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LI WANG

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SEMILINEAR STOCHASTIC DIFFERENTIAL EQUATIONS IN HILBERT SPACES DRIVEN BY NON-GAUSSIAN NOISE AND THEIR ASYMPTOTIC PROPERTIES

By

Li Wang

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ABSTRACT

SEMILINEAR STOCHASTIC DIFFERENTIAL EQUATIONS IN HILBERT SPACES DRIVEN BY NON-GAUSSIAN NOISE AND THEIR ASYMPTOTIC PROPERTIES

By

Li Wang

A class of stochastic evolution equations with additive noise (compensated Poisson random measures) in Hilbert spaces is considered. We first show existence and uniqueness of a mild solution to the stochastic equation with Lipschitz type coefficients. The properties (homogeneity, Markov, and Feller) of the solution are studied. We then study the stability and exponential ultimate boundedness properties of the solution by using Lyapunov function technique. We also study the conditions for the existence and uniqueness of an invariant measure associated to the solution. At last, an example is given to illustrate the theory.

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To my parents, my sisters, and Lan

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Contents

Chapter 1

Introduction

The area of stochastic differential equations (SDE's) in the infinite dimensional Hilbert space with Gaussian noise was motivated by the study of stochastic partial differential equations. Early contributions were made by Pardoux [35], Krylov and Rozovski [23], Metivier [29], Viot [42], and Kallianpur et al. [19] motivated by Zakai's [46] equation arising in filtering problems. These problems are described in Da Prato and Zabczyk [6], who take the approach through semigroup methods as in Ichikawa [17]. The first study of the non-Gaussian noise case was done by Kallianpur and Xiong [20]. Their approach was to study SDE's in duals of nuclear spaces where all bounded sets are compact, but solutions turn out to be "generalized functions". More general equations are studied in Gawareski, Mandrekar, and Richard [10] extending the work of Gikhman and Skorokhod [13].

In this thesis, we take the approach of Da Prato and Zabczyk and Ichikawa to study the non-Gaussian case. We first show that under the growth and Lipschitz conditions, the uniqueness of solution in $D([0,T],H)$. We show that the solution is homogeneous, Markovian and the transition semigroup is Feller. However, this is a mild solution. In order to apply the recent Itö's formula (Rüdiger and Ziglio [38]), we need to approximate this solution by strong solutions in $C(0,T;L_2^H(\Omega,\mathcal{F},P))$. Here we adapt Ichikawa's technique to generalize to our case Ichikawa's work.

In the second part of the thesis, we study the asymptotic properties of the mild solutions of these SPDE's. We then study the Lyapunov function method first introduced by Khasminski and Mandrekar [22] for the stability in the context of strong solutions as in the works of Pardoux, Krylov and Rozovski. For semilinear equations with Gaussian noise, this method was extended by Liu and Mandrekar [24]. They also studied exponential ultimate boundedness of the strong as well as mild solutions in Gaussian noise case. They showed that the ultimate boundedness can be used to study the existence of invariant measure for strong solutions. In the second part of the thesis, we study the extension of their work for non-Gaussian noise. We also study existence and uniqueness of invariant measures for the case of mild solutions in case the noise is non-Gaussian.

We begin in the next chapter by giving the definition of the Fréchet derivative and stating Taylor's theorem, followed by stochastic integral with respect to compensated Poisson noise as presented in Rüdiger [37]. Rüdiger defines these stochastic integrals for Banach space valued non-anticipative functions under some restriction. These restrictions are removed in Mandrekar and Rüdiger [28] where existence and uniqueness of solutions of Banach space valued SDE's with compensated Poisson noise are studied.

In Chapter 3, we study the existence and uniqueness of a mild solution to the semilinear SDE's under the growth and Lipschitz type conditions on the nonlinearities. We then study the homogeneity, Markov, and Feller properties of the mild solution. In Chapter 4, we study the approximating system of the semilinear SDE's. We prove that there exists a unique strong solution to the approximating system, and the strong solution converges to the mild solution in $C(0,T;L_2^H(\Omega,\mathcal{F},P))$. In Chapter 5, we study the stability and exponential ultimate boundedness in the m.s.s. of the mild solution by using the Lyapunov function technique. We use Ito formula and the approximating systems as tools. We first prove that the existence of the Lyapunov function is sufficient for the stability and exponential ultimate boundedness in the m.s.s. of the mild solution. Conversely, we construct the Lyapunov function for the linear case, and construct the Lyapunov function for the nonlinear case by using the first order approximation of the coefficients. In Chapter 6, we first give the conditions for the existence and uniqueness of an invariant measure associated to the solution. Finally, we give an example to illustrate our theory.

Chapter 2

Preliminaries

In this chapter we recall some definitions and theorems that we will use in our theory.

2.1 Fréchet derivative

The results below are adapted from Schwartz's work [39].

Let H be a real separable Hilbert space, we use $\langle \cdot, \cdot \rangle$ and $\|\cdot\|_H$ to represent the inner product and norm in H. $L(H)$ denotes the set of bounded linear operators from H to H. Assume $f\colon H \to \mathbb{R}$ is a map and $x, y \in H$.

Definition 2.1.1. We say that f is *first order Fréchet differentiable* at x, if there exists an $f'(x) \in H$, such that

$$f(x+y) = f(x) + \langle f'(x), y \rangle + o(\|y\|_H).$$

We say that f is *second order Fréchet differentiable* at x, if there exists an $f''(x) \in L(H)$, such that

$$f'(x+y) = f'(x) + f''(x)y + o(\|y\|_H).$$

Definition 2.1.2. A function $f : H \to \mathbb{R}$ is said to be in class C^2 on H, written $f \in C^2(H)$, iff $f'(x)$ and $f''(x)$ exists at every point of H and the maps $x \to f'(x)$ and $x \to f''(x)$ are continuous.

Theorem 2.1.1 (Schwartz [39]). *Suppose that $f \in C^2(H)$. Then*

$$f(x+y) = f(x) + \langle f'(x), y\rangle + \int_0^1 (1-t)\langle f''(x+ty)y, y\rangle\, dt.$$

Corollary 2.1.1 (Schwartz [39]). *Suppose that $f \in C^2(H)$. Then there exists a bounded bilinear function R_2 from H to $\mathbb{R}$ such that*

$$f(x+y) = f(x) + \langle f'(x), y\rangle + R_2(y).$$

2.2 Gronwall's inequality

Theorem 2.2.1 (Evans [9]). *If for $t_0 \leq t \leq t_1, \phi(t) \geq 0$ and $\psi(t) \geq 0$ are continuous functions such that the inequality*

$$\phi(t) \leq K + L\int_{t_0}^t \psi(s)\phi(s)\, ds$$

holds on $t_0 \leq t \leq t_1$, with K and L positive constants, then on $t_0 \leq t \leq t_1$,

$$\phi(t) \leq K \exp\Big(L\int_{t_0}^t \psi(s)\, ds\Big).$$

2.3 Poisson random measure

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a *filtered probability space* satisfying the "usual hypotheses":

1. $\mathcal{F}_t$ contains all null sets of $\mathcal{F}$, for all $t \in [0, \infty)$,

2. $\mathcal{F}_t = \mathcal{F}_t^+$, where $\mathcal{F}_t^+ = \bigcap\limits_{u > t} \mathcal{F}_u$, for all $t \in [0, \infty)$.

Definition 2.3.1. Let $(X, \mathcal{X})$ be a measurable space. A map: $N : \Omega \times \mathcal{X} \to \mathbb{R}$ is called a *random measure* if

1. $N(\omega, \cdot)$ is a measure on $(X, \mathcal{X})$ for each $\omega \in \Omega$,

2. $N(\cdot, B)$ is a random variable for each $B \in \mathcal{X}$.

Definition 2.3.2. A random measure N is called *independently scattered* if for any disjoint $B_1, \ldots, B_n \in \mathcal{X}$, the random variables $N(\cdot, B_1), \cdots, N(\cdot, B_n)$ are independent.

Let $(E, \mathcal{E})$ be a measurable space (E is a complete separable metric space), and let the map: $N : \Omega \times (\mathcal{E} \times \mathcal{B}(\mathbb{R}_+)) \to \mathbb{R}$ be a random measure, with $X = E \times \mathbb{R}_+$ and $\mathcal{X} = \mathcal{E} \bigotimes \mathcal{B}(\mathbb{R}_+)$.

Definition 2.3.3. The random measure N is *adapted* if $N(\cdot, B)$ is $\mathcal{F}_t$-measurable for $B \subset E \times [0, t]$. N is *σ-finite* if there exists a sequence E_n increasing to E such that $E|N(\cdot, E_n \times [0, t])| < \infty$ for each $n \in \mathbb{N}$ and $0 < t < \infty$.

Definition 2.3.4. The random measure N is called a *martingale random measure* if for fixed $A \in \Gamma_N := \{ A \in \mathcal{E} : E|N(A \times [0, t])| < \infty, \ \forall\, 0 < t < \infty \}$, the stochastic process $N(A \times [0, t])$ is martingale adapted to $\{ \mathcal{F}_t \}_{t \geq 0}$.

Let $\mathcal{A}$ denote the collection of all $\mathcal{F}_t$-adapted processes whose sample paths are of finite variations on any finite intervals.

Definition 2.3.5. A σ-finite adapted random measure N is said to be in the class *(QL)* if there exists a unique σ-finite predictable random measure $\widehat{N}$ such that $\widetilde{N} := N - \widehat{N}$ is a martingale random measure and for any $A \in \Gamma_N$, $\widehat{N}(A \times [0,t]) \in \mathcal{A}$ and is continuous in t. The random measure $\widehat{N}$ is called the *compensator* of N.

Definition 2.3.6. Let v be a σ-finite measure on $\mathcal{B}(E \times \mathbb{R}_+)$. The *Poisson random measure* is a random measure $N : \Omega \times \mathcal{B}(E \times \mathbb{R}_+) \to \mathbb{R}$, such that:

1. It is a independently scattered nonnegative integer-valued adapted random measure.

2. If for any $B \in \mathcal{B}(E \times \mathbb{R}_+)$ such that $v(B) < \infty$, $N(\cdot, B)$ is a Poisson random variable with mean $v(B)$.

Definition 2.3.7. Let β be a σ-finite measure on $(E, \mathcal{E})($ with $\beta(\{0\}) = 0$ and $\beta(B) < \infty$, if $B \in \mathcal{B}(E)$ and $0 \notin \overline{B}$, $\overline{B}$ represents the closure of B). If we suppose $v(A \times [0,t]) = \beta(A)t$ for any $A \in \mathcal{E}$, then β is called the *characteristic measure.*

It is clear that any Poisson random measure N is in class (QL) with the compensator $\widehat{N}(A \times [0,t]) = \beta(A)t$ for any $A \in \mathcal{E}$. And

$$\widetilde{N}(\omega, A \times [0,t]) := N(\omega, A \times [0,t]) - \widehat{N}(\omega, A \times [0,t])$$

is called *compensated Poisson random measure* (CPRM, for short).

2.4 Stochastic integral with respect to CPRM

In this chapter, we will introduce the stochastic integral with respect to CPRM following Rüdiger [37].

Let $F_t := \mathcal{B}((E\backslash\{0\}) \times \mathbb{R}_+) \otimes \mathcal{F}_t$ be the product σ-algebra generated by the semi-ring $\mathcal{B}((E\backslash\{0\}) \times \mathbb{R}_+) \times \mathcal{F}_t$ of the product sets $B \times F$, $B \in \mathcal{B}((E\backslash\{0\}) \times \mathbb{R}_+)$, $F \in \mathcal{F}_t$. (A ring is a non-empty class of sets which is closed under the formation of unions and differences)

As before let H be a real separable Hilbert space, and let $\langle \cdot, \cdot \rangle$ and $\| \cdot \|_H$ represent the inner product and norm on H separately.

Let $T > 0$ and let

$M^T(E/H) :=$

$$\{f\colon (E\backslash\{0\}) \times \mathbb{R}_+ \times \Omega \to H, \text{ such that } f \text{ is } F_T/\mathcal{B}(H)\text{-measurable}$$
$$\text{and } f(x,t,\omega) \text{ is } \mathcal{F}_t\text{-adapted } \forall x \in E\backslash\{0\},\ \forall t \in (0,T]\}$$

2.4.1 Stochastic integral for simple functions

Definition 2.4.1. A function f belongs to the set $\sum(E/H)$ of *simple functions*, if $f \in M^T(E/H)$, $T > 0$ and there exist $n \in \mathbb{N}$, $m \in \mathbb{N}$, such that

$$f(x,t,\omega) = \sum_{k=1}^{n-1} \sum_{l=1}^{m} 1_{A_{k,l}}(x) 1_{F_{k,l}}(\omega) 1_{(t_k,\, t_{k+1}]}(t) a_{k,l}$$

where $A_{k,l} \in \mathcal{B}(E)$ $(0 \notin \overline{A_{k,l}})$, $t_k \in (0,T]$, $t_k < t_{k+1}$, $F_{k,l} \in \mathcal{F}_{t_k}$, $a_{k,l} \in H$. For all $k = 1, \ldots, n-1$ fixed, $A_{k,\, l_1} \times F_{k,\, l_1} \cap A_{k,\, l_2} \times F_{k,\, l_2} = \phi$, if $l_1 \neq l_2$.

Definition 2.4.2. For the simple function $f \in \sum(E/H)$, we define the *stochastic integral with respect to CPRM* by

$$\int_0^T \int_A f(x,t,\omega) \widetilde{N}(dx\, dt)(\omega) = \sum_{k=1}^{n-1} \sum_{l=1}^{m} a_{k,l} 1_{F_{k,l}}(\omega) \widetilde{N}(A_{k,l} \cap A \times (t_k, t_{k+1}] \cap (0,T])(\omega)$$

for all $A \in \mathcal{B}(E\backslash\{0\})$, $T > 0$.

2.4.2 Stochastic integral for functions in H_2

To define the stochastic integral for more general functions than simple functions, we give some definitions first.

Definition 2.4.3. Let $L_2^H(\Omega, \mathcal{F}, P)$ be the space of H-valued random variables, such that $E\|Y\|_H^2 = \int \|Y\|_H^2 \, dP < \infty$. We denote by $\|\cdot\|_2$ the norm given by $\|Y\|_2 = (E\|Y\|_H^2)^{1/2}$. Given $(Y_n)_{n\in\mathbb{N}}, Y \in L_2^H(\Omega, \mathcal{F}, P)$, we write $\lim^2_{n\to\infty} Y_n = Y$ if $\lim\limits_{n\to\infty} \|Y_n - Y\|_2 = 0$.

Definition 2.4.4. Let $f : (E\backslash\{0\}) \times \mathbb{R}_+ \times \Omega \to H$ be given. A sequence $\{f_n\}_{n\in\mathbb{N}}$ of $F_T/\mathcal{B}(H)$-measurable functions is L^2-*approximating* f on $A \times (0,T] \times \Omega$ w.r.t. $\beta \otimes Leb \otimes P$, if f_n is $\beta \otimes Leb \otimes P$-a.s. converging to f, when $n \to \infty$, and

$$\lim_{n\to\infty} \int_0^T \int_A E\|f_n(x,t,\omega) - f(x,t,\omega)\|_H^2 \, \beta(dx)\, dt = 0;$$

i.e., $\|f_n - f\|$ converges to zero in $L^2(A \times (0,T] \times \Omega, \beta \otimes Leb \otimes P)$, when $n \to \infty$.

Next, we define the class of functions on which we will define the stochastic integral.

$H_2 :=$

$\{ f(x,t,\omega) : (E\backslash\{0\}) \times \mathbb{R}_+ \times \Omega \to H,$ such that f is $F_T/\mathcal{B}(H)$-measurable and $f(x,t,\omega)$ is $\mathcal{F}_t$-measurable for $\forall x \in E\backslash\{0\}$ and $\forall t \in (0,T]$, also $E\int_0^T \int_E \|f(x,t,\omega)\|_H^2 \, \beta(dx)\, dt < \infty \}$.

Remark. Observe that if $f \in H_2$, then for each $t \in [0,T]$, $1_{[0,t]} f \in H_2$. When it's clear from the context, we write elements of H_2 as $f(x,t)$.

Let $f \in H_2$. Then there exists simple functions $\{ f_k \} \in H_2$, such that

$$E \int_0^t \int_E \| f_k(x, s, \omega) - f(x, s, \omega) \|_H^2 \, \beta(dx) \, ds \to 0 \quad \text{as} \quad k \to \infty,$$

And the simple functions $\{ f_k \}$ satisfy, for each $t \in [0, T]$

$$E \| I_t(\varphi_k) - I_t(\varphi_j) \|_H^2 = E \int_0^t \int_E \| f_k(x, s) - f_j(x, s) \|_H^2 \, \beta(dx) \, ds$$

$$\to 0, \text{ as } k, j \to \infty.$$

(See Rüdiger [37])

Now for $f \in H_2$, we define the *stochastic integral with respect to CPRM* as,

$$I_t(f) = \int_0^t \int_E f(x, s) \, \widetilde{N}(dx \, ds) = \lim^2_{k \to \infty} I_t(f_k).$$

The following results are known about the stochastic integral with respect to CPRM.

Theorem 2.4.1 (Rüdiger [37]). *Let $I_t(f)$ be defined as above. Then we have*

1. *The sample paths of $I_t(f) = \int_0^t \int_E f(x, s) \, \widetilde{N}(dx \, ds)$ are cádlàg,*

2. *$I_t(f)$ is a mean 0 martingale with respect to $\mathcal{F}_t$.*

Chapter 3

Existence and uniqueness of mild solutions to Semilinear SDE's and their properties

In this chapter, we will consider the following semilinear stochastic differential equation (SDE, for short) on $[0, T]$.

$$d\,Z(t) = (AZ(t) + F(Z(t)))\,dt + \int_E B(v, Z(t))\,\widetilde{N}(dv\,dt),\ Z(0) = \varphi \in H. \tag{3.1}$$

Here A is the generator of a C_0-semigroup $S(t)$ on H, satisfying $\|S(t)\| \le e^{\alpha t}, \alpha \in \mathbb{R}$. $\widetilde{N}$ is a CPRM on $E \times \mathbb{R}_+$. F and B are, in general, nonlinear mappings, $F : H \to H$, $B : E \times H \to H$.

Remark. Partial differential equation can be written as Hilbert space valued linear equation.

In this chapter, we will prove the existence and uniqueness of the mild solution to the system under some conditions on the coefficients, and we will also prove that the solution has homogeneity, Markov and Feller properties.

3.1 Mild and strong solutions of equation (3.1)

Let $\mathbb{T} = [0, T]$, we first introduce two types of solutions.

Definition 3.1.1. A stochastic process $Z(t)$, $t \in \mathbb{T}$, is a *mild solution* of (3.1) if

(i) $Z(t)$ is adapted to $\mathcal{F}_t$,

(ii) $Z(t)$ is measurable and $\int_0^T E\|Z(t)\|_H^2\, dt < \infty$,

(iii) $Z(t) = S(t)\varphi + \int_0^t S(t-s)F(Z(s))\, ds + \int_0^t \int_E S(t-s)B(v, Z(s))\, \widetilde{N}(dv\, ds)$ for all $t \in \mathbb{T}$, *w.p.* 1.

Definition 3.1.2. A stochastic process $Z(t)$, $t \in \mathbb{T}$, is a *strong solution* of (3.1) if

(i) $Z(t)$ is adapted to $\mathcal{F}_t$,

(ii) $Z(t)$ is cádlàg in t, *w.p.* 1,

(iii) $Z(t) \in \mathcal{D}(A)$ almost everywhere on $\mathbb{T} \times \Omega$, and $\int_0^T \|AZ(t)\|_H\, dt < \infty$, *w.p.* 1,

(iv) $Z(t) = \varphi + \int_0^t AZ(s)\, ds + \int_0^t F(Z(s))\, ds + \int_0^t \int_E B(v, Z(s))\, \widetilde{N}(dv\, ds)$ for all $t \in \mathbb{T}$, *w.p.* 1.

3.2 Mild solutions of semilinear SDE's with Lipschitz non-linearities

We impose the following assumptions on the coefficients of equation (3.1)

$(A1)$ F and B are jointly measurable,

$(A2)$ There exists a constant l, such that $\forall x \in H$,

$$\|F(x)\|_H^2 + \int_E \|B(v,x)\|_H^2 \,\beta(dv) \le l(1 + \|x\|_H^2),$$

$(A3)$ For all $x, y \in H$, there exists a constant k, such that

$$\|F(x) - F(y)\|_H^2 + \int_E \|B(v,x) - B(v,y)\|_H^2 \,\beta(dv) \le k\|x - y\|_H^2.$$

We prove the existence and uniqueness of a mild solution to stochastic equation (3.1) under the above conditions. We follow the ideas from the work of Gikhman and Skorokhod [13], and adapt them to our case.

Observe that for $\phi \in H_2$, $\int_0^t \int_E S(t-s)\phi(v,s)\,\widetilde{N}(dv\,ds)$ exists because

$$\int_0^t \int_E \|S(t-s)\phi(v,s)\|_H^2 \,\beta(dv)\,ds < \infty.$$

The following lemma comes from Ichikawa [16].

Lemma 3.2.1. *Let N be Poisson random measure and $S(t)$ be a pseudo-contraction semigroup. Assume $\phi \in H_2$. If τ is a stopping time, then*

$$E \sup_{0 \le t \le T\wedge\tau} \left\| \int_0^t \int_E S(t-s)\phi(v,s)\,\widetilde{N}(dv\,ds) \right\|_H^2 \le b_1 E \int_0^{T\wedge\tau} \int_E \|\phi(v,t)\|_H^2 \,\beta(dv)\,dt.$$

The constant b_1 depends only on T and α as in the bound $\|S(t)\| \le e^{\alpha t}$.

Proof. Let $M_s = \int_0^s \int_E \phi(v,r)\, \widetilde{N}(dv\, dr)$ and $y_t = \int_0^t S(t-s)\, dM_s$. From Ichikawa [16], we know if $\alpha \le 0$, then

$$E(\sup_{t\le\tau} \|y_t\|_H^2) \le (3+\sqrt{10})^2 E\langle M\rangle_\tau.$$

Hence

$$\begin{aligned} &E\left(\sup_{t\le\tau}\left\|\int_0^t\int_E S(t-s)\phi(v,s)\,\widetilde{N}(dv\,ds)\right\|_H^2\right)\\ \le&(3+\sqrt{10})^2 E\int_0^\tau\int_E \|\phi(v,s)\|_H^2\,\beta(dv)\,ds. \end{aligned}$$

If $\alpha > 0$, we use the results in Ichikawa [16] and get

$$E\left(\sup_{t\le\tau}\|y_t\|_H^2\right) \le e^{2\alpha\tau_\infty}\left(3+\sqrt{10}\right)^2 E\langle M\rangle_\tau.$$

Here

$$\tau_\infty = \begin{cases} \text{ess sup}\,\tau & \text{if it exists} \\ \infty & \text{otherwise.} \end{cases}$$

Now if we use $T\wedge\tau$ in place of τ and notice that $(T\wedge\tau)_\infty \le T$ and let $b_1 = e^{2|\alpha|T}(3+\sqrt{10})^2$, we get

$$E\left(\sup_{0\le t\le T\wedge\tau}\left\|\int_0^t\int_E S(t-s)\phi(v,s)\,\widetilde{N}(dv\,ds)\right\|_H^2\right) \le b_1 E\int_0^{T\wedge\tau}\int_E \|\phi(s,v)\|_H^2\,\beta(dv)\,ds.$$

□

Now for an adapted process $\xi(\cdot) \in D([0,T],H)$, satisfying $E\sup_{0\le s\le T}\|\xi(s)\|_H^2 < \infty$, we let

$$I(t,\xi(t)) = \int_0^t S(t-s)F(\xi(s))\,ds + \int_0^t\int_E S(t-s)B(v,\xi(s))\,\widetilde{N}(dv\,ds).$$

Remark. The above integral exists under $(A1)$ and $(A2)$.

Lemma 3.2.2. *If $F(x)$ and $B(v,x)$ satisfy (A1) and (A2), $S(t)$ is a pseudo-contraction semigroup, then for a stopping time τ,*

$$E\left(\sup_{0\le s\le t\wedge\tau}\|I(s,\xi(s))\|_H^2\right)\le b_2\left(t+\int_0^t E\sup_{0\le u\le s\wedge\tau}\|\xi(u)\|_H^2\,ds\right)$$

with the constant b_2 depending on α, T and l.

Proof. We note that

$$\begin{aligned}\sup_{0\le s\le t\wedge\tau}\|I(s,\xi(s))\|_H^2\le&2\sup_{0\le s\le t\wedge\tau}\left(\left\|\int_0^s S(s-u)F(\xi(u))\,du\right\|_H^2\right.\\&\left.+\left\|\int_0^s\int_E S(s-u)B(v,\xi(u))\,\widetilde{N}(dv\,du)\right\|_H^2\right).\end{aligned}$$

There exists a bound for the expectation of the first term, using $\|S(t)\|\le e^{\alpha t}$, the Schwartz inequality and $(A2)$,

$$\begin{aligned}&E\sup_{0\le s\le t\wedge\tau}\left\|\int_0^s S(s-u)F(\xi(u))\,du\right\|_H^2\\\le&E\sup_{0\le s\le t\wedge\tau}\left\{\int_0^s\|S(s-u)F(\xi(u))\|_H\,du\right\}^2\\\le&E\Big(\int_0^{t\wedge\tau}e^{\alpha t}\|F(\xi(u))\|_H\,du\Big)^2\\\le&e^{2\alpha t}E(t\wedge\tau)\int_0^{t\wedge\tau}\|F(\xi(u))\|_H^2\,du\\\le&e^{2\alpha t}tlE\int_0^{t\wedge\tau}(1+\|\xi(u)\|_H^2)\,du\\\le&e^{2\alpha t}tl\Big(t+\int_0^t E\sup_{0\le u\le s\wedge\tau}\|\xi(u)\|_H^2\,ds\Big).\end{aligned}$$

From Lemma 3.2.1 and $(A2)$, for the second term,

$$\begin{aligned}&E\sup_{0\le s\le t\wedge\tau}\left\|\int_0^s\int_E S(s-u)B(v,\xi(u))\,\widetilde{N}(dv\,du)\right\|_H^2\\\le&b_1E\int_0^{t\wedge\tau}\int_E\|B(v,\xi(s))\|_H^2\,\beta(dv)\,ds\end{aligned}$$

$$
\begin{aligned}
&\leq b_1 l E \int_0^{t\wedge\tau} (1 + \|\xi(s)\|_H^2)\, ds \\
&\leq b_1 l \Big(t + \int_0^t E \sup_{0\leq u\leq s\wedge\tau} \|\xi(u)\|_H^2\, ds \Big).
\end{aligned}
$$

Let $b_2 = 2e^{2|\alpha|T}Tl + 2e^{2|\alpha|T}(3+\sqrt{10})^2$. We complete the proof by combining the inequalities for both terms. □

Lemma 3.2.3. *Let conditions (A1) and (A3) be satisfied and $S(t)$ be a pseudo-contraction semigroup. Then*

$$
E \sup_{0\leq s\leq t} \|I(s,\xi_1(s)) - I(s,\xi_2(s))\|_H^2 \leq b_3 \int_0^t E \sup_{0\leq u\leq s} \|\xi_1(u) - \xi_2(u)\|_H^2\, ds
$$

with the constant b_3 depending on α, T and k.

Proof. We begin with the following estimate.

$$
\begin{aligned}
&E \sup_{0\leq s\leq t} \|I(s,\xi_1(s)) - I(s,\xi_2(s))\|_H^2 \\
\leq &2E \sup_{0\leq s\leq t} \Bigg(\Big\| \int_0^s S(s-u)(F(\xi_1(u)) - F(\xi_2(u)))\, du \Big\|_H^2 \\
&+ \Big\| \int_0^s \int_E S(s-u)[B(v,\xi_1(u)) - B(v,\xi_2(u))]\, \widetilde{N}(dv\, du) \Big\|_H^2 \Bigg).
\end{aligned}
$$

As before,

$$
\begin{aligned}
&E \sup_{0\leq s\leq t} \Big\| \int_0^s S(s-u)(F(\xi_1(u)) - F(\xi_2(u)))\, du \Big\|_H^2 \\
\leq &E \sup_{0\leq s\leq t} \Big(\int_0^s \|S(s-u)\|\, \|F(\xi_1(u)) - F(\xi_2(u))\|_H\, du \Big)^2 \\
\leq &e^{2\alpha t} t E \sup_{0\leq s\leq t} \int_0^s \|F(\xi_1(u)) - F(\xi_2(u))\|_H^2\, du \\
\leq &k e^{2\alpha t} t E \sup_{0\leq s\leq t} \int_0^s \|\xi_1(u) - \xi_2(u)\|_H^2\, du \\
\leq &k e^{2\alpha t} t \int_0^t E \sup_{0\leq u\leq s} \|\xi_1(u) - \xi_2(u)\|_H^2\, ds.
\end{aligned}
$$

And,

$$
\begin{aligned}
&E \sup_{0\le s\le t} \left\| \int_0^s \int_E S(s-u)[B(v,\xi_1(u)) - B(v,\xi_2(u))]\, \widetilde{N}(dv\, du) \right\|_H^2 \\
\le & b_1 E \int_0^t \int_E \|B(v,\xi_1(s)) - B(v,\xi_2(s))\|_H^2\, \beta(dv)\, ds \\
\le & k b_1 \int_0^t E \sup_{0\le u\le s} \|\xi_1(u) - \xi_2(u)\|_H^2\, ds.
\end{aligned}
$$

Let $b_3 = 2e^{2|\alpha|T}Tk + 2ke^{2|\alpha|T}(3+\sqrt{10})^2$. We complete the proof by combining the inequalities for both terms. $\square$

Remark. Lemma 3.2.2 and Lemma 3.2.3 follow from the arguments similar to the arguments for Brownian motion case by Gawarecki et al. [10].

Now we prove the existence and uniqueness of the mild solution.

Theorem 3.2.1. *Let the coefficients F and B satisfy conditions $(A1), (A2)$ and $(A3)$, assume that $S(t)$ is a pseudo-contraction semigroup. Then the stochastic equation (3.1) has a unique mild solution $Z(t)$ satisfying*

$$
Z(t) = S(t)\varphi + \int_0^t S(t-s)F(Z(s))\, ds + \int_0^t \int_E S(t-s)B(v, Z(s))\, \widetilde{N}(dv\, ds)
$$

in the space

$$
\mathcal{H}_2 := \{\xi(\cdot) \in D([0,T], H),\ \textit{such that } E \sup_{0\le s\le T} \|\xi(s)\|_H^2 < \infty\}.
$$

Proof. We follow Picard's method. Let I be defined as before. By Lemma 3.2.2, $I\colon \mathcal{H}_2 \to \mathcal{H}_2$. The solution can be approximated by the sequence $Z_0(t) = \varphi, \cdots,$ $Z_{n+1}(t) = I(t, Z_n(t))$, $n = 0, 1, \cdots$. Let $\mathcal{V}_n(t) = E\sup_{0\le s\le t} \|Z_{n+1}(s) - Z_n(s)\|_H^2$. Then $\mathcal{V}_0(t) = E\sup_{0\le s\le t} \|Z_1(s) - Z_0(s)\|_H^2 \le \mathcal{V}_0(T) \equiv V_0$, and using Lemma 3.2.3, we obtain

$$\begin{aligned}\mathcal{V}_1(t) &= E \sup_{0\le s\le t} \|Z_2(s) - Z_1(s)\|_H^2 \\ &= E \sup_{0\le s\le t} \|I(s, Z_1(s)) - I(s, Z_0(s))\|_H^2 \\ &\le b_3 \int_0^t E \sup_{0\le u\le s} \|Z_1(u) - Z_0(u)\|_H^2 \, ds \le b_3 V_0 t.\end{aligned}$$

By induction, $\mathcal{V}_n(t) \le b_3 \int_0^t \mathcal{V}_{n-1}(s)\, ds \le \dfrac{V_0(b_3t)^n}{n!}$. Next, similar to the proof of Gikhman and Skorokhod [11], we show that $\sup_{0\le t\le T} \|Z_n(t) - Z(t)\|_H \to 0$, *a.s.* for some $Z \in \mathcal{H}_2$. If we let $\varepsilon_n = (V_0(b_3T)^n/n!)^{1/3}$, then, using Chebyshev's inequality, we arrive at

$$P(\sup_{0\le t\le T} \|Z_{n+1}(t) - Z_n(t)\|_H > \varepsilon_n) \le \Big(\frac{V_0(b_3T)^n}{n!}\Big)\Big/\Big(\frac{V_0(b_3T)^n}{n!}\Big)^{2/3} = \varepsilon_n. \qquad (3.2)$$

Because $\sum_{n=1}^\infty \varepsilon_n < \infty$, the series $\sum_{n=1}^\infty \sup_{0\le t\le T} \|Z_{n+1}(t) - Z_n(t)\|_H$ converges a.s., showing that $Z_n(\cdot)$ converges to some $Z(\cdot)$ a.s.. By (3.2)

$$\sum_n P(\sup_{0\le t\le T} \|Z_{n+1}(t) - Z_n(t)\|_H > \varepsilon_n) < \infty.$$

By Borel-Cantelli lemma, we have $\sup_{0\le t\le T} \|Z_{n+1}(t) - Z_n(t)\|_H < \varepsilon_n$ a.s.. So we have $\|Z(t) - Z_n(t)\|_H \to 0$, a.s.. Because $Z_n(t)$ has the cádlàg property, the sample paths of $Z(t)$ are cádlàg.

Moreover,

$$\begin{aligned}& E \sup_{0\le t\le T} \|Z(t) - Z_n(t)\|_H^2 \\ =& E \lim_{m\to\infty} \sup_{0\le t\le T} \|Z_{n+m}(t) - Z_n(t)\|_H^2 \\ =& E \lim_{m\to\infty} \sup_{0\le t\le T} \left\| \sum_{k=n}^{n+m-1} (Z_{k+1}(t) - Z_k(t)) \right\|_H^2\end{aligned}$$

$$\leq E \lim_{m\to\infty} \Big(\sum_{k=n}^{n+m-1} \sup_{0\leq t\leq T} \|Z_{k+1}(t) - Z_k(t)\|_H \Big)^2$$
$$= \lim_{m\to\infty} E\Big(\sum_{k=n}^{n+m-1} \sup_{0\leq t\leq T} \|Z_{k+1}(t) - Z_k(t)\|_H \, k \cdot \frac{1}{k} \Big)^2$$
$$\leq \sum_{k=n}^{\infty} E \sup_{0\leq t\leq T} \|Z_{k+1}(t) - Z_k(t)\|_H^2 k^2 \Big(\sum_{k=n}^{\infty} k^{-2} \Big),$$

and hence the second series converges. The first series is bounded by observing

$$\sum_{k=n}^{\infty} \mathcal{V}_k(T) k^2 \leq \sum_{k=n}^{\infty} \frac{V_0 (b_3 T)^k k^2}{k!} \to 0, \text{as } n \to \infty.$$

To justify that $Z(t)$ is a mild solution to equation (3.1), we note that a.s. $F(Z_n(s)) \to F(Z(s))$ uniformly in s. Therefore

$$\int_0^t S(t-s) F(Z_n(s))\, ds \to \int_0^t S(t-s) F(Z(s))\, ds \text{ a.s..}$$

Using the fact proved above, then

$$E \sup_{0\leq t\leq T} \|Z(t) - Z_n(t)\|_H^2 \to 0.$$

Thus we obtain

$$E \Big\| \int_0^t \int_E S(t-s)[B(v, Z(s)) - B(v, Z_n(s))]\, \widetilde{N}(dv\, ds) \Big\|_H^2$$
$$\leq b_1 E \int_0^t \int_E \|B(v, Z(s)) - B(v, Z_n(s))\|_H^2 \, \beta(dv)\, ds$$
$$\leq b_1 k T E \sup_{0\leq t\leq T} \|Z(t) - Z_n(t)\|_H^2 \to 0, \text{ as } n \to \infty.$$

This solution is unique. If $Z(t), Z'(t)$ are two solutions to equation (3.1), then we define $V(t) = E \sup_{0\leq s\leq t} \|Z(s) - Z'(s)\|_H^2$. By Lemma 3.2.3,

$$V(t) \leq b_3 \int_0^t V(s)\, ds \leq \cdots \leq E \sup_{0\leq s\leq T} \|Z(s) - Z'(s)\|_H^2 \frac{(b_3 t)^n}{n!} \to 0,$$

as $n \to \infty$, giving $V(t) = 0$. $\square$

Remark. We know that there exists for each $T < \infty$ an unique mild solution of (3.1). We can define $Z(t)$ on $[0, +\infty)$, such that for any T, $E \sup_{0 \le s \le T} \|Z(s)\|_H^2 < \infty$.

3.3 Homogeneity, Markov, and Feller properties of the mild solution

Notation. From now on, we will use $Z^\varphi(t)$ instead of $Z(t)$ to represent the mild solution of (3.1) to emphasize that the the solution depends on the initial value φ.

Lemma 3.3.1. *The mild solution of (3.1) $Z^\varphi(t)$ is continuous in the initial value φ (w.r.t. the strong topology on H).*

Proof. Let φ_1 and $\varphi_2 \in H$ be the initial values. Suppose that the two mild solutions are $Z^{\varphi_1}(t)$ and $Z^{\varphi_2}(t)$. Then we have

$$Z^{\varphi_1}(t) = S(t)\,\varphi_1 + \int_0^t S(t-s)F(Z^{\varphi_1}(s))\,ds + \int_0^t \int_E S(t-s)B(v, Z^{\varphi_1}(s))\,\widetilde{N}(dv\,ds),$$

and

$$Z^{\varphi_2}(t) = S(t)\,\varphi_2 + \int_0^t S(t-s)F(Z^{\varphi_2}(s))\,ds + \int_0^t \int_E S(t-s)B(v, Z^{\varphi_2}(s))\,\widetilde{N}(dv\,ds).$$

Thus

$$\begin{aligned}
&Z^{\varphi_1}(t) - Z^{\varphi_2}(t) \\
=&S(t)\,(\varphi_1 - \varphi_2) + \int_0^t S(t-s)F(Z^{\varphi_1}(s))\,ds \\
&+ \int_0^t \int_E S(t-s)B(v, Z^{\varphi_1}(s))\,\widetilde{N}(dv\,ds) \\
&- \Big(\int_0^t S(t-s)F(Z^{\varphi_2}(s))\,ds + \int_0^t \int_E S(t-s)B(v, Z^{\varphi_2}(s))\,\widetilde{N}(dv\,ds) \Big).
\end{aligned}$$

From Lemma 3.2.3 we have

$$\begin{aligned}
&E \sup_{0\le s\le t} \|Z^{\varphi_1}(s) - Z^{\varphi_2}(s)\|_H^2 \\
\le& E \sup_{0\le s\le t} (2\|S(s)(\varphi_1-\varphi_2)\|_H^2 + 2\|I(s, Z^{\varphi_1}(s)) - I(s, Z^{\varphi_2}(s))\|_H^2) \\
\le& 2e^{2\alpha t}\|\varphi_1-\varphi_2\|_H^2 + 2b_3\int_0^t E \sup_{0\le u\le s} \|Z^{\varphi_1}(u) - Z^{\varphi_2}(u)\|_H^2\, ds.
\end{aligned}$$

By Gronwall's inequality we have

$$E \sup_{0\le s\le t} \|Z^{\varphi_1}(s) - Z^{\varphi_2}(s)\|_H^2 \le 2e^{2\alpha t}\|\varphi_1-\varphi_2\|_H^2 e^{2b_3 t} = 2e^{(2\alpha+2b_3)t}\|\varphi_1-\varphi_2\|_H^2.$$

So the solution is continuous in the initial value. □

Theorem 3.3.1. *The mild solution of (3.1) is homogeneous in t and it has the Markov property.*

Proof. Fix s. Let us denote by $(Z^{s,\varphi}(t))_{t\ge s}$ the solution of

$$Z^{s,\varphi}(dt) = (AZ^{s,\varphi}(t) + F(Z^{s,\varphi}(t)))\,dt + \int_E B(v, Z^{s,\varphi}(t))\,\widetilde{N}(dv\,dt),\ \ Z^{s,\varphi}(s) = \varphi.$$

Following Theorem 3.2.1, it can be checked that such a solution exists and is unique up to stochastic equivalence. Let us remark that the compensated Poisson random measure is translation invariant in time; i.e., if $h>0$, $\mathcal{L}(\widetilde{N}(v,s+h) - \widetilde{N}(v,s)) = \mathcal{L}\widetilde{N}(v,h)$.

It follows that

$$\begin{aligned}
Z^{s,\varphi}(s+h) =& S(h)\varphi + \int_s^{s+h} S(s+h-u)F(Z^{s,\varphi}(u))\,du \\
&+ \int_s^{s+h}\int_E S(s+h-u)B(v, Z^{s,\varphi}(u))\,\widetilde{N}(dv\,du) \\
=& S(h)\varphi + \int_0^h S(h-u)F(Z^{s,\varphi}(s+u))\,du
\end{aligned}$$

$$+\int_0^h \int_E S(h-u)B(v, Z^{s,\varphi}(s+u))\,\widetilde{\widetilde{N}}(dv\,du). \tag{3.3}$$

Here $\widetilde{\widetilde{N}}(v,u) = \widetilde{N}(v, s+u) - \widetilde{N}(v,s)$.

From Theorem 3.2.1, we have

$$\begin{aligned} Z^{0,\varphi}(h) =& S(h)\varphi + \int_0^h S(h-u)F(Z^{0,\varphi}(u))\,du \\ &+ \int_0^h \int_E S(h-u)B(v, Z^{0,\varphi}(u))\,\widetilde{N}(dv\,du). \end{aligned} \tag{3.4}$$

As the solution of (3.3) and (3.4) are unique up to stochastic equivalent and $\widetilde{N}(dvds)$ and$\widetilde{\widetilde{N}}(dvds)$ are equally distributed, it follows that $\{ Z^{0,\varphi}(h) \}_{h\geq 0}$ and $\{ Z^{s,\varphi}(s+h) \}_{h\geq 0}$ are stochastic equivalent.

We have proved that $\{ Z^{0,\varphi}(t) \}_{t\geq 0}$ is cádlàg. Let $T \geq 0$. We denote by Q^{φ} the distribution induced by $\{ Z^{0,\varphi}(t) \}_{t\in[0,T]}$ on the Skorokhod space $D([0,T], H)$ and by E_{φ} the corresponding expectation. We also remark that the σ-algebra $\mathcal{F}_t^{\varphi} = \sigma\{ Z^{0,\varphi}(s), s \leq t \} \subseteq \mathcal{F}_t$, where $\{ \mathcal{F}_t \}_{t\geq 0}$ denotes the natural filtration of the compensated Poisson random measure $\widetilde{N}(dvds)$ and $\sigma\{ Z^{0,\varphi}(s), s \leq t \}$ is the σ-algebra generated by $\{ Z^{0,\varphi}(s) \}_{s\leq t}$.

Let us consider now the solution $\{ Z(r) \}_{r\in[t,T]}$ of

$$Z(r) = Z(t) + \int_t^r S(r-u)F(Z(u))\,du + \int_t^r \int_E S(r-u)B(v, Z(u))\,\widetilde{N}(dv\,du).$$

From Theorem 3.2.1, it follows that $\{ Z(r) \}_{r\in[t,T]}$ is stochastic equivalent to $\{ Z^{t,Z(t)}(r) \}_{r\in[t,T]}$. Let $H(z,t,r,\omega) := Z^{t,z}(r)$, $r \in [t,T]$. We remark that $H(z,t,r,\omega)$ is independent of $\mathcal{F}_t$. Let γ be a bounded, real valued measurable function on H. Then we can write

$$E_z[\, \gamma(Z(t+h)) \mid \mathcal{F}_t \,] = E[\, \gamma(H(Z(t), t, t+h, \omega)) \mid \mathcal{F}_t \,] \tag{3.5}$$

and

$$E_{z(t)}[\,\gamma(Z(h))\,] = E[\,\gamma(H(z,0,h,\omega))\,]_{z=Z(t)} \tag{3.6}$$

where $E[\,\cdot\,]_{z=Z(t)} := E[\,\cdot \mid Z(t) = z\,]$. We shall prove that

$$E[\,\gamma(H(Z(t),t,t+h,\omega)) \mid \mathcal{F}_t\,] = E[\,\gamma(H(z,t,t+h,\omega))\,]_{z=Z(t)}. \tag{3.7}$$

It then follows from (3.6) and the homogeneous property that

$$\begin{aligned} E[\,\gamma(H(Z(t),t,t+h,\omega)) \mid \mathcal{F}_t\,] &= E[\,\gamma(H(z,t,t+h,\omega))\,]_{z=Z(t)} \\ &= E[\,\gamma(H(z,0,h,\omega))\,]_{z=Z(t)} \\ &= E_{Z(t)}[\,\gamma(Z(h))\,]. \end{aligned}$$

Then use (3.5), it follows $E_z[\,\gamma(Z(t+h)) \mid \mathcal{F}_t\,] = E_{z(t)}[\,\gamma(Z(h))\,]$ and, since $\mathcal{F}_t^{Z(t)} \subseteq \mathcal{F}_t$, this gives $E_z[\,\gamma(Z(t+h)) \mid \mathcal{F}_t^{Z(t)}] = E_{z(t)}[\gamma(Z(h))]$.

Proof of (3.7): Put $g(z,\omega) = \gamma(H(z,t,t+h,\omega))$. Clearly $g(z,\cdot)$ is measurable in ω, and $z \to g(z,\omega)$ is continuous by the continuity with respect to the initial condition. Thus $g(z,\omega)$ is separately measurable, since H is separable. By a theorem of Mackey [26], we can find a function equal $dz \otimes dP$ a.e. to $g(z,\omega)$, which is jointly measurable. We again call it $g(z,\omega)$. Clearly g is bounded. We can approximate g pointwise boundedly by functions of the form $\sum_{k=1}^{m} \phi_k(z)\psi_k(\omega)$.

$$\begin{aligned} E[g(Z(t),\omega) \mid \mathcal{F}_t] &= \lim_{m\to\infty} \sum_{k=1}^{m} \phi_k(Z(t))E(\psi_k(\omega) \mid \mathcal{F}_t) \\ &= \lim_{m\to\infty} \sum_{k=1}^{m} E[\phi_k(z)\psi_k(\omega) \mid \mathcal{F}_t]_{z=Z(t)} = E[g(z,\omega)]_{z=Z(t)}. \end{aligned}$$

where in the first inequality we used that $\phi_k(Z(t))$ is $\mathcal{F}_t$-measurable. □

Definition 3.3.1. For a Markov process $\xi(t)$, defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ with state space χ let $P(\eta, t, \Gamma) = P(\xi(t) \in \Gamma \mid \xi(0) = \eta)$ (transition probability function). We say that $P(\eta, t, \Gamma)$ has the *Feller* property if for any bounded continuous Borel-measurable function ϕ on χ, $(P_t\phi)(\eta) = \int_\chi \phi(\gamma)\, P(\eta, t, d\gamma)$ is continuous in (t, η) for $t > 0, \eta \in \chi$.

Theorem 3.3.2. *The mild solution of (3.1) has Feller property.*

Proof. Let $h \in C_b(H)$ (bounded continuous functions on H) and let $\varphi_n \to \varphi, \varphi_n, \varphi \in H$. Then we know $Z^{\varphi_n}(t) \to Z^{\varphi}(t)$ in probability as $n \to \infty$. Then $E[h(Z^{\varphi_n}(t))] \to E[h(Z^{\varphi}(t))]$ since otherwise there would exist $\varepsilon > 0$ and a subsequence, denoted by n, such that $\|E[h(Z^{\varphi_n}(t))] - E[h(Z^{\varphi}(t))]\|_H > \varepsilon$ and $Z^{\varphi_n}(t) \to Z^{\varphi}(t)$, *a.s.*, which yields a contradiction. □

Chapter 4

Approximating system

We shall show in this chapter that every mild solution can be approximated by a strong solution.

4.1 Sufficient conditions for a mild solution to be a strong solution

We start with a Fubini type theory.

Proposition 4.1.1. *Let* $\mathbb{T} = [0, T]$ *and let* $B : \mathbb{T} \times \mathbb{T} \times E \times \Omega \to H$ *be measurable, and* $B(s,t,v)$ *is* $\mathcal{F}_t$*-measurable for each* s*, and* $\int_0^T \int_0^T \int_E E\|B(s,t,v)\|_H^2 \, \beta(dv)\, dt\, ds < \infty$*. Then*

$$\int_0^T \int_0^T \int_E B(s,t,v)\, \widetilde{N}(dv\, dt)\, ds = \int_0^T \int_E \int_0^T B(s,t,v)\, ds\, \widetilde{N}(dv\, dt).$$

Proof. (Sketch) We first prove that for the B given above, there exists simple functions

B_n having the form

$$B_n(s,t,x,\omega) = \sum_{j=1}^{p-1}\sum_{k=1}^{n-1}\sum_{l=1}^{m} 1_{A_{jkl}}(x) 1_{F_{jkl}}(\omega) 1_{(t_{jk},\, t_{jk+1}]}(t) 1_{(s_j,\, s_{j+1}]}(s) a_{jkl}. \quad (4.1)$$

Here $A_{jkl} \in \mathcal{B}(E/\{0\})$ $(0 \notin \overline{A_{jkl}}), t_{jk} \in (0,T], t_{jk} < t_{jk+1}, s_j \in (0,T], s_j < s_{j+1}, F_{jkl} \in \mathcal{F}_{t_{jk}}, a_{jkl} \in H$. For all $j \in 1,\ldots,p-1$ and $k \in 1,\ldots,n-1$ fixed, $A_{jkl_1} \times F_{jkl_1} \cap A_{jkl_2} \times F_{jkl_2} = \phi$, if $l_1 \neq l_2$. Also B_n L^2-approximating to B w.r.t. $ds \otimes v \otimes P$. The proof follows almost exactly from Theorem 4.2 in Rüdiger and Ziglio [38]. Now for simple functions in (4.1), we have

$$\begin{aligned}
&\int_0^T \int_0^T \int_E B(s,t,v)\, \widetilde{N}(dv\, dt)\, ds \\
=&\sum_{j=1}^{p-1}\sum_{k=1}^{n-1}\sum_{l=1}^{m} (s_{j+1} - s_j) a_{jkl} 1_{F_{jkl}}(\omega) \widetilde{N}((t_{jk}, t_{jk+1}] \cap (0,T] \times A_{jkl} \cap E) \\
=&\int_0^T \int_E \int_0^T B(s,t,v)\, ds\, \widetilde{N}(dv\, dt).
\end{aligned}$$

Also by the inequality

$$\begin{aligned}
&E\left\| \int_0^T \int_E \int_0^T B(s,t,v)\, ds\, \widetilde{N}(dv\, dt) \right\|_H^2 \\
=&\int_0^T \int_E E\left\| \int_0^T B(s,t,v)\, ds \right\|_H^2 \beta(dv)\, dt \\
\leq& T \int_0^T \int_0^T \int_E E\|B(s,t,v)\|_H^2\, ds\, \beta(dv)\, dt,
\end{aligned}$$

we know as $n \to \infty$, w.p. 1, we have

$$\int_0^T \int_0^T \int_E B(s,t,v)\, \widetilde{N}(dv\, dt)\, ds = \int_0^T \int_E \int_0^T B(s,t,v)\, ds\, \widetilde{N}(dv\, dt). \qquad \square$$

Proposition 4.1.2. *Suppose that:*

(a) $\varphi \in \mathcal{D}(A)$, $S(t-r)F(y) \in \mathcal{D}(A)$, $S(t-r)B(v,y) \in \mathcal{D}(A)$, *for each* $y \in H, v \in E$ *and* $t > r$,

(b) $\|AS(t-r)F(y)\|_H \leq g_1(t-r)(1+\|y\|_H)$, $g_1 \in L_1(0,T)$,

(c) $\int_E \|AS(t-r)B(v,y)\|_H^2 \, \beta(dv) \leq g_2(t-r)(1+\|y\|_H^2)$, $g_2 \in L_1(0,T)$.

Then a mild solution $Z(t)$ is also a strong solution.

Proof. By the above conditions, we have

$$\int_0^T \int_0^t \|AS(t-r)F(Z(r))\|_H \, dr \, dt < \infty, \ w.p.1,$$

and

$$\int_0^T \int_0^t \int_E E\|AS(t-r)B(v,Z(r))\|_H^2 \, \beta(dv) \, dr \, dt < \infty.$$

Thus by Fubini's theorem, we have

$$\begin{aligned}
&\int_0^t \int_0^s AS(s-r)F(Z(r)) \, dr \, ds \\
&= \int_0^t \int_r^t AS(s-r)F(Z(r)) \, ds \, dr \\
&= \int_0^t S(t-r)F(Z(r)) \, dr - \int_0^t F(Z(r)) \, dr.
\end{aligned}$$

By Proposition 4.1.1, we also have

$$\begin{aligned}
&\int_0^t \int_0^s \int_E AS(s-r)B(v,Z(r)) \, \widetilde{N}(dv\,dr) \, ds \\
&= \int_0^t \int_E \int_r^t AS(s-r)B(v,Z(r)) \, ds \, \widetilde{N}(dv\,dr) \\
&= \int_0^t \int_E S(t-r)B(v,Z(r)) \, \widetilde{N}(dv\,dr) - \int_0^t \int_E B(v,Z(r)) \, \widetilde{N}(dv\,dr).
\end{aligned}$$

Hence $AZ(t)$ is integrable w.p. 1 and

$$\begin{aligned}
\int_0^t AZ(s) \, ds =& S(t)\varphi - \varphi + \int_0^t S(t-r)F(Z(r)) \, dr - \int_0^t F(Z(r)) \, dr \\
&+ \int_0^t \int_E S(t-r)B(v,Z(r)) \, \widetilde{N}(dv\,dr) - \int_0^t \int_E B(v,Z(r)) \, \widetilde{N}(dv\,dr)
\end{aligned}$$

$$=Z(t) - \varphi - \int_0^t F(Z(r))\,dr - \int_0^t \int_E B(v, Z(r))\,\widetilde{N}(dv\,dr).$$

Thus

$$Z(t) = \varphi + \int_0^t AZ(s)\,ds + \int_0^t F(Z(s))\,ds + \int_0^t \int_E B(v, Z(s))\,\widetilde{N}(dv\,ds).$$

So $Z(t)$ is a strong solution. □

4.2 Approximation part

We now study the approximating system, which has the form

$$d\,Z(t) = AZ(t)\,dt + R(n)F(Z(t))\,dt + \int_E R(n)B(v, Z(t))\,\widetilde{N}(dv\,dt),\ \ Z(0) = R(n)\varphi \tag{4.2}$$

where $n \in \rho(A)$, the resolvent set of A ($\rho(A) := \{\lambda \in \mathbb{C}\colon \lambda - A\colon D(A) \to H$ is bijective$\}$), $R(n) = nR(n, A)$, and $R(n, A) = \int_0^\infty e^{-nt} S(t)\,dt$. We begin with a theorem on the mild solution of the stochastic equation.

Theorem 4.2.1. *The mild solution of (3.1) is in* $C(0, T; L_2^H(\Omega, \mathcal{F}, P))$.

Proof. Let $Z(t)$ be the solution of (3.1). We know that

$$\begin{aligned} Z(s+t) =& S(s+t)\,\varphi + \int_0^{s+t} S(s+t-u)F(Z(u))\,du \\ &+ \int_0^{s+t} \int_E S(s+t-u)B(v, Z(u))\,\widetilde{N}(dv\,du), \end{aligned}$$

and

$$Z(s) = S(s)\,\varphi + \int_0^s S(s-u)F(Z(u))\,du + \int_0^s \int_E S(s-u)B(v, Z(u))\,\widetilde{N}(dv\,du).$$

So

$$\begin{aligned}
Z(s+t)-Z(s) =& S(s+t)\,\varphi - S(s)\,\varphi + \int_s^{s+t} S(s+t-u)F(Z(u))\,du \\
&+ \int_0^s (S(s+t-u)-S(s-u))F(Z(u))\,du \\
&+ \int_s^{s+t}\int_E S(s+t-u)B(v,Z(u))\,\widetilde{N}(dv\,du) \\
&+ \int_0^s\int_E (S(s+t-u)-S(s-u))B(v,Z(u))\,\widetilde{N}(dv\,du).
\end{aligned}$$

We have

$$E\|Z(s+t)-Z(s)\|_H^2 \le 5(EI_1+EI_2+EI_3+EI_4+EI_5),$$

where

$$\begin{aligned}
EI_1 =& E\|S(s+t)\varphi - S(s)\varphi\|_H^2 \to 0, \text{as } t\to 0, \\
EI_2 =& E\left\|\int_s^{s+t} S(s+t-u)F(Z(u))\,du\right\|_H^2 \\
\le& tE\int_s^{s+t} \|S(s+t-u)F(Z(u))\|_H^2\,du \\
\le& tM^2E\int_s^{s+t} \|F(Z(u))\|_H^2\,du \to 0, \text{as } t\to 0,
\end{aligned}$$

$$\begin{aligned}
EI_3 =& E\left\|\int_0^s (S(s+t-u)-S(s-u))F(Z(u))\,du\right\|_H^2 \\
\le& sE\int_0^s \|(S(s+t-u)-S(s-u))F(Z(u))\|_H^2\,du \to 0, \ \text{as } t\to 0, \\
EI_4 =& E\left\|\int_s^{s+t}\int_E S(s+t-u)B(v,Z(u))\,\widetilde{N}(dv\,du)\right\|_H^2 \\
\le& M^2E\int_s^{s+t}\int_E \|B(v,Z(u))\|_H^2\,\beta(dv)\,du \to 0, \ \text{as } t\to 0, \\
EI_5 =& E\left\|\int_0^s\int_E (S(s+t-u)-S(s-u))B(v,Z(u))\,\widetilde{N}(dv\,du)\right\|_H^2 \\
\le& M^2E\int_0^s\int_E \|(S(t)-I)B(v,Z(u))\|_H^2\,\beta(dv)\,du \to 0, \ \text{as } t\to 0.
\end{aligned}$$

By the fact that $\|S(t)\| \leq M$, for $\forall t \in [0,T]$, $\|S(s+t)x - S(s)x\|_H \to 0$ as $t \to 0$, for $\forall x \in H$, and Lebesgue dominated convergence theorem. So we have

$$E\|Z(s+t) - Z(s)\|_H^2 \to 0, \text{ as } t \to 0.$$

Also by Theorem 3.2.1, we know the mild solution of (3.1) is in $C(0,T; L_2^H(\Omega, \mathcal{F}, P))$.

$\square$

The following is the main result of this chapter, which generalizes Ichikawa [17].

Theorem 4.2.2. *The stochastic differential equation (4.2) has a unique strong solution $Z(t,n)$ which lies in $C(0,T; L_2^H(\Omega, \mathcal{F}, P))$ for all T and $Z(t,n)$ converges to the mild solution of the stochastic equation (3.1) in $C(0,T; L_2^H(\Omega, \mathcal{F}, P))$ as $n \to \infty$ for all T.*

Proof. We know $AR(n)$ is a bounded operator and suppose that $|AR(n)| \leq M_1$. The first part is an immediate consequence of the existence of a mild solution and Proposition 4.1.2. Observe, by the growth condition and $|S(t)| \leq e^{\alpha t}$,

$$\|AS(t-r)R(n)F(y)\|_H \leq e^{\alpha(t-r)} M_1 \sqrt{l}(1 + \|y\|_H),$$

and similarly,

$$\int_E \|AS(t-r)R(n)B(v,y)\|_H^2 \beta(dv) \leq e^{2\alpha(t-r)} M_1^2 l(1 + \|y\|_H^2).$$

To prove the second part, we consider

$$\begin{aligned} Z(t) - Z(t,n) =& S(t)[\varphi - R(n)\varphi] + \int_0^t S(t-r)[F(Z(r)) - R(n)F(Z(r,n))]\, dr \\ &+ \int_0^t \int_E S(t-r)[B(v, Z(r)) - R(n)B(v, Z(r,n))]\, \widetilde{N}(dv\, dr) \end{aligned}$$

$$
\begin{aligned}
&= \int_0^t S(t-r)R(n)[F(Z(r)) - F(Z(r,n))]\,dr \\
&+ \int_0^t \int_E S(t-r)R(n)[B(v,Z(r)) - B(v,Z(r,n))]\,\widetilde{N}(dv\,dr) \\
&+ \left\{ S(t)[\varphi - R(n)\varphi] + \int_0^t S(t-r)[I - R(n)]F(Z(r))\,dr + \right. \\
&+ \left. \int_0^t \int_E S(t-r)[I - R(n)]B(v,Z(r))\,\widetilde{N}(dv\,dr) \right\}.
\end{aligned}
$$

We have,

$$
E \sup_{0\le s\le t} \|Z(s) - Z(s,n)\|_H^2 \le 3E \sup_{0\le s\le t} [I_1 + I_2 + I_3],
$$

where

$$
\begin{aligned}
E \sup_{0\le s\le t} I_1 =& E \sup_{0\le s\le t} \left\| \int_0^s S(s-r)R(n)[F(Z(r)) - F(Z(r,n))]\,dr \right\|_H^2 \\
\le& 4ke^{2\alpha t} t \int_0^t E \sup_{0\le r\le s} \|Z(r) - Z(r,n)\|_H^2\,ds, (\text{ by Lemma 3.2.3}) \\
E \sup_{0\le s\le t} I_2 =& E \sup_{0\le s\le t} \left\| \int_0^s \int_E S(s-r)R(n)[B(v,Z(r)) - B(v,Z(r,n))]\,\widetilde{N}(dv\,dr) \right\|_H^2 \\
\le& 4kb_1 \int_0^t E \sup_{0\le r\le s} \|Z(r) - Z(r,n)\|_H^2\,ds, \text{ (by Lemma 3.2.3)}
\end{aligned}
$$

and

$$
\begin{aligned}
E \sup_{0\le s\le t} I_3 =& E \sup_{0\le s\le t} \left\| S(s)[\varphi - R(n)\varphi] + \int_0^s S(s-r)[I - R(n)]F(Z(r))\,dr \right. \\
&+ \left. \int_0^s \int_E S(s-r)[I - R(n)]B(v,Z(r))\,\widetilde{N}(dv\,dr) \right\|_H^2. \\
\le& 3E \sup_{0\le s\le t} (I_{31} + I_{32} + I_{33}).
\end{aligned}
$$

We now estimate each term in I_3, because $|R(n) - I| \to 0$, as $n \to \infty$,

$$
E \sup_{0\le s\le t} I_{31} = E \sup_{0\le s\le t} \|S(s)[\varphi - R(n)\varphi]\|_H^2 \le |R(n) - I|^2 e^{\alpha t} \|\varphi\|_H^2 \to 0 \text{ as } n \to \infty,
$$

$$E \sup_{0\le s\le t} I_{32}$$

$$=E \sup_{0\le s\le t} \left\| \int_0^s S(s-r)[I-R(n)]F(Z(r))\,dr \right\|_H^2 \quad \text{(by Lemma 3.2.2)}$$

$$\le |R(n)-I|^2 e^{2\alpha t} tl\Big(t + \int_0^t E\|Z(r)\|_H^2\,dr\Big) \to 0 \text{ as } n\to\infty,$$

$$E \sup_{0\le s\le t} I_{33}$$

$$=E \sup_{0\le s\le t} \left\| \int_0^s \int_E S(t-r)[I-R(n)]B(v,Z(r))\,\widetilde{N}(dv\,dr) \right\|_H^2 \quad \text{(by Lemma 3.2.2)}$$

$$\le |R(n)-I|^2 b_1 l\Big(t + \int_0^t E\|Z(r)\|_H^2\,dr\Big) \to 0 \text{ as } n\to\infty.$$

Thus we get

$$E \sup_{0\le s\le t} \|Z(s)-Z(s,n)\|_H^2 \le c_4 \int_0^t E \sup_{0\le r\le s} \|Z(r)-Z(r,n)\|_H^2\,dr + |R(n)-I|^2 c_5.$$

Here c_4 and c_5 are positive constants.

By Gronwall's inequality, we have

$$E \sup_{0\le s\le t} \|Z(s)-Z(s,n)\|_H^2 \le |R(n)-I|^2 c_5 e^{c_4 t}.$$

So

$$0 \le \lim_{n\to\infty} \sup_{0\le s\le T} E\|Z(s)-Z(s,n)\|_H^2 \le \lim_{n\to\infty} E \sup_{0\le s\le T} \|Z(s)-Z(s,n)\|_H^2 = 0. \qquad \square$$

Chapter 5

Stability properties of the mild solution

5.1 Ito formula

Let $C_S^2(H)$ denote the space of all real-valued functions ψ on H with properties:

(i) $\psi(x)$ is twice (Fréchet) differentiable,

(ii) $\psi'(x)\colon H \to H$ is uniformly continuous on bounded set,

(iii) $\psi''(x)\colon H \to L(H)$ is uniformly continuous in strong operator topology on bounded set.

By $C_b^{2,loc}(H)$ denote the space of all functions in $C_S^2(H)$ with properties:

(i) There exists m_1, such that $\|\psi'(x)\|_H \leq m_1\|x\|_H$, for $\forall x \in H$,

(ii) ψ'' are independent of x.

We recall Ito formula from Rüdiger and Ziglio [38].

Theorem 5.1.1. *If $\psi(t,x)$ is once continuously differentiable in t, $\psi(t,\cdot) \in C_S^2(H)$ for any given t, also the second Fréchet derivatives ψ_{tt}, ψ_{tx}, ψ_{xt} exist and are uniformly bounded on on $[0,t] \times B(0,R)$ (centered ball of radius R), for all $R \geq 0$. Let $\{ Z^\varphi(t), t \geq 0 \}$ be the strong solution of (3.1), then*

$$\begin{aligned}
\psi(t, Z^\varphi(t)) = & \psi(0,\varphi) + \int_0^t \psi_t(s, Z^\varphi(s))\, ds \\
& + \int_0^t \langle \psi_x(s, Z^\varphi(s)), AZ^\varphi(s) + F(Z^\varphi(s)) \rangle \, ds \\
& + \int_0^t \int_E \Big[\psi(s, Z^\varphi(s) + B(v, Z^\varphi(s))) \\
& - \psi(s, Z^\varphi(s)) - \langle \psi_x(s, Z^\varphi(s)), B(v, Z^\varphi(s)) \rangle \Big] \beta(dv)\, ds \\
& + \int_0^t \int_E \Big[\psi(s, Z^\varphi(s-) + B(v, Z^\varphi(s))) - \psi(s, Z^\varphi(s-)) \Big] \widetilde{N}(dv\, ds).
\end{aligned}$$

Here ψ_t and ψ_x are Fréchet derivatives with respect to t and x respectively.

We have another form of Ito formula if ψ is independent of t.

Theorem 5.1.2. *Suppose $\psi \in C_S^2(H)$ and $\{ Z^\varphi(t), t \geq 0 \}$ is a strong solution of (3.1). Then*

$$\begin{aligned}
\psi(Z^\varphi(t)) = & \psi(\varphi) + \int_0^t \mathcal{L}\psi(Z^\varphi(s))\, ds \\
& + \int_0^t \int_E \{ \psi(Z^\varphi(s-) + B(v, Z^\varphi(s))) - \psi(Z^\varphi(s-)) \} \widetilde{N}(dv\, ds)
\end{aligned}$$

where

$$\mathcal{L}\psi(z) = \langle \psi'(z), Az + F(z) \rangle + \int_E [\psi(z + B(v,z)) - \psi(z) - \langle \psi'(z), B(v,z) \rangle] \beta(dv)$$

is the infinitesimal generator of the Markov process given by the solution of (3.1).

Since Ito formula is only applicable to the strong solution, we will use approximating method to study the stability properties of the mild solution to stochastic equation (3.1). We recall the approximating systems here for convenience.

$$\begin{cases} dZ(t) = \{\, AZ(t) + R(n)F(Z(t)) \,\}\, dt + \int_E R(n)B(v, Z(t))\, \tilde{N}(dv\, dt) \\ Z(0) = R(n)\varphi \end{cases} \tag{5.1}$$

where $n \in \rho(A)$, the resolvent set of A, and $R(n) = nR(n, A)$. The infinitesimal generator $\mathcal{L}_n$ corresponding to this equation is

$$\begin{aligned} \mathcal{L}_n\psi(z) =& \langle \psi'(z), Az + R(n)F(z) \rangle \\ &+ \int_E [\psi(z + R(n)B(v, z)) - \psi(z) - \langle \psi'(z), R(n)B(v, z) \rangle]\, \beta(dv). \end{aligned}$$

5.2 Exponential stability in the m.s.s.

Following Khasminski and Mandrekar [22], we define stochastic stability first.

Definition 5.2.1. Let $Z^{\varphi}(t)$ be the mild solution of (3.1), we say that it is *exponentially stable in the m.s.s.* if there exist positive constants c, θ, such that

$$E\|Z^{\varphi}(t)\|_H^2 \leq ce^{-\theta t}\|\varphi\|_H^2, \text{for all } \varphi \in \text{H and } t > 0. \tag{5.2}$$

The following gives a sufficient condition for exponential stability in m.s.s..

Theorem 5.2.1. *The mild solution $Z^{\varphi}(t)$ of (3.1) is exponentially stable in the m.s.s. if there exists a function $\psi : H \to R$ and $\psi \in C_b^{2,loc}(H)$ satisfying the following conditions:*

$$c_1\|x\|_H^2 \leq \psi(x) \leq c_3\|x\|_H^2 \tag{5.3}$$

$$\mathcal{L}\psi(x) \le -c_2\psi(x) \tag{5.4}$$

for $\forall x \in H$, *where* c_1, c_2, c_3 *are positive constants.*

Proof. Apply Ito formula to $e^{c_2 t}\psi(x)$ and $Z_n^\varphi(t)$ and take expectation, where $Z_n^\varphi(t)$ is the strong solution of (5.1). Then we have

$$e^{c_2 t} E\psi(Z_n^\varphi(t)) - \psi(Z_n^\varphi(0)) = E\int_0^t e^{c_2 s}(c_2 + \mathcal{L}_n)\psi(Z_n^\varphi(s))\,ds.$$

Here

$$\begin{aligned}\mathcal{L}_n\psi(x) =&\langle \psi'(x), Ax + R(n)F(x)\rangle \\ &+ \int_E [\psi(x + R(n)B(v,x)) - \psi(x) - \langle \psi'(x), R(n)B(v,x)\rangle]\,\beta(dv) \\ \mathcal{L}\psi(x) =&\langle \psi'(x), Ax + F(x)\rangle \\ &+ \int_E [\psi(x + B(v,x)) - \psi(x) - \langle \psi'(x), B(v,x)\rangle]\,\beta(dv).\end{aligned}$$

By (5.4),

$$\begin{aligned}&c_2\psi(x) + \mathcal{L}_n\psi(x) \le -\mathcal{L}\psi(x) + \mathcal{L}_n\psi(x) \\ =&\langle \psi'(x), (R(n) - I)F(x)\rangle \\ &+ \int_E \left[\begin{array}{c} [\psi(x + R(n)B(v,x)) - \psi(x) - \langle \psi'(x), R(n)B(v,x)\rangle] \\ -[\psi(x + B(v,x)) - \psi(x) - \langle \psi'(x), B(v,x)\rangle] \end{array} \right] \beta(dv).\end{aligned} \tag{5.5}$$

So we have

$$e^{c_2 t} E\psi(Z_n^\varphi(t)) - \psi(Z_n^\varphi(0)) \le E\int_0^t e^{c_2 s}(I_1(Z_n^\varphi(s)) + I_2(Z_n^\varphi(s)))\,ds, \tag{5.6}$$

where

$$I_1(h) = \langle \psi'(h), (R(n) - I)F(h)\rangle,$$

$$I_2(h) = \int_E \left[\begin{array}{c} [\psi(h + R(n)B(v,h)) - \psi(h) - \langle \psi'(h), R(n)B(v,h)\rangle] \\ -[\psi(h + B(v,h)) - \psi(h) - \langle \psi'(h), B(v,h)\rangle] \end{array} \right] \beta(dv).$$

Now, we will prove that the right hand side of (5.6) $\to 0$, as $n \to \infty$.

First the integral in (5.5) makes sense, because by Theorem 2.1.1, there exists a bounded bilinear form R_2, $|R_2| \le M'$ (a positive constant), such that the integrand satisfies

$$\begin{aligned}
&|\psi(x + R(n)B(v,x)) - \psi(x) - \langle \psi'(x), R(n)B(v,x)\rangle \\
&\quad - [\psi(x + B(v,x)) - \psi(x) - \langle \psi'(x), B(v,x)\rangle]| \\
=&|R_2(R(n)B(v,x)) - R_2(B(v,x))| \\
\le&|R_2(R(n)B(v,x))| + |R_2(B(v,x))| \\
\le&M'\|R(n)B(v,x)\|_H^2 + M'\|B(v,x)\|_H^2 \\
\le&M''\|B(v,x)\|_H^2.
\end{aligned}$$

Here M'' is a positive constant. So the integration makes sense.

We know that $\lim_{n\to\infty} \sup_{t\in[0,T]} E\|Z^\varphi(t) - Z_n^\varphi(t)\|_H^2 = 0$ by Theorem 4.2.2 and $\|Z_n^\varphi(t)\|_H^2 \le 2\|Z_n^\varphi(t) - Z^\varphi(t)\|_H^2 + 2\|Z^\varphi(t)\|_H^2$, we have $\sup_n \int_0^T E\|Z_n^\varphi(s)\|_H^2\, ds < \infty$. By the Schwartz inequality, we get $\sup_n \int_0^T E\|Z_n^\varphi(s)\|_H\, ds < \infty$. So $\{\, \|Z_n^\varphi(s)\|, n = 1, 2, \ldots\}$ is uniformly integrable on $\Omega \times [0,T]$ with respect to the measure $P \times Leb$.

Now we prove the right hand side of (5.6) $\to 0$, as $n \to \infty$.

$$\begin{aligned}
&\left|E\int_0^t e^{C_2 s} I_1(Z_n^\varphi(s))\, ds\right| \\
=&\left|E\int_0^t e^{C_2 s}\langle \psi'(Z_n^\varphi(s)), (R(n) - I)F(Z_n^\varphi(s))\rangle\, ds\right| \\
\le&E\int_0^t |e^{C_2 s}\langle \psi'(Z_n^\varphi(s)), (R(n) - I)F(Z_n^\varphi(s))\rangle|\, ds
\end{aligned}$$

$$\leq E \int_0^t e^{C_2 s} \|\psi'(Z_n^\varphi(s))\|_H |R(n) - I| \|F(Z_n^\varphi(s))\|_H \, ds$$

$$\leq |R(n) - I| E \int_0^t e^{C_2 Cs} m_1 \|Z_n^\varphi(s)\|_H \sqrt{l}(1 + \|Z_n^\varphi(s)\|_H) \, ds$$

$$\leq M_3 |R(n) - I| \ (M_3 \text{ is a positive constant})$$

$$\rightarrow 0, \ \text{as } n \rightarrow \infty.$$

By letting

$$I_n(x) = \psi(x + R(n)B(v,x)) - \psi(x) - \langle \psi'(x), R(n)B(v,x) \rangle$$

$$I(x) = \psi(x + B(v,x)) - \psi(x) - \langle \psi'(x), B(v,x) \rangle,$$

one has

$$\begin{aligned}
&\left| E \int_0^t e^{C_2 S} I_2(Z_n^\varphi(s)) \, ds \right| \\
\leq & E \int_0^t \int_E e^{C_2 S} |I_n(Z_n^\varphi(s)) - I(Z_n^\varphi(s))| \, \beta(dv) \, ds \\
= & E \int_0^t \int_E e^{C_2 S} |R_2(R(n)B(v, Z_n^\varphi(s))) - R_2(B(v, Z_n^\varphi(s)))| \, \beta(dv) \, ds.
\end{aligned}$$

Here R_2 is a bounded bilinear form by Theorem 2.1.1. By the result in Yosida [45], there exists bounded linear operator $C \colon H \rightarrow H$, such that

$$R_2(R(n)B(v, Z_n^\varphi(s))) = \langle CR(n)B(v, Z_n^\varphi(s)), R(n)B(v, Z_n^\varphi(s)) \rangle,$$

and

$$R_2(B(v, Z_n^\varphi(s))) = \langle CB(v, Z_n^\varphi(s)), B(v, Z_n^\varphi(s)) \rangle.$$

For simplicity, let

$$Q(x) = E \int_0^t \int_E e^{C_2 S}(x) \, \beta(dv) \, ds.$$

We have

$$
\begin{aligned}
&\left|E\int_0^t e^{C_2 s} I_2(Z_n^\varphi(s))\,ds\right| \\
\leq& Q\left(\left|\begin{array}{c}\langle CR(n)B(v,Z_n^\varphi(s)),R(n)B(v,Z_n^\varphi(s))\rangle \\ -\langle CB(v,Z_n^\varphi(s)),B(v,Z_n^\varphi(s))\rangle\end{array}\right|\right) \\
=& Q\left(\left|\begin{array}{c}\langle C(R(n)-I)B(v,Z_n^\varphi(s)),R(n)B(v,Z_n^\varphi(s))\rangle \\ -\langle CB(v,Z_n^\varphi(s)),(R(n)-I)B(v,Z_n^\varphi(s))\rangle\end{array}\right|\right) \\
\leq& Q\left(\begin{array}{c}|C||R(n)-I|\|B(v,Z_n^\varphi(s))\|_H|R(n)|\|B(v,Z_n^\varphi(s))\|_H \\ +|C|\|B(v,Z_n^\varphi(s))\|_H|R(n)-I|\|B(v,Z_n^\varphi(s))\|_H\end{array}\right) \\
\leq& E\int_0^t\int_E e^{C_2 s}(m_2|R(n)-I|2\|B(v,Z_n^\varphi(s))\|_H^2 \\
&+m_2|R(n)-I|\|B(v,Z_n^\varphi(s))\|_H^2)\,\beta(dv)\,ds \\
\leq& |R(n)-I|E\int_0^t\int_E 3m_2 e^{C_2 s}\|B(v,Z_n^\varphi(s))\|_H^2\,\beta(dv)\,ds \\
\leq& |R(n)-I|E\int_0^t 3m_2 e^{C_2 s}(1+\|Z_n^\varphi(s)\|_H^2)\,ds \\
\leq& M_4|R(n)-I|,
\end{aligned}
$$

with M_4 a positive constant. Thus we know the right hand side of (5.6) $\to 0$, as $n\to\infty$. By the Lebesgue dominated convergence theorem, we have

$$e^{C_2 t}E\psi(Z^\varphi(t))\leq\psi(\varphi).$$

By condition (5.4), we have

$$c_1E\|Z^\varphi(t)\|_H^2\leq c_3\|\varphi\|_H^2 e^{-c_2 t}.$$

So

$$E\|Z^\varphi(t)\|_H^2\leq\frac{c_3}{c_1}\|\varphi\|_H^2 e^{-c_2 t}. \qquad \square$$

The function $\psi(x) \in C_b^{2,loc}(H)$ and satisfy the conditions (5.3) and (5.4) in the above theorem is a Lyapunov function. Now we want to construct a Lyapunov function if the solution $Z^{\varphi}(t)$ of (3.1) is exponentially stable in the m.s.s..

First, let us consider the following linear case. Suppose $F = 0$ and $B = B_0$ is linear. Then equation (3.1) has the form

$$\begin{cases} dZ(t) = AZ(t)\,dt + \int_E B_0(v)Z(t)\,\widetilde{N}(dv\,dt) \\ Z(0) = \varphi. \end{cases} \tag{5.7}$$

We assume $\int_E \|B_0(v)y\|_H^2\,\beta(dv) \le d\|y\|_H^2$ and the solution of this equation is $Z_0^{\varphi}(t)$. The infinitesimal generator $\mathcal{L}_0$ corresponding to this equation is

$$\mathcal{L}_0\psi(z) = \langle \psi'(z), Az\rangle + \int_E [\psi(z + B_0(v)z) - \psi(z) - \langle \psi'(z), B_0(v)z\rangle]\,\beta(dv).$$

Theorem 5.2.2. *If the solution $Z_0^{\varphi}(t)$ of equation (5.7) is exponentially stable in the m.s.s., then there exists a function $\psi_0 \in C_b^{2,loc}(H)$ satisfying (5.3) and (5.4) with $\mathcal{L}$ replaced by $\mathcal{L}_0$.*

Proof. Let

$$\psi_0(\varphi) = \int_0^\infty E\|Z_0^{\varphi}(t)\|_H^2\,dt + w\|\varphi\|_H^2, \tag{5.8}$$

where w is a constant to be determined later. Since $Z_0^{\varphi}(t)$ is exponentially stable in the m.s.s., $\int_0^\infty E\|Z_0^{\varphi}(t)\|_H^2\,dt$ is well defined and there exists a symmetric and nonnegative operator $R \in L(H)$ (Prato and Zabczyk [6]), such that $\phi(\varphi) := \int_0^\infty E\|Z_0^{\varphi}(t)\|_H^2\,dt = \langle R\varphi, \varphi\rangle$. Hence

$$\psi_0(\varphi) = \langle R\varphi, \varphi\rangle + w\|\varphi\|_H^2. \tag{5.9}$$

It is obvious that $\psi_0 \in C_b^{2,loc}(H)$ and $w\|\varphi\|_H^2 \le \psi_0(\varphi) \le (|R| + w)\|\varphi\|_H^2$. This proves ψ_0 satisfies (5.3). To prove it also satisfies (5.4) with $\mathcal{L}$ replaced by $\mathcal{L}_0$, we note that

A is the infinitesimal generator of a C_0-semigroup $S(t)$ satisfying $\|S(t)\| \leq e^{\alpha t}$. There exists a constant λ (without loss of generality, we assume it is positive) such that $\langle z, Az\rangle \leq \lambda \|z\|_H^2$ (Ichikawa [17]). Hence we have

$$\mathcal{L}_0\|z\|_H^2 = 2\langle z, Az\rangle + \int_E \|B_0(v)z\|_H^2\, \beta(dv) \leq (2\lambda + d)\|z\|_H^2. \tag{5.10}$$

Also,

$$\begin{aligned}\mathcal{L}_0\phi(\varphi) =& \frac{d}{dr}(E\phi(Z_0^\varphi(r)))|_{r=0} = \lim_{r\to 0} \frac{E\phi(Z_0^\varphi(r)) - \phi(\varphi)}{r} \\ =& \lim_{r\to 0} -\frac{1}{r}\int_0^r E\|Z_0^\varphi(s)\|_H^2\, ds = -\|\varphi\|_H^2.\end{aligned}$$

Therefore

$$\begin{aligned}\mathcal{L}_0\psi_0(z) =& \mathcal{L}_0\langle Rz, z\rangle + w\mathcal{L}_0\|z\|_H^2 \\ \leq& -\|z\|_H^2 + w(2\lambda + d)\|z\|_H^2 \\ =& \{ -1 + w(2\lambda + d) \}\, \|z\|_H^2. \end{aligned} \tag{5.11}$$

Therefore, if w is small enough, (5.4) holds with $\mathcal{L}$ replaced by $\mathcal{L}_0$. This proves the theorem. □

For the nonlinear equation (3.1), to assume zero is a solution, we need to assume $F(0) = 0$ and $B(v, 0) = 0$. If the solution $Z^\varphi(t)$ is exponentially stable in the m.s.s, we can still construct a Lyapunov function as (5.8),

$$\psi(\varphi) = \int_0^\infty E\|Z^\varphi(t)\|_H^2\, dt + w\|\varphi\|_H^2.$$

But it may not be in $C_b^{2,loc}(H)$. If we assume it is in $C_b^{2,loc}(H)$, we claim that it satisfies (5.3) and (5.4). Now we prove this claim. Since $Z^\varphi(t)$ is exponentially stable in the m.s.s, we assume it satisfies (5.2). Hence

$$\int_0^\infty E\|Z^\varphi(t)\|_H^2\,dt \le \frac{c}{\theta}\|\varphi\|_H^2$$

for all $\varphi \in H$. Therefore, $w\|\varphi\|_H^2 \le \psi(\varphi) \le (\frac{c}{\theta} + w)\|\varphi\|_H^2$. This proves (5.3). To prove (5.4), let $\phi(\varphi) = \int_0^\infty E\|Z^\varphi(t)\|_H^2\,dt$. Observe that

$$E\phi(Z^\varphi(r)) = E\int_0^\infty E(\|Z^{Z^\varphi(r)}(s)\|_H^2 \mid Z^\varphi(r))\,ds.$$

But by the Markov property of the solution of (3.1), this equals

$$\int_0^\infty E\left[E\left(\|Z^{Z^\varphi(r)}(s)\|_H^2 \mid \mathcal{F}_r^\varphi\right)\right]\,ds,$$

where $\mathcal{F}_r^\varphi = \sigma\{\,Z^\varphi(t), t \le r\,\}$. The uniqueness of the solution implies

$$E(\|Z^{Z^\varphi(r)}(s)\|_H^2 \mid \mathcal{F}_r^\varphi) = E(\|Z^\varphi(s+r)\|_H^2 \mid \mathcal{F}_r^\varphi).$$

Hence

$$E\phi(Z^\varphi(r)) = \int_0^\infty E\|Z^\varphi(r+s)\|_H^2\,ds = \int_r^\infty E\|Z^\varphi(s)\|_H^2\,ds. \tag{5.12}$$

By the continuity of $t \longrightarrow E\|Z^\varphi(t)\|_H^2$, we get

$$\begin{aligned}\mathcal{L}\phi(\varphi) =& \frac{d}{dr}(E\phi(Z^\varphi(r)))|_{r=0} = \lim_{r\to 0}\frac{E\phi(Z^\varphi(r)) - \phi(\varphi)}{r}\\ =& \lim_{r\to 0} -\frac{1}{r}\int_0^r E\|Z^\varphi(s)\|_H^2\,ds = -\|\varphi\|_H^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{L}\psi(\varphi) =& \mathcal{L}\phi(\varphi) + w\mathcal{L}\|\varphi\|_H^2\\ =& -\|\varphi\|_H^2 + w(2\langle\varphi, A\varphi + F(\varphi)\rangle + \int_E \|B(v,\varphi)\|_H^2\,\beta(dv))\\ \le& -\|\varphi\|_H^2 + 2w\lambda\|\varphi\|_H^2 + w(2\langle\varphi, F(\varphi)\rangle + \int_E \|B(v,\varphi)\|_H^2\,\beta(dv)).\end{aligned}$$

Since we assume $F(0) = 0, B(v, 0) = 0$, using the Lipschitz condition, we get $\mathcal{L}\psi(\varphi) \leq -\|\varphi\|_H^2 + w(2\lambda + 2\sqrt{k} + k)\|\varphi\|_H^2$.

Hence if w is small enough, $\psi(\varphi)$ satisfies(5.4). Therefore we have the following theorem.

Theorem 5.2.3. *If the solution $Z^\varphi(t)$ of (3.1) is exponentially stable in the m.s.s., $F(0) = 0, B(v, 0) = 0$ and $\phi(\varphi) = \int_0^\infty E\|Z^\varphi(t)\|_H^2\, dt$ is in $C_b^{2,loc}(H)$, then the function $\psi(\varphi)$ constructed above satisfies (5.3) and (5.4).*

Since we have difficulty showing $\phi(\varphi) \in C_b^{2,loc}(H)$, we turn to use the first order approximation to study the exponential stability in the m.s.s. of the solution of the nonlinear equation (3.1).

Theorem 5.2.4. *Suppose the solution $Z_0^\varphi(t)$ of equation (5.7) is exponentially stable in the m.s.s., and it satisfies (5.2). Then the solution $Z^\varphi(t)$ of (3.1) is exponentially stable in the m.s.s. if*

$$2\|z\|_H\|F(z)\|_H + \int_E \|B(v, z) - B_0(v)z\|_H\|B(v, z) + B_0(v)z\|_H\, \beta(dv) < \frac{\theta}{c}\|z\|_H^2. \quad (5.13)$$

Proof. Let $\psi_0(z) = \langle Rz, z\rangle + w\|z\|_H^2$ as defined in (5.9). Since $Z_0^\varphi(t)$ satisfies (5.2), $\|R\| < \frac{c}{\theta}$. Since $\psi_0(z) \in C_b^{2,loc}(H)$ and satisfies (5.3), if we can show that $\psi_0(z)$ satisfies (5.4), then by using Theorem 5.2.1, we are done. Since

$$\begin{aligned}
&\mathcal{L}\psi_0(z) - \mathcal{L}_0\psi_0(z) \\
&= \langle \psi_0'(z), F(z)\rangle + \int_E [\psi_0(z + B(v, z)) - \psi_0(z) - \langle \psi_0'(z), B(v, z)\rangle]\, \beta(dv) \\
&\quad - \int_E [\psi_0(z + B_0(v)z) - \psi_0(z) - \langle \psi_0'(z), B_0(v)z\rangle]\, \beta(dv)
\end{aligned}$$

$$
\begin{aligned}
=&2\langle (R+w)z, F(z)\rangle \\
&+ \int_E \left(\langle (R+w)B(v,z), B(v,z)\rangle - \langle (R+w)B_0(v)z, B_0(v)z\rangle\right)\beta(dv) \\
\leq& 2(\|R\|+w)\|z\|_H\|F(z)\|_H \\
&+ (\|R\|+w)\int_E \|B(v,z)-B_0(v)z\|_H\|B(v,z)+B_0(v)z\|_H\,\beta(dv) \\
=&(\|R\|+w)(2\|z\|_H\|F(z)\|_H \\
&+ \int_E \|B(v,z)-B_0(v)z\|_H\|B(v,z)+B_0(v)z\|_H\,\beta(dv)),
\end{aligned} \tag{5.14}
$$

by (5.11) and the assumption (5.13), $\mathcal{L}\psi_0(z)$ satisfies (5.4) if we choose w small enough. □

The following example shows that the usual Lyapunov function is not bounded below.

Example 5.2.1. *Consider the SPDE*

$$d_t u(t,x) = \left(\alpha^2\frac{\partial^2 u}{\partial x^2} + \gamma u\right)dt + \int_{|v|\leq 1} uv\,\widetilde{N}(dv\,dt)$$

with initial condition $u(0,x) = \varphi(x) \in L^2(-\infty,+\infty)\cap L^1(-\infty,+\infty)$, $\widetilde{N}$ is the compensated Poisson random measure.

Let $H = L^2(-\infty,+\infty)$, $Au = \alpha^2\frac{\partial^2 u}{\partial x^2} + \gamma u$, $B(u,v) = uv$, $|u| = (\int_{-\infty}^{\infty} u^2\,dx)^{1/2}$. Now we compute $E|u^{\varphi}(t)|^2$ explicitly. Taking the Fourier transformation of the SPDE, we get

$$
\begin{aligned}
d_t\widehat{u}(t,\lambda) =&(-\alpha^2\lambda^2\widehat{u}(t,\lambda) + \gamma\widehat{u}(t,\lambda))\,dt + \int_{|v|\leq 1}\widehat{u}(t,\lambda)v\,\widetilde{N}(dv\,dt) \\
=&(-\alpha^2\lambda^2+\gamma)\widehat{u}(t,\lambda)\,dt + \int_{|v|\leq 1}\widehat{u}(t,\lambda)v\,\widetilde{N}(dv\,dt).
\end{aligned}
$$

So we have

$$\widehat{u}(t,\lambda) = \widehat{\varphi}(\lambda) + \int_0^t (-\alpha^2\lambda^2 + \gamma)\widehat{u}(s,\lambda)\,ds + \int_0^t \int_{|v|\le 1} \widehat{u}(t,\lambda) v\,\widetilde{N}(dv\,ds).$$

Now using Ito formula on $|\widehat{u}(t,\lambda)|^2$ and taking expectation, we have

$$\begin{aligned}
&E|\widehat{u}(t,\lambda)|^2 \\
=&|\widehat{\varphi}(\lambda)|^2 + 2(-\alpha^2\lambda^2+\gamma)\int_0^t E|\widehat{u}(s,\lambda)|^2\,ds + \int_0^t\int_{|v|\le 1} v^2\,\beta(dv) E|\widehat{u}(s,\lambda)|^2\,ds \\
=&|\widehat{\varphi}(\lambda)|^2 + \left[2(-\alpha^2\lambda^2+\gamma) + \int_{|v|\le 1} v^2\,\beta(dv)\right]\int_0^t E|\widehat{u}(s,\lambda)|^2\,ds.
\end{aligned}$$

So solving this equation, we get

$$E|\widehat{u}(t,\lambda)|^2 = |\widehat{\varphi}(\lambda)|^2 e^{\{-2\alpha^2\lambda^2 + 2\gamma + \int_{|v|\le 1} v^2\,\beta(dv)\}t}.$$

By the Plancheral theorem, with $H = L^2(-\infty,+\infty)$,

$$|u^\varphi(t,\cdot)|^2 = |\widehat{u}^\varphi(t,\cdot)|^2,$$

and hence we have

$$\begin{aligned}
E|u^\varphi(t)|^2 =& E|\widehat{u}^\varphi(t)|^2 = E\int_{-\infty}^{\infty}|\widehat{u}(t,\lambda)|^2\,d\lambda \\
=& \int_{-\infty}^{\infty}|\widehat{\varphi}(\lambda)|^2 e^{\{-2\alpha^2\lambda^2 + 2\gamma + \int_{|v|\le 1} v^2\,\beta(dv)\}t}\,d\lambda.
\end{aligned}$$

If we assume $2\gamma + \int_{|v|\le 1} v^2\,\beta(dv) < 0$, then we get

$$E|u^\varphi(t)|^2 \le |\varphi|^2 e^{\{2\gamma + \int_{|v|\le 1} v^2\,\beta(dv)\}t}.$$

Hence the solution of the SPDE is stable. But

$$\int_0^\infty E|u^\varphi(t)|^2\,dt = \int_{-\infty}^{\infty}|\widehat{\varphi}(\lambda)|^2 \frac{1}{2\alpha^2\lambda^2 - (2\gamma + \int_{|v|\le 1} v^2\,\beta(dv))}\,d\lambda.$$

Thus the usual Lyapunov function $\int_0^\infty E\,|u^\varphi(t)|^2\,dt$ is not bounded below.

5.3 Stability in probability

Definition 5.3.1. Let $Z^\varphi(t)$ be the mild solution of (3.1). We say that the zero solution of the equation is *stable in probability* if

$$\lim_{\|\varphi\|_H \to 0} P\Big(\sup_t \|Z^\varphi(t)\|_H > \varepsilon\Big) = 0 \text{ for each } \varepsilon > 0.$$

Theorem 5.3.1. *Let $Z^\varphi(t)$ be the solution of equation (3.1). If there exists a function $\psi(x) \in C_b^{2,loc}(H)$ having the properties:*

(i) $c_1\|x\|_H^2 \le \psi(x) \le c_2\|x\|_H^2$, where c_1 and c_2 are positive constants,

(ii) $\inf_{\|x\|_H > \varepsilon} \psi(x) = \lambda_\varepsilon > 0$,

(iii) $\mathcal{L}\psi(x) \le 0$, for $\forall x \in H$,

then the zero solution of equation (3.1) is stable in probability.

Proof. We first obtain the inequality

$$P\Big(\sup_t \|Z^\varphi(t)\|_H > \varepsilon\Big) \le \frac{\psi(\varphi)}{\lambda_\varepsilon}, \quad for\ \varphi \in H.$$

To prove this, let $O_\varepsilon = \{x \in H : \|x\|_H < \varepsilon\}, T_\varepsilon = \inf\{t : \|Z^\varphi(t)\|_H > \varepsilon\}$. Now consider the process $Z^\varphi(t \wedge T_\varepsilon)$. Using Ito formula on $\psi(x)$ and $Z_n^\varphi(t \wedge T_\varepsilon)$ and taking expectation, we have $E\psi(Z_n^\varphi(t \wedge T_\varepsilon)) - \psi(Z_n^\varphi(0)) = E\int_0^{t \wedge T_\varepsilon} \mathcal{L}_n\psi(Z_n^\varphi(s \wedge T_\varepsilon))\,ds$. Now using a technique similar to that used in Theorem 5.2.1 and

$$\mathcal{L}_n\psi(Z_n^\varphi(s \wedge T_\varepsilon)) \le -\mathcal{L}\psi(Z_n^\varphi(s \wedge T_\varepsilon)) + \mathcal{L}_n\psi(Z_n^\varphi(s \wedge T_\varepsilon)),$$

we can get $E\psi(Z^\varphi(t \wedge T_\varepsilon)) \le \psi(\varphi)$, so

$$\psi(\varphi) \ge E\psi(Z^\varphi(t \wedge T_\varepsilon)) > \lambda_\varepsilon P(T_\varepsilon < t).$$

This proves the inequality. Now let $\varphi \to 0$, and we get the assertion. $\square$

The function constructed in Theorem 5.2.2 for the linear equation (5.7) satisfies the condition of Theorem 5.3.1. Hence we get the following theorem.

Theorem 5.3.2. *The solution $Z_0^\varphi(t)$ of the linear equation (5.7) is stable in probability if it is exponentially stable in the m.s.s..*

For the stability in probability of the zero solution of the nonlinear equation (3.1), we have the following Theorem.

Theorem 5.3.3. *If the solution $Z_0^\varphi(t)$ of the linear equation (5.7) is exponentially stable in the m.s.s. and*

$$2\|x\|_H\|F(x)\|_H + \int_E \|B(v,x) - B_0(v)x\|_H\|B(v,x) + B_0(v)x\|_H\,\beta(dv) < w\|x\|_H^2 \quad (5.15)$$

for some w small enough in a sufficient small neighborhood of $x = 0$, then the zero solution of the nonlinear equation (3.1) is stable in probability.

Proof. Since the solution $Z_0^\varphi(t)$ of the linear equation (5.7) is exponentially stable in the m.s.s, we define $\psi_0(x) = \langle Rx, x\rangle + w\|x\|_H^2$ as in (5.9). By (5.14) and assumption (5.15), we get $\mathcal{L}\psi_0(x) \leq 0$. Obviously, $\psi_0(x)$ satisfies the other condition of Theorem 5.2.4. Therefore our assertion holds. $\square$

5.4 Exponential ultimate boundedness in m.s.s.

In this section, we study exponentially ultimate boundedness properties of the mild solution of (3.1). We will give a necessary and sufficient condition in terms of a

Lyapunov function for the linear case and use the first order approximation to study the nonlinear case.

Definition 5.4.1. The solution $Z^{\varphi}(t)$ of (3.1) is *exponentially ultimately bounded in the m.s.s.* if there exist positive constants c, θ, M such that

$$E\|Z^{\varphi}(t)\|_H^2 \leq ce^{-\theta t}\|\varphi\|_H^2 + M \tag{5.16}$$

for $\forall \varphi \in H$.

Definition 5.4.2. The solution $Z^{\varphi}(t)$ of (3.1) is *ultimately bounded in the m.s.s.* if there exist positive constant K such that

$$\overline{\lim}_{t\to\infty} E\|Z^{\varphi}(t)\|_H^2 \leq K \tag{5.17}$$

for $\forall \varphi \in H$.

Definition 5.4.3. A stochastic process $\{\, \xi(t),\, t > 0 \,\}$ is said to be *bounded in probability* if the random variables $|\xi(t)|$ are bounded in probability uniformly in t; i.e.,

$$\sup_{t>0} P\{\, |\xi(t)| > R \,\} \to 0, \text{ as } R \to \infty.$$

Remarks.

(1) If $M = 0$, we get that the zero solution is exponentially stable in the m.s.s..

(2) It is clear that exponentially ultimately boundedness implies ultimately boundedness.

(3) Ultimately boundedness implies bounded in probability (by using Chebyshev's inequality).

Theorem 5.4.1. *The mild solution $Z^{\varphi}(t)$ of (3.1) is exponentially ultimately bounded in the m.s.s. if there exists a function $\psi : H \to R$, also $\psi \in C_b^{2,loc}(H)$ satisfying the conditions:*

$$c_1\|x\|_H^2 - k_1 \le \psi(x) \le c_3\|x\|_H^2 - k_3 \tag{5.18}$$

$$\mathcal{L}\psi(x) \le -c_2\psi(x) + k_2 \tag{5.19}$$

for $\forall x \in H$, where $c_1 > 0, c_2 > 0, c_3 > 0$, k_1, k_2, k_3 are constants.

Proof. The proof of this theorem is similar to that of Theorem 5.2.1. Applying Ito formula to $e^{c_2 t}\psi(x)$ and $Z_n^{\varphi}(t)$ and taking expectation, we get

$$e^{c_2 t}E\psi(Z_n^{\varphi}(t)) - \psi(Z_n^{\varphi}(0)) = E\int_0^t e^{c_2 s}(c_2 + \mathcal{L}_n)\psi(Z_n^{\varphi}(s))\,ds,$$

where

$$c_2\psi(x) + \mathcal{L}_n\psi(x) \le -\mathcal{L}\psi(x) + k_2 + \mathcal{L}_n\psi(x).$$

As before, we get

$$e^{c_2 t}E\psi(Z_n^{\varphi}(t)) - \psi(Z_n^{\varphi}(0)) \le \int_0^t e^{c_2 s}k_2\,ds = \frac{k_2}{c_2}(e^{c_2 t} - 1);$$

i.e.,

$$e^{c_2 t}E\psi(Z_n^{\varphi}(t)) \le \psi(\varphi) + \int_0^t e^{c_2 s}k_2\,ds. \tag{5.20}$$

So

$$Ec_1\|Z^{\varphi}(t)\|_H^2 - k_1 \le e^{-c_2 t}(c_3\|\varphi\|_H^2 - k_3) + \frac{k_2}{c_2}(1 - e^{-c_2 t});$$

i.e.,

$$\begin{aligned} E\|Z^{\varphi}(t)\|_H^2 \le& \frac{1}{c_1}\left[c_3 e^{-c_2 t}\|\varphi\|_H^2 - k_3 e^{-c_2 t} + k_1 + \frac{k_2}{c_2}(1 - e^{-c_2 t})\right] \\ \le& \frac{c_3}{c_1}e^{-c_2 t}\|\varphi\|_H^2 - \frac{k_3}{c_1}e^{-c_2 t} + \frac{k_1}{c_1} + \frac{k_2}{c_1 c_2}(1 - e^{-c_2 t}). \end{aligned}$$

So there exist positive constants c, θ and M, such that

$$E\|Z^{\varphi}(t)\|_H^2 \le ce^{-\theta t}\|\varphi\|_H^2 + M, \text{ for } \forall \varphi \in H.$$

So $Z^{\varphi}(t)$ is exponentially ultimate bounded in m.s.s.. □

From (5.20), we have the following result.

Corollary 5.4.1. *Suppose all the conditions in Theorem (5.4.1) hold except condition (5.18) is changed to*

$$c_1\|x\|_H^2 - k_1 \le \psi(x).$$

Then the the mild solution $Z^{\varphi}(t)$ of (3.1) is ultimately bounded in the m.s.s., so it is bounded in probability.

Remark. The above corollary is a generalization of Skorokhod's work ([40], Theorem 25, p.70).

For the converse problem, we first look at the linear equation (5.7) and have the following result.

Theorem 5.4.2. *If the solution $Z_0^{\varphi}(t)$ of equation (5.7) is exponentially ultimately bounded in the m.s.s., then there exists a function $\psi_0 \in C_b^{2,loc}(H)$ satisfying (5.18) and (5.19) with $\mathcal{L}$ replaced by $\mathcal{L}_0$.*

Proof. Suppose the solution $Z_0^{\varphi}(t)$ of equation (5.7) is exponentially ultimately bounded in the m.s.s.. We suppose (5.16) holds. Let

$$\psi_0(\varphi) = \int_0^T E\|Z_0^{\varphi}(s)\|_H^2\, ds + w\|\varphi\|_H^2, \tag{5.21}$$

where T is a positive constant to be determined later. First let's show $\psi_0 \in C_b^{2,loc}(H)$. Let $\varphi_0(x) = \int_0^T E\|Z_0^x(t)\|_H^2\,dt$. Using (5.16),

$$\varphi_0(x) \le \int_0^T (ce^{-\theta t}\|x\|_H^2 + M)\,dt \le \frac{c}{\theta}\|x\|_H^2 + MT. \tag{5.22}$$

If $\|x\|_H^2 = 1$, then $\varphi_0(x) \le \frac{c}{\theta} + MT$. Since $Z_0^x(t)$ is linear in x, for any positive constant k, we have $Z_0^{kx}(t) = kZ_0^x(t)$. Hence $\varphi_0(kx) = k^2\varphi_0(x)$. Therefore, for any $x \in H$,

$$\varphi_0(x) = \|x\|_H^2\varphi_0\left(\frac{x}{\|x\|_H}\right) \le \left(\frac{c}{\theta} + MT\right)\|x\|_H^2.$$

Let $c' = \frac{c}{\theta} + MT$. Then $\varphi_0(x) \le c'\|x\|_H^2$ for $\forall x \in H$. Let

$$T(x,y) = \int_0^T E\langle Z_0^x(s), Z_0^y(s)\rangle\,ds$$

for $\forall x, y \in H$. Then T is a bilinear form on H, and by using the Schwartz inequality, we get

$$\begin{aligned}
|T(x,y)| =& \left|\int_0^T E\langle Z_0^x(s), Z_0^y(s)\rangle\,ds\right| \\
\le& \int_0^T (E\|Z_0^x(s)\|_H^2)^{1/2}(E\|Z_0^y(s)\|_H^2)^{1/2}\,ds \\
\le& \left(\int_0^T E\|Z_0^x(s)\|_H^2\,ds\right)^{1/2}\left(\int_0^T E\|Z_0^y(s)\|_H^2\,ds\right)^{1/2} \\
=& \varphi_0(x)^{1/2}\varphi_0(y)^{1/2} \le c'\|x\|_H\|y\|_H.
\end{aligned}$$

Hence there exists a continuous linear operator $C \in L(H,H)$ (Yosida [45]), such that $T(x,y) = \langle Cx, y\rangle$, and $\|C\| = \sup_{\|x\|_H=1,\|y\|_H=1} |\langle Cx,y\rangle| \le c'$. Since $\varphi_0(x) = T(x,x) = \langle Cx, x\rangle$. So $\varphi_0'(x) = 2Cx$ and $\varphi_0''(x) = 2C$. Hence $\varphi_0 \in C_b^{2,loc}(H)$ and $\psi_0(\varphi) \in C_b^{2,loc}(H)$. By (5.22) and the fact that $\psi_0(\varphi) \ge w\|\varphi\|_H^2$, (5.18) is satisfied. By the continuity of $t \to E\|Z_0^\varphi(t)\|_H^2$, we get

$$\mathcal{L}_0\varphi_0(\varphi) = \frac{d}{dr}(E\varphi_0(Z_0^\varphi(r)))\Big|_{r=0} = \lim_{r\to 0}\frac{E\varphi_0(Z_0^\varphi(r)) - E\varphi_0(\varphi)}{r}$$

$$
\begin{aligned}
&= \lim_{r\to 0}\left(-\frac{1}{r}\int_0^r E\|Z_0^\varphi(s)\|_H^2\,ds + \frac{1}{r}\int_T^{r+T} E\|Z_0^\varphi(s)\|_H^2\,ds\right)\\
&= -\ \|\varphi\|_H^2 + E\|Z_0^\varphi(T)\|_H^2 \le -\|\varphi\|_H^2 + ce^{-\theta T}\|\varphi\|_H^2 + M\\
&\le (-1 + ce^{-\theta T})\|\varphi\|_H^2 + M.
\end{aligned}
$$

Therefore using (5.10),

$$
\begin{aligned}
\mathcal{L}_0\psi_0(\varphi) =& \mathcal{L}_0\varphi_0(\varphi) + 2\mathcal{L}_0\|\varphi\|_H^2\\
&\le (-1 + ce^{-\theta T})\|\varphi\|_H^2 + w(2\lambda + d)\|\varphi\|_H^2 + M. \qquad (5.23)
\end{aligned}
$$

Therefore, if $T > \ln\frac{c}{\theta}$, then we can choose α small enough such that $\psi_0(\varphi)$ satisfies (5.19) with $\mathcal{L}$ replaced by $\mathcal{L}_0$. □

For the solution of the nonlinear equation (3.1), if $\psi(\varphi) = \int_0^T E\|Z^\varphi(s)\|_H^2\,ds + w\|\varphi\|_H^2$ is in $C_b^{2,loc}(H)$, we can follow the proof of Theorem 5.4.2 and Theorem 5.2.3 and have the following result.

Theorem 5.4.3. *If the solution $Z^\varphi(t)$ of (3.1) is exponentially ultimate bounded in the m.s.s., and $\phi(\varphi) = \int_0^T E\|Z^\varphi(t)\|_H^2\,dt$ is in $C_b^{2,loc}(H)$ for some $T > 0$, then there exists a Lyapunov function for $Z^\varphi(t)$ which satisfies (5.18) and (5.19) .*

Now, we use the first order approximation to study the properties of exponentially ultimate bounded in the m.s.s. of the solution of the nonlinear equation based on the same property of the solution of the linear equation.

Theorem 5.4.4. *Suppose the solution $Z_0^\varphi(t)$ of (5.7) is exponentially ultimately bounded in the m.s.s., and it satisfies (5.16) . Then the solution $Z^\varphi(t)$ of the equation*

(3.1) is exponentially ultimate bounded in the m.s.s. if

$$2\|z\|_H\|F(z)\|_H + \int_E \|B(v,z) - B_0(v)z\|_H\|B(v,z) + B_0(v)z\|_H \beta(dv) < W\|z\|_H^2 + M_1 \tag{5.24}$$

for any constant M_1 and

$$W < \max_{s > \ln\frac{c}{\theta}} \frac{1 - ce^{-\theta s}}{\frac{c}{\theta} + Ms}. \tag{5.25}$$

Proof. Let $\psi_0(z)$ be the Lyapunov function as defined in (5.21) with $T > \ln\frac{c}{\theta}$ such that (5.25) attains its maximum at T. We just need to show that $\psi_0(x)$ satisfies (5.19). Since $\psi_0(z) = (Cz, z) + w\|z\|_H^2$ for some $C \in L(H, H)$ with $\|C\| \leq \frac{c}{\theta} + MT$ and w very small, following (5.14), we have

$$\begin{aligned}
&\mathcal{L}\psi_0(z) - \mathcal{L}_0\psi_0(z) \\
\leq&\|C\| + w)\Big(2\|z\|_H\|F(z)\|_H + \int_E \|B(v,z) - B_0(v)z\|_H\|B(v,z) + B_0(v)z\|_H \beta(dv)\Big) \\
\leq&\Big(\frac{c}{\theta} + MT + w\Big)\Big(W\|z\|_H^2 + M_1\Big).
\end{aligned}$$

Using (5.23)

$$\begin{aligned}
\mathcal{L}\psi_0(z) \leq&(-1 + ce^{-\theta T})\|z\|_H^2 + w(2\lambda + d)\|z\|_H^2 \\
&+ M + \left(\frac{c}{\theta} + MT + w\right)(W\|z\|_H^2 + M_1) \\
=&\left(-1 + ce^{-\theta T} + W\left(\frac{c}{\theta} + MT\right)\right)\|z\|_H^2 + w(2\lambda + d + W)\|z\|_H^2 \\
&+ M + \left(\frac{c}{\theta} + MT + w\right)M_1.
\end{aligned}$$

Since W satisfies (5.25), $-1 + ce^{-\theta T} + W(\frac{c}{\theta} + MT) < 0$, and hence we can choose w small enough such that (5.19) is satisfied. $\square$

Corollary 5.4.2. *Suppose the solution $Z_0^{\varphi}(t)$ of equation (5.7) is exponentially ultimate bounded in the m.s.s., if as $\|\varphi\|_H \to \infty$,*

$$\|F(\varphi)\|_H = o(\|\varphi\|_H)$$

$$\int_E \|B(v,\varphi) - B_0(v)\varphi\|_H \|B(v,\varphi) + B_0(v)\varphi\|_H \,\beta(dv) = o(\|\varphi\|_H^2).$$

Then the solution $Z^{\varphi}(t)$ of (3.1) is exponentially ultimately bounded in the m.s.s..

Proof. Since as $\|\varphi\|_H \to \infty$, $\|F(\varphi)\|_H = o(\|\varphi\|_H)$ and

$$\int_E \|B(v,\varphi) - B_0(v)\varphi\|_H \|B(v,\varphi) + B_0(v)\varphi\|_H \,\beta(dv) = o(\|\varphi\|_H^2),$$

for any fixed W satisfying (5.25), there exists an $K > 0$, such that

$$2\|\varphi\|_H \|F(\varphi)\|_H + \int_E \|B(v,\varphi) - B_0(v)\varphi\|_H \|B(v,\varphi) + B_0(v)\varphi\|_H \,\beta(dv) \le W\|\varphi\|_H^2$$

for all $\|\varphi\|_H \ge K$.

But for $\|\varphi\|_H \le K$, by the Lipschitz condition,

$$\begin{aligned}
&2\|\varphi\|_H \|F(\varphi)\|_H + \int_E \|B(v,\varphi) - B_0(v)\varphi\|_H \|B(v,\varphi) + B_0(v)\varphi\|_H \,\beta(dv) \\
\le& \|\varphi\|_H^2 + \|F(\varphi)\|_H^2 + \int_E (\|B(v,\varphi)\|_H + \|B_0(v)\varphi\|_H)^2 \,\beta(dv) \\
\le& \|\varphi\|_H^2 + l(1 + \|\varphi\|_H^2) + 2l(1 + \|\varphi\|_H^2) \\
\le& K^2 + 3l(1 + K^2).
\end{aligned}$$

Therefore

$$\begin{aligned}
&2\|\varphi\|_H \|F(\varphi)\|_H + \int_E \|B(v,\varphi) - B_0(v)\varphi\|_H \|B(v,\varphi) + B_0(v)\varphi\|_H \,\beta(dv) \\
\le& W\|\varphi\|_H^2 + K^2 + 3l(1 + K^2).
\end{aligned}$$

The assertion follows from Theorem 5.4.4. $\square$

Corollary 5.4.3. *Consider the following system*

$$\begin{aligned} du(t) =& Au(t)\,dt + F(u(t))\,dt + \int_E B(u(t),v)\,\widetilde{N}(dv\,dt) \\ u(0) =& \varphi. \end{aligned}$$

Suppose F and B satisfy Lipschitz and growth condition defined as before and A is an infinitesimal generator of a C_0-semigroup. Then if the solution $\{\, u^{\varphi}(t), t \geq 0 \,\}$ of

$$\begin{aligned} du(t) =& Au(t)dt \\ u(0) =& \varphi \end{aligned}$$

is exponentially stable (or even exponentially ultimately bounded), as $\|\varphi\|_H \to \infty$, $\|F(\varphi)\|_H = o(\|\varphi\|_H)$ and

$$\int_E \|B(v,\varphi)\|_H^2\,\beta(dv) = o(\|\varphi\|_H^2).$$

Then the solution of the above equation is exponentially ultimately bounded in the m.s.s..

Proof. This follows exactly from Corollary 5.4.2. $\square$

Example 5.4.1. *Let us consider the system,*

$$\begin{aligned} du(t) =& -\lambda u(t)dt + \frac{1}{1+|u(t)|}\,dt + \int_R v\,\widetilde{N}(dv\,dt), \\ u(0) =& \varphi, \end{aligned}$$

where $u(t)$ is real valued. By the above argument, we know $u(t)$ is exponentially ultimate bounded in the m.s.s..

Chapter 6

Invariant measures

We will continue to study the properties of the solution in this chapter. The conditions for the existence and uniqueness of an invariant measure associated to the solution are given, and finally an example is given to illustrate our theory.

6.1 Introduction

Let H be a real, separable Hilbert space defined before and $\mathcal{B}(H)$ be the Borel σ-algebra. Let $Z(t)$ be a Markov process with transition probability $P(t, y, B), t \geq 0, y \in H, B \in \mathcal{B}(H)$. We define $T(t)\colon M_b(H) \to M_b(H)$, its semigroup, by

$$[T(t)h](y) = \int_H h(z)P(t, y, dz), h \in M_b(H),$$

where $M_b(H)$ is the space of bounded measurable functions on H.

Definition 6.1.1. Let $C_b(H)$ ($C_w(H)$) be the space of bounded continuous (weakly continuous) functions on H. The semigroup $T(t)$ (or the Markov process $Z(t)$) is said to be *Feller* (*w-Feller*) if $T(t)C_b(H) \subset C_b(H)$ ($T(t)C_w(H) \subset C_w(H)$) for $t \geq 0$.

Definition 6.1.2. A sequence of probability measures μ_n on $\mathcal{B}(H)$ is said to be *weakly (w-weakly) convergent* to a probability measure μ if for any $h \in C_b(H)$ $(C_w(H))$

$$\lim_{n \to \infty} \int_H h(y)\, \mu_n(dy) = \int_H h(y)\, \mu(dy).$$

Definition 6.1.3. The set M of probability measures on $\mathcal{B}(H)$ is *weakly (w-weakly) compact*, if from any sequence of probability measures in M, a weakly (w-weakly) convergent subsequence can be extracted.

Definition 6.1.4. Let μ be a σ-finite measure and let $\mu T_t(A) = \int_H P(t, y, A)\, \mu(dy)$. Then μ is said to be an *invariant measure* associated to the Markov process if $\mu T_t = \mu$ for all $t \geq 0$.

6.2 Existence and uniqueness of an invariant measure

We first recall some known results.

Theorem 6.2.1. *The set M of probability measures on $\mathcal{B}(H)$ is weakly compact iff for each $\varepsilon > 0$, there exists a compact set $K \subset H$ such that* $\sup\{\, \mu(H \backslash K); \mu \in M \,\} < \varepsilon$.

Remark. This is Y. V. Prokhorov's theorem (Billingsley [2]).

Theorem 6.2.2. *The set M of probability measures on $\mathcal{B}(H)$ is weakly compact iff two conditions below hold.*

i) For any $\varepsilon > 0$, there exists $c > 0$ such that $\mu\{\, y : \|y\|_H > c \,\} < \varepsilon$ for all $\mu \in M$.

ii) The series $\sum_{k=1}^{\infty} |B_\mu^c e_k|^2$ is uniformly convergent in μ for each $c > 0$ in some orthonormal basis $\{ e_k \}$, where $|B_\mu^c z|^2 = \int_{\|y\|_H \le c} \langle z, y\rangle^2 \, \mu(dy)$ for $z \in H$.

Remark. This is a result from Gikhman and Skorokhod's book ([11]).

Theorem 6.2.3. *The set M of probability measures on $\mathcal{B}(H)$ is w-weakly compact if for each $\varepsilon > 0$, there exists a weakly compact set $K \subset H$ such that* $\sup\{ \mu(H \backslash K); \mu \in M \} < \varepsilon$.

Remark. This is Y. V. Prokhorov's theorem under the weak topology.

The following lemma is also known (Ichikawa [15]). It will be used often in what follows. We give the proof here.

Lemma 6.2.1. *Let $p > 1$ and g be a nonnegative locally p-integrable function on $[0, +\infty)$. Then for each $\varepsilon > 0$ and real d*

$$\left(\int_0^t e^{d(t-r)} g(r)\, dr \right)^p \le C(\varepsilon, p) \int_0^t e^{p(d+\varepsilon)(t-r)} g^p(r)\, dr,$$

for t large enough, where $C(\varepsilon, p) = (1 + q\varepsilon)^{p/q}$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Proof. First, we use Hölder's inequality to get

$$\begin{aligned} \int_0^t e^{d(t-r)} g(r)\, dr &= \int_0^t [e^{(d+\varepsilon)(t-r)} g(r)][e^{-\varepsilon(t-r)}]\, dr \\ &\le \left[\int_0^t [e^{(d+\varepsilon)(t-r)} g(r)]^p \, dr \right]^{\frac{1}{p}} \left[\int_0^t [e^{-\varepsilon(t-r)}]^q \, dr \right]^{\frac{1}{q}}. \end{aligned}$$

So we have

$$\left[\int_0^t e^{d(t-r)} g(r)\, dr \right]^p \le \int_0^t e^{p(d+\varepsilon)(t-r)} g^p(r)\, dr \left[\int_0^t [e^{-\varepsilon(t-r)}]^q \, dr \right]^{\frac{p}{q}}.$$

Observe that

$$\int_0^t [e^{-\varepsilon(t-r)}]^q \, dr = \int_0^t e^{-\varepsilon q(t-r)} \, dr = \frac{1}{\varepsilon q}(1 - e^{-\varepsilon qt}).$$

So for any given $\varepsilon > 0, q > 1$, we need to prove

$$\frac{1}{\varepsilon q}(1 - e^{-\varepsilon qt}) \leq 1 + q\varepsilon,$$

or we need to prove that

$$e^{-\varepsilon qt} + q\varepsilon + q^2\varepsilon^2 - 1 \geq 0.$$

Because when $\varepsilon = 0, e^{-\varepsilon qt} + q\varepsilon + q^2\varepsilon^2 - 1 = 0$, and

$$\frac{d}{d\varepsilon}(e^{-\varepsilon qt} + q\varepsilon + q^2\varepsilon^2 - 1) = -qte^{-\varepsilon qt} + q + 2q^2\varepsilon = q(-te^{-\varepsilon qt} + 1 + 2q\varepsilon),$$

when t is large enough, we have $q(-te^{-\varepsilon qt} + 1 + 2q\varepsilon) > 0$, and so $\frac{1}{\varepsilon q}(1 - e^{-\varepsilon qt}) \leq 1 + q\varepsilon$. □

The following lemma is from Liu and Mandrekar [25].

Lemma 6.2.2. *Suppose $Z^\varphi(t)$ is ultimately bounded. Then for any invariant measure m of the Markov process $Z^\varphi(t)$, we have*

$$\int_H \|y\|_H^2 \, m(dy) \leq K' < \infty.$$

Proof. Put $f(x) = \|x\|_H^2$ and $f_n(x) = I_{[0,n]}(f(x))f(x)$, where I is a characteristic function. We note that $f_n(x) \in L^1(H, m)$. From the assumption of ultimate boundedness, there is a constant K such that $\overline{\lim}_{t\to\infty} E\|Z^\varphi(t)\|_H^2 \leq K$ for any $\varphi \in H$. By the ergodic theorem for Markov process with invariant measure (Yosida [45]), the limit

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T P_t f_n(x) \, dt = f_n^*(x) \ (m - a.e.)$$

exists and $E_m f_n^* = E_m f_n$, where

$$P_t f_n(x) = \int_H f_n(y) P(t, x, dy),$$

and $E_m f_n = \int_H f_n(x)\, m(dx)$. From the inequality $f_n(x) \le f(x)$ and the assumption of ultimate boundedness, we have

$$\begin{aligned} f_n^*(x) &= \lim_{T\to\infty} \frac{1}{T}\int_0^T P_t f_n(x)\, dt \le \varlimsup_{T\to\infty} \frac{1}{T}\int_0^T P_t f(x)\, dt \\ &= \varlimsup_{T\to\infty} \frac{1}{T}\int_0^T E\|Z^{\varphi}(t)\|_H^2\, dt \le K'. \end{aligned}$$

Also from $f_n(x) \uparrow f(x)$ as $n \to \infty$ implies that

$$E_m f = \lim_{n\to\infty} E_m f_n = \lim_{n\to\infty} E_m f_n^* \le K'. \qquad \square$$

Now, we examine the uniqueness of invariant measures. Let $B_R = \{\, y : \|y\|_H \le R \,\}$. The following theorem is from Ichikawa [15].

Theorem 6.2.4. *Suppose $Z^{\varphi}(t)$ is exponentially ultimately bounded and for each $R > 0, \delta > 0$ and $\varepsilon > 0$, there exists $T_0 = T_0(R, \delta, \varepsilon) > 0$ such that*

$$P\{\, \|Z^{\varphi_0}(t) - Z^{\varphi_1}(t)\|_H > \delta \,\} < \varepsilon \text{ for any } \varphi_0, \varphi_1 \in B_R \text{ whenever } t \ge T_0. \tag{6.1}$$

Then there exists at most one invariant measure.

Proof. Let m_i, $i = 0, 1$, be invariant measures. By Lemma 6.2.2, for each $\varepsilon > 0$, there exists an $R > 0$ such that $m_i(H \backslash B_R) < \varepsilon$. Let $h \in C_w(H)$. Then there exists $T = T(\varepsilon, R, h)$ such that $|[T(t)h](\varphi_0) - [T(t)h](\varphi_1)| \le \varepsilon$, for $\varphi_0, \varphi_1 \in B_R$, if $t \ge T$. Now we prove this statement. Let K be a weakly compact set in H. Recall that the weak topology on K is equivalent to the topology defined by the metric

$$\rho(y, z) = \sum_{k=1}^{\infty} \frac{1}{2^k} |\langle e_k, (y - z)\rangle|, y, z \in K$$

for any fundamental subset $\{ e_k \}$ of H; namely, the closure of the linear span of $\{ e_k \}$ is H. First we shall show that for each $\eta > 0$ and $\varepsilon > 0$, there exists a T_2, such that $t \geq T_2$ implies

$$P\{ |h(Z^{\varphi_0}(t)) - h(Z^{\varphi_1}(t))| \leq \eta \} \geq 1 - \varepsilon$$

for all $\varphi_0, \varphi_1 \in B_R$. By exponential ultimate boundedness, we know there exists T_1 such that $t \geq T_1$ implies $P[Z^{\varphi_0}(t) \in B_R] \geq 1 - \varepsilon/3$ for any $\varphi_0 \in B_R$. Note that h on B_R is uniformly continuous with respect to the metric ρ. Hence there exists a $\delta > 0$, such that $y, z \in B_R$ and $\rho(y, z) \leq \delta$ implies $|h(y) - h(z)| \leq y$. Note also that there exists an integer J such that

$$\sum_{k=J+1}^{\infty} \frac{1}{2^k} |\langle e_k, y - z\rangle| \leq \frac{\delta}{2}, \text{ for all } y, z \in B_R.$$

Now choose $T_2 \geq T_1$ such that $t \geq T_2$ implies

$$\sum_{k=1}^{J} P[|\langle e_k, Z^{\varphi_0}(t) - Z^{\varphi_1}(t)\rangle| \leq \delta/2] \geq 1 - \varepsilon/3$$

for all φ_0, $\varphi_1 \in B_R$. This is possible by (6.1). Then for $t \geq T_2$, we have

$$\begin{aligned}
&P\{|h(Z^{\varphi_0}(t)) - h(Z^{\varphi_1}(t))| \leq \eta\} \\
\geq &P\{Z^{\varphi_0}(t), Z^{\varphi_1}(t) \in B_R, \rho(Z^{\varphi_0}(t), Z^{\varphi_1}(t)) \leq \delta\} \\
\geq &P\left\{Z^{\varphi_0}(t), Z^{\varphi_1}(t) \in B_R, \sum_{k=1}^{J} \frac{1}{2^k} |\langle e_k, Z^{\varphi_0}(t) - Z^{\varphi_1}(t)\rangle| \leq \delta/2\right\} \\
\geq &P\{Z^{\varphi_0}(t), Z^{\varphi_1}(t) \in B_R, |\langle e_k, Z^{\varphi_0}(t) - Z^{\varphi_1}(t)\rangle| \leq \delta/2, k = 1,\ 2, \ldots, J\} \\
\geq &1 - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} = 1 - \varepsilon.
\end{aligned}$$

Now for any given ε, choose T such that $t \geq T$ implies

$$P\{ |h(Z^{\varphi_0}(t)) - h(Z^{\varphi_1}(t))| \leq \varepsilon/2 \} \geq 1 - \frac{\varepsilon}{4K_0},$$

where $K_0 = \sup|h(y)| < \infty$. Then

$$E|h(Z^{\varphi_0}(t)) - h(Z^{\varphi_1}(t))| \leq \frac{\varepsilon}{2} + 2K_0\Big(\frac{\varepsilon}{4K_0}\Big) = \varepsilon.$$

Note that

$$\int_H h(y)\, m_i(dy) = \int_H [T(t)h](y)\, m_i(dy) \text{ for } i = 0,\ 1.$$

Consider

$$\begin{aligned}
&\Big|\int_H h(y_0)\, m_0(dy_0) - \int_H h(y_1)\, m_1(dy_1)\Big| \\
=&\Big|\int_H\int_H [h(y_0) - h(y_1)]\, m_0(dy_0)\, m_1(dy_1)\Big| \\
=&\Big|\int_H\int_H \Big[[T(t)h](y_0) - [T(t)h](y_1)\Big]\, m_0(dy_0)\, m_1(dy_1)\Big| \\
\leq&\int_H\int_H \Big|[T(t)h](y_0) - [T(t)h](y_1)\Big|\, m_0(dy_0)\, m_1(dy_1) \\
=&\Big(\int_{B_R} + \int_{H\backslash B_R}\Big)\Big(\int_{B_R} + \int_{H\backslash B_R}\Big)\Big|[T(t)h](y_0) - [T(t)h](y_1)\Big|\, m_0\,(dy_0)\; m_1\,(dy_1) \\
\leq&\varepsilon + 2(2K_0)\varepsilon + 2K_0\varepsilon^2, \text{ if } t \geq T,
\end{aligned}$$

where $K_0 = \sup|h(y)| < \infty$. Since ε is arbitrary, we conclude $\int_H h(y)\, m_0(dy) = \int_H h(y)\, m_1(dy)$, which implies $m_0 = m_1$. □

The following Proposition (6.2.1) gives a sufficient condition for (6.1) holds.

Remark. The condition $\langle Ay, y\rangle \leq \alpha\|y\|_H^2$ for $y \in \mathcal{D}(A)$ is equivalent to $|S(t)| \leq e^{\alpha t}$, α is real (Ichikawa [17]).

Proposition 6.2.1. *Suppose that $\langle y, Ay\rangle \leq -c_0\|y\|_H^2$, $y \in \mathcal{D}(A)$, and c_0 is the maximum value satisfying the above inequality. Also suppose k is the minimum value satisfy Lipschitz condition. Then if $a = c_0 - 3k > 0$, we have $E\|Z^{\varphi_0}(t) - Z^{\varphi_1}(t)\|_H^2 \leq e^{-2at}\|\varphi_0 - \varphi_1\|_H^2$, for t large enough.*

Proof. Let $Z^{\varphi_1}(t)$ and $Z^{\varphi_0}(t)$ be two solutions. Then as in Lemma 3.3.1, we have,

$$\begin{aligned}
&Z^{\varphi_0}(t) - Z^{\varphi_1}(t) \\
=&S(t)(\varphi_0 - \varphi_1) + \int_0^t S(t-s)[F(Z^{\varphi_0}(s)) - F(Z^{\varphi_1}(s))]\, ds \\
&+ \int_0^t \int_E S(t-s)[B(v, Z^{\varphi_0}(s)) - B(v, Z^{\varphi_1}(s))]\, \widetilde{N}(dv\, ds).
\end{aligned}$$

So

$$\begin{aligned}
&\|Z^{\varphi_0}(t) - Z^{\varphi_1}(t)\|_H^2 \\
\le& 3\|S(t)(\varphi_0 - \varphi_1)\|_H^2 + 3\Big\| \int_0^t S(t-s)[F(Z^{\varphi_0}(s)) - F(Z^{\varphi_1}(s))]\, ds \Big\|_H^2 \\
&+ 3\Big\| \int_0^t \int_E S(t-s)[B(v, Z^{\varphi_0}(s)) - B(v, Z^{\varphi_1}(s))]\, \widetilde{N}(dv\, ds)\Big\|_H^2.
\end{aligned}$$

So

$$\begin{aligned}
&E\|Z^{\varphi_0}(t) - Z^{\varphi_1}(t)\|_H^2 \\
\le& 3e^{-2c_0 t}\|\varphi_0 - \varphi_1\|_H^2 + 3E\Big| \int_0^t \|S(t-s)[F(Z^{\varphi_0}(s)) - F(Z^{\varphi_1}(s))]\|_H\, ds \Big|^2 \\
&+ 3\int_0^t \int_E E\|B(v, Z^{\varphi_0}(s)) - B(v, Z^{\varphi_1}(s))\|_H^2\, \beta(dv)\, ds \\
\le& 3e^{-2c_0 t}\|\varphi_0 - \varphi_1\|_H^2 + 3kE\Big(\int_0^t e^{-c_0(t-s)}\|Z^{\varphi_0}(s) - Z^{\varphi_1}(s)\|_H\, ds\Big)^2 \\
&+ 3\int_0^t kE\|Z^{\varphi_0}(s) - Z^{\varphi_1}(s)\|_H^2\, ds \\
\le& 3e^{-2c_0 t}\|\varphi_0 - \varphi_1\|_H^2 + 3k(1+2\epsilon)\int_0^t e^{2(-c_0+\epsilon)(t-s)} E\|Z^{\varphi_0}(s) - Z^{\varphi_1}(s)\|_H^2\, ds \\
&+ 3k\int_0^t E\|Z^{\varphi_0}(s) - Z^{\varphi_1}(s)\|_H^2\, ds \text{ (by Lemma 6.2.1, } \epsilon \text{ is small positive)} \\
\le& 3e^{-2c_0 t}\|\varphi_0 - \varphi_1\|_H^2 + 3k\int_0^t E\|Z^{\varphi_0}(s) - Z^{\varphi_1}(s)\|_H^2\, ds \\
&+ 3k\int_0^t E\|Z^{\varphi_0}(s) - Z^{\varphi_1}(s)\|_H^2\, ds.
\end{aligned}$$

Letting $\epsilon \to 0$ and $e^{2(-2c_0 + \epsilon)(t-s)} < 1$, we have

$$E\|Z^{\varphi_0}(t) - Z^{\varphi_1}(t)\|_H^2 \le 3e^{-2c_0 t}\|\varphi_0 - \varphi_1\|_H^2 + 6k\int_0^t E\|Z^{\varphi_0}(s) - Z^{\varphi_1}(s)\|_H^2\, ds.$$

So By Gronwall's inequality, we have

$$E\|Z^{\varphi_0}(t) - Z^{\varphi_1}(t)\|_H^2 \le 3e^{-2c_0 t}\|\varphi_0 - \varphi_1\|_H^2\, e^{6kt} \le 3e^{(-2c_0 + 6k)t}\|\varphi_0 - \varphi_1\|_H^2. \quad \square$$

We consider H with weak topology.

Theorem 6.2.5. *Suppose $T(t)$ is w-Feller and that*

$$\frac{1}{t}\int_0^t E\|Z^{y_0}(r)\|_H^2\, dr \le M(1 + \|y_0\|_H^2),\ M > 0 \text{ and for any } t \ge t_0 > 0. \tag{6.2}$$

Then there exists an invariant measure.

Proof. For integers $n \ge t_0$, define $m_n(B) = \frac{1}{n}\int_0^n P(r, y_0, B)\, dr,\ B \in \mathcal{B}(H)$. Then m_n is a probability measure and $\int_H \|y\|_H^2\, m_n(dy) \le M(1 + \|y_0\|_H^2)$. Hence for each $\varepsilon > 0$, there exists $R > 0$, such that $m_n(B_R) > 1 - \varepsilon, B_R = \{y : \|y\|_H \le R\}$.

By Theorem 6.2.3, $\{m_n\}$, $n \ge t_0$ is w-weakly compact and there exists a subsequence, again denoted by m_n, which is w-weakly convergent to some probability measure $m_0(\cdot)$. Let $h \in C_w(H)$ be arbitrary. Then

$$\begin{aligned}
\int_H [T(t)h](y)\, m_0(dy) &= \lim_{n\to\infty}\int_H [T(t)h](y)\, m_n(dy)(\text{ since } T(t) \text{ is } w\text{-Feller}) \\
&= \lim_{n\to\infty}\left(\frac{1}{n}\right)\int_0^n [T(t+r)h](y_0)\, dr \\
&= \lim_{n\to\infty}\left(\frac{1}{n}\right)\int_t^{t+n} [T(r)h](y_0)\, dr \\
&= \lim_{n\to\infty}\left(\frac{1}{n}\right)\int_0^n [T(r)h](y_0)\, dr\ (\text{ since } T(t)h \text{ is bounded}) \\
&= \int_H h(y)\, m_0(dy).
\end{aligned}$$

which implies m_0 is an invariant measure of $P(t, y, B)$. $\square$

Now, we drop the assumption that $T(t)$ is w-Feller. Assume instead that A is self-adjoint and has eigenvectors $\{ e_k \}$, $k = 1, 2, \ldots$, which form an orthonormal basis for H and eigenvalues

$$-\lambda_k \downarrow -\infty \text{ as k} \to \infty. \tag{6.3}$$

Then the semigroup $S(t)$ has the representation,

$$S(t)y = \sum_{k=1}^{\infty} e^{-\lambda_k t} \langle e_k, y \rangle e_k, \quad y \in H.$$

We have the following result.

Lemma 6.2.3. *Assume (6.2) and (6.3) hold. Then* $m_t(\cdot) = \frac{1}{t}\int_0^t P(r, \varphi_0, \cdot)\, dr$ *is weakly compact for* $t \geq t_0 > 0$.

Proof. In view of Theorem 6.2.2, it is sufficient to show $\frac{1}{T}\int_0^T \sum_{k=1}^{\infty} E[Z_k^{\varphi_0}]^2(t)\, dt$ is uniformly convergent in $T \geq t_0$, where $Z_k^{\varphi_0}(t) = \langle Z^{\varphi_0}(t), e_k \rangle$ and $\{ e_k \}$ is the orthonormal basis given in (6.3). We have

$$\begin{aligned} Z_k^{\varphi_0}(t) =& e^{-\lambda_k t}\varphi_{k0} + \int_0^t e^{-\lambda_k (t - r)} \langle e_k, F(Z^{\varphi_0}(r))\rangle\, dr \\ &+ \int_0^t \int_E e^{-\lambda_k (t - r)} \langle e_k, B(v, Z^{\varphi_0}(r))\rangle\, \tilde{N}(dv\, dr), \end{aligned}$$

where $\varphi_{k0} = \langle e_k, \varphi_0 \rangle$. Hence

$$\begin{aligned} E[Z_k^{\varphi_0}(t)]^2 \leq& 3e^{-2\lambda_k t}\varphi_{k0}^2 + 3E\Big| \int_0^t e^{-\lambda_k (t - r)} \langle e_k, F(Z^{\varphi_0}(r))\rangle\, dr\Big|^2 \\ &+ 3E\Big| \int_0^t \int_E e^{-\lambda_k (t - r)} \langle e_k, B(v, Z^{\varphi_0}(r))\rangle\, \tilde{N}(dv\, dr)\Big|^2. \end{aligned} \tag{6.4}$$

Now let K be large enough, so that $\lambda_K > 0$. By using Lemma 6.2.1, for any integer $m > 0$ we can show,

$$\sum_{k=K}^{K+m} \frac{1}{T}\int_0^T e^{-2\lambda_k t}\varphi_{k0}^2\, dt \leq \frac{1}{T}\int_0^T e^{-2\lambda_K t}\|\varphi_0\|_H^2\, dt \leq \frac{\|\varphi_0\|_H^2}{2T\lambda_K}.$$

For the second term of the right hand side of the inequality (6.4), one has

$$\begin{aligned}
&\sum_{k=K}^{K+m} \frac{1}{T}\int_0^T E\Big|\int_0^t e^{-\lambda_k(t-r)}\langle e_k, F(Z^{\varphi_0}(r))\rangle\, dr\Big|^2\, dt \\
\leq &\sum_{k=K}^{K+m} \frac{1}{T}\int_0^T E\Big(\int_0^t e^{-\lambda_k(t-r)}|\langle e_k, F(Z^{\varphi_0}(r))\rangle|\, dr\Big)^2\, dt,
\end{aligned}$$

which by Lemma 6.2.1, and $0 < \delta < \lambda_K$ is,

$$\begin{aligned}
\leq &\sum_{k=K}^{K+m} \frac{1}{T}\int_0^T E(1+2\delta)\int_0^t e^{2(-\lambda_k+\delta)(t-r)}|\langle e_k, F(Z^{\varphi_0}(r))\rangle|^2\, dr\, dt \\
\leq &\sum_{k=K}^{K+m} \frac{1}{T}\int_0^T E(1+2\delta)\int_0^t e^{2(-\lambda_K+\delta)(t-r)}|\langle e_k, F(Z^{\varphi_0}(r))\rangle|^2\, dr\, dt.
\end{aligned}$$

Since $\lambda_k \uparrow \infty$ as $k \to \infty$, the second term of (6.4) is

$$\begin{aligned}
\leq &\frac{1}{T}\int_0^T (1+2\delta)\int_0^t e^{2(-\lambda_K+\delta)(t-r)} E\sum_{k=K}^{K+m} |\langle e_k, F(Z^{\varphi_0}(r))\rangle|^2\, dr\, dt \\
\leq &\frac{(1+2\delta)}{T}\int_0^T\int_0^t e^{-2(\lambda_K-\delta)(t-r)} E\|F(Z^{\varphi_0}(r))\|_H^2\, dr\, dt,
\end{aligned}$$

which, by Fubini theorem, is

$$\begin{aligned}
\leq &\frac{(1+2\delta)}{T}\int_0^T\int_r^T e^{-2(\lambda_K-\delta)(t-r)} E\|F(Z^{\varphi_0}(r))\|_H^2\, dt\, dr \\
= &\frac{(1+2\delta)}{2(\lambda_K-\delta)T}\int_0^T (1-e^{-2(\lambda_K-\delta)(T-r)}) E\|F(Z^{\varphi_0}(r))\|_H^2\, dr \\
\leq &\frac{(1+2\delta)}{2(\lambda_K-\delta)T}\int_0^T E\|F(Z^{\varphi_0}(r))\|_H^2\, dr,
\end{aligned}$$

and by condition (A2) and (6.2),

$$\begin{aligned}
\leq &\frac{(1+2\delta)}{2(\lambda_K-\delta)T}\int_0^T El(1+\|Z^{\varphi_0}(r)\|_H^2)\, dr \\
\leq &\frac{(1+2\delta)l}{2(\lambda_K-\delta)}\Big(1+\frac{1}{T}\int_0^T E\|Z^{\varphi_0}(r)\|_H^2\, dr\Big) \\
\leq &\frac{(1+2\delta)l}{2(\lambda_K-\delta)}(1+M(1+\|\varphi_0\|_H^2)).
\end{aligned}$$

For the third term of the right hand side of the inequality (6.4), one has,

$$\begin{aligned}
&\sum_{k=K}^{K+m} \frac{1}{T}\int_0^T E\left|\int_0^t\int_E e^{-\lambda_k(t-r)}\langle e_k, B(v, Z^{\varphi_0}(r))\rangle\, \tilde{N}(dv\,dr)\right|^2 dt\\
\le&\sum_{k=K}^{K+m} \frac{1}{T}\int_0^T \left[\int_0^t\int_E Ee^{-2\lambda_k(t-r)}|\langle e_k, B(v, Z^{\varphi_0}(r))\rangle|^2\,\beta(dv)\,dr\right] dt\\
\le&\sum_{k=K}^{K+m} \frac{1}{T}\int_0^T \left[\int_0^t\int_E Ee^{-2\lambda_K(t-r)}|\langle e_k, B(v, Z^{\varphi_0}(r))\rangle|^2\,\beta(dv)\,dr\right] dt\\
\le&\frac{1}{T}\int_0^T \left[\int_0^t\int_E e^{-2\lambda_K(t-r)}E\|B(v, Z^{\varphi_0}(r))\|_H^2\,\beta(dv)\,dr\right] dt,
\end{aligned}$$

and by condition (A2), Fubini theorem, and (6.2),

$$\begin{aligned}
\le&\frac{1}{T}\int_0^T\int_0^t le^{-2\lambda_K(t-r)}(1+E\|Z^{\varphi_0}(r)\|_H^2)\,dr\,dt\\
\le&\frac{1}{T}\int_0^T\int_r^T le^{-2\lambda_K(t-r)}(1+E\|Z^{\varphi_0}(r)\|_H^2)\,dt\,dr\\
\le&\frac{l}{T}\int_0^T \frac{1}{2\lambda_K}(1-e^{-2\lambda_K(T-r)})(1+E\|Z^{\varphi_0}(r)\|_H^2)\,dr\\
\le&\frac{l}{T}\int_0^T \frac{1}{2\lambda_K}(1+E\|Z^{\varphi_0}(r)\|_H^2)\,dr\\
=&\frac{l}{2\lambda_K}(1+\frac{1}{T}\int_0^T E\|Z^{\varphi_0}(r)\|_H^2\,dr)\\
\le&\frac{l}{2\lambda_K}(1+M(1+\|\varphi_0\|_H^2)).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sum_{k=K}^{K+m} \frac{1}{T}\int_0^T E[Z_k^{\varphi_0}(t)]^2\,dt\\
\le&3\frac{\|\varphi_0\|_H^2}{2T\lambda_K}+3\frac{(1+2\delta)l}{2(\lambda_K-\delta)}(1+M(1+\|\varphi_0\|_H^2))+3\frac{l}{2\lambda_K}(1+M(1+\|\varphi_0\|_H^2))\\
\to&0 \text{ as } K\to\infty \text{ uniformly in } T\ge t_0. \qquad \square
\end{aligned}$$

Theorem 6.2.6. *Assume (6.2) and (6.3) hold. Then there exists an invariant measure for the system.*

Proof. Taking $h \in C_b(Y)$ and using Lemma 6.2.3, we can repeat the proof of Theorem 6.2.5. □

6.3 An example

The following example shows that $-\lambda_k \downarrow -\infty$ as $k \to \infty$ is not necessary.

Example 6.3.1 (Dam storage problem). *We consider the semilinear stochastic differential equation*

$$\begin{cases} d\eta(t) = A\eta(t)\,dt + d\xi(t), \quad \textit{suppose } d\xi(t) = \int_H u\,\widetilde{N}(du\,dt) \\ \eta(0) = \varphi \in H. \end{cases}$$

Here A is an infinitesimal generator of a pseudo-contraction semigroup $\{\,S(t)\,\}$.

Compared to the general case

$$\begin{cases} dZ(t) = (AZ(t) + F(Z(t)))\,dt + \int_E B(v, Z(t))\,\widetilde{N}(dv\,dt) \\ Z(0) = \varphi \in H, \end{cases}$$

we have $F = 0$, $B(v,x) = v$. The growth condition and Lipschitz condition are satisfied, so there exists an unique mild solution $\eta^\varphi(t)$, such that

$$\eta^\varphi(t) = S(t)\varphi + \int_0^t \int_H S(t-s)u\,\widetilde{N}(du\,ds).$$

For any $h \in H$,

$$\langle \eta^{\varphi_1}(t) - \eta^{\varphi_2}(t), h\rangle = \langle S(t)(\varphi_1 - \varphi_2), h\rangle = \langle \varphi_1 - \varphi_2, S^*(t)h\rangle,$$

so we have,

$$\langle \eta^{\varphi_1}(t) - \eta^{\varphi_2}(t), h\rangle \to 0, \text{ for any } h \in H \text{ as } \varphi_2 \to \varphi_1 \text{ weakly.}$$

By Ickikawa [15], we know that $\eta^\varphi(t)$ is w-Feller.

Now if $|S(t)| \le e^{-\beta t}, (\beta > 0)$, we have

$$\begin{aligned}
E\|\eta^\varphi(t)\|_H^2 \le& 2e^{-2\beta t}\|\varphi\|_H^2 + 2\Big\|\int_0^t\int_H S(t-s)u\,\widetilde{N}(du\,ds)\Big\|_H^2 \\
\le& 2e^{-2\beta t}\|\varphi\|_H^2 + 2\int_0^t\int_H \|S(t-s)u\|_H^2\,\beta(du)\,ds \\
\le& 2e^{-2\beta t}\|\varphi\|_H^2 + 2\int_0^t\int_H e^{-2\beta(t-s)}\|u\|_H^2\,\beta(du)\,ds \\
\le& 2e^{-2\beta t}\|\varphi\|_H^2 + 2\int_0^t e^{-2\beta(t-s)}\,ds\int_H \|u\|_H^2\,\beta(du) \\
\le& 2e^{-2\beta t}\|\varphi\|_H^2 + \frac{1}{\beta}(1-e^{-2\beta t})\int_H \|u\|_H^2\,\beta(du) \\
\le& 2e^{-2\beta t}\|\varphi\|_H^2 + M,
\end{aligned}$$

where M is a positive constant. So $\eta^\varphi(t)$ is exponential ultimate bounded. By Theorem 6.2.4 and Theorem 6.2.5, there exists an unique invariant measure.

Consider $Ae_i = -\lambda e_i$, $\forall i$ with $\lambda > 0$. This shows that $-\lambda_k \downarrow -\infty$ as $k \to \infty$ is not necessary.

6.4 Future research plans

First, we will study the SDE's in Hilbert spaces driven by more general noises, for example, the Lévy processes. Second, we will study the applications of our theory to protein folding problem and finance.

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