, the folds of folded symplectic manifolds look intrinsically like oriented hypersurfaces of symplectic manifolds, so the definition transfers readily to our setting.

Definition 2.6. *The fold Z of a folded symplectic manifold (X, ω) is* dynamically stable *if there exists an embedding*

$$\psi : Z \times (-\varepsilon, \varepsilon) \to U$$

extending the inclusion $\iota : Z = Z \times 0 \hookrightarrow X$ such that the flow ϕ_s of ∂_t, $t \in (-\varepsilon, \varepsilon)$ on $Z \times (-\varepsilon, \varepsilon)$ preserves the characteristic foliation on each slice $Z \times t$, i.e. if $(\phi_s)_*$ induces bundle isomorphisms

$$(\phi_s)_* : L \to L$$

where $L(z, t) = \ker(\omega|_{Z \times t})$.

A folded symplectic manifold is called stable *if its fold is dynamically stable.*

We assume that all of the folded symplectic manifolds we work with are stable. The importance of this requirement is that dynamically stable folds admit nice CR-structures; these seem essential for the analysis carried out later. We need a special 1-form on Z.

Definition 2.7. *A 1-form $\alpha \in \Omega^1(Z)$ on the fold Z of a folded symplectic manifold is called* stable *if*

$$\alpha \wedge \omega^{n-1} > 0 \tag{2.4}$$

$$\ker(\omega) \subset \ker(d\alpha). \tag{2.5}$$

The existence of a stable α turns out to be equivalent to Z being dynamically stable as was pointed out to the author by Y. Eliashberg:

Lemma 2.8. *There exists a stable 1-form α on the fold Z of a folded symplectic manifold if and only if Z is dynamically stable.*

Proof. We adopt the notation from Definition 2.6.

For every fold admitting such a 1-form α there exists such an embedding ψ by equation (2.3).

Conversely, suppose that there exists such an embedding ψ. Define the 1-form

$$\alpha = \frac{1}{t}\iota_{\partial_t}\omega$$

and let R be the unique vector field such that

$$\iota_R\omega = -t\,dt.$$

Note that this is well-defined since $\iota_{\partial_t}\omega$ vanishes on the fold $\{(z,t)|t=0\}$.

Then $\alpha(R) = 1$ and $R \in L$. Also $\mathcal{L}_{\partial_t}R = f \cdot R \in L$ for some function f since the flow of ∂_t preserves L. Also

$$\mathcal{L}_R\omega = d\iota_R\omega + \iota_R d\omega = d(-t\,dt) = 0$$

and $\mathcal{L}_R\frac{1}{t} = 0$ since R is tangent to the level surfaces of t.

Recall the formula

$$\mathcal{L}_X \circ \iota_Y = \iota_Y \circ \mathcal{L}_X + \iota_{[X,Y]}$$

operating on forms. Then

$$\iota_R d\alpha = \mathcal{L}_R\alpha = \mathcal{L}_R\left(\frac{1}{t}\iota_{\partial_t}\omega\right) = \frac{1}{t}\left(\iota_{\partial_t}\mathcal{L}_R\omega + \iota_{[R,\partial_t]}\omega\right) = -\frac{1}{t}f\iota_R\omega = f\,dt$$

which vanishes on vectors tangent to the slices $t = const$.

So $L = \ker(\omega|_Z) \subset \ker(d\alpha)$ and $\alpha|_Z$ is a 1-form with the desired properties. $\square$

It is not clear under exactly what conditions a folded symplectic manifold (X,ω) is stable, but there are important cases in which it is: (X,ω) with fold Z is stable if

(a) Z is an S^1 bundle with the vertical subspaces being the characteristic foliation and α is a connection 1-form,

(b) (Z, α) is contact with contact form α such that $d\alpha = \omega_Z$, or

(c) α is exact.

Following the definitions of [BEH+03] we assume that α is chosen to be stable. Such an α determines a canonical section R of L by the requirement that

$$\iota_R \omega \;=\; 0, \qquad \alpha(R) = 1, \tag{2.6}$$

called the *characteristic vector field.*

Note that in case (a) the flow generated by R defines a free S^1 action on Z preserving ω and α. This motivates the following

Definition 2.9. *A folded symplectic manifold has an* S^1-invariant fold *if we can choose a 1-form α on Z that is non-vanishing on L, such that the flow of the associated characteristic vector field defines a free S^1 action on Z that preserves ω and α.*

A stable α defines a symplectic subbundle $F = \ker(\alpha) \subset TZ$ over Z such that

$$T_Z X = E \oplus F. \tag{2.7}$$

Now choose a background metric g' on X such that the splitting (2.7) is g'-orthogonal. Mimicking the standard procedure to generate a compatible triple on a symplectic manifold using the background metric g' we obtain a folded triple (ω, g, J) on $X \setminus Z$ satisfying the compatibility conditions

$$\begin{aligned} J^*\omega &= \omega \\ g(u, v) &= \omega(u, Jv). \end{aligned}$$

The details are given in Appendix A.

Definition 2.10. *Let*

$$\mathcal{J}(X,\omega,\alpha)$$

be the set of almost complex structures J on $X \setminus Z$ obtained by the above construction with the splitting (2.7) induced by ω and α.

Each $J \in \mathcal{J}(X,\omega,\alpha)$ determines a compatible triple (ω, J, g_J) and allows us to extend the splitting (2.7) to a neighborhood of the fold such that it is J-invariant and g_J and ω orthogonal. Restricting (ω, J, g_J) to F yields a smooth compatible triple on F and the restriction to E satisfies

$$\begin{aligned} \omega_E &= r\mu \\ g_E &= |r|h \\ J_E &= \operatorname{sign}(\omega^2)\tilde{J} \end{aligned} \tag{2.8}$$

where $(\mu, h, \tilde{J})$ is a smooth compatible triple on E. The details are given in Lemma A.2.

The complex structure $\tilde{J}$ allows us to define a complement K of L in E by $K = \tilde{J}L$, so we can refine the splitting (2.7) over Z to

$$T_Z X = K \oplus L \oplus F. \tag{2.9}$$

Fix a non-negative bump function β depending on r and supported in a Darboux tubular neighborhood of the fold that is equal to 1 on the fold and vanishing outside this neighborhood of the fold. Then define the non-degenerate metric

$$\tilde{g} = g + \beta \cdot h. \tag{2.10}$$

By choosing a background metric g' that makes the splitting $T_Z X = K \oplus L \oplus F$ orthogonal, we may assume that Z as well as each leaf of the characteristic foliation are totally geodesic with respect to h and therefore also $\tilde{g}$. Henceforth we assume that $\tilde{g}$ has these properties.

In the case that we have an S^1 invariant (ω, α) we can arrange (by starting from an S^1-invariant background metric g') for the compatible triple to also be S^1 invariant over U. Therefore we will assume that in case of S^1 invariant folds, the compatible triple is also chosen to be S^1 invariant near the fold.

Equations (2.8) show that J is discontinuous across the fold in the E directions. However, on U we may define two smooth complex structures, denoted by $J^{\pm}$, such that $J^{\pm}|_{X_{\pm}} = J$ by choosing $J^{\pm}_E = \pm\tilde{J}$.

As observed in [BEH+03], (Z, F, J) defines a CR structure on Z.

2.5 Circle Invariant folds

Folded symplectic manifolds with S^1-invariant folds are especially easy to work with. They also occur frequently. The standard folded symplectic structure on the spheres (described in the next section) is of this type. Connected sums of symplectic 4-manifolds always have folded symplectic structures of this type. More generally, one can arrange for the operation of Theorem 2.5, which forms the connect sum along symplectic submanifolds of any dimension, to produces symplectic structures with S^1-invariant folds in the case that the submanifolds are of codimension 2. This can be seen by choosing Z more carefully as for example done in the symplectic connect sum construction in [MW94] or [IPar].

S^1-invariant folds have a special structure, as described in the following Lemma. We will use this lemma repeatedly in later sections.

Lemma 2.11. *In the case of an S^1 invariant fold, Z is an $S^1 = \mathbb{R}/\mathbb{Z}$ bundle over a symplectic manifold (V, ω_V) with projection $\pi_V : Z \to V$ such that*

1. $\omega_Z = \pi_V^* \omega_V$

2. there exists an ω_V *compatible almost complex structure* j_v *on* V *such that* $d\pi_V|_F$ *is* (J, j_v) *linear.*

Moreover α *may be chosen such that*

$$d\alpha = C \cdot \omega_Z \qquad C = c_1(Z)/vol(V) \tag{2.11}$$

where c_1 *is the first Chern class of the circle bundle* $Z \to V$ *and* $vol(V)$ *is w.r.t.* ω_V.

Proof. Since the S^1 action on Z is free we can exhibit Z as an S^1 bundle

$$\pi_V : Z \to V$$

over a closed $(2n-2)$-dimensional manifold V. Since ω is S^1 invariant and its kernel coincides with the vertical subspace there exists a 2-form ω_V on V such that $\omega = \pi_V^*\omega_V$. One readily checks that ω_V is closed.

The complex structure $J|_F$ induces an almost complex structure j on the quotient V in the following way. Because $\ker(d\pi_V)$ is transverse to L,

$$d\pi_V|_F(z) : F \to T_{\pi_V(z)}S$$

is an isomorphism for each $z \in Z$. Since the complex structure J on F is invariant under the S^1-action this map induces a complex structure j on V so that $d\pi_V|_F$ is (J, j) linear.

To see equation (2.11) let α_0 be a connection 1-form on Z, i.e. α_0 is invariant under the S^1-action and satisfies $\alpha_0(R) = 1$. Then $\iota_R d\alpha_0 = \mathcal{L}_R\alpha_0 = 0$, so $d\alpha_0 = \pi_V^*\omega_0$ is the pullback of a 2-from ω_0 on V which is just the curvature of the connection α_0 and therefore represents $c_1(Z)$.

Since ω_V is a volume form on V there exists a constant $c \in \mathbb{R}$ with

$$\int_V (c\,\omega_V - \omega_0) = 0,$$

so $(c\,\omega_V - \tilde{\omega}) = d\beta$ for some 1-form β on V. With the gauge transformation $\alpha = \alpha_0 + \pi_V^* \beta$ we still have $\alpha(R) = 1$, and

$$d\alpha = d\alpha_0 + \pi_V^* d\beta = \pi_V^* \omega_0 + \pi_V^*(c\omega_V - \omega_0) = c\,\pi_V^* \omega_V = c\,\omega_Z.$$

□

3 Motivating Example

To motivate what follows we will investigate possibilities to define pseudo-holomorphic maps into S^4 with canonical folded symplectic structure as defined in Section 2.2. One overruling principle is that we want to obtain well-behaved moduli spaces of such maps. More precisely we are looking for a notion of pseudo-holomorphic maps in folded symplectic manifolds such that the linearized equations at a solution give rise to a Fredholm operator and that the solutions are stable under perturbations (away from the fold) of the structures involved.

Recall that S^4 does not admit any symplectic form since its second cohomology is trivial. Moreover, S^4 does not admit an almost complex structure. To see this let (M, J) be an almost complex 4-manifold. Recall that its first Pontryagin class is given by

$$p_1(TM) = -c_2(TM \otimes_{\mathbb{R}} \mathbb{C}) = -c_2(TM \oplus \overline{TM})$$

and the total Chern class of $TM \oplus \overline{TM}$ satisfies

$$c(TM \oplus \overline{TM}) = (1 + c_1(M) + c_2(M))(1 - c_1(M) + c_2(M)) = 1 + 2c_2(M) - (c_1(M))^2$$

Therefore $p_1(TM) = -2c_2(M) + (c_1(M))^2$. If M has vanishing second cohomology like S^4, then $p_1(TM) = -2c_2(M)$.

Since TS^4 is stably trivial we conclude that $p_1(TS^4) = 0$ by the Whitney sum formula. But $c_2(TS^4) = 2e(TS^4) = 4$, so S^4 cannot be almost complex.

Therefore the answer to the question how to generalize pseudo-holomorphic maps to this setting is far from obvious. We let ourselves be guided by the folded symplectic structure ω. Since ω is non-degenerate on $X \setminus Z$, or $S^4 \setminus S^3$ in this particular case, the usual procedure to construct a compatible triple will yield an almost complex structure J there. Then it is clear what a pseudo-holomorphic map from a Riemann surface (Σ, j) into $X \setminus Z$ is, namely a map with (j, J)-linear differential. Since the fold Z separates X into $X_\pm$ this means that maps from a connected domain into $X \setminus Z$ will have image in only one side X_+ or X_- of the fold. The question then is how to allow maps to "cross the fold" i.e. have image on both sides of the fold.

One way to do this is to choose an almost complex structure on $X \setminus Z$ that degenerates along Z in a way that $X_\pm$ has "cylindrical ends" in the sense of [EGH00]. This then reduces to the fairly standard problem of holomorphic curves relative to closed characteristics as discussed in [IP03], [EGH00] and [BEH$^+$03]. In effect this is treating the two sides $X_\pm$ as separate manifolds with boundary and qualitatively not different from studying holomorphic curves in a symplectic manifold where the complex structure on the target has been degenerated. In [IPar] it was shown how to reverse the process of the degeneration of complex structure to recover holomorphic curves in the symplectic manifold with smooth almost complex structure.

We try and find the analogue of holomorphic curves with non-degenerate almost complex structure for folded symplectic manifold, while being guided by the hope that these will limit to the relative curves discussed above as we degenerate the almost complex structure. To do this we define an almost complex structure on S^4 that is non-degenerate in the sense that its norm w.r.t. a metric on S^4 stays bounded, but is (necessarily) discontinuous across the fold. One way to do this is to use a background metric to construct a compatible triple as in the symplectic case. This process is described in more detail in Section A.

We describe the folded symplectic structure and a compatible almost complex structure with nice symmetry properties on S^4.

View S^4 as the unit sphere in $\mathbb{R}^5$. Then we have

- the restriction of the coordinate projection $\mathbb{R}^5 \approx \mathbb{R} \times \mathbb{R}^4 \to \mathbb{R}^4$ to S^4, and
- the stereographic projections $\sigma_\pm : S^4 \setminus (\pm 1, 0, 0, 0, 0) \to \mathbb{R}^4$.

Let ω_0 and J_0 be the standard symplectic and complex structures on $\mathbb{R}^4$. Then $\omega = \pi^*\omega_0$ and $J = \pi^* J_0$ give a folded compatible triple (J, ω, g) on S^4 with $g(u, v) = \omega(u, Jv)$. The orientation induced by the folded symplectic form agrees (disagrees) with the canonical orientation on S^4 on the upper (lower) hemisphere S^4_+ (S^4_-), the fold $Z = S^3$ is the intersection of S^4 with the equatorial plane $\{(x_0, x_1, x_2, x_3, x_4) | x_0 = 0\}$. The choice of J is compatible with choosing α to be the canonical contact structure on S^3, i.e. α is the restriction of the canonical 1-form

$$\alpha = \frac{1}{2} (x_1 dx_2 - x_2 dx_1 + x_3 dx_4 - x_4 dx_3)$$

to S^3. Therefore $d\alpha = \omega_Z$.

The map

$$\begin{aligned} \tau &: S^4 \to S^4 \\ &(x_0, x_1, x_2, x_3, x_4) \mapsto (-x_0, x_1, x_2, x_3, x_4) \end{aligned}$$

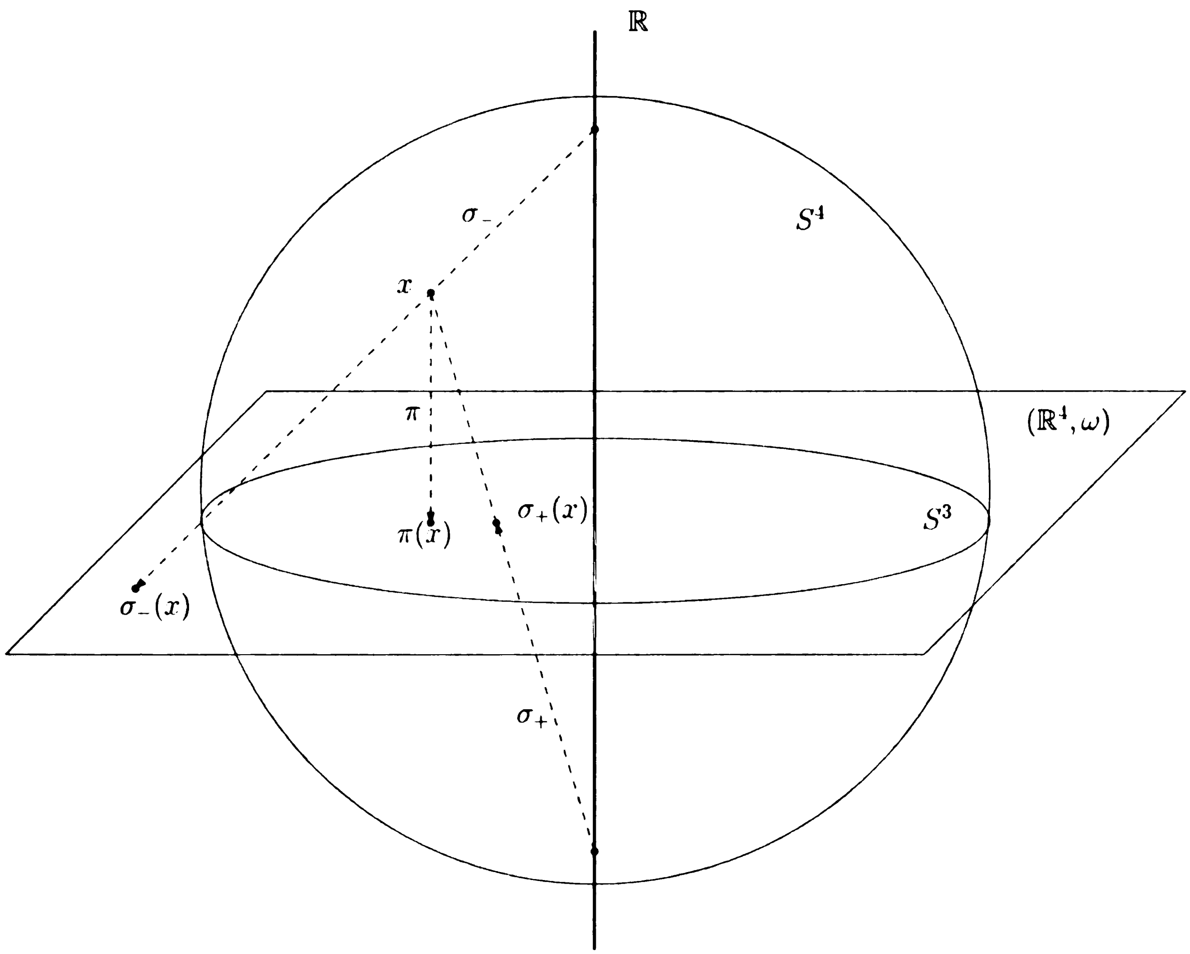

Figure 1: Folded Structures on S^4

is a biholomorphic involution on S^4 exchanging the upper and lower hemisphere and fixing the fold.

The bundle $F = \ker(\alpha)$ is given by the contact planes of the fold S^3 and E is spanned by the characteristic direction given by the vertical subspaces of the Hopf fibration and the "radial" direction.

There are a couple of straightforward observations about (non-)possibilities of holomorphic curves. First note that any non-trivial J-holomorphic has to cross the fold, since each side is biholomorphic to $B^4 \subset \mathbb{C}^2$ which has vanishing homology and therefore does not admit non-trivial holomorphic curves by their energy minimizing property.

For the following fix a Riemann surface (Σ, j) as domain.

Lemma 3.1. *There are no smooth J-holomorphic maps into S^4, i.e. there is no smooth map $u : \Sigma \to S^4$ such that $\bar{\partial}_J u = 0$ on $\Sigma \setminus u^{-1}(Z)$.*

Proof. Set $\sigma = u^{-1}(Z)$ and $\Sigma_\pm = u^{-1}(S^4_\pm)$. Recall that the fold S^3 is a pseudo-convex boundary of $B^4 \subset \mathbb{C}^2$ (cf. e.g. [AHar]) so any J-holomorphic map $u_\pm : \Sigma_\pm \to S^4_\pm$ with limit on $Z = S^3$ has to be transverse to Z by the strong maximum principle. Therefore u is transverse to the fold and σ is a smooth separating submanifold of Σ. But for $0 \neq \eta \in T_\sigma \Sigma$

$$\pi_E\, du_+(j\eta) = \pi_E\, du_-(j\eta) = J_- \pi_E\, du_+(\eta) = -J_+ \pi_E\, du_+(\eta) = -\pi_E\, du_+(j\eta)$$

so $\pi_E\, du = 0$ along σ, contradicting that u is transverse to the fold. □

The proof shows that a stronger result is true, namely there are no continuous J-holomorphic maps into S^4 that have C^1-smooth one-sided limits on the fold.

We see that the reason for the failure of maps to be smooth is essentially due to the change in orientation of the almost complex structure on the bundle E which is transverse to the fold. One can remedy this by the following equivalent modifications:

1. Ask that maps are holomorphic at points that have image in S^4_+ and anti-holomorphic at points that have image in S^4_-.

2. Work with domains Σ with oriented separating submanifold σ separating $\Sigma\backslash\sigma = \Sigma_+ \cup \Sigma_-$ where the orientation of $\partial\Sigma_+$ ($\partial\Sigma_-$) agrees (disagrees) with that of σ. Then demand that maps $u : \Sigma \to S^4$ send $\Sigma_\pm$ into $S^4_\pm$ and work with complex structures $j_\pm$ on $\Sigma_\pm$ such that there exists a smooth complex structure j on Σ with $\pm j_\pm = j|_{\Sigma_\pm}$.

3. Modify the almost complex structure on S^4 so that it is continuous on the transverse bundle E and discontinuous on the bundle $F \subset TZ$.

4. Choose a new almost complex structure $\tilde{J}$ on S^4 by replace the almost complex structure J on S^4_- defined above by $-J$ while leaving the structure on S^4_+ unchanged, i.e.

$$\tilde{J}(x) = \begin{cases} J(x) & x \in S^4_+ \\ -J(x) & x \in S^4_- \end{cases}$$

We only discuss the last modification.

Lemma 3.2. *If u is a smooth $\tilde{J}$-holomorphic map into S^4, then its intersection with S is a collection of closed characteristics.*

Proof. S^4_+ has still a pseudo-convex boundary, so for the same reasons as in Lemma 3.1 u is necessarily transverse to the fold and $\sigma = u^{-1}(Z)$ is a smooth separating submanifold. Let $p \in \sigma$. Then

$$\pi_F \, du(p) = 0$$

by a similar argument as in Lemma 3.1. Since u is transverse to the fold, $\hat{u} = u|_\sigma : \sigma \to Z$ is an immersion, and since $\pi_F \, \hat{u} = 0$ it is tangent to the characteristic foliation. Since σ is necessarily closed, each component wraps a closed characteristic non-trivially. □

Examples of such smooth $\tilde{J}$-holomorphic maps into S^4 are restrictions of lines in $\mathbb{C}^2$ that pass through the origin. In fact it is not difficult to see that any smooth $\tilde{J}$-holomorphic map is of that form (or a multiple cover of such a map). But these solutions are extremely unstable – when one perturbs $\tilde{J}$ at the north or south pole in a generic way, all of these solutions die.

Expanding on this example, one might see that asking for maps to be smooth is too restrictive. It seems more natural to consider maps that are smooth up to the fold but only continuous across the fold. A similar argument as in Lemma 3.1 shows that there are no such J-holomorphic maps. But there are many such $\tilde{J}$-holomorphic maps as the following Lemma shows.

Lemma 3.3. *Let (Σ, j) be a Riemann surface with boundary $\partial\Sigma$. Then $u : (\Sigma, \partial\Sigma) \to (S^4_+, S^3)$ is $\tilde{J}$-holomorphic if and only if $\tau \circ u : (\Sigma, \partial\Sigma) \to (S^4_-, S^3)$ is $-\tilde{J}$-holomorphic.*

Proof. τ is a biholomorphism for J so it is an anti-holomorphic involution for $\tilde{J}$. □

Now let $(\tilde{\Sigma}, \tilde{j})$ be the double of (Σ, j), i.e.

$$\tilde{\Sigma} = \Sigma \sqcup_{\partial\Sigma} \overline{\Sigma}$$

with complex structure j on Σ and $-j$ on $\overline{\Sigma}$. Then if $u : (\Sigma, \partial\Sigma) \to (S^4_+, S^3)$ is $\tilde{J}$-holomorphic the map

$$\tilde{u}(x) = \begin{cases} u(x) & x \in \Sigma \\ \tau \circ u(x) & u \in \overline{\Sigma} \end{cases}$$

is a continuous $\tilde{J}$-holomorphic map.

By the unique continuation theorem for holomorphic maps all continuous $\tilde{J}$-holomorphic maps into S^4 are of this form. Therefore such maps are in bijective correspondence to holomorphic maps

$$u : (\Sigma, \partial\Sigma) \to (B^4 \subset \mathbb{C}^2, S^3).$$

But the space of such maps is infinite dimensional and each map has an infinite dimensional space of holomorphic deformations, even if we quotient out reparametrizations. To see this note that for well behaved maps we may find a totally real subbundle $W \subset TZ \subset T_Z S^4$ with $u(\eta) \subset W$ for all $\eta \in T\sigma$. Furthermore, by choosing W appropriately one may arrange for the Maslov index of u with respect to W to be large enough so that the index of the linearized operator at u operating on sections with boundary values in W is positive. So for each such choice of W there exists non-empty finite dimensional space of nearby solutions that also map the boundary $\partial\Sigma$ into Z. But there are uncountably many choices of such totally real W, so there exists uncountably many nearby solutions.

This shows that considering holomorphic curves with "continuous images" cannot lead to a Fredholm problem. The condition of "continuous images" does not pose an elliptic boundary value problem for the linearized Cauchy-Riemann operator.

One possible remedy is to impose additional constraints to cut down the solutions to a finite dimensional space. But there is no evident way of doing that which is stable under perturbations.

Another approach is to allow discontinuous images. We think of this as holomorphic curves that leave the fold at a location that is different from where they enter, the relation between these is given by a tunneling map in Z. This way we define a Fredholm problem for discontinuous pseudo-holomorphic maps into folded symplectic manifolds. Roughly speaking, a folded holomorphic map consists of

- a domain (Σ, j) with submanifold separating Σ into Σ_+ and Σ_-
- a J-holomorphic map $u_+ : \Sigma_+ \to X_+$
- a J-holomorphic map $u_- : \Sigma_+ \to X_-$
- and a tunneling map connecting $u_+(\partial\Sigma_+)$ to closed characteristics and then continuing on to connect to $u_-(\partial\Sigma_-)$.

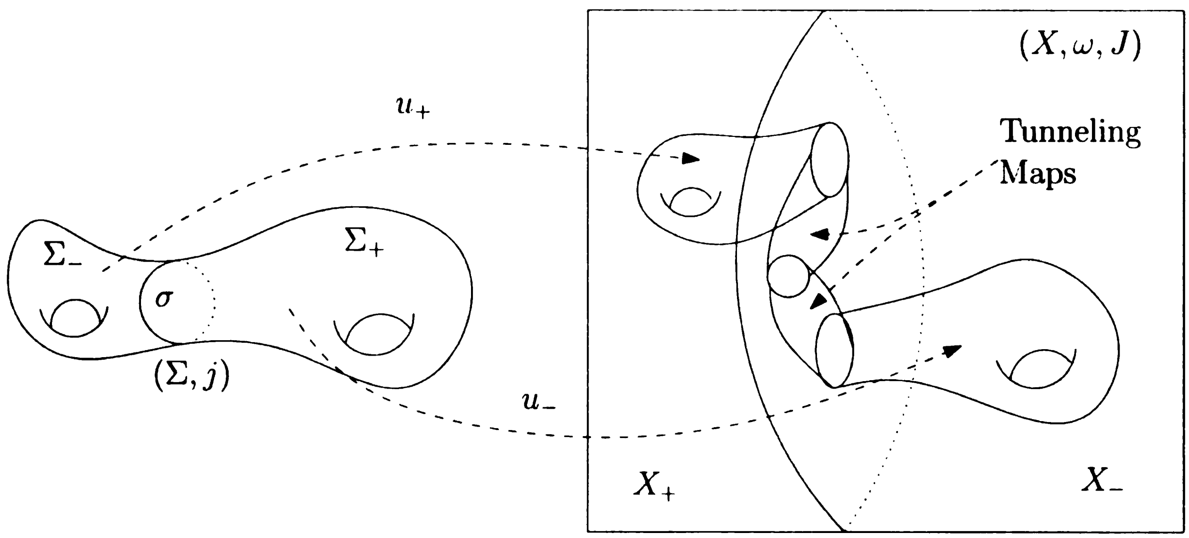

Figure 2: The map tunnels through the fold, exiting the fold at a location that is different from where it entered.

We will make this precise in the following sections. But first we will give a trivial example of discontinuous folded holomorphic maps.

Consider the complex elliptic fibration $X = E(1)$ and let $T \hookrightarrow E(1)$ be a regular fiber. There exists a tubular neighborhood X_- of T in $E(1)$ biholomorphic to $X_- = D \times T$, where D is the closed unit disk in $\mathbb{C}$ with the canonical complex structure. Set $X_+ = E(1) \setminus X_-$.

Consider the following two self-maps of the boundary $Z = \partial X_\pm = S^1 \times T$

$$\begin{aligned} \iota(z,w) &= (z,w) \\ \iota_F(z,w) &= (\bar{z},w). \end{aligned}$$

Then we have the manifolds

$$\begin{aligned} E(1) &= X_+ \sqcup_\iota X_- \\ E^F(1) &= X_+ \sqcup_{\iota_F} \overline{X_-} \end{aligned}$$

with almost complex structure inherited from each piece.

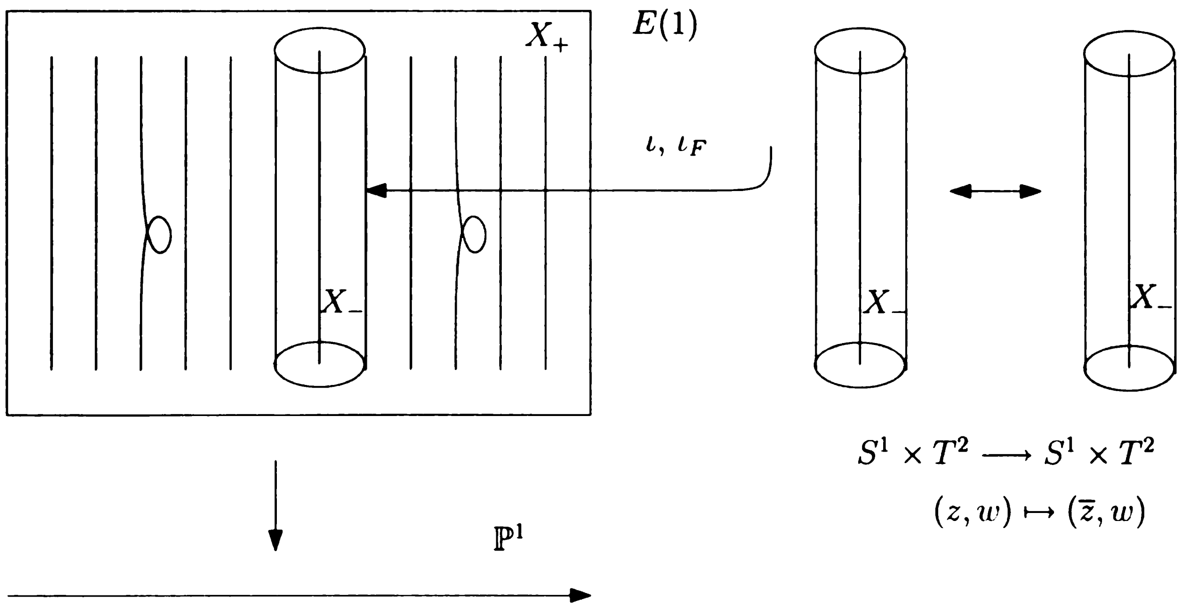

Figure 3: The manifolds $E(1)$ and $E^F(1)$.

The construction of Section 2.3 exhibits $E^F(1)$ as a folded symplectic manifold. (With a little more care we may define smooth structure on $E^F(1)$.) In particular, there exists a biholomorphism

$$\Psi : E(1) \setminus Z \to E^F(1) \setminus Z$$

given by the identity map on each piece, X_+ and X_-.

This setup suggest the following definition for folded holomorphic maps into the folded symplectic manifolds $E^F(1)$:

Definition 3.4. *A folded holomorphic map $u : \Sigma \to E^F(1)$ is a map such that $\Psi \circ u$ is (the restriction to the domain of definition of) a holomorphic map in $E(1)$.*

It is clear from the definition that this will yield a well-defined moduli space, although the maps are necessarily discontinuous.

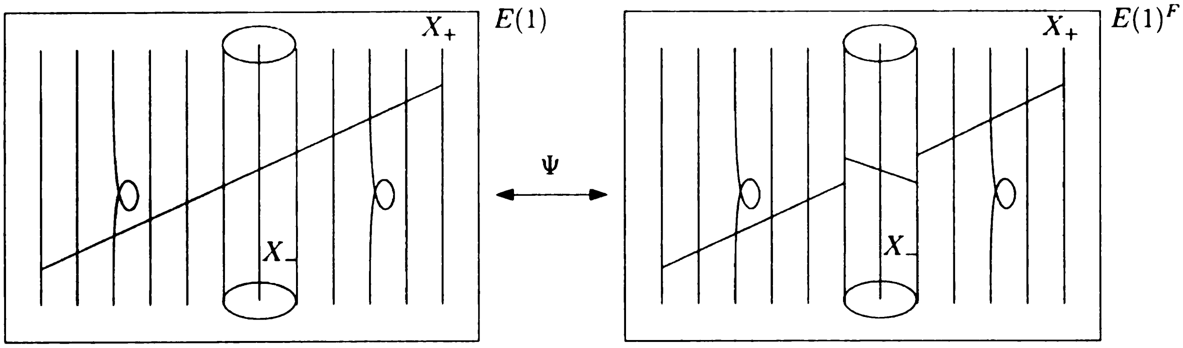

Figure 4: Folded holomorphic maps in $E^F(1)$.

The intuition behind this definition is that $E^F(1)$ is a folded symplectic manifold that was glued up in the wrong way. In general, there is no global way to cut a folded symplectic manifold into the pieces X_+ and X_- and glue them back together to obtain a symplectic manifold. In the following we will show how to nevertheless define folded holomorphic maps into more general folded symplectic manifolds that will reduce to Definition 3.4 in the case of $E^F(1)$.

Note that the pieces u_+ and u_- of a folded holomorphic map into $E^F(1)$ have boundary values in

$$\Delta^Z = \{(\hat{u}_+, \hat{u}_-) \in \mathrm{Map}(\sigma, S^1 \times T^2) \times \mathrm{Map}(\sigma, S^1 \times T^2) | \hat{u}_- = \Psi \circ \hat{u}_+\}.$$

We call Δ^Z the *folded diagonal.* The above definitions work in the case where the fold Z has the structure of a trivial S^1-bundle with the characteristic foliation being vertical. To define folded holomorphic maps into more general folded symplectic manifolds we need to generalize the folded diagonal in a way such that it continues to give elliptic boundary conditions for the pieces u_+ and u_-.

4 Folded Holomorphic Maps

Here we will define folded holomorphic maps and lay the functional analytic foundation for the later sections. We start by describing the domains of folded holomorphic maps, then we set up the Sobolev spaces and lastly we set up the PDE.

4.1 Folded Domains

Definition 4.1 (Folded Domain). *A* folded domain $\mathcal{D}$ *consists of*

(i) two closed Riemann surfaces $(\Sigma_0, j_0, \mathbf{p}_0) \in \mathcal{M}_{g_0,n_0}$ *and* $(\Sigma_1, j_1, \mathbf{p}_1) \in \mathcal{M}_{g_1,n_1}$

(ii) functions $\tau_i : \Sigma_i \to \mathbb{R}$ *with zeros of at most finite order such that*

$$\mathbf{p}_1 \subset \Sigma_1^-, \qquad \text{where } \Sigma_i^\pm = \{x \in \Sigma_i | \pm \tau_i(x) \geq 0\},\ i = 1, 2$$

(iii) a function $g : \Sigma_0^+ \to \mathbb{R}$ *and a diffeomorphism*

$$\psi : (\Sigma_0^+, j_0) \to (\Sigma_1^+, j_1),$$

satisfying the conditions

$$\psi^* j_1 = j_0, \qquad \psi^* \tau_1 = e^g \tau_0. \tag{4.12}$$

We set $\mathbf{p}_0^{\pm} = \mathbf{p}_0 \cap \Sigma_0^{\pm}$ to be the marked points contained in $\Sigma_0^{\pm}$. The marked points $\mathbf{p}_1$ are called *punctures.* The zero sets $\sigma_0 = \tau_0^{-1}(0)$ and $\sigma_1 = \tau_1^{-1}(0)$ are called the *domain folds.* To simplify notation we set

$$\Sigma_{\pm} = \Sigma_0^{\pm}, \qquad S = \Sigma_1^{-}, \qquad \dot{S} = S \setminus \mathbf{p}_1.$$

Moreover, when no confusion can occur we will drop the subscripts on j_i and τ_i.

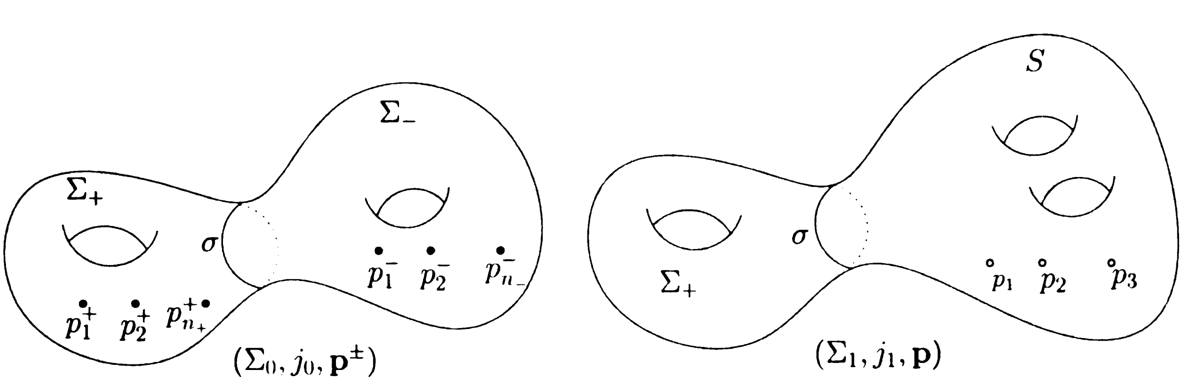

Figure 5: Folded Domains

The purpose of the functions τ_1 and τ_2 is to give possible locations of the domain fold σ, but we are not interested at the specific values of τ_i away from there zero set.

The space of functions $f : \Sigma_1 \to \mathbb{R}$, acts on a folded domain $\mathcal{D}$ by

$$\tau_1 \mapsto \tau_1' = e^{f_1}\tau_1$$

$$g \mapsto g' = g - \psi^* f_1.$$

Then

$$\psi^*\tau_1' = e^{g+-\psi^* f_1}\psi^*(e^{f_1}\tau_1) = e^{g+-\psi^* f_1}e^{\psi^* f_1}\psi^*\tau_1 = e^g\tau_0$$

so this is well-defined.

Moreover, $(\phi_0, \phi_1) \in \mathrm{Diff}^+(\Sigma_0) \times \mathrm{Diff}^+(\Sigma_1)$ acts on a folded domain $\mathcal{D}$ by

$$
\begin{aligned}
j_i &\mapsto \phi_i^* j_i = d\phi_i^{-1} \circ j_i \circ d\phi_i \\
\tau_i &\mapsto \tau_i \circ \phi_i \\
\mathbf{p}_i &\mapsto \phi_i(\mathbf{p}_i) \\
\psi &\mapsto \phi_1^{-1} \circ \psi \circ \phi_0 \\
g &\mapsto g \circ \phi_0.
\end{aligned}
$$

To see this we check equations (4.12).

$$
\begin{aligned}
\psi'^* j_1' &= d\psi'^{-1} \circ j_1' \circ d\psi' \\
&= d\phi_0^{-1} d\psi \circ d\phi_1 \circ d\phi_1^{-1} \circ j_1 \circ d\phi_1 \circ d\phi_1^{-1} \circ d\psi \circ d\phi_0 \\
&= d\phi_0^{-1} d\psi \circ j_1 \circ d\psi \circ d\phi_0 \\
&= d\phi_0^{-1} j_0 \circ d\phi_0 \\
&= j_0' \\
\psi'^* \tau_1' &= \tau_1' \circ \psi' \\
&= \tau_1 \circ \phi_1 \circ \phi_1^{-1} \circ \psi \circ \phi_0 \\
&= \tau_1 \circ \psi \circ \phi_0 \\
&= (e^g \tau_0) \circ \phi_0 \\
&= e^{g \circ \phi_0} \tau_0 \circ \phi_0 \\
&= e^{g'} \tau_0'.
\end{aligned}
$$

This leads us to the following

Definition 4.2. *The group*

$$\mathcal{G} = \mathrm{Map}(\Sigma_0, \mathbb{R}) \times \mathrm{Diff}^+(\Sigma_0) \times \mathrm{Diff}^+(\Sigma_1) \tag{4.13}$$

is called the gauge group.

Next we recall (see e.g. [BEH$^+$03]) a natural compactification of a Riemann surface Σ (possibly with boundary) with punctures p_k, $k = 1, \dots K$, that we need in our context.

Definition 4.3 (Radial Compactification). *The punctured disk $\mathbb{D} \setminus 0$ with canonical metric is isometric to the open annulus $(0, \epsilon) \times S^1$ by the map given in polar coordinates by $(r, e^{i\theta}) \mapsto (\frac{1}{2} r^2 e^{i\theta})$, where we equip the annulus with the canonical volume form $ds \wedge d\theta$. After choosing local holomorphic coordinates around each puncture p_k in S, that diffeomorphism defines a compactification of $\hat{S}$, that is a manifold with one boundary component Γ_k (the circle corresponding to $\{0\} \times S^1$ under the diffeomorphism) for each puncture.*

When we consider metrics on the radial compactification $\hat{S}$, we will always assume that they coincide with the metric induced by the one on the cylinder, so the curves Γ_k have length one.

The canonical projection $\pi : \hat{S} \to S$ sends each Γ_k to p_k and is a diffeomorphism from $\hat{S} \setminus \bigcup_k \Gamma_k \to S \setminus \bigcup_k \{p_k\} = \dot{S}$.

Remark 4.4. 1. The sets $\mathbf{p}_0^{\pm}$ are disjoint unless some of the marked points lie on σ_0.

2. Let be $\hat{S}$ the radial compactification of $\dot{S}$ as in Definition 4.3. Then we define the associated 2-dimensional C^0-cycles with boundary $(\tilde{\Sigma}_{\pm}, \mathbf{p_1})$ by gluing Σ_+ to $\hat{S}$ to get $\tilde{\Sigma}_+ = \hat{\Sigma}_1$ or by gluing Σ_- to $-\hat{S}$ to get $\tilde{\Sigma}_-$, where the gluing is done along σ.

Note that $\tilde{\Sigma}_+$ is actually a smooth Riemann surface. for $\tilde{\Sigma}_-$ this is only true in the case that the domain fold σ is a manifold.

Definition 4.5 (Stable Folded Domain). *A folded domain $\mathcal{D}$ as in Definition 4.1 is called* stable *if for each irreducible component Σ of Σ_0, $\mathrm{Aut}(\Sigma, j_0, \mathbf{p}_0)$ is finite dimensional and each connected component of S contains at least one puncture.*

The space of stable folded domains with genera g_i of Σ_i and n_0 marked points on Σ_0 and n_1 punctures on Σ_1, modulo gauge, is denoted by

$$\mathcal{M}^F_{g_0,g_1,n_0,n_1}. \tag{4.14}$$

Note that this space is infinite dimensional.

4.2 Sobolev Spaces of Maps

To proceed we need to give a precise definition of the Sobolev spaces we plan to use. Recall the non-degenerate metric $\tilde{g}$ from equation (2.10) and fix a folded domain $\mathcal{D}$ as in Definition 4.1. We follow the definitions from [BBW93] for Sobolev spaces on manifolds with boundary.

Definition 4.6. *Fix a Riemannian metric in the conformal class of j on Σ_0 and positive integers k, p with $kp > 2$. Let $U_\pm \subset \Sigma_0$ be open subsets properly containing $\Sigma_\pm$ and that is of the same homotopy type as $\Sigma_\pm$. Then let $W^{k,p}(\Sigma_\pm, X_\pm)$ be the smooth Banach manifold consisting of maps $f : \Sigma_\pm \to X_\pm$ that are restrictions of maps $\tilde{f} : U_\pm \to X$ of class $W^{k,p}$ that send the domain fold σ into Z.*

Then standard theory gives that $W^{k,p}(\Sigma_\pm, X_\pm)$ is a smooth separable Banach manifolds modeled locally at a maps $u_\pm \in W^{k,p}(\Sigma_\pm, X_\pm)$ on the space $W^{k,p}(u_\pm^* TX)$.

Next we define the Banach manifolds of maps from punctured surfaces into Z. Here we differ from the traditional treatment found in [Sch95], [Bou02], [HWZ99] and

[Dra04] in the basic definitions as we do not a priory specify the asymptotics at the punctures but rather allow the maps to converge to arbitrary closed characteristics. We also avoid using the auxiliary $\mathbb{R}$-factor in the "symplectization" as it does not add any information.

We believe that this approach gives a more natural setup for the Fredholm theory needed in our case.

For a closed Riemann surface Σ with finitely many punctures $\{p_k\}$ and $\dot{\Sigma} = \Sigma \backslash \{p_k\}$ we let $W^{k,p}_{loc}(\Sigma, Z)$ be the space of maps from $\dot{\Sigma}$ to Z that, in local coordinates, are in $W^{k,p}_{loc}(\mathbb{R}^2, \mathbb{R}^3)$.

For a Riemann surface Σ with boundary and finitely many punctures we assume that $\Sigma \subset \Sigma'$ for some open Riemannian manifold Σ' of the same homotopy type and we set $W^{k,p}_{loc}$ to be the space of maps from $\dot{\Sigma}$ to Z that are restrictions of maps in $W^{k,p}_{loc}(\dot{\Sigma}', Z)$.

Let $(\Sigma, j, \{p_k\})$ be a Riemann surface (possibly with boundary) with conformal structure j and punctures $\{p_k\}$. Set $\dot{\Sigma} = \Sigma \setminus \{p_k\}$. For $r \in \mathbb{R}$ define the half-infinite cylinder

$$C_r = [r, \infty) \times S^1$$

with coordinates $s \in [r, \infty)$, $t \in S^1 = \mathbb{R}/\mathbb{Z}$ complex structure j with $j\partial_s = \partial_t$ and measure $d\mathrm{vol} = ds \wedge dt$. Set $C = C_0$. Then at each puncture p_k we have local conformal coordinates $\sigma_k : C \to \dot{\Sigma}$.

Fix a constant $\delta > 0$. Since we will be only interested in δ close to zero we assume throughout that δ is bounded by some constant M.

Definition 4.7 (Asymptotic Energy). *For maps $v \in W^{k,p}_{loc}(C, Z)$ we define the* asymptotic energy

$$E_r(v) = \int_{C_r} \left(|v^*\alpha(\partial_s)|^2 + |d(v^*\alpha(\partial_t))|^2 + |\pi_F\, dv|^2\right) e^{\delta s}\, \mathrm{dvol}. \tag{4.15}$$

Definition 4.8. *Let $W^{k,p}_\delta(C,Z)$ be the space of finite asymptotic energy $W^{k,p}_{loc}(C,Z)$ maps. Similarly, let $W^{k,p}_\delta(\dot\Sigma,Z)$ be the space of $W^{k,p}_{loc}(\dot\Sigma,Z)$ maps that are in $W^{k,p}_\delta(C,Z)$ in some local conformal coordinates at each puncture.*

Next we exhibit the Banach manifold structure on $W^{k,p}_\delta(\dot S,Z)$.

Definition 4.9. *For $v \in W^{k,p}_\delta(C,Z)$ let $W^{k,p}_\delta(C,v^*TZ)$ to be the space of sections $\zeta \in W^{k,p}_{loc}(C,v^*TZ)$ that have finite asymptotic energy, i.e. that satisfy*

$$E_r(\zeta) = \int_{C_r} e^{\delta s}\left(|\pi_F(\nabla\zeta)|^2 + |\alpha(\nabla_s\zeta)|^2 + |d(\alpha(\nabla_t\zeta))|^2\right) \mathrm{dvol} < \infty. \tag{4.16}$$

Analogously, for $v \in W^{k,p}_\delta(\dot\Sigma,Z)$ we define $W^{k,p}_\delta(\dot\Sigma,v^*TZ)$ to be the space of sections $\zeta \in W^{k,p}_{loc}(\dot\Sigma,v^*TZ)$ that are in $W^{k,p}_\delta(C,v^*TZ)$ in local conformal coordinates at each puncture.

With these definitions we have that $W^{k,p}_\delta(C,Z)$ is a separable Banach manifold, locally at a map $v \in W^{k,p}_\delta(C,Z)$ modeled on a neighborhood of the zero section in $W^{k,p}_\delta(C,v^*TZ)$. This is proved in Lemma C.1.

4.3 Folded Maps and Folded Holomorphic Maps

Definition 4.10 (Space of Folded Maps). *Fix a positive integers k and $p \in \mathbb{R}$ with $kp > 2$, non-negative integers g_0, g_1, n_0, n_1 and relative homology classes $A_\pm \in H_2(X_\pm,\mathcal{R};\mathbb{Z})$. Then a folded map (u_+,u_-) with respect to $A_\pm$ consists of*

(i) a stable folded domain $\mathcal{D} \in \mathcal{M}^F_{g_0,g_1,n_0,n_1}$ and

(ii) $u_\pm \in W^{k,p}(\Sigma_\pm, X_\pm)$

such that there exist maps $v_\pm \in W^{k,p}_\delta(\dot{S}, Z)$ *with*

$$u_\pm^* \det(\omega) = \tau \tag{4.17}$$

$$u_\pm|_\sigma = v_\pm|_\sigma \tag{4.18}$$

$$[u_\pm \sqcup_\sigma v_\pm] = A_\pm. \tag{4.19}$$

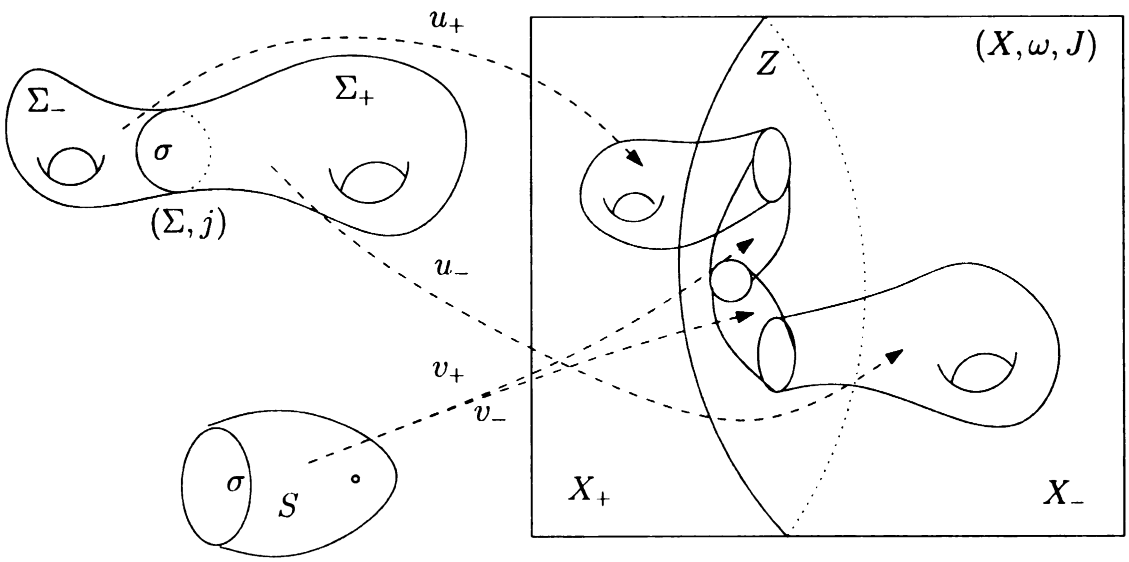

Figure 6: Folded (Holomorphic) Maps

The key to defining folded holomorphic maps lies in the definition of the "matching condition". It is specified by the subspace

$$\Delta^Z \subset \mathrm{Map}(\sigma, Z) \times \mathrm{Map}(\sigma, Z),$$

called the *folded diagonal.* Intuitively, we view Δ^Z as a scattering function that takes boundary conditions of "incoming" holomorphic curves on the "+" side and transforms them into boundary conditions for "outgoing" holomorphic curves on the "-" side. To make sense out of this we actually don't need to define this scattering function for every element of $\mathrm{Map}(\sigma, Z)$ but it suffices to define it on the maps that are possible boundary conditions of holomorphic maps. We will make use of this observation later on.

The definition of the folded diagonal Δ^Z is rather involved and we postpone it until Section 5.

Definition 4.11 (Folded Holomorphic Maps). *A folded map (u_+, u_-) is called* folded holomorphic *if $\bar{\partial}_J u_\pm = 0$ and $(u_+, u_-)|_\sigma \in \Delta^Z$, where the folded diagonal Δ^Z is defined in 5.10*

Lemma 4.12. *The gauge group $\mathcal{G}$ defined in Definition (4.2) acts on the space of folded holomorphic maps by precomposition of $u_\pm$ by the element in* $\mathrm{Diff}^+(\Sigma_0)$, *on $v_\pm$ by the element in* $\mathrm{Diff}^+(\Sigma_1)$ *and acts on the folded domain as described above.*

Proof. The action preserves solutions to the holomorphic map equation and the condition that $\tau_0 = u_\pm^* \det(\omega)$, since τ_0 and $u_\pm$ are acted on by precomposition with the same diffeomorphism. Moreover it preserves the identification of the domain folds σ_0 and σ_1 as for $p \in \sigma$

$$\begin{aligned} u'_\pm(p) &= u_\pm \circ \phi_0(p) \\ &= v_\pm \circ \psi \circ \phi_0(p) \\ &= v'_\pm \circ \phi_1^{-1} \circ \psi \circ \phi_0(p) \\ &= v'_\pm \circ \psi'(p). \end{aligned}$$

So this action preserves the set of folded holomorphic maps as long as it preserves the folded diagonal. We will postpone this part of the proof until Remark 5.11. □

Ultimately we will be interested in the space of folded holomorphic maps modulo gauge.

We make an important observation about the orientations of maps along the fold.

Remark 4.13. Let $p \in \sigma$ be a point where τ vanishes transversely and let $\eta \in T_p\Sigma$ be an outward normal vector to σ at p, i.e. $j\eta \in T_p\sigma$ and $\eta(\tau) < 0$. Then

$$\tilde{g}(du_\pm(\eta), J_+R) > 0$$

since $\tau = u^* \det(\omega)$ and J_+R points from X_+ to X_- so $d(\det(\omega))(J_+R) < 0$. Then

$$0 < \tilde{g}(du_\pm(\eta), J_+R) = \pm\tilde{g}(du_\pm(j\eta), R) = \pm u_\pm^*\alpha(j\eta)$$

so the values of $u_\pm^*\alpha$ on tangent vectors have opposite sign. In particular, if the map (u_+, u_-) is transverse to the fold, i.e. if τ vanishes transversely so σ is a manifold we have for non-negative real constants c_1 and c_2 that do not both vanish that

$$(c_1\, u_+^*\alpha - c_2\, u_-^*\alpha)|_{T\sigma} \neq 0. \tag{4.20}$$

This turns out to be the crucial observation in showing that the folded diagonal poses an elliptic boundary condition and that therefore the linearized operator is Fredholm.

5 Tunneling Maps

Throughout this section we assume that X has dimension 4 and that the function τ vanishes transversely, so the domain fold σ is a manifold.

Tunneling maps give the matching conditions for folded maps into X, add homological data to maps, and ensure that families of folded holomorphic maps have constant energy. They are central in proving regularity of solutions and they guarantee that the linearized operator is Fredholm. In this section we define all relevant structures and discuss their properties.

5.1 Definition of Tunneling Maps and the Folded Diagonal

Tunneling maps are maps into the fold Z that connect the images of the folded maps into X_+ and X_-. Throughout this section we assume that we have fixed a folded domain $\mathcal{D}$ as in Definition 4.1. All tunneling maps will have domains $(\dot{S}, j)$ of the form $(\Sigma_1^- \setminus \{p_k\}, j_1)$ with boundary $\partial S = \sigma$.

Tunneling maps satisfy an equation that depends only on the CR structure (Z, F, J) on the fold Z and the 1-form α. First we need this

Definition 5.1 ($\mathcal{H}$-Holomorphic Maps). *A map $v : \dot{S} \to Z$ is called* $\mathcal{H}$-holomorphic *if*

$$0 = \bar{\partial}_J^F v = \frac{1}{2}\left(\pi_F\, dv + J\, \pi_F\, dv\, j\right) \tag{5.21}$$

$$0 = d(v^*\alpha \circ j). \tag{5.22}$$

$\mathcal{H}$-holomorphic maps are essentially given by families of J-holomorphic maps into the symplectization parametrized by $\mathcal{H}_n$ which we define below. The detailed discussion of this relation can be found in Section D.

Definition 5.2.

$$\mathcal{H}_n(S, j) = \{\lambda \in \Omega^1(S) | d\lambda = 0,\ d^*\lambda = 0,\ \text{and } \lambda(j\eta) = 0\, \forall \eta \in T\partial S\}.$$

Definition 5.3 (Tunneling Maps). *Fix a positive integer k and real $p > 0$ such that $kp > 4$. A tunneling map is a $\mathcal{H}$-holomorphic map of class $W_\delta^{k,p}(\dot{S}, Z)$ such that $\pi_F\, dv$ does not vanish identically.*

We make a simple, but essential observation about tunneling maps.

Lemma 5.4. *Let v be a tunneling map. Then the periods of $v^*\alpha \circ j$ vanish in a neighborhood of each puncture.*

Proof. Fix local conformal coordinates $C = [0,\infty) \times S^1$ at a puncture as in Definition 4.8. We will show that $v^*\alpha \circ j$ vanishes on $S^1_r = \{r\} \times S^1$. Since $v^*\alpha \circ j$ is closed, the value of

$$\int_{S^1_r} v^*\alpha \circ j = \int_{S^1_r} v^*\alpha(\partial_s)dt$$

does not depend on r. Remembering that $A_r = C_{r,r+1}$ has unit area and using equation (4.16) we compute

$$\begin{aligned} \left| \int_{S^1_r} v^*\alpha \circ j \right| &\leq \int_{A_r} |v^*\alpha(\partial_s)| \, \mathrm{dvol} \\ &\leq e^{-\delta r/2} \left(\int_{A_r} e^{\delta s} |v^*\alpha(\partial_s)|^2 \, \mathrm{dvol} \right)^{\frac{1}{2}} \\ &\leq e^{-\delta r/2} \sqrt{E_r(v)}, \end{aligned}$$

showing that the periods of $v^*\alpha \circ j$ are arbitrarily small and therefore vanish. □

Tunneling maps are very well-behaved. They satisfy an elliptic system of PDEs and are therefore smooth on the interior of $\dot{S}$. Moreover, they have nice limits at the punctures as Theorem C.12 shows, namely they converge to closed characteristic exponentially fast. Therefore they extend to continuous maps from the radial compactification of the domain. We will blur the distinction between a tunneling map and its continuous extension to the radial compactification.

In the following we will restrict our discussion to tunneling maps v such that $v|_\sigma$ is the boundary value of a J-holomorphic map u into X_+ or X_-.

Lemma 5.5. *Suppose v is a tunneling map such that $\hat{v} = v|_\sigma$ is the boundary value of a J-holomorphic map u into X_+ or X_- and suppose that the domain fold σ is smooth. Then $\pi_F\, d\hat{u}$ and $\pi_F d\hat{v}$ are smooth and have zeros of at most finite order.*

Proof. Recall from Section 2.4 that there exists a smooth J-invariant extension of F to a neighborhood of the fold Z in X, also denoted by F. u is smooth on the interior of its domain by elliptic regularity and the same is true for v by Theorem C.12. We restrict u to a domain such that u^*F is well-defined and fix totally real boundary conditions on the newly acquired boundary component of the domain of u.

Consider the operator D_v^F operating on sections of v^*F and the corresponding operator D_u^F operating on sections of u^*F obtained by linearizing the operator $\bar{\partial}^F$ at u.

The boundary value problem given by the diagonal in $\hat{u}^*F \oplus \hat{v}^*F$ is elliptic and therefore we may employ Theorem 19.1 and 20.8 of [BBW93] in conjunction with Corollary C.10 to conclude that $\pi_F d\hat{u}$ and $\pi_F d\hat{v}$ are smooth and either vanish identically (the possibility of which we excluded in Definition 5.3) or have zeros of at most finite order. $\square$

Remark 5.6. The gauge group $\mathcal{G}$ from Definition 4.2 acts on the space of tunneling maps by precomposition. Recall that $S \subset \Sigma_1$, so if $\phi_1 : \Sigma_1 \to \Sigma_1$ is a diffeomorphism with $\phi_1(p_k) = p_k$, then this preserves the $\mathcal{H}$-holomorphic map equation.

Also note that the asymptotic energy is invariant under diffeomorphisms of the domain, so the action of $\mathcal{G}$ preserves $W_\delta^{k,p}(\dot{S}, Z)$.

Let $\mathcal{R}$ be the set of closed characteristics, i.e. the set of embedded circles $Y \subset Z$ with $TY = L$ along Y. Each $Y \in \mathcal{R}$ has a minimal period T_Y:

$$T_Y = \inf_{T>0} \{T | \varphi_T(y) = y, \; \forall \, y \in Y\},$$

where φ_t is the time-t flow of the characteristic vector field R. Note that the minimal period is always positive by definition, in the S^1-invariant case we have that $\mathcal{R} = \sqcup\{\text{fibers of } \pi_V\}$, parametrized by $p \in V$ and $T_Y = 1$ for all $Y \in \mathcal{R}$.

5.2 Conjugate Tunneling Maps

Tunneling maps connect boundary values of J-holomorphic maps in $X_\pm$ to closed characteristics. To get a scattering function of incoming boundary values from a map u_+ into X_+ to outgoing boundary values of a map u_- into X_-, we need to define a relation between tunneling maps v_+ connecting to u_+ and v_- connecting to u_-. Intuitively this relation is given by reflection through a "horizontal tunneling map". The definitions in this section become a lot clearer in the S^1-invariant case which we explain in the following section.

Horizontal tunneling maps have the same features as Abbas solutions (cf. [Abb] and [ACH04]).

Definition 5.7. *Let $(\Sigma_h, j, \{p_k\})$ be a punctured compact Riemann surface. A* horizontal tunneling map *is a $\mathcal{H}$-holomorphic $W^{k,p}_\delta$ map $v_h : \dot{\Sigma}_h \to Z$ with*

1. *$\pi_F \, dv_h$ is injective on $\dot{\Sigma}_h$,*
2. *all punctures are positive, i.e. the associated asymptotic charges are positive,*
3. *v converges to the eigenvector corresponding to the lowest eigenvalue of the asymptotic operator S_∞ (see [HWZ96a] and [HWZ96b]), and*
4. *each closed characteristic $Y \in \mathcal{R}$ either gets wrapped by a puncture or intersects v_h in forward and backward time.*

A tunneling map $v_0 : \dot{S} \to Z$ is called a horizontal covering *if it factors as $v = v_h \circ w$, where $w : \dot{S} \to \dot{\Sigma}_h$ is holomorphic.*

Note that in the S^1-invariant case v_h is a horizontal tunneling map if and only if its projection into V extends over the punctures to an embedding. Thus, in this case, horizontal tunneling maps are $\mathcal{H}$-holomorphic sections of the bundle Z on the complement of some points. Abbas solutions are generalizations of these sections. On a contact manifold with Giroux contact form they give $\mathcal{H}$-holomorphic parametrizations of a Giroux open book decomposition. The existence of Abbas solution in this case is shown in [Abb].

Once and for all fix a horizontal tunneling map $v_h : \dot{\Sigma}_h \to Z$.

For each tunneling map $v : \dot{S} \to Z$ and puncture $p_k \in S$ we fix local conformal coordinates $\sigma_k : C \to \dot{S}$ at p_k and define the map

$$v^k : S^1 \to Z \qquad v^k(t) = \lim_{s \to \infty} v(s,t)$$

and the vector

$$\begin{aligned} \chi_{v^k}(t) &\in S^1(F_{v^k(t)}) \\ \chi_{v^k}(t) &= \lim_{s\to\infty} \frac{\pi_f \, dv(s,t)(\partial_s)}{|\pi_F \, dv(s,t)(\partial_s)|} \end{aligned}$$

This is well-defined by Theorem C.12 and since zeros of $\pi_F \, dv$ cannot accumulate at p_k. To see this we work locally at a puncture p_k. Take a non-trivial holomorphic section η of TS that vanishes at p_k and consider $\chi = \pi_F \, dv(\eta)$. Then χ satisfies

$$D_v^F(\chi) = v^*\alpha(\eta) T^{0,1}(R, dv) \tag{5.23}$$

Using an isometric embedding of Z into $\mathbb{R}^N$ we trivialize v^*TZ and since χ decays exponentially we can use a conformal change of coordinates from the half-infinite cylinder to the punctured disk $\dot{D}$ and conclude that χ is a smooth function from the punctured disk with all derivatives bounded (using Theorem C.12) and that $\chi(0) = 0$. Therefore we have that in particular $\chi \in W^{2,2}(D, \mathbb{R}^N)$.

Now suppose the zeros of $\pi_F \, dv$ accumulate at $0 = p_k$. Then χ has zeros that accumulate at 0, so it vanishes to infinite order at 0.

On the other hand note that the coefficients of equation (5.23) are smooth with all derivatives bounded on D so we may square the equation to conclude that there exists a constant $c > 0$ such that

$$|\Delta\chi(z)| \leq c(|\chi(z)| + |\partial_x\chi(z)| + |\partial_t\chi(z)|$$

for almost all $z = x + iy \in \mathbb{D}$. But then χ vanishes identically by Aronszajn's Unique Continuation Theorem ([Aro57]), implying that $\pi_F\, dv$ vanishes identically, the possibility of which was excluded in the definition of tunneling maps 5.3.

An alternative argument can be made by suspending v into the complex cylinder over Z and using the results in [HWZ95].

Definition 5.8 (Conjugate Tunneling Maps). *A pair of tunneling maps* (v_+, v_-) *is called* v_h-conjugate *or just* conjugate *if there exists a horizontal covering* v_0 *of* v_h *such that*

$$v_0^*\omega = v_+^*\omega = v_-^*\omega \tag{5.24}$$

$$\lambda|_{T\sigma} = 0 \qquad \text{where} \qquad \lambda = v_+^*\alpha \circ j + v_-^*\alpha \circ j - 2v_0^*\alpha \circ j, \tag{5.25}$$

with the finite dimensional constraints that at each puncture p_k, $1 \leq k \leq K$,

$$v_0^k(0) = v_+^k(t) + v_-^k(t) - 2v_0^k(t) \qquad \forall t \in S^1 \tag{5.26}$$

$$\chi_{v_0^k}(t) = \chi_{v_+^k}(t) \qquad \text{whenever} \qquad v_0^k(t) = v_+^k(t) \tag{5.27}$$

$$\chi_{v_0^k}(t) = \chi_{v_-^k}(t) \qquad \text{whenever} \qquad v_0^k(t) = v_-^k(t) \tag{5.28}$$

and the zero dimensional homological constraint

$$\#((v_+)_*[\Gamma], (v_h)_*[\hat{\Sigma}_h]) = -\#((v_-)_*[\Gamma], (v_h)_*[\hat{\Sigma}_h]) \qquad \forall\, \Gamma \in H_1(S; \mathbb{Z}). \tag{5.29}$$

Note that equations (5.26), (5.27) and (5.28) do not depend on the choice of parametrization σ^k.

Remark 5.9. We want to comment on the role of the individual equations in the above definition. Equation (5.24) relates the F-components of the differentials of the tunneling maps. Remembering that the F-components of tunneling maps are (j, J)-linear this fixed the F components up to a constant, which is supplied by equations (5.27) and (5.28).

The 1-form λ defined in equation (5.25) is closed since v_+, v_- and v_0 are $\mathcal{H}$-holomorphic. Under the simplifying assumption that $v_\pm^* d\alpha = v_0^* d\alpha$, it is also co-closed. Equation (5.26) implies that λ extends over the punctures, so $\lambda \in \mathcal{H}_n(S, j)$ and thus λ is determined by its absolute periods. The role of equation (5.29) is to fix these periods.

Equation (5.26) is a zeroth order constraint that fixed the relation between the tunneling map along the characteristic foliation.

Moreover note that this data relates the multiplicities $m_\pm$ of $v_\pm$ and m_0 of v_0 at a puncture p as

$$d = m_+ - m_0 = -(m_- - m_0) \tag{5.30}$$

so the difference of multiplicities of $v_\pm$ and v_0 have opposite sign. The integer d is called the *degree* of the conjugate pair of tunneling maps at the puncture.

In the case that the tunneling maps are boundary values for holomorphic maps into $X_\pm$, we have that the degree must be non-zero at at least one puncture on each connected component of S. Otherwise we have $v_+ = v_-$ on that component, contradicting equation (4.20). If $d \neq 0$ at p_k we have that the set of t such that $v_\pm^k(t) = v_0^k(t)$ is non-empty.

In the S^1-invariant case this definition simplifies considerably as we will explain shortly.

We are now prepared for the main definition.

Definition 5.10 (Folded Diagonal). *Fix a horizontal section v_h and a folded domain in $\mathcal{M}^F_{g_0,g_1,n_0,n_1}$ with domain fold σ. The* folded diagonal *is the subset in* $\mathrm{Map}(\sigma, Z) \times \mathrm{Map}(\sigma, Z)$ *defined by*

$$\Delta^Z_{v_h} = \left\{ (v_+|_\sigma, v_-|_\sigma) \middle| (v_+, v_-) \text{ are } v_h\text{-conjugate} \right\}.$$

This definition might seem somewhat artificial, but this is a natural generalization of the situation where smooth J-holomorphic maps into a (non-folded) symplectic manifold intersect a stable hypersurface as we explain in Section E.

Remark 5.11. Note that the folded diagonal is invariant under the action of the gauge group $\mathcal{G}$. Indeed, if (v_+, v_-, j) is a conjugate pair of tunneling maps, and $\phi_1 : \Sigma_1 \to \Sigma_1$ is a diffeomorphism, then $(\phi_1^* v_+, \phi_1^* v_-, \phi_1^* j)$ is also a pair of conjugate tunneling maps with domain $\phi_1^{-1}(S)$. This concludes the proof of Lemma 4.12.

5.3 The S^1-Invariant Case

We will explain Definition 5.8 in the case of S^1-invariant folds as defined in 2.9. This case is inherently simpler due to its symmetries, and the definition simplifies greatly.

Lemma 5.12. *Let $v : \dot{S} \to Z$ be a tunneling map. Then $\pi_V\, v : \dot{S} \to Z$ extends to a smooth holomorphic map $\pi_V\, v : S \to Z$ over the puncture.*

Proof. By Lemma 2.11 ω pulls back from a symplectic form ω_V on V and there exists a complex structure j_V on V such that the projection $\pi_V|_F$ is (J, j_V)-linear. Therefore $\pi_V\, v$ is holomorphic and has finite ω_V-energy since v has finite ω-energy. Then the removal of singularity theorem shows that $\pi_V\, v$ extends smoothly over the punctures. □

Next we prove some properties about special tunneling maps in the S^1-invariant case.

Suppose v_1 and v_2 are tunneling maps with $\pi_V v_1 = \pi_V v_2$. Then define the "difference" function

$$g : \dot{S} \to S^1 = \mathbb{R}/\mathbb{Z}, \qquad v_2(z) = \quad (z) * v_1(z) = \varphi_{g(z)}(v_1(z))$$

where φ_t is the time-t characteristic flow, i.e.

$$\varphi_t : Z \to Z, \qquad \frac{d}{dt}\varphi_t = R.$$

The derivative of φ_t is given by

$$(\varphi_t)_* : T_z Z \to T_{\varphi_t(z)} Z$$

and it satisfies

$$(\varphi_t)_* R = R \qquad \text{and} \qquad \varphi_t^* \alpha = \alpha \circ (\varphi_t)_* = \alpha$$

since α is stable. Then

$$dv_2 = (\varphi_g)_* dv_1 + R \cdot dg$$

so

$$v_2^*\alpha = \alpha \circ dv_2 = \alpha \circ (\varphi_g)_* dv_1 + \alpha(R \cdot dg) = (\varphi_g^*\alpha) \circ dv_1 + dg = \alpha \circ dv_1 + dg = v_1^*\alpha + dg.$$

From this we conclude

Lemma 5.13. *Let v_1 and v_2 be maps from $\dot{S}$ into Z such that there exists a function $g : \dot{S} \to S^1$ with $v_2 = g * v_1$. Assume v_1 is a tunneling map. Then v_2 is a tunneling map if and only if dg is a harmonic 1-field.*

Proof. Note that

$$d(dg) = d(v_2^*\alpha - v_1^*\alpha)$$

$$d^*(dg) = - * d(v_2^*\alpha \circ j - v_1^*\alpha \circ j).$$

Thus if v_0 is a tunneling map, then the first equation vanishes since v_0 and v_1 must have the same projections to the base V, and $d\alpha$ pulls back from the base. The second equation vanishes by the definition of tunneling maps.

Conversely, if dg is a harmonic 1-field both equation vanish, showing that $d(v_2^*\alpha \circ j) = 0$. Noting again that the projections of v_1 and v_2 to V agree, so $\bar{\partial}^F v_2 = \bar{\partial}^F v_1 = 0$. Thus v_2 is a tunneling map. □

Lemma 5.14. *Assume the fold is S^1-invariant and let v_+ and v_- be tunneling maps. Then (v_+, v_-) are v_h-conjugate if and only if there exists a horizontal covering map v_0 of v_h, and a harmonic function $g : \dot{S} \to S^1 = \mathbb{R}/\mathbb{Z}$ with*

$$v_\pm = (\pm g) * v_0 = \varphi_{\pm g}(v_0), \tag{5.31}$$

where φ_t is the time-t characteristic flow.

Proof. Suppose $(v_+, v_-$ are conjugate and let v_0 be the associated horizontal covering map. By Lemma 5.12 we obtain holomorphic maps

$$\begin{aligned} \tilde{v}_0 &= \pi_V \circ v_0 \\ \tilde{v}_+ &= \pi_V \circ v_+ \\ \tilde{v}_- &= \pi_V \circ v_- \end{aligned}$$

that extend over the punctures. Using equation (5.24) and the fact that the maps are holomorphic we conclude that they satisfy the pointwise identity

$$|d\tilde{v}_0| = |d\tilde{v}_+| = |d\tilde{v}_-|.$$

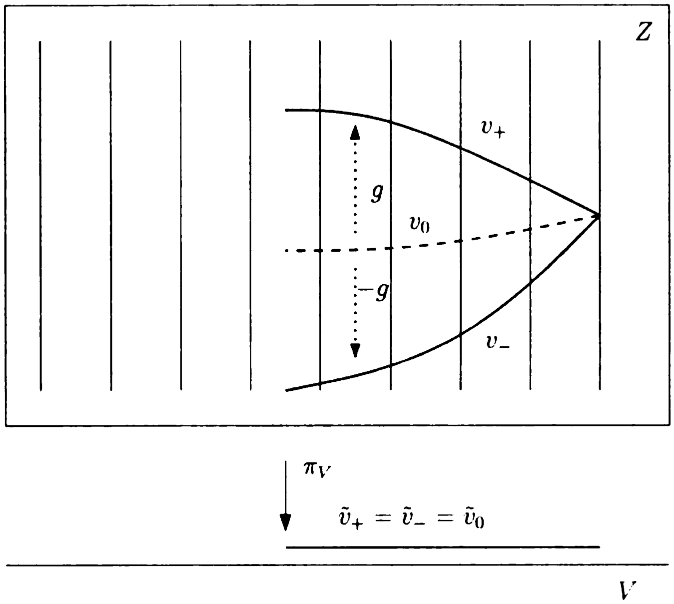

Figure 7: Conjugate Tunneling Maps

Thus there exists an S^1-valued holomorphic functions $f_\pm$ with

$$d\tilde{v}_\pm = f_\pm d\tilde{v}_0.$$

But holomorphic functions with image contained in a 1-dimensional submanifold are necessarily locally constant by the identity theorem, so $f_\pm$ are locally constant.

By equation 5.26 the images of the punctures of $\tilde{v}_0$, $\tilde{v}_+$ and $\tilde{v}_-$ agree and by equations (5.27) and (5.27), $d\tilde{v}_0 = d\tilde{v}_+ = d\tilde{v}_-$ at the punctures, so $f_\pm = 1$ since each connected component of the domain contains at least one puncture. Thus $\tilde{v}_0 = \tilde{v}_+ = \tilde{v}_-$.

Then we may define the functions

$$\begin{aligned} g_\pm &: \dot{S} \to S^1 \\ v_\pm &= g_\pm * v_0. \end{aligned}$$

Note that g is harmonic by Lemma 5.13. Equation (5.25) guarantees that $dg_+ + dg_- = 0$ on $j \cdot T\sigma$ so $dg_+ + dg_-$ is determined by its absolute period integrals by Theorem

D.2.

In this setting equation (5.29) simplifies to

$$\#((g_+)_*(\Gamma), 0) = -\#((g_-)_*(\Gamma), 0)$$

or equivalently that all period integrals of $dg_+ + dg_-$ vanish, so we conclude that $dg_+ + dg_- = 0$. Thus $g_+ + g_-$ is constant and by equation (5.26) that constant must be zero and $g_+ = - + g_-$, recovering equation (5.31).

Conversely it is straightforward to check that in case there exist harmonic functions $g_\pm$ and horizontal covering v_0 satisfying equation (5.31), then $v_\pm = g_\pm * v_0$ satisfy the requirements of Definition 5.8. □

In this case, given a tunneling map v_+ we can use the S^1 action to show that there always exists a conjugate tunneling map v_-.

Lemma 5.15. *For every tunneling map $v_+ : \dot{S} \to Z$ there exists a unique conjugate tunneling map $v_- : \dot{S} \to Z$.*

Proof. Since $\pi_V v_h$ is an embedding, there exists a unique map $w : \dot{S} \to \Sigma_h$ such that $\pi_V(v_h \circ w) = \pi_V v_+$. Set $v_0 = v_h \circ w$, so $\pi_V v_+ = \pi_V v_0$. Therefore there exists a function

$$g : \dot{S} \to S^1 = \mathbb{R}/\mathbb{Z}, \qquad v_+ = g * v_0 = \varphi_g \circ v_0$$

where φ_t is the time-t characteristic flow.

By Lemma 5.13 g is harmonic and we define

$$v_- = (-g) * v_0.$$

□

6 Properties of the Folded Diagonal

In this section we study the properties of the folded diagonal by looking at its deformation space. In short, we find that the deformations of the folded diagonal at a pair of conjugate tunneling maps restricted to σ $(\hat{v}_+, \hat{v}_-)$ is given by the graph of a function from sections of $\hat{v}_+^* TZ$ to sections of $\hat{v}_-^* TZ$, which we will describe.

For most of this section we will restrict the discussion to S^1-invariant folds.

6.1 Properties of Tunneling Maps

In this section we study the space of conjugate tunneling maps and the deformations of the folded diagonal.

We start with a remark about the relation between infinitesimal gauge transformations that leave the structures on Σ_0 invariant to deformations of complex structure on S. To this end, let h be a deformation of complex structure j_1 on S. Any such deformation can be achieved by an infinitesimal gauge transformation $\eta \in \Gamma(T\Sigma_1)$ of Σ_1 with $\mathcal{L}_\eta j_1 = h$. This infinitesimal gauge transformation induces an infinitesimal deformation of τ_1, ψ and g as directed by the action of the gauge group, but it does not change the structure τ_0, j_0 on Σ_0. Thus when looking for tunneling maps matching given $u_\pm$, we may vary the complex structure on the domain of the tunneling maps $S \subset \Sigma_1$ in arbitrary ways, while $u_\pm$ remain solutions. We will use this infinitesimal gauge action repeatedly during this sections.

Roughly speaking, the system of equations for tunneling maps are made up of a 1-dimensional equation of second order and a 2-dimensional equation of first order. Thus we can expect to solve the equations after imposing 2-dimensional boundary conditions, one dimension for the second order equation and half a dimension for each dimension of the first order equation. We will show below that we can find deformations with arbitrary (3-dimensional) boundary conditions. To achieve this, we need to make use of the gauge action.

Lemma 6.1. *The linearizations of equations (5.21) and (5.22) are*

$$D^F_{(v,j)}(\xi, h) = \nabla^{0,1}(\pi_F\xi) + \frac{1}{2}J\,\pi_F\,dv\,h + \frac{1}{2}\pi_F(\nabla_\xi J)dv\circ j = 0 \in \Omega^{0,1}(F) \quad (6.32)$$

$$D^L_{(v,j)}(\xi, h) = d\left[d(\alpha(\xi))\circ j + v^*(\iota_\xi d\alpha)\circ j + v^*\alpha\circ h\right] = 0 \in \Omega^2(S) \quad (6.33)$$

where and ∇ is the Levi-Civita connection as defined in Section B and

$$\nabla^{0,1} = \frac{1}{2}\{\nabla + J\nabla\circ j\}.$$

Proof. Let v_t be a family of tunneling maps with complex structures j_t, $\xi = \frac{d}{dt}\big|_{t=0} v_t$ and $h = \frac{d}{dt}\big|_{t=0} j$. Then

$$\begin{aligned} D^F_{(v,j)}(\xi, h) &= \frac{d}{dt}\bigg|_{t=0} \frac{1}{2}\pi_F\{dv_t + J\,dv_t\,j_t\} \\ &= \frac{1}{2}\pi_F\{\nabla_t dv_t + J\nabla_t(dv_t)\circ j\}_{t=0} + \frac{1}{2}\pi_F(\nabla_\xi J)dv\circ j + \frac{1}{2}J\,\pi_F\,dv\,h \\ &= \frac{1}{2}\pi_F\{\nabla\xi + J\nabla\xi\circ j\} + \frac{1}{2}\pi_F(\nabla_\xi J)dv\circ j + \frac{1}{2}J\,\pi_F\,dv\,h \\ &= \nabla^{0,1}(\pi_F\xi) + \frac{1}{2}\pi_F(\nabla_\xi J)dv\circ j + \frac{1}{2}J\,\pi_F\,dv\,h. \end{aligned}$$

For the second equation we compute

$$\frac{d}{dt}\bigg|_{t=0} d(v_t^*\alpha\circ j_t) = (v^*\mathcal{L}_\xi\alpha)\circ j + v^*\alpha\circ h = d(\alpha(\xi))\circ j + v^*(\iota_\xi d\alpha)\circ j + v^*\alpha\circ h.$$

□

Note J is integrable in the S^1-invariant case by Lemma B.4. In this case the system of PDEs (6.32) and (6.33) is upper triangular. We will assume for the remainder of this section that this is the case.

We set

$$D = D^F \oplus D^L.$$

We will refer to D as the linearized operator.

Infinitesimal deformations of tunneling maps with given boundary conditions are not unique. Even when restricting attention to deformations perpendicular to the gauge action this may not be the case.

If v is a tunneling map in the S^1-invariant case, then the space of sections of v^*F plays a special role as we explain next.

Theorem 6.2. *Given $\chi \in \Gamma(v^*F)$ and a function $\hat{\zeta} : \sigma \to \mathbb{R}$, there exists an infinitesimal gauge transformation $\eta \in \Gamma(T\Sigma_1)$ such that $D^F(\chi, h) = 0$, where $h = \mathcal{L}_\eta j$ is uniquely determined on S. Moreover there is and a unique real-valued function ζ on $\dot{S}$ agreeing with $\hat{\zeta}$ on σ such that $D^L(\chi + \zeta \cdot R, h) = 0$.*

Proof. Set $\tilde{v} = \pi_V v$ and let $\{q_k\} \subset S$ be the points where $d\tilde{v}$ is not an immersion. These points are isolated and the zeros are of finite order since $\tilde{v}$ is holomorphic and by Lemma 5.5.

At each q_k fix local conformal coordinates z and let β be a holomorphic vector field on $S \backslash \{q_k\}$ with poles at q_k of the same order m_k as the zeros of $d(\pi_V v)$ such that $\chi - \pi_F dv(\beta) \in O(|z|^{m_k})$ at q_k. Then there exists an infinitesimal gauge transformation $\eta \in \Gamma(T\Sigma_1)$ with

$$\pi_F dv(\eta) = \chi - \pi_F dv(\beta).$$

Since β was chosen holomorphic we have that $\mathcal{L}_\beta j = 0$, so with the deformation of complex structure $h = \mathcal{L}_\eta j$ induced by η we have $\bar{\partial}\eta + \frac{1}{2}jh = 0$ on S.

Then

$$D^F(\chi, h) = D^F(\pi_F\, dv(\eta + \beta), h) = 0. \tag{6.34}$$

To see that $h|_S$ is the unique consider deformation of complex structure h' such that $D^F(\chi, h') = 0$. Then equation (6.32) shows that the difference $h - h'$ of the deformations of complex structure satisfies $\pi_F\, dv(h - h') = 0$ on $\dot{S}$, so $h = h'$.

Now let ζ be the unique real-valued function on $\dot{S}$ solving the Poisson equation with Dirichlet boundary conditions

$$\begin{aligned} * \Delta\zeta &= d(v^*(\iota_\chi d\alpha) + v^*\alpha \circ h) \\ \zeta|_\sigma &= \hat{\zeta}. \end{aligned} \tag{6.35}$$

Then

$$\begin{aligned} D^L(\chi + \zeta \cdot R, h) &= d(d\zeta \circ j) + v^*(\iota_{\chi+\zeta\cdot R} d\alpha) + v^*\alpha \circ h \\ &= - * \Delta\zeta + d(v^*(\iota_\chi d\alpha) + v^*\alpha \circ h) = 0. \end{aligned}$$

□

To make a statement about the uniqueness of deformations of tunneling maps we need the following

Definition 6.3. *For a tunneling map v, the space*

$$\Gamma(v^*F, 0) = \{\eta \in \Gamma(v^*F) | v_\sigma = 0\}$$

is called the space of twisted gauge transformations.

Theorem 6.2 immediately yields that the twisted gauge transformations are isomorphic to the kernel of the linearized operator with fixed boundary values.

Corollary 6.4. *With the notation of the above lemma, the map*

$$\Psi : \Gamma(v^*F, 0) \to \ker(D_{(v,j)}) \cap (\Gamma(v^*TZ, 0) \times \mathcal{J}_j(S))$$

is an isomorphism.

Proof. Theorem 6.2 shows that the map is well-defined and injective. To see that Ψ is onto, suppose that $D_{(v,j)}(\xi, h) = 0$. Then set $\chi = \pi_F \xi$ and let h' be the unique deformation of complex structure and ζ' the unique function from Theorem 6.2 such that

$$\begin{aligned} D_{(v,j)}(\chi + \zeta \cdot R, h) &= 0 \\ (\chi + \zeta \cdot R)|_{\partial S} &= 0. \end{aligned}$$

By the uniqueness part of Theorem 6.2 he conclude that $h' = h$ and $\chi + \zeta \cdot R = \xi$. □

In particular this shows that we can always arrange for infinitesimal deformations to vanish at the punctures in the F-directions. This constraint fixes the closed characteristics that tunneling maps wrap. We will impose this constraint when we fix relative homology classes that maps represent.

We also conclude that any given deformation of the boundary of a tunneling map extends to a deformation of the tunneling map.

Corollary 6.5. *Let $\hat{\xi}$ be a section of v^*TZ over σ. Then there exists an extension ξ of $\hat{\xi}$ to S and a deformation h of complex structure j such that (ξ, h) is in the kernel of the linearization $D_{(v,j)}$. Moreover, (ξ, h) is unique on S up to $\Gamma(v^*F, 0)$.*

Proof. The existence follows from Theorem 6.2 and the uniqueness is a direct consequence of Corollary 6.4. □

Note that $(\varphi_{\pm g})_* : v_\pm^* F \to v_0^* F$ gives an isomorphism of $v_\pm^* F = v_0^* F$. Moreover, since the horizontal section v_h is an immersion, we have that for each $\chi \in v_0^* F$ there exists a unique section $\zeta \in v_0^* L$ with $\chi + \zeta$ tangent to the image of v_h. In particular this shows that ζ is given by $\zeta = v_0^* \alpha \circ (\pi_F \, dv_0)^{-1}(\chi)$ and is well-defined.

Lemma 6.6. *Let (v_+, v_-) be conjugate tunneling maps and (ξ_+, ξ_-, h) a conjugate deformation of conjugate tunneling maps. Then the linearizations of equation 5.31 are*

$$
\begin{aligned}
(\varphi_g)_*(\pi_F \xi_+) &= (\varphi_{-g})_*(\pi_F \xi_-) \qquad \text{on } \dot{S} && (6.36)\\
\alpha(\xi_+) - \zeta &= -(\alpha(\xi_-) - \zeta), && (6.37)
\end{aligned}
$$

where $\zeta : \dot{S} \to \mathbb{R}$ is the unique function such that $(\varphi_g)_* \pi_F \xi_+ + \zeta \cdot R$ is tangent to the image of v_h.

Proof. L is vertical for π_V, giving (6.36) and equation (6.37) follows from the condition that v_0 has to cover v_h. □

Twisted gauge transformations don't only generate deformations of tunneling maps, but they moreover generate conjugate deformations of conjugate tunneling maps as we show next.

Lemma 6.7. *Let (v_+, v_-) be a conjugate pair of tunneling maps. Then the twisted vector fields $\Gamma(v_0^* F, 0)$ give conjugate deformations that vanish on σ.*

Proof. Given $\chi \in \Gamma(v_0^* F, 0)$, there exists by Corollary 6.4 unique $\xi_\pm \in \Gamma(v_\pm^* TZ, 0)$ and deformation h of complex structure that give deformations of the tunneling maps $v_\pm$ with $\pi_F \xi_\pm = (\varphi_{\pm g})_* \chi$. This immediately gives equation (6.36). Now let $\zeta \in v_0^* L$ be the unique section such that $\chi + \zeta$ is tangent to the image of v_h. Then

$$
\alpha(\xi_+) = \alpha(\xi_-) = \alpha(\zeta) = 0 \qquad \text{on } \sigma,
$$

so (ξ_+, ξ_-) also satisfy equation (6.37) and vanish on σ. $\square$

In particular this shows that although for a fixed boundary loop $v_+|_\sigma$ there might be a family of tunneling maps v_+^t with the boundary conditions $v_+^t|_\sigma = v_+|_\sigma$, and therefore a family of conjugate tunneling maps (v_+^t, v_-^t), the conjugate tunneling maps satisfy $v_-^t|_\sigma = v_-^{t_0}|_\sigma$.

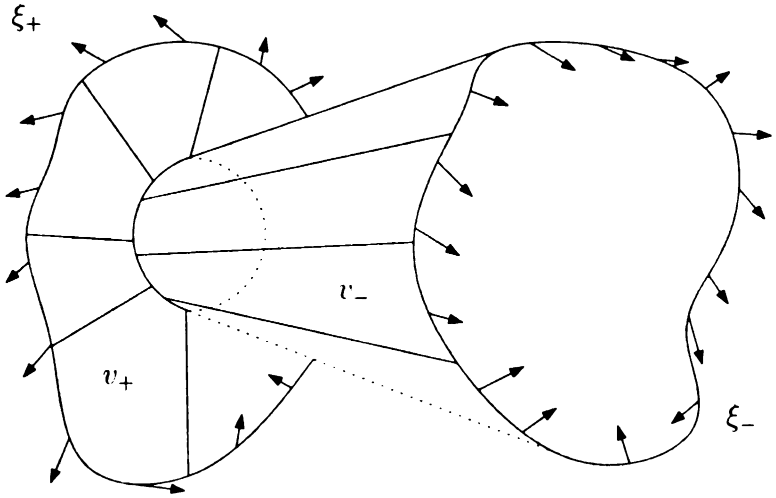

Figure 8: Conjugate deformations restricted to the boundary are unique.

The next theorem characterizes deformations of the folded diagonal.

Theorem 6.8. *Let (v_+, v_-) be a pair of conjugate tunneling maps. Then for any section $\hat{\xi}_+$ of $v_+^*TZ|_\sigma$ there exists a unique section $\hat{\xi}_-$ of $v_-^*TZ|_\sigma$ such that the pair $(\hat{\xi}_+, \hat{\xi}_-)$ is the restriction of a conjugate deformation of conjugate tunneling maps.*

Proof. Let v_0 be the horizontal tunneling map and g be the harmonic function satisfying $v_+ = g * v_0$. Set $\hat{\chi} = (\varphi_{-g})_* \pi_F \hat{\xi}_+$ and let $\hat{\zeta}$ be the unique section such that $\hat{\chi} + \hat{\zeta}$ is tangent to the image of v_h.

Define

$$\hat{\xi}_- = (\varphi_{-2g})_* \pi_F \xi_+ + (2\alpha(\hat{\zeta}) - \alpha(\hat{\xi}_+)) \cdot R. \tag{6.38}$$

By Corollary 6.5 there exist extensions $\xi_\pm$ of $\hat{\xi}_\pm$ to all of $\dot{S}$ that are deformations of the tunneling maps $v_\pm$. These are unique up to $\Gamma(v_0^*F, 0)$, so we may assume without loss of generality that $(\varphi_{-g})_*\pi_F\xi_+ = (\varphi_g)_*\pi_F\xi_-$ so (ξ_+, ξ_-) are conjugate deformations.

Moreover, $\pi_F\hat{\xi}_-$ is unique since ξ_- is unique up to $\Gamma(v_-^*F, 0)$ by Corollary 6.5 and $\alpha(\hat{\xi}_-)$ is uniquely defined by the conjugation condition (6.37), so $\hat{\xi}_-$ is the unique section with the desired properties. $\square$

6.2 Deformations of the Folded Diagonal

Using the results of the previous section we describe the deformations of the folded diagonal. We show that these are given by the graph of a bundle map.

Definition 6.9. *Given conjugate tunneling maps (v_+, v_-), let $d\Delta^Z$ be the space of deformations of conjugate tunneling maps, restricted to σ and set $\hat{v}_\pm = v_\pm|_\sigma$. Thus*

$$
\begin{aligned}
d\Delta^Z \;=\; \{(\hat{\xi}_+, \hat{\xi}_-) \in \Gamma(\hat{v}_+^*TZ \oplus \hat{v}_-^*TZ) \,|\, \exists\, \xi_\pm \in \Gamma(v_\pm^*TZ), h \in T_j\mathcal{J}(S) \text{ with } \xi_\pm|_{\partial S} = \hat{\xi}_\pm \\
\textit{satisfying equations } D_{(v_\pm, j)}(\xi_\pm, h) = 0, \textit{ (6.36) and (6.37).}\}
\end{aligned}
$$

By the Definition 4.10 of folded maps we identify $\sigma = \sigma_0 = \sigma_1$ and therefore, with the convention that $\hat{u}_\pm = u_\pm|_\sigma$ we may identify $\hat{v}_\pm^*TZ$ with $\hat{u}_\pm^*TZ$, when (u_+, u_-) is a folded map with tunneling maps (v_+, v_-). Therefore we may also view space of deformations of folded diagonal $d\Delta^Z$ as a subset of

$$\Gamma(\hat{u}_+^*TZ \oplus \hat{u}_-^*TZ) \subset \Gamma(\hat{u}_+^*TX \oplus \hat{u}_-^*TX).$$

Note that if $(\hat{\xi}_+, \hat{\xi}_-) \in d\Delta^Z$, and $\eta \in T_{\mathrm{Id}}\mathrm{Diff}(\Sigma_1, \sigma)$ is an infinitesimal gauge transformation that is tangent to the domain fold σ, then also $(\hat{\xi}_+ + du_+(\eta), \hat{\xi}_- + du_-(\eta)) \in d\Delta^Z$. This defines an action

$$T_{\mathrm{Id}}\mathrm{Diff}(\Sigma, \sigma) \times d\Delta^Z \to d\Delta^Z. \tag{6.39}$$

It will prove convenient to extend the definition of deformations of folded diagonal to sections of $\hat{u}_+^*TX \oplus \hat{u}_-^*TX$ in such a way that the action (6.39) extends to all infinitesimal gauge transformations $T_{\mathrm{Id}}\mathrm{Diff}(\Sigma)$ of the map domain, including the ones that move the fold. This will greatly simplify taking the quotient by the gauge action later. The following definitions and lemmas facilitate this.

Definition 6.10. *Let $(\Sigma, j, \sigma, u_+, u_-)$ be a folded holomorphic map. Recall the splitting $T_ZX = E \oplus F = K \oplus L \oplus F$ from equations (2.7) and (2.9). Define the subspaces*

$$
\begin{aligned}
F_\pm &= \hat{u}_\pm^* F \subset u_\pm^* TX|_\sigma \\
E_\pm &= \hat{u}_\pm^* E \subset u_\pm^* TX|_\sigma
\end{aligned}
$$

Let

$$
\begin{aligned}
A_F &: F_+ \to F_- \\
A_E &: E_+ \to E_-
\end{aligned}
$$

be the (J_+, J_-)-linear bundle maps defined by

$$
\begin{aligned}
A_F &= (\pi_F\, du_-) \circ (\pi_F\, du_+)^{-1} \\
A_E &= (\pi_E\, du_-) \circ (\pi_E\, du_+)^{-1}
\end{aligned}
$$

Note that A_F is well-defined even if $\pi_F\, du_+$ is not injective, since the locations and orders of the zeros of $\pi_F\, dv_+$ and $\pi_F\, dv_-$ agree and $\pi_F\, du_\pm = \pi_F\, dv_\pm$ on σ. Similarly A_E is well defined since $\pi_E\, du_+$ and $\pi_E\, du_-$ have the same order of vanishing on the fold, namely the order of vanishing of τ.

Also define the "correction map"

$$
\begin{aligned}
A_c &: E_+ \oplus F_+ \to E_- \\
A_c(\xi^E \oplus \xi^F) &= v_0^*\alpha\left((\pi_F\, du_+)^{-1}(\xi^F) - (\pi_E\, du_+)^{-1}(\xi^E)\right) \cdot R
\end{aligned}
$$

To see that A_c is well-defined note that $\pi_L dv_0 \circ (\pi_F\, du_+)^{-1}$ is the unique vector $\zeta \in v_0^* L$ such that $\chi + \zeta$ is tangent to the image of v_h, where $\chi = (\pi_F dv_0)(\pi_F dv_+)^{-1}$ in $v_0^* F$.

As we will see, A_c turns out to have no significant impact on the results of this section, but merely makes the computations more cumbersome.

Definition 6.11. *Let $(\Sigma, \sigma, u_+, u_-)$ be a folded map and let (S, j) be a Riemann surface with boundary $\partial S = \sigma$. Using the splitting (2.7) define the bundle isomorphism*

$$\begin{aligned} B &: \hat{u}_+^* TX|_\sigma \to \hat{u}_-^* TX|_\sigma \\ B(\xi^F \oplus \xi^E) &= A_F(\xi^F) \oplus A_E(\xi^E) + 2A_c(\xi^E + \xi^F). \end{aligned} \tag{6.40}$$

Lemma 6.12. *The deformations of the folded diagonal are given by the graph of the operator B defined in (6.40), restricted to $u_+^* TZ$:*

$$d\Delta^Z = \mathrm{graph}(B|_{u_+^* TZ})$$

Proof. Recall that $u_\pm|_\sigma = v_\pm|_\sigma$ and both $du_\pm$ and $\pi_F\, dv_\pm$ are $(j, J_\pm)$-linear. Therefore

$$\pi_F\, du_\pm = \pi_F\, dv_\pm \qquad \text{over } \sigma.$$

Writing $\xi = \xi^F \oplus \xi^L \in v^* TZ = v^* F \oplus v^* L$ and setting

$$\begin{aligned} \eta_F &= (\pi_F\, du_+)^{-1} \xi^F \\ \eta_E &= (\pi_E\, du_+)^{-1} \xi^L, \end{aligned}$$

which are defined almost everywhere, we note that $\eta_E \in T\sigma$ and therefore

$$\pi_E\, du_\pm(\eta_E) = u_\pm^* \alpha(\eta_E) R = v_\pm^* \alpha(\eta_E) R.$$

Then we find

$$
\begin{aligned}
B(\xi) &= A_F\xi^F \oplus A_E\xi^L + 2A_c\xi \\
&= \pi_F\, du_-(\eta_F) \oplus \pi_E\, du_-(\eta_E) + 2v_0^*\alpha(\eta_F - \eta_E)R \\
&= \pi_F\, dv_-(\eta_F) \oplus v_-^*\alpha(\eta_E)R + 2v_0^*\alpha(\eta_F - \eta_E)R \\
&= \pi_F\, dv_-(\eta_F) \oplus -v_+^*\alpha(\eta_E)R + 2v_0^*\alpha(\eta_F)R \\
&= \pi_F\, dv_-(\eta_F) \oplus -\xi^L + 2v_0^*\alpha(\eta_F)R,
\end{aligned}
$$

which is defined everywhere and agrees with equation (6.38). □

Using B we extend the definition of the folded diagonal to a subspace $\Gamma(\hat{u}_+^*TX \oplus \hat{u}_-^*TX)$ in the obvious way:

Definition 6.13. *The space of extended deformations $d\Delta^X$ of the folded diagonal is*

$$d\Delta^X = \mathrm{graph}(B).$$

Lemma 6.14. *The space of extended deformations $d\Delta^X$ of the folded diagonal is invariant under the full infinitesimal gauge group of Σ_1 (not just the subgroup that preserves σ as a set).*

Proof. Let η be a section of $T_\sigma\Sigma$ and note that

$$A_c(du_+(\eta)) = v_0^*\alpha(\eta - \eta) \cdot R = 0 \tag{6.41}$$

and therefore

$$B(du_+(\eta)) = A_E(\pi_E\, du_+(\eta)) \oplus A_F(\pi_F\, du_+(\eta)) = \pi_E\, du_-(\eta) \oplus \pi_F\, du_-(\eta) = du_-(\eta).$$

□

7 The Moduli Space of Folded Holomorphic Maps

Now we come to the result that justifies the definitions and lemmas pertaining to tunneling maps. We show that they give elliptic boundary values.

Let (u_+, u_-, j) be a folded holomorphic map with domain fold $\sigma \subset \Sigma$. Set $H_\pm = u_\pm^* TX$ and $\hat{H}_\pm = H_\pm|_\sigma$.

Theorem 7.1. *Assume that the map (u_+, u_-) is transverse to the fold, so σ is a manifold. Then the map*

$$\begin{aligned} R : \hat{H}_+ \oplus \hat{H}_- &\to \hat{H}_- \\ \xi_+ \oplus \xi_- &\mapsto \xi_- - B(\xi_+) \end{aligned}$$

poses elliptic boundary conditions for the folded holomorphic map (u_+, u_-, j), i.e. R restricted to the range of the principle symbol p of the Calderón projector P for the complexified Cauchy-Riemann operator $D_{u_+} \times D_{u_-}$ on $H_+ \otimes \mathbb{C} \times H_- \otimes \mathbb{C}$

$$R|_{\text{range}(p)} \to \hat{H}_- \otimes \mathbb{C}$$

is an isomorphism.

Proof. Let $\hat{H}_\pm^{\mathbb{C}} = \hat{H}_\pm \otimes \mathbb{C}$ denote the complexification of $\hat{H}_\pm$ and let $\hat{H}'_\pm$ ($\hat{H}''_\pm$) denote the $(i, J_\pm)$-linear (antilinear) subspace of $\hat{H}_\pm^{\mathbb{C}}$. Recall that $\sigma = \partial\Sigma_+ = -\partial\Sigma_-$ inherits the orientation from Σ_+. Then

$$\begin{aligned} p \quad &: \quad \hat{H}_+^{\mathbb{C}} \oplus \hat{H}_-^{\mathbb{C}} \to \hat{H}'_+ \oplus \hat{H}''_- \\ p(v, w) \quad &= \quad \frac{1}{2}(\mathrm{Id} - iJ_+) \oplus \frac{1}{2}(\mathrm{Id} + iJ_-) \end{aligned}$$

is the projection onto the $(i, J_+ \oplus -J_-)$ linear subspace.

Let $v \in H'_+$ then $A_E(\pi_E v) + A_F(\pi_F v) \subset H''_-$ since A_E and A_F are (J_+, J_-)-linear. We write the complexified map

$$A_c = A'_c \oplus A''_c$$

where A'_c and A''_c are the projections of A_c into H'_- and H''_- respectively, i.e.

$$\begin{aligned} A'_c(v) &= \frac{1}{2}\{A_c + iJ_-A_c\}(v) \\ A''_c(v) &= \frac{1}{2}\{A_c - iJ_-A_c\}(v). \end{aligned}$$

In particular

$$\begin{aligned} 2A''_c(v) &= \{A_c - iJ_-A_c\}(v) \\ &= \{A_c(v) - J_-A_c(J_+v)\} \\ &= v_0^*\alpha\left((\pi_F\, du_+)^{-1}(\pi_F dv) - (\pi_E\, du_+)^{-1}(\pi_E dv)\right)\cdot R \\ &\quad -v_0^*\alpha\left((\pi_F\, du_+)^{-1}(J_+\pi_F dv) - (\pi_E\, du_+)^{-1}(J_+\pi_F dv)\right)\cdot \partial_r \\ &= v_0^*\alpha\left((\pi_F\, du_+)^{-1}(\pi_F dv) - (\pi_E\, du_+)^{-1}(\pi_E dv)\right)\cdot R \\ &\quad -v_0^*\alpha\circ j\left((\pi_F\, du_+)^{-1}(\pi_F dv) - (\pi_E\, du_+)^{-1}(\pi_F dv)\right)\cdot \partial_r. \end{aligned}$$

Now let $w \in H'_-$ and suppose that $w - B(v) = 0$. Then

$$\begin{aligned} w - 2A'_c(v) &= 0 \\ A_F(\pi_F v) + A_E(\pi_E v) + 2A''_c(v) &= 0. \end{aligned}$$

Since A_c has image in E_- and A_F is an isomorphism we conclude that $\pi_F v = 0$. Then with $\eta = (\pi_E\, du_+)^{-1}\pi_E v$ and $\eta = \eta_1 + j\eta_2$, $\eta_{1,2} \in T\sigma$

$$\begin{aligned} A_E(\pi_E v) + 2A''_c(\pi_E v) &= \pi_E\, du_-(\eta) - v_0^*\alpha(\eta)R + v_0^*\alpha(j\eta)\partial_r \\ &= u_-^*\alpha(\eta)R + u_-^*\alpha(\eta)\partial_r - v_0^*\alpha(\eta)R + v_0^*\alpha(j\eta)\partial_r \\ &= \left((u_-^*\alpha(\eta) - v_0^*\alpha(\eta)\right)R - \left(u_-^*\alpha(j\eta) - v_0^*\alpha(j\eta)\right)\partial_r \end{aligned}$$

which vanishes if and only if $\eta = 0$ or

$$u_-^*\alpha - v_0^*\alpha = 0.$$

But for $0 \neq \mu \in T\sigma$

$$\begin{aligned} u_-^*\alpha(\mu) - v_0^*\alpha(\mu) &= u_-^*\alpha(\mu) - \frac{1}{2}(u_+^*\alpha(\mu) + u_-^*\alpha(\mu)) \\ &= \frac{1}{2}(u_-^*\alpha(\mu) - u_+^*\alpha(\mu)) \end{aligned}$$

which is nonzero since the map is transverse to the fold by assumption and by equation (4.20).

This shows that $\eta = 0$ and therefore $\pi_E dv = 0$. Thus $v = 0$ and consequently also $w = 0$, so we showed that

$$R(v \oplus w) = w \oplus -B(v)$$

is an isomorphism from $\hat{H}'_+ \oplus \hat{H}'_-$ to $\hat{H}_-^{\mathbb{C}}$. $\square$

Remark 7.2. Note that since B is a vector bundle map, the condition that graph(B) gives elliptic boundary conditions is equivalent to asking that graph(B) is totally real in

$$(\hat{H}_+ \oplus \hat{H}_-, J_+ \oplus -J_-).$$

We have to use the complex structure $-J_-$ on $\hat{H}_-$ since σ is oriented as the boundary of Σ_+, so its orientation is opposite from that of $\partial\Sigma_-$.

Now standard theory shows that the linearized operator is Fredholm. For this next theorem we fix the complex structure on the domain $\Sigma = \Sigma_0$.

Theorem 7.3. *For any $s > \frac{1}{2}$, the operator*

$$\begin{aligned} D_B^s \ : \ & (H^s(\Sigma_+, H_+) \times H^s(\Sigma_-, H_-)) \to \\ & H^{s-1}(\Sigma_+, \Lambda^{0,1}T^*\Sigma_+) \times H^{s-1}(\Sigma_-, \Lambda^{0,1}T^*\Sigma_-) \times H^{s-\frac{1}{2}}(\sigma, \hat{H}_-) \end{aligned}$$

given by

$$(\xi_+, \xi_-) \mapsto (D_{u_+}\xi_+, D_{u_-}\xi_-, R(\rho_+(\xi_+), \rho_-(\xi_-)))$$

is Fredholm with real Fredholm index (in the case that $\pi_F\, du|_\sigma \neq 0$)

$$index(D^s_B) = \mu(u_+, K_+) + \mu(u_-, K_-) + 2\chi(\Sigma) \tag{7.42}$$

where $\rho_\pm : H^s(\Sigma_\pm, H_\pm) \to H^{s-\frac{1}{2}}(\sigma, \hat{H}_\pm)$ the restriction map and $K_\pm = JL \oplus du_\pm(T\sigma)$ and μ is the relative Maslov index.

Moreover the kernel of D^s_B is independent of choice of $s > \frac{1}{2}$ and consists only of smooth solutions.

Proof. First note that the linearized Cauchy-Riemann operator $D_{u_\pm}$ on $H_\pm$ is a generalized Dirac operator. Then it follows directly from Theorem 19.1 and 20.8 of [BBW93] and Theorem 7.1 above that the operator D^s_B is Fredholm with kernel independent of $s > \frac{1}{2}$ consisting only of smooth solutions.

To see the index formula, we first homotope $B = B_0$ to the (J_+, J_-)-linear bundle isomorphism B_1 via

$$B_t(\xi) \;\; = \;\; A_E(\pi_E\xi) \oplus A_F(\pi_F\xi) + tA_c(\xi).$$

We need to check that graph(B_t) gives elliptic boundary values so that we obtain a corresponding family of Fredholm operators which then will all have the same index.

This comes down to repeating the arguments in the proof of Theorem 7.1 and we need to show that the quantity

$$u_-^*\alpha - t\, v_0^*\alpha$$

does not vanish. But for $0 \neq \mu \in T_\sigma$

$$\begin{aligned} u_-^*\alpha(\mu) - t\, v_0^*\alpha(\mu) &= u_-^*\alpha(\mu) - \frac{t}{2}\left(u_+^*\alpha(\mu) + u_-^*\alpha(\mu)\right) \\ &= \frac{2-t}{2} u_-^*\alpha(\mu) - \frac{t}{2} u_+^*\alpha(\mu) \end{aligned}$$

which again does not vanish by equation (4.20) if the map (u_+, u_-) is transverse to the fold.

Now

$$B_1(du_+(T\sigma)) = B_0(du_+(T\sigma)) = du_-(T\sigma)$$

since A_c vanishes on the push-forward of vector fields on the domain as we saw in equation (6.41). Also for $\zeta \in JL$ we have that

$$B_1(\zeta) = A_E(\zeta) \subset JL.$$

The last step is to connect the boundary condition graph(B_1) to $K_+, K_-)$ so that the associated operators remain Fredholm. Note that if $C : \mathbb{C} \to \mathbb{C}$ is complex linear, then

$$\Lambda_t = \{(u + (1-t)iv, C[(1-t-it)((1-t)u + iv)])|u, v \in \mathbb{R}\}$$

is a homotopy of elliptic boundary conditions, or equivalently totally real subspaces, in $\mathbb{C}$ with

$$\begin{aligned} \Lambda_0 &= \text{graph}(C) \\ \Lambda_1 &= (\mathbb{R}, C(\mathbb{R})). \end{aligned}$$

Now choose an appropriate basis of E_+ and $T_+ = du_+(T_\sigma\Sigma)$ and apply the above homotopy on E_+ and T_+.

Putting these homotopies together we connected the Fredholm operator with boundary values graph(B) with the Fredholm operator with boundary values (K_+, K_-), proving the statement about the index. □

Note that the above argument shows that the totally real subbundles $K_\pm$ of $\hat{H}_\pm$ may be replaced by $(K_+, K_-) = (K_+, B_t(K_-))$ where K_+ is a totally real subbundle of $\hat{H}_+$ and B_t is any member of the above family of bundle homomorphisms.

To visualize the construction and the results up to here consider the following. As seen in Section 3, the diagonal in $\mathrm{Map}(\sigma, Z) \times \mathrm{Map}(\sigma, Z)$ does not yield elliptic boundary conditions. In the language of [Nic97] we may say that given holomorphic maps (u_+, u_-) (with $u_+|_\sigma = u_-|_\sigma$), the subspace

$$\{(\hat{\xi}_+, \hat{\xi}_-) \in L^2(\sigma, u_+^* TX) \times L^2(\sigma, u_-^* TX) | \exists\, \xi_\pm \in \ker(D_{u_\pm}),\ \xi_\pm|_\sigma = \hat{\xi}_\pm\}$$

is not a Fredholm pair. But the folded diagonal gives elliptic boundary conditions, or given a folded holomorphic map (u_+, u_-) the subspace

$$\{(B(\hat{\xi}_+), \hat{\xi}_-) \in L^2(\sigma, u_-^* TX) \times L^2(\sigma, u_-^* TX) | \exists\, \xi_\pm \in \ker(D_{u_\pm}),\ \xi_\pm|_\sigma = \hat{\xi}_\pm\}$$

is a Fredholm pair.

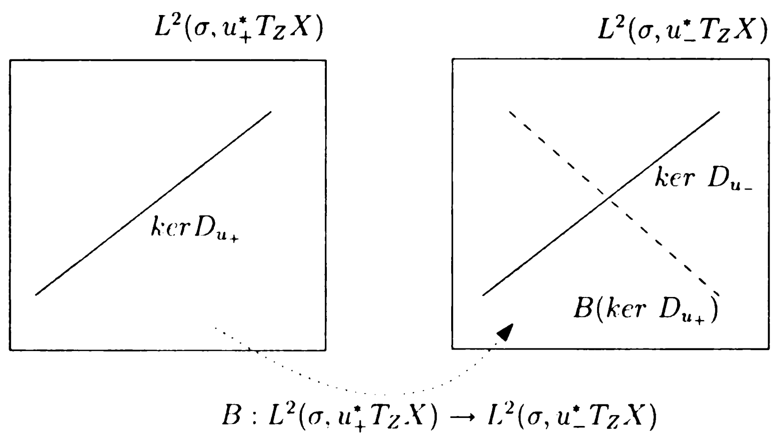

Figure 9: The map B induces a Fredholm pair.

Remark 7.4. We can generalize Theorem 7.3 by incorporating variations of the folded domain. Here are some brief comments on how this can be done.

The first step is allowing variations in j_0 and τ_0, modulo $Diff^+(\Sigma_0)$. Note that τ_0 is determined by the map by equation 4.17. Since the folded diagonal is invariant under the gauge action of $\mathrm{Diff}^+(\Sigma_0)$, Theorem 7.3 holds in this case with the index raised by the dimension of Teichmüller space $-3\chi(\Sigma_0)$.

Next consider variations in τ_1, j_1, ψ and g, modulo the action of the remaining factors of the gauge group $\mathrm{Diff}^+(\Sigma_1) \times \mathrm{Map}(\Sigma_1, \mathbb{R})$. First note that for fixed domain location of the domain fold σ_1, the space of holomorphic diffeomorphisms $\psi^{-1} : \Sigma_0^+ \to \Sigma_1^+$ sending σ_0 to σ_1 (with j_0 fixed and j_1 varying), has dimension $2\chi(\Sigma_1^+) + (1 - 3)\chi(\Sigma_1^+) = 0$. Thus variations in ψ and j_1 on Σ_1^+ do not change the dimension count.

Finally, note that g is determined by choice of τ_1 and ψ by equation (4.12), and that the deformations of complex structure j_1 on S are fixed by the tunneling map as explained in Theorem 6.2. Thus the freedom left in choosing τ_1, j_1, ψ, g is exactly given by the remaining part $\mathrm{Diff}^+(\Sigma_1) \times \mathrm{Map}(\Sigma_1, \mathbb{R})$ of the gauge group $\mathcal{G}$. In conclusion, when varying the folded domain and taking the quotient by the gauge group, the index is

$$\mu(u_+, K_+) + \mu(u_-, K_-) + (2 - 3)\chi(\Sigma). \tag{7.43}$$

7.1 Homological Data

A folded map (u_+, u_-) together with a pair of conjugate tunneling maps (v_+, v_-) gives rise to two relative homology classes

$$\begin{aligned} A_\pm &\in H_2(X^\pm, \mathcal{R}; \mathbb{Z}) \\ A_\pm &= (u_\pm, v_\pm)_*[\tilde{\Sigma}_\pm]. \end{aligned}$$

Since the map gluing the tunneling domain $\hat{S}$ to Σ_+ (Σ_-) is orientation preserving (reversing), and the ω-energies of the tunneling maps agree by Definition (5.8), we obtain the energy identities

$$\begin{aligned}
E_\omega(u_+) + E_\omega(v_+) &= \omega \cdot A_+ = \text{const} \\
E_\omega(u_-) - E_\omega(v_-) &= \omega \cdot A_- = \text{const} \\
E_\omega(u_+) + E_\omega(u_-) &= \omega \cdot (A_+ + A_-) = \text{const.}
\end{aligned}$$

Therefore the space of folded holomorphic maps breaks up into components labeled by the relative homology classes $A_\pm$, and the sum of the ω-energies of the maps u_+ and u_- is constant in families.

Remark 7.5. In case of S^1-invariant structures we have a natural isomorphism $H_2(X_\pm, \mathrm{R}; \mathbb{Z}) = H_2(\hat{X}_\pm, \mathbb{Z}) \times V$, where $\hat{X}_\pm$ is the symplectic cut of $X_\pm$.

7.2 Transversality

Here we show that the cokernel of the linearized operator at a simple map (u_+, u_-, j) vanishes for a generic $J \in \mathcal{J}(X, \omega, \alpha)$. To see this we recall standard definitions (see e.g. [MS04]) that have straightforward generalizations to our case.

Definition 7.6. *A folded holomorphic map* (u_+, u_-) *with folded domain*

$$(\Sigma_0, j_0, \tau_0, \Sigma_1, j_1, \tau_1)$$

is called multiply covered *if there exist a folded holomorphic map* (u'_+, u'_-) *with folded domain*

$$(\Sigma'_0, j'_0, \tau'_0, \Sigma'_1, j'_1, \tau'_1)$$

and holomorphic branched coverings

$$\begin{aligned}\psi_i &: (\Sigma_i, j_i) \to (\Sigma_i', j_i') \qquad i = 1, 2 \\ \psi_i^*(\tau_i') &= \tau_i\end{aligned}$$

with

$$(u_+, u_-) = (u_+' \circ \psi_0, u_- \circ \psi_0) \qquad \deg(\psi_0) > 1.$$

A map (u_+, u_-) *is called* simple *if it is not multiply covered.*

Definition 7.7. *Given a folded symplectic manifold* (X, ω) *with stable 1-form* α *on* Z*, relative homology classes* $A_\pm \in H_2(X_\pm, \mathcal{R}; \mathbb{Z})$ *and an almost complex structure* $J \in \mathcal{J}(X, \omega, \alpha)$*, let*

$$\mathcal{M}_{g_0, g_1, n_0, n_1}((X, \omega, \alpha), A_+, A_-, \Sigma, J)$$

be the moduli space of simple folded J*-holomorphic maps with domain in* $\mathcal{M}^F_{g_0, g_1, n_0, n_1}$ *so that* $u_\pm \sqcup_\sigma v_\pm$ *represent* $A_\pm$ *and that are transverse to the fold, modulo gauge* $\mathcal{G}$.

Denote the subset of simple maps by

$$\mathcal{M}^*((X, \omega, \alpha), A_+, A_-, \Sigma, J).$$

Definition 7.8. *Let* (u_+, u_-, j) *be a folded holomorphic map. A point* $p \in \Sigma$ *is called an* injective point *if*

$$du(p) \neq 0 \qquad u^{-1}(u(p)) = \{p\}.$$

Let

$$NI(u_+, i_-) = \{p \in \Sigma | du(p) = 0 \text{ or } u^{-1}(u(p)) \setminus \{p\} \neq \emptyset\}$$

be the complement of the set of injective points.

Lemma 7.9. *Let (u_+, u_-, j) be a simple map, and assume that $u_\pm|_\sigma$ are embeddings. Then the set $NI(u_+, u_-)$ of non-injective points is at most countable and can only accumulate at the critical points of (u_+, u_-).*

Proof. This proof follows exactly the lines of the proof of the corresponding result in the standard theory (see e.g. Proposition 2.5.1 from [MS04]. □

Lemma 7.10. *There exists a set $\mathcal{J}_{reg}(X, \omega, \alpha) \subset \mathcal{J}(X, \omega, \alpha)$ of second category such that if (u_+, u_-, j) is a simple folded holomorphic map then the cokernel of the linearized operator D_B vanishes.*

Proof. First note that if (u_+, u_-, j) is simple there exist an neighborhood U of Z such that $T = u^{-1}(X \setminus \overline{U})$ is an open non-empty subset of Σ. By Lemma 7.9 there exits an open non-empty subset T' of T consisting only of non-injective points.

Now standard results (see e.g. Theorem 3.1.5 of [MS04]) show that it is enough to vary the almost complex structure in the immediate vicinity of $u(p)$ for some (arbitrary) $p \in T'$ to achieve transversality of D_B. Since $u(p) \notin U$ by construction, we may vary J in such a way that it remains unchanged in U, so the resulting almost complex structure is still in $\mathcal{J}_{reg}(X, \omega, \alpha)$. □

We immediately obtain the following

Theorem 7.11. *If $J \in \mathcal{J}_{reg}(X, \omega, \alpha)$ then the space*

$$\mathcal{M}^*((X, \omega, \alpha), A_+, A_-, \Sigma, J)$$

is a smooth manifold of the expected dimension.

8 Compactness – First Considerations

We want to show that the moduli space of folded holomorphic maps modulo diffeomorphisms is compact. The key issue is to understand the compactness properties of tunneling maps.

8.1 Energy Estimates

We have seen that the total ω-energy of a folded holomorphic map (u_+, u_-, j) is a topological invariant. But the folded symplectic form ω degenerates along the fold, so the ω energy density $e(u_\pm) = \frac{1}{2}u_\pm^*\omega$ cannot be used to estimate the area with respect to a non-degenerate metric near the fold. But we can still use the ω-energy to estimate how much area can accumulate near the fold.

First we state the standard monotonicity lemma for pseudo-holomorphic curves in symplectic manifolds. Proofs are given in [PW93] or [Tau98].

Lemma 8.1 (Monotonicity). *Let X be a smooth 4-manifold with symplectic form ω and almost complex structure J and compatible Riemannian metric g. Let E be a positive number and $K \subset X$ a compact set. Then there exists a constant $c = c(E, K)$ with the following significance. Let u be a finite energy pseudo-holomorphic map with image $C =$ image(u) and $\int_C \omega \leq E$. Suppose that $x \in K \cap C$ and for $r > 0$ let $B(x, r) \subset X$ denote the g-ball of radius r with center x. Then*

$$c^{-1}r^2 \leq \int_{C\cap B(r,x)} \omega \leq cr^2 \tag{8.44}$$

if $r \leq c^{-1}$.

We cannot apply this result in a neighborhood of the fold, but a similar result still holds true for folded holomorphic curves near the fold of a folded symplectic manifold.

By equation (2.3)

$$\omega = \omega_Z + d(r^2\alpha)$$

in a neighborhood $U = [-\varepsilon, \varepsilon] \times Z$ of the fold Z in X. Here ω_Z is a (non-degenerate) symplectic form on an extension of F to U. Therefore the ω-energy of the map u can be used to estimate the F-components of du. But this is not the case in the E components of du, since ω degenerates there. We focus on estimating the energy on the "+" side of the fold, the "−" side is completely analogous.

Let Σ be a Riemann surface with boundary $\partial\Sigma$ and let $u_+ : \Sigma \to X_+$ be a holomorphic map sending $\partial\Sigma$ into Z and denote its image by $C = u(\Sigma)$. Let $\beta : [0, \infty) \to [0, 1]$ be a cutoff function with $\beta(x) = 1$ if $x \leq 1$, $\beta(x) = 0$ if $x \geq 2$ and $0 \leq -\beta' < 2$. Then define

$$\begin{aligned} E(s) &= \int_C \beta(r/s) d(r^2\alpha) \text{ and} \\ A(s) &= \int_C \beta(r/s) d(r\alpha). \end{aligned}$$

Then $E(s)$ gives a bound on the symplectic E-energy in a neighborhood of size s of the fold and $A(s)$ bounds the E-area. These two are easily related by

$$\begin{aligned} A(s) &\leq \int_C \frac{r}{s}\beta(r/s)d(r\alpha) \\ &= \frac{1}{s}\int_C \beta(r/s)d(r^2\alpha) - \frac{1}{s}\int_C \beta(r/s)rdr \wedge \alpha) \\ &\leq \frac{1}{s}\int_C \beta(r/s)d(r^2\alpha) \\ &= \frac{1}{s}E(s) \end{aligned} \tag{8.45}$$

since $dr \wedge \alpha$ is positive on the image of a holomorphic map near Z.

The following Lemma employs standard energy estimates (see e.g. [Tau98]) to bound the area that can accumulate near the fold.

Lemma 8.2. *Let C be the image of a finite ω-energy folded holomorphic map. Then the function $E(s)$ is differentiable on $(0, r_0/2)$ there exists a constant c independent of C and s such that*

$$E(s) \;\leq\; cs^2 \tag{8.46}$$

$$A(s) \;\leq\; cs. \tag{8.47}$$

Proof. Set $\eta = 1 - \beta$. Then for s fixed, the function

$$E(s,t) \;=\; \int_C \eta(r/t)\beta(r/s)d(r^2\alpha)$$

is decreasing in t with

$$\lim_{t\to 0} E(s,t) \;= E(s) = \textstyle\int_C \beta(r/s)d(r^2\alpha)$$

The limit is uniform in s since for given $\varepsilon > 0$ there exists t_ε such that for all $t_1, t_2 \leq t_\varepsilon$ and for all s we have

$$|E(s,t_1) - E(s,t_2)| < \varepsilon,$$

since C has finite ω-energy. Furthermore $E(s,t)$ is continuous in s and therefore $E(s)$ is also continuous (as it is the uniform limit of $E(s,t)$).

Similarly we have that $E(s)$ is continuously differentiable with

$$E'(s) \;=\; \frac{1}{s}\int_C \frac{r}{s}(-\beta'(r/s))d(r^2\alpha)$$

Here the right hand side is bounded by $2/sE(2s)$.

We can now derive a differential inequality for $E(s,t)$ as follows.

$$\begin{aligned}
E(s,t) &= \int_C \eta(r/t)\beta(r/s)d(r^2\alpha) \\
&= -\int_C \eta(r/t)d\beta(r/s)\wedge r^2\alpha - \int_C \beta(r/s)d\eta(r/t)\wedge r^2\alpha \\
&\leq \int_C \frac{r}{s}(-\beta'(r/s))r\,dr\wedge\alpha - \int_C \beta(r/s)\frac{r}{t}(-\beta'(r/t))rdr\wedge\alpha \\
&= \frac{s}{2}E'(s) - \frac{1}{2}\int_C \frac{r}{s}(-\beta'(r/s))r^2d\alpha - \int_C \beta(r/s)\frac{r}{t}(-\beta'(r/t))rdr\wedge\alpha. \\
&\leq \frac{s}{2}E'(s) - \int_C \beta(r/s)\frac{r}{t}(-\beta'(r/t))rdr\wedge\alpha
\end{aligned}$$

where we used that $(-\beta') \geq 0$ and $d\alpha > 0$ on C since u was holomorphic. Also, as $t \to 0$ the last term vanishes, since

$$\frac{1}{2}\int_C \beta(r/s)\frac{r}{t}(-\beta'(r/t))rdr\wedge\alpha \;\leq\; \frac{1}{2}\int_C (-\beta'(r/t))\omega \leq \int_C \beta(r/2t)(1-\beta(2r/t)\omega.$$

Therefore

$$E(s) \leq \frac{s}{2}E'(s).$$

Integrating this gives (8.46) with $c = E(s_0)/s_0^2$, $s_0 = r_0/2$ independent of s and C. Now (8.47) follows from equations (8.46) and (8.45). $\square$

9 Examples of Folded Holomorphic Maps

We give examples of folded holomorphic maps in two special cases.

9.1 Folded Holomorphic Maps into Folded $E(1)$

We come back to the example of $E(1)$ from Section 3 and show that Definitions 4.11 and 3.4 coincide in this case.

As horizontal section we choose

$$v_h : T \to Z = S^1 \times T \qquad w \mapsto (1, w).$$

Then equation 5.31 gives that incoming boundary conditions are scattered to outgoing boundary conditions by the map

$$\Psi : S^1 \times T \to S^1 \times T \qquad (z, w) \mapsto (\bar{(z)}, w),$$

in the case that there exists a tunneling map capping off that boundary condition. This reproduces Definition 3.4 and we just need to show that tunneling maps exist.

Given a folded holomorphic map $u : \Sigma \mapsto E^F(1)$ in the sense of Definition 3.4 fix the folded domain $\Sigma_0 = \Sigma_1 = \Sigma$ and set $S = \Sigma_-$. Let S' denote the components of S that are mapped entirely into $0 \times T = T$ and set $\dot{S} = S \setminus u^{-1}(0 \times T)$.

Let $\pi : \dot{D} \times T \mapsto S^1 \times T$ be the radial projection.

Define

$$v_\pm : S' \cup \dot{S} \mapsto Z = S^1 \times \mathbb{R}$$

by

$$\begin{aligned} v_-(x) &= \begin{cases} \pi \circ u_-(x) & x \in \dot{S} \\ z_0 \times u_-(x) & x \in S' \end{cases} \\ v_+(x) &= \psi \circ v_- \end{aligned}$$

Then (v_+, v_-) are conjugate tunneling maps with the desired boundary conditions.

The different choices of horizontal tunneling maps correspond to choosing a different gluing map ι_F by composing with a rotation of the S^1.

9.2 Folded Holomorphic Rational Curves in S^4

We give examples of folded holomorphic curves by utilizing the symmetries of the folded symplectic and complex structure on S^4 defined in Section 3. Essentially these curves come from pseudo-holomorphic curves in $\mathbb{P}^2$.

As shown in [HWZ03], the pseudo-holomorphic cylinder over S^3 with its standard $\mathbb{R}$-invariant structure is biholomorphic to $\mathbb{C}^2 \setminus \{0\}$ via

$$\Phi : \mathbb{R} \times S^3 \to \mathbb{C}^2 \setminus \{0\}, \qquad (t, z) \mapsto e^{2t} z.$$

For the rest of this section we fix homogeneous coordinates $[x : y : z]$ on $\mathbb{P}^2$ and a corresponding embedding $\mathbb{C}^2 \subset \mathbb{P}^2$, $(z, w) \mapsto [x : w : 1]$, whose complement is denoted by $\mathbb{P}^1_\infty$. Using this we can view finite energy pseudo-holomorphic maps in $\mathbb{R} \times S^3$ as maps in $\mathbb{P}^2$. Conversely, pseudo-holomorphic maps in $\mathbb{P}^2$ that have no components that lie entirely in $\mathbb{P}^1_\infty \cup \{0\}$ can be viewed as (punctured) pseudo-holomorphic maps into $\mathbb{R} \times S^3$ by restriction. A straightforward calculation reveals that punctured finite energy (as defined in [HWZ03]) pseudo-holomorphic maps into $\mathbb{R} \times S^3$ extend over the punctures to pseudo-holomorphic maps into $\mathbb{P}^2$, and that conversely maps into $\mathbb{R} \times S^3$ that are restrictions of pseudo-holomorphic maps into $\mathbb{P}^2$ have finite energy. By finite energy pseudo-holomorphic maps into $\mathbb{C}^2$ we mean a map with no components mapping entirely to $\{0\}$ and whose restriction to the preimage of $\mathbb{C}^2 \setminus \{0\}$ is a finite energy map into the cylinder over S^3.

First we need to fix a horizontal tunneling map. Any horizontal tunneling map projected into the base S^2 has to be an embedding, extending over the punctures. Therefore the domain of the horizontal tunneling map has to be S^2 and the composition with the projection has degree 1. Since $H^1(S^1, \mathbb{R})$ is trivial, the horizontal tunneling map is the projection of a holomorphic map into the symplectization, so it has to be the restriction of a degree 1 map into $\mathbb{P}^2$. Thus the map has only one puncture wrapping a closed characteristic with multiplicity 1 and the only freedom in choosing the horizontal tunneling map is the choice of closed characteristic and slice of the S^1-action. Since two of such choices can be mapped onto one another by a diffeomorphism of S^3 preserving the CR structure and α, all these choices are equivalent.

We choose the horizontal tunneling map

$$\begin{aligned} v_h &: \mathbb{C} \to \mathbb{Z} \\ v_h(z) &= \frac{1}{\sqrt{1+|z|^2}}(z, 1). \end{aligned}$$

Now we choose homology classes for our maps. Note that

$$H_2(B^4, \mathcal{R}; \mathbb{Z}) = S^2 \times \mathbb{Z}$$

where the isomorphism is given by specifying the closed characteristic and the multiplicity. We fix the as closed characteristic the one that the tunneling map wraps. Then the only homological information is the multiplicity with which we wrap the characteristic. Equation (5.30) relates the difference of multiplicities

$$d = m_+ - m_0 = -(m_- - m_0)$$

of the maps v_+, v_- and v_0. So given the choices we have made so far, the only choice we have left is that of the integer d, which we call the *degree* of our map into S^4.

We look for degree 1 maps with domain $\Sigma_0 = \Sigma_1 = S^2$ and domain fold an embedded circle.

For the examples we construct it suffices to consider $\Sigma_+ = \overline{\mathbb{D}} = \{z \in \hat{\mathbb{C}} | \, |z| \leq 1\}$ and $S = \Sigma_- = \mathbb{D}^c = \{z \in \hat{\mathbb{C}} | \, |z| \geq 1\}$, with the puncture of S at $\infty \in \hat{\mathbb{C}}$. As parameter space for our family we take $\dot{\mathbb{D}} = \{z \in \mathbb{C} | \, 0 < |z| < 1\}$.

Fix $a \in \dot{\mathbb{D}}$ and set $c = (1 - |a|^2)^{-\frac{1}{2}}$, so $|a|^2 + 1/|c|^2 = 1$. We define the maps

$$\begin{aligned} u_\pm &: \Sigma_\pm \to S^4_\pm \\ u_+(z) &= \sigma_+^{-1}(az^2, z/c) \\ u_-(z) &= \sigma_-^{-1}(a, \frac{1}{cz}) \end{aligned}$$

and the tunneling maps

$$\begin{aligned} v_\pm &: \dot{S} \to Z \\ v_+(z) &= \frac{1}{\sqrt{|az^2|^2 + |z|^2/c^2}}(az^2, \frac{z}{c}) \\ v_-(z) &= \frac{1}{\sqrt{|a|^2 + |cz|^{-2}}}(a, \frac{1}{cz}). \end{aligned}$$

The corresponding horizontal covering is given by

$$\begin{aligned} v_0 &: \dot{S} \to Z \\ v_0(z) &= \frac{1}{\sqrt{1 + |acz|^2}}(acz, 1) = v_h(acz). \end{aligned}$$

To see that this data defines folded holomorphic maps note that if $|z| = 1$,

$$\begin{aligned} v_+(z) &= \frac{1}{\sqrt{|a|^2 + 1/c^2}}(az^2, z/c) = (az^2, z/c) = u_+(z) \\ v_-(z) &= \frac{1}{\sqrt{|a|^2 + |c|^{-2}}}(a, \frac{1}{cz}) = (a, \frac{1}{cz}) = u_-(z). \end{aligned}$$

Note that $\ln(\frac{z}{|z|})$ is purely imaginary, so with $g(z) = \Im\left(\ln(\frac{|z|}{z})\right)$,

$$\begin{aligned} g(z) * v_0(z) &= e^{ig(z)} v_0(z) = \frac{z}{|z|}\frac{(acz, 1)}{\sqrt{1 + |acz|^2}} = \frac{(az^2, z/c)}{\sqrt{|z/c|^2 + |az^2|^2}} = v_+(z) \\ -g(z) * v_0(z) &= e^{-ig(z)} \cdot v_0(z) = \frac{|z|}{z}\frac{(acz, 1)}{\sqrt{1 + |acz|^2}} = \frac{(a, \frac{1}{cz})}{\sqrt{|cz|^{-2} + |a|^2}} = v_-(z). \end{aligned}$$

We give another way to visualize the family.

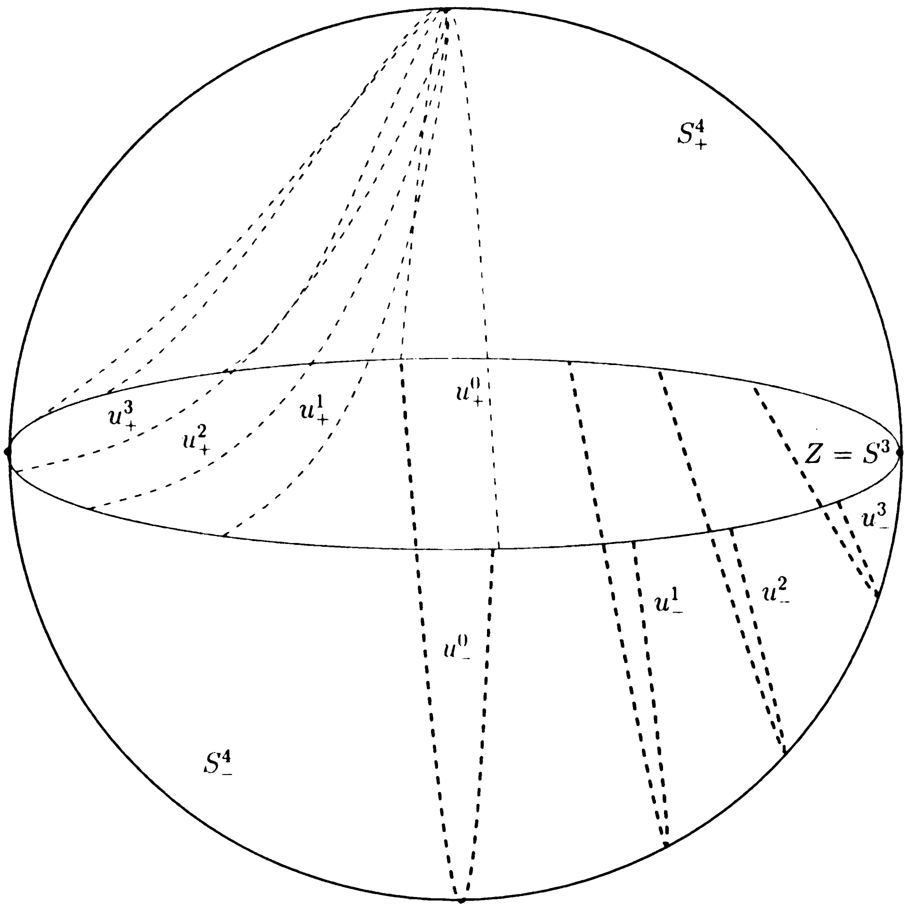

Figure 10: Several members of the above family of maps into S^4. The map (u^0_-, u^0_+) corresponds to the extreme case $a = 0$. Then the maps u^t_- loose energy as they disappear into the fold whereas the maps u^t_+ gain energy. Then tunneling maps go back to the closed characteristic. They sweep out the image of a holomorphic map in Z.

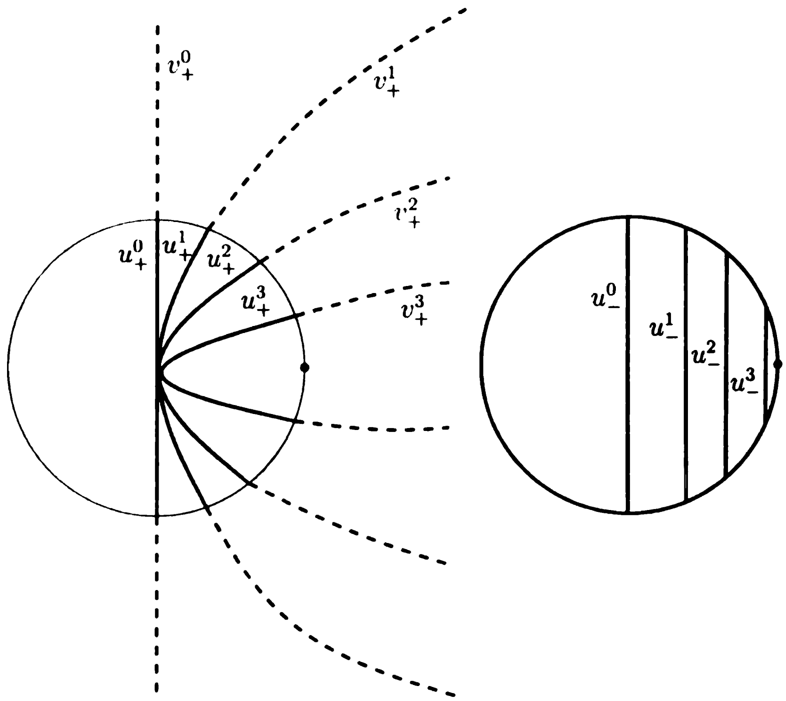

Figure 11: Here we visualize the family as maps into $\mathbb{C}^2$. We also suspend the tunneling map to (generalized) holomorphic maps into $\mathbb{C}^2$. The maps $u_\pm$ have image in the unit ball, whereas v_+ has image outside the unit ball. v_- coincides with u_-. When projected into S^3, $v_-(0)$ lies on the closed characteristic. As the family of maps approaches (u_+^0, u_-^0), the tunneling maps become longer and longer and in the limit bubble off to form an $\mathcal{H}$-holomorphic map from $\dot{S}^2$ into the fold.

Note that all of these maps send $0 \in \hat{\mathbb{C}} = S^2$ to the north pole N in S^4. Moreover the derivative of u_+ at 0 always has image in 1 2-dimensional subbundle, as

$$d(\sigma_+ \circ u_+(0)) = (0, 1/c)dz, \tag{9.48}$$

so the maps in this family satisfy a 4-dimensional constraint.

Observe that for the above family the sum of the ω-energies of u_+ and u_- remains constant.

All the above maps extend to the case when $|a| = 1$ and thus $\frac{1}{c} = 0$ as well as $a = 0$ and $c = 1$. These are boundary points in the moduli space we defined, as $\pi_F dv_\pm$ vanishes identically in these cases.

If $a = 0$, the maps u_+ and u_- intersect the fold in a closed characteristic and the tunneling maps $v_\pm$ formed a bubble in the fold, i.e. they have a closed punctured Riemann surface as domain (S^2 with two punctures in this case) and are non-trivial $\mathcal{H}$-holomorphic maps into the fold.

In the case that $|a| = 1$ the map u_- and the tunneling maps are point maps and u_+ carries the entire energy.

This family as the property that it sweeps out S^4_- in a way reminiscent of how lines sweep out $\mathbb{P}^2$. To see this we mark three points, one in S^2_+4 $p_1 = 0$ in the above parametrization), one on the domain fold σ ($p_2 = (1 - |a|^2)^{-\frac{1}{2}}$ in the above parametrization) and one in S^2_-. These points are characterized by the property that p_1 be mapped to the north pole and the map satisfies the tangency condition (9.48) at that point, and that p_2 be the unique point with $u_+(p_2) = u_-(p_2)$.

Lemma 9.1. *Marking a third point $p \in S^2_- = D^c \subset \hat{\mathbb{C}}$ we obtain an isomorphism*

$$\begin{aligned} \Psi &: \mathbb{D} \times D^c \to S^4_- \\ \Psi(a, p) &= u_a(p). \end{aligned}$$

Proof. To see injectivity, suppose $\Psi(a,p) = \Psi(b,q)$. Then if $p, q \in \mathbb{D}$

$$(a, \sqrt{1-|a|^2}/p) = (b, \sqrt{1-|b|^2}/q)$$

so $a = b$ and $p = q$.

To see surjectivity, let $\sigma_-(z,w) \in S^4_-$. Then set

$$a = z \qquad \frac{1}{p} = w/\sqrt{1-|a|^2}.$$

Then

$$u_a(p) = \sigma_-(a, \sqrt{1-|a|^2}/p) = \sigma_-(z,w).$$

□

Next we compute the index of the linearized operator at one of the maps in the above family. Note that we can homotope the boundary conditions (F_+, F_-) to $\pi_1(v_* T\partial D) \times \pi_2(v_* T\partial D)$ where π_1 and π_2 are the projections on the first and second factor of $\mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$, respectively. To do this note that $\mathbb{R}$ and $\pi_1\mathbb{R}$ are transverse to both $v_* T\partial D$ and $\pi_1(v_* T\partial D)$, so we may homotope the boundary conditions to $\pi_1 \mathbb{R} \times \pi_2(v_* T\partial D)$ and the to the desired boundary conditions. Then it follows that the Maslov indices $\mu_\pm$ for the maps $u_\pm$ are given by $\mu_- = 2$ and $\mu_+ = 6$ and therefore we have

$$index = 8 + 2\chi(S^2) = 12.$$

Cutting down by the 6-dimensional group of automorphisms and the 4-dimensional constrained mentioned above we are left with the 2-dimensional family exhibited above.

A Folded Compatible Triples

Recall the following result from linear algebra.

Lemma A.1. *Let V be a $2n$-dimensional real vector space and g an inner product on V, i.e. a positive definite symmetric bilinear form. Let A be a skew-adjoint endomorphism. Then A has purely imaginary eigenvalues that appear in complex conjugate pairs and the eigenspaces to eigenvalues with different modulus are orthogonal.*

In the following let (X, ω) be an oriented folded symplectic manifold of real dimension $2n$ with compact folding hypersurface Z with defining function t. Furthermore define $\sigma : X \setminus Z \to \{\pm 1\}$, by $\sigma(x) = \pm 1$ if $x \in X^{\pm}$. Also denote by E_Z the real 2-plane bundle over Z defined by $E_Z = \ker(\omega)$.

Lemma A.2. *Given a background Riemannian metric g there exists a canonical smooth folded compatible triple (ω, J, g_J) defined on $X \setminus Z$, where*

(i) $J^2 = -Id$,

(ii) $\omega(u, v) = g_J(Ju, v)$.

Furthermore there exists a neighborhood U of Z and a splitting $TU = E \oplus F$ such that

(a) *along Z* $E = \ker(\omega) = \ker(g)$

(b) *E and F are J-invariant and perpendicular with respect to both g_J and ω*

(c) *for any defining function t of Z there exist a symplectic form μ and a positive definite symmetric bilinear form h on E_Z with*

(i) $g|_E = |t| \cdot h$

(ii) $w|_E = t \cdot \mu$

(d) the almost complex structures on $J^\pm$ *on* TU *defined by* $J^\pm = J_F \oplus \sigma J_E$ *are smooth.*

Proof. Let g be a Riemannian metric on X and define the skew-endomorphism A of TX by $\omega(u,v) = g(Au,v)$. This exists and is unique since g is positive definite. Define $E_Z^\perp = E_Z^{\perp_g}$ and observe that $A|_{E_Z} = 0$, and $A|_{E_Z^\perp}$ is non-degenerate. Since Z is compact there exists $\alpha \in \mathbb{R}^+$ with $|\lambda| > \alpha$ for all eigenvalues of $A|_{E_Z^\perp}$. Therefore there exists a neighborhood U of Z such that $|\lambda_1| < \alpha/2 < |\lambda|$, over U, where λ_1 is the smallest eigenvalue of A and λ is the second smallest eigenvalue of A (after λ_1 and $\bar{\lambda}_1$). Define E to be the 2-plane bundle over U given by the eigenspace to the smallest eigenvalue of A and set $F = E^{\perp_g}$. Then for $u \in E$ and $v \in F$ we have $Au = \tilde{u} \in E$ and

$$\omega(u,v) = g(Au,v) = g(\tilde{u},v) = 0.$$

Therefore ω splits as $\omega_E \oplus \omega_F$. Furthermore, by Lemma A.1 the other eigenspaces of A are contained in F, so A also splits as $A = A_E \oplus A_F$. Since A_F is non-degenerate, the usual polarization procedure produces a canonical compatible triple there.

Since ω is folded symplectic, $\omega_E = t \cdot \mu$ for some non-degenerate 2-form μ. Therefore $A_E = t \cdot \tilde{A}_E$, where $\tilde{A}_E = g_E^{-1}\mu$ is a non-degenerate skew-endomorphism of E. By polarization of $\tilde{A}_E$ we obtain a smooth compatible triple $(\mu, \tilde{J}, h)$ on E, and using A_E instead we get a compatible triple (ω_E, J_E, g_{JE})on $E \setminus E_Z$ which relates to the above one by

$$\begin{aligned} \omega &= t \cdot \mu, \\ J_E &= \frac{t}{|t|}\tilde{J} = \sigma \tilde{J}, \\ g_{JE} &= |t| \cdot h. \end{aligned}$$

$\square$

The proof of the lemma motivates the following

Definition A.3. *A local folded hermitian trivialization of $TX|_U$ w.r.t. a folded compatible triple (ω, J, g) (and defining function t of Z) near a point $p \in Z$ is given by an ω and g orthogonal splitting $TU = E \oplus F$ and local sections $u_1, v_1, \ldots u_n, v_n$ such that $u_2, v_2, \ldots, u_n, v_n$ is a hermitian basis of F and u_1, v_1 is a basis of E with*

(i) $g(u_1, u_1) = g(v_1, v_1) = |t|$,

(ii) $\omega(u_1, v_1) = t$,

(iii) $\sigma J u_1 = v_1$ *and* $\sigma J v_1 = -u_1$.

Corollary A.4. *Given a folded compatible triple on a folded symplectic manifold, there exist local folded hermitian trivializations of the tangent bundle.*

Remark A.5. If ω and the background metric g are invariant under a smooth group action G on U, then the resulting splitting $TU = E \oplus F$ and the triple (ω, g_J, J) will also be invariant under this action. To see this, note that the skew-endomorphism A above will be invariant under the action, so the eigenspaces and eigenvalues of A will be. We conclude that the resulting J and therefore also g_J will be invariant under the action.

Definition A.6. *In a folded symplectic manifold (X^{2n}, ω), a component Z of the fold is of* contact type, *if there exists a 1-form α on Z such that*

1. $\alpha \wedge \omega^{n-1} \neq 0$ *and*

2. $d\alpha = \omega|_Z$.

The component is called completely integrable *if there exists a 1-form α on Z such that*

1. $\alpha \wedge \omega^{n-1} \neq 0$ *and*

2. $d\alpha = 0$.

Lemma A.7. *Suppose a component Z of the fold in a folded symplectic manifold is of contact type or completely integrable. Then, in (global) Darboux coordinates $Z \subset U = [-\varepsilon, \varepsilon] \times Z$ near the component Z of the fold associated to the given 1-form α. Then we may arrange for the bundles E and F and the complex structures $J^{\pm}$ to be invariant under translation along the normal direction in a (possible smaller) neighborhood of the fold.*

Proof. Over Z define $E = \ker(\omega)$ and $F = \ker(\alpha)$ and use the canonical parallel transport in Darboux coordinates to extend this splitting to a neighborhood of the fold.

Choose a background metric g' making this splitting orthogonal and note that in Darboux coordinates

$$\begin{aligned} \omega &= d(r^2\alpha) + \omega_Z \\ &= 2rdr \wedge \alpha + a\omega_Z \end{aligned} \tag{A.49}$$

where a is a positive function. Choose the background metric

$$g' = dr \otimes dr \oplus \alpha \otimes \alpha \oplus a \cdot g'_F$$

with some metric g'_F on F that is independent of r.

Since the splitting $TU = E \oplus F$ is orthogonal with respect to both g' and ω we may carry out the construction of the associated skew-endomorphism A and the compatible triple separately on each summand, immediately yielding that the given translation invariant splitting is also J-invariant. Moreover a short computation shows that the resulting J is also translation invariant. $\square$

Remark A.8. The essential part in the above proof was that the translates of E has symplectic complements that were translates of one another. This will happen precisely when ω can be written as in equation (A.49). But this happens only if there exists a 1-form α on Z with $d\alpha$ a multiple of ω which is equivalent to the condition $\mathcal{L}_R\alpha = 0$, in which case the above lemma still holds true. (See also Lemma 2.8.)

B Properties of the Levi-Civita Connection

Recall that the non-degenerate metric $\tilde{g}$ on Z is given by

$$\tilde{g} = \alpha^2 \oplus g_F$$

where g_F is the inner product on F induced by J and ω. Then the splitting $TZ = L \oplus F$ is $\tilde{g}$-orthogonal.

First we want to state some properties of the Levi-Civita connection ∇ associated with $\tilde{g}$.

Lemma B.1. *Let X be a vector field on Z and Y a section of F. Then*

$$\nabla_X R \subset F \tag{B.50}$$

$$\nabla_R Y \subset F \tag{B.51}$$

$$\nabla_R R = 0 \tag{B.52}$$

$$\nabla_R \alpha = 0 \tag{B.53}$$

$$\nabla_R (JY) = J\nabla_R Y \tag{B.54}$$

$$\nabla_R \omega = 0 \tag{B.55}$$

$$\nabla_R d\alpha = 0. \tag{B.56}$$

Proof. For the first equation note that

$$0 = \nabla_X \tilde{g}(R, R) = 2\tilde{g}(\nabla_X R, R). \tag{B.57}$$

Next note that $[R, Y] \in F$ since α is stable and therefore

$$\alpha([R, Y]) = d\alpha(R, Y) - R \cdot \alpha(Y) + Y \cdot \alpha(R) = 0.$$

Now equation (B.51) follows from $\nabla_R Y = \nabla_Y R + [R, Y] \subset F$ where we used equation (B.50) and that ∇ is torsion free. Then using this result we get

$$0 = \nabla_R \tilde{g}(R, Y) = \tilde{g}(\nabla_R R, Y) + \tilde{g}(R, \nabla_R Y) = \tilde{g}(\nabla_R R, Y)$$

which, when combined with equation (B.50), yields equation (B.52).

For equation (B.53) use again that α is stable to compute

$$(\nabla_R \alpha)X = \nabla_R(\alpha(X)) - \alpha(\nabla_R X) = \alpha(\mathcal{L}_R X) - \alpha(\nabla_X R + [R, X]) = -\alpha(\nabla_X R)$$

which vanishes by equation (B.50).

For equation (B.54) note that $\nabla_R(JY) \in F$ by equation (B.51). Then

$$\begin{aligned}
g(Y, \nabla_R(JY)) &= -g(\nabla_R Y, JY) = g(Y, J\nabla_R Y) \\
g(JY, \nabla_R(JY)) &= \frac{1}{2}\nabla_R g(JY, JY) = \frac{1}{2}\nabla_R g(Y, Y) = g(Y, \nabla_R Y) = g(JY, J\nabla_R Y)
\end{aligned}$$

and we conclude that $\nabla_R(JY) = J\nabla_R Y$ for all non-zero sections Y of F. Since $\nabla_R(JY) - J\nabla_R Y$ is tensorial in Y it holds for all sections Y.

Now for sections X and Y of TZ

$$\begin{aligned}
(\nabla_R \omega)(X, Y) &= \nabla_R \omega(X, Y) - \omega(\nabla_R X, Y) - \omega(X, \nabla_R Y) \\
&= \nabla_R g(X, JY) - g(\nabla_R X, JY) - g(X, J\nabla_R Y) \\
&= \nabla_R g(X, JY) - g(\nabla_R X, JY) - g(X, J\nabla_R Y) \\
&= g(\nabla_R X, JY) + g(X, \nabla_R JY) - g(\nabla_R X, JY) - g(X, J\nabla_R Y) \\
&= g(X, \nabla_R JY) - g(X, J\nabla_R Y) \\
&= 0
\end{aligned}$$

where we used equation (B.54) in the last step, proving equation (B.55).

For equation (B.56) recall that α is stable, so there exists a function $f : Z \to \mathbb{R}$ such that $d\alpha = f\omega$. Then $R \cdot f = 0$ since $\mathcal{L}_R d\alpha = \mathcal{L}_R \omega = 0$. Thus

$$\nabla_R d\alpha = \nabla_R f\omega = (R \cdot f)\omega + f\nabla_R \omega = 0.$$

$\square$

Next we compute some components of ∇J.

Lemma B.2. *Let $p \in Z$ and $X, Y \in F_p$. Then*

$$\begin{aligned}
\alpha((\nabla J)R) &= 0 && \text{(B.58)}\\
\alpha((\nabla_X J)Y) &= -\omega(\nabla_X R, Y) && \text{(B.59)}\\
\pi_F \circ (\nabla J) \circ \pi_F &= 0 && \text{(B.60)}\\
(\nabla_X J)Y &= (\nabla_Y J)X && \text{(B.61)}\\
\pi_F \circ (\nabla J) &= -J(\nabla R) \otimes \alpha && \text{(B.62)}
\end{aligned}$$

Proof. For equation (B.58)

$$\alpha((\nabla J)R) = \alpha(\nabla(JR) - J\nabla R) = 0.$$

For equation (B.59)

$$\begin{aligned}
\alpha((\nabla_X J)Y) &= \alpha(\nabla_X(JY))\\
&= g(R, \nabla_X(JY))\\
&= -g(\nabla_X R, JY)\\
&= -\omega(\nabla_X R, Y).
\end{aligned}$$

For equation (B.60) let W be any section of TZ and let χ be a local unit section of F. We compute the components of $\nabla_W J$ in the trivialization given by χ and $J\chi$.

$$\begin{aligned}
g(\chi, (\nabla_W J)\chi) &= g(\chi, \nabla_W(J\chi)) - g(\chi, J\nabla_W \chi)\\
&= -g(\nabla_W \chi, J\chi) + g(J\chi, \nabla_W \chi) = 0\\
g(J\chi, (\nabla_W J)\chi) &= g(J\chi, \nabla_W(J\chi)) - g(J\chi, J\nabla_W \chi)\\
&= \frac{1}{2}\nabla_W\, g(J\chi, J\chi) - \frac{1}{2}\nabla_W\, g(\chi, \chi) = 0.
\end{aligned}$$

The vanishing of the remaining components follows after replacing χ by $J\chi$ in the above argument.

For equation (B.61) we only have to show that $\alpha((\nabla_X J)Y = (\nabla_Y J)X)$ in view of equation B.60). Using equation (B.59) we compute

$$\begin{aligned}
\alpha((\nabla_X J)Y - (\nabla_Y J)X) &= -\omega(\nabla_X R, Y) + \omega(\nabla_Y R, X) \\
&= -\omega(\nabla_X R, Y) - \omega(X, \nabla_Y R) \\
&= -\omega(\nabla_R X, Y) + \omega(\mathcal{L}_R X, Y) - \omega(X, \nabla_R Y) + \omega(X, \mathcal{L}_R Y) \\
&= -R \cdot \omega(X, Y) + R \cdot \omega(X, Y) \\
&= 0,
\end{aligned}$$

where we used that $\mathcal{L}_R \omega = 0$ and that $\nabla_R \omega = 0$. by equation (B.55).

For equation (B.62) let V and W be sections of TZ. Then, using equation (B.60),

$$\begin{aligned}
\pi_F \circ (\nabla_W J)V &= \pi_F \circ (\nabla_W J)(V - \pi_F V) \\
&= \pi_F \circ (\nabla_W J)R \cdot \alpha(V) \\
&= \pi_F \circ \nabla_W (JR \cdot \alpha(V)) - \pi_F \circ J(\nabla_W (R \cdot \alpha(V))) \\
&= -J(\nabla_W R)\alpha(V).
\end{aligned}$$

$\square$

We can summarize the findings of the above lemma as follows. We may view

$$\nabla J : (R \oplus F) \otimes (R \oplus F) \to R \oplus F$$

in block form, where the only non-zero components are the "cross-terms"

$$\begin{aligned}
\nabla J|_{F \otimes F} &: F \otimes F \to L \\
\nabla J|_{F \otimes F} &= (-\iota_{\nabla R}\omega) \cdot R,
\end{aligned}$$

which is symmetric in its arguments and

$$\begin{aligned}
\nabla J|_{F\otimes L} &: F \otimes L \to F \\
\nabla J|_{F\otimes L} &= -J(\nabla R) \otimes \alpha.
\end{aligned}$$

We can prove some results about ∇R.

Lemma B.3. *For X in F_p,*

$$g(X, \nabla_X R) = -g(JX, \nabla_{JX} R) \tag{B.63}$$

$$g(X, \nabla_{JX} R) = -g(JX, \nabla_X R) - g(X, (\mathcal{L}_R J)X. \tag{B.64}$$

Proof. In a neighborhood of p, extend X to a section of F of constant length. Then

$$\begin{aligned}
g(JX, \nabla_{JX} R) &= g(JX, \nabla_R(JX) - \mathcal{L}_R(JX)) \\
&= g(JX, J\nabla_R X) - g(JX, \mathcal{L}_R(JX)) \\
&= g(X, \nabla_R X) - \omega(\mathcal{L}_R(JX), X) \\
&= \omega(JX, \mathcal{L}_R X) \\
&= -g(X, \nabla_R X) + g(X, \mathcal{L}_R X) \\
&= -g(X, \nabla_X R)
\end{aligned}$$

and

$$\begin{aligned}
g(X, \nabla JXR) &= g(X, \nabla_R(JX) - \mathcal{L}_R(JX)) \\
&= g(X, J\nabla_R X - (\mathcal{L}_R J)X - J\mathcal{L}_R X) \\
&= g(X, J(\nabla_X R) - (\mathcal{L}_R J)X) \\
&= -g(JX, \nabla_X R) - g(X, (\mathcal{L}_R J)X).
\end{aligned}$$

$\square$

Lemma B.4. . *In the S^1-invariant case we have*

$$\nabla R = 0 \tag{B.65}$$

$$\nabla J = 0 \tag{B.66}$$

$$\nabla \alpha = 0. \tag{B.67}$$

Proof. Since in this case the metric g is invariant under the characteristic flow, so is ∇ and parallel transport along characteristics is given by pushing vectors forward using the characteristic flow.

Since J is also invariant under the characteristic flow we have

$$\nabla_R J = 0.$$

Let $X \in F$ at p and extend X to a vector field on a slice transverse to L near p. Set $Y = JX$ and transport this frame to a neighborhood of p using the characteristic flow. Thus $\nabla_R X = \nabla_R Y = 0$, and

$$\nabla_X R = \nabla_R X - \mathcal{L}_R X = 0.$$

We conclude that $\nabla R = 0$.

Thus $(\nabla J)\pi_E = 0$ and $\pi_E(\nabla J) = 0$, and we conclude in conjunction with equation B.62 that $\nabla J = 0$.

Lastly, by choosing normal coordinates with $X, Y \in F$ at p,

$$(\nabla_X \alpha)Y = \alpha(\nabla_X Y) = 0.$$

Together with Lemma B.1 and that $(\nabla_X \alpha)R = 0$ we conclude $\nabla\alpha = 0$. □

C Properties of the Sobolev Spaces of Maps

The purpose of this section is to prove some of the properties of the Sobolev spaces used. These results are needed to show that tunneling maps have a well-defined Fredholm theory. The issue here is the behavior of the maps near the punctures.

Define the half-infinite cylinder

$$C_r = [r, \infty) \times S^1$$

and set $C = C_0$.

First we show that $W^{k,p}_\delta(C, v^*TZ)$ is a Banach manifold.

Lemma C.1. *$W^{k,p}_\delta(C, Z)$ is a separable Banach manifold, locally at a map $v \in W^{k,p}_\delta(C, Z)$ modeled on a neighborhood of the zero section in $W^{k,p}_\delta(C, v^*TZ)$.*

Proof. Embed Z isometrically into $\mathbb{R}^N$ for some N. Let $v \in W^{k,p}_\delta(C, Z)$ and $\zeta \in W^{k,p}(C, v^*TZ)$. Set $\tilde{v} = \exp_v(\zeta)$ where we parallel transport using the Levi-Civita connection ∇. Then

$$d\tilde{v} = dv + \int_0^1 \frac{d}{d\tau} d\exp_v(\tau\zeta) d\tau = dv + \int_0^1 \nabla\zeta_\tau d\tau = dv + \nabla\zeta$$

where we use the canonical parallel transport in $\mathbb{R}^n$.

$$\begin{aligned} E(\tilde{v}) &= \int_C e^{\delta s}\left(|\pi_F d\tilde{v}|^2 + |\tilde{v}^*\alpha(\partial_s)|^2 + d(\tilde{v}^*\alpha(\partial_t))|^2\right) d\text{vol} \\ &\leq \int_C e^{\delta s}\left(|\pi_F dv|^2 + |v^*\alpha(\partial_s)|^2 + d(v^*\alpha(\partial_t))|^2\right) d\text{vol} \\ &\quad + \int_C e^{\delta s}\left(|\pi_F \nabla\zeta|^2 + |\alpha(\nabla_s\zeta)|^2 + d(\alpha(\nabla_t\zeta))|^2\right) d\text{vol} \\ &= E(v) + E(\zeta) \end{aligned}$$

So $\tilde{v}$ has finite asymptotic energy if ζ does. Conversely

$$\begin{aligned} E(\zeta) &= \int_C e^{\delta s}\left(|\pi_F \nabla\zeta|^2 + |\alpha(\nabla_s\zeta)|^2 + d(\alpha(\nabla_t\zeta))|^2\right) \text{dvol} \\ &\leq E(\tilde{v}) + E(v) \end{aligned}$$

so ζ has finite asymptotic energy if $\tilde{v}$ does. $\square$

The following theorem motivates the definition of asymptotic energy. For $r \geq 0$ set $C_r = [r, \infty) \times S^1$.

Theorem C.2. *Let $v \in W^{k,p}_\delta(C, Z)$ with $\delta > 0$, $kp > 2$. Then there exists a constant $T \in \mathbb{R}$ called the* asymptotic charge *of v and a constant C, independent of r and δ, such that*

$$\int_{C_r} e^{\delta s}|dv - T(R \otimes dt)|^2 \, \text{dvol} \leq C\, E_r(v)\left(1 + \frac{E_r(v)e^{-\delta r}}{1 - e^{\delta}}\right). \tag{C.68}$$

Proof. For any non-negative function $f : C \to \mathbb{R}$

$$\int_{C_r} f \, \text{dvol} = \int_{C_r} f e^{\delta s} e^{-\delta s} \, \text{dvol} < e^{-\delta r} \int_{C_r} f e^{\delta s} \, \text{dvol}$$

so

$$\int_{C_r} \left(|v^*\alpha(\partial_s)|^2 + |d(v^*\alpha(\partial_t))|^2 + |\pi_F dv|^2\right) \text{dvol} \quad < \quad E_r(v)e^{-\delta r} \tag{C.69}$$

Recalling that α is stable and that $|d\alpha| < M$ on Z we have

$$|v^*d\alpha| < |d\alpha|\, |\pi_F\, dv|^2 < M|\pi_F\, dv|^2,$$

and hence for $0 \leq r \leq r'$

$$\left| \int_{S^1_r} v^*\alpha - \int_{S^1_{r'}} v^*\alpha \right| \leq \int_{C_{r,r'}} |v^*d\alpha| \, \text{dvol} \leq M \int_{C_r} |\pi_F\, dv|^2 \, \text{dvol} \leq M E_r(v) e^{-\delta r} \tag{C.70}$$

where we used equation (C.69) in the last step. So we may define the *asymptotic charge* $T \in \mathbb{R}$

$$T = \lim_{r \to \infty} \int_{S^1_r} v^*\alpha.$$

Define the function $u : C \to \mathbb{R}$

$$u = v^*\alpha(\partial_t) - T$$

In view of equation (C.70) u satisfies

$$\left|\int_{S^1_r} u\,dt\right| \leq cE_r(v)e^{-\delta r}.$$

Then $du = d(v^*\alpha(\partial_t))$ so by equation (C.69)

$$\int_{C_r} e^{\delta s}|du|^2 \leq E_r(v).$$

For $r \geq 0$, define the cylinder of unit volume

$$A_r = \{(s,t) \in C | r \leq s \leq r+1\}.$$

and set $\bar{u}_r = \int_{A_r} u\,\mathrm{dvol}$, so

$$|\bar{u}_r| \leq \int_r^{r+1}\left|\int_{S^1} u\,dt\right| ds \leq c\int_r^{r+1} e^{-\delta s}E_s(v)ds \leq c\,e^{-\delta r}E_r(v).$$

Since A_r has area 1,

$$\int_{A_r}(u - \bar{u}_r)\,\mathrm{dvol} = 0.$$

so by the Poincaré Inequality (Theorem 3.6.5 of [Mor66]) there exists a constant $c > 0$ such that

$$\int_{A_r}|u - \bar{u}_r|^2\,\mathrm{dvol} \leq c\int_{A_r}|d(u-\bar{u}_r)|^2\,\mathrm{dvol} = c\int_{A_r}|du|^2\,\mathrm{dvol}.$$

Therefore

$$\int_{A_r}|u|^2\,\mathrm{dvol} \leq \int_{A_r} 2\left(\bar{u}_r^2 + |u-\bar{u}_r|^2\right)\,\mathrm{dvol} \leq 2\bar{u}_r^2 + c\int_{A_r}|du|^2\,\mathrm{dvol} \tag{C.71}$$

Combining this with the pointwise bound

$$|dv - T(R\otimes dt)|^2 \leq |\pi_F dv|^2 + |v^*\alpha - T dt|^2 \leq |\pi_F dv|^2 + |u|^2 + |v^*\alpha(\partial_s)|^2,$$

we have

$$\int_{A_r} |dv - T(R \otimes dt)|^2 \,\mathrm{dvol} \leq 2\bar{u}_r^2 + \int_{A_r} \left(|\pi_F dv|^2 + c|du|^2 + |v^*\alpha(\partial_s)|^2\right) \mathrm{dvol}.$$

Summing this we obtain

$$\begin{aligned}
\int_{C_r} e^{\delta s}|dv - T(R \otimes dt)|^2 \,\mathrm{dvol} &= \sum_{n\in\mathbb{N}} \int_{A_r} e^{\delta s}|dv - T(R \otimes dt)|^2 \,\mathrm{dvol} \\
&\leq \sum_{n\in\mathbb{N}} e^{\delta(r+n+1)} \int_{A_{r+n}} |dv - T(R \otimes dt)|^2 \,\mathrm{dvol} \\
&\leq 2\sum_{n\in\mathbb{N}} e^{\delta(r+n+1)} \bar{u}_r^2 \\
&\quad + \sum_{n\in\mathbb{N}} e^{\delta(r+n+1)} \int_{A_{r+n}} \left(|\pi_F dv|^2 + c|du|^2 + |v^*\alpha(\partial_s)|^2\right) \mathrm{dvol} \\
&\leq ce^{\delta} \sum_{n\in\mathbb{N}} e^{\delta(r+n)} E_r^2(v) e^{-2\delta(r+n)} \\
&\quad + c\, e^{\delta} \int_{C_r} e^{\delta s} \left(|\pi_F dv|^2 + |d(v^*\alpha(\partial_t))|^2 + |v^*\alpha(\partial_s)|^2\right) \mathrm{dvol} \\
&\leq C\, E_r(v) \left(1 + E_r(v) e^{-\delta r} \sum_{n\in\mathbb{N}} e^{-\delta n}\right) \\
&\leq C\, E_r(v) \left(1 + \frac{E_r(v) e^{-\delta r}}{1 - e^{\delta}}\right).
\end{aligned}$$

□

We immediately obtain the following

Corollary C.3. *Let $v \in W_\delta^{k,p}(\dot{\Sigma}, Z)$ with $\delta > 0$, $kp > 2$. Then for each puncture p_l there exists a constant $T_l \in \mathbb{R}$ called the* asymptotic charge *of p_l and local conformal coordinates $\sigma_l : C \to \dot{\Sigma}$ at each puncture p_l such that v and a constant C, independent of r and δ, such that*

$$\int_{C_r} e^{\delta s}|dv \circ \sigma_l - T(R \otimes dt)|^2 \,\mathrm{dvol} \leq C\, E_r(v \circ \sigma) \left(1 + \frac{E_r(v \circ \sigma) e^{-\delta r}}{1 - e^{\delta}}\right). \quad \text{(C.72)}$$

In the following we will make repeated use of the properties of ∇ as stated in Lemma B.1. The proofs of the results below follow largely the same pattern as the proof of Theorem C.2.

Lemma C.4. *If $\zeta \in W^{k,p}_{\delta}(C, v^*TZ)$ then there exists a constant $C > 0$, independent of $\delta < M$ and r such that*

$$\int_{C_r} e^{\delta s} |\nabla \zeta|^2 \, d\mathrm{vol} \leq C E_r(v).$$

Proof. The proof follows similar lines as the previous one. First note that

$$\int_{S^1_r} \alpha(\nabla_t \zeta) dt = \int_{S^1_r} \nabla_t(\alpha(\zeta)) dt = \int_{S^1_r} d(\alpha(\zeta)) = 0.$$

Therefore

$$\int_{A_r} \alpha(\nabla_t \zeta) \, d\mathrm{vol} = 0$$

and we conclude by Poincaré Inequality that there exists a constant C independent of r and δ such that

$$\int_{A_r} |\alpha(\nabla_t \zeta)|^2 \, d\mathrm{vol} \leq C \int_{A_r} |d(\alpha(\nabla_t \zeta))|^2 \, d\mathrm{vol}.$$

With this and equation (4.16) we integrate

$$
\begin{aligned}
\int_{C_r} e^{\delta s}|\nabla\zeta|^2 \,\mathrm{dvol} &\leq E_r(\zeta) + \int_{C_r} e^{\delta s}|\alpha(\nabla_t\zeta)|^2 \,\mathrm{dvol} \\
&= E_r(\zeta) + \sum_{n\in\mathbb{N}} \int_{A_{r+n}} e^{\delta s}|\alpha(\nabla_t\zeta)|^2 \,\mathrm{dvol} \\
&\leq E_r(\zeta) + \sum_{n\in\mathbb{N}} e^{\delta(r+n+1)} \int_{A_{r+n}} |\alpha(\nabla_t\zeta)|^2 \,\mathrm{dvol} \\
&\leq E_r(\zeta) + \sum_{n\in\mathbb{N}} e^{\delta(r+n+1)} C \int_{A_{r+n}} |d\alpha(\nabla_t\zeta)|^2 \,\mathrm{dvol} \\
&\leq E_r(\zeta) + \sum_{n\in\mathbb{N}} e^{\delta} C \int_{A_{r+n}} e^{\delta s}|d\alpha(\nabla_t\zeta)|^2 \,\mathrm{dvol} \\
&\leq E_r(\zeta) + C \int_{C_r} e^{\delta s}|d\alpha(\nabla_t\zeta)|^2 \,\mathrm{dvol} \\
&\leq C\, E_r(\zeta)
\end{aligned}
$$

yielding equation (C.72). $\square$

Lemma C.5. *If $\zeta \in W^{k,p}_\delta(C, v^*TZ)$, then there exists a constant C independent of r and $\delta < M$ such that*

$$
\int_{A_r} |\zeta|^2 \,\mathrm{dvol} < C(1+\delta^{-1}) \tag{C.73}
$$

Proof. First note that

$$
\begin{aligned}
|\zeta(s,t)| &\leq \left|\zeta(0,t) + c\int_0^s \nabla_s\zeta ds\right| \\
&\leq |\zeta(0,t)| + c\int_0^\infty e^{-\delta s/2}\left(e^{\delta s/2}|\nabla\zeta|\right) ds \\
&\leq C + c\left(\int_0^\infty e^{-\delta s} ds\right)^{\frac{1}{2}} \left(\int_0^\infty e^{\delta s}|\nabla\zeta|^2 ds\right)^{\frac{1}{2}} \\
&\leq C + c\delta^{-\frac{1}{2}} \left(\int_0^\infty e^{\delta s}|\nabla\zeta|^2 ds\right)^{\frac{1}{2}}.
\end{aligned}
$$

where $c = \sup_{t \in S^1} |\zeta(0,t)|$. Therefore

$$\begin{aligned}
\int_{A_r} |\zeta|^2 \, d\mathrm{vol} &\leq \int_{A_r} \left(2C^2 + 2c^2\delta^{-1} \int_0^\infty e^{\delta s} |\nabla\zeta|^2 ds\right) d\mathrm{vol} \\
&\leq 2C^2 + 2c^2\delta^{-1} \int_r^{r+1} \left(\int_C e^{\delta s} |\nabla\zeta|^2 \, d\mathrm{vol}\right) ds \\
&\leq 2C^2 + cE(\zeta)\delta^{-1} \\
&\leq C(1+\delta^{-1}).
\end{aligned}$$

□

We can strengthen this result.

Lemma C.6. *There exists constants $S \in \mathbb{R}$ and $C > 0$ independent of δ and r such that for $\zeta \in W^{k,p}_\delta(C, v^*TZ)$*

$$\int_{C_r} |\alpha(\zeta) - S|^2 \, d\mathrm{vol} < C\left(1 + \frac{\delta^{-1}}{1-e^\delta}\right) E_r(\zeta) e^{-\delta r}. \tag{C.74}$$

Proof. First note that

$$\begin{aligned}
\left|\int_{S^1_r} \alpha(\zeta) dt - \int_{S^1_{r'}} \alpha(\zeta) dt\right| &\leq \int_{C_{r,r'}} |\nabla_s(\alpha(\zeta))| \, d\mathrm{vol} \\
&\leq \int_{C_r} |\alpha(\nabla\zeta)| \, d\mathrm{vol} \\
&\leq \delta^{-\frac{1}{2}} e^{-\delta r/2} \left(\int_{C_r} e^{\delta s} |\alpha(\nabla\zeta)|^2 \, d\mathrm{vol}\right)^{\frac{1}{2}} \\
&\leq C\,\delta^{-\frac{1}{2}} \sqrt{E_r(\zeta)} e^{-\delta r/2}
\end{aligned}$$

so $S = \lim_{r\to\infty} \int_{S^1_r} \alpha(\zeta) dt$ exists and $u = \alpha(\zeta) - S$ satisfies

$$\left|\int_{S^1_r} u \, dt\right| \leq C\,\delta^{-\frac{1}{2}} \sqrt{E_r(\zeta)} e^{-\delta r/2}.$$

Set $u_r = \int_{A_r} u \, d\mathrm{vol}$, so

$$|u_r| = \left|\int_{A_r} u \, d\mathrm{vol}\right| \leq C\,\delta^{-\frac{1}{2}} \sqrt{E_r(\zeta)} e^{-\delta r/2}.$$

Also $\int_{A_r}(u-u_r)\,d\mathrm{vol}=0$ so applying the Poincaré Inequality to $u-u_r$ yields

$$\int_{A_r}|u|^2\,d\mathrm{vol}\le 2\int_{A_r}|u-u_r|^2\,d\mathrm{vol}+2u_r^2\le c\int_{A_r}|du|^2\,d\mathrm{vol}+2u_r^2.$$

By summing we obtain

$$\begin{aligned}
\int_{C_r}|\alpha(\zeta)-S|^2\,d\mathrm{vol} &= \int_{C_r}|u|^2\,d\mathrm{vol}\\
&= \sum_{n\in\mathbb{N}}\int_{A_{r+n}}|u|^2\,d\mathrm{vol}\\
&\le \sum_{n\in\mathbb{N}}\left(c\int_{A_{r+n}}|du|^2\,d\mathrm{vol}+2u_r^2\right)\\
&\le \sum_{n\in\mathbb{N}}\left(c\int_{A_{r+n}}|\nabla(\alpha(\zeta))|^2\,d\mathrm{vol}+C\,\delta^{-1}E_{n+r}(\zeta)e^{-\delta(n+r)}\right)\\
&\le C\left(\int_{C_r}|\alpha(\nabla\zeta)|^2\,d\mathrm{vol}+\delta^{-1}E_r(\zeta)e^{-\delta r}\sum_{n\in\mathbb{N}}e^{-\delta n}\right)\\
&\le C\left(1+\frac{\delta^{-1}}{1-e^{\delta}}\right)E_r(\zeta)e^{-\delta r}
\end{aligned}$$

□

Now assume that we are in the S^1-invariant case. Then note that $(\varphi_t)_*:TZ\mapsto TZ$ is given by parallel transport along characteristics. This is clearly true vectors in L so we concentrate on showing this for $X_0\in T_pF$. Extend X_0 to a vector field X along the characteristic through p using the derivative of the characteristic flow. We need to show that $\nabla_R X=0$. Note that $\nabla_X R=0$ since R is parallel by Lemma B.4 and $[R,X]=\mathcal{L}_R(X)=0$. Then

$$\nabla_R X=\nabla_X R+[R,X]=0.$$

In the S^1-invariant case 1 is an eigenvalue of the Poincaré return map $(\varphi_T)_*|_F$ at a closed characteristic, so there exists a parallel non-vanishing section $\chi_\infty=\chi_\infty(t)$ of v_0^*F.

Now parallel transport $\chi_\infty(t)$ along the lines $s \mapsto (s,t)$ to obtain a section χ. Then

$$\begin{aligned}\nabla_s \chi &= 0 \\ \nabla_t \chi(r,t) &= \mathcal{O}\left(\int_r^\infty |v_s(s,t)|ds\right) = \mathcal{O}(\sqrt{E_r(v)}e^{-\delta r/2}).\end{aligned}$$

We write $v^*F = \mathbb{C} \times C$

$$(z = x+iy, (s,t)) \mapsto x\chi(s,t) + y\, J\chi(s,t).$$

Lemma C.7. *In the S^1-invariant case, there exists a constant $Q \in \mathbb{C}$ and a constant $C > 0$ independent of δ and r such that for $\zeta \in W_\delta^{k,p}(C, v^*TZ)$*

$$\int_{C_r} e^{\delta s}|\pi_F(\zeta) - Q\chi|\,\mathrm{dvol} < C\left(1 + \frac{e^{-\delta r}E_r(\zeta)}{1-e^{\delta}}\right)E_r(\zeta). \tag{C.75}$$

Proof. Write $\pi_F\zeta = g\chi$,

$$\left|\int_{S_r^1} g\,dt - \int_{S_{r'}^1} g\,dt\right| \le \int_{C_{r,r'}} |\partial_s g|\,\mathrm{dvol} = \int_{C_{r,r'}} |\nabla_s(g\chi)|\,\mathrm{dvol} \le C\,E_r(\zeta)e^{-\delta r}$$

so

$$Q = \lim_{r\to\infty}\int_{S_r^1} g\,dt$$

exists and $u = g - Q$ satisfies

$$\left|\int_{S_r^1} u\,dt\right| \le C\,E_r(\zeta)e^{-\delta r}.$$

Set $u_r = \int_{A_r} u\,\mathrm{dvol}$, so

$$|u_r| = \left|\int_{A_r} u\,\mathrm{dvol}\right| \le C\,E_r(\zeta)e^{-\delta r}.$$

Also $\int_{A_r}(u-u_r)\,d\mathrm{vol}=0$ so we may apply the Poincaré Inequality to $u-u_r$ and

$$\begin{aligned}
\int_{A_r}|u|^2\,d\mathrm{vol} &\leq 2\int_{A_r}|u-u_r|^2\,d\mathrm{vol}+2u_r^2\\
&\leq c\int_{A_r}|du|^2\,d\mathrm{vol}+2u_r^2\\
&\leq c\left(\int_{A_r}|dg\chi+g\nabla\chi|^2\,d\mathrm{vol}+\int_{A_r}|g\nabla\chi|^2\,d\mathrm{vol}+E_r^2(\zeta)e^{-2\delta r}\right)\\
&\leq c\left(\int_{A_r}|\nabla(\pi_F\zeta)|^2\,d\mathrm{vol}+\int_{A_r}|g\nabla\chi|^2\,d\mathrm{vol}+E_r^2(\zeta)e^{-2\delta r}\right).
\end{aligned}$$

By summing we obtain

$$\begin{aligned}
\int_{C_r}e^{\delta s}|\pi_F\zeta-Q\chi|^2\,d\mathrm{vol} &= \int_{C_r}e^{\delta s}|u|^2\,d\mathrm{vol}\\
&\leq \sum_{n\in\mathbb{N}}e^{\delta(r+n+1)}\int_{A_{r+n}}|u|^2\,d\mathrm{vol}\\
&\leq \sum_{n\in\mathbb{N}}e^{\delta(r+n+1)}c\Big(\int_{A_r}|\nabla(\pi_F\zeta)|^2\,d\mathrm{vol}\\
&\qquad+\int_{A_r}|g\nabla\chi|^2\,d\mathrm{vol}+E_r^2(\zeta)e^{-2\delta r}\Big)
\end{aligned}$$

□

Corollary C.8. *In the S^1-invariant case, for $\zeta\in W_\delta^{k,p}(C,v^*TZ)$ there exist constants $S\in\mathbb{R}$ and $Q\in\mathbb{C}$ and constant $C=C(\delta)$ such that with $\zeta_\infty=S\cdot R+Q\chi$*

$$\int_C e^{\delta s}\left(|\nabla(\zeta-\zeta_\infty)|^2+|\zeta-\zeta_\infty|^2\right)\,d\mathrm{vol}<C. \tag{C.76}$$

Observe that if $v\in W_\delta^{k,p}(C,Z)$ is $\mathcal{H}$-holomorphic, then the linearized operator D_v (for fixed complex structure j) is asymptotically translation invariant in the sense of [LMO85] by Theorem C.12. To see this note that the terms that are not translation invariant in equations (6.32) and (6.33) are linear in $\pi_F dv$.

Let $A:W_\delta^{k,p}(C,v^*TZ)\mapsto W^{k-\frac{1}{2},p}(S_0^1,H)$ be an elliptic boundary condition for D_v. Set $\tilde{W}_\delta^{k-1,p}(C,v^*TZ)$ to be the space of sections ζ of class $W_{loc}^{k-1,p}$ that additionally satisfy

$$\tilde{E}(\zeta) = \int_C e^{\delta s}|\zeta - \zeta_0|^2 \, d\mathrm{vol} < \infty$$

for some $\zeta_0 = S \cdot R + Q\chi$.

Then we use Theorem 6.3 of [LMO85] to conclude the following

Corollary C.9. *The operator*

$$D_A : W^{k,p}_\delta(C, v^*TZ) \to \tilde{W}^{k-1,p}_\delta(C, v^*TZ) \times W^{k-\frac{1}{2},p}(S^1_0, H)$$

is Fredholm for almost all $\delta \in \mathbb{R}$.

Similarly we obtain for arbitrary domains

Corollary C.10. *The operator*

$$D_A : W^{k,p}_\delta(\dot{S}, v^*TZ) \to \tilde{W}^{k-1,p}_\delta(\dot{S}, v^*TZ) \times W^{k-\frac{1}{2},p}(\partial S, H)$$

is Fredholm for almost all $\delta \in \mathbb{R}$.

C.1 Elliptic Estimates

In this chapter we will sharpen and expand on the estimates from Theorem C.2 for $W^{k,p}_\delta$ maps in the case that they are also $\mathcal{H}$-holomorphic. We show that such maps limit to close characteristics and investigate the asymptotics of solutions near the puncture. For these issues it suffices to consider tunneling maps with domain the half-infinite cylinder $C_r = [r, \infty) \times S^1$.

Note that the equations for $\mathcal{H}$-holomorphic maps define a system of PDE's, the first equation being of first order, the second one of second order. Therefore it is easier to study not the map itself, but its suspension as defined in Definition D.5. Note that $\mathcal{H}$-holomorphic $W^{k,p}_\delta(C, Z)$ maps v have the property that the periods of

$v^*\alpha \circ j$ vanish by Lemma 5.4. Moreover, generalized holomorphic maps with domain the punctured disk C reduce to ordinary holomorphic maps since $\mathcal{H}_s(D) = \{0\}$.

Our goal is to show that maps limit to a closed characteristic.

Definition C.11. *A* parametrization of a closed characteristic *is a smooth map*

$$v_0 : S^1 \to Z$$

with image tangent to L *and* $v_0^*\alpha = \mathrm{const} \cdot d\theta$.

Theorem C.12. *Let* $v \in W_\delta^{k,p}(C, Z)$ *be a* $\mathcal{H}$*-holomorphic map. Then either the puncture is removable or* v *is asymptotic to a closed characteristic. More specifically, either* v *extends to a map smooth across the puncture or there exists a closed characteristic parametrized by* v_0 *and constants* $c_n > 0$ *such that*

$$\lim_{s \to 0} v(s,t) = v_0(s,t) \tag{C.77}$$

$$|\nabla^n (dv - dv_0)(s,t)| \leq c_n \sqrt{E_s(v)} e^{-\delta s/2} \qquad \forall\, n \in \mathbb{N}. \tag{C.78}$$

Proof. For $\mathcal{H}$-holomorphic $v : C \to Z$ let $\tilde{v} = (a, v) : C \to \mathbb{R} \times Z$ be a suspension of v.

Define the $v^*T(\mathbb{R} \times Z)$-valued 1-form

$$\beta = dv - T(ds \otimes \partial_r + dt \otimes R)$$

Then $\beta \in \Omega^{1,0}(\tilde{v}^*T(\mathbb{R} \times Z))$ as

$$\beta \circ j = dv \circ j - (-dt \otimes \partial_r + ds \otimes R) = J dv - JT(ds \otimes \partial_r + dt \otimes R) = J \circ \beta.$$

Moreover note that,

$$\int_{C_r} e^{\delta s} |\beta|^2 \, \mathrm{dvol} \leq C\, E_r(v) \tag{C.79}$$

by Theorem C.2 and

$$da - Tds = v^*\alpha \circ j - Tds = (v^*\alpha - Tdt) \circ j.$$

We follow [PW93] to establish the regularity estimates we need.

We work with the almost complex metric connection

$$\tilde{\nabla} X = \nabla X - \frac{1}{2} J (\nabla J) X.$$

on $\mathbb{R} \times Z$. Recall that the complexified tangent spaces of C and $\mathbb{R} \times Z$ split as $T_{\mathbb{C}}C = T^{1,0}C \oplus T^{0,1}C$ and $T_{\mathbb{C}}(\mathbb{R} \times Z) = T^{1,0}(\mathbb{R} \times Z) \oplus T^{0,1}(\mathbb{R} \times Z)$ into the sum of the holomorphic and anti-holomorphic tangent bundles. For a map $v : C \to \mathbb{R} \times Z$ the connection induces an operator

$$d^{\tilde{\nabla}} : \Omega^k(v^*T(\mathbb{R} \times Z)) \longrightarrow \Omega^{k+1}(v^*T(\mathbb{R} \times Z)).$$

If v is J-holomorphic then the complexification of dv preserves the type (holomorphic or anti-holomorphic) of the tangent vectors. Therefore we may split the complexification of dv as $dv = (dv)^{1,0} \oplus (dv)^{0,1}$ and $(dv)^{0,1}(\bar{\eta}) = \overline{(dv)^{1,0}(\eta)}$, so $(dv)^{0,1}$ is determined by $(dv)^{1,0}$.

Set

$$H = v^*T^{1,0}(\mathbb{R} \times Z).$$

The operator $d^{\tilde{\nabla}}$ extends to the complexification $\Omega^k_{\mathbb{C}}(H)$ and denote the composition with the projection onto the holomorphic and anti-holomorphic tangent spaces by $\partial^{\tilde{\nabla}}$ and $\bar{\partial}^{\tilde{\nabla}}$, respectively. so $\partial^{\tilde{\nabla}} = \frac{1}{2}(1 - iJ)d^{\tilde{\nabla}}$ and $\bar{\partial}^{\tilde{\nabla}} = \frac{1}{2}(1 + iJ)d^{\tilde{\nabla}}$.

If v is J-holomorphic these operators fit in the Dolbeault complex

$$0 \xrightarrow{\partial^{\tilde{\nabla}}} \Omega^{0,0}(H) \xrightarrow{\partial^{\tilde{\nabla}}} \Omega^{1,0}(H) \xrightarrow{\partial^{\tilde{\nabla}}} \Omega^{2,0}(H) = 0$$

with Hermitian L^2 adjoint $(\partial^{\bar{\nabla}})^* = -*\bar{\partial}^{\bar{\nabla}}*$, where $*$ is the Hodge-star operator. This is well-defined since J-holomorphic maps preserve the Dolbeault splitting.

The operator

$$D = \partial^{\bar{\nabla}} \oplus (\partial^{\bar{\nabla}})^* \quad : \quad \Omega^{1,0}(H) \to \Omega^{2,0}(H) \oplus \Omega^0(H)$$

is elliptic. For $\zeta \in \Omega^{0,1}(H)$ we have

$$\begin{aligned}
\partial^{\bar{\nabla}}\zeta &= 0 \\
(\partial^{\bar{\nabla}})^*\zeta &= -*\bar{\partial}^{\bar{\nabla}}*\zeta = i*\bar{\partial}^{\bar{\nabla}}\zeta = i*d^{\bar{\nabla}}\zeta = i(d^{\bar{\nabla}}\zeta)(\partial_s,\partial_t).
\end{aligned}$$

Then

$$\begin{aligned}
(\partial^{\bar{\nabla}})^*(dv)^{1,0} &= i\left(d^{\bar{\nabla}}\frac{1}{2}(1-iJ)dv\right)(\partial_s,\partial_t) \\
&= i\left(\frac{1}{2}(1-iJ)d^{\bar{\nabla}}dv\right)(\partial_s,\partial_t) \\
&= i\frac{1}{2}(1-iJ)\left(\tilde{\nabla}_s dv(\partial_t) - \tilde{\nabla}_t dv(\partial_s) - dv([\partial_s,\partial_t])\right) \\
&= i\frac{1}{2}(1-iJ)v^*T(\partial_s,\partial_t) \\
&= iv^*T^{1,0}(\partial_s,\partial_t).
\end{aligned}$$

With

$$\nu \;=\; (ds\otimes\partial_r + dt\otimes R)$$

we have that $\tilde{\nabla}\nu = 0$ since R and ∂_r are parallel. Using that $T\circ\pi_E = T\circ\pi_F = 0$ we compute

$$
\begin{aligned}
\left|(\partial^{\tilde\nabla})^*(dv)^{1,0}\right| &= |T^{1,0}(dv(\partial_s), dv(\partial_t))| \\
&\leq \left(|T^{1,0}(\pi_F\, dv(\partial_s), dv(\partial_t))| + |T^{1,0}(\pi_E\, dv(\partial_s), dv(\partial_t))|\right) \\
&= \left(|T^{1,0}(\pi_F\, dv(\partial_s), dv(\partial_t))| + |T^{1,0}(\pi_E\, dv(\partial_s), \pi_F\, dv(\partial_t))|\right) \\
&\leq 2|T||dv|\,|\pi_F\, dv| \\
&\leq c|\pi_F\, dv| \\
\left|(\partial^{\tilde\nabla})^*(\nu)^{1,0}\right| &= 0
\end{aligned}
$$

where we used that $|T|$ and $|dv|$ are bounded on Z. Therefore

$$|D\beta| \leq c|\pi_F\, dv|. \tag{C.80}$$

Since D is elliptic we have the standard elliptic estimate

$$||\beta||_{1,2} \leq c(||D\beta||_{0,2} + ||\beta||_{0,2}),$$

so in particular

$$\int_{A_r} |\tilde\nabla\beta|^2\, \mathrm{dvol} \leq c\, E_r(v)e^{-\delta r}$$

by equations (C.79) and (C.80)

By elliptic bootstrapping arguments we conclude that there are constants c_n such that

$$\int_{A_r} |\tilde\nabla^n\beta|^2\, \mathrm{dvol} \leq c_n\, E_r(v)e^{-\delta r}.$$

By the Sobolev embedding theorems we conclude that β together with all its derivatives vanishes faster than $e^{-\delta s/2}$ at ∞, i.e. there are constants c_n such that

$$|\tilde\nabla^n\beta| \leq c_n\,\sqrt{E_r(v)}e^{-\delta s/2}.$$

pointwise, proving equation (C.78).

Therefore

$$v_0(t) = \lim_{s\to\infty} v(s,t)$$

exists and is smooth with $\frac{d}{dt}v_0(t) = T \cdot R$. If $T = 0$ then v_0 maps to some point $p \in Z$ and for $\delta > 0$

$$\int_{C_r} |dv|^2 \, d\mathrm{vol} = \int_{C_r} |\beta|^2 \, d\mathrm{vol} \leq cE_r(v)e^{-\delta r}$$

by Theorem (C.2). So the area of the image of C_r under $\tilde{v}$ goes to zero uniformly as $r \to \infty$. Using the biholomorphism of C_0 with the punctured unit disk in $\mathbb{C}$ we may employ the usual Removal of Singularity Theorem (cf. [PW93]) to conclude that $\tilde{v}$, and therefore also v, extend to a smooth map over the disk.

If $T \neq 0$, then $dv_0^*\alpha = Tdt \cdot R$, so v_0 is an immersion with image a closed characteristic, parametrized by a constant multiple of the characteristic vector field. Then equation (C.77) follows from this and equation (C.78). □

D $\mathcal{H}$-Holomorphic Maps vs. Generalized Holomorphic Maps

We explain the close relation between $\mathcal{H}$-holomorphic maps and generalized holomorphic maps.

In [ACH04], generalized holomorphic maps were defined for domains that are closed Riemann surfaces Σ with punctures. They are families of J-holomorphic maps into the symplectization with parameter space $H^1(\Sigma, \mathbb{R})$. This generalized readily to our setting when the domains in question have boundary. We will give a precise definition of these maps describe how they relate to our setting.

Recall from equation (2.9) that the tangent bundle of X over Z splits as

$$T_Z X = K \oplus L \oplus F$$

and $L \oplus F = TZ$. This suggest the following model for an infinitesimal neighborhood of the fold.

Definition D.1. *The* cylinder over Z *is the manifold* $\mathbb{R} \times Z$. *We extend the CR structure to the cylinder by translation. Moreover we identify* K *with* $T\mathbb{R} \subset T(\mathbb{R} \times Z)$.

The subbundle $E = K \oplus L \subset T(\mathbb{R} \times Z)$ *inherits two different complex structures,* J_+ *and* J_-*, depending whether we take the almost complex structure limiting from* X_+ *or* X_-. *Together with the almost complex structure* J *on* F *we obtain two different almost complex structures, also denoted by* J_+ *and* J_- *on the cylinder over* Z.

In the case that the 1-form α is a contact form, the cylinder over Z is sometimes also called the "symplectization" of Z. But this terminology is misleading in our case, since we make no use of the induced symplectic form on $\mathbb{R} \times Z$. There is a (J_+, J_-)-linear involution on $\mathbb{R} \times Z$, given by $(r, z) \mapsto (-r, z)$, so without loss of generality we may consider only $J = J_+$-holomorphic maps. The subbundle $E \subset T(\mathbb{R} \times Z)$ is canonically trivial and isomorphic to $\mathbb{C}$.

This allows us to talk about J-holomorphic maps into the cylinder over Z. As mentioned above, generalized holomorphic maps are parametrized by $H^1(\Sigma, \mathbb{R})$.

We need the following theorem from [DS52]:

Theorem D.2. *Let* (S, j) *be a Riemann surface with boundary. There exists a unique harmonic 1-field on* S *with a given admissible normal boundary value and given periods on* $b_1(S)$ *linearly independent absolute 1-cycles in* $H_1(S; \mathbb{R})$.

Recall that a harmonic 1-field on S is a 1-form $\lambda \in \Omega^1(S)$ satisfying

$$d\lambda = 0 \quad \text{and} \quad d^*\lambda = 0.$$

Note that if S has boundary this is not equivalent to λ being a harmonic 1-form, i.e. λ satisfying $\Delta\lambda = 0$. Also not that the boundary value that λ vanishes on vectors normal to the boundary is admissible in the sense of Theorem D.2. This motivates the definition of $\mathcal{H}_n(S,j)$ from Definition 5.2 and we conclude by Theorem D.2 that $\mathcal{H}_n(S,j) \approx (H_1(S,\mathbb{Z}))^*$, where the isomorphism is given by the absolute period integrals.

We need a complexified version of this space. Set

$$\mathcal{H}_n^{0,1}(S,j) = \{\frac{1}{2}(\delta + i\delta \circ j) | \delta \in \mathcal{H}_n(S,j)\} \subset \Omega^1(S,\mathbb{C}).$$

With the help of this definition we obtain the following characterization of generalized holomorphic maps in the sense of [ACH04], generalized to domains with boundary.

Definition D.3. *A map $\tilde{v} : \dot{S} \to \mathbb{R} \times \mathbb{Z}$ is called* generalized holomorphic *if*

$$\bar{\partial}_J \tilde{v} = \frac{1}{2}\{d\tilde{v} + J\, d\tilde{v}\, j\} \in \mathcal{H}_n^{0,1} \oplus 0 \in \tilde{v}^*(E \oplus F).$$

There is an obvious $\mathbb{R}$-action on the space of generalized holomorphic maps given by translation along the $\mathbb{R}$-factor on each connected component of the domain S.

We explain how to obtain a $\mathcal{H}$-holomorphic map from a generalized holomorphic map.

Lemma D.4. *Let $\tilde{v} : \dot{S} \to \mathbb{R} \times Z$ be a generalized holomorphic map. Then its projection $v = \pi_Z \tilde{v}$ into Z is $\mathcal{H}$-holomorphic and the periods of $v^*\alpha \circ j$ vanish on a tubular neighborhood U of the punctures, i.e.*

$$\int_{\partial U} v^*\alpha \circ j \;=\; 0. \tag{D.81}$$

Proof. Note that the map v defined above satisfies equation (5.21). Also

$$v^*\alpha \circ j = \tilde{v}^*\alpha \circ j = da + \delta$$

where $a = \pi_{\mathbb{R}}\tilde{v}$ is the projection onto the $\mathbb{R}$-factor and $\delta \in \mathcal{H}(S)_n$. Therefore v also satisfies equation (5.22) and is therefore $\mathcal{H}$-holomorphic. To see equation (D.81) note that

$$\int_{\partial U} v^*\alpha \circ j = \int_{\partial U} (da + \delta) = 0$$

U is contractible so $\delta|_U$ is exact. □

Now if v satisfies equations (5.22) and (D.81), we can suspend v to a map $\tilde{v} : \dot{S} \to \mathbb{R} \times Z$ in the following way.

Definition D.5. *Let $v : \dot{S} \to Z$ be a $\mathcal{H}$-holomorphic map with vanishing periods at the punctures. Let $\delta \in \mathcal{H}_n(S, j)$ be uniquely defined by having the same periods integrals as $v^*\alpha \circ j$. Then their difference is exact and we choose a function*

$$\begin{aligned} a &: \dot{S} \to \mathbb{R} \\ da &= v^*\alpha \circ j - \delta. \end{aligned}$$

Then the map

$$\tilde{v} : \dot{S} \to \mathbb{R} \times Z, \qquad \tilde{v}(z) = (a(z), v(z))$$

is called a suspension *of v in $\mathbb{R} \times Z$.*

Suspension are unique up to an overall constant on each connected component of S.

Lemma D.6. *Let $v : \dot{S} \to Z$ be $\mathcal{H}$-holomorphic and assume that the periods of $v^*\alpha \circ j$ vanishing at the punctures. Then every suspension $\tilde{v} : \dot{S} \to \mathbb{R} \times Z$ of v is a generalized holomorphic map.*

Proof. We compute

$$\begin{aligned}
\bar{\partial}_J \tilde{v} &= \frac{1}{2}\{(da, v^*\alpha) + J(da, v^*\alpha)\circ j\} \\
&= \frac{1}{2}\{(da, v^*\alpha) + (-v^*\alpha\circ j, da\circ j)\} \\
&= \frac{1}{2}(da - v^*\alpha\circ j, da\circ j + v^*\alpha) \\
&= \frac{1}{2}(-\delta, -\delta\circ j) \in \mathcal{H}.
\end{aligned}$$

□

Also note that the suspension composed with the projection into Z is the identity map.

To establish equivalence of $\mathcal{H}$-holomorphic maps and generalized holomorphic maps we need to take the respective function spaces into account. Tunneling maps are required to be of class $W^{k,p}_\delta(\dot{S}, Z)$, whereas generalized holomorphic maps are required to have finite energy in the following sense (see [HWZ96a], [BEH+03] and [ACH04]).

Definition D.7 (Energy). *The* energy *of a generalized holomorphic map* $v : \dot{S} \to Z$ *is the sum*

$$E(v) = E_\omega(v) + E_\alpha(v)$$

of the ω*-energy*

$$E_\omega(v) = \int_S v^*\omega,$$

and the α*-energy*

$$E_\alpha(v) = \sup_{\phi\in\mathcal{A}} \int_{\dot{S}} (\phi\circ a)\, da\wedge v^*\alpha$$

where a denotes the $\mathbb{R}$-component of v and the supremum is taken over all positive functions ϕ in the space

$$\mathcal{A} = \{\phi \in C^\infty(\mathbb{R}, [0,\infty)) | \int_{\mathbb{R}} \phi = 1\}.$$

Lemma D.8. *The suspension of an $\mathcal{H}$-holomorphic $W^{k,p}_\delta(\dot{S}, Z)$ map has finite energy in the sense of Definition D.7.*

Proof. It suffices to show this for $\dot{S} = C$.

Note that with $M = \sup_Z |\omega|$

$$E_\omega(v) = \int_C v^*\omega \leq M \int_C |\pi_F\, dv|^2 < \infty$$

by equation (4.16).

Let a be an integral of $v^*\alpha \circ j$ with $\min_C a = 0$. By Theorem C.12 we know that $|da| = |v^*\alpha|$ is bounded on C. Using that $\int_C \phi\, d\mathrm{vol} \leq 1$ for $\phi \in \mathcal{A}$ we obtain

$$\int_C \phi \circ a\, da \wedge v^*\alpha \leq \int_C \phi(s)|da|\,|da|\,|v^*\alpha|\, d\mathrm{vol} < \infty.$$

Thus

$$E_\alpha(v) = \sup_{\phi\in\mathcal{A}} \int_C \phi \circ a\, da \wedge v^*\alpha < \infty.$$

□

For the converse of this theorem we need a result by Hofer (see e.g. [BEH+03]). This applies to generalized holomorphic maps since the proof is done locally in a neighborhood of each puncture. So we may assume that the domain is the punctured disk (half-infinite cylinder), but there are no non-zero harmonic 1-fields on the disk that vanish on vectors normal to the boundary. So generalized holomorphic maps from the punctured disk are just J-holomorphic maps.

Theorem D.9. *Given a generalized holomorphic map v, puncture p and local conformal coordinates $z = e^{s+2\pi i\theta}$, at p. Then either v extends smoothly over the puncture p or v is asymptotic to a closed characteristic $Y \in \mathcal{R}$ in the sense that there exists $m \in \mathbb{Z}$ and $y \in Y$ such that*

$$\lim_{s \to -\infty} v(e^{s+2\pi i\theta}) = \varphi_{(mT_Y\theta)}(y) \qquad \text{in } C^\infty(S^1), \tag{D.82}$$

where φ_t is the time-t characteristic flow. The integer m is called the multiplicity *of the puncture. If the puncture is removable we say that the multiplicity is* 0.

This as the following immediate

Corollary D.10. *Let $\tilde{v} : \dot{S} \to \mathbb{R} \times Z$ be a finite energy generalized holomorphic map. Then its projection $v = \pi_Z \tilde{v}$ into Z is of class $W^{k,p}_\delta(\dot{S}, Z)$.*

In summary we have shown

Theorem D.11. *Finite energy generalized holomorphic maps $\tilde{v} : \dot{S} \to \mathbb{R} \times Z$ modulo $\mathbb{R}$ action on each connected component of S are in bijective correspondence to $\mathcal{H}$-holomorphic maps $v : \dot{S} \to Z$ of class $W^{k,p}_\delta(\dot{S}, Z)$ via $v = \pi_Z \tilde{v}$.*

E Tunneling Maps in Symplectic Manifolds

Here we want to briefly explain how tunneling maps come up in the usual symplectic setting. From this point of view, tunneling maps appear as tools for studying J-holomorphic curves relative to a codimension 1 hypersurface in a symplectic manifold. We transfer our definitions to this case:

Let (X, ω) be a symplectic manifold and $f : X \to \mathbb{R}$ a smooth function with transverse zeros. Then $Z = f^{-1}$ is a smooth hypersurface, separating X into two parts labeled X_+ and X_- by the sign of f on them. Assume Z is dynamically stable. Choose a stable 1-form α on Z and a compatible almost complex structure $J \in \mathcal{J}(X, \omega, \alpha)$.

Fix a folded domain as in Definition 4.1 and let $u_\pm : \Sigma_\pm \to X_\pm$ be J-holomorphic with $\tau = u^* f$. Assume τ vanishes transversely and $\sigma = \tau^{-1}(0) \neq \emptyset$, so σ is a smooth non-empty compact submanifold of Σ, separating Σ.

Now, just like in the folded symplectic case we may look for tunneling maps in Z that connect the image of $u|_\sigma$ to closed characteristics, i.e. an $\mathcal{H}$-holomorphic map $v : \dot{S} \to Z$ with $v|_\sigma = u|_\sigma$.

As opposed to the folded symplectic case, the complex structures $J_\pm$ induced on $T_Z X$ coming from X_+ and X_- agree in the symplectic setting. Thus the argument in the discussion of the sign of $u^*\alpha$ in Remark 4.13 has to be modified and the equivalent of equation (4.20) reads in this case

$$(c_1 u_+^* \alpha + c_2 u_-^* \alpha)|_{T\sigma} \neq 0.$$

Therefore the folded diagonal Δ^Z does not pose Fredholm boundary conditions in the symplectic case.

To understand this better we will take another look at tunneling maps in the folded symplectic setting. Let (v_+, v_-) be conjugate tunneling maps and consider the suspension $\tilde{v}_\pm$ of tunneling maps $v_\pm$ into $\mathbb{R} \times Z$, where $v_\pm$ is a generalized holomorphic map w.r.t. $J_\pm$. Also let $\tilde{v}_0^\pm$ be generalized $J_\pm$-holomorphic suspensions of the associated horizontal covering. Assume for simplicity that the resulting maps are actually $J_\pm$-holomorphic and that each connected component of S has only one boundary component. Then the equation (5.25) of the definition of conjugate tunneling maps can be rewritten in terms of the $\mathbb{R}$-components a_+, a_- and $a_0^\pm$ of the corresponding tunneling maps. Then

$$\begin{aligned} \tilde{v}_+^*\alpha\circ j + \tilde{v}_-^*\alpha\circ j - 2\tilde{v}_0^*\alpha\circ &= da_+ - da_- - da_0^+ + da_0^- \\ &= (da_+ - da_0^+) - (da_- - da_0^-) \end{aligned}$$

and using the fact that the $\mathbb{R}$-component is only determined up to a constant, equation (5.25) says

$$a_+ - a_0^+ = a_- - a_0^- \qquad \text{on } \sigma.$$

Written in this way the equation carries over verbatim to the (non-folded) symplectic setting.

If $J_+ = J_-$ we observe that $a_0^+ = a_0^-$, so the horizontal covering map drops out the equation simplifies to

$$a_+ = a_- \qquad \text{on } \sigma$$

We can follow the above transformations backward and obtain the replacement of equation (5.25) for the symplectic setting:

$$v_+^*\alpha\circ j = v_-^*\alpha\circ j \text{ on } T\sigma$$

for $\mathcal{H}$-holomorphic maps $v_\pm$.

But this, together with the remaining equations in Definition 5.8 implies that $v_+ = v_-$. Thus in the symplectic case, the analogue of the folded diagonal is the actual diagonal

$$\Delta = \{(\hat{v},\hat{v}) | \hat{v} : \sigma \to Z\} \subset \mathrm{Map}(\sigma, Z) \times \mathrm{Map}(\sigma, Z).$$

Viewing this the other way, the folded diagonal Δ^Z in the folded symplectic setting is analogue of the actual diagonal Δ in the symplectic setting.

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