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ARISTOTLE AS SECONDARY MATHEMATICS TEACHER EDUCATOR: METAPHORS AND STRENGTHS

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ABSTRACT

ARISTOTLE AS SECONDARY MATHEMATICS TEACHER EDUCATIOR: METAPHORS AND STRENGTHS

BY

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This dissertation examines secondary mathematics teacher candidates' discussion, in a required methods class, of Aristotle's argument that a line is not composed of points. The conceptual framework for this research is based on the insistence that students' mathematical conceptualizations are cognitive strengths rather than replete with misconceptions. An analytical framework deriving from Lakoff and Núñez's *Where Mathematics Comes From* is used to tease out students' conceptualizations of foundational issues – most urgently, the point-line relationship and the nature of numbers – in terms of the metaphorical structure of their expressed thinking. This framework uses image schemas and conceptual metaphor to demonstrate that the students' actions and abilities are elements that are natural and necessary when doing mathematics. The paper suggests that if students' abilities can be seen in this way this may provide another avenue for improving student learning.

To my mother and father,
Annie Louise Johnson
and
Thelman Johnson
who instilled the value of education in me.
Thank you for the opportunity.

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educator and I pray that you will take full advantage of what the world has to offer you and that you will give back to it your uniqueness and creativity.

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CHAPTER 1

INTRODUCTION

Introduction to the study

I began this study wanting to find a way to express to others what I saw as strengths in my preservice teacher education students' mathematical thinking and doing. After examining the transcript of conversations they had in the context of a teacher education class, I have found connections to the text Where Mathematics Comes From by George Lakoff and Rafael E. Núñez. At the outset, I wasn't quite sure that I would be able to accomplish my goal of identifying students' strengths, but I think I have.

Lakoff and Núñez's argument rests on the premise that mathematicians use everyday human cognitive abilities in the creation and doing of mathematics. They argue that mathematics doesn't have any neural connections or structures that are particular to it alone. It instead uses the same conceptual mechanisms that humans use for all other areas of thinking; this primarily consisting of image schemas, aspectual schemas, conceptual metaphor and conceptual blends. It is through conceptual metaphor that abstract thought is made possible. Mathematicians make extensive use of conceptual metaphor in the creation of mathematics. However, the teaching of conceptual metaphor and blends are not included in the teaching of mathematics.

Using Lakoff and Núñez's text has been quite beneficial to me. Not being a cognitive scientist myself, I don't necessarily agree with all of the details concerning image and aspectual schemas and how they are embedded in metaphors. However, the simple statement that the authors make about being explicit with students about what it is they are learning and its construction has won me over. Their use of conceptual metaphor and blends also have impressed me to the point that I think they could be effectively used in improving instruction to secondary mathematics students, in particular prospective teachers, and eventually improving their learning experiences and their future teaching and the learning experiences of their future students.

The conversations I created with my preservice teacher education students centered on a text by Aristotle, a piece from the *Metaphysics*. The use of the Aristotle article was quite productive for us. It allowed students to raise questions, question each other and further one another's thinking. This piece also assisted me in seeing how the Lakoff and Núñez text can be useful and work in tandem with the Aristotle piece and other historical mathematical texts. By bringing the two materials, the portion of the *Metaphysics* and Lakoff and Núñez's text, together I was able to see that there existed moments for students to do some mathematical work that isn't a part of traditional mathematics classes. These opportunities could allow for deeper mathematical study and also open the door for discussion of conceptual metaphor and blends in mathematics. This combination worked well together and, with the historical aspect, produced the following opportunities.

- It brought the students' questions to the surface for them to work on and to hear from one another.
- It provided a context for students to bring up mathematics that they have previously or were currently studying. This provided the opportunity for us to revisit mathematical ideas from other classes that overlap or connect to the topics

in their current class and build a deeper understanding by removing the focus from the details of the subject and allowing students to first think more broadly.

- It provided the opportunity for students to deepen their understanding about the mathematical topics that they raise. It allowed students to clear up misunderstandings and to build deeper understandings about the topics.
- Conversations about the article create many places in which an instructor could introduce the relevant issues from cognitive science in relation to mathematics. Here the students demonstrated
- skills that are often used in mathematics. They also demonstrated the importance of definition in the doing of mathematics and how every aspect must be attended to;
- the ability to generate and to look across several examples, isolate the essential elements and then use this information in the doing of mathematics;
- that discretized space is inherent to the students' thinking – in particular, the repeated use of speaking of geometrical figures as made out of points;
- that the Space-Set blend and the final metaphor for discretized space (to be described later on) are present in the students' thinking;
- S-P-G schema and Container Schema in their thinking

Connecting to the Mathematical Preparation of Teachers

Although I agree with Liping Ma (1999) that we have to work simultaneously on the education of prospective teachers and the education of K–12 students, this study is focused on the former. I see my work as being at the intersection among teaching, learning, and mathematics. I think we as educators have the most to gain if we can find ways to look at these three areas simultaneously. In The Mathematical Preparation for Teachers, Part I, a document written for mathematicians, the authors often state that what prospective teachers need is a "deep understanding of school mathematics concepts and procedures." They make a point of saying that "deep understanding" refers to the mathematics but acknowledge that these future teachers also need "...mathematical knowledge for teaching." This knowledge allows teachers to assess their students' work, recognizing both the sources of student errors and their students' understanding of the mathematics being taught. They also can appreciate and nurture the creative suggestions of talented students. Additionally, these teachers see the links between different mathematical topics and make their students aware of them" (p. 13). In their attempts to bridge mathematics with the education of mathematics teachers, they have managed to still keep the two separate. Overall, while the authors are acknowledging that due to the changes in the high school curriculum and the lack of success that high school students are having in mathematics and the problems that appear with undergraduate math majors, there needs to be some action taken on the part of mathematics departments. They make the following recommendations and state how they see them as addressing the issues.

To meet these needs and to address the concerns discussed above, the education of prospective high school mathematics teachers should develop:

- deep understanding of the fundamental mathematical ideas in curricula for Grades 9–12 and strong technical skills for application of those ideas;

- knowledge of the mathematical understandings and skills that students acquire in their elementary and middle school experiences, and how these affect learning in high school;
- knowledge of the mathematics that students are likely to encounter when they leave high school for collegiate study, vocational training or employment;
- mathematical maturity and attitudes that will enable and encourage continued growth of knowledge in the subject and its teaching. (Conference Board of the Mathematical Sciences, 2001)

This report recommends two main ways that mathematics departments can attain these goals. First, core mathematics major courses can be redesigned to help future teachers make insightful connections between the advanced mathematics they are learning and the high school mathematics they will be teaching. Second, mathematics departments can support the design, development, and offering of a capstone course sequence for teachers in which conceptual difficulties, fundamental ideas and techniques of high school mathematics are examined from an advanced standpoint.

Although I agree with the premise of the suggestions above, it doesn't seem to be very different from those which began in the 1980s. Thus my experiences and those of my classmates and students that I discuss later in this chapter have benefited from these changes to some extent. Looking at the recommendations within the different subject areas within mathematics, what has been done is to scale up the changes that have been made at the secondary level to the undergraduate level. Also, the subject areas are still quite isolated from one another and only come together in the suggestion of a six-credit capstone course for prospective teachers.

> This [course] sequence is an opportunity for prospective teachers to look deeply at fundamental ideas, to connect topics that often seem unrelated, and to further develop, the habits of mind that define mathematical approaches to problems. By including the historical development of major concepts and examination of conceptual difficulties, this capstone sequence connects individual mathematics courses with school mathematics and contributes to the mathematical understanding and pedagogical skills of teachers. (p. 46)

In The Mathematical Preparation for Teachers, Part II, the authors give more detail for each subject area. My study stands apart in that it begins to look at how we can combine many of these aspects into one course perhaps leaving the capstone course to primarily make a strong connection back to the high school classroom. In Part I, the authors state that the quality of the mathematics is more important than the quantity of mathematics studied. There is no recommendation for a decrease in the quantity of courses. What they may be suggesting and I am in support of is to build a solid foundation of undergraduate mathematics, with the changes that I have hinted at in this document, and to give a capstone sequence focusing on one area of mathematics and its connection to the secondary curriculum. Issues of pedagogy, appropriate lesson planning, finding suitable texts for secondary students, etc., can be addressed in the capstone course.

In order to understand some of my concerns about the Mathematical Preparation of Teachers document, in the next section I use my own learning experiences to present a mathematical learning experience from a student's perspective.

My Learning Experiences

I consider myself to be a survivor of my preparation in mathematics. My bachelor and master's degrees are both in mathematics. Although the experiences in my bachelor's degree moved me to study for a master's, they were quite stifling to me as a thinker and as someone who wanted to possess and develop my own mathematical ideas. It wasn't that my professors took control of my thinking. It was more that some important things were missing. Throughout these experiences, I had few opportunities to understand how what I was learning was connected. And, many of the questions that I had were left unanswered. Yet, unlike many of my peers, I continued actively to think about mathematics and at times, when alone, to do mathematics in ways that were sensible and aesthetically pleasing to me. It is in this regard that I consider myself a survivor of my mathematical preparation. I managed to survive my education and continue to do mathematics. The obstacles before me—the demeanor of my professors in class, their lack of patience in office hours, the lecture structure of their classes, their lack of interest in my own thinking about crucial mathematical ideas—led me to a posture of outwardly conforming to their way of doing mathematics, while inwardly chafing at how I was being asked to learn.

I vividly recall from my undergraduate experience my first course in analysis (I subsequently took other analysis courses as a master's student). Analysis, at least in the places that I've studied, is one of the courses that weeds out math majors. It is a hurdle that keeps some from being successful mathematics majors and becoming teachers. Even for those who jump over this hurdle, their GPAs and their egos may have been bruised as a result.

Analysis is where math majors might learn about the underpinnings of the real number system and why the relationship between lines and points is supposed to mirror the relationship between real numbers and the real number system. Where I studied, analysis was one of the first "theoretical" courses in which the focus is on proving theorems, rather than simply solving problems. There was an important companion change in classroom practice. Prior courses, particularly in the calculus sequence, emphasized procedural aspects of problem solving, while in analysis there were few mechanical procedures to carry out. Unlike my experience in these earlier courses, in order to be successful in analysis, it was necessary to pay close attention to the definitions and theorems, and to learn how to prove statements. There seemed to be a consensus amongst my professors, as well as the junior and senior math majors, that this was the course where students begin to study mathematics–not that we discussed what mathematics is and what it means to study it.

The following quote from an undergraduate analysis text suggests that my experiences were not unique. In Mathematical Analysis-An Introduction (Scott & Tims, 1966), the authors state that the purpose of their book is "...to widen and deepen his [the student's] understanding of the fundamental notions on which they [mathematical results, theorems, etc.] rest, and to clarify the logical connections between them" (p. 1). In reference to limits, the mathematical construct at the foundation of analysis, the authors write, "[T]he reader will almost certainly have had some manipulative practice with 'limits', and very likely will have used the theorems of 5.2.4 without question; but this is a different thing from understanding what the definition of a limit is, from knowing how the theorems may be proved from that definition and from realizing that such proofs are desirable" (p. 1).

We began the course with a discussion of sequences and finding bounds for them. The professor introduced the terms *least upper bound* (lub) and *greatest lower bound* (glb) and their definitions. We were also given the definition of an *infinite sequence* as an ordering of a set wherein each element in the set is associated with one of the natural numbers. A set, when ordered by the natural number associated with each term, is called a sequence. For example, the sequence $\{s_n\}$: {-1, 1, -2, 2, -3, 3, ...} is one ordering of the integers (leaving out 0). It is a sequence where the first term in the sequence, named s_1, is –1, the second term s_2 is 1, etc. From what I had gathered in class, we were concerned with the difference between *bounded* and *unbounded* infinite sequences. A bound for a sequence is a number that the sequence does not go beyond. The above sequence has no bounds. It doesn't have any lower bounds because the negative terms of the sequence continue to decrease in an unlimited way; it lacks upper bounds because the positive terms of the sequence increase in the same way. In class, we were told that a set is bounded (bdd) if is bounded both above and below. For instance, the sequence {-1, 1, -1, 1, ...} is bounded above by 1 and below by -1. Therefore it is a bounded sequence.

This early part of the course was fine, but already I began to have questions. Given a set, I could produce a bound for it, if of course it had one. But, since we were simply discussing whether sets were bounded or not, I didn't see what the purpose was in seeking the smallest upper bound or the greatest lower bound. As the class days passed, my question remained unanswered; the question gnawed at me.

We moved on. One day, my teacher began what appeared to me to be a new topic, the definition of a *convergent sequence*. This states that a sequence $\{s_n\}$ converges to its limit s if for every $\varepsilon > 0$ there exists an N such that for all $n > N$, $|s_n - s| < \varepsilon$. Upon first reading this definition, I was struck by the amount of notation that was present. There are many variables involved and they don't all function in the same way. Initially, I was very confused. I believe I had the following thoughts:

- The $\varepsilon > 0$ was any fixed positive number that according to my professor should be as small as possible.
- The n in $\{s_n\}$ was a means of naming any given term in the sequence.
- The N caused a lot of confusion for me. Although I knew it was related to the subscript and that it picked out a particular term in the sequence, I wasn't sure why it was doing this.
- In $|s_n - s| < \varepsilon$, I knew that the s in the absolute value was the limit of the sequence.

Yet I didn't understand the entire statement.

According to the research literature, these conceptions are common for students who begin to study the ideas of analysis (Cornu, 1991; Sierpinska, 1990). Later, I was able to develop a pictorial sense of the definition, a picture of what was supposed to happen for every value chosen for ε. For any given epsilon, in order for the sequence to converge, the associated N is supposed to divide the sequence into two parts, a finite part that is outside the band "s plus or minus ε" and an infinite part whose elements lie inside that band.

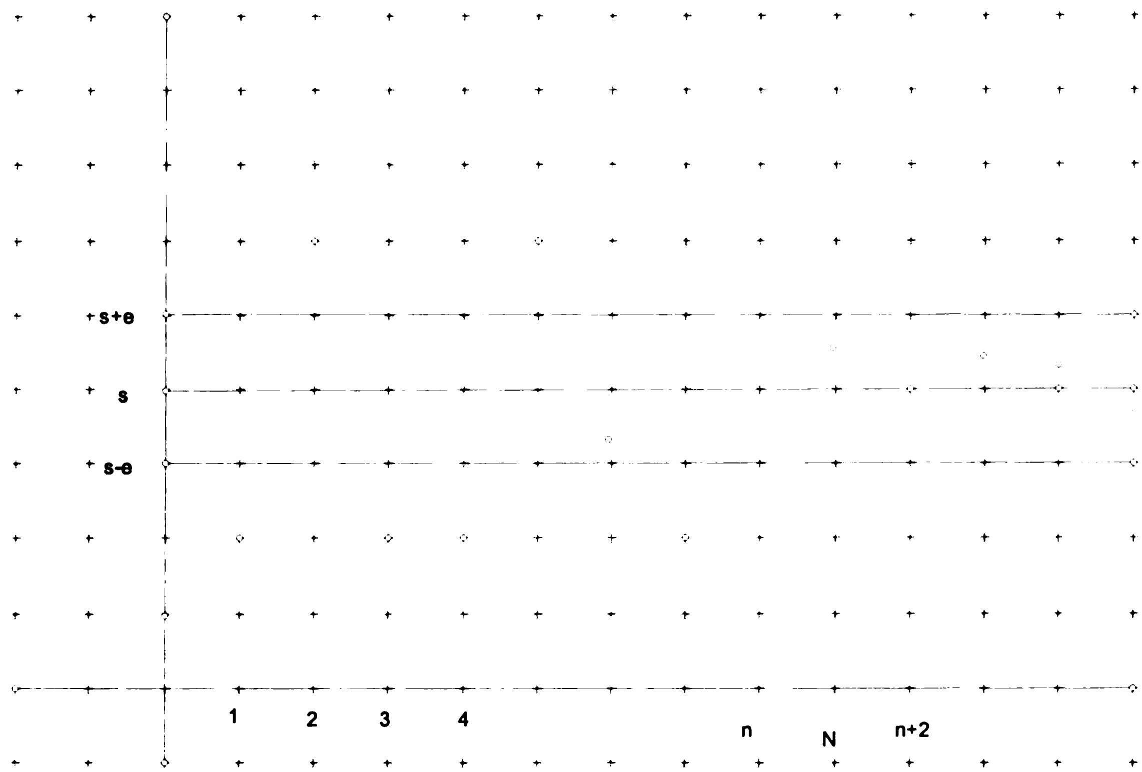

Figure 1
Convergent Sequence

For any given ε, as we move further along in the sequence—or, in other words as the values for n (the subscripts) increase—there must be a place, N, at which the terms of the sequence are at a distance no more than ε (e in the sketch) away from s. In order for the sequence to converge, it must be true that for all choices of ε, there must be places in the sequence, N, for which all of the subsequent values of the sequence, after s_N, are inside the interval (s-e, s+e). The picture above represents what must happen for <u>every</u> possible value of ε. Thus no matter how wide or how small the width of the interval centered on s is, we must be able to find a N, for which all of the terms in the sequence after s_N lie within the interval. Otherwise, the sequence does not converge.

In contrast to what I first believed, I came to realize that in terms of the variables in this definition:

- The subscript *n* is used to denote the ordering or labeling of the terms in the sequence, but it is also used to speak about the sequence in different stages. Thus there is an element of time or change involved with sequences.
- $\varepsilon > 0$ isn't a **fixed** non negative number. In fact, it represents or takes on all small positive values.
- N does pick out a particular term in the sequence, and in this way it appears to be fixed. However, it changes as the value for ε changes. Thus every epsilon has its very own N associated with it. Although the term in the sequence that it specifies isn't important in itself, N identifies a cut-off point in the sequence. This term divides the sequence into a finite portion, which can be ignored, and an infinite portion that stays close to the limit.
- $|s_n - s| < \varepsilon$ is a crucial statement. It states that the limit and each of the terms in the infinite portion of the sequence are not more than ε apart.

So, there was a lot to learn about the definition of a convergent sequence. But I did not learn these things at the time. Separate from all of the ways in which I was going to have to learn to think differently about the variables in this definition, I had many unanswered questions. I was missing a sense of purpose. To reiterate, we began the semester finding bounds for sequences. I could do this, but I still didn't understand <u>why</u> one would want to do such a thing or why the bounds were useful. When the professor moved from finding bounds to finding the glb or the lub, I again was bothered because I didn't see the need for these quantities even though I wasn't having much trouble finding them. Then we moved to the convergence of sequences; my classmates and I did not know why this was the next thing one would want to do with sequences. I wondered why the course began with bounds if the professor wasn't going to make any connection for us between bounds and convergence, since it appeared that convergence was going to be a dominant theme.

And if all of this was not enough, then came the issue alluded to in the quotes from Scott and Tims. We began to use the definition of convergence to prove that sequences had limits. This seemed illogical to me. In calculus classes, we had computed the limits for many of the sequences we were looking at in analysis. We had made use of these limits, but now somehow it seemed as if we had been reckless in calculus. On the other hand, our capacity to compute limits was recognized in analysis, because whenever we began a proof it wasn't necessary to first compute the limit; the professor always gave the limit to us.

There are important subtleties here. At the time, I did not understand why it was necessary to prove that a sequence for which one has calculated a limit indeed has a limit. Neither did I see a difference between calculating that a particular number is a good candidate for the limit of a sequence and proving that the limit of a sequence indeed exists. And without a good understanding of the definition of convergence, our proofs seemed funny. Our proofs consisted of finding a way to compute N as a function of epsilon.[1] It was not clear to me at the time why this constituted a proof that the limit existed. As I indicated earlier, the nature of this course was different from ones I had taken in the past, but none of these differences were stated or clarified. Without any attention being paid on the part of the professor to the differences in the nature of the mathematics courses, this made learning the material a difficult chore.

I needed at least to try and receive help in understanding what was occurring in this class. Not too long after the introduction of the definition of convergence, I went to visit my analysis professor in his office. In our office-hour discussion, he repeated his statements from class, but they were of little use to me. If I had understood them originally, why would I continue in my confusion and push for more explanation? In hindsight, I had all sorts of misunderstandings of the uses of the variables present in the definition, but these were not getting clarified. I left with no greater an understanding than I had entered. But, at least I had demonstrated my interest in the course. The interaction would probably have a positive impact on my grade.

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[1] A proof in class that this sequence $\left\{1+\frac{1}{n}\right\}$ converges to 1 would have gone as follows. $\left|1+\frac{1}{n}-1\right|=\left|\frac{1}{n}\right|=\frac{1}{n}$. *Let* $\frac{1}{n}=\varepsilon \Rightarrow n=\frac{1}{\varepsilon}$ and thus we should take N=$\frac{1}{\varepsilon}$.

The Research Questions

The previous sections provide my rationale for studying the following questions; I began with my own experiences with mathematics at the university level, connected these to my evolving understandings of the role of mathematics preparation in preservice teacher education, and then described an experience I had with the history of mathematics that provided resources shaping the current study. During each of the conversations about the Aristotle article, my students invariably raised many examples in order to process the text. Some of these examples were mathematical and contained ideas on the geometry of the real line; some were from other areas of mathematics not directly related to the geometry of the real line. I saw the examples that were seemingly unrelated to mathematics at all to be of great interest. In these examples along with the others I observed the students tackling very deep mathematical issues. My observations and prior experiences led me to the following research questions for this study.

- Are students engaged in mathematical activity when they grapple with an historical text?
    - What opportunities to learn are available when attempting to interpret Aristotle's position on the continuous and the indivisible?
    - In the context of bringing examples from their own experience to bear on understanding Aristotle's claim that lines cannot be made of points, what schemas and metaphors do the students utilize?
    - What does their use of schemas and metaphors say about their capacity to do mathematics?

CHAPTER 2

CONCEPTUAL FRAMEWORK

Introduction

In their book, Where Does Mathematics Come From (2000), I see Lakoff and Núñez's argument as having two pieces. First, they argue that all mathematics is embodied. Then, they explain the role of cognitive mechanisms, conceptual metaphors and blends for example, in the creation and doing of mathematics. They then detail many of the metaphors that have led to the creation of modern mathematical thinking. In this chapter, I outline my understandings of their argument since their thinking is the framework that I use to analyze my students' example use in discussing Aristotle's position on points and lines.

Cognitive Science of the Embodied Mind

In the first part of their argument Lakoff and Núñez argue that all of mathematics is embodied. In this section I set out to explain embodied mathematics to the reader and to prepare for the following section in which I explain how conceptual metaphor is used to create what Lakoff and Núñez call discretized space; this is an example of the second part of the authors' argument.

The authors' goal in this book is to look closely at the structure of mathematical ideas. Their belief is that mathematical understanding is achieved by implementation of cognitive mechanisms used in everyday nonmathematical thought. They define mathematical cognition as "the way we implicitly understand mathematics as we do it or talk about it" (p. 28). In short, the following is a partial list of everyday cognitive mechanisms that are used in the corresponding mathematical ideas.

Technical Mathematical Ideas	Cognitive Mechanisms
Mathematical class	Collection of objects in a bounded region of space
Recursion	Repeated action
Complex arithmetic	Rotation
Derivatives	Motion, approaching a boundary, etc.

Table 1
Cognitive Mechanisms in Mathematical Ideas

There are several conceptual mechanisms central to everyday thought that are also central to mathematical thought. These are image schemas, aspectual schemas, conceptual metaphor and conceptual blends. "Image schemas are the link between language [and reasoning] and spatial perception" (p. 31). Aspectual schemas provide a connection between our motor control system and the manner in which we reason about events. Conceptual metaphors and blends are our means for thinking abstractly.

Images schemas are so named because they "...use neural structures in our visual system" (p. 33). They are also either perceptual or conceptual. For the purpose here the focus will be on conceptual image schemas. These image schemas arise from neural structures in our brains (Terry Reiger, 1996). Image schemas that are central to mathematics are as follows

Image Schemas that Characterize Concepts Important for Mathematics	
1	Centrality
2	Contact
3	Closeness
4	Balance
5	Straightness...
6	Containment
7	Orientation (pp.33-34)

Table 2
Image Schemas

Containment and orientation are the two image schemas that are central to Reiger's work and that Lakoff and Núñez identify as central to mathematics. Containment is important because its ideas are embedded in the ideas of boundedness and closed sets and orientation because it is important for direction and rotations.

Aspectual schemas are important to this work because of the connection between the motor control system and our structuring of events. Srini Narayanan created a motor-control aspectual schema.

Narayanan's Motor Control Super-Structure

1. Readiness: Before you can perform a bodily action, certain conditions of readiness have to be met (e.g., you may have to reorient your body, stop doing something else, rest for a moment, and so on).
2. Starting up: You have to do whatever is involved in beginning the process (e.g., to lift a cup, you first have to reach for it and grasp it).
3. The main process: Then you begin the main process.
4. Possible interruption and resumption: While you engage in the main process, you have an option to stop, and if you do stop, you may or may not resume.
5. Iteration or continuing: When you have done the main process, you can repeat or continue it.
6. Purpose: If the action was done to achieve some purpose, you check to see if you have succeeded.
7. Completion: You then do what is needed to complete the action.
8. Final State: At this point, you are in the final state, where there are results and consequences of the action. (pp. 34 – 35)

Table 3
Narayanan's Model

Narayanan's work documents that event structures are parallel to the motor control schema. Just as our motor control system defines how our bodies move, it also provides the links through which we are able to carry out reasoning and thinking of events. Thus this cognitive function may also be embodied. The fact that the general neural control system can give data to muscles in order to complete a bodily movement but can also perform a rational inference lends credit to the viability of the idea that mathematics may be embodied (p. 35).

In embodied mathematics, mathematical symbols such as 27, π, and $e^{\pi i}$ are meaningful by virtue of the mathematical concepts that attach to them. Those mathematical concepts are given in cognitive terms (e.g., image schemas; imagined geometrical shapes; metaphorical structures, like the number line; and so on), and those cognitive structures will ultimately require a neural account of how the brain creates them on the basis of neural structure and bodily and social experience. To understand a mathematical symbol is to associate it with a concept—something meaningful in human cognition that is ultimately grounded in experience and created via neural mechanisms. Ultimately, mathematical meaning is like everyday meaning. It is part of embodied cognition. This has important consequences for the teaching of mathematics. Rote learning and drill is not enough. It leaves out understanding.

Aspect is the general structuring of events. "Imperfective aspect focuses on the internal structure of the main process and perfective aspect conceptualizes the event as a whole, not looking at the internal structure of the process, and typically focusing on the completion of the action. These represent the two ways in which processes that have completions can be conceptualized: either (1) internal to the process or (2) external to the process" (p. 36). Living and breathing are imperfective actions. The completions of the these activities, not breathing or dying, are unfortunately not a part of them. Jumping, on the other hand, is a perfective action. You cannot continue to jump unless you land in between each one. Thus the completion of the action is a part of the process.

Reasoning about space seems to be done directly in spatial terms, using image schemas rather than symbols, as in mathematical proofs and deductions in symbolic logics" (pp. 31-33). The most prominent schema to Lakoff and Núñez appears to be the container schema. Below are the three parts of this schema which make no sense without the whole that they comprise.

1 Interior

2 Boundary

3 Exterior

Another important schema to mathematics is the Source-Path-Goal Schema (S-P-G). "It is the principal image schema concerned with motion, and it has the following elements (or roles)

1 A trajectory that moves

2 A source location (the starting point)

3 A goal—that is, an intended destination of the trajectory

4 A route from the source to the goal

5 The actual trajectory of motion

6 The position of the trajectory at a given time

7 The direction of the trajectory at that time

8 The actual final location of the trajectory, which may or may not be the intended destination (p. 38)."

The Source-Path-Goal schema also has an internal spatial logic and built-in inferences. The first item on the list above appears often in the students' conversation.

> Woody?: Okay, think of like 1 and 2. Just to count 1, 2 – going straight from one and landing on 2. But if you consider the points in the middle – if you travel along them like this, that's between….

Here Woody is trying to explain Aristotle's definition of between. Built into the second part of his comment is the idea that if you travel from one to two by moving along a number line, then you will encounter all of the points (corresponding to numbers) in between one and two. Assuming that Lakoff and Núñez are correct, this comment indicates that my students are demonstrating one of the essential image schemas that is necessary for the doing of mathematics.

1 If you have traversed a route to a current location, you have been at all previous locations on that route.

2 If you travel from A to B and from B to C, then you have traveled from A to C.

3 If there is a direct route from A to B and you are moving along that route toward B, then you will keep getting closer to B.

4 If X and Y are traveling along a direct route from A to B and X passes Y, then X is further from A and closer to B than Y is (p. 38).

Conceptual metaphor is "...[t]he mechanism by which the abstract is comprehended in terms of the concrete..." (p. 5). Conflation is the biological aspect of this process and is a part of embodied cognition. "It is the simultaneous activation of two distinct areas of our brains, each concerned with distinct aspects of our experience, like the physical experience of warmth and the emotional experience of affection. In conflation the two kinds of experiences occur inseparably. The co-activation of two or more parts of the brain generates a single complex experience—.... It is via such conflations that neural links across domains are developed—links that often result in conceptual metaphor, in which one domain is conceptualized in terms of the other" (pp. 41-42).

An example from the authors is the measuring stick metaphor in which we think of arithmetic in terms of using a measuring stick. Here is their metaphorical mapping chart.

The Measuring Stick Metaphor		
Source Domain		Target Domain
The Use of a Measuring Stick		Arithmetic
Physical segments (consisting of ultimate parts of unit length)	→	Numbers
The basic physical segment	→	One
The length of the physical segment	→	The size of the number
Longer	→	Greater
Shorter	→	Less
Acts of physical segment placement	→	Arithmetic operations
A physical segment	→	The result of an arithmetic operation
Putting physical segments together end-to-end with other physical segments to form longer physical segments	→	Addition
Taking shorter physical segments from larger physical segments to form other physical segments	→	Subtraction

(p. 46)

Table 4
Measuring Stick Metaphor

Conceptual metaphors have a structure to them. Each conceptual metaphor consists of two domains, the source domain and the target domain, and a function mapping items from the source to the target. Usually the target domain contains the more abstract item and the source domain the more concrete item. The mapping allows us to think of the abstract target domain item in terms of the more familiar and tangible source domain item. The structure of image schemas is preserved by conceptual metaphorical mappings. If a source domain item has a particular schema structure that structure will map intact onto the corresponding target domain.

Conceptual blends "are …the conceptual combination of two distinct cognitive structures with fixed correspondences between them. When the fixed correspondences in a conceptual blend are given by metaphor, we call it a *metaphorical blend*"(p. 48). Conceptual blends are created from conceptual metaphors. The blends keep active the target and the source domains simultaneously. Characteristics in the source domain are given meaning in the target domain and vice versa (p. 30). .

Line segments in space are considered as physical segments in the above metaphor. The results of this blend are "line segments with numbers specifying their length" and the blend is referred to as the Number/Physical Segment blend (p. 70).

In metaphors and in blends ideas from the source domain can introduce new elements into the target domain. These new elements are known as entailments of the metaphor. In the above example we are using what is known about using a measuring stick to make sense out of arithmetic. For every number there is a physical unit to associate with it. When we take the blend of this metaphor we also gain the idea that for every physical segment there exists a number to associate with it. This is the blend that gave rise to the creation of the irrational numbers.

The Measuring Stick metaphor, along with three other metaphors are titled the four grounding metaphors and will be discussed further in chapter five. These metaphors are central in understanding how the innate arithmetic abilities of humans is extended and develops characteristics that innate arithmetic does not have. In the next section linking metaphors will be discussed and explored. These metaphors play a central role in what the authors name mathematical idea analysis. This is the process of revealing the cognitive mechanisms used to mathematize ordinary, everyday concepts (p. 29). This paper suggests that if we include this process in the teaching and learning of mathematics then students will receive credit for the mathematical activity and their understanding of mathematics may greatly increase.

Discretized Space

To the authors to understand a mathematical idea is "...associate it with a concept—something meaningful in human cognition that is ultimately grounded in experience and created via neural mechanisms" (p.49). They continue by saying, "Rote learning and drill is not enough. It leaves out understanding. Similarly, deriving theorems from formal axioms via purely formal rules of proof is not enough. It, too, can leave out understanding" (p.49). The focus of this section is the metaphors that are central to understanding the creation of mathematical space.

The measuring stick metaphor at the end of the previous section is an example of a *grounding metaphor* according to Lakoff and Núñez. Grounding metaphors "...ground our understanding of mathematical ideas in terms of everyday experience" (p. 150). There are other types of conceptual metaphors in cognitive science that are central to the human creation of mathematics. These are *linking metaphors*, which allow for the conceptualization of one area of mathematics in terms of another, and *redefinitional metaphors*, which replace ordinary concepts with more formal and technical concepts (p. 150). At the center of the discretization program is a redefinitional metaphor, "A Space is a Set of Points."

The discretization program describes the process by which mathematicians eliminated naturally continuous space from mathematics. It began with the work of René Descartes, who initiated the development of analytic geometry by linking arithmetic and algebra with geometry. First naturally continuous space (NCS) was replaced by thinking of space (and spatial objects) as a set of points. This was accomplished by "Space is a Set of Points" metaphor, whose metaphorical chart is as follows.

A Space Is a Set of Points		
Source Domain A Set With Elements		*Target Domain* Naturally Continuous Space With Point Locations
A set	→	An n-dimensional space—for example, a line, a plane, a 3 dimensional space
Elements are members of the set.	→	Points are locations in the space.
Members exist independently of the sets they are members of.	→	Point-locations *are* inherent to the space they are located in.
Two set members are distinct if they are different entities.	→	Two point-locations are distinct if they are different locations
Relations among members of the set (p. 263)	→	Properties of space

Table 5
Space is a Set of Points Metaphor

This metaphor allows every naturally continuous spatial object to be conceptualized as being composed of a set of elements with specific relations among them. As with sets and their members, the members exist independently of the sets, but the sets don't exist without the members that make them up. Thus if we now conceptualize spaces (all spatial objects included) as a set of points, they no longer exist without those points. So whereas for naturally continuous spatial objects, points were locators/indicators in those spaces, and if you removed a point from one you didn't alter the spatial object at all in discretized space, if you remove a point from a spatial object there is now a hole in that space. The object remaining is no longer the same because the set is now different. Here we get our first metaphorical conception (definition) of a point. It can be any "mathematical entity" that can be a member of a set. Thus it may not be spatial at all, i.e., being thought of as an object that occupies a portion of space as we naturally conceptualize space.

The next step in the discretized program had a three part agenda.

1. Pick out the necessary properties of naturally continuous space that can be modeled in a discretized fashion and model them.
2. Model enough of those necessary properties to do classical mathematics as it was developed using naturally continuous space.
3. Call the discretized models 'spaces,' and create new discretized mathematics replacing naturally continuous space with such 'spaces.' (p. 274)

The necessary properties of NCS chosen by mathematicians are

1. The metric property
2. The neighborhood property
3. The limit point property
4. The accumulation point property
5. The open set property.

According to the authors, if you assume the metric property then the Infinite Nesting Property for sets, which is defined by the Basic Metaphor of Infinity (BMI), can be used to conceptualize the other four properties. This will complete the first two steps of the discretization program. Instead of turning to step three, I'd like to focus now on the point and the line in discretized mathematics.

In mathematics there are two ways of conceptualizing a point. The first is as a disc of zero diameter and the second is as an infinitesimal point. Both perspectives use the BMI in their creation.

To understand how a point can be conceptualized as a disc of zero diameter you begin with a disc with a radius of one. You then perform a sequential chain of actions by shrinking the diameter by a factor of 1/n, n a fixed positive integer, at each step. Allowing the sequential chain of actions to be the BMI will eventually produce a disc of zero diameter; the metaphorical result of the BMI.

To create a point of infinitesimal diameter we again begin with a disc of diameter one. We shrink the disc as before but this time at each stage we add to the process a requirement that the diameter remain greater than zero. As before allow the BMI to be the sequential process and the metaphorical result will be a disc of infinitesimal diameter, according to the authors. As an aside, I take issue with the authors on this one point. I don't agree with their development of the infinitesimals at this time. It would appear to me that the result from the BMI would be a disc whose diameter is as small as you would like it to be but not an infinitesimal.

Looking at the point as a disc with zero diameter from the cognitive perspective requires the use of frames. Frames are a way of describing the essentials of different processes that produce the same results (Filmore, 1982, 1985). This case uses the frame for a disc and one for a line segment. The frame for a disc is as follows.

The Frame for a Disc

Roles: Center, Circumference, Interior, Diameter, where Center ≠ Circumference ≠ Interior
Parts: Interior, Center, Circumference
Constraints: (a) Center is in Interior. (b) Distance from Center to the Circumference is the same for all points of the Circumference.
(p. 266)

Table 6
Disc Frame

The frame for a line segment would look similar.

The Frame for a Line Segment

Roles: Endpoint A, endpoint B, center, interior and length where endpoint A ≠ endpoint B ≠ center ≠ interior
Parts: Interior, Center, Endpoints
Constraints: (a) Center is in Interior. (b) Distance from center to an endpoint is the same for either endpoint A or endpoint B.

Table 7
Line Segment Frame

As with metaphors we can form a blend with frames. In this case the two frames create "The Disc/Line Segment Blend."

The Disc/Line Segment Blend

Element 1 The Disc Frame		Element 2 The Line-Segment Frame
A disc, with roles: center, circumference, interior, and diameter, where center ≠ circumference ≠ interior	↔	A line segment, with roles: endpoint A, endpoint B, center, interior, and length, where endpoint A ≠ endpoint B ≠ center ≠ interior
Diameter	↔	Length
Center	↔	Center
Opposite points on circumference	↔	Endpoints A and B

(p. 267)

Table 8
Discretized Line Segment Blend

At each stage of the sequential chain of actions there is a disc whose diameter is the line segment from the Line-Segment blend. At each stage there is the disc whose diameter is the same as the length of the line segment. At the final stage the metaphorical result produces a disc, which by, the constraints within its frame will always have a center distinct from the points on the circumference. Thus it will always have some positive area. However the diameter of the disc in the Line-Segment blend will have a length equal to zero in the final resultant state. These two instances occurring at the same time produces a conceptual problem. It is not possible to have the length of the diameter equaling zero and still maintain a disc with positive area.

For the second conceptionalization, the infinitesimal disc, there is a different problem. Here you begin with a disc and at each stage including the final metaphorical stage you have a disc with positive area. In mathematics this disc is then called a point and students are asked to think of a point as an object with zero length, zero width, and breadth. Earlier work by Núñez has shown that when children were taken through this process and asked about the final product they concluded that the point would have some positive volume to it (Núñez, 1993).

Both of these conceptions of points are used by Aristotle to argue that a line cannot be made out of points.

At the crux of Aristotle's argument that a line cannot be made out of points is the relationship between the points on the line. The following appears in chapter three.

> If one were to try and create a line out of a set of points, one of two major problems would occur, both making it impossible for a line to be composed of points. In the first case he argues that points are not continuous nor can they be in contact with one another. Points, being indivisibles, cannot have one side that is distinct from another side. If this were the case then they would have a middle area and would no longer be an indivisible. Thus the points would have to be in contact with one another whole to whole and this would not produce any measurable length because the points would become the same point.

Aristotle's thinking about a point here exemplifies the conceptual issues that Lakoff and Núñez raise with both zero-diameter discs and with infinitesimal discs. In their opinion these are the most common ways in which humans have of conceptualizing a point, and both cause cognitive difficulties. Furthermore in discussions with mathematicians, Lakoff and Núñez found that the most common response to whether or not points on a line touch is no. The reason given is that points, being zero-dimensional objects, would become one point if they were to touch. Here it can be seen that what has been preserved in discretized space had some beginnings in the thoughts of Aristotle. However, the point in discretized space is still quite different from Aristotle's point and mathematicians will have to devise a way of thinking about what it means for mathematical points to touch one another.

Now that space has been discretized the relationship of touching between what we would commonly think of as spatial objects is also quite different. Recall that the first step in creating discretized space was to think of any spatial object as a set of points by using the "A Space is a Set of Points" metaphor.' Given a space, a plane for instance, it is necessary then to define what a geometrical figure would entail. A circle would be the set of points equidistant from a chosen point. This definition alone shows the nature of discretized space. Visually imagine the plane as consisting of a grid of points. Then if a metric is specified, the points for determining the circle can be identified. Note that the points of the circle are a subset of the points of the plane. Now envision a discretized line tangent to the circle at a specified point, i.e., the discretized line touching the circle at exactly one point. The nature of touching here is quite different from the manner in which humans usually think of two objects touching. In this example the line, which is also a set of points, touches the circle if the two objects share a common point. In other words, touching is defined by examining the point set for the circle and the point set for the line to see if they have a common point between them. If they share a point in their sets, then they touch.

What makes this so interesting is its contrast to our everyday way of thinking. Students have a great amount of experience with what it means to touch. Touch is one of the most central acts for newborn babies. There are many conceptual metaphors that humans have to draw on to understand what it means for two or more things to touch; however, not many of them (I dare to say none of them) is quite like the definition of touching in discretized space. For instance, the touch between humans can be as simple as holding hands. It isn't the case that the two hands share anything in common; they are simply in contact with one another.

Furthermore, when space is conceptualized in this way, there is another dramatic difference to be seen between it and NCS. In naturally continuous space when a geometrical figure is drawn it is conceptualized as being in NCS. Thus the figure is almost superimposed on top of the plane (if working in two dimensions). In discretized space when a figure is drawn it is a part of the background space. The figure consists of the same points that constitute discretized space; there are no other points in the background. If you were to remove or erase the figure in discretized space you would also erase a part of the supporting medium, leaving holes where the points were.

Both of the descriptions of points, although mathematical, contain very tangible ways of thinking about a point. A true discretized point, however, has no real physical representation at all. Due to the "Space is a set of Points" metaphor' (refer to table 5), points in discretized space are any elements in the set under consideration. As long as the elements of the sets have the proper relations amongst its members that can be mapped onto the properties of the spatial object under consideration they are considered to be points in discretized space. This is why this is a major rub for mathematicians when asked, "Do the points on the line touch?" It isn't only the problem with the conceptualization of points as objects without dimension; it is also because a point can be anything. Given a set with members of any kind and the rules that determine their existence and the relationship between all of the members are given, the 'point' metaphorically stands in for the elements of a set. There is no requirement that the elements have any physical nature at all. As Hilbert said "It must always be possible to substitute 'table', chair' and 'beer mug' for 'point', line' and 'plane' in a system of geometrical axioms."

Considering the dilemmas that exist with points, it is now a bit easier to see that there are important issues with lines, number lines in particular, that need to be discussed. There are two different number lines in mathematics. Both use the "Numbers are Points on a Line" metaphor.'

Numbers Are Points On A Line
{For Naturally Continuous Space}

Source Domain Points on a Line		*Target Domain* A Collection of Numbers
A point *P* on a line	→	A Number *P'*
A point *O*	→	Zero
A point *I* to the right of *O*	→	One
Point *P* is to the right of point *Q*	→	Number P' is greater than number *Q'*
Point *Q* is to the left of point *P*	→	Number Q' is less than number *P'*
Point *P* is in the same location as point *Q*	→	Number P' equals number *Q'*
Points to the left of *O*	→	Negative numbers
The distance between *O* and *P*	→	The absolute value of number *P'*

(p. 279)

Table 9
Numbers Are Points on a Line Metaphor

This metaphor describes the number line that is introduced to students in elementary school. It is a naturally continuous line with the numbers spread (using a metric of course) along the line. This line is not made out of points but instead has a point-location in every place that there is a number. However, since this is a naturally continuous line, there are an infinite number of other points on the line that do not correspond to any number. Thus if the naturally continuous line were removed, leaving only the points that corresponded to a number, remaining would be a discontinuous line or a set of points in which no two points were in contact with one another.

The second type of line is the most prominent one used in mathematics—discretized line. As a part of discretized space the "Space is a Set of Points" metaphor' is essential to understanding it. The blend of this metaphor, "The Space Set Blend," is below.

The Space-Set Blend

Target domain Naturally Continuous Space With Point-Locations Special Case: The Line		*Source Domain* A set with Elements
The line	↔	A set
Points are locations on the line.	↔	Elements are members of the set.
Point-locations are inherent to the line they are located on.	↔	Members exist independently of the sets they are members of.
Two point-locations are distinct if they are different locations.	↔	Two set members are distinct if they are different entities.
Properties of the line	↔	Relations among members of the set

(p. 279)

Table 10
Space Set Blend

The metaphor allows us to think of a naturally continuous line as a set of points. The blend gives the opportunity to think of the set of points as a line. Thus attributes such as continuity of a naturally continuous line find life in the set of points domain because of the blend. To complete the metaphor for completely discretized space this blend is then metaphorically mapped onto numbers, giving the following chart.

Numbers Are Points On A Line (Fully Discretized Version)				
Source *Domain*		*Target Domain*		
The Space-Set Blend				
Naturally Continuous Space: The Line		Sets		Numbers
The line	↔	A set	→	A set of numbers
Point-locations	↔	Elements of the set	→	Numbers
Points are locations on the line.	↔	Elements are members of the set.	→	Individual numbers are members of the set of numbers.
Point-locations are inherent to the line they are located on.	↔	Members exist independently of the sets they are in.	→	Numbers exist independently of the sets they are in.
Two point-locations are distinct if they are different locations.	↔	Two set members are distinct if they are different entities.	→	Two numbers are distinct if there is a nonzero difference between them.
Properties of the line	↔	Relations among members of the set	→	Relations among numbers
A point *O*	↔	**An element "0"**	→	**Zero**
A point *I* to the right of *O*	↔	**An element "1"**	→	**One**
Point *P* is to the right of point *Q*.	↔	**A relation "$P > Q$"**	→	**Number *P* is greater than number *Q***
Points to the left of *O*	↔	**The subset of elements x, with $0 > x$**	→	**Negative numbers**
The distance between *O* and *P*	↔	**A function d that maps ($0 < P$) onto an element x, with $x > 0$**	→	**The absolute value of number *P***

(p. 281)

Table 11
Fully Discretized Space Metaphor

The lines in bold print are entailments of the metaphor. Once a set of numbers is chosen for the target domain of the larger metaphor, a one-to-one correspondence is set up between the numbers in the set and a set of points. The key is that there only exists a point for which there is a number. Thus the real line cannot ever be continuous in the way that humans naturally think of continuity. For that matter, no number line can.

In the conversation with the students, they speak of a number line representing the numbers in Z_3. I at first thought that this was a very odd thing to do but also that it would be quite difficult to convince others that this was an example of a mathematical strength. With the use of Lakoff and Núñez it is just as sensible to speak of a number line in Z_3 as it is to speak of the rational line, as Dedekind does, or even the real line. None of these lines can ever be a naturally continuous line. All of the sets of numbers have elements and axioms that define them and thus the necessary correspondences can be carried out by the final metaphor.

The metaphors in this section describe a space very different from the everyday naturally continuous space that all students experience. Thus mathematical space is a very specialized idea. Objects and relations in discretized space behave and operate differently than objects and relations in naturally continuous space. The two have many commonalities and students often conflate them which will be seen in the analysis of the data for the study. Before moving to an analysis the next chapter will discuss some ideas of Aristotle and Richard Dedekind. The section on Aristotle gives a review of the article that was read with the students. The following section on Dedekind discusses his ideas on the continuum and is a preface for some ideas that the students raise in their conversation.

CHAPTER 3

CONCISE HISTORY OF THE MATHEMATICS INVOLVED

Introduction

The data for this study is a transcript of a conversation from November 11 1999. The participants are students in a secondary mathematics methods course taught by the author of this paper. The instructor chose a portion of Aristotle's Metaphysics which focused on the nature of a line, a divisible object, and its relationship to points or indivisibles. This section will review the argument made by Aristotle and give a summary of the students' thinking on his argument.

Aristotle

Aristotle on points and lines

In this time, we think of Aristotle primarily as a philosopher but in Aristotle's times the distinctions we now make between philosopher mathematician and scientist were not so clearly drawn. Aristotle's contributions to mathematics were largely in the area of logic and the structure of mathematics. He took an interest in the paradoxes of Zeno (which survive on in the context of his attempts to refute them!), which led to his discussions in the *Physics* and *Metaphysics* on the relationships between points and lines. He argues that just as lines cannot be composed of points, any length of time cannot be composed of 'nows'. With this being the essential element in Zeno's argument in his Arrow Paradox that motion is impossible, Aristotle is able to refute it.

In the piece from the *Metaphysics* and *Physics* that I chose for the senior math majors/preservice teachers to read, Aristotle largely seems to be making a geometrical argument. For Aristotle quantity consisted of two categories: number, the discrete; magnitude, the continuous. Continuous magnitudes have the ability to be infinitely divisible into smaller continuous magnitudes. Thus they couldn't be constructed out of indivisible objects, points, and therefore a line cannot be constructed out of points (for the argument he makes see Calinger, 1998, pp. 85-86).

In constructing his argument, Aristotle introduces many terms and their definitions. The key ones in the piece we read are as follows: "...things being 'continuous', if their extremities are one, 'in contact' if their extremities are together, and 'in succession' if there is nothing of their own kind intermediate between them—nothing that is continuous can be composed of indivisibles..." (Calinger, 1998, p. 86). To Aristotle, lines and points are essentially of two different species. The line is from the continuous or the infinitely divisible class, and the point is from the non-continuous or the indivisible class. Objects that are in the continuous class can be infinitely subdivided into parts that possess the same properties of the original object. Thus any segment of a line can be further divided and never reach a place where it ceases to be divisible.

In contrast, points or objects in the indivisible class do not possess this property. Points cannot ever be divided. Aristotle describes them as things without parts and for this reason can never be divided. If one were to try and create a line out of a set of points, one of two major problems would occur, both making it impossible for a line to be composed of points. In the first case he argues that points are not continuous nor can they be in contact with one another. Points, being indivisibles, cannot have one side that is distinct from another side. If this were the case then they would have a middle area and would no longer be an indivisible. Thus the points would have to be in contact with one another, whole to whole, and this would not produce any measurable length because the points would become the same point. The second case addresses composing the line out of points. In order to do this the points would have to be in succession to one another (what in modern lingo we might call well-ordered). This produces a problem because between any two points a line can exist and on that line one can find another point. (Aristotle does not distinguish between what we call a line and what call a line segment). Thus the two original points would not be successive because, as stated above, two things are successive if there is nothing of their own kind between them. Having exhausted all of the possibilities Aristotle concludes that a line cannot be composed out of points.

The students' understandings of Aristotle's argument

The argument that Aristotle gives involving lines and points is a fascinating and challenging one for future high school mathematics teachers. It contradicts what they were taught as students and what they know they will have to teach as teachers. However, the reasoning on which it is based accords with how people think now. One cannot construct a line out of a countably infinite set of points.

The conversation with my students that I analyze in this dissertation begins by my asking the students of their thoughts or reactions to the Aristotle reading. Some describe what they have read as Aristotle constructing an argument by building up a hierarchy of definitions. Baron gives a summary of what he thinks Aristotle is saying.

> **Baron:** The way I took it was he was saying that – his argument was basically that a line is of a different species than a point. It's not, a line is not composed of points – it contains an infinite number of points, but from points alone you can't generate a line...
> Because it's like if you take two points and put them next to each other, because there's no they can't if they're touching, then since they have no dimensions, they're essentially the same point. And if they're not touching then there's a space between them so then it's not a continuous thing because they're not touching. And so then no matter

> how many you know and infinite number of points that you stack side by side, you're still staying at the same point so it's not like you're gonna be like going along 0, 0, 0, 0, 4.

Baron's account of Aristotle's argument is impressively accurate. He hits on some very key points.

1. Lines and points are distinctly different types of objects.
2. Lines and points do have a relationship between them. Lines can contain points but you cannot create a line from a set of points.
3. Because points are dimensionless, if you try to place two points side by side they would become the same point.
4. If two points cannot touch and remain distinct, then there must be a space between them. Thus the line would be discontinuous, which contradicts its nature.

Other students in the class also had immediate understandings of the text that were not as detailed as Baron's account. For instance, when addressing the definitions of Aristotle's terms, Jim says,

> I think he also defines it a little bit because he defines them in a way—not necessarily in the way we've seen it before, and I think he defines them a little different than I would define as the truth.

After Baron makes his statement above, Jim states, I must say you got a lot more out of it than I did.

For Jim, it wasn't clear what Aristotle was arguing for; however, his statement does indicate his disposition toward mathematical knowledge, Jim is suggesting that for him there are mathematical truths and Aristotle is incorrect in his thinking. As the conversation progresses Jim is one of the main participants in the conversation and works diligently on his understanding of the piece.

Lakoff and Núñez on Aristotle on points and lines

In addition to the students' understandings of Aristotle's argument, Lakoff and Núñez's work can also be used to analyze and interpret Aristotle's argument. Again I look at the definition of between, *"That at which the changing thing, if it changes continuously according to its nature, naturally arrives before it arrives at the extreme into which it is changing is between."* Here is evidence of Lakoff and Núñez's argument. Embedded in this definition from which all of Aristotle's others are built is motion in the form of the Source-Path-Goal schema (SPG), the most prominent schema discussed by Lakoff and Núñez. As discussed in the prior chapter, this schema has distinct parts to it. One of these parts is its internal spatial logic, described by the following statements.

1. If you have traversed a route to a current location, you have been at all previous locations on that route.
2. If you travel from A to B and from B to C, then you have traveled from A to C.
3. If there is a direct route from A to B and you are moving along that route toward B, then you will keep getting closer to B.

4. If X and Y are traveling along a direct route from A to B and X passes Y, then X is further from A and closer to B than Y is.

Aristotle's definition of between is implied by statements 1 and 3.

Using Lakoff and Núñez argument about the sources of mathematical ideas it can be seen that Aristotle is not working with an understanding or belief in discretized space. According to the time of his work, the authors would place him in the mathematical period operating with naturally continuous space. In his writing of the *Metaphysics* Aristotle was concerned with the nature of things as they are and not with the creation of new mathematical ideas or entities. The Space is a Set of Points metaphor is a part of the larger discretization program. It was the creation of mathematicians to further the progress of rigor and precision in mathematics. It aided mathematicians in their efforts to maintain the elite stature of mathematics over other areas of science. During Aristotle's time mathematics also held a superior position to other bodies of knowledge but it was also the source for reasoning in other subject areas. There was no need nor was there any stimulus for Aristotle to think in terms of discretized space. Sets weren't brought into mathematical thinking until centuries later. Set theory wasn't developed until the mid 1800s. Thus Aristotle did not have access to the main ideas embedded in the Space is a Set of Points metaphor and he had no reason to create it for himself.

Aristotle argues for the continuous nature of spatial objects and their relationship to discrete or indivisible objects. For him the continuous is "...found in things out of which a unity naturally arises in virtue of their contact. ...Nor again can a point be in succession to a point or a moment to a moment in such a way that length can be composed of points or time of moments; ..."

Opportunities for learning from interacting with Aristotle

As described in the previous chapter there is a distinct difference between naturally continuous space and discretized space. In his Metaphysics Aristotle made an argument based on naturally continuous space which upon reading introduced the students to this notion. The students were unaware of conceptual metaphors and the creation of discretized space using these metaphors. If they had both of these elements at the time of the conversation it would have provided them more tools to argue against what they intuitively knew was false; this being that a line cannot be composed of points.

The use of non-traditional texts holds promise for teaching secondary mathematics students. They possess a number of characteristics that might provide opportunities for students to increase their mathematical understanding of topics previously taught to them and to learn new ideas. This article in particular was, in its time, written to other learned individuals. It style of writing is different than today's style causing the students to work at understanding it. This allowed for a deeper level of engagement. The text also was very authoritative during Aristotle's time. Although his ideas are not promoted in current scientific thinking Aristotle is still respected for his work. Thus this text has an implicit authority which the students seemed to respect. The historical characteristic and the authority opens the possibility for students to connect their understanding and modern day mathematics to the past. All of these components make the use of this text valuable to use with undergraduate mathematics majors. In the next section another historical text, *The Theory of Numbers* will be discussed and later in chapter five connected to the analysis.

Dedekind

As I continued to think about Aristotle's argument I came across the work of Richard Dedekind. Although I didn't assign his text for reading in the seminar with the seniors his ideas added to my understanding about the real line and Lakoff and Núñez found his work to be an essential part of the discretization program. I'll begin this section with my understanding of his argument concerning the continuity of the real numbers and end it with the authors' analysis of his work.

Richard Dedekind, who developed his fundamental insights in the context of his teaching, is a pivotal figure in the development of our current understandings of the real line. He worked hard at developing ways of understanding the real numbers in connection with his understandings of the nature of lines. However, in the 2000+ years between Aristotle and Dedekind, how people understood the nature of lines greatly shifted. As I outlined in the previous section, Aristotle's argument is primarily based on the inconsistencies between the physical nature of points and lines.

Dedekind however took a very different approach. Like Aristotle he began with the general straight line as a motivation for his argument. Second, he wasn't arguing whether a continuous spatial object can be created from discrete spatial objects. He was interested in making the ideas surrounding the continuity of spatial objects more rigorous and precise. He found the geometrical arguments and ways of thinking to be imprecise and ambiguous and sought an argument based on the arithmetic properties of the real numbers which at that time were more precise and more rigorous than geometry. Instead of basing his argument on the geometry of the point and line, he assumed that a line was composed out of points and chose to argue on the basis of the 'arithmetic properties' of the real numbers.

Dedekind says that the question "Is space continuous or not?" is immaterial. For one reason, if it is discontinuous or has a finite or infinite number of gaps, then humans can through thought fill in all of those holes. (Dedekind, 1948)

> If space has at all a real existence it is not necessary for it to be continuous; many of its properties would remain the same even were it discontinuous. And if we knew for certain that space was discontinuous there would be nothing to prevent us, in any case we so desired, from filling up its gaps, in thought, and thus making it continuous; this filling up would consist in a creation of new **point individuals** [(not number points)] and would have to be effected in accordance with the above principle (p. 12).

Second, Dedekind says that looking back over the history of geometry, the continuity of space again doesn't matter. All of Euclid's results would still hold in discontinuous space. It was a geometry that only needed the spatial objects (points, lines, etc.) that Euclid referred to in his axioms, theorems, postulates and proofs.

> ...If we select three non-collinear points A, B, C at pleasure, with the single limitation that the ratios of the distances AB, AC, BC are algebraic numbers, and regard as existing in space only those points M, for which the ratios of AM, BM, CM to AB are likewise algebraic numbers, then is the space made up of points M, as is easy to see, everywhere discontinuous; but in spite of this discontinuity, and despite the existence of gaps in this space, all constructions that occur in Euclid's Elements, can so far as I can see, be just as accurately effected as in perfectly continuous space; the discontinuity of this space would not be noticed in Euclid's science, would not be felt at all (p. 37).

He also states that it is with our minds that we see a line as continuous. This intuitive, natural pure sensation that looking at a line suggests to us is what he wants reflected in the number system, in which he also finds a certain beauty. Dedekind sees geometry as a motivation or inspiration for his work with numbers, but by no means is the system of numbers (the numbers and their operations, addition, subtraction, multiplication and division) dependent upon any geometry. He uses the geometry and the images surrounding continuity to create an understanding of his argument about the nature of real numbers. His creation of Dedekind cuts arithmetically captures the absence of gaps that we perceive when we look at a line or curve drawn in the plane. He states,

> What is meant by this is sufficiently indicated by my use of expressions borrowed from geometric ideas; but just for this reason it will be necessary to bring out clearly the corresponding purely arithmetic properties in order to avoid even the appearance as if arithmetic were in need of ideas foreign to it (p.5).
>
> If now, as is our desire, we try to follow up arithmetically all phenomena in the straight line, the domain of rational numbers is insufficient and it becomes absolutely necessary that the instrument R constructed by the creation of the rational numbers be essentially improved by the creation of new numbers such that the domain of numbers shall gain the same completeness, or as we may say at once, the same continuity, as the straight line (p. 9)

These "new numbers" which Dedekind is referring to are the irrational numbers. Before constructing the irrationals out of the raw material of the rationals, Dedekind describes the arithmetic relationships between rational numbers, >, <, and =, and likens these to the positional relationships between points on a line, right, left and occupying the same place.

Rational Numbers	**A Line with points**
Law I: If $a > b$ and $b > c$ then $a > c$ and we say b is between a and c	Law I: If p lies to the right of q and q to the right of r then p lies to the right q (i.e. q to the left of p) or q lies to the right of p (i.e. p to the left of q)
Law II: If $a \neq c$ then there are infinitely many numbers between a and c	Law II: If p lies to the right of q and q to the right of r then p lies to the right of r and we say q lies between the points p and r
Law III: If a is a fixed number then all numbers of R fall into two infinite classes A_1 and A_2 where A_1 is comprised of (not contains) all numbers $a_1 < a$ and A_2 comprises all numbers $a_2 > a$ and a is either a part of A_1 or A_2 but not both. *note this implies that all numbers in A_1 are less than all numbers of A_2*	Law III: If p is a fixed point on L then all points in L fall into 2 classes P_1 and P_2 each of which contains infinitely many individuals; where P_1 contains all the points p_1 that lie to the left of p and the 2nd class P_2 contains all of the points p_2 that lie to the right of p and p is either a part of P_1 or P_2 but not both.

Table 12
Rational Numbers and Points

The most interesting similarity is in Law III, which in both cases (rational numbers and lines) cannot hold without Laws I and II. It is this law that motivates the famous Dedekind cut.

Dedekind continues on to strengthen this relationship between the rationals and the line by fixing a point on the line, calling it the origin and defining a unit of measure. Then he lays off every rational length and determines exactly one point for each rational number. He then takes into account that there are irrational lengths and that the prior mapping does not establish points for these numbers. He alludes to having more points on the line than he has rational numbers when he says:

> If we lay off such a length [an incommensurable length] from the point of [the origin] upon the line we obtain an end-point which corresponds to no rational number. Since further it can be easily shown that there are infinitely many lengths which are incommensurable with the unit of length, we may affirm: The straight line *L* is infinitely richer in point-individuals than the domain ***R*** of rational numbers in number-individuals. (p. 9).

If the rationals are to possess the same beauty of continuity that the line has, they will need to be filled in and completed in some way. From here Dedekind goes on to arithmetically create Dedekind cuts, which he uses to define irrational numbers. It is again important to note that he is trying to model in number what he sees in the geometry of a line, and not argue that the real numbers will have the identical physical makeup that the naturally continuous line has.

> 15a) Whenever, then, we have to do with a cut (A_1, A_2) produced by no rational number, we create a new, an irrational number α, which we regard as completely defined by this cut (A_1, A_2); we shall say that the number α corresponds to this cut, or that it produces this cut.

By comparing cuts he can determine whether or not the two numbers are the same; if they produce the same cut then they are the same number.

> From now on, therefore, to every definite cut there corresponds a definite rational or irrational number, and we regard two numbers as different or unequal always and only when they correspond to essentially different cuts (p.15).

Dedekind has indeed accomplished a major feat. He has provided mathematicians a way of thinking about continuity that does not depend on pictures, graphs or any geometry. He was able to disconnect the mathematics from the geometry yet his ideas are consistent with the geometry. Although this can be a source for confusion it is also useful to learners who are more in tune with the geometry than they are the pure mathematical ideas. This connection (not a dependent connection) is useful but not necessary in his work as he describes below.

> "...but the fact that in the course of this exposition my name happens to be mentioned, not in the description of the purely arithmetic phenomenon of the cut but when the author discusses the existence of a measurable quantity corresponding to the cut, might easily lead to the supposition that my theory rests upon the consideration of such quantities. Nothing could be further from the truth; rather have I in Section III, of my paper advanced several reasons why I wholly reject the introduction of measurable quantities; indeed, at the end of the paper I have pointed out with respect to their existence that for a great part of the science of space the continuity of its configurations is not even a necessary condition, quite aside from the fact that in works on geometry arithmetic is only casually mentioned by name but is never clearly defined and therefore cannot be employed in demonstrations." (p.37)

Lakoff and Núñez's view of Dedekind's work

The discussion given in the prior section is my understanding of Dedekind's argument in the Theory of Numbers. Lakoff and Núñez have a different position.. They find Dedekind's work to depend on the geometry of the line. This line wasn't the naturally continuous line as I understood it to be, but instead a discretized line as described in the previous chapter. Dedekind sought to free the calculus from its geometrical roots. This was accomplished through a series of metaphors in which Dedekind begins with the geometry of the line and the Space is a Set of Points metaphor and moves to the arithmetization of the real numbers. More than ever it is imperative to keep in mind that the authors are giving a cognitive account of Dedekind's work and not an accounting of the mathematical development of his ideas.

In his search to understand continuous curves in terms of arithmetic Dedekind set out to define continuity in arithmetic terms. He reasoned that if continuous curves were images produced from the real numbers then the real numbers must be continuous. There was already in existence a development of the rational numbers out of the natural numbers and Dedekind sought a similar development for all of the real numbers. Thus he wanted to define the irrational numbers in terms of the rational numbers which were already defined in terms of the natural numbers. According to the authors the following statement begins his argument.

> This analogy between rational numbers and the point of a straight line, as is well known, becomes a real correspondence when we select upon the straight line a definite origin or zero-point and a definite length for the measurement of segments. (pp. 7-8)

Lakoff and Núñez take this statement as his construction of the discretized number line blend detailed in chapter two. He is identifying a number for every point and a point for every number. Next he notes that this process of locating a point for all measurable segments leaves points unaccounted for.

He continues by arguing that if the real numbers are to have the same continuity as the curve then "...it becomes absolutely necessary that the instrument *R* constructed by the creation of the rational numbers be essentially improved by the creation of new numbers such that the domain of all numbers shall gain the same completeness, or as we may say at once, the same continuity, as the straight line." (p.9)

Pausing for a moment, here is what the authors are arguing. Looking at the discretized number line blend

Numbers Are Points On A Line (Fully Discretized Version)

Source *Domain*				*Target Domain*
The Space-Set Blend				
Naturally Continuous Space: The Line		Sets		Numbers
The line	↔	A set	→	A set of numbers
Point-locations	↔	Elements of the set	→	Numbers
Points are locations on the line.	↔	Elements are members of the set.	→	Individual numbers are members of the set of numbers.
Point-locations are inherent to the line they are located on.	↔	Members exist independently of the sets they are in.	→	Numbers exist independently of the sets they are in.
Two point-locations are distinct if they are different locations.	↔	Two set members are distinct if they are different entities.	→	Two numbers are distinct if there is a nonzero difference between them.
Properties of the line	↔	Relations among members of the set	→	Relations among numbers
A point *O*	↔	**An element "0"**	→	**Zero**
A point *I* to the right of *O*	↔	**An element "1"**	→	**One**
Point *P* is to the right of point *Q*.	↔	**A relation "$P > Q$"**	→	**Number *P* is greater than number *Q***
Points to the left of *O*	↔	**The subset of elements *x*, with $0 > x$**	→	**Negative numbers**
The distance between *O* and *P*	↔	**A function d that maps ($0 < P$) onto an element *x*, with $x > 0$**	→	**The absolute value of number *P***

(p. 281)

Table 13
Fully Discretized Space Metaphor

the source domain consists of a blend between naturally continuous space and sets. In the source domain for the blend the line is naturally continuous. This line is viewed as a set of points by the Space-Set blend. This blend is then mapped onto numbers by the Numbers are Points on a line (The fully discretized version). This larger one to one mapping of the points from the blend to the numbers is how Dedekind is able to establish a gap. According to the authors, here is where Dedekind is able to argue that if the line, which is composed of points, is continuous then the numbers that are being associated with the points must also be continuous because the mapping preserves the structure of the source domain. Thus the set of rational numbers must be added to in order to obtain for numbers the continuity that he sees in the line.

In the number line blend that Dedekind was creating he unofficially begins with the number line blend discussed in chapter two. This blend contains the naturally continuous line with numbers spread along it (not in a haphazard fashion). "Since this conceptual blend of space and arithmetic is used for measurement and built into measuring instruments (like rulers), it is taken for granted as objectively true: There is a correspondence between points and numbers, as seen in the act of measuring and in the instruments for doing so." (p.297) In effect Dedekind extends the physical idea of a ruler to a more conceptual tool; a tool which includes all of the real numbers which have no tangible, physical meaning as length has. By doing so he can make the observation that the line has points that have not been assigned to any rational number. Therefore if the set of rational numbers is to have the same feeling of continuity that the line possesses other numbers will have to be introduced to fill in the gaps on the line. This is where he introduces the Dedekind cuts as a means for defining irrational numbers and associating them with points on the line. He never proves that all of the points on the naturally continuous line have been accounted for. Nor does he ever prove that the naturally continuous line is composed of points. Instead he uses another metaphor 'Continuity is Numerical Completeness' to move our thinking from the points on the line an arithmetic conception of continuity. He maintains the strong visual elements of continuity suggested by the geometry and then uses completeness as the tool which provides the continuity.

Lakoff and Núñez continue to give a more detailed mathematical idea analysis of Dedekind's ideas. There are three essential conceptual pieces to this analysis. They are the Cut Frame, the Geometric metaphor and finally the Arithmetic metaphor. The cut frame states

Dedekind's Cut Frame
The Number-Line blend for the rational numbers, with a point *C* (the "cut") on the line, dividing all the rationals into two disjoint sets, *A* and *B*, such that every member of *A* is to the left of, and hence less than, every member of *B*.

Table 14
Dedekind's Cut Frame

This frame designates where Dedekind's thinking begins. Again it isn't with the nature of the relationships between points and lines but with the assumption that every rational number can be associated with a distinct point on a given line, creating the Number-Line blend. He then uses this Number-Line blend along with his prior work with generic points and lines to define a structure on the set of rational numbers. Next he introduces irrational numbers and creates a metaphor which extends the Rational Number Line-blend to the Real Number-line blend. The first line of the chart below detailing Dedekind's Geometric Cut Metaphor is the same as before. We think of R, a rational number as the point that cuts the rationals into two disjoint sets A and B where all numbers in A are less than all numbers in B. The second line defines irrational numbers in a similar fashion. I, an irrational number is to be thought of as the point or cut that divides the line into two disjoint sets A and B where A has no smallest rational and B has no largest rational. Uniqueness is provided by the one to one correspondence between numbers and points in the Number-Line blend which guarantees that there will be a point for every irrational number. Also the Archimedian principle which is embedded in the Number-Line blend prevents the number from being an infinitesimal and lastly the density of the rationals on the line guarantees the uniqueness the irrational number.

Dedekind's Geometric Cut Metaphor

Source Domain		Target Domain
The Rational-Number Line Blend with the Cut Frame	→	The Real-Number Line Blend, with the Cut Frame
Case 1: A has a largest rational R, or B has a smallest rational R. C (the "cut")	→	R
Case 2: A has no largest rational and B has no smallest rational. C(the "cut")	→	I, an irrational number
(p. 300)		

Table 15
Dedekind's Geometric Cut Metaphor

Although this was a great triumph in itself Dedekind still had the irrational numbers defined or dependent on the geometry of the line. The total elimination of all geometry was obtained by one more metaphor.

Dedekind's Arithmetic Cut Metaphor		
Source Domain	→	*Target Domain*
Rational Numbers and Sets		The Real-Number-Line Blend, with the Geometric Cut Metaphor Defining Irrationals
An ordered pair (A , B) of sets of rational numbers where A ⋃ B contains all of the rationals and every member of A is less than every member of B	→	The point C (the "cut") on the line dividing all the rationals into two sets A and B, such that every member of A is to the left of, and hence less than, every member of B
Case 1: A has a largest rational R, or B has a smallest rational R. (A , B) = *R*	→	Case 1: A has a largest rational *R*, or B has a smallest rational *R*. C (the "cut") = *R*
Case 2: A has no largest rational and B has no smallest rational. (A , B) = *I*, not a rational	→	Case 2: A has no largest rational, and B has no smallest rational C = *I*, an irrational number

(p. 302)

Table 16
Dedekind's Arithmetic Cut Metaphor

This metaphor replaces all notions of geometry, the line and points with sets and numbers.

It is through the use of the above metaphors and blends that Lakoff and Núñez analyze Dedekind's thinking concerning the continuity of the real numbers. Chapter four will discuss the methods and methods of analysis for the study. The ideas in this chapter will be revisited in chapter five where the data is analyzed. Again, the students did not read this piece by Richard Dedekind. However they do make comments which connect to Dedekind's ideas.

CHAPTER 4

METHODS OF DATA COLLECTION AND ANALYSIS

Methods

Uniqueness of the Study

This study stands out from other studies in a variety of ways. It was conducted in a very non-traditional manner. Instead of fully conceptualizing ahead of time what I wanted to study, forming research questions, developing a construct to study a situation and then a method of analysis, I forged ahead with an idea and decided to study it afterwards. I had as a basis my own mathematical experiences coupled with those of friends and students of mine, along with ideas of a different environment that I thought would be useful. As I mentioned in the introduction, my own mathematical preparation left me with many questions; there was never any place in classes for them to be raised or for them to be addressed as part of the curriculum. I had also taught mathematics to many students (teaching from standardized syllabi) and found that the restrictions of time and the goals of the course for which I was teaching didn't allow me to address difficulties that the students were having – difficulties that I often attributed to the narrowness of the curriculum and not their lack of mathematical ability. When I thought about helping to prepare students to be future secondary teachers, I didn't want to contribute to the production of what I saw as a problem. I wanted to help provide experiences that I at the time hoped would affect my students' views towards mathematics and the way in which they thought about teaching it.

Why I chose to use the Aristotle article as a pedagogical tool

I decided to use the Aristotle piece in the secondary methods course that I had been given the opportunity to teach. I was hoping that the students would not only ask questions about the article itself but also begin to raise out loud some other questions about the real numbers and the real line that they may have had at the time – both questions similar to my own and others I could not possibly anticipate in advance. I also wanted them to see that they weren't the only ones to have these questions and that addressing them would be a worthwhile effort. Because of the lack of communication and focus on solving problems or proving theorems in mathematics courses, students learn to push aside their current questions that might provide meaning for them. They learn that this isn't a part of what it means to learn mathematics or become a 'good' student of mathematics. I thought that they viewed the environment for learning in their mathematics courses as I did and that they would welcome the chance to work in a more open environment accepting of their questions and what others might label as misconceptions. The traditional environment was much more a part of their identity at that time than I had anticipated; however, once we pushed through their reluctance they were quite personally involved and interested in the conversations that we had. I had hoped that the students would see this as an environment that they would want to provide for their future students and hoped that we could spend the second semester working on how to do this.

With all of my hopes in hand I set about teaching the course and devoting some time to the reading and discussion of Aristotle. Unfortunately I didn't have too much more at the time. I wasn't at all sure on how to structure the conversations. I felt that this would be a useful thing to do but at the same time I didn't want to impose a structure that would inhibit the students from speaking and freely asking questions. Although this approach worked very well for some students, it wasn't as productive for those who were less vocal and were used to and needed more structure. Thus in the data that I examine here the conversations are heavily dominated by three individuals which happen to also be male. There were two females present and they were active in terms of their listening, their thinking as I perceive it, and the few times that they participated in the conversation, but I do feel that my lack of pedagogical forethought hampered their activity.

Another area that I wish I had prepared more ahead of time was to weave a more coherent direction for the mathematics involved. I hadn't done this ahead of time primarily because I wanted the students questions to arise and for us to follow them. I didn't want to introduce items that were important and or interesting to me and have them take over. Also, I still had many of my own questions surrounding these issues and I was hoping that we would begin to explore them together.

At this time I also had no plans that this was going to be the data for my dissertation. I was hoping to have very good conversations from which I could learn. Good being thought of as pedagogical fruitful (Haroutunian-Gordon & Tartakoff, 1996; Sfard, Nesher, Streefland, Cobb, & Mason, 1998). I was surprised at the quality of the conversations that we did have and decided that I wanted to use them for my dissertation. Although this has made for a very interesting and informative study it has made writing it very difficult. As mentioned before, I didn't have a definitive plan of action and I had little idea of how I wanted to analyze my data. Whichever method or direction I chose I wanted to be sure to use my analysis to show to others that these students possess hidden qualities that we would want mathematics majors and in particular those who have chosen teaching as a profession to have. To work towards this goal in a credible fashion, I have had to create my own way of analyzing the data and drawing conclusions.

Data Collection

I collected the data for this study in three consecutive years of the same preservice secondary mathematics class (1997-98, 1998-99, and 1999-2000) offered at a land-grant institution in the midwestern United States. This class is a two semester eleven credit (five in the fall and 6 in the spring) seminar taken by teacher candidates in their senior year. (A subsequent yearlong internship is required for certification.) I read the Aristotle chapter with each cohort of "the seniors." Each year, there was one class period solely dedicated to the chapter. However, the discussion of Aristotle continued alongside other activities for the remainder of the year.

The data for this study were collected over a three year period. I was teaching the equivalent of methods for prospective secondary mathematics teachers. At this institution the students are divided by subject matter, thus all of my students were math majors. I taught these students for three years. I audio taped the two-hour class twice a week for 30 weeks each year. This produced 180 taped conversations. Of these conversations approximately 20 included some discussion of the Aristotle article.

Although the data from the other years were interesting and inspired me to continue using the article for two subsequent years I chose to use the data from the 1999 – 2000 academic year for analysis in this study. This narrowed the number of conversations down to approximately six. During this year I had divided the class into two sections in order to make the conversations more manageable. I chose to look at only one of the sections which narrowed my choice for conversations to three. As I listened to the remaining six tapes (there were two tapes per conversation) I decided to focus on the first conversation because it captured the emphasis on making sense out of the article.

The conversation took place on November 11, 1999. I had separated the class into two sections and this is the second half. I chose this particular half because I was very interested in the interactions between Baron and Schroeder. This was the first conversation around the Aristotle reading. We didn't have any more formal conversations about the reading but we continued to refer to the article throughout the remainder of the year. We began the conversation approximately 2/3 of the way through the semester. We had already discussed topics in infinity and whether or not .999... = 1.

Students

The students in this study were mostly senior math majors at the time. Three of the sixteen students were juniors, two of whom were in this half of the class. I separated the class in hopes of making the conversations more manageable and to give each student more time to contribute to the conversation. When dividing the students I tried to balance those who appeared to me to be more 'comfortable' and 'successful' in their mathematical thinking with those who were less so. I also wanted to distribute the number of females in both sections. Both sections had two females. Overall one section wasn't any more 'mathematically talented' than the other. I chose to analyze the section with Baron and Schroeder 1) because Schroeder wasn't only very well versed in mathematics but he had shown to have done a lot of thinking in other subject areas as well. The conversation of this half of the class also had more to offer in terms of analysis than the other section did.

Data Analysis

The data for this dissertation has worn many faces. I wont' describe all of the earlier attempts to make sense out of the conversation except to say that by doing so I gained a priceless familiarity with the data. I also spent a great amount of time discussing pieces of the transcript with my advisor, Daniel Chazan, and a devoted colleague Bill Rosenthal. Having spent a year with these students I felt that I knew them quite well. Once I decided to use the Lakoff and Núñez text as an essential part of the conceptual and analytical framework I did take particular actions to re-orient my mind meshing of the data and the text together.

I went through the entire transcript just making notes of items that struck me and why. Some of the items that stood out to me were references to time and motion. I have since decided to disregard these as part of the analysis because they seem to come more from the article than from my students. I then went back through the entire transcript and separated it into sections and matched them to particular sections of the Aristotle article. As I went through this time I also looked more closely for instances where the students discussed number, geometry, the infinite, infinitesimal or infinite divisibility and any connections within this list.

I eventually moved to cutting and pasting all of the comments from each individual student into their own documents. With these I read them to see how much of the conversation they were a part of. Also I was looking to see if they raised any specific examples and why; I was looking to see the nature in which they participated, did they ask questions more than make statements or arguments. I did all of these things with Lakoff and Núñez in the background of my thinking.

I spent a lot of time reading Lakoff and Núñez and trying to understand what made their work so different than others. I summarized different parts of their text and eventually chose the ones that were most relevant to my data. As I read through the text and made the summaries I generated my own mathematical examples to test my understanding of what they were writing along with testing the validity of what they were writing for me. At the times when I was most puzzled I would consult with Dan and Bill for their input. They would read sections of the text simultaneously with me.

Without a more specific, detailed plan of analysis in hand I had to find a way to make sense to others what was becoming clear to me. I returned to Lakoff and Núñez to synthesize their 'argument' to myself. I took what I saw as the essential parts and used those to look through as I thought about the data. I went through the transcript again. I found evidence of the different pieces that I identified in Lakoff and Núñez in the students comments and examples.

After reading the text I have found there to be two essential parts to the argument of Lakoff and Núñez and also corresponds to my data. First is their notion that mathematics is embodied. They attempt to carefully explain how physical and mental human actions build particular neural structures in our brains. They describe the role of these structures in human cognition as schemas. All of human thought can eventually be traced back to a schema in the human brain. There are different levels of complexity in human thought. In order to do more abstract thinking (and other types), for instance higher level mathematics, humans use conceptual metaphors and blends. Thus the analysis of my data is in two sections. In the first section I take the reader through nearly the entire transcript. First this is so that the reader can get a sense of the flow of the conversation. Second I wanted to highlight portions of the students thinking as it connected to schemas and for this it would have difficult to take their comments out of context. The next layer of analysis consists of portions of the transcripts in which the students are unknowingly using the conceptual metaphors and blends.

CHAPTER 5

DATA PRESENTATION AND ANALYSIS

The Nature of the Conversation

Overview

The conversation that I will now analyze took place on a mid-November day in 1999. On this day, half of the class, consisting of eight students, two female and six male (Sally, Lucy, Baron, Schroeder, Jim, Franklin, Penny and Jason), met for a two-hour conversation. During this conversation, we spent our time coming to grips with the Aristotle reading described earlier. The kinds of student behaviors exhibited in this conversation are representative of the other conversations of this cohort, as well as conversations of the other two cohorts. I am focusing on this single conversation in order to have a manageable dataset to analyze using Lakoff and Núñez's framework.

The transcript of this session is a 45-page, single-spaced document. I have structured this document according to terms introduced in the reading. The conversation began with the students expressing their general thoughts on Aristotle's argument. Upon my suggestion and in order to simplify and focus the conversation, the students set off by discussing each term individually in the order that they were presented in the article. The following table is organized according to the term under discussion (in boldface type). Underneath each term àre the examples introduced by the students as they try to understand its meaning. The last column indicates sub-terms from the article that the students are also discussing.

Structure of the Conversation by Terms and Examples		
Transcript Page	**Between**	Sub-terms
3	Points A and B. 1 and 2 and the points in the middle.	
4	Sequence of Integers. "certain progression."	Continuously according to its nature (CAN)
5	Traveling from 1 to 10. Euclidean circle.	
6	Continuously according to its nature (CAN) and numbers and order. CAN and a plane.	
	Successive	Sub-terms
12	Euclidean Circle. Circle with radian measure.	Contrary in place (CIP), Extremes
13	Paths and circuits (roundtrip to a house). Comparison of a line to a circle. Dimension.	
14	The Earth	
16	Football (continuing the discussion of CIP)	
17	Trip (CIP)	
	Contiguous	
19	People around the table. Dimension and coordinates again.	
20	Circuit: square. Z_3.	
21	Equivalence and circle with radian measure. Equivalence and numbers in Z_3.	
24	Integers.	
25	Townhouses.	
26	Contiguous states.	
27	Reals, rationals and irrationals.	
28	Blocks.	
29	The real line as contiguous.	

Table 17 (cont'd)

	Continuous
32	Real number line
36	Limit points.
37	1 through 10 as continuous.
39	Numbers on a string.

Table 17
Transcript by Terms and Examples

The analysis in this section is arranged according to the following claims and titles.

Section Structure	
Claim	**Section Title**
Although three students do the majority of the speaking, there is evidence of engagement by all of the students.	Engagement
There is evidence that the students refer back to earlier parts of the conversation and stay focused on mathematical issues.	1) Making connections and staying focused: The circle 2) Making connections and staying focused: Numbers

Table 18
Transcript by Claims

The chosen pieces of transcript for each of these sections cut across the definitions in the "Structure of the Conversation by Terms and Examples" table. This table provides a way of locating the chosen selections in the larger conversation.

Engagement

Reading through the conversation, one is struck by the amount of text that was produced during this time and by the prevalence of three male voices throughout the transcript. The examination of the transcript begins with an attempt to describe the nature of the interaction in this conversation and of students' engagement. I will argue in this section that other students are engaged in the conversation besides the three who speak regularly. In the following section, I will support the claim that the students are staying on focus in this two-hour conversation. Third, I will examine particular incidents in the students' comments. These excerpts demonstrate that the students are interested and invested in the conversation by how they handle the complexity of the ideas, and that the ideas do not overwhelm or frustrate them.

Upon reading the entire transcript, the conversation appears to be taking place among three male students (Baron, Schroeder, and Jim), while there are five other individuals who are a part of this class. Since there are only two females, Sally and Lucy, in this section of the class, if each member of the class participated equally the female comments would still be outweighed three to one. However, although the ratio of male comments to female comments is far greater than three to one, the females' engagement is by no means of lesser caliber. First I will consider Lucy's engagement in detail, then Sally's.

Lucy's comments in the conversation are quite different from Sally's. Sally participates in more conversational interactions than Lucy. Lucy, on the other hand, appears to intently listen and is very selective about posing a question or making a comment. Her comments aren't great in length, thus not occupying much time, but they do occur throughout the entire conversation, showing her attentiveness to the conversation. It is apparent that Lucy is engaged in sense making from the beginning of the conversation. On page four she begins to try and form a question that she has (the tape doesn't pick up her entire question). Her participation in the conversation can be seen in several comments.

> 4: Somebody...
> Is he only talking the ...

Here on page 12 she again begins to ask a question or make a statement which I notice but cannot hear:

> 12: I: Lucy?
> Huh?
> I: You were going to say something?
> No, I'm just, I think I'm...

Baron, Schroeder and Penny are in the midst of discussing what makes a figure two dimensional as opposed to one dimensional. Lucy's last comment before the one immediately above was on page 12, where she is heard once more drawing the gentlemen's attention to the portion of the text in which Aristotle refers to lines as being straight. This again demonstrates her engagement and attentiveness. Her second comment is in reference to Schroeder's introduction of a football as an example. The students are discussing the meaning of "contrary in place" and Schroeder asks which two points on a football would be "contrary in place." Lucy is engaged enough to understand what Schroeder is referring to and to give a response. Although she isn't speaking very much and at most times not even very loudly, she is definitely an active part of the conversation.

> 16: Wait, wait. At the top of page 145 – (quotes the reading). So therefore I assume the rest of the article to mean a straight line.
>
> I would think that the line, whatever line you draw through it though. If you draw it from endpoint to endpoint then yes, that is the most distant point. But if you draw up to down, then this point is the most distant…

The discussion of "contrary in place" continues on page 17. I have introduced an example of traveling along the segment of the real line with endpoints 1 and 5. Lucy's replies indicate that she comprehends my comments and also is very clear in expressing her understanding of the phrase.

> 17: It just depends on which line.
>
> Because you've never changed the direction..
>
> You've just paused your time—you never changed your direction. So I would say 1 and 5.

Lucy's first comment on page 18 is a carryover from the discussion on page 17. Again the students are discussing paths and endpoints to help them determine when one point is successive to another, mostly focusing on the first and last point. Baron and Schroeder are having a short dialogue concerning whether Aristotle is only discussing lines or might be considering other objects as well. Lucy's second comment here on page 18 doesn't interrupt the gentlemen's discussion but when they appear to be done she asks a question that reminds them of the original reason they began the entire discussion with paths. Again there is powerful evidence here that Lucy is not only participating in the brief side conversations that the class is having, but there that she is actively making sense out of the larger conversation as well.

18: I think that is a different path.

So what's successive?

On page 19 the conversation moves to the discussion of Aristotle's next term, contiguous. Baron begins the conversation with an example using the people in the room at the time. Since the definition of contiguous is an extension of the definition of successive, the students return to the discussion of successive. Schroeder introduces Z_3 as an example and the students discuss this for three pages. When Lucy asks her question concerning the integers, the conversation on contiguous has started to wind down. Although she didn't verbally participate in the discussion of contiguous, her question reveals the deep level on which she is participating. The integers were not a part of the discussion of contiguous. No one had offered them as an example at this time. The discussion of Z_3 is the closest that the students came. In their discussion of between and successive the students used segments of the real line to argue their points and clarify their positions. Yet Lucy's question is a larger one and again demonstrates that she is thinking about both the original dilemma of whether a line is made out of points, along with the Aristotle's individual terms, and the more specific example of using the real line as her replacement for Aristotle's line.

24: Um, so are the integers contiguous?

Are there any like number lines that…

27: Are the integers contiguous or not?

The problem the students are having is what it means for numbers to touch. The primary example is the one that Baron introduced of the students in the room moving close together so that their arms were in contact with one another. Lucy comments on how this isn't possible with numbers. By page 32 the conversation has moved to include continuous. The general consensus in the room is that if they are to follow Aristotle's reasoning then there are no number lines in which the points/numbers are contiguous so they are also not continuous. This is bothersome to many of the students in the class because it seems to contradict what they have believed for a very long time. Lucy, in her comment on page 32, is saying that although she isn't convinced that they the integers, aren't continuous, she can't explain why. She has shown her understanding of Aristotle's argument – but this statement shows that it isn't enough for her to believe it.

> 28: Because people are objects. They're not the same thing as numbers.
>
> But just at that point…
>
> Only at that one point that your skin is touching.
>
> 32: That woulda been easier, huh?
>
> I'm not sold on the fact that (unclear) I can't explain why they are but (unclear)
>
> Well, yeah, I think, I think they are, but-
>
> 34: If they're equal, they're not contiguous. They're (unclear). But if they're not equal, then they could…
>
> 43: I don't know; if I'm going to reflect on this for the next two years, I want to have some sort of idea of what everybody else thinks.

At the very end of Lucy's comments she is again looking at the definitions of contiguous and how they fit with her knowledge of numbers in general. She is still trying to make sense out of the discussion and ends with a response to a statement that I have made. I have told the students that the discussion of this piece usually carries over for the remaining of that current academic year and also into the next year. Instead of dismissing what they have done, she makes it known that she wants to know what everyone else is thinking about these ideas. This to me shows that she is interested in having the discussion for a more sustained period of time.

Moving on to Sally, the first evidence of her engagement comes in her comments that begin the discussion. She is the first to attempt to unravel Aristotle's use of the word between. As readers we need to appreciate the courage and confidence that Sally shows in her comments. This conversation takes place in mid-November after the course began in September. Schroeder, Baron and Jim have always been the most vocal students in the classroom. Although they at times try to make room for others to speak and to sincerely listen to their thoughts and ideas, it was a difficult atmosphere for student who aren't accustomed to or don't want to forcefully assert themselves. Sally is the first to attempt to summarize Aristotle's argument and also the only student to say that it contradicts her own beliefs up until this time that a line is made out of points. She says,

> Something that I thought was interesting was that I thought you know that the lines were made up of like an infinite number of points and he says that that's not even possible, yeah.

Although it is most likely that Sally may have thought that everyone in the class thought that a line is composed of points because it is taught to students beginning at a very young age and reiterated throughout their mathematical lives, I am still struck by her bravery to make the statement and to connect it to her own mathematical understanding. This statement demonstrates that, at least in the beginning, Sally is participating in the conversation and is engaged in the ideas of the article.

She again makes her belief in the ambiguity of Aristotle's statements apparent to everyone when introducing the word "between." After Franklin oversimplifies her questioning of the definition and implies that she doesn't know what the word means in this apparently mathematical context (apparently to the students at this time), Sally very strongly responds back by asserting,

> I understand what between is!,

and then goes on to say,

> I know but it's kind of the way you were saying it, it's like, "Yeah, well I know that's what between is," but just understanding why he's saying it the way he's saying it.

This last comment of Sally's again demonstrates her engagement but also it shows her awareness to the fact that Aristotle is crafting something and that there is a purpose to the way in which he is choosing and defining his words.

From this point on in the conversation Sally's comments diminish. She is next heard on page seven. The focus of the discussion has changed to relationships between a set and what 'continuously according to its nature' means in connection to the set. Again her comments aren't timid and they are meaningful and helpful in this portion of the discussion. The disappearance of her verbal comments from the conversation does not indicate that she is no longer participating. She continued to take notes throughout the conversation and her facial expressions and side comments (which weren't obtained by the tape recorder) to others indicated to me as the instructor that I still had her attention and interest.

Making connections and staying focused: The circle

Aristotle's argument centers on the introduction and defining of the words "Between, Successive, Contiguous and Continuous" in this particular order. Beginning with the definition of "between," defining each subsequent word is done in terms of the definition of the previous word. As noted in the chart in the overview section at the beginning of this chapter, the students introduce particular examples as they make sense out of the meaning of Aristotle's definitions. The use of these examples in the discussion of more than one term indicates that the students are attentive throughout the conversation, that they understand that there is consistency involved in 'mathematical meaning' and that they students are focused on the task of making sense out this argument.

The most prominent example that illustrates how the students refer to items throughout the conversation and stay on focus is raised Schroeder in the discussion of the word between. He states,

> Like I was thinking of a circle. Like let's say you're on the perimeter of a circle and you travel around the circle, you can't all the sudden just cut across the diameter and say that the center of the circle is between two points in the outside. I think, I think that's the way I read it anyway. Because that's not, then you're not, there is no path. Yeah, you're jumping off and going not according to the circle's ...

In this instance Schroeder is using the circle to demonstrate what "between" cannot mean. It appears that he has chosen the circle and a direction for moving around it. To leave this "natural" way of moving by cutting across the diameter of the circle would be a violation of moving, in Aristotelian terms, according to the circle's nature. Schroeder next raises the example in the discussion of the word successive. He states,

> Or if your class is a circle, would your first point be successive?

Here Schroeder has carried over the example of the circle to the next term. Although he first used it to show what "between" isn't, the circle wasn't found to be a poor example for the word and thus it wasn't discarded. In this case Schroeder uses the circle to raise an issue to help clarify the meaning of successive. It appears that the interesting behavior for him lies at the endpoints of the object. Thus this example is to focus everyone's attention on the first and last point of the circle. Unlike in the first use of the circle as an example where there wasn't much discussion following its introduction, this time the example raises some questions. The first issue is how there can be a first and last point on a circle. Schroeder addresses this issue by saying,

> That's why I'm trying to be because he says that is successive which is after the beginning? Well, what if you are in a circle like you said and you pick your beginning point, go around the circle and get to your last point, then the beginning point would be successive to your last point.

He continues,

> I think the key words are it says "the order being determined by the position or form or in some other way" so I would say if you're talking about a circle, if you have had the liberty to determine how you want to talk about um order. So you could say, once you get to the end you just start over again. Or you could say—well, I think, I think it—I think when he says "position or form or in some other ways" it's um a little shaky sentence…

Jim is listening quite closely to Schroeder and is bothered by the nature of the endpoints. He doesn't agree that the first point can be successive to the last and tries to demonstrate this theory by changing Schroeder's example to a circle with radian measure. His idea is that the first point will have a value of 0 and the last point will have a value greater than 0 – thus how could it be that 0 could be successive to 2π? Schroeder and Jim never reach an agreement at this juncture and Baron enters the dialogue to suggest that Aristotle is restricting himself to one dimension and thus the circle may not be a feasible example.

Although it is difficult not to pay attention to the details of discussion and the difference in positions that the students take, what is most important here is the general nature of the conversation. Schroeder begins by introducing the circle as an example in the discussion of the word between. He reintroduces it in the discussion of successive and uses it as a tool to test Aristotle's definition to see if it can have any meaning. Jim then enters the conversation to make sure that the use of the example is not violating the definition. At one point he states,

> That's what I'm saying—this says no. "That is successive which is after the beginning." How can zero be after itself?

These excerpts are evidence that the students are closely focused on the article and trying to make sense out of Aristotle's terms. The excerpts also show that the students are not only paying attention to each definition alone but they are in the process of making sense out of the entire argument. The evidence for this assertion is in how they examine the same examples as the discussion moves on. There is an expectation that as the argument grows by moving through the terms, so should the examples.

Due to Baron's statement that Aristotle was only considering one-dimensional objects the conversations move in this direction. The students begin to speak about traveling along paths and circuits. In referring to trips Baron states,

> Which even if it's screwy, as long as it can be—there's a place you're starting from and place you're going and they're not the same.
>
> Jim: What if go to your house and back? Is that a circle?...
>
> Schroeder: How is a circle different than a line, though, other than what we've been reading so far? ...I mean, you can describe any part of a circle by just one coordinate, as long as you have a starting—you know, you can just, just do the circumference...

Schroeder is not yet ready to concede on the use of the circle as an example of successive. Baron attempted to end the dispute between Jim and Schroeder by redirecting their attention to one dimension. Instead of relenting, Schroeder takes the example of a path and tries to draw parallels from it to the circle. It isn't that he is unaware of the definition of a circle. In this instance what is important to him is defending the circle as a viable example. There is no need for Schroeder to argue according to definition of a circle; it is of no use here. These excerpts instead show that he has quite a bit of flexibility in his thinking about mathematical objects and he demonstrates the ability to apply his thinking across a variety of mathematical contexts. He is doing mathematical thinking even though he isn't following what are the more traditional means of argument.

The conversation continues and eventually moves to the discussion of the next term, contiguous. Baron gives the following example.

> Huh?... Like right now, even though this end trigger says two dimensions (laughing) we're all pretty much successive. But if we all held hands, we'd be contiguous. Like if we were all like shoulder to shoulder.

Although this passage highlights very well the essence of contiguous, it introduces a circuit, and Jim and Schroeder strongly object and turn to discussing dimension and the endpoints again. Jim takes the focus back to the circle with radian measure. He states,

> [B]ecause I think—well, see, 2π is not equal to 0. They are when you're talking about measuring … a circle, yes, they're the s same, we do sines and cosines and yes, they're the same but zero is not equal to 2π. So I guess to rebut our own argument, I guess what we can say is this number line Z_3 goes 0, 1, 2 but then when we get to three, that is 3 it's not 0, I mean they're equivalent in the sense of adding, subtracting, that kind of stuff, but three is different than zero. So that's how when we get …

The students are examining here the relationship between the first and last point on the circle. Aristotle says in his definition, that is successive which is after the beginning. If the last point is distinct from the first, as on a line or a non-closed path, then there isn't a problem. But if the object that one is traveling on or the path is closed, then can the first point be successive to the last point? The discussion concerns discussing equivalence and in what sense the first and last point of a circle are equivalent to one another, and in what sense not. The important idea is that the example continues to arise and to be examined in light of the new definition. Also, the students continue to revisit and re-examine what they have said earlier in the conversation.

Making connections and staying focused: Numbers

The students use numbers in a variety of examples throughout the conversation. Numbers appear in the discussion of each of the terms from the article. In many cases the students choose them as a means for demonstrating what Aristotle is referring to. At first, their understanding of numbers appears to be taken for granted, but by the end of the conversation they begin to examine their own knowledge more closely.

The first example using numbers doesn't say very much about the students' understanding. Baron is the first to introduce them in his interpretation of Aristotle's argument. He states,

> Baron: Because it's like if you take two points and put them next to each other, because there's no they can't, if they're touching, then since they have no dimension, they're essentially the same point. And if they're not touching then there's a space between them so then it's not a continuous thing because they're not touching. And so then no matter how many you know an infinite number of points that you stack side by side, you're still staying at the same point so it's not like you're gonna be like going along 0-0-0-0-4.

What is of most value in this example is the inherent connection that Baron makes between points on a line and numbers. This statement occurs at the very beginning of the conversation (on the second page) and number had not been connected to the article at this point.

The second reference to number is given by Franklin. He states,

> Okay, think of like 1 and 2. Just to count 1, 2—going straight from one and landing on 2. But if you consider the points in the middle—if you travel along them like this, that's between.

The topic of conversation is the meaning of "between." Franklin's example again identifies a point on a line for each number. It also begins to introduce notions of continuity and discreteness. Baron follows with

> Well he's differentiating between a continuous thing and a thing of units. So like he was saying numbers are units, you go from 1 to 2 instantaneously, there's no between, if you consider like there's one thing and then there are 2 things and there's no in between. But if you're measuring something, then or if you're like crossing a distance then that's when it's possible to be in between. Not when you're talking about units.

This statement of Baron's strengthens Franklin's idea by connecting numbers to the act of measuring. This is explained in more detail in the following section, "Mathematical Sense Making."

A part of Aristotle's definition of between is the phrase "…if it changes continuously according to its nature… ." This is an important aspect to Schroeder, who comments,

> The way I read that was like you can't take a sequence of integers and go like you know um like 1, 2, 3—I can't think of an example, but I think what he, the way I read it was "continuously according to its nature" meant that if it's supposed to have like a certain progression that you're supposed to do, you can't like loop around or something and then say it's between, when I think that's what I was thinking then when I read it. I don't know. I can't think of an example.

As the students continue to introduce examples containing numbers, more is revealed in their comments about number. Here we see in Schroeder's comments that the natural numbers have a natural progression to them. If taken out of this order, "between" would have no meaning because the order has been violated. By identifying order with the phrase “continuously according to its nature” Schroeder appears to think of order as inherent to the numbers or to humans’ thinking of them. He follows up on this thinking on page 6 when he says,

> I was just thinking so you’re talking about numbers—it seems that naturally they have some order. And so I was thinking that maybe that might be the way that we think of numbers as being greater than or less than so to follow to progress through numbers naturally would be to go from smaller to larger—that’s the way I was thinking. Or to be—we talk about um a plane, then it’s confusing to me because I don’t know what’s the natural way to address a plane.

This statement shows that Schroeder certainly sees a structure for numbers. The phrase “that we think of numbers as being greater than or less than…” is evidence that he may see the structure as lying in the human mind and not in the numbers themselves. This conceptualization is also shown in his ending remark, “…I don’t know what’s the natural way to address a plane.” Here Schroeder is noting the absence of a structure that can correspond to CAN, but it is unclear whether he is perceiving the plane itself as lacking the structure or saying the plane has a structure that his understanding is missing. Whichever the case, certainly Schroeder is thinking deeply about some very important mathematical ideas. He is pondering the nature of the set of natural numbers and also the nature of the plane.

As the conversation moves on to the word “successive,” the natural numbers are no longer at the center. The issue with successive is the first and last points and the question, Can the first point ever be successive to the last point? Jim introduces a circle with radian measure to show that since 2π is greater than 0, the last point cannot be successive to the first when traveling around the circle in a clockwise direction. This belief is shown in the following interchange with Schroeder.

> Jim: That's what I'm saying—this says no. That is successive which is after the beginning. How can zero be after itself?
>
> Schroeder: Because it's the beginning then it's also the end. So you could have—you could just think of it as the end because it's at the beginning...
>
> Jim: Is that even—that's why I'm confused, because it says that is successive which is after the beginning, so if you define zero as your beginning, something after it, 2π....

In the above exchange Jim is focusing on the order relationship between the measures of the angles. Schroeder is considering this as well as the fact that the first point is also the last point. He is holding the order relationship in mind while also noticing that the values 0 and 2π correspond to the same point. Thus the first point can be successive to the last since in terms of points, once you complete a revolution you arrive at the same point but not at the beginning.

Up until this moment the students have spoken of numbers as locations (places on a number line) and as physical objects – points that can sit side by side one another on the line. In their discussion of the term "contiguous," the students begin to examine these conceptions of number. Baron starts the discussion with the following statement (also referred to in the previous section).

> Huh? ...Like right now, even though this end trigger says two dimensions (laughing) we're all pretty much successive. But if we all held hands, we'd be contiguous. Like if we were all like shoulder to shoulder.

This immediately takes the discussants back to dimensions and the difference between the endpoints in two-dimensional objects as opposed to one-dimensional objects. To try and solve the problem Schroeder introduces a new example:

> So what if you're talking about like Z_3 or something, where you go 0, 1, 2; 0, 1, 2.—you just keep going over and over again, so...

The idea here is that it appears that Z_3 would represent a circuit but the beginning point and endpoint aren't the same. Thus you could label the points as you would on the line and you would not have the problem of re-labeling an earlier point as it happens on the circle. Schroeder then draws a parallel to geometric objects.

> That's the same as sitting in a triangle or something, or a square, or a triangle I guess it would be. You go around this way, and then, but you...

What is hidden in the students' comments up until this time is an implicit assumption that numbers are continuous. No one has spoken about it as of yet but it appears to be embedded in Schroeder's example. He moves from Z_3, where there are discretely three elements, to a triangle, which when drawn is taken to be continuous. He maintains the continuity but also gains distinct endpoints.

The conversation about Z_3 continues. In response to Schroeder's comment equating Z_3 with a triangle, Jim states,

> But if you look at a number line in Z_3, it goes 0, 1, 2, 3—that's a straight line.

Baron also tries to clarify and chimes in,

> But zero, real – I mean I'm sure you could justify it some way, I'm sure I mean in his own head he could rationalize

> it and just say well 0, 1 and 2—any thing else is just and equivalence class to 0, 1, and 2—so 0, 1, and 2 are really the only elements in there. And so you would go 0, 1, 2 and then you're done.

No one objects to this notion of a number line in Z_3; in fact it seems that the students prefer it to what Schroeder is suggesting. They are more bothered by the inclusion of 3 in Z_3 and move on to discuss equivalence classes of numbers in this set. Going back to the example of the circle with radian measure, Jim states,

> ...because I think—well, see, 2π is not equal to 0. They are when you're talking about measuring ... a circle, yes, they're the same, we do sines and cosines and yes, they're the same but zero is not equal to 2π. So I guess to rebut our own argument, I guess what we can say is this number line in Z_3 goes 0, 1, 2 but then when we get to 3, that is 3; it's not 0. I mean they're equivalent in the sense of adding, subtracting, that kind of stuff, but 3 is different than zero. So that's how when we get...

The students notice the similarity between equivalence on a circle with radian measure and the equivalence between numbers in Z_3. They see that the two notions are not exactly the same and continue to work on their understandings. They begin to discuss which numbers are members of Z_3 and how they are represented there.

> Jim: I can say 10 in Z_3....
>
> Baron: You can just say this {writes, [[3]]}. Now that happens to also equal the equivalence class of 0. Yeah, but the number inside there is now, is now an integer. It's not the number in Z_3, that 3 that you wrote inside of the brackets....
>
> Schroeder: It's an integer.... But it's, that's not the same as the symbol that's inside those brackets. That's an element of the integers. But I think we got off track here, I don't know. We we're talking about um—no wait, you're saying

though—3, Z, 3 in Z_3 is still an integer, it's just an integer in Z_3.

> Baron: But I think it's the same thing. 2π—it's, depends on how you think of it. 2π in terms of a measure of an angle is 0.
>
> Jim: the same as 0, but they're not the same thing.
>
> Baron: So it's just a matter, if you're taking, if you take it in the real number line they're not the same. But if you take it in a different context, they are....
>
> Jim: So we have to try to take everything that's in a circle and a triangle and a square and somehow go to a line.

As stated earlier, the more examples the students raise and the more they continue to discuss their examples, the more insightful comments they make about what they know and the more questions their pose about what they know. The degree to which they are aware of what it is they are doing can be seen in Jim's last statement. He hasn't lost sight of the earlier portions of the discussion and the problems that arose there. His statement ultimately shows that the discussion began with the creation of a line and a set of points; it indicates the need to take all that he and his peers have discussed and decided on thus far and apply that back to the original argument.

At this point I redirect their attention back to discussing "contiguous." There is more conversation about the closed circuits and non-closed circuits, and as the discussion of contiguous is seemingly coming to an end, Lucy asks,

> Lucy: Um, so are the integers contiguous?

An apparently simple question but the students' comments reveal how complex it is. Baron says no; Schroeder thinks they are and Jim says it depends on how they look at it. They begin tracing back through all of the terms – between, successive, contiguous – in

order to make an argument. "Between" and "successive" don't cause a problem, but contiguous does because the students cannot decide what it would mean for two numbers to touch. Here is where they begin to become aware of how they have been speaking of numbers.

> R (undeterminable speaker): Touch, the definition of touch means things whose extremes are together—"touch," that's what it says. Things whose extremes are together touch.
>
> Franklin: But then you get to the later argument where he's saying points don't have ...
>
> Baron: ... and I think the extremes of 1 and 2 and 3 and 4 are points—they don't have dimension.... There's not this end of five. Five is five.

Franklin's statement shows again how the students freely think of numbers as points. Baron's statement reinforces that these points would be dimensionless. They begin to use other examples to examine contiguous again and then return to numbers.

> Baron: They're {in reference to townhouses} objects and you're putting them next to each other. So they're touching, and there's no distance between them there's nothing of any class that you can put between them, but...
>
> Schroeder: So how is that different then the integers then? Why can't the integers be...
>
> Baron: I can put a real number between integers. I can put something of another class between them. I can't put, if it's contiguous there's nothing there's no class which you can shove in.

Although Baron's paraphrasing of Aristotle's definition isn't exactly accurate, he does begin to successfully apply it to the numbers. To make his point, however, he has to move from the integers to the reals. This maneuver begins to push the students' thinking even more about the nature of numbers. As they continue to compare objects to numbers, someone begins to ask,

> R: Well why can't you stick something in there?

This is in reference to an example raised by Baron to demonstrate contiguity by two hands touching. Lucy responds,

> Because people are objects. They're not the same thing as numbers.

This is a key realization for Lucy, and once she has stated it to the class it appears to be one for everyone else also. Were her statement obvious to her from the outset, then she wouldn't have posed the question, "Are the integers contiguous?" If she had seen at that time that numbers aren't physical objects, then she would likely have realized that it would not have been possible for them to be contiguous. Penny's agreement with Lucy's realization shows in his stating,

> When you are talking about touching hands and all that, I don't think that relates to 1 and 2,

to which Schroeder chimes in,

> Yeah, I agree with ya. I don't think so either.

Shortly afterwards (nine lines), Schroeder proposes another example.

> Well think—what if you think of integers of like as like blocks like one like there's a block that's 1, 2, 3, … you just stick them together.

Although it is difficult for him to accept speaking of numbers as objects and applying Aristotle's terms to numbers as if they were objects, Schroeder seems to be compelled to continue speaking of them in this way. The students resolve the issue by deciding that

Aristotle wasn't applying his argument to numbers and that it all is just an analogy. It appears the conversation is over when Schroeder asks,

> So you think the integers are not contiguous then?

The discussion of whether or not the integers can be contiguous begins anew. Although the students temporarily agreed that treating numbers as physical objects was just an analogy, their questions don't disappear. The problem, which is on the horizon, is that if the integers or real numbers are not contiguous, they also are not continuous. This goes against their intuition.

With this realization, Jim requests an example of a line that is continuous. They are now moving on to the final term, "continuous," in Aristotle's argument. Aristotle's definition states, "The continuous is a species of the contiguous or of that which touches: two things are called continuous when the limits of each, with which they touch and are kept together, become one and the same, so that plainly the continuous is found in the things out of which a unity naturally arises in virtue of their contact" (p. 86). Again the problem is plain: If numbers cannot be contiguous with one another, they cannot be continuous. Not totally convinced, the students attempt to find an example of numbers in contact with one another. Recalling a previous conversation in which we discussed whether or not $.99\bar{9} = 1$, Penny says,

> So (unclear) saying that $.99\bar{9}\, and\, 1$ are not equal, wouldn't they be considered contiguous?

When we had this conversation, there were some students who believed that the two numbers were equal and others who disbelieved it. Penny is suggesting that if they are not equal and there isn't anything between them, then they should be contiguous.

Baron challenges him and asks for a proof, but Penny isn't able to provide one. No one else enters the conversation to assist Penny. It appears that at this time different people are pondering different things. Baron is willing to continue the conversation to pursue what leads to either equality or inequality between $.99\bar{9}\, and\, 1$. Lucy is wanting to go for the case of equality; Jim is thinking about the relationship between contiguity and equality; Schroeder wants to focus on the continuity of the reals. I ask them to summarize what it is that we know so far about continuity. The focus moves to their understanding of continuity. Schroeder and Penny each have a bit to say on the matter.

> Schroeder: Well, isn't—if you're talking about like a, the real numbers being continuous, wouldn't that mean that every point is a limit point? That's pretty much what continuous means to real numbers, right? We're not talking about like a function being continuous, and that has like epsilon delta definition, but if you're just talking about the real line being continuous, it's that every, every point is a limit point, right? …
>
> Penny: I don't know, I guess, I always look as continuous as (unclear). This may or may not be what's usually referred to, like I always say that one to ten, one, two, three, four, five, six, seven, eight, nine, ten, is continuous. Well, one to ten going on, two and then skipping to five then to seven then to eight, I always thought that was not continuous. Even though we can put numbers in between, I still say that was not continuous. So therefore even if I'm going one, two, three, four, five, six, seven, eight, nine ten that would be continuous whether or not I put .(unclear). I would still say that one to ten is continuous.

The students begin to talk about continuity in terms of patterns, the natural numbers versus the reals, and how it may be possible for one set to be more continuous than another. What is striking about this point of the conversation is that while still considering what it might mean for the real line to be continuous, they are spending more effort on their own understanding of these very complex ideas. They have arrived at a place where they are questioning what they thought they knew. This isn't happening because Aristotle is insisting they are incorrect and they believe him. Not long before, they collectively dismissed the connection between their ideas and his ideas as just an analogy. Baron often reminded everyone to keep Aristotle's argument in its proper historical context. Even with this reminder, he and others continued to discuss and apply his reasoning to their own understanding. Schroeder later suggests,

> I don't know, I, I, I think, I thought I was hearing like different, a bunch of different things, and I, I was just thinking that we've all been exposed to the definition of continuity so many times that I, I , why don't we just use that definition instead of (unclear).

No one really accepts this proposal and acts on it. Finally Baron offers his own conception of the real numbers and Schroeder gives a response.

> Baron: Like, I was thinking about, I don't know, I'm just throwing this out, I was thinking about, like if you put, if you put the natural numbers in a row, you've cut 'em, like if you put 'em along the line and have like a unit between them so like here's one, here's two, here's three, here's four. You could like cut that line in half, it was like a string, you could cut that string in half in between and not be on one of the natural numbers. But if that represented the real number line, there's nowhere you could cut and not be and not be cutting and not be cutting a real number. Like on an exact real number. So to me, that seems to be a reason why the real numbers are continuous. Not with his definition, but just in general. There's nowhere you can't, you can't find—there are functions with holes that are continuous, but...
>
> Schroeder: That's what I'm thinking too, but I was just wondering if there's a mathematical term that's different from continuous that would describe that property of real numbers. That's what I was asking. I don't know. Seems like there should be. Probably is.

No one requests a proof of Baron's idea, but Jim and Penny do ask for clarification. They are unclear on the difference between locating a number on the line (or string) and cutting at that number, and just making an arbitrary cut and "knowing" that the cut was at a real number. Baron's idea certainly suggests that he is thinking of the real numbers as being continuous in some sense. He appears to be describing a Dedekind cut although it isn't clear where he has encountered these ideas, nor is the depth of his understanding revealed in this conversation. The way in which the students continue to push their thinking deeper and deeper; the way in which they keep track of the larger argument in spite of the numerous side conversations indicate that they are seriously focused on the task at hand.

Encountering complexity with interest and not frustration

To say there was no resistance to engaging in the conversation would not be accurate; however, there wasn't very much. After the discussion of between Jim suggested that we not continue on because everyone wasn't in agreement at that moment. This occurred the year before with an earlier class, and based on the experiences there I predicted the resistance would shortly disappear. It did. The students very quickly became engaged in the conversation and showed that they had a personal stake and interest in continuing. There are three ways in which the students demonstrate that they are interested and engaged in the conversation. These are by:

- the examples that they pose;
- how they continue to talk even when the discussion could have been ended;

- deciding to move away from the article and to pursue a discussion of their own mathematical understanding.

Aristotle makes an abstract argument as to why a line cannot be composed of points. The students, in discussing this argument, bring fifteen examples to the conversation in order to improve their own understanding even though they are accustomed to working on mathematical arguments with limited given information. As impressive as the number of examples the students raise are the types of examples that they have chosen. These examples connect number or one of Aristotle's terms to another context and are summarized in the chart below.

Involving Humans	Involving Physical Objects	Involving other mathematical ideas.
People in a circle holding hands to demonstrate contiguous	Sidewalk	Paths
	Football	Circles and angle measure
	Blocks	Z_3
	Hours	Dimension
	Townhouses	Spaces and lines
	Garages and houses	Measuring units
	Earth	Euclidean circle

Table 19
Students' Examples by Category

The students bring a variety of contextual examples to the conversation to assist in their sense making. The existence of these examples shows that the students are confronting the difficulty of Aristotle's writing. They move beyond the language in the article and use the examples to make the argument more accessible to them. The examples are also used by the students to argue for their particular understanding of a term. This can be seen in the examples of the Euclidean circle and the circle with radian measure that Schroeder and Jim raise.

The conversation could have come to an end in many instances. Baron argued that the time period of the original text makes Aristotle's argument not pertinent to modern day mathematical thinking. He makes this argument three times and in none of these instances does anyone ever suggest that the conversation end. In all cases, the conversations continue to move forward by another example or question introduced by a student.

The number of side conversations that take place is further evidence that the students are willing to confront difficult and complex mathematical ideas. They also are not embarrassed or afraid to examine their own knowledge or thinking. The exploration of the meaning o f mathematical dimension, the representation of numbers in Z_3, and the discussions about equivalence in the context of circles with radian measure, Euclidean circles and numbers in Z_3 are examples of this. They pursue their ideas attempting to reach a final understanding.

The most significant example is the dialogue at the end of the conversation. The students are just beginning to explore what they know and believe about real numbers. Even though they accept that Aristotle's ideas may not apply they realize that they don't have a concrete way of stating or proving what they intuitively know – that the real numbers are "continuous." The realization of the difficulty of this idea and what it actually means becomes apparent to them. No one suggests that the conversation ends; no one attempts to change the topic. Instead, everyone is thinking quite deeply about how to argue what they feel they know. They move back through Aristotle's argument;

they try and make their own arguments; and near the end of the class they ask what mathematicians say about this. They are interested and invested in the ideas and they seek a resolution to their dilemma.

Mathematical Sense Making

As described in Chapter 2, there are primarily two pieces to the argument that Lakoff and Núñez offer in contending that mathematics is embodied. The first piece deals with the cognitive ideas and how it is that mathematics can be embodied. The second uses mathematical idea analysis to demonstrate how some mathematical ideas have come into being. This section draws on these ideas from the authors and identifies the portions that indicate that fruitful mathematical activity is taking place. Evidence that the Container and Source-Path-Goal schemas are in use suggest that the students have the necessary mechanisms that "...link...language and spatial perception" (p. 31). Evidence of the grounding metaphors indicate that mathematics is grounded in their experiences and still continues to be. Evidence of linking metaphors (not necessarily the ones treated by the authors) suggests that students engage in making connections across mathematical disciplines as a way of making sense for themselves. Lastly, evidence of the folk theory of essences demonstrates that students have incorporated over the time of their mathematical studies the general idea of what it means to do mathematics or think mathematically.

Lakoff and Núñez on Grounding Metaphors

Conceptual metaphor allows humans to do mathematics beyond the innate abilities of subitizing and counting (in the range of 0 to 4) that they are born with. As stated in Chapter 2, there are two types of these metaphors important to mathematics. These are "linking metaphors," which connect different subject areas in mathematics to one another, and "grounding metaphors," which, as the authors say, "allow you to project from everyday experiences (like putting things into piles) onto abstract concepts (like addition)....The grounding metaphors yield *basic, directly grounded ideas*. Examples: addition as adding objects to a collection, subtraction as taking objects away from a collection, sets as containers, members of a set as objects in a container. These usually require little instruction" (pp. 52-53; emphasis in the original). There are four grounding metaphors, which Lakoff and Núñez refer to as the 4Gs. The first, Arithmetic Is Object Collection, is as follows.

Arithmetic Is Object Collection

Source Domain Object Collection		*Target Domain* Arithmetic
Collections of objects of the same size	→	Numbers
The **size** of the collection	→	The size of the number
Bigger	→	Greater
Smaller	→	Less
The **smallest** collection	→	The unit (One)
Putting collections together	→	Addition
Taking a smaller collection from a larger collection	→	Subtraction

(p. 55)

Table 20
Arithmetic is Object Collection Metaphor

In this metaphor, "[E]veryday experiences of subitizing, addition and subtraction with small collections of objects involve correlations between addition and adding objects to a collection and between subtraction and taking objects away from a collection. Such regular correlation, we hypothesize, result in neural connections between sensorymotor physical operations like taking away objects from a collection and arithmetic operations like the subtraction of one number from another. Such neural connections, we believe, *constitute a conceptual metaphor* at the neural level—in this case, the metaphor that Arithmetic Is Object Collection" (pp. 54-55; emphasis in the original). Also note here that within this metaphor a number is conceptualized as a collection and the size of the number as the size of the collection. Humans have the ability to understand small collections, ones of four items or fewer, from birth. It is this metaphor that allows us to handle numbers larger than four. Note the words that are in bold type in the chart. These are words that are regularly used to describe objects and collections. These words and phrases are used again when speaking of numbers, which are abstract entities. This is one way in which we begin to speak of numbers as actual physical objects with their own existence.

The Arithmetic Is Object Collection metaphor has a long list of entailments. These entailments arise by taking "truths" about collections and mapping those onto corresponding statements about numbers. There is one metaphor that is worth mentioning. Under the Arithmetic Is Object Collection metaphor, there isn't anything in the source domain to map onto the number zero. For this a new metaphor is needed. The Zero Collection Metaphor allows us to think of having no collection of objects as a physical entity or collection. Given this metaphor, the empty collection can then be mapped onto the number zero by the Arithmetic is Object Collection metaphor. By this metaphor zero is thought of as emptiness.

The Zero Collection Metaphor		
The lack of objects to form a collection	→	The empty collection

(p. 64)

Table 21
Zero Collection Metaphor

The second grounding metaphor is Arithmetic Is Object Construction.

Arithmetic Is Object Construction		
Source Domain Object Construction		*Target Domain* Arithmetic
Objects (consisting of ultimate parts of unit size)	→	Numbers
The smallest whole object	→	The unit (one)
The size of the object	→	The size of the number
Bigger	→	Greater
Smaller	→	Less
Acts of object construction	→	Arithmetic operations
A constructed object	→	The result of an arithmetic operation
A whole object	→	A whole number
Putting objects together with other objects to form larger objects	→	Addition
Taking smaller objects from larger objects to form other objects	→	Subtraction

(pp. 65-66)

Table 22
Arithmetic is Object Construction Metaphor

This metaphor, Arithmetic Is Object *Construction*, is very similar to Arithmetic Is Object *Collection*. Here number is a whole thing made up of parts. Again words that appear in the source domain are words that are commonly used when speaking of objects and this language allows us to speak of numbers as though they were objects also. This metaphor has all of the inferences of the previous one with two more. Since a whole object can be divided into its parts this metaphor introduces fractions. The whole corresponds to a unit and the parts of that whole would correspond to pieces of the whole or fractions of the unit. It also has the entailments of the prior metaphor with an additional one allowing us to decompose a number into other numbers. The following (unnamed) metaphor is needed for this.

Whole objects are composites of their parts, put together by certain operations	→	Whole numbers are composites of their parts, put together by certain operations.
(p. 68)		

Table 23
Fraction Metaphor

As in the Arithmetic Is Object Collection metaphor, an additional metaphor is needed to account for zero. With the Arithmetic Is Object Construction metaphor, zero will be thought of as nothingness or the lack of an object.

The Zero Object Metaphor		
The Lack of a Whole Object	→	Zero
(p. 67)		

Table 24
Zero Object Metaphor

The third grounding metaphor is the Measuring Stick Metaphor (briefly discussed in Chapter 2).

The Measuring Stick Metaphor		
Source Domain The Use of a Measuring Stick		*Target Domain* Arithmetic
Physical segments (consisting of ultimate parts of unit length)	→	Numbers
The basic physical segment	→	One
The length of the physical segment	→	The size of the number
Longer	→	Greater
Shorter	→	Less
Acts of physical segment placement	→	Arithmetic operations
A physical segment	→	The result of an arithmetic operation
Putting physical segments together end-to-end with other physical segments to form longer physical segments	→	Addition
Taking shorter physical segments from larger physical segments to form other physical segments (pp. 68-69)	→	Subtraction

Table 25
Measuring Stick Metaphor

According to this metaphor, numbers are conceptualized as actual physical objects. The size of the number is thought of as the length of the segment. Words used when discussing segments – longer, shorter, putting segments end–to-end, and taking segments away from larger ones – all are mapped onto words for number, greater, less, addition and subtraction. This metaphor says that for any number (which refers to the positive rational numbers) there is a segment which is mapped onto it. It is as an entailment of the blend of this metaphor that the irrational numbers are created. The blend requires that for any segment there needs to exist a number for the segment to map onto. Thus numbers needed to be created for incommensurable segments, like the hypotenuse of the right triangle with legs of length one. Zero is conceptualized as the lack of a physical segment.

The last grounding metaphor is Arithmetic Is Motion Along a Path.

Arithmetic Is Motion Along a Path		
Source Domain Motion Along a Path		*Target Domain* Arithmetic
Acts of moving along the path	→	Arithmetic operations
A point-location on the path	→	The result of an arithmetic operation
The origin, the beginning of the path	→	Zero
Point-locations on a path	→	Numbers
The unit location, a point location distinct from the origin	→	One
Further from the origin than	→	Greater than
Closer to the origin than	→	Less than
Moving from a point-location A away from the origin, a distance that is the same as the distance from the origin to a point-location B	→	Addition of B to A
Moving toward the origin from A, a distance that is the same as the distance from the origin to B (p. 72)	→	Subtraction of B from A

Table 26
Arithmetic is Motion Along a Path

This metaphor is similar to the Measuring Stick metaphor. However, here numbers are thought of as point-locations on a line. Words used for motion, further and closer, are mapped onto words for number, greater than and less than. Unlike the other metaphors no additional metaphor is needed to accommodate zero. The point in the source domain corresponding to the origin of the motion is the natural choice for the pre-image of zero under the mapping. And zero is thought of as an originating place for motion.

The four grounding metaphors are significant in understanding how human beings are able to do and create mathematics. First they are the means by which humans extend the innate mathematical abilities they are born with. It is an extension of the innate abilities because the mappings in the metaphor preserve inferences. Thus what is true in the source domains is also true in the abstract target domain of arithmetic. For example, if an object is placed in a collection of five objects the result is six objects would now be in the collection; this carries over to arithmetically adding one to five and obtaining six as the result. Furthermore, the source domains contain activities that humans do naturally from the time they are small children. These activities become conflated with the innate arithmetic of children and are easily extended to the natural numbers by the grounding metaphors.

There are more cognitive mechanisms, other metaphors for example, that help to embody mathematics. These four grounding metaphors are highlighted here as one piece of the tool that is used to analyze the data in this study. These are important because they account for how some prominent neural connections that support the doing and creating of mathematics are established in the brains of human beings.

Lakoff and Núñez on the Folk Theory of Essences

How mathematics was done today and what mathematics is today extends from mathematics as done by the Greeks. They certainly were not the only ones in history to do mathematics or to create mathematical systems per se; however, they were the ones to have the axiomatic method at the core, and this foundation has translated over into modern-day mathematics. This transition was no accident. European philosophy, which was founded on Greek philosophy, inherited the spirit of Greek philosophy. Thus the ideas spread into other areas of European thought; mathematics was one of those. Lakoff and Núñez name this spirit "the folk theory of essences." It has five main pieces:

- Every specific thing is a kind of thing.
- Kinds are categories, which exist as entities in the world
- Everything has an essence—a collection of essential properties—that makes it the kind of thing it is.
- Essences are causal; essences—and only essences—determine the natural behavior of things.
- The essence of a thing is an inherent part of that thing.

(pp. 107-108)

In Greek philosophy, Aristotle's theory of categories parallels the folk theory of essences. In Aristotle's theory categories are defined "by a set of necessary and sufficient conditions." As a result, Aristotle "...defined 'definition' in terms of essences: A definition is a list of properties that are both *necessary* and *sufficient* for something to be the kind of thing it is, and from which all its natural behavior flows" (p. 109; emphasis in the original).

In tracing the incorporation of this theory of essences into modern day mathematical thinking, the authors move from Aristotle to Euclid and his claim that all of plane geometry develops from five postulates. These five postulates describe what plane geometry is—i.e., the essence of plane geometry. This idea of Euclid's was extended to all other subject areas within mathematics. In other words, this is the beginning of the notion that any area of mathematics can be characterized by a small set of axioms from which all other theorems can be deduced.

Analysis of the Students' Examples: Overview

Lakoff and Nunez describe what they think occurs in mathematical sense making. In their view, mathematical sense making involves the use of schemas and metaphors to build up from human capabilities to cultural creations like mathematics. In the transcript of the November Aristotle conversation, there is much evidence to support the claim that in Lakoff and Núñez's terms, the students in the conversation are involved in mathematical sense making. There is:

- Behavior or activity that demonstrates the embodiment of mathematics, for example, use of the Container schema or the Source-Path-Goal schema;
- Behavior or activity that demonstrates significant features of the metaphors (grounding and/or linking) and blends that Lakoff and Núñez set forth; and
- Behavior or activity that illustrates the folk theory of essences.

The chart below indicates what information from the bullets above can be found in the examples that the students raised. Each column corresponds to a piece of the mathematical sense making framework. The folk theory of essences is not as easy to summarize in the table and will be discussed at the end of this section. The grounding metaphor column indicates which grounding metaphor is being referred to. The 'Xs' indicate that a schema, linking metaphor or discretized space is present in the example. Some examples contain more than one element of the sense-making framework I have developed.

Example	Container Schema	Source-Path-Goal Schema	Grounding Metaphors	Linking Metaphor	Disc. Space
Baron's summary of Aristotle's argument			Motion		X
Baron and Franklin's explanation of between			Motion		
Baron and numbers as sets			Measuring and motion		
Schroeder and progression			Motion		X
Baron and traveling from 1 to 10		X	Motion		
Jim and between			Motion		
Schroeder and circle					X
Jim and sets of numbers	X				
Confusion between points and what they represent			Motion	X	X
Points on a circle labeled in radian measure			Motion		X
Schroeder and dimension			Motion		
Dimension with coordinates				X	
Z_3 and equivalence Classes				X	
Baron's statements relating to Dedekind cuts			Measuring		

Table 27
Categorization of Examples

The remaining portions of this chapter consist of a detailed analysis of the students' examples using the different aspects of the mathematical sense-making framework outlined in the above table.

Schemas in the Students' Examples

Image schemas, which are fully explained in chapter two, are central to human cognition. They are the bridge between spatial relations and language. Although linguistic structures, grammar systems for instance, differ across languages, all of the structures reduce to a universal set of image schemas. There are many different types of image schemas; however, two, the Container schema and the Source-Path-Goal schema (SPG), are important to mathematics. Evidence of these schemas can be found in the students' conversation. Here are excerpts from the transcript.

1) Baron: Yeah, kind of. I mean if you're traveling, if I'm going from 1 to 10 and I'm passing 2 through 9, so I can arrive at 2, 3, 4, 5, 6, 7—I can arrive at 7, and I'm not to 10 yet but I've arrived at a specific, at something I can. (SPG)

2) Jim: I think what he's trying to—I guess I think maybe what we're all trying to say is that that redefines your class as the set—you have 0 to 10 and it says there's nothing to prevent a thing from some other class from being between. So if 2, 3, 4 up through 9 could be between 1 and 10 but it's not between in the sense of [unclear]. (Container)

3) Jim: And there's another set outside of that with the real numbers with something between one and two if you move, if you look at integers as a subset of the real. And you have something in between. But they're not between—I think that's why he defines between in the sense of a class. So if I say you know, ".5 is between 1 and 2" [should be 0 and 1], I can't say that they are in the sense of the class of integers. I have to define what between—I have to define what my class is if I say something is between. So if I say, ".5 is between 1 and 2," I have to say—I have to define if integers is my class. (Container)

4) Baron: Which even if its screwy, as long as it can be—there's a place you're starting from and place you're going and they're not the same. (SPG)

The Container schema has three parts to it: the interior, exterior and the boundary. This image schema is the foundation for understanding classes and sets in mathematics. A class is a conceptual container created by the grounding metaphor 'Classes Are Containers'. Classes are thought of as containers in which their elements are located. The 'Sets are Objects metaphor' makes a distinction between a class and a set. Classes can be subsets of other classes but they cannot be a member of a class because they do not exist as objects as other members of the class do. The 'Sets Are Objects' metaphor moves a mathematical class to a mathematical object which 'exists' on its own. This new object can now be placed within other sets or within itself and can possess the mathematical relationship of membership in other sets. The students in their conversation don't make this fine of a distinction between sets and classes. Aristotle uses the word class but the students don't use this word any differently than the word set.

Statements #2 and #3 above demonstrate the use of the Container schema by the students. In #2, the meaning of "between" is under discussion. The example given comes from the question: If you take the whole numbers and move through them in multiples of ten, starting with 0, would 4, for example, be between 0 and 10? Jim here is saying that if the moving in multiples of ten redefines the set from the whole numbers to only the set containing multiples of ten. Thus although 1 through 9 are between 0 and 10 in one sense, they aren't a part of the set under consideration. In this example, Jim is defining what his container schema consists of. For him the inside the container would be multiples of ten and, although 1 through 9 are positive integers, they are in the exterior of this particular container.

In Jim's second statement, #3, he has two containers, the reals and the integers. He speaks of the integers as a subset of the reals. This requires use of the Sets as Objects metaphor. Lastly he demonstrates what "between" would mean in terms of the sets. This reasoning is consistent with the logic of the schema. If your container is the reals, then with the Sets as Objects metaphor it is true that the integers are also inside of this container. So .5 can be between 0 and 1. However, if your container is the integers this cannot be the case: .5 is not an integer and cannot be located in that container, so it cannot be between 0 and 1.

The Source-Path-Goal schema has three essential parts to it also: the source, originating place for the motion; the goal, the destination place for the motion; and the path, the trajectory of the motion. The students quite often speak of moving through sets of numbers and around geometrical figures. Statements #1 and #4 above are two examples of this. In the first statement, Baron is speaking of moving through the numbers 1 through 10 and encountering the integers in between. The source is 1, the goal is 10, and the path consists of the numbers 1 through 10. In the discussion of successive, Schroeder later introduces the circle as an example. He is moving through the points on the circle and wants to argue that the origin can be successive to the goal. This leads to a lengthy discussion and at one point Baron clarifies the difference with statement #4 above. In this statement he lays out the distinction that the source and the goal need to be different. This isn't to meet the requirements of the schema but instead to satisfy the definition of "successive."

Grounding Metaphors in the Students' Examples

As described above, the grounding metaphors are attached to our everyday experiences. The four primary ones, Arithmetic as Object Collection, Arithmetic as Object Construction, Arithmetic as Motion Along a Path and the Measuring Stick Metaphor, are all grounded in human experiences as young children. From these we can project our understandings onto more abstract domains like mathematics. Evidence of these metaphors can be found in how the students speak of numbers. As seen in the description of the grounding metaphors, numbers do not objectively exist in the world. It is through metaphor, these and others, that numbers begin to take on a physical existence. Thus when students are heard speaking in these terms it isn't a weakness or deficit but a very ordinary human activity. The following excerpts students speak of numbers as physical objects.

<u>Motion</u>

1) Baron: Because it's like if you take two points and put them next to each other, because there's no they can't, if they're touching, then since they have no dimension, they're essentially the same point. And if they're not touching then there's a space between them so then it's not a continuous thing because they're not touching. And so then no matter how many you know an infinite number of points that you stack side by side, you're still staying at the same point so it's not like you're gonna be like going along 0-0-0-0-4.

2) Baron: He's just saying that if I'm going from Point A to Point B, if I've arrived at any point other than B before I get to B then I'm between A and B.

   Franklin: Okay, think of like 1 and 2. Just to count 1, 2—going straight from one and landing on 2. but if you consider the points in the middle—if you travel along them like this, that's between.

3) Baron: Well he's differentiating between a continuous thing and a thing of units. So like he was saying numbers are units, you go from 1 to 2 instantaneously, there's no between, if you consider like there's one thing and then there are two

things and there's no in between. But if you're measuring something, then or if you're like crossing a distance then that's when it's possible to be in between. Not when you're talking about units.

4) Baron: Yeah, kind of. I mean, if you're traveling, if I'm going from 1 to 10, and I'm passing 2 through 9, so I can arrive at 2, 3, 4, 5, 6, 7—I can arrive at 7, and I'm not to 10 yet but I've arrived at a specific, at something I can…[unclear]

5) Schroeder: [W]hat I was thinking of "according to its nature" and the word [unclear]…I was just thinking so you're talking about numbers—it seems that naturally they have some order. And so I was thinking that maybe that might be the way that we think of numbers as being greater than or less than so to follow to progress through numbers naturally would be to go from smaller to larger—that's the way I was thinking. Or to be—we talk about um a plane then it's confusing to me because I don't know what's the natural way to address a plane…

6) Jim or Schroeder: That's why I'm trying to be because he says that is successive which is after the beginning? Well, what if you are in a circle like you said and you pick your beginning point, go around the circle and get to your last point, then the beginning point would be successive to your last point

7) Schroeder: That's a good example because once you get around to 2π you add a little bit more, that angle is the same the way that we find angles its you know, just take subtract 2π and it's gonna be the same…as your um—like 0 is the same as 2π so maybe that is the one you could do by…

8) Schroeder: I think you can but you're, you're um your space or your set or whatever is defined in that same path. So like you still only need one coordinate to determine where you are on that path. Because it's—you can't travel off that path, so all you have to say is okay, I'm gonna travel 68 units and then I'll tell you exactly where on your square your pencil is. Because you know that after you go so far you have to turn at a right angle um—that's what I think.

Many of the excerpts relate to the Arithmetic as Motion along a Path metaphor. Although the students aren't making any references to arithmetic, their conception of number fits the conception of number in the metaphor; that is, that numbers are thought of as point-locations along a path. The students in these examples are speaking about traversing through sets of numbers, along a geometrical figure or along a path. There are many references to motion in the students' comments. Baron's comments include "staying at the same point" and arriving at intermediary points, going from 1 to 2 instantaneously and crossing a distance. These are all phrases that humans often use when speaking of moving from one place to another.

Schroeder's comments are especially interesting. He also uses phrases that refer to motion in his talk, making direct connections to mathematical objects as does Baron. Schroeder speaks of traveling around a circle point by point and incorporates the association of a radian measure with the points. He sees that motion can correspond to moving through the numbers according to their order; this may be a natural thing to do. It isn't as clear to him what it might mean to traverse a plane "according to its nature," but he does consider applying the idea to that mathematical structure of a plane as well.

<u>Measuring</u>

1) Baron: Well he's differentiating between a continuous thing and a thing of units. So like he was saying numbers are units, you go from 1 to 2 instantaneously, there's no between, if you consider like there's one thing and then there are two things and there's no in between. But if you're measuring something, then or if you're like crossing a distance then that's when it's possible to be in between. Not when you're talking about units.

2) Baron: Like, I was thinking about, I don't know, I'm just throwing this out, I was thinking about, like if you put, if you put the natural numbers in a row, you've cut 'em, like, if you put 'em along the line and have like a unit between them so like here's 1, here's 2, here's 3, here's 4. You could like cut that line in half, it was like a string, you could cut that string in half in between and not be on one of the natural numbers. But if that represented the real number line, there's nowhere you could cut and not be and not be cutting and not be cutting a real number. Like on an exact real number. So to me, that seems to be a reason why the real numbers are continuous. Not with his definition, but just in general. There's nowhere you can't, you can't find—there are functions with holes that are continuous, but ...

3) Baron: You know, you might still make that argument because no matter where you're gonna cut, it's going to be some proportion of the string....

4) Baron: So, I mean if you have a string, I'm not saying that like, like you know the square root of two or like π or something you can like find π and cut it there.

> But I'm saying that if I cut it, if I cut the string, there's some real number that represents that length that I've cut, so I've cut on a real number.

In the Measuring Stick Metaphor numbers are thought of as physical segments. The essence of the metaphor is the connection between number and length and the act of measuring. In the first statement above, Baron uses the act of measuring to describe what it means for one number to be between two others. In this there is the notion of having to traverse all of the space in between the two numbers, which is also obtained if you think of numbers as the actual segments. The second, third, and fourth examples bring in another interesting aspect. Here Baron is discussing his example relating to Dedekind cuts. He has likened the real numbers to a string and speaks of the string as being covered by these numbers. He recognizes that you can't locate every number on the string—π and the square root of two are the main examples—however, it is a fact, to him, that if you were to cut the string you would indeed be cutting on some real number. It is the Measuring Stick blend and the Arithmetic as Motion along a Path metaphor that support this thinking. From the blend we obtain segments for the natural numbers, 0, and the rational numbers. The blend then provides a number for other segments. When the Pythagoreans found the square root of two, they declared that it wasn't a number. The blend requires that there be a number corresponding to all lengths. The Arithmetic is Motion Along a Path metaphor allows humans to then conceptualize these new numbers as points on a line. In this sense the metaphor extends reality to abstract objects. There is no way to measure an irrational length however they are spoken of and treated as lengths just as rational numbers are. Baron, in his example, points out these elements.

Linking Metaphors in the Students' Examples

Linking metaphors allow humans to think of one area of mathematics in terms of another. Unlike grounding metaphors, where the target domain is some abstract concept and the source domain is an experiential activity, both the source and target domains of linking metaphors are mathematical ideas or branches of mathematics. For example, thinking of the natural numbers in terms of sets would be a linking metaphor. In the conversation, my students are not found to be creating new linking metaphors; they have not been exposed to the terminology or these ideas in this setting. Instead they can be viewed as uncovering for themselves possible linking metaphors; they appear to be in the act of isolating elements that would correspond to one another if a linking metaphor chart were to be written.

In one particular part of the discussion that is essential to their understanding of "successive," the students use a variety of examples to try and make explicit their understanding of this term as well as "contiguous." The definition given is "that is successive which is after the beginning (the order being determined by position or form or in some other way) and has nothing of the same class between it and that which it succeeds…" (Calinger, p. 86). The first example is offered by Schroeder. He states,

> Or if your class is a circle, would your first point be successive?

After receiving some challenges to the nature of the points on a circle he continues on,

> That's why I'm trying to be because he says that is successive which is after the beginning? Well, what if you are in a circle like you said and you pick your beginning point, go around the circle and get to your last point, then the beginning point would be successive to your last point.

Jim changes the context from a general circle to a circle with radian measure. This maneuver provides the students with labels for the points on the circle and seemingly a way of speaking about the first and last point. The context is then framed in terms of beginning at 0, moving around the circle, and arriving at 2π. The question is, Would 0 then be successive to 2π? No agreement is reached at this point. Schroeder wants to argue that 0 is successive to 2π because the point corresponding to 0 and 2π is both the beginning and the end. Jim, on the other hand, objects because the definition states, "that which is after the beginning...," and if 0 is the beginning it can't be successive.

Baron then changes the context again. He says that Aristotle is discussing one-dimensional objects only and thus they shouldn't be considering circles. The topic changes to lines and paths. Baron says about lines,

> Which even if it's screwy, as long as it can be—there's a place you're starting from and place you're going and they're not the same.
>
> Jim: What if I go to your house and back? Is that a circle?
>
> Baron: What's what I'm saying, that's two, that's two tracks....
>
> Schroeder: How is a circle different than a line, though, other than what we've been reading so far?...I mean, you can describe any part of a circle by just one coordinate, as long as you have a starting—you know, you can just, just do the circumference...

These exchanges indicate elements of linking metaphors. There are three different mathematical objects under consideration: the Euclidean circle, the circle with degree measure, and paths. If the circle with radian measure is thought to reside within trigonometry then each of these objects can be representative of three different subject areas (Euclidean geometry, trigonometry and graph theory) within mathematics. The students are not at this time looking at large-scale connections between and among the subject areas. This would involve them in full blown mathematical idea analysis of these areas, which is not the task before them. Nevertheless, their ability to focus on essential elements (the nature of the endpoints of a path and a circle), how these elements operate in different areas of mathematics, and their purposeful playing with the definitions of objects ("what makes a circle") all indicate their ability to do this type of work. They not only display knowledge of the different domains but also show that they can relate objects and relations in one area to corresponding objects and relations in another. Looking at the relations is essential in doing mathematical idea analysis.

In discussing the word "contiguous," another example arises. The definition from the text is given as "That which, being successive, touches, is contiguous…" (p. 86). Baron begins the discussion:

> Huh? Like right now, even though this end trigger says two dimensions (laughing) we're all pretty much successive. But if we all held hands, we'd be contiguous. Like if we were all like shoulder to shoulder.
>
> Schroeder: See now, that was seen before—I don't think people—I don't know, maybe I'm getting myself in trouble here, but I don't think we would have to consider us sitting here as two dimensions, because all you have to say is—the sixth person, and go 1, 2, 3, 4, 5, 6 and you know exactly who I'm talking 'bout. You only have to give one coordinate….
>
> Baron: So is he the 11th per—is he the first and the 11th? I mean, if you go around there's 10, 11, 12.
>
> Schroeder: What's the difference between one and two dimensions that affect that? It doesn't….
>
> Schroeder: So what if you're talking about like Z_3 or something, where you go 0, 1, 2, 0, 1, 2—you just keep going over and over again, so…
>
> Jim: That's a good point.
>
> Schroeder: That's the same as sitting in a triangle or something, or a square, or a triangle I guess it would be. You go around this way, and then, but you…

Although the word under consideration has changed, the item under inspection has not. The students are still concerned with the relationship between the endpoints on two- and one-dimensional objects. There are two mathematical domains, geometry and algebra, under consideration here. The students have matched the objects (points on the circles) to numbers in Z_3 and are comparing the order in each. As before, they are in the beginning stages of mathematical idea analysis.

Another noteworthy aspect here is Baron's use of an example involving humans and the relationship of touching to describe what Aristotle may be meaning by contiguous. With the four grounding metaphors, Lakoff and Núñez show how our innate mathematical abilities are grounded in physical activities. They make no mention of connections between more advanced mathematics and physical activities. It appears that Baron is involved in taking what appears to be an abstract definition at the outset and trying to locate it in an experiential context—a context that has more meaning to him and others. This is a type of metaphor which the authors would label as extraneous; it is similar to their example of a step function as an extraneous metaphor (p.53). I am tempted to disagree because Baron's use is to provide a rewording of a mathematical definition. There is a clear contrast to the metaphor of a step function, which is done simply because of the picture of the graph that is made and is not at all connected to the mathematical definition of the function. Baron's example considers the meaning of the definition. Perhaps it is somewhere in between a "legitimate" metaphor and an extraneous one.

Discretized Space

1) Baron: Like, I was thinking about, I don't know, I'm just throwing this out, I was thinking about, like if you put, if you put the natural numbers in a row, you've cut 'em, like, if you put 'em along the line and have like a unit between them so like here's 1, here's 2, here's 3, here's 4. You could like cut that line in half, it was like a string, you could cut that string in half in between and not be on one of the natural numbers. But if that represented the real number line, there's nowhere you could cut and not be and not be cutting and not be cutting a real number. Like on an exact real number. So to me, that seems to be a reason why the real numbers are continuous. Not with his definition, but just in general. There's nowhere you can't, you can't find—there are functions with holes that are continuous, but...

2) Baron: So, I mean if you have a string, I'm not saying that like, like you know the square root of two or like π or something you can like find π and cut it there. But I'm saying that if I cut it, if I cut the string, there's some real number that represents that length that I've cut, so I've cut on a real number.

As described in Chapter 3, discretized space is of a very different nature from naturally continuous space. Here those ideas will be revisited and portions of the transcript chosen to demonstrate that the students do have a working knowledge of discretized space. However they are struggling with a meaningful aspect of the Space is a Set of Points metaphor. In particular, they are working hard on the last line of the metaphor chart in trying to understand what properties of a line (with points) correspond to which properties of a set. According to Lakoff and Núñez, if this correspondence could have been made more explicit to the students much of their confusion may have been alleviated (p. 49).

Naturally continuous space is gapless and fluid as sensed and experienced by human beings. In the realm of mathematics prior to the discretization program, spatial objects such as lines, planes, etc. also possessed these attributes. Thus there were no points in naturally continuous space as a part of the space itself. In fact, Lakoff and Núñez tell us that a point was defined by Euclid as "that which has no part", and points existed only to indicate a particular location or intersection between spatial objects (p. 265). Discretized space, however, is quite different. It is composed of points, and those points aren't physical or spatial in nature but are representative of elements of a given set. If a line is to be composed of points in some way, we as human beings must have a cognitive way of making this happen. One of the problems is that there is no way of thinking that doesn't lead to a paradox or incongruence.

Using the Basic Metaphor of Infinity, the authors construct two ways in which human beings are capable of understanding what a point is (pp. 267-269). One is that a point is a disc with infinitesimal diameter and the second is as a disc of zero diameter. Consider the discs with infinitesimal diameter. If two of these were to be in contact with one another, how would that contact be described? The only possibility would be for it to be a point on the next smaller infinitesimal order—but then the same question could be asked of two of these points. This argument could continue indefinitely and there would never be any fixed definition for a point. The second case, a point with zero diameter, is a paradox within itself. By definition a disc has to have a diameter of positive length and thus this construction doesn't make sense if all of the information is drawn upon at the same time. Further problems arise when considering either of these two characterizations and asking if two points can be in contact with one another. For the infinitesimal disc, the side of one point would be in contact with the sides of the next. However, this does not fit with dimensionless conception of a point. For the zero-diameter discs, two of these can only be in contact if they are in fact the same point. In this case no measurable length could ever be created. In thinking of the question "Can a line be composed of points?" some combination of the above problems comes in to play. The transcript shows that the students are struggling with these questions along with fitting number into their thinking.

The relationships that we would want to ascribe amongst the points, touching for example, will have to be redefined in mathematical terms as relationships between the elements of the set. Consider the line as composed of points, the set of real numbers and the relationship of touching. Lakoff and Núñez argue that the real number line is a discretized mathematical object bearing little resemblance to the naturally continuous line. Through the work of Dedekind the real line only has points on it that correspond to a real number. Although the set of real numbers is uncountably infinite it is still incomparably small to the possible number of points on any given line. Regardless of the amount of points one may have, it will never be enough to fill or recreate the naturally continuous line (pp. 282-283). Furthermore, it is not possible for the points on a line to physically touch. As Aristotle and Lakoff and Núñez argue, if two points were to physically touch they would become the same point and thus not create any measurable length. Thus the notion of "touching" that is paradoxically meaningful in the naturally continuous world becomes alive in the "set with elements" world and has to be defined there. Recall that in a metaphorical blend, the Space-Set blend in this case, both the target and source domains are simultaneously active and characteristics in one take on new meaning in the other. In other words, "touching" as a relationship between points has to be defined as a relationship between the numbers within the set of real numbers. Instead of moving directly to this example there are examples that show that it is reasonable to infer that the students' thinking does include ideas of discretized space.

There is evidence within the transcript that the students think of spatial objects as being composed of points in some way. On the first page of the transcript, Sally finds Aristotle's argument opposing an idea that she has believed to be true.

> Something that I thought was interesting was that I thought you know that the lines were made up of like an infinite number of points and he says that that's not even possible, yeah.

Shortly afterwards Baron demonstrates a more complex understanding of the number line.

> The way I took it was he was saying that—his argument was basically that a line is of a different species than a point. It's not, a line is not composed of points—it contains an infinite number of points, but from points alone you can't generate a line.

This statement demonstrates Baron's understanding of Aristotle's argument. It also agrees with Lakoff and Núñez's distinction between naturally continuous space and discretized space. In the last sentence of his last statement, Baron introduces number. He draws a parallel between the points on a naturally continuous line and some set of numbers.

I: Okay.

Baron: Because it's like if you take two points and put them next to each other, because there's no they can't, if they're touching, then since they have no dimension, they're essentially the same point. And if they're not touching then there's a space between them so then it's not a continuous thing because they're not touching. And so then no matter how many you know an infinite number of points that you stack side by side, you're still staying at the same point so it's not like you're gonna be like going along 0-0-0-0-4.

In this portion of transcript it can be seen that the students do think of spatial objects as composed as points. There is some questioning as how to identify the points but no questioning of the circle being made up of points:

Schroeder: Or if your class is a circle, would your first point be successive?

Jim?: How do you have a first point on a circle?

R: We can give a circle a starting point. Would that point be successive to the last point?

R: I think it would be dependent on how you defined your circle.

This last snippet of transcript shows that now it may be quite natural for these students to think of geometrical figures as being composed of points. For them there appears to be inherent relationships between points and numbers, and also clear that a set of points may have a corresponding number line.

Schroeder: So what if you're talking about like Z_3 or something, where you go 0, 1, 2, 0, 1, 2 – you just keep going over and over again, so –

Jim: That's a good point.

Schroeder: That's the same as sitting in a triangle or something, or a square, or a triangle I guess it would be. You go around this way, and then, but you...

Jim: But if you look at a number line in Z_3, it goes 0, 1, 2, 3 – that's a straight line.

Penny: Well, we were talking about a straight line, yeah?

The above examples demonstrate some working knowledge of the first four lines of the A Space Is a Set of Points metaphor. The introduction of numbers by the students also alludes to the use of the final discretized space metaphor in which the source domain is the Space-Set Blend and the target domain is the set of numbers. The last line in the A Space Is a Set of Points metaphor is more difficult to illustrate.

A Space Is a Set Of Points

Source Domain A Set with Elements		Target Domain Naturally Continuous Space With Point Locations
A set	→	An n-dimensional space—for example, a line, a plane, a 3-dimensional space
Elements are members of the set	→	Points are locations in the space
Members exist independently of the sets they are members of.	→	Point-locations are inherent to the space they are located in.
Two set members are distinct if they are different entities.	→	Two point-locations are distinct if they are different locations.
Relations among members of the set	→	**Properties of space**

(p. 263)

Table 28
Space is a Set of Points Metaphor

Lakoff and Núñez do not explain the last line in the above metaphorical chart for their audience and I cannot do so either. However, the transcripts provide instances demonstrating that the students see these properties and relations as a worthwhile and necessary task. Looking at what it means for points to touch in naturally continuous space and how the students grapple with what this may mean in the set of numbers is an example of how a property of space is to be thought of in terms of a relation amongst members of a set. This is seen in the transcript where the students are in the midst of discussing the term "contiguous" from the Aristotle reading.

Baron begins the discussion with his interpretation of the definition.

> Huh?....[L]ike right now, even though this end trigger says two dimensions (laughing) we're all pretty much successive. But if we all held hands, we'd be contiguous. Like if we were all like shoulder to shoulder.

His example expresses very well what it would mean for two objects to touch in naturally continuous space. All that is necessary is for a side of one object to be in contact with the side of another object. He is using as an example all of the people in the room who are seated around a table. After a lengthy discussion on "successive," Baron agrees to a way of thinking that allows the arrangement of people to be situated successively. By suggesting that we all move closer together and hold hands or have the shoulder of one person touch the shoulder of the next, we would all become contiguous.

The conversation moves back to a discussion of "succession. In doing so the students discuss Z_3, the endpoints, and the starting and ending point of a circle, all to determine the meaning of contiguous, which depends on the interpretation of successive. At the end Lucy asks,

> Um, so are the integers contiguous?

Baron's immediate response is no and Schroeder's immediate response is yes. Schroeder immediately moves back to applying the definitions of successive and contiguous to the integers.

> You can't—if you're talking about the integers, you can't get from one to two with—okay, there is no integer between one and two, so they're contiguous.

Having not convinced anyone else, or himself, Schroeder continues on.

> Okay, okay, here it is! "That is successive which is after the beginning." So if we're talking about let's say the natural numbers, um, and we're talking about 3, so "that is successive which is after the beginning" so 3 is after the beginning, "...and has nothing of the same class between it and that which it succeeds." So um, we're not denying that...

The remaining students then press him to say how the integers are contiguous and he responds by saying,

> Because they're successive...because they, because that would mean successive, and they touch—1 and 2 are next to each other, they touch. Or are you saying you could have 1 and 2 next to each other [and not touch]?

The problem that Schroeder is having is that there is no way for him or the other students to think of what it would mean for two numbers to touch one another. It appears that they are using Aristotle's definitions of successive and contiguous in order to help make sense out of their present understandings of the number line. They are not aware that Aristotle is operating in naturally continuous space with an everyday meaning and use of "touching." Their thinking surrounding the number line was formed by using discretized space so they will need a different set of tools to work with. The conversation continues.

> R: Touch, the definition of touch means things whose extremes are together—"touch", that's what it says. Things whose extremes are together touch.
>
> Matt S: But then you get to the later argument where he's saying points don't have...can't touch.

> Baron: …and I think the extremes of 1 and 2 and 3 and 4 are points—they don't have dimensions.

Here the students have added in the physical nature of number, if it has one at all. If natural numbers are going to touch one another, then they have to be physical entities. This is seen in the re-reading of the definition of touch and taking this definition back to the argument on points touching. They are operating with the fully discretized version of the A Space Is a Set of Points metaphor. They are blending what they know about numbers with what they are reading about points and their relationships. This is seen most prominently in the phrase "…the extremes of 1 and 2 and 3 and 4 are points."

What the students are not doing above is taking the relationship of touching and trying to find a mathematical equivalent. This line of conceptualizing comes in when Baron brings in an example using townhouses and the following exchange takes place.

> Baron: They're objects and you're putting them next to each other. So they're touching, and there's no distance between them there's nothing of any class that you can put between them, but…
>
> Schroeder: So how is that different than the integers then? Why can't the integers be [contiguous]?
>
> Baron: I can put a real number between integers. I can put something of another class between them. I can't put, if it's contiguous there's nothing there's no class which you can shove in.

Baron's first statement is another that Lakoff and Núñez would argue exemplifies how mathematicians think. To define touching between two objects as there being zero distance between them is a mathematical act and is not the only option. This is the first attempt by the students to define in some mathematical way what it means to touch. The problem is that it too will lead to gaps in the line. Given any two distinct numbers, the distance between them is never zero; thus they never touch and the line should be riddled with gaps or holes. Later on Penny makes the following statement. This is his attempt to make sense out of "touching" for numbers.

> Okay, so if you want to be technical, you're using decimals
> to touch each other's hands, so...

This statement has many layers. First is the connection or substitution of hands for points on the number line. Next would be the thinking of numbers as points on the line. Penny appears to be working on how the numbers can be in contact with one another so that there aren't any gaps between them. Again he is working on what it could mean for two numbers to touch one another.

The Folk Theory of Essences in the Students' Examples

It is this folk theory that I find evident in the conversation between my students. In their discussions of the terms of Aristotle and their analysis of the real number line, they pay close attention to the definitions of Aristotle and to connecting the ideas throughout the conversation. They don't always look for what is necessary and/or sufficient however they do make a strong and generally successful effort to obtain the essence of what is being said and of what they understand. I am using "essence" here in the way in which Lakoff and Núñez use it and I describe above, and not as a more general aesthetic understanding.

As described at the beginning of the Mathematical Sense Making section, the folk theory of essence is a particular way of thinking. It includes

- Every specific thing is a kind of thing.
- Kinds are categories, which exist as entities in the world.
- Everything has an essence—a collection of essential properties—that makes it the kind of thing it is.
- Essences are causal; essences—and only essences—determine the natural behavior of things
- The essence of a thing is an inherent part of that thing.

This section argues that if the evidence of this type of thinking can be found in the students' comments, then they have embodied this folk theory as a way of operating when they do mathematics.

1) Jim: And there's another set outside of that with the real numbers with something between 1 and 2 if you move, if you look at integers as a subset of the reals. And you have something in between. But they're not between—I think that's why he defines between in the sense of a class. So if I say you know, ".5 is between 1 and 2" [should be 0 and 1], I can't say that they are in the sense of the class of integers. I have to define what between—I have to define what my class is if I say something is between. So if I say, ".5 is between 1 and 2," I have to say—I have to define if integers is my class...

2) Schroeder: Well, I was just thinking that her—to me the major problem with it—you say, "Okay, my set I'm gonna consider is the positive real numbers and then I'm only gonna go in multiples of ten," then you're not going to be continuous in terms of real numbers. You're jumping. So I would say then you can't talk about between because you're not moving continuously or if you want to talk about between then you have to some how create a set you're moving continuously through. And to do that, you have to and like make a set, and call it "multiples of ten" and then you're moving continuously and so then you can talk about between units. So you can say 10 is between 20 and 0, but I don't think you could say 5 is between 0 and 10 because it's not in the set that you're moving continuously through.

3) Jim: So we have to try to take everything that's in a circle and a triangle and square and somehow go to a line.

4) Lucy: Um, so are the integers contiguous?

5) Franklin: But then you get to the later argument where he's saying points don't have…

6) Baron: and I think the extremes of 1 and 2 and 3 and 4 are points—they don't have dimensions.

7) Baron: There's not this end of 5. Five is 5.

8) Lucy: Because people are objects. They're not the same thing as numbers.

The first two statements above demonstrate that the students do speak of mathematical entities as objects with a set of inherent characteristics. Jim uses his knowledge of the integers to demonstrate Aristotle's meaning of "between." Schroeder does him one better. First he speaks of continuity as a property of the reals. Next he goes on to define how moving "continuously according its nature" could be defined in terms of a set. The important thing to note is the students' ability to identify the particular characteristics that a set of numbers may have that are inherent because of the numbers that they are. One can't move continuously through the integers in multiples of 10, nor can this be done with the reals.

The third statement is a point in the conversation where Jim pauses and states the task at hand. This comment demonstrates that he hasn't lost track of the overall argument and that he believes the students need to go back and reapply what they have done to lines. It also shows that the comments around circles, squares and triangles were in contexts in which they argued with elements true to those figures. Jim points out that now it is time to move back and find the commonalities amongst the essences of the figures. In other words, there are things that are true for triangles and not squares and circles and vice versa. He is aware that even though this is the case there still may be some benefit to finding what can be generalized across all figures and in particular applied to lines. It is the recognition of the essences of each of the figures along with the awareness of how to make an argument that makes this an instance of the folk theory of essences.

The last five statements begin with Lucy's question about the contiguity of the integers. Embedded in this statement is the thinking of these abstract objects as physical objects that can be in contact with one another. Before the question was raised, the students had not acknowledged that they had been speaking as if objects were physical entities with their own essence. The raising of the question brings this conceptualization to their attention and they begin to analyze this notion for themselves. This analysis is shown in the association between numbers and points and how points are dimensionless. It ends with Lucy stating that numbers aren't objects in the same way that people are. Again, this excerpt shows that the students see mathematical entities in a manner consistent with the folk theory. The folk theory is somehow embedded into their thinking but it doesn't hinder them. They are able, once it has been brought to their attention, to work on these understandings as well.

When taken as a whole, this chapter shows that the students do possess many mathematical strengths. They are able and willing to engage complicated mathematical ideas; they are attentive enough in their discussion to investigate a variety of examples and take what was learned there back to the larger discussion; their conversations are populated with that the aspects that Lakoff and Núñez state as the means by which humans are able to do mathematical thinking. In fact, it does appear that mathematics has been embodied in the students in the sense of the authors' subtitle "How the Embodied Mind Brings Mathematics into Being."

CHAPTER 6

CONCLUSION

Final Thoughts

To recapitulate, the analysis presented in the last chapter used Lakoff and Núñez's conception of mathematical activity to argue that the students in this conversation were doing mathematics and possess mathematical strengths. In particular, using the work of the authors the following three elements can be found in the students' comments. These elements indicate that the students possess the ability to do mathematical thinking and also that they engage in this type of thinking while making sense out of foreign mathematical situations.

- Behavior or activity that demonstrates the embodiment of mathematics, for example, use of the Container schema or the Source-Path-Goal schema;
- Behavior or activity that demonstrates significant features of the metaphors (grounding and/or linking) and blends that Lakoff and Núñez set forth; and
- Behavior or activity that illustrates the folk theory of essences.

The fact that the students were able to do all of this around a historical text suggests possibilities for developments in mathematics teacher education.

The possibilities illuminated by this dissertation are some of its major results. These possibilities lie in the use of a non-traditional text, in this case a portion of Aristotle's *Metaphysics*, to access students' mathematical thinking. Using texts of this nature provides opportunities to work on students mathematical conceptions in connection with the chosen text. It provides the opportunity for students to think and make connections across different subject areas within mathematics. Conversations with students surrounding the content of a non-traditional text can allow instructors access to students' mathematical thinking; this can provide instructors more information when assessing students' strengths.

There are also possibilities created by using the work from cognitive science in particular that of Lakoff and Nunez. The ideas involved in the embodiment of mathematics provide a very plausible view of how humans are able to do and create abstract mathematics. The fact that mathematics is embodied in all human beings in the same manner may help to work against some students' beliefs that they are unable to do mathematics. Cognitive mechanisms, conceptual metaphor and blends, can also aid in helping students to believe in their ability to do mathematics. The use of conceptual metaphors and blends provides another lens into the mathematics that students may be studying at a given time. They assist students in differentiating between their own intuitions and specialized mathematically created entities (student's ideas concerning a naturally continuous line versus a discretized line for example). Being able to see mathematics as a created product can help a student separate their success or failure from right or wrong answers. If one can see the metaphors that are producing the mathematical ideas then right or wrong answers depend on the metaphors and not the innate mathematical ability of the student doing the mathematics.

The use of the *Metaphysics* allowed the students to bring to a conversation mathematical ideas that they held as truths and questions that they had. I can only speculate at this point but there are a few characteristics of the text that produced these results. The first is that the students were dealing with an argument that contradicted something that they believed to be true. This raised questions for individual students about the veracity of their thinking versus that of Aristotle's. Overall, the students appeared to be compelled to show that Aristotle was incorrect; tackling quite complicated mathematical ideas was no deterrent to them. This moved them into a discussion in which they needed to put forth their own mathematical thinking and understanding. The text was also written in a manner that was difficult for the students to readily understand. In trying to refute Aristotle they first had to understand the argument that he was making. It was in the discussion of Aristotle's terms that the students brought examples to bear that revealed their mathematical thinking. It also brought their questions and assumptions to the discussion and to the students' own awareness. Near the end of the conversation the students become aware of how they have been speaking of numbers as objects. Having this awareness for themselves could allow them to continue to examine their own mathematical understanding in the future. It moves their understanding from intuition and something that is taken for granted to an object that can be explored and studied.

The dissertation also shows that around a non-standard text like this one there may be possibilities to work on students' understanding of mathematics specific to the text being used and also to other topics that are introduced by the students in the course of discussing an article. Conversations can take place across different areas of mathematics that students have previously studied. Equivalence classes, number and its representation and mathematical definition are some of the topics that arose in this conversation in November and that could have been explored in more detail.

This sort of possibility is quite different from what happens in the mathematics courses now prevalent as part of teacher education. Having the students' thoughts and questions in the forefront allows an instructor to make use of them. Most mathematics programs are structured into separate courses with no obvious overlap. It is only recently that capstone courses have been introduced into programs and what these should consist of is still under debate. Using non-traditional texts with students appears to be one way that connections across subject areas within mathematics can be embedded into programs. Non-traditional texts create an environment in which students may naturally volunteer their own thinking about prior courses, and these ideas can be used in the immediate teaching and learning situation. One benefit to embedding this activity throughout programs instead of waiting until the end of a program might be that the practice of examining one's understanding as a part of the learning process will be something that students are learning to do and are likely to continue to do. Leaving this type of thinking until the end of a program may suggest to them that such examination isn't a valuable part of mathematics or the doing of mathematics.

At the beginning of this document I have used my own personal experiences to offer an image of some of the frustrations that students experience when studying mathematics. Some of these experiences I attribute to the nature of the subject; however, based on the excavation of students' strengths that Lakoff and Núñez have afforded me, I'd like to suggest that if we as educators can change our stance towards teaching and students' mathematical abilities, then there will be benefits for everyone.

On the utility of Lakoff and Núñez

Throughout the last chapter, I used the thinking of Lakoff and Núñez to articulate my preservice students' mathematical strengths. But, these ideas have utility for me beyond their use in analyzing the student conversation. When I began reading the Aristotle article with the students I had many questions about the real numbers and the real line and wanted to explore these ideas with my students. My thoughts and questions were as followed:

- Given a line and a unit of measurement it is easy to locate the rational numbers whose decimal expansions terminate on the line.
- Using this unit, how can the irrational numbers be located or placed?
- If each number has a place on the line and is connected to its length, how can an irrational number (or a non-terminating rational number) exist on the line? It seemed that an irrational number would need more space than a rational number because it is an unending number, i.e., an unending length.
- The assumption that the line is made out of points (an assumption I held at the time of reading the article) implies that all of the real numbers could be ordered in a way that would allow them to sit side by side on the line. This being the case, it made sense that I should be able to determine the points/numbers that would lie on each side of any given number, and this I knew wasn't possible.
- How is it possible to locate the number on the line when that number is unending, for example, locating π?

After reading Lakoff and Núñez my confusion has lifted and I am now able to pursue the understanding of the real numbers and the real line. What the authors have helped me to understand is that my thinking was combining two distinct conceptualizations of space. I was mixing naturally continuous space (NCS) and mathematical space, and my questions could not be answered in NCS. I became aware of my own thinking and learned the following:

- A line cannot be composed of points. Aristotle convinced me of this in NCS; however, I could not transfer the conception to my understanding of the real line because the opposite had been taught to me and continually used by my professors.
- A number is not a physical object. Reading the authors' text it became clear to me how I was able to think and speak of a number as a physical object in many different instances, and this was never problematic. The four grounding metaphors have helped me to understand where these ideas come from. Having this awareness may allow me to understand situations in the future where I may become confused.
- Even though points and numbers are not physical entities there is still a difference between a point and a number. I had not only associated a point with a number but at times thought of points as numbers and numbers as points; the authors refer to these as number-points. Understanding the difference between the two and where the overlap comes from helps me to separate ideas that are true for points from ideas that are true for numbers. It isn't that these two worlds are always distinct; in fact much of mathematics is located within their overlap. Knowing that the ideas are a combination of two things helps me to keep straight what is true for one but not the other and when we as humans use one to make sense or move forward in the other.

- When we speak of points on a line touching they do not do so in a way consistent with physical objects – points sharing a common space. The authors' metaphors detailing discretized space are very helpful. They highlight the difference between what I intuitively know as a human being and what I am working with in formal mathematics. Knowing that mathematical space has been reconceptualized as a set of points (discretized space) provided me with a different terrain in which to work out my mathematical thinking. It lessens my expectations for ideas to make sense with what I know to be true about NCS through my embodied experiences. Instead of searching for examples based on NCS, I can now focus my attention on objects and relations that exist in this new space. Being aware of the two, NCS and discretized space, and that they will overlap allows me in those moments to know that I am trying to apply a tool that isn't made for that particular situation (the natural idea of touching to the ideas of numbers on a line).
- Mathematical points can be any mathematical element. Although we often use points to indicate a position or object in mathematics, the mathematical point in discretized space is a representative of some other mathematical entity in a set.

This new perspective provided to me by Lakoff and Núñez allows me to make sense out of my own thoughts. I summarize the changes in the following table.

My Thinking Prior to Lakoff and Núñez	My Thinking After Lakoff and Núñez
Given a line and a unit of measurement, it is easy to locate the rational numbers whose decimal expansions terminate on the line.	This is an act that makes sense in NCS and is very much connected to the Measuring Stick metaphor.

Table 29 (cont'd)

Using this unit, how can the irrational numbers be located or placed?	Locating the irrational numbers in this way does not make sense in NCS. Extending to irrationals what is known and experienced as true with rational numbers arises as an entailment of the blend of the Measuring Stick metaphor.
• If each number has a place on the line and it is connected to its length, then how can an irrational number (or a non-terminating rational number) exist on the line? It seemed that it would need more space than a rational number because it is an unending number, i.e. an unending length. • How is it possible to locate a number on the line when that number is unending, π, for example?	Points on a line are not the actual numbers or the endpoint of the segment of its length. The point is merely a representative of the number and has no characteristics in common with the number. Thus there has to be some other relationship that determines where on the continuum/line the point for π will lie.
The assumption that the line was made out of points (an assumption I held at the time of reading the article) implies that all of the real numbers could be ordered in a way that would allow them to sit side by side on the line. This being the case, it made sense that I should be able to determine the points/numbers that would lie on each side of any given number, and this I knew wasn't possible.	The distinction between NCS and discretized space allows me to see that my notion of a line being composed of points is a separate conceptualization than the real line being composed out of the real number-points. I wanted the numbers to behave exactly as I thought the points did. What Lakoff and Núñez have done for me is to show that the real line is an image of the geometric (Euclidean) line. Just as there is a relation detailing the behavior of points on a line, there is a parallel relationship detailing numbers, order and "touching." This relation is a mathematical relationship which now that I am aware of its existence and its role I can pursue an understanding of it if I choose.

Table 29
Change in Author's Thinking

The above table describes the changes in my thinking about points, lines, the real numbers and the real line. I can now differentiate between NCS and discretized space. I no longer have an expectation that numbers on the real line will be situated as points can be on a Euclidean line. The notion of ordering that Dedekind describes (see Chapter 2) for points on a line is mirrored in the real numbers but by some other mathematical relationship. As there is a natural way to order points, there is a mathematical way of ordering numbers, and there are different types of orderings, partially ordered sets and well ordered sets. Pursuing these mathematical ideas may be the place where more answers will lie for me. Whether they are there or not isn't as important as my awareness and understanding of what to look for and where I might find more answers.

Assuming that college students in mathematics are unaware of these ideas from Lakoff and Núñez (a likely prospect), my experience raises the question of whether introducing these elements of "mathematical embodiment" to the students would be useful to them. It might allow them to look at mathematics through a different lens and allow them to analyze their own understanding in a context different from trying to understand particular mathematical ideas. They would then have the opportunity to see where their current thinking fits into a body of knowledge and then decide for themselves where their confusion lies or what questions they still have. This might provide an opportunity for students to make choices and/or moves in order to improve their own mathematical understanding. They would also be able to perceive their so-called misconceptions as reasonable and even mathematically correct.

To conclude, as we imagine changing the mathematical experiences of preservice teachers, we may find important and valuable resources in unexpected places, like the mathematical thinking of noted philosophers or the psychological theories of cognitive psychologists. These sources have the potential of moving the thinking of educators concerning what may count for positive and fruitful mathematical activity. They also have the potential of creating a more encompassing learning experience for preservice mathematics students. As a learner of mathematics I was disheartened by the lack of opportunities to have mathematics make sense to me. As a student of education I was disheartened by the evaluation of mathematics students' abilities to think and do mathematics. This study has provided me with hope. I now have hope that there are ways in which mathematics can be made more accessible and meaningful throughout all stages of mathematics programs. I also have hope that if educators continue to look to areas of cognitive science that the assessment of students' abilities can change for the better. This may lead to an ultimate goal of increasing the mathematical comprehension and abilities of students.

APPENDIX

From the Metaphysics (1068b – 1069a) and Physics (230a – 240a)*

(On the Continuous and Zeno's Paradoxes)

—ARISTOTLE

*SOURCE: From W. D. Ross, ed., Aristotle: Selections (1995), 88 – 89.

Things which are in one place (in the strictest sense) are *together in place*, and things which are in different places are *apart*. Things whose extremes are together *touch*. That at which the changing thing, if it changes continuously according to it nature, naturally arrives before it arrives at the extreme into which it is changing, is *between*. That which is most distant in a straight line is *contrary in place*. That is *successive* which is after the beginning (the order being determined by position or form or in some other way) and has nothing of the same class between it and that which it succeeds, e.g., lines succeed a line, units a unit, or one house another house. (There is nothing to prevent a thing of some *other* class from being between.) For the successive succeeds something and is something later; "one" does not succeed "two," nor the first day of the month the second. That which, being successive, touches, is *contiguous*. Since all change is between opposites, and these are either contraries or contradictories, and there is no middle term for contradictories, clearly that which is between is between contraries. The *continuous* is a species of the contiguous or of that which touches; two things are called continuous when the limits of each, with which they touch and are kept together, become one and the same, so that plainly the continuous is found in the things out of which a unity naturally arises in virtue of their contact. And plainly the successive is the first of these concepts; for the successive does not necessarily touch, but that which touches is successive. And if a thing is continuous, it touches, but if it touches, it is not necessarily continuous; and in things in which there is no touching, there is no organic unity. Therefore a point is not the same as a unit; for contact belongs to points, but not to units, which have only succession; and there is something between two of the former, but not between two of the latter....

If the terms "continuous," "in contact," and "in succession" as defined above—things being "continuous" if their extremities are one, "in contact" if their extremities are together, and "in succession" if there is nothing of their own kind intermediate between them—nothing that is continuous can be composed of indivisibles; e.g., a line cannot be composed of points, the line being continuous and the point indivisible. For the extremities of two points can neither be *one* (since of an indivisible there can be no extremity as distinct from some other part) *nor together* (since that which has no parts can have no extremity, the extremity and the thing of which it is the extremity being distinct).

Moreover if that which is continuous is composed of points, these points must be either *continuous* or *in contact* with one another: and the same reasoning applies in the case of all indivisible. Now for the reason given above they cannot be continuous: and one thing can be in contact with another only if whole is in contact with whole or part with part or part with whole. But since an indivisible has no parts, they must be in contact with one another as whole with whole. And if they are in contact with one another as whole with whole, they will not be continuous; for that which is continuous has distinct parts; and these parts into which it is divisible are different in this way, i.e., spatially separate.

Nor again can a point be in *succession* to a point or a moment to a moment in such a way that length can be composed of points or time of moments; for things are in succession if there is nothing of their own kind intermediate between them, whereas that which is intermediate between points is always a line and that which is intermediate between moments is always a period of time.

Again, if length and time could thus be composed of indivisibles, they could be divided into indivisible, since each is divisible into the parts of which it is composed. But, as we saw, no continuous thing is divisible into things without parts. Nor can there be any thing of another kind intermediate between the points or between the moments; for if there could be any such thing it is clear that it must be either indivisible or divisible, and if it is divisible it must be divisible either into indivisibles or into divisibles that are infinitely divisible, in which case it is continuous.

Moreover it is plain that everything continuous is divisible into divisibles that are infinitely divisible; for if it were divisible into indivisibles, we should have an indivisible in contact with an indivisible, since the extremities of things that are continuous with one another are one, and such things are therefore in contact. (Calinger, pp.85 – 87).

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