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ON A CLASS OF NONLOCAL EVOLUTION EQUATIONS

By

Guangyu Zhao

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ABSTRACT

ON A CLASS OF NONLOCAL EVOLUTION EQUATIONS

By

Guangyu Zhao

The thesis includes three parts. In the first part, we study a nonlocal evolution equation which describes the seed dispersal of single species and prove the existence, uniqueness and stability of positive steady state solution to this equation. In the second part, we study the principal eigenvalue problem and given a sufficient condition that ensures the existence of coexistence state to a nonlocal evolution system. Then we consider a competition model which involves two similar species. The existence of coexistence states and their stability are investigated. In the third part, we establish the existence, uniqueness and continuous dependence on initial values for the solutions to a nonlocal phase field system. We also discuss the asymptotic behavior of the solution and prove the global boundedness of the solutions.

To my parents

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CHAPTER 1

Introduction

The spatial dispersal of cell or organisms is central to biology, it has important effects both from ecological and genetic point of view in a variety of situations. Obviously the mechanism of dispersal is of great importance in this context and has received much attention recently. It is now a major focus of theoretical interest. The models for this can be roughly placed into two categories. The first consists of models continuous in both time and space. Most continuous models related to dispersal are based upon reaction-diffusion equations, which have been extensively studied, (see [19, 30, 34, 38] and [39]). The exact nature of dispersal and its theoretical treatment has became an important question and in particular there is a variety of reasons that suggest that the class of reaction-diffusion models, with its fundamental assumption that motion is governed by a random walk, is too restrictive to model the seed dispersal. These consideration led to the derivation [33], based on a variation of a position-jump process, to the evolution equations of the form

$$\frac{\partial u}{\partial t} = \gamma[\int_{\Omega} k(x-y)u(y)dy - u(x)] + f(u,x)u, \quad \Omega \subset \mathbb{R}^n \tag{1.1}$$

under the constraint that $k > 0$.

To gain a insight of the derivation of the model (1.1). We quote the section 2.1 of [33]. Consider a single species in an n-dimensional habitat where the population can be modelled by a single function $u(x,t)$, which is the density at position x and at time t. To establish the continuous model, we shall focus on the case $n = 1$ for clarity of exposition (It is straightforward to generalized to arbitrary dimension). Divide the habitat $\mathbb{R}$ into contiguous sites, each of length Δx. Discretize time into steps of size Δt. Let $u(i,t)$ be the density of individuals in site i at time t. With the assumption that the rate at which individuals are leaving site i and going to site j is constant, the total number should be proportional to: the population in the interval i, which is $u(i,t)\Delta x$; the size of the target site, which is Δx; and the amount of time during which the transit is being measured, Δt. Let $\alpha(j,i)$ be the proportionality constant. Then, the number of individuals leaving site i during the interval $[t, t+\Delta t]$ is

$$\sum_{j=-\infty j\neq i}^{\infty} \alpha(j,i)u(i,t)(\Delta x)^2\Delta t. \tag{1.2}$$

It is biologically reasonable to believe that the mean and variance of the distances moved are finite. Hence

$$\sum_{j=-\infty j\neq i}^{\infty} \alpha(j,i)\Delta x, \qquad \sum_{j=-\infty j\neq i}^{\infty} |j-i|\alpha(j,i)(\Delta x)^2 \qquad \text{and} \sum_{j=-\infty j\neq i}^{\infty} |j-i|^2\alpha(j,i)(\Delta x)^3$$

will all be assumed finite.

During the same time interval, the number of arrivals to site i from elsewhere is

$$\sum_{j=-\infty j\neq i}^{\infty} \alpha(j,i)u(i,t)(\Delta x)^2\Delta t. \tag{1.3}$$

Finally, with each site we allow for the birth and death of individuals. Let $f(u(i,t),i)$ denote the per capita net reproduction rate at site i at the given population density. We assume that this rate is constant over the time interval. Then the number of new individuals at site i is

$$f(u(i,t),i)u(i,t)\Delta x\Delta t.$$

With regard to f, the following assumption are made:

$$f(0,x) > 0, \qquad \frac{\partial f}{\partial u} < 0$$

With (1.2) and (1.3), we deduce that the population density at location i and time $t + \Delta t$ is given by

$$u(i, t + \Delta t) = u(i, t) + \sum_{j=-\infty j \neq i}^{\infty} \alpha(i, j) u(j, t) \Delta x \Delta t$$

$$- \sum_{j=-\infty j \neq i}^{\infty} \alpha(j, i) u(i, t) \Delta x \Delta t + f(u(i, t), i) u(i, t) \Delta x \Delta t.$$

Let both $\Delta x \to 0$ and $\Delta t \to 0$, we obtain

$$\frac{\partial u}{\partial t}(x, t) = \int_{-\infty}^{\infty} [\alpha(x, y) u(y, t) - \alpha(y, x) u(x, t)] dy + f(u(x, t), x) u(x, t).$$

Assume that the rate of transition between the various patches, $\alpha(x, y)$, is homogeneous and only depends on the distance between patches i.e. upon $|x - y|$. Above equation can be written as

$$\frac{\partial u}{\partial t} = \gamma [\int_{-\infty}^{\infty} k(x - y) u(y) dy - u(x)] + f(u, x) u \tag{1.4}$$

where k is an even function with

$$\int_{-\infty}^{\infty} k(s) ds = 1$$

and

$$\gamma := \int_{-\infty}^{\infty} \alpha(|s|) ds$$

Notice that the dispersal rate γ, which represents the total number of the dispersing organism per unit time, play an important role in the model. It is worth to notice that Bates, Fife, Ren and Wang [14] studied a dissipative model with nonlocal interaction which is derived from the point of view of certain continuum limits in dynamic Ising models. It is similar to (1.4) but with the bistable nonlinearity.

So far the habitat has been considered to be infinite in the extent. Somehow, it is biologically unrealistic. With this in mind (1.4) may be modified to apply to a habitat in $\mathbb{R}$ of length h by two natural ways which lead to the equations

$$\frac{\partial u}{\partial t} = \gamma [\int_{0}^{h} k(x - y) u(y) dy - u(x)] + f(u, x) u \tag{1.5}$$

with either $\int_0^h k(s)ds \leq 1(\neq 1)$ or $\int_0^h k(s)ds \equiv 1$. (see [33] for details) Although the approach used to obtain (1.5) has similarities with the classical derivation of the Laplacian via random walk, it is not assumed that individuals move from a given patch with a binomial distribution. In contrast to the Laplacian, the integral operator J in equation (1.5) is not a local operator, thus (1.5) should be considered as a model with long-range interaction. With the assumption that (1.5) has a positive steady solution $\widetilde{u}$ and the nonlinearity $f(x, u) = u(a(x) - u)$, where $a(x)$ is C^1 continuous, in [33] it is shown that $\widetilde{u}$ is the global attractors for all solutions whose initial data is non-trivial and nonnegative.

Based on (1.5), The authors also proposed a competitive system of the form

$$\frac{\partial u}{\partial t} = \gamma_1[\int_0^h k(x-y)u(y)dy - u(x)] + f(u+v, x)u, \tag{1.6}$$

$$\frac{\partial v}{\partial t} = \gamma_2[\int_0^h k(x-y)v(y)dy - v(x)] + f(u+v, x)v \tag{1.7}$$

to try to understand how competition drives selection of γ. Here u and v stand for the densities of two different species. They discovered an interesting fact that the slowest disperser as measured by dispersal rate is always selected. Their study also reveals that both (1.5) and (1.8) can display very rich dynamics and hence gives rise to many interesting mathematics issues. As pointed out in [33], the mathematical analysis of (1.1) appears to be difficult even though the dispersal is represented by a bounded operator. Unlike the reaction-diffusion equations, (1.1) no longer has a regularizing effect.

In the study of classical dispersal models which are often based upon reaction-diffusion equations, many useful results on the global dynamics of diffusive equations were established in terms of principal eigenvalues of scalar elliptic eigenvalue problems. In [23], these linear elliptic eigenvalue problems are carefully explored. The authors obtain several important properties of the principal eigenvalue which were then used to study the global dynamics of logistic models. A trichotomy of the global asymptotics was also established. Meanwhile, by using monotone dynamical systems theory, in [24] the authors derived similar results for some quasimonotone reaction-diffusion systems with delays. Even though their approaches are not immediately applicable to (1.1) due to the lack of compactness, the importance of the principal eigenvalue is evident. Those approaches strongly suggest that an analogous idea using the principal eigenvalue should be developed, particularly for the case where the reaction term is sublinear. In [33] the authors prove the existence of a principal eigenvalue for the integral operator $Lu := J * u + b(x)u$ under certain conditions, where $J * u = \int_\Omega J(x,y)u(y)dy$. We shall use those ideas combined with a comparison argument to study the steady solutions of (1.1) and their stability.

In 1997, Bates [9] proposed the study of a nonlocal phase-field equation in which motility of phase boundaries is temperature-dependent and temperature varies according to a heat equation with phase change becoming a heat source or sink through the latent heat of fusion. This system, when considered on a finite region, has the form

$$\frac{\partial u}{\partial t} = \int_\Omega J(x-y)u(y)dy - \int_\Omega J(x-y)dyu(x) - f(u) + l\theta, \tag{1.8}$$

$$\frac{\partial(\theta + lu)}{\partial t} = \triangle\theta \tag{1.9}$$

in $(0,T) \times \Omega$, with initial and boundary conditions

$$u(0,x) = u_0(x), \;\; \theta(0,x) = \theta_0(x), \tag{1.10}$$

$$\frac{\partial\theta}{\partial n}|_{\partial\Omega} = 0, \tag{1.11}$$

where $T > 0$, $\Omega \subset \mathbb{R}^n$ is a bounded domain. Here θ represents temperature, u is an order parameter, l is a latent heat coefficient, the interaction kernel satisfies $J(-x) = J(x)$, and f is bistable.

For this system, one expect to see spinodal decomposition or the spontaneous creation of a fine-scaled patterned structure from initial data that is close to homogeneous but with in a certain range.

The dissertation is organized as follows: In chapter 2, we adopt the sub-super solution methods and local bifurcation theory to study the positive steady solution to (1.1) and its uniqueness. We investigate the stability of this positive steady state and the long time behavior of solutions to (1.1). In chapter 3, we are more interested in the coexistence states of the competition system

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1[\displaystyle\int_0^h J(x,y)u(y)dy - b(x)u(x)] + u[\lambda l(x) + f(x,u) + F(x,u,v)v], \\ \dfrac{\partial v}{\partial t} = d_2[\displaystyle\int_0^h J(x,y)v(y)dy - b(x)v(x)] + v[\gamma m(x) + g(x,v) + G(x,u,v)u] \end{cases} \quad x \in [0,h] \tag{1.12}$$

where $h > 0, d_i > 0 (i = 1,2)$ and $\lambda, \gamma \in \mathbb{R}$. By studying the principal eigenvalue problem, we are able to apply abstract results from the bifurcation theory to obtain the existence of coexistence states. We also consider a special form of (1.12) which involves two similar species. The existence of coexistence states and their stability are investigated. In chapter 4, we focus our attention to the system (1.8)-(1.11) and prove the existence, uniqueness and continuous dependence on initial data of the solution, in this case, we require initial data $u_0 \in L^\infty(\Omega)$, and $\theta_0 \in L^\infty(\Omega) \cap W^{1,2}(\Omega)$. We also discuss the asymptotic behavior of the solution.

CHAPTER 2

Existence,Uniqueness and Stability of Steady Solution for a Nonlocal Evolution Equation

In this chapter we study the existence,uniqueness and stability of positive steady state solutions of nonlocal evolutionary problem of the form

$$\begin{cases} \dfrac{\partial u}{\partial t} = \displaystyle\int_{\Omega} J(x,y)u(y)dy + b(x)u(x) + f(x,u) \\ u(x,0) = \phi(x) \end{cases} \tag{2.1}$$

where J, b and f are sufficiently smooth functions and J is positive.

2.1 Existence and uniqueness of positive steady solution

Let Ω be a bounded domain of $\mathbb{R}^n$ of class C^γ for some $\gamma \in (0,1)$. Let H be the Hilibert space $L^2(\Omega)$ with inner product $(\cdot,\cdot)$. Let $X := C(\overline{\Omega})$ be the Banach space of real continuous functions on $\overline{\Omega}$. Throughout this paper, X is considered as an ordered Banach space with a positive cone X_+, where $X_+ = \{u \in X | u \geq 0\}$. It is well known that X_+ is generating, normal and has nonempty interior, which we denote by $intX_+$, (see [1] for more details). For $\phi, \varphi \in X$, we write $\phi \leq \varphi$ if $\varphi - \phi \in X_+$, $\varphi \gg \phi$ if $\varphi - \phi \in intX_+$ and $\varphi \ll \phi$ if $\phi - \varphi \in intX_+$. An operator $T : X \to X$ is called positive if $TX_+ \subseteq X_+$.

Definition 2.1.1 *An operator A is said to be resolvent positive if the resolvent set $\rho(A)$ of A contains an interval (α, ∞) and $(\lambda I - A)^{-1}$ is positive for sufficiently large $\lambda \in \rho(A) \cap \mathbb{R}$.*

We also denote the spectral bound of an operator A by

$$\mathfrak{s}(A) = \sup\{Re\lambda : \lambda \in \sigma(A)\}$$

where $\sigma(A)$ is the spectrum of A.

Consider the linear operator L on X defined by

$$Lu(x) := \int_\Omega J(x,y)u(y)dy + b(x)u(x) \tag{2.2}$$

where we assume that

(H1) $J(x,y) \in C(\overline{\Omega} \times \overline{\Omega}, \mathbb{R}^+)$ is symmetric.

(H2) $J(x,y) > 0$, for any $x, y \in \overline{\Omega}$.

Lemma 2.1.2 *Let L be given by (2.2). Assume that (H1) and (H2) are satisfied and $b \in X$. Then L is a resolvent positive operator on X and $\mathfrak{s}(L) \in \sigma(L)$. If there exist $\Lambda \in \mathbb{R}$ and a continuous function $\phi \in X_+ \backslash \{0\}$ such that $L\phi = \Lambda\phi$. Then $\Lambda = \mathfrak{s}(L)$. Moreover, $\mathfrak{s}(L)$ is an isolated eigenvalue and* $\ker(L - \mathfrak{s}(L)I) = span\{\phi\}$.

Proof. First, we prove that L is a resolvent positive operator on X. In fact, L is a bounded, linear operator on X. Thus, $\rho(L)$ contains $(||L||, \infty)$. Choose $\omega \in \mathbb{R}^+$ such that $\omega > \sup_{x\in\overline{\Omega}} |\int_\Omega J(x,y)dy| + \sup_{x\in\overline{\Omega}} |b(x)|$. Obviously, $\lambda \in \rho(L)$ whenever

$\lambda \geq \omega$. To prove that $(\lambda I - L)^{-1}$ is positive for all $\lambda \geq \omega$, it is sufficient to show that $(L - \lambda I)v \leq 0$ implies $v \geq 0$ for all $\lambda \geq \omega$. Let $v = v^+ - v^-$, where $v^- = \max\{-v, 0\}$ and $v^+ = \max\{v, 0\}$. If $v^- \neq 0$, then straight calculation yields that

$$0 \leq (J * v^+, v^-) \leq ((L - \lambda)v^-, v^-)$$

and so

$$0 \leq ((L - \lambda)v^-, v^-) \leq (\sup_{x \in \overline{\Omega}} |\int_\Omega J(x, y)dy| + \sup_{x \in \overline{\Omega}} |b| - \lambda)(v^-, v^-) < 0.$$

The contradiction shows that $v^- \equiv 0$ in Ω, and so $v \geq 0$ in $\overline{\Omega}$. Since $\mathfrak{s}(L) > -\infty$, by ([45],Theorem 3.5), $\mathfrak{s}(L) \in \sigma(L)$.

Now, assume that there exist $\phi \in X_+ \setminus \{0\}$ and $\Lambda \in \mathbb{R}$ such that

$$J * \phi = (\Lambda - b(x))\phi.$$

It is easy to see that $J * \phi \gg 0$. Consequently, we have that $\phi \gg 0$ and $\Lambda - b(x) \gg 0$. If $\mathfrak{s}(L) > \Lambda$, then the linear operator K, defined by $Ku = (\mathfrak{s}(L) - b(x))u$, is continuous and bijective on X because that $\mathfrak{s}(L) - b(x)$ is bounded and $\mathfrak{s}(L) - b(x) \gg 0$. Note that $(L - \mathfrak{s}(L)I)u = J * u - Ku$. We infer that the linear operator $L - \mathfrak{s}(L)I$ is Fredholm of index zero because that $J : u \longrightarrow J * u$ is compact on X (see [46], Theorem5.C, p295). By ([32], Proposition 2.3 and 2.4 p.151), (see also [36]), $\mathfrak{s}(L)$ is an eigenvalue with finite algebraic multiplicity and there exists a positive eigenfunction $\varphi \in X_+ \setminus \{0\}$ associated with $\mathfrak{s}(L)$. Since L is self-adjoint when considered as an operator on H, its eigenfunctions corresponding to distinct eigenvalues are orthogonal, hence we have $(\phi, \varphi) = 0$. But this contradicts the fact that both ϕ and φ are strictly positive. Thus, $\mathfrak{s}(L) = \Lambda$.

We now show that $\ker(L - \mathfrak{s}(L)) = \operatorname{span}\{\phi\}$. Suppose this is not true, then there exists an eigenfunction ψ associated with $\mathfrak{s}(L)$ such that $\psi \neq t\phi$ for all $t \in \mathbb{R}$. Since $\phi \gg 0$, there exist t such that $t\phi + \psi \geq 0$. Let $\bar{t} = \inf\{t \in \mathbb{R} | t\phi + \psi \geq 0\}$. Note that we have $J * (\bar{t}\phi + \psi) = (\mathfrak{s}(L) - b)(\bar{t}\phi + \psi)$ and $\bar{t}\phi + \psi \neq 0$. Again, $J * (\bar{t}\phi + \psi) \gg 0$ implies $\bar{t}\phi + \psi \gg 0$, which violates the definition of $\bar{t}$. The contradiction yields the desired conclusion.

To prove that $\mathfrak{s}(L)$ is isolated, we assume to the contrary that there exists a sequence $\{\mu_n\}_{n=1}^{\infty} \subset \sigma(L)$ such that $lim_{n\to\infty}\mu_n = \mathfrak{s}(L)$. By ([46], Proposition1,p300), it is evident that $L - \mu_n I$ is a Fredholm operator of index zero and hence μ_n is an eigenvalue of L on X if n is sufficiently large. On the other hand, thanks to $\mathfrak{s}(L) - b(x) \gg 0$, we have $\mu_n - b(x) \geq \delta$ for some $\delta > 0$ provided n sufficiently large. Let θ_n be the corresponding eigenfunction with $||\theta_n||_X = 1$. Due to the compactness of J and the fact that the sequence $\{\theta_n\}$ is bounded in X, along some subsequence, still labelled n,

$$lim_{n\to\infty}||(\mu_n - b(x))^{-1}J * \theta_n - \vartheta||_X = 0$$

for some $\vartheta \in X$. From $(\mu_n - b(x))^{-1}J * \theta_n = \theta_n$, it follows that

$$(L - \mathfrak{s}(L)I)\vartheta = 0 \text{ and } ||\vartheta||_X = 1.$$

Thus ϑ is an eigenfunction associated with $\mathfrak{s}(L)$. Since ϑ does not change sign over $\overline{\Omega}$ and is bounded away from zero, the convergence of $(\mu_n - b(x))^{-1}J * \theta_n$ implies $(\theta_n, \vartheta) > 0$ and we arrive at a contradiction again. Therefore, $\mathfrak{s}(L)$ is isolated and the proof is completed.

Lemma 2.1.3 *Let all assumptions in Lemma 2.1.2 be satisfied. Then the following three statements are equivalent.*

(i) There exists a $\overline{u} \in X_+\setminus \{0\}$ such that $-L\overline{u} \in X_+\setminus \{0\}$.

(ii) $\mathfrak{s}(L) < 0$.

(iii) For each $f \in X$, $Lu = f$ has exactly one solution in X. Moreover, if w is a solution to $Lu = f$ and $f \leq 0$, then $w \geq 0$.

Proof. (i)$\Rightarrow$(ii). Suppose $\mathfrak{s}(L) \geq 0$. Let $g = L\overline{u} - \mathfrak{s}(L)\overline{u}$. Obviously, $g \leq 0$ and $g \neq 0$. Let ϕ be the positive eigenfunction associated with $\mathfrak{s}(L)$, then $(\phi, g) < 0$. On the other hand, we have

$$(\phi, g) = (\phi, (L - \mathfrak{s}(L)I)\overline{u}) = ((L - \mathfrak{s}(L)I)\phi, \overline{u}) = 0,$$

which is a contradiction. Thus, $\mathfrak{s}(L) < 0$.

(ii)$\Rightarrow$ (i). This is trivial since the eigenfunction $\phi \gg 0$ and satisfies $L\phi \ll 0$.

(iii)$\Rightarrow$(ii). Clearly, $0 \in \rho(L)$, thus $\mathfrak{s}(L) < 0$.

(ii)$\Rightarrow$(iii). The existence of a unique solution is ensured by the fact that $0 \in \rho(L)$. Suppose w is the solution to $Lu = f$ and $f \leq 0$ with $w \notin X_+$. Let ϕ be the positive eigenfunction associated with $\mathfrak{s}(L)$. There exists $t > 0$ such that $t\phi + w \geq 0$. Once again, we let $\overline{t} = \inf\{t \in \mathbb{R}^+ | t\phi + w \geq 0\}$. Obviously, $\overline{t}\phi + w \neq 0$ and $L(\overline{t}\phi + w) \leq 0$. Now, let $x_0 \in \overline{\Omega}$ be a point such that $\overline{t}\phi(x_0) + w(x_0) = 0$, then we have

$$0 \leq \int_\Omega J(x_0, y)(\overline{t}\phi + w)dy = L(\overline{t}\phi + w)(x_0) \leq 0.$$

Since $J(x_0, y) > 0$ and $\overline{t}\phi + w$ is nonnegative, we have $\overline{t}\phi + w \equiv 0$. The contradiction leads to $\overline{t}\phi + w \gg 0$, which of course violates the definition of $\overline{t}$. Consequently, $w \geq 0$ and the proof is completed.

Lemma 2.1.4 *Assume (H1) and (H2). Suppose that $b_1, b_2 \in X$ with $b_1 \geq b_2$ and $b_1 \neq b_2$. If $L_i\phi_i = \mu_i\phi_i, i = 1, 2$, where $L_iu := J * u + b_iu$ and $\phi_i \in X_+ \setminus \{0\}, i = 1, 2$. Then $\mu_1 = \mathfrak{s}(L_1) > \mu_2 = \mathfrak{s}(L_2)$.*

Proof.

From Lemma 2.1.2, it follows that $\mu_1 = \mathfrak{s}(L_1)$ and $\mu_2 = \mathfrak{s}(L_2)$. Furthermore, $J * \phi_i = (\mu_i - b_i)\phi_i, i = 1, 2$ and $J * \phi_i \gg 0$ indicate $\phi_i \gg 0, i = 1, 2$. Now, if $\mu_1 \leq \mu_2$, then

$$(L_2 - \mu_2 I)\phi_1 = L_1\phi_1 + (b_2 - b_1)\phi_1 - \mu_2\phi_1 = (\mu_1 - \mu_2)\phi_1 + (b_2 - b_1)\phi_1 \leq 0.$$

According to Lemma 2.1.3, $\mathfrak{s}(L_2 - \mu_2) < 0$. This is impossible, because $\mathfrak{s}(L_2 - \mu_2) = 0$. Therefore, $\mathfrak{s}(L_1) > \mathfrak{s}(L_2)$.

Next we consider the existence of solutions to

$$\int_\Omega J(x,y)u(y)dy + b(x)u(x) + f(x,u) = 0 \tag{2.3}$$

For the remainder of this paper , we assume that

(H3) $f(x,s) \in C(\overline{\Omega} \times \mathbb{R}^+, \mathbb{R})$. $\partial_S f(x,s) \in C(\overline{\Omega} \times \mathbb{R}^+, \mathbb{R})$ and $f(x,0) \equiv 0$.

(H4) $f(x,\cdot)$ is strictly sublinear,i.e., for any $\alpha \in (0,1)$, $f(x,\alpha s) > \alpha f(x,s)$, where $s > 0$.

(H5) $f(\cdot,s)$ and are continuous, uniformly for s in bounded sets.

We also let

$$g(x,u) = \begin{cases} \dfrac{f(x,u)}{u} \text{ for } u > 0 \\ \partial_U f(x,0) \text{ for } u = 0. \end{cases} \tag{2.4}$$

It is clear that $g \in C(\overline{\Omega} \times \overline{\Omega}, \mathbb{R})$ and $f(x,u) = g(x,u)u$.

Definition 2.1.5 *A function in X is said to be a subsolution of (2.3) if*

$$\int_\Omega J(x,y)u(y)dy + b(x)u(x) + f(x,u) \geq 0. \tag{2.5}$$

A supersolution is defined similarly by reversing the inequality.

Theorem 2.1.6 *suppose that (H1)-(H5) are satisfied. If (2.3) has a supersolution $\overline{u}$ and subsolution $\underline{u}$ in $X_+ \backslash \{0\}$ such that $\underline{u} \leq \overline{u}$. Then (2.3) has a unique solution in $intX_+$.*

Proof. We define $V = \{u \in C(\overline{\Omega}) | \underline{u} \leq u \leq \overline{u}\}$. By condition (H3), we see that $\dfrac{\partial f}{\partial u}(x,u)$ is uniformly bounded on $\overline{\Omega} \times V$ and

$$\frac{\partial f}{\partial u}(x,u) + b(x) + \beta \gg 0 \tag{2.6}$$

for $(x,u) \in \overline{\Omega} \times V$ provided β is sufficiently large. We define the mapping for $\beta > 0$ large as follows: $v = Tu$ if

$$\int_\Omega J(x,y)v(y)dy + b(x)v(x) - \beta v(x) = -[f(x,u) + \beta u]. \tag{2.7}$$

We also define

$$\mathcal{J}v(x) := \int_\Omega J(x,y)v(y)dy + b(x)u - \beta v(x).$$

Since $\mathcal{J}$ is invertible on X if β is sufficiently large and the right hand side of (2.7) belongs to X for each $u \in V$, $T : V \to X$ is well defined. Next, we show that T is monotone in the sense that $w_1 \leq w_2$ implies $Tw_1 \leq Tw_2$, provided both w_1 and w_2 belonging to V. In fact, if $w_1 \leq w_2$ then

$$Fw_1 \equiv f(x, w_1) + \beta w_1 \leq Fw_2 \equiv f(x, w_2) + \beta w_2,$$

thanks to (2.6). Notice that

$$\mathcal{J}(Tw_i) = -Fw_i,$$

thus, we have

$$\mathcal{J}(Tw_2 - Tw_1) \leq 0.$$

By virtue of Lemma 2.1.2, $(-\mathcal{J})^{-1}$ is a positive operator as long as β is sufficiently large. Hence we obtain

$$Tw_1 \leq Tw_2$$

From this, we deduce that the sequence defined inductively by

$$u_1 = T\overline{u} \quad \text{and} \quad u_n = Tu_{n-1}$$

is monotone decreasing. Similarly,

$$v_1 = T\underline{u} \quad \text{and} \quad v_n = Tv_{n-1}$$

define a monotone increasing sequence. Furthermore, we can show by induction that

$$\underline{u} \leq v_1 \leq \cdots \leq v_n \cdots \leq u_n \leq \cdots \leq u_1 \leq \overline{u}$$

Because the sequence $\{u_k\}$ and $\{v_k\}$ are monotone, the pointwise limits

$$u^* = \lim_{k\to\infty} u_k(x) \quad \text{and} \quad v^* = \lim_{k\to\infty} v_k(x)$$

both exist. Obviously, $\underline{u} \le v^* \le u^* \le \overline{u}$. By the monotone convergence theorem, we have

$$\lim_{k\to\infty}\int_\Omega J(x,y)u_k(y)dy = \int_\Omega J(x,y)u^*(y)dy, \lim_{k\to\infty}\int_\Omega J(x,y)v_k(y)dy = \int_\Omega J(x,y)v^*(y)dy.$$

On the other hand, by the continuity of f, we see

$$\lim_{k\to\infty} f(x,u_k(x)) + b(x)u_{k+1}(x) = f(x,u^*) + b(x)u^*(x),$$

$$\lim_{k\to\infty} f(x,v_k(x)) + b(x)v_{k+1}(x) = f(x,v^*) + b(x)v^*(x).$$

Since

$$\int_\Omega J(x,y)u_k(y)dy = -[b(x)u_k(x) + f(x,u_{k-1}(x))] + \beta[u_k(x) - u_{k-1}(x)],$$

it follows that

$$\int_\Omega J(x,y)u^*(y)dy + b(x)u^* + f(x,u^*) = 0.$$

Similarly,

$$\int_\Omega J(x,y)v^*(y)dy + b(x)v^* + f(x,v^*) = 0.$$

Next, we show that both u^* and v^* belong to X_+, i.e, (2.3) has at least one positive continuous solution. To this end, we make the following observations. First, we see $J * u^* \in intX_+$. This together with (H3) implies that both u^* and v^* are bounded away from zero. Second, due to the fact that $J * u^* + [b(x) + g(x,u^*)]u^* = 0$ and u^* is strictly positive and bounded on $\overline{\Omega}$, we may conclude that there is $\delta > 0$ such that

$$b(x) + g(x,u^*) \le -\delta, \quad \text{for all } x \in \overline{\Omega}.$$

Furthermore, for any $x_1, x_2 \in \overline{\Omega}$, we find that

$$J * u^*(x_1) - J * u^*(x_2) + [b(x_1) - b(x_2)]u^*(x_1) + [f(x_1,u^*(x_1)) - f(x_2,u^*(x_1))] \quad (2.8)$$

$$= -[b(x_2) + \partial_u f(x_2, \theta u^*(x_1) + (1-\theta)u^*(x_2))](u^*(x_1) - u^*(x_2)),$$

where $0 \leq \theta \leq 1$. Without loss generality, we may assume $u^*(x_1) \geq u^*(x_2)$. Since (H4) gives that $g(x, \cdot)$ is decreasing and $\partial_u f(x, s) \leq g(x, s)$ for all $s > 0$, the following inequalities are true

$$\partial_u f(x_2, \theta u^*(x_1) + (1-\theta)u^*(x_2)) \leq g(x_2, \theta u^*(x_1) + (1-\theta)u^*(x_2)) \leq g(x_2, u^*(x_2))$$

Hence, we have

$$-[b(x_2) + \partial_u f(x_2, \theta u^*(x_1) + (1-\theta)u^*(x_2))] \geq \delta. \tag{2.9}$$

From (2.8) and (2.9), we conclude that u^* is continuous.

We now show the uniqueness of positive solutions in X_+. We shall argue by contradiction. Let $\varphi_1 \neq \varphi_2$ be two positive continuous solutions of (2.3). Then it is easy to see that $k\varphi_1$ is a supersolution of (2.3) which is greater than both φ_1 and φ_2 provided k is sufficiently large. Actually k can be chosen so that $k\varphi_1 > \varphi_1 + \varphi_2$. Hence we may assume without loss of generality that $\varphi_2 \ll \varphi_1$. On the other hand, the fact that $g(x, \varphi_1) < g(x, \varphi_2)$ and Lemmas 2.1.2 and 2.1.4 yield

$$0 = \mathfrak{s}(L + g(x, \varphi_1)) < \mathfrak{s}(L + g(x, \varphi_2)) = 0$$

This contradiction completes the proof.

Next we assume that (2.3) can be written as

$$\int_\Omega J(x, y)u(y)dy - u\int_\Omega J(x, y)dy + f(x, u, \mu) = 0, \tag{2.10}$$

as is the case when it arises as the Euler-Lagrange equation. The existence of a positive solution will be established by Crandall and Rabinowitz's bifurcation theorem which allows less restriction on f than is considered in [33].

In the following, we define

$$G(\mu, u) = \int_\Omega J(x, y)u(y)dy - u\int_\Omega J(x, y)dy + f(x, u, \mu) \tag{2.11}$$

and

$$\Sigma = \{(\mu, u) \in \mathbb{R} \times X | G(\mu, u) = 0\}$$

$$\Sigma_+ = \left\{\mu \in \Sigma | \text{ for some } u_\mu \in X_+ \backslash \{0\}, (\mu, u_\mu) \in \Sigma\right\}.$$

Clearly, (2.10) has a positive solution if Σ_+ is not empty.

We also assume that

(H6) $f(x, \cdot, \cdot) \in C^2$ uniformly for all $x \in \overline{\Omega}$. $f(x, 0, \mu) \equiv 0$ and $\partial_u f(x, 0, 0) \equiv 0$.

(H7) $\int_\Omega \partial_u \partial_\mu f(x, 0, 0) dx \neq 0$.

Theorem 2.1.7 *Suppose that (H1),(H2),(H6) and (H7) are satisfied. Then* $(0, 0)$ *is a bifurcation point of* $G(\mu, u) = 0$ *and* Σ_+ *is not empty.*

Proof. First, we see that $G \in C^2(\mathbb{R} \times X, X)$ and $G(\mu, 0) \equiv 0$ for all $\mu \in \mathbb{R}$. Moreover,

$$D_u G(0, 0) u = \int_\Omega J(x, y) u(y) dy - u \int_\Omega J(x, y) dy$$

and

$$D_\mu D_u G(0, 0) = \partial_u \partial_\mu f(x, 0, 0).$$

Since $D_u G(0, 0)\mathbf{1} = 0$, Lemma 2.1.2 implies that

$$\mathfrak{s}(D_u G(0, 0)) = 0, \quad \ker D_u G(0, 0) = \text{span}\{\mathbf{1}\},$$

and $D_u G(0, 0)$ is a Fredholm operator on X whose index is zero. Here $\mathbf{1}$ stands for constant function whose value is 1. According to Crandall-Rabinowitz's theorem (cf [28]), to show that $(0, 0)$ is a bifurcation point of $G(\mu, u) = 0$, we only need to verify that

$$D_\mu D_u G(0, 0)\mathbf{1} \notin \text{range} D_u G(0, 0).$$

In fact, if this is not true, then there exists $w \in X$ such that

$$D_u G(0, 0) w = \partial_u \partial_\mu f(x, 0).$$

Since $D_u G(0,0)$ is a bounded, self-adjoint operator on H, the Fredholm alternative yields

$$(\mathbf{1}, \partial_u \partial_\mu f(x,0)) = \int_\Omega \partial_u \partial_\mu f(x,0)dx = 0.$$

This contradicts the given condition. Therefore, $(0,0)$ is a bifurcation point. By the Crandall-Rabinowitz theorem, there is a nontrivial continuously differentiable curve $(\mu(s), u(\cdot,s))$ through $(0,0)$ such that $(\mu(s), u(\cdot,s)) \in \Sigma$, where $s \in (-\delta, \delta)$ for $\delta > 0$ and $(\mu(0), u(\cdot,0)) = (0,0)$. Furthermore, $u(\cdot,s) = s\mathbf{1} + o(s)$ and so Σ_+ is not empty.

Corollary 2.1.8 *Assume (H1) and (H2). In addition, assume $f(x,u) = \mu a(x)u + h(x,u)u$, where $a \in X$ and $h \in C^2(\overline{\Omega} \times \overline{\Omega}, R)$, $h(x,0) \equiv 0$, $\partial_u h(x,0) < 0$. Then $\Sigma_+ \subset \mathbb{R}^+$ for $s \in (0,\delta)$ provided $\int_\Omega a^+(x)dx > \int_\Omega a^-(x)dx$, where $a^- = \max\{-a, 0\}$ and $a^+ = \max\{a,0\}$. Moreover, if $a(x) \gg 0$ in $\overline{\Omega}$, then $(0,0)$ is the unique bifurcation point for positive solutions.*

Proof. Let $(\mu(s), u(\cdot,s))$ be the continuously differentiable curve ensured by Theorem 2.1.7. Notice that, supposing the dependence on s,

$$\int_\Omega [J * u(x) - \int_\Omega J(x,y)dyu(x)]dx = -\int_\Omega [\mu a(x) + h(x,u)]udx.$$

The symmetry of J implies

$$\int_\Omega [\mu a(x) + h(x,u)]udx = 0. \tag{2.12}$$

If s is sufficiently small, we have

$$\int_\Omega [\mu a(x) + h(x,u)]udx = s\int_\Omega [\mu(s)(a^+ - a^-) + \partial_u h(x,0)s]dx + o(s^2), \tag{2.13}$$

and from (2.12) and (2.13), we deduce that $\mu(s) > 0$ as $s \in (0,\delta)$.

Now, assume $a(x) \gg 0$, suppose that μ_1 is another bifurcation point. Then there is a sequence $(\mu_n, u_n) \in R^+ \times X_+$ such that $(\mu_n, u_n) \to (\mu_1, 0)$. If $\mu_1 = 0$, nothing needs to be done. Therefore, we assume $\mu_1 > 0$. Let $u_n(x^n) = \min_{\overline{\Omega}} u_n(x)$. Note that

$$J * u_n(x^n) - \int_\Omega J(x^n,y)dyu_n(x^n) \geq 0.$$

Consequently, $\mu_n a(x^n)+h(x^n,u_n(x^n)) \le 0$. On the other hand, we may choose $N > 0$ such that $\mu_n a(x^n) \ge \frac{1}{2}\mu_1 \min_{\overline{\Omega}} a(x)$ and $|h(x^n,u_n(x^n))| \le \frac{1}{3}\mu_1 \min_{\overline{\Omega}} a(x)$ whenever $n > N$. The contradiction shows that $(0,0)$ is the unique bifurcation point.

Throughout the rest of this section, We shall focus on the case that $\Omega \subset \mathbb{R}$ and establish the necessary and sufficient condition for the existence of a steady state solution to (2.3). Besides the condition (H1)-(H5), we assume that

(H8) $\Omega = (0,l), 0 < l < \infty$.

(H9) $\partial_u f(x,0)$ is Lipschitz continuous on $\overline{\Omega}$.

(H10) $f(\cdot,s)$ is Lipschitz continuous, uniformly for s in bounded sets.

First, we need the following lemma which can be found in [33].

Lemma 2.1.9 *Assume that (H1),(H2) and (H8) hold and that $b \in X$ is Lipschitz continuous. Then L is a bounded, self-adjoint operator on H and has a simple eigenvalue λ_0 given by*

$$\lambda_0 = \max_{||u||_{L^2}=1} (Lu,u).$$

The maximum is attained by a strictly positive eigenfunction $\phi \in X$. Also $\sigma(L) \subset (-\infty,\lambda_0]$.

Lemma 2.1.10 *Suppose that (H1)-(H4) and (H8)-(H9) hold. In addition, b is Lipschitz continuous on $\overline{\Omega}$. If $\mathfrak{s}(L+g(x,0)) \le 0$. Then* (2.3) *does not possess any solution in $X_+ \setminus \{0\}$.*

Proof. Suppose this is not true. Let w be a solution in $X_+ \setminus \{0\}$. By Lemma 2.1.2, we infer that $\mathfrak{s}(L+g(x,w)) = 0$. On the other hand, Lemma 2.1.2 and 2.1.9 ensure that $\mathfrak{s}(L+g(x,0))$ is a simple eigenvalue having an eigenfunction in $intX_+$, by $(H4)$ and Lemma 2.4, we have

$$0 = \mathfrak{s}(L+g(x,w)) < \mathfrak{s}(L+g(x,0)) \le 0.$$

The contradiction gives the desired conclusion.

Lemma 2.1.11 *Suppose that (H1)-(H4) and (H8)-(H9) hold and that $b \in X$ is Lipschitz continuous. The following statements are equivalent.*

(i) (2.3) admits a subsolution $\underline{u} \in X_+ \setminus \{0\}$ such that the inequality (2.4) is not identical to zero.

(ii) (2.3) admits an arbitrarily small subsolution in $X_+ \setminus \{0\}$.

(iii) $\mathfrak{s}(L + g(x, 0)) > 0$.

Proof. (i)$\Rightarrow$ (ii). It follows from the fact that

$$L\epsilon\underline{u} + f(x, \epsilon\underline{u}) \geq L\epsilon\underline{u} + \epsilon f(x, \underline{u}) \geq 0$$

for any $0 < \epsilon < 1$.

(ii)$\Rightarrow$ (i). This is trivial.

(i)$\Rightarrow$ (iii). Because of (i) and (H4),

$$(L + g(x, 0))\underline{u} \geq L\underline{u} + g(x, \underline{u})\underline{u} = L\underline{u} + f(x, \underline{u}),$$

that is, $(L + g(x, 0))(-\underline{u}) \leq 0$. As a consequence of Lemma 2.1.3, we have $\mathfrak{s}(L + g(x, 0)) \geq 0$. We next show $\mathfrak{s}(L + g(x, 0)) \neq 0$. Suppose this is not the case. Let $\psi \in intX_+$ be an eigenfunction corresponding to the eigenvalue 0, then

$$0 = ((L + g(x, 0))\psi, -\underline{u}) = (\psi, (L + g(x, 0))(-\underline{u})) < 0.$$

The contradiction shows $\mathfrak{s}(L + g(x, 0)) > 0$.

(iii)$\Rightarrow$ (i). Again, let $\psi \in intX_+$ be an eigenfunction associated with the eigenvalue $\mathfrak{s}(L + g(x, 0))$, then $L\epsilon\psi + g(x, 0)\epsilon\psi \gg 0$ for any $\epsilon > 0$. By the continuity of $g(x, \cdot)$, we have

$$L\epsilon\psi + f(x, \epsilon\psi) = L\epsilon\psi + g(x, \epsilon\psi)\epsilon\psi \geq 0$$

for sufficiently small $\epsilon > 0$. Hence, $\epsilon\psi$ is a subsolution of (2.3).

Theorem 2.1.12 *Suppose that (H1)-(H5) and (H8)-(H9) hold and that $b \in X$ is Lipschitz continuous. If (2.3) has a positive supersolution $\tilde{u} \in X$, then (2.3) has a unique positive continuous solution if and only if* $\mathfrak{s}(L + g(x, 0)) > 0$

Proof. We first prove the necessity. Let $\varphi \in X_+ \setminus \{0\}$ be a positive solution to (2.3). Then we have, in fact, $\varphi \gg 0$ and $\mathfrak{s}(L + g(x, \varphi)) = 0$. Consequently,

$$\mathfrak{s}(L + g(x, 0)) > \mathfrak{s}(L + g(x, \varphi)) = 0.$$

Now, suppose that $\mathfrak{s}(L + g(x, 0)) > 0$. Let ψ be the positive eigenfunction associated with $\mathfrak{s}(L + g(x, 0))$, whose existence is guaranteed by the condition (H9) and Lemma 2.1.9. In addition, the proof of Lemma 2.1.11 shows that $\varepsilon\psi$ is a subsolution of (2.3) for arbitrarily small ε. Also, $\widetilde{u} \in X_+$ and $-(b(x) + g(x, \widetilde{u})\widetilde{u} \geq J * \widetilde{u} \gg 0$ force $\widetilde{u} \gg 0$, and hence $\varepsilon\psi \ll \widetilde{u}$ for some $\varepsilon > 0$. It follows that (2.3) has a solution u in X_+ with $\varepsilon\psi \leq u \leq \widetilde{u}$.

Corollary 2.1.13 *Assume (H1)-(H4) and (H8)-(H10) and that $b \in X$ is Lipschitz continuous. Suppose $\mathfrak{s}(L + g(x, 0)) > 0$. Then the following statements are equivalent*

(i) Problem (2.3) has a positive solution in X.

(ii) Problem (2.3) has a positive supersolution in X.

(iii) Problem (2.3) has an arbitrarily large positive supersolution in X.

(iv) There exists $v \in intX_+$, which is Lipschitz, such that $\mathfrak{s}(L + g(x, v)) \leq 0$

Proof. The equivalence of (i) and (ii) is an immediate consequence of Theorem 2.1.12 and the fact that

$$L\alpha\omega + f(x, \alpha\omega) \leq \alpha(L\omega + f(x, \omega)) = 0,$$

where $\alpha > 1$ and $\omega \in X_+$ is a solution of (2.3). The equivalence of (ii) and (iii) comes from the fact that if ψ is a supersolution of (2.3), then

$$Lk\psi + f(x, k\psi) \leq Lk\psi + kf(x, \psi) \leq 0, \text{ for all } k > 1.$$

If (2.3) has a positive solution w in X, then $\mathfrak{s}(L + g(x, w)) = 0$. Let $v \in intX_+$ be Lipschitz continuous. Due to (H10), $b(x) + g(x, v)$ is Lipschitz continuous. Lemmas

2.1.2 and 2.1.9 imply that $\mathfrak{s}(L+g(x,v)I)$ is an eigenvalue of the linear and bounded operator $L+g(x,v)I$ on X and there is a strictly positive eigenfunction associated with $\mathfrak{s}(L+g(x,v)I)$. Choose v such that $v \gg w$, By condition (H4), $g(x,\cdot)$ is nonincreasing, it follows from Lemma 2.1.4 that

$$\mathfrak{s}(L+g(x,v)I) < \mathfrak{s}(L+g(x,w)).$$

Therefore, (i) implies (iv). We now complete the proof by showing that (iv) implies (ii). Let ϕ be the strictly positive eigenfunction corresponding to $\mathfrak{s}(L+g(x,v))$. Then we have

$$Lk\phi + f(x,k\phi) = Lk\phi + kg(x,k\phi)\phi \leq Lk\phi + kg(x,v)\phi \leq 0$$

for sufficiently large k. Thus, $k\phi$ is the desired supersolution of (2.3).

2.2 The Existence and asymptotic behavior

In this section, we establish the basic existence and uniqueness results for (2.1) and study the long time behavior of the solution to (2.1). We shall first establish local existence and uniqueness in X.

For $t_1 > 0$, define $\widehat{X} = C([0,t_1],X)$ with norm $||\phi||_{\widehat{X}} = \max_{t\in[0,t_1]} ||\phi||_X$.

Theorem 2.2.1 *Assume that (H1)-(H3) hold and $b \in X$. For each $\phi_0 \in X$, there exists $t_1 > 0$ such that (2.1) has a unique solution in $\widehat{X}$.*

Proof. We take the semigroup approach used in [9] to show the existence and uniqueness. As usual, we define the linear operator L on X by $L = J * u + b(x)u$. For each $\phi \in \widehat{X}$, we define mapping $S\phi = u$ where

$$u = e^{Lt}\phi(0) + \int_0^t e^{L(t-s)} f(x,\phi(s))ds \tag{2.14}$$

and e^{Lt} is the uniformly continuous semigroup on X generated by L because L is bounded on X. Now the equation (2.1) is reduced to (2.14). Since $f(x,\cdot)$ is locally

Lipschitz, with an argument similar to that in [9], one can show that S is a contraction mapping on $\widehat{X}$ for the suitable t_1. Therefore, the existence and uniqueness of the solution to (2.1) follows from Banach's fixed point Theorem.

Definition 2.2.2 *Let $S_T = \overline{\Omega} \times (0,T)$ for $0 < T \leq \infty$. A function $u \in C^1([0,\infty), X)$ is said to be a subsolution of (2.1) in S_T if*

$$u_t \leq \int_\Omega J(x,y)u(y)dy + b(x)u(x) + f(x,u). \tag{2.15}$$

A supersolution is defined similarly by reversing the inequality.

Proposition 2.2.3 *Assume (H1), (H2),(H3) and (H4) are satisfied and that $b \in X$. Then (2.1) has a global solution $u(\cdot, x, \psi)$ for each $\psi \in X_+$.*

Proof. Let $\widehat{u}$ be the solution to

$$\begin{cases} u_t = \int_\Omega J(x,y)u(y)dy + b(x)u(x) + g(x,0)u \\ u(x,0) = \psi, \end{cases}$$

where $g(x,0)$ is given by (2.4). Since $L + g(x,0)$ is a bounded linear operator on X, we have

$$||\widehat{u}(x,t,\psi)||_X \leq ||e^{(L+g(x,0))t}||||\psi||_X \leq e^{||L+g(x,0)||t}||\psi||_X.$$

This indicates that $\widehat{u}$ is a global solution. Furthermore, $\widehat{u} \in X_+$ according to the comparison principle, see [33]. Due to (H5)

$$f(x,u) = g(x,u)u \leq g(x,0)u,$$

whenever $u \geq 0$. Thus, $\widehat{u}$ is supersolution of (2.1). By the comparison principle,

$$0 \leq u(x,\cdot,\psi) \leq \widehat{u}(x,\cdot,\psi),$$

where $u(x,\cdot,\psi)$ is the solution of (2.1) with $u(0) = \psi$. the desired conclusion follows immediately.

Now, we are ready to give the main result in this section.

Theorem 2.2.4 *Assume that (H1)-(H4) and (H8)-(H10) are satisfied and that $b \in X$ is Lipschitz continuous. Then one of following statement holds.*

(i) If $\mathfrak{s}(L + g(x, 0)) < 0$, then the zero solution of (2.1) is globally asymptotically stable in X_+.

(ii) If $\mathfrak{s}(L + g(x, 0)) = 0$, then the solution of (2.1) $u(x, t, \psi)$ with $\psi \in X_+ \setminus \{0\}$ satisfies

$$\lim_{t \to \infty} u(\cdot, t, \psi) = 0 \ a.e.$$

(iii) If $\mathfrak{s}(L + g(x, 0)) > 0$, then (2.1) admits at most one stationary solution in $intX_+$. If (2.1) has a stationary solution $\widetilde{u} \in intX_+$, then $\widetilde{u}$ is globally asymptotically stable in X_+.

(iv) if $\mathfrak{s}(L + g(x, 0)) > 0$ and (2.1) has no positive stationary solution, then

$$\lim_{t \to \infty} ||u(\cdot, t, \phi_0)||_X = \infty, \textit{ for all } \phi_0 \in X_+ \setminus \{0\}$$

Proof. (i) Let φ be the eigenfunction associated with $\mathfrak{s}(L + g(x, 0))$. According to Proposition 2.2.3, $0 \leq u(x, t, \phi) \leq e^{(L + g(x,0))t}\phi$, where $u(x, t, \phi)$ is solution to (2.1) with $\phi \in X_+ \setminus \{0\}$. Since $\mathfrak{s}(L + g(x, 0)) < 0$, for some $M, \alpha > 0$

$$||e^{t(L + g(x,0))}||_X \leq Me^{-\alpha t}$$

(see [31] Theorem 1.3.4). Thus, we find $\lim_{t\to\infty} ||u(\cdot, t, \phi)||_X = 0$.

(ii) Because of $\mathfrak{s}(L + g(x, 0)) = 0$, $k\varphi$ is a supersolution of (2.1) if $k > 0$. By the comparison principle given in [33], we have

$$u(\cdot, t + h, k\varphi) = u(\cdot, t, u(\cdot, h, k\varphi)) \leq u(\cdot, t, k\varphi)$$

for each $h > 0$,that is, the function $t \mapsto u(\cdot, t, k\varphi)$ is nonincreasing. This ensure that the pointwise limit

$$\widehat{u} = \lim_{t \to \infty} u(\cdot, t, k\varphi) \tag{2.16}$$

exists. By The monotone convergence theorem and the continuity of f, $L\widehat{u} = f(x, \widehat{u})$ and $0 \le \widehat{u} \le k\varphi$. In fact, we have $\widehat{u} = 0$ a.e. Otherwise, with $J * \widehat{u} \gg 0$, we may argue as in Theorem 2.1.6 to deduce that $\widehat{u} \in intX_+$ which contradicts the fact that zero is the only non-negative steady state of (2.1) which is ensured by Lemma 2.1.10 Let $\phi_0 \in X_+ \backslash \{0\}$. Then there exists $k > 0$ such that $0 \le \phi_0 \ll k\varphi$. Again, the comparison principle gives $0 \le u(\cdot, t, \phi_0) \le u(\cdot, t, k\varphi)$, for all $t \ge 0$. (ii) follows from this fact together with (2.16).

(iii) By Lemma 2.1.11 and Corollary 2.1.13, (2.1) has an arbitrarily small subsolution $\epsilon\varphi$ and an arbitrarily large supersolution $k\widetilde{u}$, where φ is the strictly positive eigenfunction corresponding to $\mathfrak{s}(L + g(x, 0))$ and $0 < \epsilon < 1, k > 1$. With the reasoning similar to that for (ii), we find that the function $u(x, \cdot, \epsilon\varphi)$ is nondecreasing while $u(x, \cdot, k\widetilde{u})$ is nonincreasing. Furthermore, since (2.1) has an arbitrarily small subsolution, The comparison principle implies that there exist $\delta(\epsilon) > 0$ and $\delta(k) > 0$ such that $u(\cdot, t, \epsilon\varphi) \ge \delta(\epsilon)$ and $u(\cdot, t, k\widetilde{u}) \ge \delta(k)$ for all $t \ge 0$. The uniqueness of the positive stationary solution of (2.1) together with previous argument yields that the pointwise convergence

$$\lim_{t \to \infty} u(\cdot, t, \epsilon\varphi) = \widetilde{u}, \ \lim_{t \to \infty} u(\cdot, t, k\widetilde{u}) = \widetilde{u}$$

both hold true. Because $\widetilde{u}$ is continuous, Dini's Theorem gives

$$\lim_{t \to \infty} ||u(\cdot, t, \epsilon\varphi) - \widetilde{u}||_X = 0, \ \lim_{t \to \infty} ||u(\cdot, t, k\widetilde{u}) - \widetilde{u}||_X = 0 \tag{2.17}$$

(see [20]). For $\phi \in X_+ \backslash \{0\}$, by the comparison principle, there exists $h^* > 0$ such that $u(\cdot, h^*, \phi) \gg 0$. Moreover, we have that $u(\cdot, t, \underline{\gamma}\varphi) \le u(\cdot, t + h^*, \phi)$ for some $0 < \underline{\gamma} < 1$ and $u(\cdot, t, \phi) \le u(\cdot, t, \overline{\gamma}\widetilde{u})$ for some $\overline{\gamma} > 1$. Therefore, (iii) follows from (2.17).

(iv) Corollary 2.1.13 suggests

$$\mathfrak{s}(L + g(x, v)) > 0$$

for each $v \in intX_+$ which is Lipschitz. It is well known that $L + g(x, v)I$ generates a uniformly continuous semigroup $e^{t(L + g(x,v))I}$ on X. Let $\psi^* \in intX_+$ be an eigenfunction corresponding to $\mathfrak{s}(L + g(x, v)I)$. By the spectral mapping theorem, $\sigma(e^{t(L + g(x,v))}) = e^{t\sigma(L + g(x,v))I)}$ and $e^{t(L + g(x,v)I)}\psi^* = e^{t\mathfrak{s}(L + g(x,v))}\psi^*$. The comparison principle gives

$$e^{t(L + g(x,v)I)}\phi_0 \geq e^{t(L + g(x,v)I)}\rho\psi^* = \rho e^{t\mathfrak{s}(L + g(x,v)I)}\psi^*,$$

where $\phi_0 \in intX_+$ and ρ is a positive number such that $\rho\psi^* \leq \phi_0$. Since $\mathfrak{s}(L + g(x, v)I) > 0$

$$\lim_{t\to\infty} ||e^{t(L + g(x,v)I)}\phi_0||_X = \infty \tag{2.18}$$

for any $\phi_0 \in intX_+$. Note that (2.18) also holds for $e^{t(L + g(x,v)I)}\varphi_0$ with $\varphi_0 \in X_+ \setminus \{0\}$ because that $e^{(t+h)(L + g(x,v)I)} \gg 0$ for some $h > 0$ (see [33]). In the following, we shall adopt an idea in [23] to complete the proof. Suppose there is $w_0 \in X_+ \setminus \{0\}$ such that

$$||u(\cdot, t, w_0)||_X \leq c < \infty$$

for all $t \geq 0$, where $c > 0$ is constant. Therefore, we may choose positive constant $m > c$ such that

$$0 \leq u(\cdot, t, w_0) \leq m$$

for all $t \geq 0$. Consequently,

$$\begin{aligned} u_t(\cdot, \cdot, w_0) &= Lu(\cdot, \cdot, w_0) + f(x, u(\cdot, \cdot, w_0)) \\ &= Lu(\cdot, \cdot, w_0) + g(x, u(\cdot, \cdot, w_0))u(\cdot, \cdot, w_0) \\ &\geq Lu(\cdot, \cdot, w_0) + g(x, m)u(\cdot, \cdot, w_0) \end{aligned}$$

in $\overline{\Omega} \times (0, \infty)$. This fact and the comparison principle imply that

$$e^{t(L + g(x,m))}w_0 \leq u(\cdot, t, w_0)$$

for $t \geq 0$. Thus, (2.18) yields

$$\lim_{t \to \infty} ||u(\cdot, t, w_0)||_X = \infty.$$

The contradiction completes the proof.

CHAPTER 3

Existence and Stability of Coexistence States for a Nonlocal Evolution System

In this chapter we study the existence of component-wise positive steady state solutions of nonlocal evolutionary problem of the form

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 Lu + u[\lambda l(x) + f(x,u) + F(x,u,v)v], \\ \dfrac{\partial v}{\partial t} = d_2 Lv + v[\gamma m(x) + g(x,v) + G(x,u,v)u] \end{cases} \quad x \in [0,h] \tag{3.1}$$

where $Lu := \displaystyle\int_0^h J(x,y)u(y)dy - b(x)u(x)$ and $h > 0$. $d_i > 0 (i = 1,2)$ and $\lambda, \gamma \in \mathbb{R}$,

We also assume

(H1) $J(\cdot,\cdot) \in C([0,h]\times[0,h], \mathbb{R}^+)$ is symmetric and $J(x,y) > 0$ for any $x, y \in [0,h]$.

(H2)$\int_0^h |J(x+z,y) - J(x,y)|dy \le \widetilde{C}|z|$ for some $\widetilde{C} > 0$.

(H3) $b,\ l,\ m \in X_+ \backslash \{0\}$ are Lipschitz.

(H4) $f(x,w)$ and $g(x,w)$ are defined on $[0,h] \times [0,\infty)$, and C^1 continuous in both x and w, such that

$$f(x,0) = g(x,0) = 0, \quad \partial_w f(x,\cdot) < 0, \quad \partial_w g(x,\cdot) < 0.$$

(H5) $F(x,u,v)$ and $G(x,u,v)$ are defined on $[0,h]\times[0,\infty)^2$ and C^1 continuous in x and (u,v).

The system (3.1) admits three types of componentwise nonnegative solution couples in $X \times X$ that are independent of time. Namely, the trivial one $(0,0)$, the semitrivial positive solutions $(u,0)$ and $(0,v)$,where u, v are positive solutions of

$$d_1Lu + u[\lambda l(x) + f(x,u)] = 0, \quad x \in [0,h], \tag{3.2}$$

$$d_2Lv + v[\gamma m(x) + g(x,v)] = 0, \quad x \in [0,h] \tag{3.3}$$

respectively, and the coexistence states, which are the solution couples (u,v) with both components positive. $(0,0)$ is always a solution of (3.1). Moreover, (3.1) has a semitrivial coexistence state of the form $(u,0)$ (resp.$(0,v)$) if and only if (3.2) (resp.(3.3)) has a positive solution in X.

3.1 The existence of coexistence states

We now consider the one-parameter family of eigenvalue problems

$$(L+\lambda K)u = \mu u, \quad \lambda \in \mathbb{R}, u \in X \tag{3.4}$$

where the operator L is given by (3.1), $b \in X$ is Lipschitz. $K \in \mathcal{L}(X)$ is the multiplication operator by $k \in X$. $L(\lambda) = L + \lambda K$. we shall denote the principal eigenvalue of $L(\lambda)$ by $\mu(\lambda)$, that is, an eigenvalue associated with an eigenfunction $u \in X_+$.

Lemma 3.1.1 *Suppose that $k \in X$ is Lipschitz, then the linear operator $L(\lambda)$ has a unique principal eigenvalue equal to $\mathfrak{s}(L(\lambda))$ and the mapping $\lambda \mapsto \mathfrak{s}(L(\lambda))$ is analytic. Moreover, it is convex.*

Proof. The existence and uniqueness of an principal eigenvalue for $L(\lambda)$ is an consequence of Lemma 2.1.2 and 2.1.9. Moreover, the principal eigenvalue is equal to

$\mathfrak{s}(L(\lambda))$ for fixed λ. Therefore we only need to show that $\lambda \mapsto \mathfrak{s}(L(\lambda))$ is analytic. Let $\widetilde{\lambda} \in \mathbb{R}$ be an arbitrarily fixed number. Note that $\lambda \mapsto L(\lambda)$ is analytic from $(\widetilde{\lambda}-1, \widetilde{\lambda}+1)$ into $\mathcal{L}(X)$ and $\mathfrak{s}(L(\widetilde{\lambda}))$ is a simple eigenvalue of $L(\widetilde{\lambda})$ with eigenvector $\psi(\widetilde{\lambda}) \in intX_+$. By the Proposition 4.5.8 in [15] , there exists $\epsilon > 0$ and analytic curve $\left\{\pi(s), \psi(s) : s \in (\widetilde{\lambda}-\epsilon, \widetilde{\lambda}+\epsilon)\right\} \subset \mathbb{R} \times X$ such that$(\pi(0), \psi(0)) = (\mathfrak{s}(L(\widetilde{\lambda})), \psi(\widetilde{\lambda}))$,

$$(L(s))\psi(s) = \pi(s)\psi(s) \text{ and } \psi(s) = \psi(\widetilde{\lambda}) + \eta(s),$$

where $\eta(s) \in \text{range}(L(\widetilde{\lambda}) - \mathfrak{s}(L(\widetilde{\lambda}))I)$ and $\pi(s)$ is a simple eigenvalue of $L(s)$. For sufficiently small ϵ, $\psi(s) \in intX_+$ if $|s| < \epsilon$. Lemma 2.1.2 implies $\mathfrak{s}(L(s)) \equiv \pi(s)$. Since $\widetilde{\lambda}$ is arbitrary, the desired result follows. Finally, μ is convex with respect to λ because $\mu(\lambda) = \sup_{||u||_H=1}(L + \lambda K u, u)$ is affine for fixed u.

Proposition 3.1.2 *Suppose that $k \in X$ is Lipschitz. Let $\Gamma_+ = \{x \in [0,h] | \; k(x) > 0\}$, $\Gamma_- = [0,h]\backslash\Gamma_+$. Then $\mu(\lambda) \to \infty$ as $\lambda \to \infty$ if $\Gamma_+ \neq \emptyset$ and $\mu(\lambda) \to \infty$ as $\lambda \to -\infty$ if $\Gamma_- \neq \emptyset$.*

Proof. We only give a proof for $\Gamma_+ \neq \emptyset$. The case that $\Gamma_- \neq \emptyset$ can be treated similarly. Choose $x' \in \Gamma_+$ and $\delta > 0$ such that $(x'-\delta, x'+\delta) \subset \Gamma_+$ and $\min_{[x'-\delta, x'+\delta]} k(x) > c$ for some positive constant c. Let $\widetilde{\theta} \in C_0([0,h]) \cap X_+$ such that $\text{supp}\widetilde{\theta} \subset (x'-\delta, x'+\delta)$ and $\widetilde{\theta} \equiv 1$ in $(x' - \frac{1}{2}\delta, x' + \frac{1}{2}\delta)$. A Straightforward calculation gives

$$((L+\lambda K)\widetilde{\theta}, \widetilde{\theta}) \geq - \max_{x \in [0,h]} \int_0^h J(x,y)dy - ||b||_{L^1} || ||\widetilde{\theta}||^2_{L^\infty} + \lambda c\delta.$$

Consequently,

$$\mu(\lambda) = \max_{||u||_H = 1} ((L+\lambda K)u, u) \to \infty \quad as \; \lambda \to \infty.$$

Next, we consider the eigenvalue problem

$$(L + \lambda K)\phi = 0, \tag{3.5}$$

denoting a principal eigenvalue by $\lambda(k)$ and distinguish two cases

$$(D)\ \int_0^h J(x,y)dy \le (\neq) b(x) \qquad\qquad (N)\ \int_0^h J(x,y)dy \equiv b(x)$$

Remark 3.1.3 *By the definition, $\lambda(k)$ is a root of $\mu(\cdot)$.*

Lemma 3.1.4 *Assume (case D)*

(i) (3.5) has two principal eigenvalues $\lambda_1(k) < 0 < \lambda_2(k)$ if $\Gamma_+ \neq \emptyset$ and $\Gamma_- \neq \emptyset$.

(ii) (3.5) has a unique principal eigenvalue $\lambda(k) > 0$ if $\Gamma_- = \emptyset$ and $\lambda(k) < 0$ if $\Gamma_+ = \emptyset$.

Proof. (i) Note that $L(0)1 \le 0$. Lemma 2.1.3 implies $\mu(0) < 0$. On other hand, Proposition 3.1.2 shows that $\mu(M) > 0$ and $\mu(-M) > 0$ for some $M > 0$. By the continuity of μ, the result follows.

(ii) Suppose $\Gamma_- = \emptyset$. By (i), We see that (3.5) has a principal eigenvalue $\lambda^*(k) > 0$. Since k is nonnegative, by Lemma 2.1.4, $\mu(\lambda)$ is monotone with respect to λ. Therefore, uniqueness follows.

Lemma 3.1.5 *Assume (case N). Suppose $\Gamma_+ \neq \emptyset$ and $\Gamma_+ \neq \emptyset$, then (3.5) has a principal eigenvalue $\lambda(k) > 0$ if $\int_0^h k(x)dx < 0$; $\lambda(k) < 0$ if $\int_0^h k(x) > 0$ and 0 is the unique principal eigenvalue if $\int_0^h k(x) = 0$ and $k(x) \neq 0$.*

Proof.

Let $(\mu(\lambda), \phi(\lambda))$ be the eigenvalue-eigenfunction pair for (3.4). Differentiating (3.4) with respect to λ and noting $\mu(0) = 0$ and $\phi(0) = 1$, we find

$$L\phi'(0) + K1 = \mu'(0)\phi(0),$$

where $' := \dfrac{d}{d\lambda}$. Integrating both side , we have

$$\int_0^h k(x)dx = h\mu'(0).$$

If $\int_0^h k(x)dx < 0$ then $\mu'(0) < 0$ and by the convexity of μ, $\mu(\lambda) > 0$ for any $\lambda < 0$. Furthermore, by Proposition 3.1.2, μ has a positive zero. The case that $\int_0^h k(x)dx > 0$ follows similarly. Next, we assume that $\int_0^h k(x) = 0$ and $k(x) \neq 0$. Clearly, $\mu'(0) = 0$ and $L\phi'(0) = -k(x)$. Taking the derivative of (3.4) twice gives

$$L\phi''(0) + K\phi'(0) = \mu''(0).$$

Consequently, we have

$$h\mu''(0) = \int_0^h k(x)\phi'(0)dx = -\int_0^h L\phi'(0)\phi'(0)dx > 0. \tag{3.6}$$

The last inequality is true because $\sup_{||u||_H=1}(Lu, u) = 0$ and the supreme is attained only by constant. The convexity of μ and (3.6) implies $\mu(\lambda) > 0$ for any $\lambda \neq 0$.

Throughout the remainder of this section, $\sigma[dL, K]$ will stand for the principal eigenvalue of $(dL + \lambda K)\phi = 0$

Proposition 3.1.6 *Assume (H1)-(H5), then*

$$d_1 Lw + w(\lambda l(x) + f(x, w)) = 0 \quad in\ [0, h] \tag{3.7}$$

has a unique positive continuous solution defined on $[0, h]$ if and only if $\lambda > \sigma[d_1 L, \lambda l]$. Furthermore, if θ_λ is a positive continuous solution to (3.7), then $\theta_\lambda \in int\ X_+$ is Lipschitz and the operator $L_{1,\lambda}$ defined by

$$L_{1,\lambda} = d_1 L + \lambda l(x) + \theta_\lambda D_w f(x, \theta_\lambda) + f(x, \theta_\lambda)$$

is invertible.

Proof. The existence of a unique positive Lipschitz continuous solution is an consequence of Theorem 2.1.6. To prove the other part, we observe that $\mathfrak{s}(d_1 L + (\lambda l(x) + f(x, \theta_\lambda)I)) = 0$ and $\theta_\lambda D_w f(x, \theta_\lambda) < 0$, which implies $\mathfrak{s}(L_{1,\lambda}) < 0$. Then Lemma 2.1.3 gives the desired conclusion.

Theorem 3.1.7 *Assume (H1)-(H5). Let $\theta_\lambda \in X_+$ be the solution to (3.7). Consider*

$$\gamma_\lambda := \sigma[d_2 L, \gamma m + G(x, \theta_\lambda, 0)\theta_\lambda].$$

Then a continuum $C^+ \subset \mathbb{R} \times intX_+ \times intX_+$ of coexistence states of (3.1) emanates from the point $(\gamma_\lambda, \theta_\lambda, 0)$.

Proof. We fix $\lambda \in \mathbb{R}$ and treat γ as the bifurcation parameter. We first show that $(\gamma_\lambda, \theta_\lambda, 0)$ is a bifurcation point to a branch of coexistence states. To do that, we use the theorem by Crandall and Rabinowitz in [28]. Consider the operator $\mathcal{F}$: $\mathbb{R} \times X \times X \to X \times X$ defined by

$$\mathcal{F}(\gamma, u, v) = \begin{pmatrix} d_1 Lu + u[\lambda l(x) + f(x, u) + F(x, u, v)v] \\ d_2 Lv + v[\gamma m(x) + g(x, u) + G(x, u, v)u] \end{pmatrix}.$$

$\mathcal{F}$ is an operator of class C^2 in (u, v) and analytic in γ and the zeros of $\mathcal{F}$ are the solutions to (3.1). Since θ_λ solves (3.1), $\mathcal{F}(\gamma, \theta_\lambda, 0) = 0$ for all $\gamma \in \mathbb{R}$ and hence $(\gamma, \theta_\lambda, 0)$ can be regarded as the known branch of solution to (3.1). The linearization of $\mathcal{F}$ at $(\gamma_\lambda, \theta_\lambda, 0)$ with respect to (u, v) is given by

$$\mathcal{J}(\gamma) := D_{(u,v)}\mathcal{F}(\gamma, \theta_\lambda, 0) = \begin{pmatrix} L_{1,\lambda} & F(x, \theta_\lambda, 0)\theta_\lambda \\ 0 & d_2 L + \gamma m(x) + g(x, \theta_\lambda) + G(x, \theta_\lambda, 0)\theta_\lambda \end{pmatrix},$$

where $L_{1,\lambda}$ is the operator defined in Proposition 3.1.6. Let $\varphi \gg 0$ denote the principal eigenfunction of $d_2 L + \gamma m(x) + g(x, \theta_\lambda) + G(x, \theta_\lambda, 0)\theta_\lambda$,which is unique up to multiplicative constants. Then the null space of $\mathcal{J}(\gamma_\lambda)$ is

$$N(\mathcal{J}(\gamma_\lambda)) = \text{span}\left\{ (L_{1,\lambda}^{-1}(F(x, \theta_\lambda, 0)\theta_\lambda \varphi), -\varphi)^T \right\}. \tag{3.8}$$

Since $L_{1,\lambda}$ is invertible and $R[d_2 L + \gamma_\lambda m(x) + g(x, \theta_\lambda) + G(x, \theta_\lambda, 0)\theta_\lambda]$ has codimension one, the range of $\mathcal{J}(\gamma_\lambda)$

$$R(\mathcal{J}(\gamma)) = X \times R[d_2 L + \gamma_\lambda m(x) + g(x, \theta_\lambda) + G(x, \theta_\lambda, 0)\theta_\lambda]$$

has codimension one. Furthermore, the operator $\mathcal{J}(\gamma)$ is a polynomial in $\gamma - \gamma_\lambda$ and can be written as

$$\mathcal{J}(\gamma) = \mathcal{J}(\gamma_\lambda) + (\gamma - \gamma_\lambda)\mathcal{J}_1, \quad \mathcal{J}_1 = \begin{pmatrix} 0 & 0 \\ 0 & m(x) \end{pmatrix}. \tag{3.9}$$

In particular, it is analytic in γ. From (3.8) and (3.9), we find that

$$\mathcal{J}_1(N(\mathcal{J}(\gamma_\lambda))) = \text{span}\left\{(0, m(x)\varphi)^T\right\}.$$

We claim that

$$m(x)\varphi \notin R[d_2L + \gamma_\lambda m(x) + g(x, \theta_\lambda) + G(x, \theta_\lambda, 0)\theta_\lambda]. \tag{3.10}$$

To show this, we argue by contradiction assuming that there exists $u \in X$ such that

$$(d_2L + \gamma_\lambda m(x) + g(x, \theta_\lambda) + G(x, \theta_\lambda, 0)\theta_\lambda)u = m(x)\varphi.$$

Since $d_2L + \gamma_\lambda m(x) + g(x, \theta_\lambda) + G(x, \theta_\lambda, 0)\theta_\lambda$ is a self-adjoint operator on H, we obtain $\int_0^h m(x)\varphi^2 dx = 0$, which is impossible because the integrand is nonnegative and nontrivial, and so (3.10) holds. Thus,

$$(0, m(x)\varphi)^T \notin R(\mathcal{J}(\gamma_\lambda))$$

and the following transversality condition holds

$$\mathcal{J}_1(N(\mathcal{J}(\gamma_\lambda))) \oplus R(\mathcal{J}(\gamma_\lambda)) = X \times X.$$

Therefore, it follows from Crandall-Rabinowitz's Theorem that γ_λ is a bifurcation point. Furthermore, there exists $\varepsilon > 0$ and a curve $s \to (\gamma(s), u(s), v(s))$, $|s| < \varepsilon$, of class C^1 such that

$$(\gamma(s), u(s), v(s)) = (\gamma_\lambda + O(s), \theta_\lambda + O(s), s\varphi + O(s^2)), \quad s \to 0,$$

and

$$\mathcal{F}(\gamma(s), u(s), v(s)) = 0, \quad |s| < \varepsilon.$$

As $(\theta_\lambda, \varphi) \in intX_+ \times intX_+$, $(u(s), v(s))$ is component-wise positive if $s > 0$ is sufficiently small and hence (3.1) has a coexistence state for some γ near γ_λ.

3.2 Two similar competing species

In this section, we consider a special case of (3.1) where u and v stand for the densities of two competing species. We assume that two species are very similar. Namely, in what follows, we shall focus our attentions on the system

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mu L_N u + u(\alpha(x) + \tau\beta(x) - u - v), \\ \dfrac{\partial v}{\partial t} = \mu L_N v + v(\alpha(x) - u - v) \end{cases} \quad \text{in } [0, h] \tag{3.11}$$

where $L_N w := \int_0^h J(x,y)w(y)dy - \int_0^h J(x,y)dyw$, α, $\beta \in X$ are Lipschitz in $[0, h]$, and $\int_0^h \alpha(x)dx > 0$. Assume that τ is positive but very small. Clearly, when $\tau = 0$, u and v play identical roles in this system. For each fixed μ, there is a set of nonnegative equilibria $\{(s\widehat{v}, (1-s)\widehat{v}) | 0 \le s \le 1\}$. Here $\widehat{v}$ is the unique positive continuous solution to the equation

$$\mu L_N v + v(\alpha(x) - v) = 0. \tag{3.12}$$

$\widehat{v}$ depends smoothly on μ. (see the appendix for more details)

Following the approach employed in [34], we let $M : (0, \infty) \to \mathbb{R}$ be defined by

$$M(\mu) = \int_0^h \beta(x)\widehat{v}(x, \mu)dx, \quad \mu > 0, \tag{3.13}$$

For $\tau \approx 0$, we look for triples (u, v, μ) that satisfy

$$\begin{cases} \mu L_N u + u(\alpha(x) + \tau\beta(x) - u - v) = 0, \\ \mu L_N v + v(\alpha(x) - u - v) = 0 \end{cases} \quad \text{in } [0, h] \tag{3.14}$$

and that are close to the curve $\Gamma_{\widehat{\mu}} \times \{\widehat{\mu}\}$ for some $\widehat{\mu} > 0$, where

$$\Gamma_{\widehat{\mu}} = \{(s\widehat{v}(\cdot\widehat{\mu}), (1-s)\widehat{v}(\cdot, \widehat{\mu}))|0 \le s \le 1\}.$$

Note that for any μ, $\Gamma_\mu \times \{\mu\}$ is a curve of solutions of (3.14) for $\tau = 0$. Also, for any small τ, (3.14) has the semitrivial solutions

$$(0, \widehat{v}(\cdot, \mu), \mu) \quad \text{(independent of } \tau)$$
$$(\widehat{u}(\cdot, \mu, \tau), 0, \mu),$$

where $\widehat{u}(\cdot, \mu, \tau)$ is the positive solution of (3.12) with $\alpha(x)$ being replaced by $\gamma(x) = \alpha(x) + \tau\beta(x)$. For our purpose, we introduce the following function space.

$$\begin{aligned} Y &= X \times X \\ X_2 &= \left\{(w, z) \in Y : \int_0^h (w(x) - z(x))\widehat{v}(x, \widehat{\mu})dx = 0\right\}. \end{aligned}$$

Clearly, X_2 is a closed subspace. As will become clear through the analysis presented in the section, X_2 is the complement of the subspace spanned by $(\widehat{v}(x, \widehat{\mu}), -\widehat{v}(x, \widehat{\mu}))^T$, which is the kernal of a Fredholm operator we discuss in Proposition 3.2.1

Proposition 3.2.1 *Assume $\int_0^h \alpha(x)dx > 0$. Then for any $\widehat{\mu}$ there exists a neighborhood U of the curve $\Gamma_{\widehat{\mu}} \times \{\widehat{\mu}\}$ in $Y \times (0, \infty)$ and $\delta > 0$ with the following properties*

(i) If $M(\widehat{\mu}) \neq 0$, then for $\tau \in (0, \delta)$ there are no solutions of (3.14) in U other than the semitrivial solutions of (3.14).

(ii) If $M(\widehat{\mu}) = 0$ and $M'(\widehat{\mu}) \neq 0$, then for $\tau \in (0, \delta)$ the set of solutions of (3.14) in U consists of the semitrivial solutions and of the set $\Sigma \cap U$, where Σ is a smooth curve given by

$$\Sigma = \{((u(\tau,s), v(\tau,s), \mu(\tau,s)) : -\delta \le s \le 1+\delta\}. \tag{3.15}$$

Here $(\tau,s) \mapsto (u(\tau,s), v(\tau,s)) \in Y$ and $(\tau,s) \mapsto \mu(\tau,s) \in (0,\infty)$ are smooth functions on $[0,\delta) \times (-\delta, 1+\delta)$ satisfying the following relations

$$(u(\tau,0), v(\tau,0)) = (0, \widehat{v}(\cdot, \mu(\tau,0))), \tag{3.16}$$

$$(u(\tau,1), v(\tau,1)) = (\widehat{u}(\cdot, \mu(\tau,1), \tau), 0), \tag{3.17}$$

$$(u(0,s), v(0,s), \mu(0,s)) = (s\widehat{v}(\cdot, \widehat{\mu}), (1-s)\widehat{v}(\cdot, \widehat{\mu}), \widehat{\mu}). \tag{3.18}$$

Remark 3.2.2 *In other words, a branch of coexistence states bifurcates from the branch of semitrivial equilibria* $(\widehat{u}, 0)$ *at* $\mu = \mu(\tau, 1)$ *and meets the other branch of semitrivial equilibria* $(0, \widehat{v})$ *at* $\mu = \mu(\tau, 0)$. *For* $\tau = 0$ *the branch coincides with* $\Gamma_{\widehat{\mu}}$.

Note that from (3.18), it follows that the function $u(\tau,s), v(\tau,s)$ *and* $\mu(\tau,s)$ *have the following expansion for* $-\delta \le s \le 1+\delta$ *and* $|\tau| \ll 1$:

$$u(\tau,s) = s\widehat{v}(\cdot, \widehat{\mu}) + \tau u_1(s) + O(\tau^2), \tag{3.19}$$

$$v(\tau,s) = (1-s)\widehat{v}(\cdot, \widehat{\mu}) + \tau v_1(s) + O(\tau^2), \tag{3.20}$$

$$\mu(\tau,s) = \widehat{\mu} + \tau\widehat{\mu}_1(s) + O(\tau^2), \tag{3.21}$$

where $(u_1, v_1) \in Y$ *and* $\widehat{\mu} \in \mathbb{R}$ *are smooth function of* s.

Proof. We look for any triple (u, v, μ) near $\Gamma_{\widehat{\mu}} \times \{\widehat{\mu}\}$ such that (u, v) can be written as

$$(u, v) = (s\widehat{v}(\cdot, \mu), (1 - s)\widehat{v}(\cdot, \mu)) + (w, z), \tag{3.22}$$

where $s \in \mathbb{R}$ and $(w, z) \in X_2$, and they are in or near $[0, 1]$ and $\{(0, 0)\}$, respectively. Rewrite (3.22) as

$$(u, v - \widehat{v}(\cdot, \mu)) = s(\widehat{v}(\cdot, \mu), -\widehat{v}(\cdot, \mu)) + (w, z).$$

We shall find s and (w, z) from

$$\begin{aligned} (w, z) &= Q(\mu)(u, v - \widehat{v}(\cdot, \mu)), \\ s(\widehat{v}(\cdot, \mu), -\widehat{v}(\cdot, \mu)) &= (I - Q(\mu))(u, v - \widehat{v}(\cdot, \mu)), \end{aligned}$$

where I is the identity on Y and $Q(\mu)$ is the projection of Y onto X_2 along the subspace

$$X_1(\mu) := \text{span } \{(\widehat{v}(\cdot, \mu), -\widehat{v}(\cdot, \mu))\}.$$

(Note that $X_1(\mu)$ is a complement of X_2 in Y for $\mu \approx \widehat{\mu}$.) In particular, we find the following values of s and (w, z) for semitrivial equilibria:

$$(0, \widehat{v}(\cdot, \mu)) = (0, \widehat{v}(\cdot, \mu)) + (0, 0), \tag{3.23}$$

$$(\widehat{u}(\cdot, \mu, \tau), 0) = (s\widehat{v}(\cdot, \widehat{\mu}), (1 - s)\widehat{v}(\cdot, \widehat{\mu})) + (\eta(\tau, \mu), \zeta(\tau, \mu)), \text{with} s = \sigma(\tau, \mu), \tag{3.24}$$

where (η, ζ) and σ are smooth functions of (τ, μ) taking values in X_2 and $\mathbb{R}$, respectively. Clearly,

$$\sigma(0, \mu) = 1, \quad (\eta(0, \mu), \zeta(0, \mu)) = (0, 0), \tag{3.25}$$

since $\widehat{u}(\cdot, \mu, 0) = v(\cdot, \mu)$.

For a small $\delta > 0$ let F be the map on $Y \times (-\delta, \delta) \times (-\delta, 1 + \delta) \times (\widehat{\mu} - \delta. \widehat{\mu} + \delta)$ defined by

$$F(w,z,\tau,s,\mu) =$$

$$\begin{bmatrix} \mu L_N w - (w+z)s\widehat{v}(\cdot,\mu) + (\alpha - \widehat{v}(\cdot,\mu))y - (w+z)w + \tau\beta s\widehat{v}(\cdot,\mu) + \tau\beta w \\ \mu L_N z - (w+z)(1-s)\widehat{v}(\cdot,\mu) + (\alpha - \widehat{v}(\cdot,\mu))z - (w+z)z \end{bmatrix}.$$

It is clear that F is well defined and smooth (in fact polynomial) as a Y-valued map. By plugging (3.22) into (3.14), we see that, in order to finds solution to (3.14), we need to solve the equation

$$F(w,z,\tau,s,\mu) = 0, \tag{3.26}$$

with $(w,z) \in X_2$. By examining the properties of $F(w,z,\tau,s,\mu)$ for $(w,z) \in X_2$. we find

$$F(0,0,0,s,\mu) \equiv 0 \quad (s \in (-\delta, 1+\delta), \mu \in (\widehat{\mu} - \delta, \widehat{\mu} + \delta)), \tag{3.27}$$

$$F(0,0,\tau,0,\mu) \equiv 0 \quad (\tau \in (-\delta,\delta), \mu \in (\widehat{\mu} - \delta, \widehat{\mu} + \delta)), \tag{3.28}$$

$$F(\eta(\tau,\mu), \zeta(\tau,\mu), \tau, \sigma(\tau,\mu), \mu) \equiv 0 \quad (\tau \in (-\delta,\delta), \mu \in (\widehat{\mu} - \delta, \widehat{\mu} + \delta)). \tag{3.29}$$

Define

$$E(s,\mu) := D_{(w,z)}F(0,0,s,\mu) \in \mathcal{L}(Y,Y),$$

that is,

$$E(s,\mu) = \begin{bmatrix} \mu L_N - s\widehat{v} + (\alpha - \widehat{v}) & -s\widehat{v} \\ -(1-s)\widehat{v} & \mu L_N - (1-s)\widehat{v} + (\alpha - \widehat{v}) \end{bmatrix}. \tag{3.30}$$

We may rewrite (3.30) as

$$E(s,\mu) = K(\mu) - V(s,\mu),$$

where

$$K(\mu)(u,v)^T := \begin{bmatrix} \mu J * u & 0 \\ 0 & \mu J * v \end{bmatrix}$$

and

$$V(s,\mu) := \begin{bmatrix} \int_0^h J(x,y)dy + s\widehat{v} + (\widehat{v}-\alpha) & s\widehat{v} \\ (1-s)\widehat{v} & \int_0^h J(x,y)dy + (1-s)\widehat{v} + (\widehat{v}-\alpha) \end{bmatrix}.$$

Since $\widehat{v} \in intX_+$ is a solution of (3.12), The fact that $J * \widehat{v} = (\int_0^h J(x,y)dy + \widehat{v} - \alpha)\widehat{v}$ immediately implies $\int_0^h J(x,y)dy + \widehat{v} - \alpha > 0$ for any $x \in [0,h]$. Therefore, $V(s,\mu)$ is an invertible operator on Y and it is a standard consequence of the compactness of $K(\mu)$ that $E(s,\mu)$ is a Fredholm operator of index zero. By the definition of F, the vector $(\widehat{v}(\cdot,\mu), -\widehat{v}(\cdot,\mu))^T$ is in the kernel of $E(s,\mu)$, or equivalently it is an eigenfunction corresponding to the eigenvalue 0 of $E(s,\mu)$. By the positivity of $\widehat{v}$, zero must be a simple eigenvalue of $E(s,\mu)$ and

$$\mathrm{Ker}E(s,\mu) = \mathrm{span}\left\{(\widehat{v}(\cdot,\mu), -\widehat{v}(\cdot,\mu))^T\right\} = X_1(\mu).$$

Indeed, let $(\overline{u},\overline{v})$ be an eigenfunction associated with 0. Plugging it into the equation

$$E(s,\mu)(u,v)^T = 0,$$

we find

$$\mu L_N(\overline{u}+\overline{v}) + (\alpha - 2\widehat{v})(\overline{u}+\overline{v}) = 0.$$

By virtue of Lemma 2.1.4, $s(\mu L_N + (\alpha - 2\widehat{v})I) < 0$. Consequently, $\mu L_N + (\alpha - 2\widehat{v})I$ is invertible. It follows that $\overline{u}+\overline{v} \equiv 0$ and $(\mu L_N + \alpha - \widehat{v})\overline{u} = 0$. By Lemma 2.1.2, one sees that $\overline{u} = c\widehat{v}$ for some constant c. As will become clear late, $\ker E(s,\mu) \oplus R(E(s,\mu)) = Y$. This will confirm that zero is a simple eigenvalue.

Now, let $P(s,\mu)$ be the continuous linear projection of Y onto $X_1(\mu)$ along the range of $E(s,\mu)$ (the range $R(E(s,\mu))$ is a closed subspace of Y of codimension one). $P(s,\mu)$ can be explicitly written as follows:

$$P(s,\mu)(w,z)^T = \frac{(1-s)\int_0^h \widehat{v}(\cdot,\mu)w - s\int_0^h \widehat{v}z}{\int_0^h \widehat{v}^2(\cdot,\mu)}(\widehat{v}(\cdot,\mu), -\widehat{v}(\cdot,\mu))^T. \tag{3.31}$$

In fact, we have

$$R(P(s,\mu)) = X_1(\mu), \quad (P(s,\mu))^2 = P(s,\mu), \quad \text{and } P(s,\mu)E(s,\mu) = 0.$$

These properties can be verified by a straightforward computation. Formula (3.31) in particular implies

$$Y_2(s,\mu) := R(E(s,\mu)) = \left\{(w,z) \in Y : (1-s)\int_0^h \widehat{v}(\cdot,\mu)w - s\int_0^h \widehat{v}(\cdot,\mu)z = 0\right\}.$$

Also note that $(s,\mu) \mapsto P(s,\mu)$ is smooth (in the operator norm). Following the Lyapunov-Schmidt scenario, we now consider the system

$$P(s,\mu)F(w,z,\tau,s,\mu) = 0, \tag{3.32}$$

$$(I - P(s,\mu))F(w,z,\tau,s,\mu) = 0, \tag{3.33}$$

where $(w,z) \in X_2$ and I is the identity on Y. If μ is sufficiently close to $\widehat{\mu}$ (and we make δ small enough for that to hold for all $\mu \in (\widehat{\mu} - \delta, \widehat{\mu} + \delta)$, then

$$\ker E(s,\mu) \cap X_2 = \{0\}.$$

It follows that $E(s,\mu)$ is an isomorphism of X_2 onto $Y_2(s,\mu)$. By the implicit function theorem, we can thus solve (3.32), which leads to the following conclusion. There exists $\delta_1 > 0$, a neighborhood U of $(0,0) \in X_2$, and a smooth function

$$(\tau,s,\mu) \mapsto (w(\tau,s,\mu), z(\tau,s,\mu)) : (-\delta_1,\delta_1) \times (-\delta_1, 1+\delta_1) \times (\widehat{\mu} - \delta_1, \widehat{\mu} + \delta_1) \to X_2$$

such that $(w(0,s,\mu), z(0,s,\mu)) = (0,0)$ and $(w,z,\tau,s,\mu) \in U \times (-\delta_1,\delta_1) \times (-\delta_1, 1+\delta_1) \times (\widehat{\mu} - \delta_1, \widehat{\mu} + \delta_1)$ satisfies (3.26) if and only if $(w,z) = (w(\tau,s,\mu), z(\tau,s,\mu))$ and (τ,s,μ) solves the bifurcation equation

$$P(s,\mu)F(w(\tau,s,\mu), z(\tau,s,\mu)), \tau, s, \mu) = 0.$$

By (3.28) and (3.29), w and z satisfy

$$(w(\tau,0,\mu),z(\tau,0,\mu))=(0,0), \tag{3.34}$$

$$(w(\tau,\sigma(\tau,\mu),\mu),z(\tau,\sigma(\tau,\mu),\mu)))=(\eta(\tau,\mu),\zeta(\tau,\mu)). \tag{3.35}$$

Now, defining $\xi(\tau,s,\mu)$ by

$$\xi(\tau,s,\mu)(\widehat{v}(\cdot,\mu),-\widehat{v}(\cdot,\mu))^T=P(s,\mu)F(w(\tau,s,\mu),z(\tau,s,\mu)),\tau,s,\mu),$$

the bifurcation equation is equivalent to

$$\xi(\tau,s,\mu)=0. \tag{3.36}$$

We immediately have the following solution of (3.25):

$$\xi(0,s,\mu)\equiv\xi(\tau,0,\mu)\equiv\xi(\tau,\sigma(\tau,\mu),\mu)\equiv 0. \tag{3.37}$$

These identities hold because for each of the indicated values of $(\tau,s,\mu)\in(-\delta_1,\delta_1)\times(-\delta_1,1+\delta_1)\times(\widehat{\mu}-\delta_1,\widehat{\mu}+\delta_1)$, there is a solution $(w,z)\in U$ of (3.26); see (3.27)-(3.29). Recall that from these solutions, the triples $(\tau,0,\mu)$ and $(\tau,\sigma(\tau,\mu),\mu)$ correspond to semitrivial equilibria of (4.10); see (3.23),(3.24) and (3.35).

It follows from (3.37) that

$$\xi(\tau,s,\mu)=s(\sigma(\tau,\mu)-s)\tau\xi_1(\tau,s,\mu)$$

for some smooth function $\xi_1(\tau,s,\mu)$. Solutions of (3.36) different from (3.37) are found by solving

$$\xi_1(\tau,s,\mu)=0. \tag{3.38}$$

Observe that

$$\partial_T\xi(0,s,\mu)\equiv s(1-s)\xi_1(0,s,\mu),$$

as $\sigma(0,\mu) \equiv 1$. The derivative on the left-hand side is computed from

$$\begin{aligned}
(\widehat{v}(\cdot,\mu), -\widehat{v}(\cdot,\mu))\partial_T\xi(0,s,\mu) &= \partial_T(P(s,\mu)F(w(\tau,s,\mu),z(\tau,s,\mu)),\tau,s,\mu))|_T = 0 \\
&= P(s,\mu)F_T(0,0,0,s,\mu) + P(s,\mu)E(s,\mu)(w_T,z_T) \\
&= P(s,\mu)F_T(0,0,0,s,\mu)
\end{aligned}$$

(recall that $R(E(s,\mu)) = \ker P(s,\mu)$). Using (3.31), we find

$$P(s,\mu)F_T(0,0,0,s,\mu) = P(s,\mu)(s\beta\widehat{v}(\cdot,\mu),0)^T = \frac{s(1-s)\int_0^h \beta\widehat{v}^2}{\int_0^h \widehat{v}^2}(\widehat{v}(\cdot,\mu), -\widehat{v}(\cdot,\mu))^T.$$

Thus

$$\partial_T\xi(0,s,\mu) = \frac{s(1-s)M(\mu)}{\int_0^h \widehat{v}^2},$$

i.e.,

$$\xi_1(0,s,\mu)|_{\mu = \widehat{\mu}} = \frac{M(\widehat{\mu})}{\int_0^h \widehat{v}^2(x,\widehat{\mu})}. \tag{3.39}$$

To complete the proof, consider first the case $M(\widehat{\mu}) \neq 0$. Making δ_1 smaller, if necessary, we infer from (3.39) that (3.38) has no solution in $(-\delta_1,\delta_1) \times (-\delta_1, 1 + \delta_1) \times (\widehat{\mu} - \delta_1, \widehat{\mu} + \delta_1)$. This implies statement (i) of Proposition 3.2.1.

Now assume $M(\widehat{\mu}) = 0$. Then

$$\partial_\mu\xi_1(0,s,\mu)|_{\mu = \widehat{\mu}} = \frac{M'(\widehat{\mu})}{\int_0^h \widehat{v}^2(x,\widehat{\mu})}.$$

If $M'(\widehat{\mu}) \neq 0$, the implicit function theorem implies that for some $\delta_2 > 0$, all solutions of (3.38) in $(-\delta_1,\delta_1) \times (-\delta_1, 1+\delta_1) \times (\widehat{\mu} - \delta_1, \widehat{\mu} + \delta_1)$ are given by

$$\mu = m(\tau,s), \quad \tau \in (-\delta_2,\delta_2), \quad s \in (-\delta_2, 1+\delta_2),$$

where $m(\tau,s)$ is a smooth function satisfying $m(0,s) \equiv \widehat{\mu}$. Thus, in addition to the solutions given by (3.37), the bifurcation equation (3.36) has the family of solutions

$$\{(\tau,s,m(\tau,s)) : \tau \in (-\delta_2,\delta_2), \quad s \in (-\delta_2, 1+\delta_2)\}. \tag{3.40}$$

In this family, the point $(\tau,0,m(\tau,0))$ is also contained in the set of solutions found in (3.37), and it corresponds to the semitrivial solution $(0,\widehat{v}(\cdot,\mu))$ of (4.10) with

$\mu = m(\tau, 0)$. Next we look for points $(\tau, s, m(\tau, s))$ in the family corresponding to the semitrivial equilibria $(\widehat{u}(\cdot, \mu, \tau), 0)$. We see that s and τ are found from the equation

$$s = \sigma(\tau, m(\tau, s)). \tag{3.41}$$

Since $\sigma(0, \mu) \equiv 1$, for $\tau \approx 0$ there is a unique solution $s = \overline{s}(\tau)$ of (3.41), and it depends smoothly on τ. Hence for each fixed $\tau \approx 0$, $(\tau, \overline{s}(\tau), m(\tau, \overline{s}(\tau)))$ is a point contained in the family (3.40) which corresponds to the semitrivial solution $(\widehat{u}(\cdot, \mu, \tau), 0)$ of (4.10) with $\mu = m(\tau, \overline{s}(\tau))$.

Using the scaled variable $\widetilde{s} = s\overline{s}(\tau)$, we define

$$\begin{aligned} u(\tau, s) &= \widetilde{s}\widehat{v}(\cdot, m(\tau, \widetilde{s})) + w(\tau, \widetilde{s}, m(\tau, \widetilde{s})), \\ v(\tau, s) &= (1 - \widetilde{s})\widehat{v}(\cdot, m(\tau, \widetilde{s})) + z(\tau, \widetilde{s}, m(\tau, \widetilde{s})), \\ \mu(\tau, s) &= m(\tau, \widetilde{s}). \end{aligned}$$

These are smooth functions of $(\tau, s) \in (-\delta, \delta) \times (-\delta, 1+\delta)$ if δ is sufficiently small, and $(u(\tau, s), v(\tau, s))$ is a solution of (3.14) for $\mu = m(\tau, s)$. These solutions together with the semitrivial equilibria contain all solutions of (3.14) in a small neighborhood of $\Gamma_{\widehat{\mu}} \times \{\widehat{\mu}\}$ for $\tau \in (-\delta, \delta)$. The relations $(w(0, s, \mu), z(0, s, \mu)) = (0, 0)$, $m(0, s) = \widehat{\mu}$, and $\overline{s}(0) = 1$ imply (3.18). The correspondences between the solution $(\tau, \overline{s}(\tau), m(\tau, \overline{s}(\tau)))$, $(\tau, 0, \mu(0, \tau))$ of (3.36) and the semitrivial equilibria, as discussed above, imply (3.16), (3.17). This completes the proof.

Next we consider the stability of coexistence states on the curve Σ given in Proposition 3.2.1. A crucial step is to analyze the corresponding linear eigenvalue problem

$$\begin{cases} \mu L_N \varphi + (\alpha + \tau\beta - 2u - v)\varphi + (-u)\psi = \lambda\varphi \\ \mu L_N \psi + (-v)\varphi + (\alpha - u - 2v)\psi = \lambda\psi \end{cases} \text{in } [0, h], \tag{3.42}$$

where (u, v) is the coexistence state of (4.10). When $\tau = 0$, we have $(u, v) = (s\widehat{v}, (1 - s)\widehat{v})$ and (3.42) has an eigenvalue $\lambda = 0$, the corresponding eigenfunction being $(\widehat{v}, -\widehat{v})$. Let $N(\tau, s, \mu)$ be the linear, bounded operator defined by

$$N(\tau,s,\mu)(\varphi,\psi)^T := \begin{pmatrix} \mu L_N\varphi + (\alpha+\tau\beta-2u-v)\varphi & u\psi \\ v\varphi & \mu L_N\psi + (\alpha-u-2v)\psi \end{pmatrix},$$

it is easy to see that 0 is an eigenvalue of $N(0,s,\mu)$ with corresponding eigenfunction being $(\widehat{v},\widehat{v})$. Moreover, 0 is (algebraically) simple and equal to $s(N(0,s,\mu))$. By spectral perturbation theory, for $|\tau| \ll 1$, $N(\tau,s,\mu)$ has a simple eigenvalue, denoted by $\lambda(\tau,s)$, such that $\lim_{\tau\to 0}\lambda(\tau,s)=0$, and $\lambda(\tau,s)$ corresponds to an eigenfunction in the interior of $\widehat{Y}_+$. Furthermore, we can show $\lambda(\tau,s)=\mathfrak{s}(N(\tau,s,\mu))$. Hence the sign of $\lambda(\tau,s)$ determines the stability of coexistence states on Σ. (For more details, see Appendix). Note that

$$\lambda(\tau,0)=\lambda(\tau,1)=0 \tag{3.43}$$

for $\mu(\tau,0)$ and $\mu(\tau,1)$ are bifurcation points (points of intersections of Σ with the branch of semitrivial solutions).

By straightforward but tedious computation, we can obtain a formula for $\lambda(\tau,s)$ (the proof is given in the appendix). For the formulation we introduce some notation. Take the linear subspace spanned by $\widehat{v}$ to be Θ and let $\Theta^\perp$be its orthogonal complement in H. We have the self-adjoint operators defined on H

$$\begin{aligned} \mathcal{L} &= \widehat{\mu}L_N + \alpha - \widehat{v} \\ \mathcal{L}-\widehat{v} &= \widehat{\mu}L_N + \alpha - 2\widehat{v}. \end{aligned}$$

Notice that $\mathcal{L}$ is a Fredholm operators of index zero on H and $0=\sup_{||w||_H=1}(\mathcal{L}w,w)$ is the principal eigenvalue of $\mathcal{L}$ (the eigenfunction is $\widehat{v}$). By Lemma 2.3 and 2.4, $\mathcal{L}-\widehat{v}$ has the bound inverse $(\mathcal{L}-\widehat{v})^{-1}$on H. Now we define $\mathcal{L}^{-1}$on $\Theta^\perp$ by setting $\mathcal{L}^{-1}\phi=\psi$ if and only $\mathcal{L}\psi=\phi$ and $\phi,\psi\in\Theta^\perp$. Then we have the following results about $\lambda(\tau,s)$.

Proposition 3.2.3 *For* $0\le s\le 1$*and* $0<\tau\ll 1$, *the following statement hold:*

(i)

$$\lambda(\tau, s) = s(s-1)\tau^2 \left\{ \frac{2}{\int_0^h \widehat{v}^2} \int_0^h \beta\widehat{v}[((\mathcal{L} - \widehat{v})^{-1} - \mathcal{L}^{-1}]\beta\widehat{v} + C(\tau, s)\tau \right\},$$

where $C(\tau, s)$ is some constant uniformly bounded for $s \in [0, 1]$ and $\tau \ll 1$.

(ii) With $\widehat{\mu}_1$ as in (3.10), one has

$$\widehat{\mu}_1(s) = \frac{1}{G'(\widehat{\mu})} \int_0^h \beta\widehat{v}[2s(\mathcal{L} - \widehat{v})^{-1}(\beta\widehat{v}) + (1 - 2s)\mathcal{L}^{-1}(\beta\widehat{v})].$$

Note that, by our assumption, $M(\widehat{\mu}) = \int_\Omega \beta\widehat{v}^2 = 0$, so that $\mathcal{L}^{-1}(\beta\widehat{v})$ is well defined.

Proof. see the appendix.

Lemma 3.2.4 *The following holds for any nontrivial* $\varphi \in \Theta^\perp$*:*

$$\int_0^h \varphi[(\mathcal{L} - \widehat{v})^{-1} - \mathcal{L}^{-1}]\varphi > 0. \tag{3.44}$$

Proof. For $t > 0$ let

$$h(t) = \int_0^h \varphi(\mathcal{L} - t\widehat{v})^{-1}\varphi.$$

Due to the fact that $\mathcal{L} - t\widehat{v}$ is invertible for $t > 0$, h is well defined. We claim that h is strictly increasing. To show this, we set $\Phi = (\mathcal{L} - \tau\widehat{v})^{-1}\varphi$. With resolvent , it is easy to see that

$$\frac{\partial\Phi}{\partial t} = (\mathcal{L} - t\widehat{v})^{-1}(\widehat{v}\Phi). \tag{3.45}$$

Then

$$\begin{aligned}
\frac{dh}{dt} &= \int_0^h \varphi\frac{\partial\Phi}{\partial t} = \int_0^h \varphi(\mathcal{L} - t\widehat{v})^{-1}(\widehat{v}\Phi) \\
&= \int_0^h ((\mathcal{L} - t\widehat{v})^{-1}\varphi)(\widehat{v}\Phi) \\
&= \int_0^h \widehat{v}\Phi^2 > 0.
\end{aligned}$$

The last inequality is strict since $\Phi \neq 0$. In the following we show that

$$\lim_{t \to 0^+} h(t) = \int_0^h \varphi \mathcal{L}^{-1} \varphi \tag{3.46}$$

for every $\varphi \in \Theta^\perp$, from which (3.44) follows. To prove (3.46), let $Su := \widehat{v}u$ for $u \in H$. We always assume that $\tau > 0$ is small and C_i are strictly positive constants independent of τ. Then we have $\mathcal{L}^{-1}\Theta^\perp \subset \Theta^\perp$ and

$$||\mathcal{L}^{-1}|| \leq C_1, \tag{3.47}$$

$$(-\mathcal{L}g, g) \geq C_2 ||g||_H^2, \quad \phi \in \Theta^\perp, \tag{3.48}$$

$$||S|| \leq C_3, \tag{3.49}$$

$$(S\psi, \psi) \geq C_4 ||\psi||_H^2, \quad \psi \in H. \tag{3.50}$$

Consider the equation

$$(\mathcal{L} - tS)\varphi = g, \quad g \in \Theta^\perp. \tag{3.51}$$

We first show that

$$||\varphi||_H \leq C_5 ||g||_H. \tag{3.52}$$

To prove this, set $\varphi = c\widehat{v} + \psi$, where $c \in \mathbb{R}$ and $\psi \in \Theta^\perp$. Substituting in (3.51), we have

$$\mathcal{L}\psi - tcS\widehat{v} - tS\psi = g. \tag{3.53}$$

Take the inner product with $\widehat{v}$ and use (3.49) and (3.50)

$$|c| = |(S\psi, \widehat{v})/(S\widehat{v}, \widehat{v})| \leq C_6 ||\psi||_H \tag{3.54}$$

Take the inner product of (3.53) with $-\psi$ and use (3.48), (3.49) and (3.54), we get

$$C_2||\psi||_H^2 \le (-\mathcal{L}\psi, \psi) = -(g, \psi) - tc(S\widehat{v}, \psi) - t(S\psi, \psi) \quad (3.55)$$

$$\le ||g||_H||\psi||_H + t(C_6 + 1)C_3||\psi||_H^2, \quad (3.56)$$

which implies that

$$||\psi||_H \le C_7||g||_H \quad (3.57)$$

if t is small enough. Estimate (3.54) and (3.57) prove (3.52).

For φ given by (3.51), we set

$$\theta = \frac{1}{t}(\varphi - \mathcal{L}^{-1}g + \frac{(S\mathcal{L}^{-1}g, \widehat{v})}{(S\widehat{v}, \widehat{v})}\widehat{v}). \quad (3.58)$$

It is easy to see that

$$(\mathcal{L} - \tau S)\theta = h \quad (3.59)$$

where

$$h = S\mathcal{L}^{-1}g - \frac{(S\mathcal{L}^{-1}g, \widehat{v})}{(S\widehat{v}, \widehat{v})}S\widehat{v}.$$

Clearly, $||h||_H \le C_8||g||_H$. Applying (3.52) to (3.59), we also find $||\theta||_H \le C_5||h||_H \le C_9||g||_H$.

Finally, for any $g \in \Theta^\perp$, from (3.58)

$$h(\tau) = (g, \varphi) = (g, \mathcal{L}^{-1}g) + t(g, \theta).$$

Since $||\theta||_H$ is bounded, we find that $\lim_{\tau\to 0^+} h(t) = (g, \mathcal{L}^{-1}g)$. This proves Lemma.

Theorem 3.2.5 *Assume* $\int_0^h \alpha(x)dx > 0$. *For any* $\widehat{\mu} > 0$, *the following statements hold true:*

(i) If $M(\widehat{\mu}) \neq 0$, then there exists $\epsilon > 0$ such that for $\mu \in (\widehat{\mu} - \epsilon, \widehat{\mu} + \epsilon)$ and $\tau \in (0, \epsilon)$ problem (4.10) has no coexistence states.

(ii) If $M(\widehat{\mu}) = 0$ and $M'(\widehat{\mu}) \neq 0$, then for any sufficiently small $\epsilon > 0$, there exists $\widehat{\tau} = \widehat{\tau}(\epsilon) > 0$ with the following property. For every $\tau \in (0, \widehat{\tau})$, there exist $\underline{\mu} < \overline{\mu}$ with $\underline{\mu}, \overline{\mu} \in (\widehat{\mu} - \epsilon, \widehat{\mu} + \epsilon)$ such that for any $\mu \in [\widehat{\mu} - \epsilon, \widehat{\mu} + \epsilon]$, (4.10) has a coexistence state if $\mu \in (\underline{\mu}, \overline{\mu})$; moreover, any coexistence state, if it exists, is asymptotically stable.

Proof. We have $\mu(\tau, s) = \widehat{\mu} + \tau\widehat{\mu}_1(s) + O(\tau^2)$. By Proposition 3.2.3 and lemma 3.2.4, for $\mu \approx \widehat{\mu}$ and $\tau \approx 0, \mu(\tau, \cdot)$ is strictly monotone. It follows that the first statement of theorem hold with

$$\underline{\mu} = \inf_{s \in [0,1]} \mu(\tau, s), \quad \overline{\mu} = \sup_{s \in [0,1]} \mu(\tau, s)$$

and that the coexistence state on the branch Σ is unique for each fixed $\mu \in (\underline{\mu}, \overline{\mu})$. By Proposition 3.2.3 and Lemma 3.2.4, $\lambda(\tau, s) > 0$ for small τ, and thus the coexistence state is stable.

3.3 Appendix

Proof of Proposition 3.2.3. The proof of proposition 3.2.3 is given here after some computational results, throughout, (u, v) will be a coexistence state of (3.14), $u_1, v_1, \widehat{\mu}$ are given by (3.19), (3.20) and (3.21) respectively, and (ζ, ς) is the solution of (3.42). Throughout the appendix, we assume that $M(\widehat{\mu}) = 0$, where M is given by (3.13). Note that $\zeta = \widehat{v}$, $\varsigma = -\widehat{v}$ for $\tau = 0$.

Lemma 3.3.1 *The following statements hold*

(i)

$$\int_0^h \beta uv = 0. \tag{3.60}$$

(ii)

$$\lambda(\tau, s) = \tau \int_0^h \beta(\zeta v + \varsigma u) / \int_0^h (\zeta v + \varsigma u). \tag{3.61}$$

Proof.

(i) Multiply (3.14a) by v, (3.14b) by u and subtracting, we obtain

$$\mu(L_N uv - L_N vu) + \tau\beta uv = 0. \tag{3.62}$$

The result follows from integrating (3.62) over $(0, h)$.

(ii) Multiplying (3.42a) by v, (3.42b) by u and subtracting, we find

$$\lambda(\tau, s)(\zeta v - \varsigma u) = v[\mu L_N \zeta + (\alpha + \tau\beta - u - v)] - u[\mu L_N \varsigma + (\alpha - u - v)]. \tag{3.63}$$

Integrating (3.63) over $(0, h)$ and using (3.14) we deduce that

$$\begin{aligned} \lambda(\tau, s) \int_0^h (\zeta v - \varsigma u) &= \int_0^h \{\zeta[\mu L_N v + (\alpha + \tau\beta - u - v)] - \varsigma[\mu L_N v + (\alpha - u - v)]\} \\ &= \tau \int_0^h \beta(\zeta v + \varsigma u). \end{aligned}$$

Set

$$A = (\mathcal{L} - \widehat{v})^{-1}(\widehat{\mu}_1 L_N \widehat{v}), \tag{3.64}$$

$$B = (\mathcal{L} - \widehat{v})^{-1}(\beta\widehat{v}), \tag{3.65}$$

$$C = \mathcal{L}^{-1}(\beta\widehat{v}), \tag{3.66}$$

and expand the eigenvalue ζ, ς in the form

$$\zeta = \widehat{v} + \tau\zeta_1(\cdot, s) + \tau^2\zeta_2(\cdot, \tau, s), \tag{3.67}$$

$$\varsigma = -\widehat{v} + \tau\varsigma_1(\cdot, s) + \tau^2\varsigma_2(\cdot, \tau, s). \tag{3.68}$$

Lemma 3.3.2 *For some $\gamma_i \in R$, we have*

(i)

$$u_1 = -s[A + sB + (1-s)C] + \gamma_1\widehat{v}, \tag{3.69}$$

$$v_1 = -(1-s)[A + sB - sC] - \gamma_1\widehat{v}, \tag{3.70}$$

(ii)

$$\zeta_1 = -A - 2sB + (2s-1)C + \gamma_2\widehat{v}, \tag{3.71}$$

$$\varsigma_1 = A + (2s-1)B - (2s-1)C - \gamma_2\widehat{v}. \tag{3.72}$$

Proof.

(i) From direct calculation, the following hold in $[0, h]$

$$\mu L_N u_1 + (\beta - \widehat{v})u_1 + s\widehat{v}(\beta - u_1 - v_1) + s\widehat{\mu}L_N\widehat{v} = 0, \tag{3.73}$$

$$\mu L_N v_1 + (\beta - \widehat{v})v_1 + (1-s)\widehat{v}(-u_1 - v_1) + (1-s)\widehat{\mu}L_N\widehat{v} = 0. \tag{3.74}$$

Multiplying (3.73),(3.74) by $(1-s)$ and s respectively, and subtracting, we find that

$$\mathcal{L}[(1-s)u_1 - sv_1] + s(1-s)\beta\widehat{v} = 0,$$

from which it follows on taking the inverse and using definition (3.66) that

$$(1-s)u_1 - sv_1 = -s(1-s)C + \gamma_3\widehat{v}. \tag{3.75}$$

Adding (3.73) and (3.74), we have in a similar manner

$$u_1 + v_1 = -A - sB. \tag{3.76}$$

Then (3.69) and (3.70) follows from (3.75) and (3.76) by straightforward manipulation.

(ii) Since $\lambda_\tau(0, s) = 0$, it is easy to check that ζ_1 and ς_1 satisfy the following

$$\mathcal{L}\zeta_1 - s\widehat{v}(\zeta_1 + \varsigma_1) + \widehat{\mu}_1 L_N \widehat{v} + \widehat{v}(\beta - u_1 - v_1) = 0, \tag{3.77}$$

$$\mathcal{L}\varsigma_1 - (1-s)\widehat{v}(\zeta_1 + \varsigma_1) - \widehat{\mu}_1 L_N \widehat{v} + \widehat{v}(u_1 + v_1) = 0. \tag{3.78}$$

Adding (3.77) and (3.78), by an argument similar to that used previously, we find

$$\zeta_1 + \varsigma_1 = -B. \tag{3.79}$$

By (3.76), $u_1 + v_1 = -A - sB$. Substituting this and (3.79) in (3.77), we obtain the equation which determines ζ_1 up to an additive term $\gamma_4\widehat{v}$. Using definition (3.64), (3.65) and (3.66), it is easy to see that ζ_1 given by (3.71) satisfies that equation, which verifies (3.71). this and (3.79) yield (3.72).

Lemma 3.3.3 *The following holds:*

$$\int_0^h \beta\widehat{v}[2A + 2sB + (1-2s)C] = 0. \tag{3.80}$$

Proof.

By (3.19)-(3.21) and (3.60),

$$0 = \int_0^h \beta uv = \tau \int_0^h \beta\widehat{v}[sv_1 + (1-s)u_1] + O(\tau^2)$$

since $\int_0^h \beta\widehat{v}^2 = 0$. Therefore

$$\int_0^h \beta\widehat{v}[sv_1 + (1-s)u_1] = 0. \tag{3.81}$$

The result follows from (3.81) together with (3.69)-(3.70).

Proof of Proposition 3.2.3 In the following, $C_i(\tau, s)$ denote quantities that are uniformly bounded for $s \in [0,1]$ and small τ. From (3.67) and (3.68),

$$\int_0^h (\zeta v - \varsigma u) = \int_0^h \widehat{v}^2 + \tau C_1(\tau, s). \tag{3.82}$$

We now use (3.69)-(3.72) to obtain

$$\begin{aligned}\int_0^h \beta(\zeta v + \varsigma u) &= (1-2s)\int_0^h \beta\widehat{v} + \tau\int_0^h \beta\widehat{v}[v_1 - u_1 + (1-s)\zeta_1 + s\varsigma_1] + \tau^2 C_2(\tau, s)\\ &= \tau\int_0^h \beta\widehat{v}[(4s-2)A + (6s^2-4s)B + (6s-1-6s^2)C] + \tau^2 C_3(\tau, s).\end{aligned}$$

From this and (3.80),

$$\int_0^h \beta(\zeta v + \varsigma u) = 2s(1-s)\tau\int_0^h \beta\widehat{v}(C-B) + \tau^2 C_4(\tau, s). \tag{3.83}$$

As a consequence of (3.61), (3.82) and (3.83), we have

$$\lambda(\tau, s) = \tau^2[\frac{2s(1-s)\int_0^h \beta\widehat{v}(B-C)}{\int_0^h \widehat{v}^2} + \tau C_5(\tau, s)]. \tag{3.84}$$

Since $\lambda(\tau, 0) = \lambda(\tau, 1) \equiv 0$ for all τ (see (3.43)), we have $C_5(0,\tau) = C_5(1,\tau) \equiv 0$. This implies that we can write C_5 as $C_5(\tau, s) = s(1-s)C_6(\tau, s)$. This proves part (i) of Proposition 3.2.3.

Part (ii) follows directly from Lemma 3.3.3 and the relation

$$A = \widehat{\mu}_1(\mathcal{L} - \widehat{v})^{-1}(L_N\widehat{v}) = -\widehat{\mu}_1\widehat{v}_\mu.$$

The latter equality is obtained by differentiating the equation (3.12) for $\widehat{v}$ with respect to μ.

In the remainder of the appendix, we simplify the notion by writing

$$N(\tau) = N(\tau, s, \mu), \quad \lambda(\tau, s) = \lambda(\tau).$$

Lemma 3.3.4 *$N(\tau)$ is a resolvent positive operator on Y and $\mathbf{s}(N(\tau)) = \lambda(\tau)$.*

Proof. We first prove that $N(\tau)$ is a resolvent positive operator on Y. Choose $\eta \in \rho(N(\tau))$ such that

$$\eta > \{\max_{0 \le x \le h} \int_0^h J(x,y)dy + 4 \max_{0 \le x \le h}(|\alpha(x)| + |\beta(x)|\}.$$

It is sufficient to show that $(N(\tau) - \lambda I)(w, z)^T \le 0$ implies $w \ge 0$ and $z \ge 0$ if $\lambda \ge \eta$. Let $w^+ = \max(0, w)$ and $w^- = \max(0, -w)$. z^+, z^- are defined similarly. Then we have

$$\mu L_N w w^- + (\alpha + \tau\beta - 2u - v)ww^- + uzw^- - \lambda ww^- \le 0 \tag{3.85}$$

and

$$\mu L_N z z^- + (\alpha - u - 2v)zz^- + vwz^- - \lambda zz^- \le 0 \tag{3.86}$$

Integrating (3.85) and (3.86), we find

$$-(\mu L_N w^-, w^-)_H - ((\alpha+\tau-2u-v)w^-, w^-)_H - (uz^-, w^-)_H + (\lambda w^-, w^-)_H \le 0 \tag{3.87}$$

and

$$-(\mu L_N z^-, z^-)_H - ((\alpha - 2u - v)z^-, z^-)_H - (vw^-, z^-)_H + (\lambda z^-, z^-)_H \le 0 \tag{3.88}$$

Adding (3.87) to (3.88) and Multiplying the both side of resulting inequality by -1, we have

$$\begin{aligned}
&(\mu L_N(w^- + z^-), (w^- + z^-))_H + ((\alpha + \tau\beta - 2u - v)w^-, w^-)_H \\
&+((\alpha - 2u - v)z^-, z^-)_H + ((u + v)(w^- + z^-), (w^- + z^-))_H \\
&-\lambda((w^- + z^-), (w^- + z^-))_H \\
&\ge 0
\end{aligned}$$

With the positivity of u and v, we get

$$\begin{aligned}(\mu L_N(w^- + z^-), (w^- + z^-))_H &+ 2((|\alpha| + |\beta| + u + v)(w^- + z^-), (w^- + z^-))_H \\ &- \lambda((w^- + z^-), (w^- + z^-))_H \geq 0\end{aligned}$$

and so

$$\{\max_{0 \leq x \leq h} \int_0^h J(x,y)dy + 4 \max_{0 \leq x \leq h} (|\alpha(x)| + |\beta(x)| - \lambda\}((w^- + z^-), (w^- + z^-))_H \geq 0.$$

This would lead to a contradiction if $w^- + z^- \not\equiv 0$ on Ω. Thus we have $w^- \equiv 0$ and $z^- \equiv 0$ on Ω and hence $w \geq 0$ and $z \geq 0$ in Ω by the continuity of w and z.

We observe that

$$N(\tau) = K(\mu) + W(\tau)$$

where

$$K(\mu)(w, z)^T := \begin{bmatrix} \mu J * w & 0 \\ 0 & \mu J * z \end{bmatrix}$$

and

$$W(\tau) = \begin{bmatrix} \alpha + \tau\beta - 2u - v - \int_0^h J(x,y)dy & u \\ v & \alpha - u - 2v - \int_0^h J(x,y)dy \end{bmatrix}.$$

When $\tau = 0$, as we proved in Proposition 3.2.1, $W(0)$ is invertible and $N(0)$ is a Fredholm operator of index zero. Moreover, 0 is an eigenvalue of $N(0)$ with corresponding eigenvalue $(\widehat{v}, \widehat{v})$. By the spectral perturbation theory, for $|\tau| \ll 1$, $N(\tau)$ has a simple eigenvalue, denoted by $\lambda(\tau)$ such that

$$\lim_{\tau \to 0} \lambda(\tau) = 0$$

and

$$\lim_{\tau \to 0} (\varphi(\tau), \phi(\tau))^T = (\widehat{v}, \widehat{v})^T \quad \text{in } Y \tag{3.89}$$

where $(\varphi(\tau), \phi(\tau))$ is an eigenfunction associated with $\lambda(\tau)$. Furthermore, due to (3.19)-(3.21), we find

$$\lim_{\tau \to 0} [(\alpha + \tau\beta - u - v - \int_0^h J(x,y)dy) - \lambda(\tau)] = \alpha - \widehat{v} - \int_0^h J(x,y)dy < 0$$

and so $W(\tau) - \omega I$ is invertible for any $\omega \geq \lambda(\tau)$ if τ is sufficiently small. On the other hand, with the same reasoning, we see that $W(\tau)$ is also a resolvent positive operator and hence $\mathfrak{s}(W(\tau)) \in \sigma(W(\tau))$. Consequently, we obtain that

$$\mathfrak{s}(W(\tau)) < \lambda(\tau) \leq \mathfrak{s}(N(\tau))$$

provided τ is sufficiently small. Since $K(\mu)$ is a positive and compact operator on Y, Theorem 4.7 in [45] shows that $\mathfrak{s}(N(\tau))$ is an eigenvalue of $N(\tau)$ associated with positive eigenfunction of $N(\tau)$.

Next we show that

$$\lambda(\tau) = \mathfrak{s}(N(\tau)).$$

Suppose this is not true. Let $(\Phi(\tau), \Psi(\tau)) \in intX_+ \times intX_+$ be an eigenfunction corresponding to $\mathfrak{s}(N(\tau))$. For any $t \in \mathbb{R}^+$, define $\varphi_t = (\varphi(\tau) - t\Phi(\tau))$ and $\phi_t = (\phi(\tau) - t\Psi(\tau))$. By virtue of (3.89), for sufficiently small τ, $\varphi(\tau) \gg 0, \phi(\tau) \gg 0$ and then there exists $\bar{t}$ such that $\varphi_t \gg 0, \phi_t \gg 0$ for $t < \bar{t}$, $\varphi_{\bar{t}} \geq 0, \phi_{\bar{t}} \geq 0$ and either $\varphi_{\bar{t}}$ or $\phi_{\bar{t}}$ has a zero in $[0,h]$. We may assume without loss of generality that there exists $x_1 \in [0,h]$ such that $\varphi_{\bar{t}}(x_1) = 0$. Then

$$(\mu L_N \varphi_{\bar{t}})(x_1) + (\alpha + \tau\beta - 2u - v)(\varphi_{\bar{t}})(x_1) + (u\phi_{\bar{t}})(x_1) - s(N(\tau))\varphi_{\bar{t}}(x_1) \leq 0$$

and so

$$\mu \int_0^h J(x,y)\varphi_{\bar{t}}(y)dy \leq 0. \tag{3.90}$$

Since $\mu \int_0^h J(x,y)\varphi_{\bar{t}}(y)dy \geq 0$, (3.90) implies $\varphi_{\bar{t}} \equiv 0$ and hence $\phi_{\bar{t}} \equiv 0$, which is impossible. Therefore, $\lambda(\tau) = \mathfrak{s}(N(\tau))$ and the proof is completed.

Proposition 3.3.5 *Assume that $\alpha(x)$ is Lipschitz continuous in $[0,h]$ with $\int_0^h \alpha(x)dx > 0$ and $\mu > 0$. Then for each μ*

$$\mu L_N u + u(\alpha(x) - u) = 0 \tag{3.91}$$

has a unique solution in $intX_+$ which continuously depends upon μ.

Proof. According to Lemma 3.1.5, $\mathfrak{s}(\mu L_N + \alpha I) > 0$. Notice that $\max\limits_{0 \le x \le h} |\alpha(x)| + 1$ is a supsolution to (3.91). Thus it follows from Theorem 2.1.12 that (3.91) has a unique solution $w(\mu)$ in $intX_+$ for each $\mu > 0$. Furthermore, in light of the proof of Theorem 2.1.6, it is evident that $w(\mu)$ is also Lipschitz continuous in $[0,h]$. To show the continuous dependence of w upon μ, we define $G : R \times X \to X$ by

$$G(\mu, u) = \mu L_N u + u(\alpha(x) - u).$$

Clearly, G is C^1 in both μ and u and each zero of G is a solution to (3.91). Now let $\widetilde{\mu}$ be an arbitrarily fixed positive number. A linearization of G with respect to $\widetilde{\mu}$ leads to the operator $A : X \to X$

$$Au := \widetilde{\mu} L_N u + u(\alpha - 2u).$$

By Lemma 2.1.3 , $\mathfrak{s}(A) < 0$. Consequently, A is invertible and so there exist a pair $(\mu, u(\mu))$ in a small neighborhood U of $(\widetilde{\mu}, w(\widetilde{\mu}))$ such that $G(\mu, u(\mu)) = 0$ and $\lim\limits_{\mu \to \widetilde{\mu}} ||u(\mu) - w(\widetilde{\mu})|| = 0$ in term of the implicit function theorem. Here $U \subset (\widetilde{\mu} - \delta, \widetilde{\mu} + \delta) \times X$ for some $\delta > 0$. By the positivity of $w(\widetilde{\mu})$, $u(\mu)$ is also positive if μ is sufficiently close to $\widetilde{\mu}$. Then the uniqueness of $w(\mu)$ implies $w(\mu) \equiv u(\mu)$. Thus the proof is completed.

CHAPTER 4

The Cauchy Problem for a Nonlocal Phase-Field Equation

Consider the following problem

$$u_t = \int_\Omega J(x-y)u(y)dy - \int_\Omega J(x-y)dyu(x) - f(u) + l\theta, \tag{4.1}$$

$$(\theta + lu)_t = \triangle\theta \tag{4.2}$$

in $(0,T)\times\Omega$, with initial and boundary conditions

$$u(0,x) = u_0(x),\ \ \theta(0,x) = \theta_0(x), \tag{4.3}$$

$$\frac{\partial\theta}{\partial n}|_{\partial\Omega} = 0, \tag{4.4}$$

where $T > 0$, $\Omega \subset \mathbb{R}^n$ is a bounded domain. Here θ represents temperature,u is an order parameter, l is a latent heat coefficient, the interaction kernel satisfies $J(-x) = J(x)$, and f is bistable.

4.1 Existence and uniqueness

In order to prove the existence, we make the following assumptions

(A_1) $M \equiv \sup \int_\Omega |J(x-y)|dy < \infty$ and $f \in C(\mathbb{R})$.

(A_2) There exist $c_1 > 0$, $c_2 > 0$, $c_3 > 0$, $c_4 > 0$ and $r > 2$ such that $f(u)u \geq c_1|u|^r - c_2|u|$, and $|f(u)| \leq c_3|u|^{r-1} + c_4$.

Note that (A_2) implies

$$F(u) = \int_0^u f(s)ds \geq c_5|u|^r - c_6|u| \tag{4.5}$$

for some positive constants c_5 and c_6.

We prove the existence of a solution to (4.1)-(4.4) by the method of successive approximation.

Define $\theta^{(0)}(t,x) := \theta_0(x)$ and for $k \geq 1$ $(u^{(k)}, \theta^{(k)})$ iteratively to be solutions to the system

$$u_t^{(k)} = \int_\Omega J(x-y)u^{(k)}(y)dy - \int_\Omega J(x-y)dyu^{(k)}(x) - f(u^{(k)}) + l\theta^{(k-1)}, \tag{4.6}$$

$$\theta_t^{(k)} - \triangle\theta^{(k)} + \theta^{(k)} = -lu_t^{(k)} + \theta^{(k-1)} \tag{4.7}$$

in $(0,T) \times \Omega$, with initial and boundary conditions

$$u^{(k)}(0,x) = u_0(x),\ \theta^{(k)}(0,x) = \theta_0(x), \tag{4.8}$$

$$\frac{\partial\theta^{(k)}}{\partial n}|_{\partial\Omega} = 0. \tag{4.9}$$

Lemma 4.1.1 *With $k = 1$, for any $T > 0$, if $u_0 \in L^\infty(\Omega)$, and $\theta_0 \in H^1 \cap L^\infty(\Omega)$, then there exists a unique solution (u, θ) to system (4.6) -(4.9). Furthermore, $u^{(1)}$, $u_t^{(1)} \in L^\infty((0,T), L^\infty(\Omega))$ and $\theta^{(1)} \in L^\infty((0,T), L^\infty(\Omega)) \cap L^2((0,T), H^2(\Omega))$.*

Proof. Since the right hand side of equation (4.6) is locally Lipschitz continuous in $L^\infty((0,T), L^\infty(\Omega))$, local existence follows from standard ODE theory. In order to prove the global existence, we prove global boundedness of the solutions. For any $p > 1$, multiplying equation (4.6) by $|u^{(1)}|^{p-1}u$ and integrating over Ω, we obtain

$$\begin{aligned} &\frac{1}{p+1}\frac{d}{dt}\int |u^{(1)}|^{p+1}dx + \int f(u^{(1)})|u^{(1)}|^{p-1}udx \\ &= \int\int J(x-y)u^{(1)}(y)|u^{(1)}|^{p-1}u^{(1)}dxdy \\ &- \int\int J(x-y)u^{(1)}(x)|u^{(1)}|^{p-1}u^{(1)}dxdy + l\int \theta^{(0)}|u^{(1)}|^{p-1}udx. \end{aligned} \tag{4.10}$$

Using Holder's and Young's inequalities and conditions (A_1) and (A_2), we have

$$\begin{aligned} &\frac{1}{p+1}\frac{d}{dt}\int |u^{(1)}|^{p+1}dx + C\int |u^{(1)}|^{p+r-1}udx \\ &\le C(p){C_1}^{p+1}, \end{aligned} \tag{4.11}$$

where C_1 is a constant independent of p and $lim_{p \to \infty} C(p)^{\frac{1}{p+1}} \le C_2$ with C_2 independent of p.

We need the following version of Gronwall's lemma (see Temam [44]):

Lemma 4.1.2 *(Uniform Gronwall's inequality) Let y be a positive absolutely continuous function on $(0, \infty)$ which satisfies*

$$y' + \nu y^m \le \delta$$

with $m > 1, \nu > 0, \delta \ge 0$. Then, for $t \ge 0$, we have

$$y(t) \le (\frac{\delta}{\nu})^{\frac{1}{m}} + (\nu(m-1)t)^{\frac{-1}{m-1}}. \tag{4.12}$$

Using this and (4.11), we have

$$||u^{(1)}||_{p+1}^{p+1} \le (C(p)C_1^{p+1})^{\frac{p+1}{p+r-1}} + (C(r-2)t)^{\frac{-(p+1)}{r-2}}. \tag{4.13}$$

Therefore,

$$||u^{(1)}||_{p+1} \le C(p)^{\frac{1}{p+1}}(C_1)^{\frac{p+1}{p+r-1}} + (C(r-2)t)^{\frac{-1}{r-2}}. \tag{4.14}$$

Letting $p \to \infty$, we have

$$||u^{(1)}||_\infty \le C. \tag{4.15}$$

for some constant C.

Also from condition (A_2) and equation (4.6), we have

$$||u_t^{(1)}||_\infty \le C. \tag{4.16}$$

Since equation (4.7) is a linear parabolic equation, by inequality (4.16) and standard parabolic theory, we have $\theta^{(1)} \in L^\infty((0,T), L^\infty(\Omega)) \cap L^2((0,T), H^2(\Omega))$.

By induction, there exist unique solution $(u^{(k)}, \theta^{(k)})$ of system (4.6)-(4.8). Furthermore, $u^{(k)}$, $u_t^{(k)} \in L^\infty((0,T), L^\infty(\Omega))$ and $\theta^{(k)} \in L^\infty((0,T), L^\infty(\Omega)) \cap L^2((0,T), H^2(\Omega))$ for every k. Now we prove that there exists a uniform bound for $u^{(k)}$, $u_t^{(k)}$ and $\theta^{(k)}$.

Multiplying equation (4.7) by $|\theta^{(k)}|^{p-1}\theta^{(k)}(x)$ for $p > \frac{n}{2}$, and integrating over Ω, we have

$$\begin{aligned}
&\int |\theta^{(k)}|^{p-1}\theta^{(k)}\theta_t^{(k)}dx + \int \nabla(|\theta^{(k)}|^{p-1}\theta^{(k)}) \cdot \nabla\theta^{(k)}dx + \int |\theta^{(k)}|^{p+1}dx \\
&= -l\int\int J(x-y)u^{(k)}(y)|\theta^{(k)}|^{p-1}\theta^{(k)}dydx + l\int f(u^{(k)})|\theta^{(k)}|^{p-1}\theta^{(k)}dx \\
&+ l\int\int J(x-y)u^{(k)}(x)|\theta^{(k)}|^{p-1}\theta^{(k)}dydx + (1-l^2)\int |\theta^{(k)}|^{p-1}\theta^{(k)}\theta^{(k-1)}dx.
\end{aligned} \tag{4.17}$$

Since

$$|\nabla|\theta|^{\frac{p+1}{2}}|^2 = \frac{(p+1)^2}{4}|\theta|^{p-1}|\nabla\theta|^2 = \frac{(p+1)^2}{4p}\nabla(|\theta|^{p-1}\theta)\cdot\nabla\theta, \tag{4.18}$$

using Holder's and Young's inequalities, we obtain

$$\begin{aligned}\frac{1}{p+1}\frac{d}{dt}\int_\Omega |\theta^{(k)}|^{p+1}dx + \frac{4p}{(p+1)^2}\int |\nabla|\theta^{(k)}|^{\frac{p+1}{2}}|^2 dx + \frac{1}{2}\int |\theta^{(k)}|^{p+1}dx \\ \leq c_1(l,p)\int |u^{(k)}|^{p+1}dx + c_2(l,p)\int |\theta^{(k-1)}|^{p+1}dx + \int |f(u^{(k)})|^{p+1}dx\end{aligned} \tag{4.19}$$

for some positive constants $c_1(l,p)$ and $c_2(l,p)$ which depend only on p and l.

Multiplying equation (4.6) by $|u^{(k)}|^{(r-1)p-1}u^{(k)}$, and integrating over Ω, we obtain

$$\begin{aligned}&\frac{1}{(r-1)p+1}\frac{d}{dt}\int |u^{(k)}|^{(r-1)p+1}dx + \int f(u^{(k)})|u^{(k)}|^{(r-1)p-1}u^{(k)}dx \\ &= \int\int J(x-y)u^{(k)}(y)|u^{(k)}|^{(r-1)p-1}u^{(k)}dxdy + l\int \theta^{(k-1)}|u^{(k)}|^{(r-1)p-1}u^{(k)}dx \\ &- \int\int J(x-y)u^{(k)}(x)|u^{(k)}|^{(r-1)p-1}u^{(k)}dxdy\end{aligned} \tag{4.20}$$

Condition (A_2) implies

$$f(u)|u|^{(r-1)p-1}u \geq c_1|u|^{(r-1)(p+1)} - c_2|u|^{(r-1)p} \tag{4.21}$$

and

$$|f(u)|^{p+1} \leq c_7|u|^{(r-1)(p+1)} + c_8 \tag{4.22}$$

for some positive constants c_7 and c_8. From equation (4.20), inequality (4.21), Holder's and Young's inequalities, we have

$$\frac{1}{(r-1)p+1}\frac{d}{dt}\int |u^{(k)}|^{(r-1)p+1}dx + \frac{c_1}{2}\int |u^{(k)}|^{(r-1)(p+1)}dx \\ \leq c(r,p) + c_1(r,p,l)\int |\theta^{(k-1)}|^{p+1}dx \tag{4.23}$$

for some positive constants $c(r,p)$ and $c_1(r,p,l)$.

Integrating (4.23) from 0 to t, we obtain

$$\begin{aligned}\frac{1}{(r-1)p+1}\int |u^{(k)}|^{(r-1)p+1}dx &+ \frac{c_1}{2}\int_0^t\int |u^{(k)}|^{(r-1)(p+1)}dx \\ &\leq c(r,p)t + c_1(r,p,l)\int_0^t\int \theta^{(k-1)}|^{p+1}dx + \int |u_0|^{(r-1)p+1} \\ &\leq c(u_0,T,r,p) + c_1(r,p,l)\int_0^t\int \theta^{(k-1)}|^{p+1}dx\end{aligned} \tag{4.24}$$

for some positive constants $c(u_0,T,r,p)$ and $c_1(r,p,l)$.

Integrating inequality (4.19) from 0 to t, using (4.22) and (4.24), we have

$$\int_\Omega |\theta^{(k)}|^{p+1}dx \leq c(u_0,\theta_0,p,r,l,T)(1+\int_0^t\int |\theta^{(k-1)}|^{p+1}dxds) \tag{4.25}$$

for some positive constant $c(u_0,\theta_0,p,r,l,T)$ which does not depend on k.

By induction, we have

$$\int_\Omega |\theta^{(k)}|^{p+1}dx \leq ce^{ct} \tag{4.26}$$

for some positive constant c which does not depend on k.

Similarly from inequalities (4.23) and (4.26), we also have

$$\int_\Omega |u^{(k)}|^{p+1}dx \leq C, \tag{4.27}$$

and

$$\int_\Omega |f(u^{(k)})|^{p+1}dx \leq C \tag{4.28}$$

for some positive constant C which does not depend on k.

Equation (4.6), inequalities (4.26)-(4.28), and Young's inequality imply

$$\int_\Omega |u_t^{(k)}|^{p+1} dx \leq C \tag{4.29}$$

for some positive constant C which does not depend on k.

This implies $-lu_t^{(k)} + \theta^{(k-1)} \in L^{p+1}((0,T), L^{p+1}(\Omega))$ and

$$|| - lu_t^{(k)} + \theta^{(k-1)}||_{p+1} \leq C \tag{4.30}$$

for some positive constant C which does not depend on k.

The following lemma may be found in [29].

Lemma 4.1.3 *Consider the following linear parabolic equation:*

$$\begin{aligned} &\theta_t - \Delta\theta + \theta = g \ \ in \ (0,T) \times \Omega, \\ &\theta(0,x) = \theta_0(x), \\ &\frac{\partial \theta}{\partial n}|_{\partial\Omega} = 0. \end{aligned} \tag{4.31}$$

If $p > \frac{n}{2}$, $g \in L^p((0,T), L^p(\Omega))$ *and* $\theta_0 \in L^\infty(\Omega) \cap W^{1,2}(\Omega)$, *then* $\theta \in L^\infty((0,T), L^\infty(\Omega))$, *and we have*

$$\sup_{0<t<T} ||\theta||_\infty \leq C \max\{||\theta_0||_\infty, (\int_0^T ||g||_p^p)^{\frac{1}{p}}\}, \tag{4.32}$$

where $||\cdot||_p$ denotes the norm of $L^p((0,T), L^p(\Omega))$.

Applying Lemma 4.1.3 to equation (4.7), and using inequality (4.29), we have

$$||\theta^{(k)}||_\infty \leq C. \tag{4.33}$$

Multiplying equation (4.7) by θ_t^k, and integrating equation (4.7) over Ω, using Holder and Young's inequalities and (4.33), we have

$$\int_0^T \int_\Omega |\theta_t^{(k)}|^2 dxdt \leq C \tag{4.34}$$

for some constant C which does not depend on k.

Equation (4.7), inequalities (4.29), (4.33), and (4.34) yield

$$\int_0^T \int_\Omega |\triangle \theta^{(k)}|^2 dxdt \leq C \tag{4.35}$$

for some constant C which does not depend on k.

Since $||\theta^{(k)}||_\infty \leq C$, using a similar argument to that in the proof of Lemma 4.1.1, we have

$$||u^{(k)}||_\infty \leq C, \tag{4.36}$$

and

$$||u_t^{(k)}||_\infty \leq C \tag{4.37}$$

for some constant C which does not depend on k.

Next we prove the convergence of $\{\theta^{(k)}\}$ in $C([0,T], L^2(\Omega))$.

From equation (4.7), we have

$$\begin{aligned}(\theta^{(k+1)} - \theta^{(k)})_t - \triangle(\theta^{(k+1)} - \theta^{(k)}) + (\theta^{(k+1)} - \theta^{(k)}) \\ = -l(u^{(k+1)} - u^{(k)})_t + (\theta^{(k)} - \theta^{(k-1)})\end{aligned} \tag{4.38}$$

Multiplying equation (4.38) by $(\theta^{(k+1)} - \theta^{(k)})$, and integrating over Ω, using Holder's and Young's inequalities, we have

$$\frac{1}{2}\frac{d}{dt}\int|\theta^{(k+1)}-\theta^{(k)}|^2dx+\int|\nabla(\theta^{(k+1)}-\theta^{(k)})|^2dx+\frac{1}{2}\int(\theta^{(k+1)}-\theta^{(k)})^2$$
$$\leq l^2\int|u_t^{(k+1)}-u_t^{(k)}|^2dx+\int|\theta^{(k)}-\theta^{(k-1)}|^2dx. \tag{4.39}$$

Since $||u^{(k)}||_\infty \leq C$, from equation (4.6), and condition (A_2), we have

$$\int|u^{(k+1)}-u^{(k)}|^2dx \leq C(T)\int|\theta^{(k)}-\theta^{(k-1)}|^2dx, \tag{4.40}$$

and

$$\int|f(u^{(k+1)})-f(u^{(k)})|^2dx=\int|f'(\lambda u^{(k+1)}+(1-\lambda)u^{(k)})(u^{(k+1)}-u^{(k)})|^2dx$$
$$\leq C(T)\int|u^{(k+1)}-u^{(k)}|^2dx. \tag{4.41}$$

Therefore, equation (4.6), and inequalities (4.40)-(4.41) imply

$$\int|u_t^{(k+1)}-u_t^{(k)}|^2dx$$
$$\leq 4\int|\int J(x-y)(u^{(k+1)}-u^{(k)})dy|^2dx+4\int(\int J(x-y)dy)^2(u^{(k+1)}-u^{(k)})^2dx$$
$$+4\int(f(u^{(k+1)})-f(u^{(k)})^2dx+4\int(\theta^{(k)}-\theta^{(k-1)})^2dx$$
$$\leq C_1(T)\int|\theta^{(k)}-\theta^{(k-1)}|^2dx \tag{4.42}$$

for some positive constant $C_1(T)$ which does not depend on k.

Inequalities (4.39)-(4.42) yield

$$\frac{d}{dt}\int|\theta^{(k+1)}-\theta^{(k)}|^2dx \leq C(T)\int|\theta^{(k)}-\theta^{(k-1)}|^2dx \tag{4.43}$$

for some positive constant $C(T)$ which does not depend on k.

By induction, this implies

$$\int |\theta^{(k+1)} - \theta^{(k)}|^2 dx \le \frac{(ct)^{(k-1)}}{(k-1)!} \int_0^t \int |\theta^1 - \theta^0| dx ds. \tag{4.44}$$

So $\theta^{(k)}$ is a Cauchy sequence in $C([0,T], L^2(\Omega))$. Therefore, there exists $\theta \in C([0,T], L^2(\Omega))$ such that $\theta^{(k)} \to \theta$ in $C([0,T], L^2(\Omega))$. From (4.33)-(4.35), we have

$$||\theta||_\infty \le C, \tag{4.45}$$

$$\int_0^T \int_\Omega |\Delta\theta|^2 dx dt \le C, \tag{4.46}$$

$$\int_0^T \int_\Omega |\theta_t|^2 dx dt \le C. \tag{4.47}$$

Also from (4.36), (4.40)-(4.42), we have

$$u^{(k)} \to u \ in \ C([0,T], L^2(\Omega)), \tag{4.48}$$

$$u_t^{(k)} \to u_t \ in \ C([0,T], L^2(\Omega)), \tag{4.49}$$

$$f(u^{(k)}) \to f(u) \ in \ C([0,T], L^2(\Omega)). \tag{4.50}$$

Therefore, letting $k \to \infty$ in equation (4.6), we have

$$u_t = \int_\Omega J(x-y)u(y)dy - \int_\Omega J(x-y)dy u(x) - f(u) + l\theta \tag{4.51}$$

for $t > 0$ and a.e. $x \in \Omega$.

Since $u_t^{(k)} \rightharpoonup u_t$, $\theta_t^{(k)} \rightharpoonup \theta_t$, $\Delta\theta^{(k)} \rightharpoonup \Delta\theta$ in $L^2((0,T), L^2(\Omega))$, letting $k \to \infty$ in the weak form of equation (4.7), we have

$$\int_0^T \int_\Omega (lu_t + \theta_t)\xi(t,x) dx dt = \int_0^T \int_\Omega \Delta\theta\xi(t,x) dx dt \tag{4.52}$$

for $\xi(t,x) \in L^2((0,T), L^2(\Omega))$.

Since it is true of $\theta^{(k)}$, we also have

$$\int_0^T \int_\Omega \eta(t)(\triangle\theta\varphi + \nabla\theta\cdot\nabla\varphi)dxdt = 0 \tag{4.53}$$

for any $\varphi \in W^{1,2}(\Omega)$ and $\eta \in L^2(0,T)$. This implies $\dfrac{\partial\theta}{\partial n} = 0$ a.e on $(0,T)\times\partial\Omega$.

Also we have

$$\begin{aligned}\int_\Omega |\theta(0,x)-\theta_0|^2dx \le 3(\int_\Omega |\theta(0,x)-\theta(t,x)|^2dx + \int_\Omega |\theta(t,x)-\theta^{(k)}(t,x)|^2dx \\ + \int_\Omega |\theta^{(k)}(t,x)-\theta_0|^2dx)\end{aligned} \tag{4.54}$$

Since $\theta^{(k)}(t,x) \to \theta$ in $C([0,T], L^2(\Omega))$, and since $\theta^{(k)}(t,x)$ and $\theta(t,x)$ are continuous with respect to t in $L^2(\Omega)$, by taking k arbitrarily large we can see that $\theta(0,x) = \theta_0$ a.e. in Ω. Similarly, $u(0,x) = u_0$ a.e. in Ω.

Equations (4.51)-(4.54) imply that u and θ are solutions of system (4.1)-(4.4) in a weak sense.

To prove uniqueness and continuous dependence on initial data, let $\theta_{i0} \in L^\infty(\Omega)\cap W^{1,2}(\Omega)$, $u_{i0} \in L^\infty(\Omega)$, and for $R > 0$, $||\theta_{i0}||_{L^\infty} \le R$, $||u_{i0}||_{L^\infty} \le R$, where $i = 1, 2$.

Let u_i and θ_i be solutions corresponding to initial data u_{i0} and θ_{i0}, then we have $||\theta_i||_{L^\infty} \le C(T,R)$, and $||u_i||_{L^\infty} \le C(T,R)$.

Denote $v = u_1 - u_2$, $w = \theta_1 - \theta_2$. We have

$$v_t = \int_\Omega J(x-y)v(y)dy - \int_\Omega J(x-y)dyv(x) - f'(\lambda u_1 + (1-\lambda)u_2)v + lw, \tag{4.55}$$

$$(w+lv)_t = \triangle w \tag{4.56}$$

in $(0,T)\times\Omega$, for some $\lambda(x,t) \in [0,1]$. We also have initial and boundary conditions

$$v(0,x) = v_0(x),\ w(0,x) = w_0(x), \tag{4.57}$$

$$\frac{\partial w}{\partial n}|_{\partial\Omega} = 0. \tag{4.58}$$

Multiplying equation (4.55) by v_t, integrating over Ω, multiplying equation (4.55) by v, integrating over Ω, multiplying equation (4.56) by w, integrating over Ω, we have

$$\begin{aligned}\int |v_t|^2 = &\int\int J(x-y)v(y)dyv_t dx - \int J(x-y)dyv(x)v_t \\ &- \int (f'(\lambda u_1 + (1-\lambda)u_2)vv_t + lwv_t)dx,\end{aligned} \tag{4.59}$$

$$\begin{aligned}\int v_t v = &\int\int_\Omega J(x-y)v(y)dyvdx - \int_\Omega J(x-y)dyv^2 \\ &- \int (f'(\lambda u_1 + (1-\lambda)u_2)v^2 + lwv)dx\end{aligned} \tag{4.60}$$

$$\int (w_t w + lv_t w) = -\int |\nabla w|^2 dx \tag{4.61}$$

Adding equations (4.59)-(4.61) together, using Holder's and Young's inequalities, we have

$$\frac{d}{dt}\int [w^2 + v^2]dx \le C_2(T,R)\int [w^2 + v^2]dx \tag{4.62}$$

for some positive constant $C_2(T,R)$.

Inequality (4.62) and Gronwall's inequality imply the uniqueness and continuous dependence on initial data of the solution of (4.6)-(4.7).

Denote $Q_t = (0,T) \times \Omega$, we have the following theorem:

Theorem 4.1.4 *If assumptions $(A_1) - (A_2)$ are satisfied, $u_0 \in L^\infty(\Omega)$ and $\theta_0 \in L^\infty \cap H^1(\Omega)$, then there exists a unique solution $(u,\theta) \in C([0,T], L^\infty(\Omega))$ to the system (4.6)-(4.9) such that $u_t \in L^\infty(Q_T)$, and u_{tt}, θ_t, $\triangle\theta \in L^2(Q_T)$.*

4.2 Asymptotic behavior of the solutions

In this section, we consider the long term behavior of the solution and prove that there exists an "absorbing set" in some affine space.

Multiplying equation (4.1) by u_t, multiplying equation (4.1) by u, multiplying equation (4.2) by θ, and integrating each over Ω, we have

$$\begin{aligned}\int_\Omega |u_t|^2 dx = &\int_\Omega\int_\Omega J(x-y)u(y)u_t(x)dydx - \int_\Omega\int_\Omega J(x-y)u(x)u_t(x)dxdy \\ &+ \int_\Omega f(u)u_t dx + l\int_\Omega \theta u_t dx,\end{aligned} \tag{4.63}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 dx = &\int_\Omega\int_\Omega J(x-y)u(y)u(x)dydx - \int_\Omega\int_\Omega J(x-y)u(x)u(x)dxdy \\ &+ \int_\Omega f(u)udx + l\int_\Omega \theta udx,\end{aligned} \tag{4.64}$$

and

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\theta|^2 dx + l\int_\Omega u_t\theta dx + \int_\Omega |\nabla\theta|^2 dx = 0. \tag{4.65}$$

Since

$$\frac{dF}{dt} = f(u)u_t, \tag{4.66}$$

$$f(u)u \geq c_1|u|^r - c_2|u|, \tag{4.67}$$

$$F(u) + c_6|u| \geq c|u|^r, \tag{4.68}$$

adding equations (4.63)-(4.65) together, and using Holder's and Young's inequalities, for any small $\epsilon > 0$ we have

$$\begin{aligned}&\int_\Omega |u_t|^2 dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 dx + c_1\int_\Omega |u|^r dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\theta|^2 dx \\ &+ \frac{d}{dt}\int_\Omega (F(u) + c_6|u|)dx + \int_\Omega |\nabla\theta|^2 dx \\ &\leq C(\epsilon)\int_\Omega |u|^2 dx + \frac{1}{2}\int_\Omega |u_t|^2 dx + \epsilon\int_\Omega |\theta|^2 dx + c.\end{aligned} \tag{4.69}$$

From Poincare's inequality $\int(\theta-\bar{\theta})^2dx \le C\int|\nabla\theta|^2dx$, where $\bar{\theta}=\frac{1}{|\Omega|}\int\theta dx$, and since $\int(\theta+lu)dx = I_0 \equiv \int(\theta_0+lu_0)dx$, we have

$$\int\theta^2dx \le C(I_0)\int u^2 + C(I_0) + C\int|\nabla\theta|^2dx. \tag{4.70}$$

Denote $Y(t) = \int_\Omega |u|^2dx + \int_\Omega |\theta|^2dx + \int_\Omega (F(u)+c_6|u|)dx$. Using Holder's and Young's inequalities again, (4.68), (4.69) and (4.70) yield

$$\frac{dY}{dt} + C_1(I_0)Y \le C_2(I_0) \tag{4.71}$$

for some positive constants $C_1(I_0)$ and $C_2(I_0)$ which do not depend on initial data.

Gronwall's inequality implies

$$Y \le C_2(I_0) + Y_0e^{-C_1(I_0)t}. \tag{4.72}$$

We have:

Theorem 4.2.1 *There exists a constant $C(I_0)$ and a time $t_0(I_0)$ which does not otherwise depend on initial data such that*

$$||u||_r \le C(I_0), \tag{4.73}$$

$$||\theta||_2 \le C(I_0) \tag{4.74}$$

for $t \ge t_0(I_0)$.

Next we estimate $||\nabla\theta||^2$.

The following lemma may be found in [44]

Lemma 4.2.2 *Let g, h, y be positive locally integrable functions on (t_0, ∞) such that y' is locally integrable on (t_0, ∞), and which satisfy*

$$\frac{dy}{dt} \leq gy + h \quad \mathit{for}\ t \geq t_0$$
$$\int_t^{t+1} g(s)ds \leq a_1, \int_t^{t+1} h(s)ds \leq a_2, \int_t^{t+1} y(s)ds \leq a_3 \tag{4.75}$$

for $t \geq t_0$, where a_1, a_2, a_3, are positive constants. Then

$$y(t+1) \leq (a_3 + a_2)\exp(a_1) \tag{4.76}$$

for any $t \geq t_0$.

Multiplying equation (4.2) by θ_t, and integrating over Ω, we obtain

$$\int(\theta_t)^2 + \frac{1}{2}\frac{d}{dt}\int|\nabla\theta|^2 dx = \int(-lu_t\theta_t)dx. \tag{4.77}$$

Holder's and Young's inequalities and (4.77) imply

$$\frac{d}{dt}\int|\nabla\theta|^2 dx \leq c\int(u_t)^2 dx. \tag{4.78}$$

By inequality (4.69), we also have

$$\int_t^{t+1}\int_\Omega|\nabla\theta|^2 dx \leq c \tag{4.79}$$

for $t \geq t_0(I_0)$.

Multiplying (4.1) by u_t, integrating over Ω, and using Holder's and Young's inequalities, we have

$$\frac{1}{2}\int(u_t)^2 dx + \frac{d}{dt}\int(F(u) + c_6|u|)dx \leq c\int u^2 dx + \int\theta^2 dx + C. \tag{4.80}$$

Integrating from t to $t+1$, and using inequality (4.69), we obtain

$$\int_t^{t+1} \int (u_t)^2 dx \le c \tag{4.81}$$

for $t \ge t_0(I_0)$.

Applying Lemma 4.2.2 to (4.78), using (4.79) and(4.81), we have

$$\int |\nabla \theta|^2 dx \le c \tag{4.82}$$

for $t \ge t_0(I_0) + 1$.

Theorem 4.2.3 *There exists constants $C(I_0)$ and $t_1(I_0) \equiv t_0(I_0) + 1$ independent of initial data such that*

$$||\nabla \theta||_2 \le C(I_0) \tag{4.83}$$

for $t \ge t_1(I_0)$.

Corollary 4.2.4 *If $n = 1$, there exists an absorbing set in an affine subspace of $L^\infty \times W^{1,2}(\Omega)$*

Proof. It follows from Theorem 4.2.1 and Theorem 4.2.3 that there exists an absorbing set in $W^{1,2}(\Omega)$ for θ. We need to prove there exists an absorbing set for u in L^∞. In fact, since n=1, $W^{1,2}(\Omega) \hookrightarrow C^\beta(\bar{\Omega})$ is compact, there exist constants $C_3(I_0)$ and $t_0(I_0)$ which do not otherwise depend on initial data such that

$$||\theta||_\infty \le C_3(I_0) \tag{4.84}$$

for $t \ge t_0(I_0)$.

Using a similar argument to that in the proof of Lemma 4.1.1, we also have

$$||u||_\infty \leq C(I_0). \tag{4.85}$$

for some constant $C(I_0)$ which does not depend on initial data and for $t \geq t_0(I_0)$ Since

$$\frac{d}{dt}E(u,e) = -\int_\Omega u_t^2 dx - \int_\Omega |\nabla\theta|^2 dx, \tag{4.86}$$

integrating equation (4.86) from 0 to t, we have

$$E(u_0, e_0) - E(u(t), e(t)) = \int_0^t \int_\Omega u_t^2 dx ds + \int_0^t \int_\Omega |\nabla\theta|^2 dx ds. \tag{4.87}$$

From (4.5), $r > 2$ and the Cauchy-Schwartz inequality, we have

$$\begin{aligned} E &= \frac{1}{4}\int\int J(x-y)(u(x)-u(y))^2 dx dy + \int (F(u(x)) + \frac{1}{2}\theta^2) dx \\ &\geq \int\int J(x-y)(u(x)-u(y))^2 dx dy + c_5 \int |u|^r dx - \int c_6 |u| dx \\ &\geq c_7 \int |u|^r dx - c_8 \end{aligned} \tag{4.88}$$

for some constants c_7 and c_8.

Therefore $E(u,e)$ is bounded below, we have $E(u_0,e_0) - E(u(t),e(t)) \leq C$ for some positive constant C which does not depend on t. This implies

$$\int_0^\infty \int_\Omega u_t^2 dx ds + \int_0^\infty \int_\Omega |\nabla\theta|^2 dx ds \leq C. \tag{4.89}$$

Multiplying equation (4.2) by $\triangle\theta$, and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\theta|^2 dx + \int_\Omega |\triangle\theta|^2 dx = -l\int_\Omega u_t \triangle\theta dx. \tag{4.90}$$

It follows from Holder's and Young's inequalities that

$$\frac{d}{dt}\int_\Omega |\nabla\theta|^2 dx + C_1\int_\Omega |\triangle\theta|^2 dx \le C_2\int_\Omega |u_t|^2 dx \tag{4.91}$$

for some constant C_1 and C_2.

Since

$$\int_\Omega |\nabla(\theta-\theta_0)|^2 dx = \int_\Omega (\theta-\theta_0)\triangle(\theta-\theta_0)dx \le ||\theta-\theta_0||_2||\triangle(\theta-\theta_0)||_2,$$

from Poincare's inequality, we also have $||\nabla\theta||_2 \le C||\triangle\theta||_2$.

The inequality (4.91) implies

$$\frac{d}{dt}\int_\Omega |\nabla\theta|^2 dx + C_3\int_\Omega |\nabla\theta|^2 dx \le C_2\int_\Omega |u_t|^2 dx \tag{4.92}$$

for some constant C_3 and C_2.

Claim: $\int_\Omega |\nabla\theta|^2 dx \to 0$ as $t \to \infty$.

Multiplying $e^{C_3 t}$ on both sides of inequality (4.92), and integrating from t to $2t$, we have

$$\begin{aligned} &e^{2tC_3}||\nabla\theta(2t,\cdot)||_2^2 - e^{tC_3}||\nabla\theta(t,\cdot)||_2^2 \\ &\le C_1\int_t^{2t} e^{C_3 s}\int_\Omega |u_t(s,x)|^2 dxds \end{aligned} \tag{4.93}$$

This implies

$$\begin{aligned} &||\nabla\theta(2t,\cdot)||_2^2 - e^{-tC_3}||\nabla\theta(t,\cdot)||_2^2 \\ &\le C_1\int_t^{2t} e^{C_3(s-2t)}\int_\Omega |u_t(s,x)|^2 dxds. \end{aligned} \tag{4.94}$$

Since $||\nabla\theta(t,\cdot)||_2^2 \le C$ and $\int_0^\infty\int_\Omega |u_t(s,x)|^2 dxds \le C$, from (4.89), letting $t \to \infty$ in inequality (4.94) gives

$$\lim_{t\to\infty} ||\nabla\theta(2t,\cdot)||_2^2 = 0. \tag{4.95}$$

Since $sup_t||\theta||_2 \le C$, by inequality (4.89), we have:

Theorem 4.2.5 *If (u,θ) is a solution of system (4.1) -(4.4), there exists a sequence $t_k \to \infty$ such that*

$$\int_\Omega |\int_\Omega J(x-y)u_k(y)dy - \int_\Omega J(x-y)dyu_k(x) - f(u_k) + l\theta_k|^2 dx \to 0, \tag{4.96}$$

$$\theta_k \to c \ in \ W^{1,2} \tag{4.97}$$

for some constant c, where $u_k(x) = u(t_k, x)$ and $\theta_k(x) = \theta(t_k, x)$.

4.3 Global boundedness of the solutions

In section 3, we proved that u and θ are uniformly bounded with respect to t for dimension $n = 1$. In this section, we shall study the global boundedness of solutions for higher space dimensions if the initial data θ_0 has better regularity. We prove that the solution $(u(t), \theta(t))$ is uniformly bounded in suitable function space on the entire interval $[0, \infty)$. We are also concerned with the regularity of solutions. Within this framework, we are able to prove a sharper result on the asymptotic behavior of (4.1)-(4.2).

Through out this section, we fix p such that $p = 2$, if $n = 1$, 2, 3 and $p > \frac{n}{2}$, if $n > 3$.

Let A be the unbounded linear operator from $L^p(\Omega)$ into itself defined by the following:

$$\begin{aligned} &Av = -\Delta v + v, v \in D(A) \\ &D(A) = \{v \in L^p(\Omega) :\ Av \in L^p(\Omega),\ \frac{\partial v}{\partial \nu}|_\Gamma = 0\}. \end{aligned} \tag{4.98}$$

It is well known that A generates an analytic semigroup on the space $L^p(\Omega)$. Also let X^α be the space $D(A^\alpha)$ endowed with the graph norms $||.||_\alpha$ of A^α for $\frac{n}{2p} < \alpha < 1$.

Define a linear operator on L^q for $q > 1$:

$$Lu :\equiv \int_\Omega J(x-y)udy - \int_\Omega J(x-y)dyu. \tag{4.99}$$

Denote $Q_T = [0,T] \times \Omega$. We have the following comparison principle.

Lemma 4.3.1 *Let $u \in C^1([0,T], L^q)$ for $(q > 1)$ and*

$$u_t \leq Lu + c(t,x)u, \tag{4.100}$$

a.e on Ω. Assume $c(t,x) \leq M_1$ on Q_T for some positive constant M_1, then $u(t,x) \leq 0$ a.e on Ω for $t \in [0,T]$.

Proof. Let $\rho = M_1 + M + 1$ and $v = ue^{-\rho t}$, where $M = sup \int_\Omega J(x-y)dy$, we have

$$v_t = u_t e^{-\rho t} - \rho u e^{-\rho t} \leq e^{-\rho t}(Lu + cu) - \rho u = Lv + cv - \rho v. \tag{4.101}$$

Multiplying inequality (4.101) by $(v^+)^{q-1}$ and integrating over Ω, we obtain

$$(\frac{1}{q}\int_\Omega (v^+)^q dx)_t \leq \int_\Omega Lv(v^+)^q dx + \int_\Omega c(v^+)^q dx - \rho \int_\Omega (v^+)^q dx. \tag{4.102}$$

Since

$$\begin{aligned}\int_\Omega Lv(v^+)^{q-1}dx &= \int_\Omega\int_\Omega J(x-y)v^+ dy(v^+)^{q-1}dx \\ &- \int_\Omega\int_\Omega J(x-y)v^- dy(v^+)^{q-1}dx - \int_\Omega\int_\Omega J(x-y)dy(v^+)^{q-1}dx \\ &\leq \int_\Omega\int_\Omega J(x-y)v^+ dy(v^+)^{q-1}dx.\end{aligned} \tag{4.103}$$

It follows from Holder's and Young's inequalities that

$$\begin{aligned}&\int_\Omega\int_\Omega J(x-y)v^+dy(v^+)^{q-1}dx \\ &\leq \left(\int_\Omega\left(\int_\Omega J(x-y)v^+dy\right)^q dx\right)^{\frac{1}{q}}\left(\int_\Omega (v^+)^q dx\right)^{\frac{q-1}{q}} \leq M_1\int_\Omega (v^+)^q dx.\end{aligned} \tag{4.104}$$

Inequalities (4.102)-(4.104) yield

$$\left(\frac{1}{q}\int_\Omega (v^+)^q dx\right)_t + (\rho - M_1 - M_2)\int_\Omega (v^+)^q dx \leq 0 \tag{4.105}$$

Integrating (4.105) from 0 to t, we have

$$||v^+(t)||_q^q \leq ||v^+(0)||_q^q. \tag{4.106}$$

$v^+(0) = 0$ implies the conclusion.

Lemma 4.3.2 *Instead of condition* (A_2), *we assume*

$(\bar{A}_2)$ There exist $a_i > 0$, $b_i > 0$, $c_i > 0$ and $r > 2$ such that

$$f(u): \begin{cases} \leq a_1|u|^{r-2}u + b_1u + c_1, & if\ u \leq 0 \\ > a_2|u|^{r-2}u - b_2u - c_2, & if\ u > 0 \end{cases} \tag{4.107}$$

Suppose that the condition (A_1) and $(\bar{A}_2)$ are satisfied, then there are positive constants n_1 and n_2 such that

$$||u(s)||_\infty \leq \left(n_1 + ||u_0||_\infty^{r-1} + n_2 \sup_{0 \leq \tau \leq s} ||\theta(\tau)||_\infty\right)^{\frac{1}{r-1}}, \tag{4.108}$$

where constants n_1 and n_2 only depend on a_i, b_i, c_i and r.

Proof. Choose any $T > 0$. Let $M(s) = \sup_{0 \leq \tau \leq s} ||\theta(\tau)||_\infty$ for each $s \in [0, T)$ and set

$$\bar{u}(\tau) \equiv (n_1 + ||u_0||_\infty^{r-1} + n_2 M(s))^{\frac{1}{r-1}}, \tag{4.109}$$

$$\underline{u}(\tau) \equiv -(n_1 + ||u_0||_\infty^{r-1} + n_2 M(s))^{\frac{1}{r-1}} \tag{4.110}$$

for each $\tau \in [0, s]$, where n_1 and n_2 are two constants which will be determined later.

Notice that both $\bar{u}$ and $\underline{u}$ are constant on the interval $[0, s]$ for fixed s. Thus

$$\bar{u}_t - \int_\Omega J(x-y)\bar{u}dy + \int_\Omega J(x-y)dy\bar{u} + f(\bar{u}) = f(\bar{u}), \tag{4.111}$$

and

$$(\underline{u})_t - \int_\Omega J(x-y)\underline{u}dy + \int_\Omega J(x-y)dy\underline{u} + f(\underline{u}) = f(\underline{u}). \tag{4.112}$$

We claim that $f(\bar{u}) \geq M(s)$ and $f(\underline{u}) \leq -M(s)$ on $[0, s]$ for suitably chosen n_1 and n_2.

In fact, from condition $(\bar{A}_2)$ and the definitions of $\bar{u}$ and $\underline{u}$, we have

$$\begin{aligned} f(\bar{u}) - M(s) \geq & a_2 n_1 - b_2 n_1^{\frac{1}{r-1}} + a_2 ||u_0||_\infty^{r-1} - b_2 ||u_0||_\infty \\ & - c_2 + (a_2 n_2 - 1) M(s) - b_2 n_2^{\frac{1}{r-1}} M(s)^{\frac{1}{r-1}}, \end{aligned} \tag{4.113}$$

and

$$\begin{aligned} -f(\underline{u}) - M(s) \geq & a_1 n_1 - b_1 n_1^{\frac{1}{r-1}} + a_1 ||u_0||_\infty^{r-1} - b_1 ||u_0||_\infty \\ & - c_1 + (a_1 n_2 - 1) M(s) - b_1 n_2^{\frac{1}{r-1}} M(s)^{\frac{1}{r-1}}. \end{aligned} \tag{4.114}$$

First, choosing $n_2 = \max_{i = 1,2}\{\frac{r-1}{a_i(r-2)} + (\frac{b_i}{a_i})^{\frac{r-1}{r-2}} + 1\}$ then $a_2n_2 - 1 - b_2n_2^{\frac{1}{r-1}} \geq 0$ and $a_1n_2 - 1 - b_1n_2^{\frac{1}{r-1}} \geq 0$.

Inequalities (4.113)-(4.114) yield

$$f(\bar{u}) - M(s) \geq a_2n_1 - b_2n_1^{\frac{1}{r-1}} - b_2(\frac{b_2}{a_2})^{\frac{1}{r-2}} - c_2 - b_2n_2^{\frac{1}{r-1}}, \tag{4.115}$$

and

$$-f(\underline{u}) - M(s) \geq a_1n_1 - b_1n_1^{\frac{1}{r-1}} - b_1(\frac{b_1}{a_1})^{\frac{1}{r-2}} - c_1 - b_1n_2^{\frac{1}{r-1}}. \tag{4.116}$$

Choosing n_1 such that

$$a_2n_1 - b_2n_1^{\frac{1}{r-1}} - b_2(\frac{b_2}{a_2})^{\frac{1}{r-2}} - c_2 - b_2n_2^{\frac{1}{r-1}} \geq 0, \tag{4.117}$$

and

$$a_1n_1 - b_1n_1^{\frac{1}{r-1}} - b_1(\frac{b_1}{a_1})^{\frac{1}{r-2}} - c_1 - b_1n_2^{\frac{1}{r-1}} \geq 0. \tag{4.118}$$

It follows from (4.111)-(4.118) that

$$\bar{u}_t - \int_\Omega J(x-y)\bar{u}(y)dy + \int_\Omega J(x-y)dy\bar{u} + f(\bar{u}) \geq l\theta, \tag{4.119}$$

and

$$\underline{u}_t - \int_\Omega J(x-y)\underline{u}dy + \int_\Omega J(x-y)dy\underline{u} + f(\underline{u}) \leq l\theta \tag{4.120}$$

on $[0, s]$. Denote $\bar{w} = u - \bar{u}$ and $\underline{w} = \underline{u} - u$, then we have

$$\bar{w}_T - \int_\Omega J(x-y)\bar{w}(y)dy + \int_\Omega J(x-y)dy\bar{w} + \bar{c}\bar{w} \leq 0. \tag{4.121}$$

and

$$\underline{w}_T - \int_\Omega J(x-y)\underline{w}(y)dy + \int_\Omega J(x-y)dy\underline{w} + \underline{c}\,\underline{w} \le 0, \tag{4.122}$$

where $\bar{c} = \int_0^1 -f'(\lambda\bar{u} + (1-\lambda)u)d\lambda$ and $\underline{c} = \int_0^1 -f'(\lambda\underline{u} + (1-\lambda)u)d\lambda$.

Lemma 4.3.1 and condition $(\bar{A}_2)$ imply

$$\underline{u} \le u \le \bar{u} \tag{4.123}$$

on $[0, s]$. Therefore,

$$||u(s)||_\infty \le (n_1 + ||u_0||_\infty^{r-1} + n_2 \sup_{0\le\tau\le s} ||\theta(\tau)||_\infty)^{\frac{1}{r-1}}. \tag{4.124}$$

Theorem 4.3.3 *Suppose that conditions (A_1) and $(\bar{A}_2)$ are satisfied and $(u_0, \theta_0) \in L^\infty \times X^\alpha$ with $\frac{n}{2p} < \alpha < 1$. Then there exists a unique solution (u, θ) to (1.1) and (1.2) which possesses the property described in Theorem 2.2. Moreover, $(u, \theta) \in L^\infty \times X^\alpha$*

$$\sup_{0\le t<\infty} ||\theta(t)||_{X^\alpha} \le C_1(a_i, b_i, c_i, ||\theta_0||_{X^\alpha}, ||u_0||_\infty) \tag{4.125}$$

$$\sup_{0\le t<\infty} ||u(t)||_\infty \le C_2(a_i, b_i, c_i, ||\theta_0||_{X^\alpha}, ||u_0||_\infty) \tag{4.126}$$

Proof. It follows from Theorem 4.1.4 that system (4.1)-(4.4) has a unique solution (u, θ) because $\theta_0 \in X^\alpha \subset W^{1,2} \cap L^\infty$. Since $u_t \in L^\infty((0,T), L^p)$, a basic regularity result implies that $\theta \in C([0,T], X^\alpha)$. Without loss of generality, we may assume that $p > r - 1 + 1$ and

$$|f(u)| \le (a_1 + a_2)|u|^{r-1} + (b_1 + b_2)|u| + (c_1 + c_2). \tag{4.127}$$

Note that

$$
\begin{aligned}
(\int_\Omega |u_t|^p dx)^{\frac{1}{p}} &= (\int_\Omega |Lu + f(u) + l\theta|^p dx)^{\frac{1}{p}} \\
&\leq (\int_\Omega |\int_\Omega J(x-y)u dy|^p dx)^{\frac{1}{p}} + (\int_\Omega |\int_\Omega J(x-y) dy u|^p dx)^{\frac{1}{p}} \\
&+ (\int_\Omega |f(u)|^p dx)^{\frac{1}{p}} + (\int_\Omega |\theta|^p dx)^{\frac{1}{p}} \\
&\leq (2M_1 + b_1 + b_2)(\int_\Omega |u|^p dx)^{\frac{1}{p}} + (a_1 + a_2)(\int_\Omega |u|^{rp} dx)^{\frac{1}{p}} \\
&+ l(\int_\Omega |\theta|^p dx)^{\frac{1}{p}} + (c_1 + c_2)|\Omega|^{\frac{1}{p}} \\
&\leq (2M_1 + b_1 + b_2 + a_1 + a_2)(||u||_\infty^{\frac{p-(r)}{p}} + ||u||_\infty^{\frac{r-1(p-1)-1}{p}})||u||_r^{\frac{r-1+1}{p}} \\
&+ l||\theta||_\infty^{\frac{p-2}{p}} ||\theta||_2^{\frac{2}{p}} + (c_1 + c_2)^p|\Omega|.
\end{aligned}
\tag{4.128}
$$

Let $\lambda = \frac{p-1}{p} - \frac{1}{rp}$. Due to inequality (4.128) and Lemma 4.3.2, we see that

$$||u_t(s,\cdot)||_p \leq C_1(\sup_{0 \leq \tau \leq s} ||\theta(\tau)||_\infty)^\lambda + C_2||\theta(s)||_\infty^\lambda + C_3. \tag{4.129}$$

Therefore,

$$\sup_{0 \leq s \leq t} ||u_t(s,\cdot)||_p \leq C_4(\sup_{0 \leq s \leq t} ||\theta(s)||_\infty)^\lambda + C_5, \tag{4.130}$$

where $C_i, (i = 1, 2, ...5)$ are positive constants which only depend on a_i, b_i, c_i, M_1, $||u_0||_\infty$, $||\theta_0||_\infty$ and Ω.

By using the operator A, (1.2) can be formulated in terms of the abstract Cauchy problem in the Banach space $L^p(\Omega)$ as follows:

$$\theta_t + A\theta = \theta - lu_t \tag{4.131}$$

$$\theta(0) = \theta_0. \tag{4.132}$$

Therefore

$$\begin{aligned}||\theta(t)||_{X^\alpha} &\le ||e^{-At}\theta_0||_{X^\alpha} + \int_0^t ||e^{-A(t-s)}(lu_t(s,\cdot) - \theta(s))||_{X^\alpha} ds \\ &\le e^{-at}||\theta_0||_{X^\alpha} + \sup_{0\le s\le t}(||u_t(s,\cdot)||_p + ||\theta(s)||_p)\int_0^t (t-s)^{-\alpha}e^{-a(t-s)}ds \\ &\le ||\theta_0||_{X^\alpha} + (\sup_{0\le s\le t} ||\theta(s)||_\infty)^\lambda C_6 \int_0^\infty s^{-\alpha}e^{-as}ds.\end{aligned} \tag{4.133}$$

Using the embedding $X^\alpha \hookrightarrow L^\infty(\Omega)$, we obtain

$$\sup_{0\le t<T_{max}} ||\theta(t)||_\infty \le C_\alpha \sup_{0\le t<T_{max}} ||\theta(t)||_{X^\alpha} \le C_8(\sup_{0\le t<T_{max}} ||\theta(t)||_\infty)^\lambda + C_9. \tag{4.134}$$

This implies

$$\sup_{0\le t<T_{max}} ||\theta(t)||_\infty \le \frac{C_8}{1-\lambda} + C_9^{\frac{1}{1-\lambda}}, \tag{4.135}$$

where C_8 and C_9 are positive constants which depend on a_i, b_i, c_i, M_1, $||u_0||_\infty$, $||\theta_0||_{X^\alpha}$ and Ω.

Since the right side of inequality (4.135) does not depend on T, we have

$$\sup_{0<t<\infty} ||\theta||_\infty \le C$$

for some positive constant C, depending on the above parameters and on $||\theta_0||_{X^\alpha}$.

Inequality (4.124) implies

$$\sup_{0<t<\infty} ||u||_\infty \le C. \tag{4.136}$$

Theorem 4.3.4 *Assume that conditions (A_1) and $(\bar{A}_2)$ are satisfied and $(u_0,\theta_0) \in L^\infty \times X^\alpha$. If (u,θ) is the solution to (1.1) and (1.2), then*

$$\theta \in C^{\alpha-\gamma}([0,\infty), X^\gamma) \cap C^{1+\alpha}([\epsilon,\infty), X) \cap C^\alpha([\epsilon,\infty), D(A)) \tag{4.137}$$

for any $0 \le \gamma \le \alpha < 1$ *and* $0 < \epsilon < \infty$. *Also*

$$u \in C^{1+\beta}([0,\infty), L^\infty)$$

for any $\beta = \alpha - \mu, \dfrac{n}{2p} < \mu < \alpha.$

Proof.

Since u_t and θ belong to $L^\infty((0,\infty), L^p)$, we have

$$\begin{aligned} ||\int_0^{t+h} e^{-A(t+h-s)}(u_t(s,\cdot)+\theta(s,\cdot))ds - \int_0^t e^{-A(t-s)}(u_t(s,\cdot)+\theta(s,\cdot))ds||_{X^\nu} \\ \le C_\nu h^{1-\nu} \sup_{0\le s<\infty}(||u_t(s)||_p + ||\theta(s)||_p) \end{aligned} \tag{4.138}$$

for any $0 < \nu < 1$.

Meanwhile

$$||(e^{-Ah} - I)e^{-At}\theta_0||_{X^\gamma} \le C_{\alpha,\gamma} h^{\alpha-\gamma}||\theta_0||_{X^\alpha}. \tag{4.139}$$

It follows that

$$\theta \in C^{\alpha-\gamma}([0,\infty), X^\gamma) \tag{4.140}$$

for any $0 \le \gamma \le \alpha$.

Choosing $\dfrac{n}{2p} < \gamma < \alpha$, from the embedding theorem, we have

$$\theta \in C^{\alpha-\gamma}([0,\infty), L^\infty(\Omega)). \tag{4.141}$$

On the other hand,

$$\begin{aligned} |u(t+h) - u(t)|_\infty &\le \int_t^{t+h} [||Lu(s)||_\infty + ||f(u(s))||_\infty + ||\theta(s)||_\infty]ds \\ &\le C(||u_0||_\infty, ||\theta_0||_\infty)h. \end{aligned} \tag{4.142}$$

Consequently,

$$\begin{aligned} ||u_t(t+h) - u_t||_\infty \le 2M_1||u(t+h) - u(t)||_\infty + ||f'(\eta)|| |u(t+h) - u(t)||_\infty \\ + ||\theta(t+h) - \theta(t)||_\infty \le C_1(||u_0||_\infty, ||\theta_0||_\infty)h^{\alpha-\gamma}. \end{aligned} \tag{4.143}$$

for $\eta = \varsigma u(t+h) + (1-\varsigma)u(t)$.

A similar argument also shows that

$$u_t \in C^{\alpha}([0,\infty), L^p). \tag{4.144}$$

This completes the proof.

Proposition 4.3.5 *If the condition $(\bar{A}_2)$ is replaced by*

$(\bar{A}_3)$ There exist $a_3 > 0$, $b_3 > 0$, $d_1 > 0$, $d_2 > 0$ and $r > 2$ such that

$$a_3|u|^{r-2}u + b_3 u - d_1 \le f(u) \le a_3|u|^{r-2}u + b_3 u + d_2. \tag{4.145}$$

and $(u(t), \theta(t))$ are solution of (1.1-1.2) with $(u_0, \theta_0) \in L^{\infty} \times X^{\alpha}$, then there is a constant K independent of u_0 and θ_0 such that

$$\lim_{t \to \infty} ||\theta - \bar{\theta}||_{W^{1,q}} = 0, \tag{4.146}$$

$$\limsup_{t \to \infty} ||\theta||_{\infty} \le K, \tag{4.147}$$

$$\limsup_{t \to \infty} ||u||_{\infty} \le K, \tag{4.148}$$

where $\bar{\theta} = \frac{1}{|\Omega|}\int_{\Omega} \theta dx$, $n < q < n + \delta$,, and $\delta \le 2n(\alpha - n/2p)$

Proof.

By interpolation theorem, we have

$$||\theta - \bar{\theta}||_{W^{1,q}} \le C||\theta - \bar{\theta}||^{\lambda}_{W^{1,2}} ||\theta - \bar{\theta}||^{1-\lambda}_{X^{\alpha}}. \tag{4.149}$$

From inequality (4.134), and $sup_t ||\theta||_{\infty} < C$, we have

$$\sup_t ||\theta - \bar{\theta}||_{X^\alpha} < C. \tag{4.150}$$

From section 3, we have

$$\lim_{t \to \infty} ||\theta - \bar{\theta}||_{W^{1,2}} = 0. \tag{4.151}$$

This implies (4.146).

Since

$$||\theta||_{W^{1,q}} \leq ||\theta - \bar{\theta}||_{W^{1,q}} + ||\bar{\theta}||_{W^{1,q}}, \tag{4.152}$$

and

$$||\bar{\theta}||_{W^{1,q}} \leq C||\theta||_2^\beta \tag{4.153}$$

for some constants C and β.

Inequality (4.74) implies

$$\limsup_{t \to \infty} ||\theta||_{W^{1,q}} \leq C(I_0), \tag{4.154}$$

where $I_0 = \int_\Omega (lu_0 + \theta_0) dx$. Embedding theorem implies (4.147).

Using a similar argument to that in the proof of Lemma 4.1.1 gives (4.148).

Remark 4.3.6 Inequalities (4.133), (4.147) and (4.148) also imply:

$$\limsup_{t \to \infty} ||\theta||_{X^\alpha} \leq C(I_0).$$

Next, we prove that there exists a subsequence t_k such that $u(t_k) \to u^*$, $\theta(t_k) \to c$, and (u^*, c) is a steady state solution of system (4.1)-(4.4).

We assume:

(A_3) $M \equiv \sup |\int_\Omega J'(x-y)dy| < \infty$, and $f \in C^1(\mathbb{R})$.

(A_4) There exists a constant $c_7 > 0$, such that $f'(u) + a(x) \geq c_7 > 0$, where $a(x) = \int_\Omega J(x-y)dy$.

We have:

Theorem 4.3.7 *Assume that conditions (A_1), $(\bar{A}_2)$, and (A_3) are satisfied, let (u, θ) be a solution of system (4.1)-(4.4), we have*

$$||u_t||_2 \to 0 \ as \ t \to \infty, \tag{4.155}$$

$$||\nabla\theta||_2 \to 0 \ as \ t \to \infty. \tag{4.156}$$

In addition, if (A_4) is also satisfied, then there exists a subsequence $\{t_k\}$, $\tilde{u} \in L^2(\Omega)$, and constant C such that

$$||u(t_k, x) - \tilde{u}(x)||_2 \to 0 \ as \ k \to \infty, \tag{4.157}$$

$$||\theta - C||_2 \to 0 \ as \ k \to \infty, \tag{4.158}$$

$$l \int_\Omega \tilde{u}dx + C|\Omega| = I_0, \tag{4.159}$$

where $I_0 = \int_\Omega (lu_0 + \theta_0)dx$. And $(\tilde{u}, C)$ is a steady state solution to system (4.1)-(4.4).

Proof.The conclusion (4.156) follows from (4.95).

Taking derivative with respect to t in equation (4.1), multiplying by u_t and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_t|^2 = \int_\Omega\int_\Omega J(x-y)u_t(y)u_t(x)dxdy - \int_\Omega\int_\Omega J(x-y)u_t^2(x)dydx \\ - \int_\Omega f'(u)u_t^2(x)dx + l\int_\Omega \theta_t u_t dx \tag{4.160}$$

By (4.136) and Holder's and Young's inequalities, we have

$$\frac{d}{dt}\int_\Omega |u_t|^2 dx \le C_1\int_\Omega |u_t|^2 dx + C_2\int_\Omega |\theta_t|^2 dx \tag{4.161}$$

Multiplying equation (4.2) by θ_t, integrating over Ω, and using Holder's and Young's inequalities, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\theta|^2 + \frac{1}{2}\int_\Omega |\theta_t|^2 dx \le C\int_\Omega |u_t|^2 \tag{4.162}$$

Inequality (4.89), (4.95), and (4.162) imply

$$\int_0^\infty\int_\Omega |\theta_t(x)|^2 dx \le C, \quad \int_0^\infty\int_\Omega |u_t(x)|^2 dx \le C \tag{4.163}$$

for some constant C.

Let $\epsilon > 0$, it follows from (4.163) that there exists $\{s_n\}$ with

$$\int_\Omega |u_t(s_n, x)|^2 dx \to 0, \tag{4.164}$$

and there exists N such that and for any $n > N$ and $t > s_n$, we have

$$\int_\Omega |u_t(s_n, x)|^2 dx < \epsilon, \tag{4.165}$$

$$\int_{s_n}^t\int_\Omega |u_t(s,x)|^2 dxds < \epsilon, \tag{4.166}$$

$$\int_{s_n}^{t}\int_{\Omega}|\theta_t(s,x)|^2dxds < \epsilon. \tag{4.167}$$

For any $t > s_n$, from (4.162), (4.165)-(4.167), we have

$$\int_{\Omega}|u_t(s,x)|^2dx < (2+c)\epsilon. \tag{4.168}$$

This implies (4.155).

Denote $w = \theta - \bar{\theta}$, since $\bar{\theta} = \frac{1}{|\Omega|}\int_{\Omega}\theta dx = \dfrac{I_0}{|\Omega|} - \dfrac{l}{|\Omega|}\displaystyle\int_{\Omega}udx = c_1 - c_2\int_{\Omega}udx$, we have $\theta = w + c_1 - c_2\int_{\Omega}udx$. By (4.156) and Poincare inequality, we also have

$$||w||_2 \to 0 \ as \ t \to \infty. \tag{4.169}$$

Using this and (4.1), we have

$$a(x)u(t,x) + f(u) = \int_{\Omega}(J(x-y) - lc_2)u(t,y)dy + lw + lc_1 - u_t. \tag{4.170}$$

Since operator K: $L^2(\Omega) \to L^2(\Omega)$ defined by

$$K(u) \equiv \int_{\Omega}(J(x-y) - lc_2)u(t,y)dy$$

is compact, there exists a sequence $\{t_k\}$ such that $K(u(t_k))$ converges to $\tilde{k}$ in $L^2(\Omega)$. This and (4.155), (4.169) imply that $a(x)u(t_k) + f(u(t_k))$ is a Cauchy sequence in $L^2(\Omega)$.

From condition (A_4), we have

$$c_7||u(t_k) - u(t_m)||_2 < ||a(x)u(t_k) + f(u(t_k)) - a(x)u(t_m) + f(u(t_m))||_2. \tag{4.171}$$

Therefore, $u(t_k)$ is a Cauchy sequence in $L^2(\Omega)$, there exists $\tilde{u}$ such that $||u(t_k) - \tilde{u}||_2 \to 0$.

The boundedness of $||\theta||_{W^{1,2}(\Omega)}$ and (4.156) imply (4.158).

Since

$$\int (lu + \theta) = I_0,$$

This, (4.157) and (4.158) imply (4.159).

Since $u(t_k)$ and $\tilde{u}$ are globally bounded, we have $||f(u(t_k) - f(\tilde{u})||_2 \to 0$. This, (4.157),(4.158), and (4.170) imply that $(\tilde{u}, C)$ is a steady state solution to system (4.1)-(4.4).

We complete the proof.

If the initial data are smooth enough, we also have:

Corollary 4.3.8 *If $u_0(x) \in W^{1,\sigma}(\Omega)$, $\theta_0(x) \in W^{2,\sigma}(\Omega)$ for $\sigma > n$, conditions (A_1), $(\bar{A}_2)$, (A_3), (A_4) are satisfied, then there exists a subsequence t_k such that*

$$u(t_k) \to u^* \ in \ C^{\gamma_1}(\bar{\Omega}), \tag{4.172}$$

$$\theta(t_k) \to c \ in \ C^{\gamma_2}(\bar{\Omega}) \tag{4.173}$$

where (u^*, c) is a steady state solution of system (4.1)-(4.4), $0 < \gamma_1$, $\gamma_2 < 1$ are two constants.

Proof.

Since $u_0(x) \in W^{1,\sigma}(\Omega)$, $\theta_0(x) \in W^{2,\sigma}(\Omega)$ for $\sigma > n$, by Theorem 4.3.3, there exist solutions $u \in L^\infty((0,\infty), L^\infty(\Omega))$, $\theta \in C([0,\infty), W^{2,\sigma}(\Omega))$. This implies

$$|\nabla\theta| \in C([0,\infty), W^{1,\sigma}(\Omega)). \tag{4.174}$$

Denote $\triangle_h^i u = \dfrac{u(x + he_i) - u(x)}{h}$.

From equation (4.1), we have

$$\begin{aligned}\triangle_{h^i} u_t = &\int_\Omega \triangle_{h^i} J(x-y)u(y)dy - \int_\Omega \triangle_{h^i} J(x-y)dy u(x) \\ &- \int_\Omega J(x-y)dy \triangle_{h^i} u - f'(\xi)\triangle_{h^i} u + l\triangle_{h^i}\theta,\end{aligned} \tag{4.175}$$

where $\xi = \lambda u(x+h) + (1-\lambda)u(x)$.

Since $||\triangle_h^i \theta||_{L^\sigma(\Omega')} \leq ||\nabla\theta||_{L^\sigma(\Omega)}$ for $h < dist(\partial\Omega', \partial\Omega)$, and $W^{1,\sigma} \hookrightarrow L^\infty$, we have

$$\sup_t ||\triangle_h^i \theta||_\infty \leq C \tag{4.176}$$

for some positive constant C which does not depend on h.

Multiplying equation (4.175) by $|\triangle_h^i u|^{\sigma-2}\triangle_h^i u$, integrating over Ω', using Holder's and Young's inequalities, (4.176), and condition (A_4), we have

$$\frac{1}{\sigma}\frac{d}{dt}\int_{\Omega'} |\triangle_h^i u|^\sigma + C\int_{\Omega'} |\triangle_h^i u|^\sigma \leq C_1^\sigma \tag{4.177}$$

for some constants C, C_1 which do not depend on h.

Gronwall's inequality implies

$$\sup_t ||\triangle_h^i u||_\sigma \leq C_2 \tag{4.178}$$

for some constant C_2 which does not depend on h. This implies

$$\sup_t ||u_{x_i}||_\sigma \leq C_2. \tag{4.179}$$

Since $||u||_\infty < C$, we have

$$\sup_t ||u||_{W^{1,\sigma}} \leq C. \tag{4.180}$$

Therefore there exists a subsequence t_k such that

$u(t_k) \rightharpoonup u^*$ in $W^{1,\sigma}(\bar{\Omega})$,

$u(t_k) \to u^*$ in $C^{\gamma_1}(\bar{\Omega})$

This and Theorem 4.2.5 yield

$$\int_\Omega J(x-y)u^*(y)dy - \int_\Omega J(x-y)dyu^*(x) - f(u^*) + lc = 0 \tag{4.181}$$

where c is a constant.

Also (4.146) and Theorem 4.2.5 imply that there exists a sequence $\{t_k\}$ such that

$$\lim_{t_k \to \infty} ||\theta_k - c||_{W^{1,q}} = 0. \tag{4.182}$$

Since $q > n$, by embedding theorem, we have

$$\lim_{t_k \to \infty} ||\theta_k - c||_{C^{\gamma_2}} = 0. \tag{4.183}$$

This completes the proof.

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