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The Existence of Global Solutions of a Variational Nonlinear Wave Equation

By

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ABSTRACT

The Existence of Global Solutions of a Variational Nonlinear Wave Equation

By

Tae-Wan Park

We analyze the existence of smooth global solutions of a variational nonlinear wave equation which originates from the modeling of orientation waves in a massive nematic liquid crystal director field. We prove that the equation has the global solution in three space dimensions and smooth solutions develop singularities in finite time in one space dimension. The method depends on detailed analysis of solutions along the nonlinear characteristics and careful selection of initial data.

To my family.

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Introduction

During the past three decades, the initial value or Cauchy problem has long played the central role in the theory of evolutionary differential equations, which describe many fundamental physical processes of interaction. The Cauchy problem has been studied extensively with considerable success. In spite of a great deal of recent activity, many physically and mathematically important difficult problems remain. In addition, mathematically quite interesting and physically significant questions remain even when global existence and unicity have been well established. Among the most interesting problems of this type are those of the asymptotic behavior, regularity and stability of solutions.

This dissertation is focused on the nonlinear wave equation

$$u_{tt} - C(u)\nabla \cdot (C(u)\nabla u) = 0, \tag{1}$$

where the wave speed C is given positive function of u. In particular, we study carefully the one dimensional version of the equation (1) which can be written as

$$u_{tt} - C(u)(C(u)u_x)_x = 0. \tag{2}$$

One motivation for studying (2) comes from the theory of liquid crystals. This equation (2) has been derived by Saxton [44] for the director field in a simplified situation of a nematic liquid crystal in the regime where inertia of the director field dominates dissipation. That was based on the previous result by Leslie [32] and Ericksen [4] which studied the strain of liquid crystal. Further details can be found in [44], [8], and [9].

We give a brief explanation of how the equation arises in that context, and how the liquid crystal problem differ from the related problem of wave maps from Minkowski space to the two-sphere. Suppose $\mathbf{n} : \mathbb{R}^3 \to \mathbf{S}^2$ maps three dimensional Euclidean space into the two sphere. We define the "energy" functional

$$\nu[\mathbf{n}] = \frac{1}{2}\int |\nabla \mathbf{n}|^2 dx.$$

Harmonic maps are solutions of the constrained variational principle

$$\frac{\delta \nu}{\delta \mathbf{n}} = 0, \quad \mathbf{n} \cdot \mathbf{n} = 1.$$

The associated Euler-Lagrange equation is

$$\Delta \mathbf{n} + |\nabla \mathbf{n}|^2 \mathbf{n} = 0.$$

There are two time dependent partial differential equations naturally associated with this functional. One is the parabolic gradient flow

$$\mathbf{n}_t = \Delta \mathbf{n} + |\nabla \mathbf{n}|^2 \mathbf{n},$$

and the other one is hyperbolic Hamiltonian system

$$\mathbf{n}_{tt} = \Delta \mathbf{n} + (|\nabla \mathbf{n}|^2 - \mathbf{n}_t^2)\mathbf{n}. \tag{3}$$

This hyperbolic partial differential equation is the Euler-Lagrange equation of the action principle

$$\frac{\delta}{\delta \mathbf{n}} \int \{\mathbf{n}_t^2 - |\nabla \mathbf{n}|^2\} dx dt = 0, \quad \mathbf{n} \cdot \mathbf{n} = 1,$$

for $\mathbf{n} : \mathbb{R}^1 \times \mathbb{R}^3 \to \mathbf{S}^2$.

A related and more general energy functional is from liquid crystal. In this case, let $\mathbf{n}$ be an unit director field. The well-known Oseen-Frank energy functional in nematic state [6] is given by

$$W[\mathbf{n}] = \frac{1}{2} \int W(\mathbf{n}, \nabla \mathbf{n}) dx,$$

where

$$W(\mathbf{n}, \nabla \mathbf{n}) = \alpha |\mathbf{n} \times (\nabla \times \mathbf{n})|^2 + \beta (\nabla \cdot \mathbf{n})^2 + \gamma (\mathbf{n} \cdot \nabla \times \mathbf{n})^2,$$

the positive constants α, β, γ are elastic constants of the liquid crystal. In the special case $\alpha = \beta = \gamma$, the Oseen-Franck energy reduces to the harmonic map energy.
The form of the Oseen-Franck energy is determined (up to a null Lagrangian) by the requirement that it is invariant under reflections $\mathbf{n} \to -\mathbf{n}$ and under simultaneous rotations R of the spatial variables and the director field, $x \to Rx, \mathbf{n} \to R\mathbf{n}$. The harmonic map energy functional has a larger symmetry group, since it is invariant under independent orthogonal transformations, $x \to Rx, \mathbf{n} \to S\mathbf{n}$, of the domain and target spaces.

There are three partial differential equations naturally associated with the Oseen-Franck energy,

$$\begin{aligned} -\frac{\delta W}{\delta \mathbf{n}} + \lambda \mathbf{n} &= 0, \\ \mathbf{n}_t &= -\frac{\delta W}{\delta \mathbf{n}} + \lambda \mathbf{n}, \\ \mathbf{n}_{tt} &= -\frac{\delta W}{\delta \mathbf{n}} + \lambda \mathbf{n}. \end{aligned} \tag{4}$$

Here, λ is a suitable Lagrange multiplier which preserves the constraint that $\mathbf{n}$ is a unit vector. Our interest here is in the hyperbolic equation, the last equation in (4). This equation is the "liquid crystal" analog of the wave map equation (3). It has the action principle

$$\frac{\delta}{\delta \mathbf{n}} \int \{\mathbf{n}_t^2 - W(\mathbf{n}, \nabla \mathbf{n})\} dxdt = 0, \quad \mathbf{n} \cdot \mathbf{n} = 1.$$

A principle theme of our work is that the qualitative properties of (4) are completely different from those of (3).

The simplest interesting class of solutions of (4) consists of planar deformations depending on a single variable x [44]. The director field then has the form

$$\mathbf{n}(x,t) = \cos u(x,t) e_x + \sin u(x,t) e_y.$$

Here, the dependent variable $\mathbf{n} \in \mathbf{S}^1$ measures the angle of the director field to the $x-$direction, and e_x and e_y are the coordinate vectors in the x and y directions, respectively. Then

$$\mathbf{n}_t^2 = |-u_t \sin u e_x + u_t \cos u e_y|^2 = u_t^2,$$

$$\nabla \cdot \mathbf{n} = -u_x \sin u,$$

and

$$\begin{aligned} \nabla \times \mathbf{n} &= (-u_z \cos u) e_x + (-u_z \sin u) e_y + (u_x \cos u + u_y \sin u) e_z \\ &= (u_x \cos u + u_y \sin u) e_z. \end{aligned}$$

So

$$(\nabla \cdot \mathbf{n})^2 = u_x^2 \sin^2 u,$$

$$\begin{aligned} (\mathbf{n} \cdot \nabla \times \mathbf{u})^2 &= (-u_z \cos^2 u - u_z \sin^2 u)^2 \\ &= (-u_z)^2 = u_z^2 = 0, \end{aligned}$$

and

$$|\mathbf{n} \times (\nabla \times \mathbf{u})|^2 = u_x^2 \cos^2 u(\sin^2 u + \cos^2 u) = u_x^2 \cos^2 u$$

Finally,

$$W(\mathbf{n}, \nabla \mathbf{n}) = \alpha u_x^2 \cos^2 u + \beta u_x^2 \sin^2 u.$$

In this case, the action principle for $\mathbf{n}$ reduces to the one space dimension version of

$$\frac{\delta}{\delta u} \int \{u_t^2 - C^2(u) u_x^2\} dx dt = 0$$

with the wave speed C given by

$$C^2(u) = \alpha \cos^2 u + \beta \sin^2 u.$$

The corresponding wave equation is (2).

Equation (1) also looks very similar to the perturbed wave equation

$$u_{tt} - C\Delta u + f(u, Du, D^2 u) = 0. \tag{5}$$

Blow-up for (5) has been studied extensively by Levine [33], John [16], Glassey [7], Sideris [47], [48], Schaeffer [45], Kato [22], Hanouzet and Joly [10], Balabane [1], and others, using integral methods. For the proof of the global existence, Klainerman ([23] and [24]) used the Nash-Moser-Hörmander iteration scheme to get approximate solutions to the Cauchy problem and establish the global convergence of approximate solutions on $t \geq 0$. Klainerman [26], Klainerman and Ponce [27] and Matdumura [41] adopted another method, namely the extension method of local solutions to get the global classical solution. This method can be essentially made in two steps. The first step is to show the local convergence of approximate solutions on a domain locally in time and get the local classical solution, and the second step is to establish some uni-form estimates for local solutions to the original nonlinear problem, and consequently the local solution can be extended to a global solution. This method is simpler than the previous one, especially in the case that the local existence is well known. Li and Chen [34], Li and Yu [37] used other methods. That method based only on the decay estimate for solutions to the corresponding linear homogeneous equation and on the existence and the energy estimate for solutions to the corresponding linear inhomogeneous equations, they got the global existence or the lower bound of the lifespan of classical solutions to the Cauchy problem with small initial data for nonlinear wave equations. Moreover, the global classical solution, if any, has the same decay property when $t \to +\infty$ as the solution to the corresponding linear homogeneous equation.

In addition, Belchev, Kepka and Zhou [2] proved the global existence of solutions to nonlinear wave equations using the Penrose conformal compactification method in 1999. They use new method which transforms the equations from Minkowski space to the Einstein universe in order to change the global existence question to the local one.

The blow-up for (1) without critical points of speed, to the first order, was studied by Glassey, Hunter, and Zheng [8] using the characteristic method. Recently, Zhang and Zheng [51] show the weak global solution of (2) with some smoothness condition on the initial data.

In 1995, Hunter and Zheng proved that the asymptotic equation

$$(u_t + uu_x)_x = \frac{1}{2}u_x^2 \tag{6}$$

has global continuous weak solutions, even though smooth solutions of (6) break down [14] and [15].

Equation (6) comes from an informal approximation of (2), when we look for a weakly nonlinear asymptotic solution of (2) of the form

$$u(x,t;\epsilon) = u_0 + \epsilon u_1(\theta,\tau) + O(\epsilon^2), \tag{7}$$

with $\theta = x - C_0 t$, $\tau = \epsilon t$, and $C_0 = C(u_0) > 0$ is the unperturbed wave speed. Looking at the coefficients of ϵ^2 in the expansion implies

$$(u_{1\tau} + C_0' u_1 u_{1\theta})_\theta = \frac{1}{2} C_0' u_{1\theta}^2. \tag{8}$$

Here, $C_0' = C'(u_0)$. Assuming that $C_0' \neq 0$, the change of variables $u = C_0' u_1, x = \theta$, and $t = \tau$ transforms (8) into (6). In (6), (x,t) are not the original space-time variables; instead x is the space variable in a reference frame moving with the unperturbed wave speed, and t is a slow time.

The results on (6) suggest that (2) also has global continuous weak solutions. However, this question remains open, and recently Zhang and Zheng show the existence of global weak solution to (2) with special Cauchy data [51].

In 1996, Glassey, Hunter, and Zheng [8] showed the singularity of the solution to (2) without the critical points of the wave speed $C(u)$. There, they used the method of characteristics instead of the energy method. Unlike energy methods, this approach only works in the case of one space dimension. That method is one of the important motivation of this dissertation, here I also use the characteristic method to analyze the behavior of solutions.

This dissertation will be organized as follows.

In Chapter 1, we discuss about the history and back ground of the perturbed wave equation, and the existence of smooth global solution of (1) with small general smooth initial data in high dimensions.

Especially we will prove the follow theorem.

Theorem 0.1 *Assume that $C(u) \in C^2(\mathbb{R})$ and $u_0 \in \mathbb{R}$ such that $C(u_0) > 0, C'(u_0) = 0$. Then the initial value problem*

$$\begin{cases} u_{tt} - C(u)\nabla \cdot (C(u)\nabla u) = 0, \\ u(0,x) = u_0 + \epsilon f(x), \quad u_t(0,x) = \epsilon g(x), \end{cases}$$

always has a global C^∞ solution if $\epsilon > 0$ is small enough, when $n \geq 3$.

In Chapter 2, we will consider the non-existence of the smooth global solution of (2) for general smooth initial data using the method of characteristics [40]. This approach only works in the case of one space dimension at the moment.
Especially we will prove the follow theorem.

Theorem 0.2 *Assume that $C(u) \in C^2(\mathbb{R})$ satisfies the following:*

(a) there exist positive constants $0 < C_0 < C_1 < \infty$ such that $C_0 \leq C(u) \leq C_1$ for all $u \in \mathbb{R}$,

(b) $C'(u_0) = 0$ and $C''(u_0) \neq 0$.

Suppose that $u(t,x) \in C^2([0,T) \times \mathbb{R})$ is a smooth solution of the initial value problem,

$$\begin{cases} u_{tt} - C(u)(C(u)u_x)_x = 0, \\ u(0,x) = u_0 + \epsilon\phi(\frac{x}{\epsilon}), \\ u_t(0,x) = -sign(C'(u_0 \pm \delta))C(u(0,x))u_x(0,x), \end{cases}$$

where $\epsilon > 0$ is sufficiently small, $\phi \in C^1(\mathbb{R})$ with $\phi \not\equiv 0, |\phi| < 1$, and $\phi''(a) \neq 0$ when $\phi'(a) = 0$.

Then $T < \infty$ for some ϕ, i.e. a global smooth solution does not exist.

This part is motivated by the work of Glassey, Hunter, and Zheng [8].

In Chapter 3, we will show that the space dimension $n = 2$ is the critical dimension of the existence of a global classical solution of a variational nonlinear wave equation. Here we will discuss the difficulties for energy method and characteristic method. Even though we could not prove the development of singularity rigorously at the moment, we indicate the possibilities. The study will be continued after the completion of this dissertation.

CHAPTER 1

The existence of global smooth solution in high space dimensions

To establish the global existence, we compare the variational wave equation (1)

$$u_{tt} - C(u)\nabla \cdot (C(u)\nabla u) = 0$$

to the perturbed wave equation (5)

$$u_{tt} - C\Delta u + f(u, Du, D^2u) = 0.$$

During the past three decades, the perturbed wave equation was studied extensively. There are lots of results about that wave equation, even though many physically and mathematically important difficult problems remain.

In the first section we mention some related existence theorems of the perturbed wave equation. And in the second section we show the existence result about the variational wave equation.

1.1 Motivation and history

Throughout this section, we will consider the initial value problem for nonlinear wave equations

$$\begin{cases} \Box u = F(u, Du, D_x(Du)), & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ u(0,x) = \epsilon\phi(x), u_t(0,x) = \epsilon\psi(x), x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where

$$\Box = \frac{\partial^2}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

is the usual wave operator,

$$D_x = (\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}), D = (\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}).$$

and $\phi, \psi \in C_0^\infty(\mathbb{R}^n)$, $\epsilon > 0$ is a small parameter.

Let

$$\hat{\lambda} = (\lambda, \beta, \gamma),$$

with $\lambda \in \mathbb{R}, \beta = (\beta_0, \beta_1, \cdots, \beta_n) \in \mathbb{R}^{n+1}, \gamma = (\gamma_{ij}), i = 0, 1, \cdots, n,$ and $j = 0, 1, \cdots, n.$

Suppose that in a neighborhood of $\hat{\lambda} = 0$, say, for $|\hat{\lambda}| \leq 1$, the nonlinear term $F = F(\hat{\lambda})$ in (1.1) is a sufficiently smooth function satisfying

$$F(\hat{\lambda}) = o(|\hat{\lambda}|^{1+\alpha}), \tag{1.2}$$

where α is an integer ≥ 1.

First consideration of the existence is the special case where the nonlinear term F in (1.1) does not explicitly depend on u;

$$F = F(Du, D_x Du). \tag{1.3}$$

Based on the decay estimates on the $L^\infty(\mathbb{R}^n)$ norm and the energy estimates for solutions to the Cauchy problem for wave equations, Klainerman [23] used the Nash-Moser-Hörmander iteration scheme in 1980 to first prove the existence and uniqueness of global classical solutions with small initial data for $\alpha = 1$, and $n \geq 6$. Furthermore, the solution possesses some decay properties when $t \to +\infty$.
Three years later, under hypothesis

$$(n-1)/2 > (1+1/\alpha)/\alpha, \tag{1.4}$$

Klainerman [24] used same method to generalize his result to the general case that α is an integer ≥ 1. In particular, when $\alpha = 1$, (1.4) reduces to $n \geq 6$. Moreover, based on the decay estimates on the $L^q(\mathbb{R}^n)$ norm ($q > 2$) of solutions to the Cauchy problem for wave equations, Klainerman and Ponce [27] used the extension method of local solutions in 1983 to recover the same result for the general case $\alpha \geq 1$. A simpler proof was given by Shatah [46] in 1982.

The relationship between n and α given by (1.4) can be explicitly expressed in the table

$\alpha =$	1	2	$3, 4, \cdots$
$n \geq$	6	3	2

When $\alpha = 1$, the restriction $n \geq 6$ on the space dimension n is not optimal, and Klainerman [26] actually improved it in 1985. Based on the Lorentz invariance of the wave operator, he successfully used the extension method of local solutions to get the global existence theorem under hypothesis for $n \geq 4$, instead of $n \geq 6$.

In a similar way as in Klainerman [26], we can prove that the lifespan $\tilde{T}(\epsilon) = +\infty$, i.e., there exists a unique classical solution to Cauchy problem (1.1) (with (1.3)) on $t \geq 0$, if

$$n > 1 + \frac{2}{\alpha}. \tag{1.5}$$

namely, there is a relationship between α and n as follows:

$\alpha =$	1	2	$3, 4, \cdots$
$n \geq$	4	3	2

This result coincides with the previous table given by (1.4) when $\alpha \geq 2$.

Generally speaking, as a restriction on the space dimension, hypothesis $n \geq 4$ is necessary. In fact, John [17] has proved that, when $n = 3$, any nontrivial C^3 solution to the Cauchy problem

$$\begin{cases} u_{tt} - \Delta u = u_t^2, \\ u(0,x) = \phi(x), u_t(0,x) = \psi(x), x \in \mathbb{R}^n \end{cases}$$

must blow up in a finite time, provided that the initial data have a compact support. Moreover, Sideris [49] has also pointed out that, if the initial data are not small, then classical solutions may blow up in finite time no matter what the space dimension is.

In the special but important case $n = 3$, since classical solutions may blow up in a finite time even for small initial data, it is important to estimate the lifespan $\tilde{T}(\epsilon)$ of classical solutions.

Using a method based on an asymptotic expansion of the solution in powers of ϵ, John [19] proved that

$$\lim_{\epsilon \to 0} \epsilon^N \tilde{T}(\epsilon) = +\infty$$

for any integer $N > 0$, namely, the lifespan increases at least like a polynomial of ϵ^{-1} as $\epsilon \to 0$. Moreover, for solutions with spherical symmetry, it has been proved, in the semilinear case $F = F(Du)$ by John [17] and Sideris [47], and in the general case $F = F(Du, D_x Du)$ by Klainerman [25], that the lifespan

$$\tilde{T}(\epsilon) \geq \exp\{a\epsilon^{-1}\}, \tag{1.6}$$

for small $\epsilon > 0$ and a positive constant a.

Finally, John and Klainerman [21] and Klainerman [26] proved that (1.6) is still valid for general solutions to the Cauchy problem under consideration, and they referred to solutions of this kind as almost global solutions.

Furthermore, for the case $n = 2$, Kovalyov [30] proved that

$$\tilde{T}(\epsilon) \geq \begin{cases} b(\epsilon \ln \epsilon)^{-2}, & \text{if } \alpha = 1, \\ \exp\{a\epsilon^{-2}\}, & \text{if } \alpha = 2, \end{cases}$$

where a and b are positive constants.

A complete analysis on the lifespan of classical solutions to Cauchy problem (1.1) with (1.3) can be found in Hömander [11] and Li Ta-tsien and Yu Xin [36].

The lower bound of the lifespan of classical solutions to (1.1) (with (1.3)) for all integers α, n with $\alpha \geq 1$ and $n \geq 1$ can be summarized as follows:

$$\tilde{T}(\epsilon) \geq \begin{cases} +\infty, & \text{if } n > 1 + \frac{2}{\alpha}, \\ \exp\{a\epsilon^{-\alpha}\}, & \text{if } n = 1 + \frac{2}{\alpha}, \\ b\epsilon^{-\alpha/(1-K_0)}, & \text{if } n < 1 + \frac{2}{\alpha}, \end{cases} \tag{1.7}$$

where $K_0 \equiv \frac{n-1}{2}\alpha$ and a, b are positive constants only depending on α and n. We may outline this result in the following table

$\tilde{T}(\epsilon) \geq$ α \ n	1	2	$\cdots$	α	$\cdots$
1	$b\epsilon^{-1}$	$b\epsilon^{-2}$	$\cdots$	$b\epsilon^{-\alpha}$	$\cdots$
2	$b\epsilon^{-2}$	$\exp\{\alpha\epsilon^{-2}\}$		$+\infty$	
3	$\exp\{\alpha\epsilon^{-1}\}$				
4,5,$\cdots$					

Thus, we recover the results mentioned above with some improvements, for instance, in the case $n = 2$ and $\alpha = 1$.

We point out that all lower bounds in this table, except the case $n = 2$ and $\alpha = 2$, are known to be sharp because of Lax [31] (for $n = 1$ and $\alpha = 1$), John [20] (for $n = 2, 3$ and $\alpha = 1$), Kong De-xing [29] (for $n = 1$ and $\alpha \geq 1$) and Zhou Yi [52] (for $n \geq 1$ and odd $\alpha \geq 1$).

Now we turn to the general case where the nonlinear term F may depend on u, *i.e.*,

$$F = F(u, Du, D_x Du).$$

Since the L^2-norm of the solution to the wave equation cannot be estimated by the standard energy method, the problem becomes more complicated and thus we need some more refined estimates and analysis in order to get the lower bound of the lifespan of classical solutions.

Matsumura used the extension method of local solutions in his thesis [41] to consider the following Cauchy problem for a special kind of quasilinear wave equations

$$u_{tt} - \Delta u = \sum_{i,j=1}^{n} b_{ij}(u, Du)u_{x_i x_j} + F(u, Du)$$

He proved the global existence of classical solutions for small $\epsilon > 0$, provided that there is a relationship between n and α as follows

$\alpha =$	2	3	$4, 5, \cdots$
$n \geq$	4	3	2

For the most important case $\alpha = 1$, by means of the conformal mapping from $\mathbb{R}^{n+1}$ to $\mathbb{R} \times \mathbb{S}^n$, Christodoulou [3] obtained in 1986 the corresponding global existence of classical solutions under the hypothesis that n is an odd integer ≥ 5.

In 1988, Li Ta-tsien and Chen Yun-mei [34] used a simple method. This method avoid the use of the Nash-Moser-Hömander technique as well as the use of the extension of local solutions. Based only on the decay estimate for solutions to the corresponding linear homogeneous equation and on the existence and the energy estimate for solutions to the corresponding linear inhomogeneous equations, one can get directly the global existence or the lower bound of the lifespan of classical solutions to the Cauchy problem with small initial data for nonlinear wave equations. Moreover, the global classical solution (if any) will have the same decay property when $t \to +\infty$ as the solution to the corresponding linear homogeneous equation. By this method, they eliminated the restriction that n must be an odd integer in the result of Christodoulou [3], and their results in the general case $n \geq 3$ and $\alpha \geq 1$ can be summarized as follows:

$\tilde{T}(\epsilon) = +\infty$, if $K \equiv \frac{n-1}{2}(1 - \frac{2}{\alpha n})\alpha > 1$. Namely, if there is a relationship between α and n as shown in the table

$\alpha =$	1	2,3,$\cdots$
$n \geq$	5	3

Recently, Hörmander [12] (for $n = 4$) and Lindblad [39] (for $n = 3$) proved respectively that the lifespan

$$\tilde{T}(\epsilon) \geq \begin{cases} b\epsilon^{-2}, & \text{if } n = 3 \text{ and } \alpha = 1, \\ \exp\{a\epsilon^{-1}\}, & \text{if } n = 4 \text{ and } \alpha = 1, \end{cases}$$

where a, b are positive constants. Moreover, they also proved that in the particular case

$$F''_{uu}(0,0,0) = 0, \tag{1.8}$$

then

$$\tilde{T}(\epsilon) \begin{cases} \geq \exp\{a\epsilon^{-1}\}, & \text{if } n = 3 \text{ and } \alpha = 1, \\ = +\infty, & \text{if } n = 4 \text{ and } \alpha = 1. \end{cases}$$

This is to say, for $n = 3, 4$, the same lower bound of the lifespan can be obtained under assumption (1.8) as in the case where F does not explicitly depend on u.

The following table summarizes all results mentioned above for all n, α with $n \geq 3$ and $\alpha \geq 1$,

<table>
<tr><td>$\tilde{\mathrm{T}}(\epsilon) \geq$ \ α / n</td><td>1</td><td>2,3, $\cdots$</td></tr>
<tr><td rowspan="2">3</td><td>$b\epsilon^{-2}$</td><td rowspan="5">$+\infty$</td></tr>
<tr><td>$\exp\{a\epsilon^{-1}\}$, if $\partial_u^2 F(0,0,0) = 0$</td></tr>
<tr><td rowspan="2">4</td><td>$\exp\{a\epsilon^{-1}\}$</td></tr>
<tr><td>$+\infty$ if $\partial_u^2 F(0,0,0) = 0$</td></tr>
<tr><td>5,6, $\cdots$</td><td></td></tr>
</table>

in which a,b are positive constants (Li Ta-tsien and Yu Xin [37]).

Besides, for $n = 1$ and all integers α with $\alpha \geq 1$, we can get that

$$\tilde{\mathrm{T}}(\epsilon) \geq \begin{cases} b\epsilon^{-\alpha/2}, & \text{in the general case,} \\ b\epsilon^{-\alpha(1+\alpha)/(2+\alpha)}, & \text{if } \int_{-\infty}^{\infty} g(x)dx = 0, \\ b\epsilon^{-\alpha}, & \text{if } \partial_u^\beta F(0,0,0) = 0, \forall 1+\alpha \leq \beta \leq 2\alpha, \end{cases}$$

where b is a positive constant [35].

For $n = 2$ and all integers α with $\alpha \geq 1$, the following results can be obtained:

<table>
<tr><td>$\tilde{\mathrm{T}}(\epsilon) \geq$ \ α / n</td><td>1</td><td>2</td><td>3,4, $\cdots$</td></tr>
<tr><td rowspan="3">2</td><td>$b\epsilon^{-2}$</td><td rowspan="2">$b\epsilon^{-6}$</td><td rowspan="3">$+\infty$</td></tr>
<tr><td>$b\epsilon^{-1}$, if $\int g dx = 0$</td></tr>
<tr><td>$b\epsilon^{-2}$, if $\partial_u^2 F(0,0,0) = 0$</td><td>$\exp\{a\epsilon^{-2}\}$
if $\partial_u^2 F(0,0,0) = 0$
$(\beta = 3,4)$</td></tr>
</table>

where $e(\epsilon)$ is defined by

$$\epsilon^2 e^2(\epsilon)\ln(1+e(\epsilon)) = 1$$

and a, b are positive constants [35].

All these lower bounds, except the case that $n = 4$ and $\alpha = 1$, are known to be sharp due to Lindblad [38], Li Ta-tsien and Chen Yunmei [35], etc.

Here we rewrite Theorem 2.10 on page 77 in [28], and the proof can be found in [34], [37] and [35].

Theorem 1.1 *Consider the wave equation* (1.1) *with the assumption* (1.2).

Then global existence of small solutions exists if

$$\frac{n-1}{2}(1-\frac{2}{\alpha n})\alpha > 1 \quad \text{and} \quad \alpha \geq 1.$$

Proof. The proof follow what we mentioned before.

When $\alpha > 3$, we will use natural energy estimation, and when $1 \leq \alpha \leq 3$, the proof consists recasting the equation as a fixed point problem for a suitable integral operator, which is shown to be a contraction in the norm

$$\sup_{t\geq 0}(1+t)^{(n-1)(1-(2/\alpha n))/2}[u]_{s_0,\alpha n} + \sup_{t\geq 0}[u(t)]_{s,2} + \sup_{t\geq 0}[Du(t)]_{s+1,2},$$

where $[u]_{m,p} = (\sum_{k\leq m}\|(1+|x|)^k D_x^k u(x)\|^2_{L^p(\mathbb{R}^n)})^{1/2}$, that is, it is the weighted norm involved in the global Sobolev inequality, $s_0 \geq n+10$, and $s_0+n+1 \leq s \leq 2s_0-9$. The details are in [34] and [35]. □

1.2 Main result

In this section, we consider the initial value problem with high dimensions

$$\begin{cases} u_{tt} - C(u)\nabla \cdot (C(u)\nabla u) = 0, \\ u(0,x) = u_0 + \epsilon f(x), \quad u_t(0,x) = \epsilon g(x), \end{cases} \tag{1.9}$$

where u_0 is constant, f, g are given smooth functions with compact support and ϵ is a small parameter.

Theorem 1.2 *Assume that $C(u) \in C^2(\mathbb{R})$ and $u_0 \in \mathbb{R}$ such that $C(u_0) > 0, C'(u_0) = 0$. Then the initial value problem (1.9) for $n \geq 3$ always has a global C^∞ solution if $\epsilon > 0$ is small enough.*

Proof. Let $u(x,t) = u_0 + v(x,t)$, then the equation (1.9) turns to

$$v_{tt} - C^2(u_0+v)\Delta v - C(u_0+v)C'(u_0+v)|\nabla v|^2 = 0. \tag{1.10}$$

And by Taylor's series at u_0,

$$\begin{aligned} C(u_0+v) &= C(u_0) + \sum_{k\geq 2} a_i v^k, \\ C^2(u_0+v) &= C^2(u_0) + \sum_{k\geq 2} b_i v^k, \\ C'(u_0+v) &= b_1 C''(u_0)v + \sum_{k\geq 2} c_i v^k. \end{aligned} \tag{1.11}$$

Equation (1.10) becomes

$$v_{tt} - C^2(u_0)\Delta v = d_1 C(u_0)C''(u_0)v|\nabla v|^2 + \sum_{k\geq 2} d_i v^k |\nabla v|^2 + \sum_{k\geq 2} b_i v^k \Delta v. \tag{1.12}$$

The right-hand side has order bigger than 3, consequently the initial value problem (1.9) has a global C^∞ solution when ϵ is very small by Theorem 1.1 for $n \geq 3$. $\square$

We should remark that for $n \geq 4$, the global solution exists for any u_0 i.e., u_0 is not critical point of wave speed $C(u)$, as long as ϵ is small.

Corollary 1.1 *Assume that $C(u) \in C^2(\mathbb{R})$ is a positive function. Then the initial value problem* (1.9) *always has a global C^∞ solution if $\epsilon > 0$ is small enough, when $n \geq 4$.*

Proof. Set $u = u_0 + v(x,t)$ then the equation for v is

$$v_{tt} - C^2(u_0+v)\Delta v - C(u_0+v)C'(u_0+v)|\nabla v|^2 = 0. \tag{1.13}$$

In this case, the Taylor expansion gives us,

$$\begin{aligned} C(u_0+v) &= C(u_0) + \sum_{k\geq 1} a_i v^k, \\ C^2(u_0+v) &= C^2(u_0) + \sum_{k\geq 1} b_i v^k, \\ C'(u_0+v) &= C'(u_0) + \sum_{k\geq 1} c_i v^k, \end{aligned} \tag{1.14}$$

the equation (1.10) turn to

$$v_{tt} - C^2(u_0)\Delta v = \sum_{k\geq 1} b_i v^k \Delta v + C(u_0)C'(u_0)|\nabla v|^2 + \sum_{k\geq 1} d_i v^k |\nabla v|^2. \tag{1.15}$$

The right hand side has order bigger then 2, Theorem 1.1 concludes that the initial value problem (1.15) has a global C^∞ solution with small initial data. $\square$

We can also consider the initial value problem in high dimensions with a nonlinear term such as

$$\begin{cases} u_{tt} - C(u)\nabla\cdot(C(u)\nabla u) = F(u, Du, D_x Du), \\ u(0,x) = u_0 + \epsilon f(x), \quad u_t(0,x) = \epsilon g(x), \end{cases} \tag{1.16}$$

where

$$D_x = (\frac{\delta}{\delta x_1}, \cdots, \frac{\delta}{\delta x_n}), \quad D = (\frac{\delta}{\delta x_t}, \frac{\delta}{\delta x_1}, \cdots, \frac{\delta}{\delta x_n})$$

f, g given smooth functions with compact support and ϵ a small parameter.

Let

$$\hat{\lambda} = (u_0, \lambda_0, \lambda_1, \cdots, \lambda_n, \lambda_{ij})$$

where $i = 0, 1, \cdots, n$ and $j = 0, 1, \cdots, n, i + j \geq 1$.

Suppose that in a neighborhood of $\hat{\lambda} = (u_0, 0, \cdots, 0)$ say, for $|\hat{\lambda}| \leq 1$, the nonlinear term $F = F(\hat{\lambda})$ in (1.16) is a sufficiently smooth function satisfying

$$F(\hat{\lambda}) = o(|\hat{\lambda}|^{1+\alpha}),$$

where α is an integer ≥ 1.

Corollary 1.2 *Assume that $C(u) \in C^2(\mathbb{R})$ and there exists $u_0 \in \mathbb{R}$ such that $C(u_0) > 0, C'(u_0) = 0$. Then the initial value problem* (1.14) *always has a global C^∞ solution if $\epsilon > 0$ is small enough when dimension $n \geq 3$ and $\alpha \geq 2$.*

Proof. It follows directly from Theorem 1.1, [34], and [37]. □

Corollary 1.3 *Assume that $C(u) \in C^2(\mathbb{R})$ is positive function. Then the initial value problem* (1.16) *always has a global C^∞ solution if $\epsilon > 0$ is small enough when dimension $n \geq 4$ and $\alpha \geq 1$.*

Proof. Again, it follows immediately from Corollary 1.1, [34], and [37]. □

where

$$D_x = (\frac{\delta}{\delta x_1}, \cdots, \frac{\delta}{\delta x_n}), \quad D = (\frac{\delta}{\delta x_t}, \frac{\delta}{\delta x_1}, \cdots, \frac{\delta}{\delta x_n})$$

f, g given smooth functions with compact support and ϵ a small parameter.

Let

$$\hat{\lambda} = (u_0, \lambda_0, \lambda_1, \cdots, \lambda_n, \lambda_{ij})$$

where $i = 0, 1, \cdots, n$ and $j = 0, 1, \cdots, n, i + j \geq 1$.

Suppose that in a neighborhood of $\hat{\lambda} = (u_0, 0, \cdots, 0)$ say, for $|\hat{\lambda}| \leq 1$, the nonlinear term $F = F(\hat{\lambda})$ in (1.16) is a sufficiently smooth function satisfying

$$F(\hat{\lambda}) = o(|\hat{\lambda}|^{1+\alpha}),$$

where α is an integer ≥ 1.

Corollary 1.2 *Assume that $C(u) \in C^2(\mathbb{R})$ and there exists $u_0 \in \mathbb{R}$ such that $C(u_0) > 0, C'(u_0) = 0$. Then the initial value problem* (1.14) *always has a global C^∞ solution if $\epsilon > 0$ is small enough when dimension $n \geq 3$ and $\alpha \geq 2$.*

Proof. It follows directly from Theorem 1.1, [34], and [37]. □

Corollary 1.3 *Assume that $C(u) \in C^2(\mathbb{R})$ is positive function. Then the initial value problem* (1.16) *always has a global C^∞ solution if $\epsilon > 0$ is small enough when dimension $n \geq 4$ and $\alpha \geq 1$.*

Proof. Again, it follows immediately from Corollary 1.1, [34], and [37]. □

CHAPTER 2

The singularity of a wave equation in 1 space dimension

In this chapter we show the formation of the singularity of a nonlinear wave equation

$$u_{tt} - C(u)(C(u)u_x)_x = 0,$$

with positive function $C(u)$.

In the sense of distributions in the space $W^{1,\infty}(\mathbb{R}^2_+)$, the energy estimation given us

$$E(u) \equiv \int_{-\infty}^{\infty} u_t^2 + C^2(u)u_x^2 dx = \text{constant}. \tag{2.1}$$

in 1-dimension case. This is well-known fact for the wave speed $C(u) \equiv C$.
For general function $C(u)$, multiplying u_t on the original differential equation and integration give us

$$\int_{-\infty}^{\infty} u_{tt}u_t - C(u)(C(u)u_x)_x u_t dx = 0.$$

Then integration by parts,

$$\int_{-\infty}^{\infty} u_{tt}u_t + (C(u)u_x)(C(u)u_t)_x dx = 0.$$

Note

$$u_t u_{tt} = \frac{1}{2}(u_t^2)_t$$

and

$$(C(u)u_x)(C(u)u_t)_x = (C(u)u_x)(C(u)u_x)_t = \frac{1}{2}[\{C(u)u_x\}^2]_t.$$

We have

$$\frac{d}{dt}\int_{-\infty}^{\infty}[(u_t^2) + (C(u)u_x)^2]dx = 0,$$

which implies the conservation of energy.

To show the non-existence of the classical global solution, we will show the blow-up to the first order derivatives u_x or u_t in finite time while the function u itself stays bounded.

In first section we will establish the boundedness of u during large time, and in second section we will show that there is no classical global solution in a variational wave equation in 1 space dimension by estimating a combination of first order derivatives.

2.1 Boundedness of u

To see the boundedness of u, we need the one special version of the Sobolev inequalities. Next two Lemmas are the part of the Sobolev inequalities in [5].

Lemma 2.1 *For any ball $B(x,r) \subset \mathbb{R}^n$, there exists a constant C, depending only on n, such that*

$$\fint_{B(x,r)} |u(y) - u(x)|dy \leq C\int_{B(x,r)} \frac{|Du(y)|}{|y-x|^{n-1}}dy. \tag{2.2}$$

Proof. Fix any point $w \in \partial B(0,1)$. Then if $0 < s < r$,

$$\begin{aligned}
|u(x+sw) - u(x)| &= |\int_0^s \frac{d}{dt} u(x+tw)dt| \\
&= |\int_0^s Du(x+tw) \cdot wdt| \\
&\leq \int_0^s |Du(x+tw)|dt.
\end{aligned}$$

Hence

$$\begin{aligned}
\int_{\partial B(0,1)} |u(x+sw) - u(x)|dS &\leq \int_0^s \int_{\partial B(0,1)} |Du(x+tw)|dSdt \\
&= \int_0^s \int_{\partial B(0,1)} |Du(x+tw)| \frac{t^{n-1}}{t^{n-1}} dSdt.
\end{aligned}$$

Let $y = x + tw$, so that $t = |x - y|$. Then converting from polar coordinates, we have

$$\begin{aligned}
\int_{\partial B(0,1)} |u(x+sw) - u(x)|dS &\leq \int_{B(x,s)} \frac{|Du(y)|}{|x-y|^{n-1}} dy \\
&\leq \int_{B(x,r)} \frac{|Du(y)|}{|x-y|^{n-1}} dy.
\end{aligned}$$

Multiply by s^{n-1} and integrate from 0 to r with respect to s:

$$\int_{B(x,r)} |u(y) - u(x)|dy \leq \frac{r^n}{n} \int_{B(x,r)} \frac{|Du(y)|}{|y-x|^{n-1}} dy.$$

□

Next Lemma is a special version of Morrey's inequality which is one of the Sovolev inequalities in Chapter 5 of [5].

Lemma 2.2 *Assume the dimension $n < p \leq \infty$. then there exists a constant C,* *depending only on p and n, such that*

$$\sup_{\mathbb{R}n} |u| \leq C\|u\|_{W1,p(\mathbb{R}n)}. \tag{2.3}$$

Proof. Fix $x \in \mathbb{R}^n$. We apply inequality (2.2) as follows:

$$\begin{aligned}
|u(x)| &\leq \fint_{B(x,1)} |u(x) - u(y)|dy + \fint_{B(x,1)} |u(y)|dy \\
&\leq C\int_{B(x,1)} \frac{|Du(y)|}{|y-x|^{n-1}}dy + C\|u\|_{L^p(B(x,1))} \\
&\leq C(\int_{\mathbb{R}n} |Du|^p dy)^{1/p}(\int_{B(x,1)} \frac{dy}{|x-y|^{(n-1)p/p-1}})^{p-1/p} + C\|u\|_{Lp(\mathbb{R}n)} \\
&\leq C\|u\|_{W1,p(\mathbb{R}n)}.
\end{aligned} \tag{2.4}$$

The last estimate holds since $p > n$ implies $(n-1)\frac{p}{p-1} < n$; so that

$$\int_{B(x,1)} \frac{1}{|x-y|^{(n-1)p/p-1}}dy < \infty.$$

As $x \in \mathbb{R}^n$ is arbitrary, inequality (2.4) implies

$$\sup_{\mathbb{R}n} |u| \leq C\|u\|_{W1,p(\mathbb{R}n)}.$$

□

Theorem 2.1 *If u is a solution of the initial value problem,*

$$\begin{cases} u_{tt} - C(u)(C(u)u_x)_x = 0, \quad 0 < t < T_1, \\ u(0,x) = u_0 + \epsilon f(x), u_t(0,x) = \epsilon g(x), \end{cases} \tag{2.5}$$

where $\epsilon > 0$ is sufficiently small, $f(x) \in C_0^\infty(\mathbb{R})$, $g(x) \in C_0^\infty(\mathbb{R})$ and there exist positive constants $0 < C_0 < C_1 < \infty$ such that $C_0 \leq C(u) < C_1$ for all $u \in \mathbb{R}$, then *for any* $0 < T' < T_1$, $u \in L^\infty([0,T') \times \mathbb{R})$.

Proof. The energy of (2.1) is

$$E(u) \equiv \int_{-\infty}^{\infty} u_t^2 + C^2(u)u_x^2 dx = \text{ constant.}$$

So, we have the boundedness of L^2 norm of u_t and u_x using

$$\int_{-\infty}^{\infty} u_t^2 dx \leq E(u) < \infty \text{ and}$$
$$C_0 \int_{-\infty}^{\infty} u_x^2 dx \leq E(u) < \infty.$$

We can conclude that $Du \in L^2$ where $D = \partial/\partial x$ or $\partial/\partial t$.

Remember that $u - u_0$ is compactly supported for each $t > 0$ from the finite propagation speed of the nonlinear wave equation. We see that $||u(t,\cdot)||_{W^{1,2}(\mathbb{R})} \leq M$ for all $t \in [0,T']$ Morrey's inequality concludes that $u \in L^\infty([0,T'] \times \mathbb{R})$. □

In next section we will show that the non-existence of the smooth global solution of (2) for general smooth initial data.

2.2 Main theorem

In this section we consider the initial value problem

$$\begin{cases} u_{tt} - C(u)(C(u)u_x)_x = 0, \\ u(0,x) = u_0 + \epsilon\phi(x/\epsilon), \\ u_t(0,x) = -sign(C'(u_0 \pm \delta))C(u(0,x))u_x(0,x), \end{cases} \tag{2.6}$$

where $\epsilon > 0$ is sufficiently small, $\phi \in C^1(\mathbb{R})$ with $\phi \not\equiv 0, |\phi| < 1$, and $\phi''(a) \neq 0$ when $\phi'(a) = 0$.

Theorem 2.2 *Assume that* $C(u) \in C^2(\mathbb{R})$ *satisfies the following:*

(a) there exist positive constants $0 < C_0 < C_1 < \infty$ *such that* $C_0 \leq C(u) \leq C_1$ *for all* $u \in \mathbb{R}$,

(b) $C'(u_0) = 0$ *and* $C''(u_0) \neq 0$.

Suppose that $u(t, x) \in C^1([0, T) \times \mathbb{R})$ *is a smooth solution of* (2.6) *in* $0 \leq t < T$. *Then* $T < \infty$ *for some* ϕ, *i.e. a global smooth solution does not exist.*

Proof. We have to prove two cases, $C(u)$ have a local maximum at u_0 and $C(u)$ have a local minimum at u_0. However the proofs for the two cases are similar, so we treat only the first case.

To prove this theorem, we will show that if u is continuous and $|u| < \infty$, then u_t and u_x must blow up in finite time.

The energy estimate of (2.6) is $E(u) \equiv \int_{-\infty}^{\infty} u_t^2 + C^2(u)u_x^2 dx =$ constant as in (2.1), and here we will use the method of characteristics [8] .

We will write the non-linear wave equation (2) as a system of first order equations by introducing new dependent variables

$$\begin{aligned} R &= u_t + C(u)u_x, \\ S &= u_t - C(u)u_x. \end{aligned} \tag{2.7}$$

Then, for smooth solutions, equation (2) is equivalent to the following system for (R, u, S),

$$\begin{aligned} R_t - CR_x &= \frac{C'}{4C}(R^2 - S^2), \\ u_t &= \frac{1}{2}(R + S), \\ S_t + CS_x &= \frac{C'}{4C}(S^2 - R^2), \end{aligned} \tag{2.8}$$

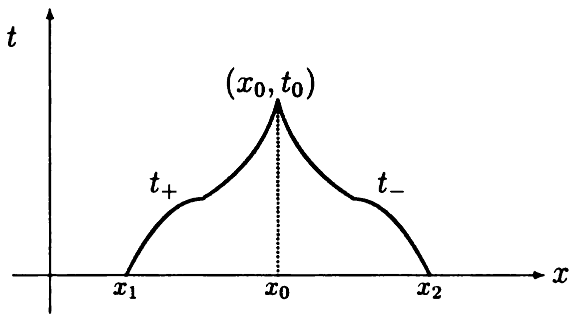

Figure 2.1. A characteristic region

with the constraint

$$u_x = \frac{R - S}{2C}. \tag{2.9}$$

These equations will help us to estimate the values of R and S along characteristics t_+ and t_-. Given any point (t_0, x_0) in the upper half plane $t > 0$, let $t_\pm(x)$, or $x_\pm(t)$, denote the plus and minus characteristics through (t_0, x_0), extended backward in time:

$$\frac{dt_\pm(x)}{dx} = \pm\frac{1}{C(u)}, \quad t_\pm(x_0) = t_0 \quad 0 \leq t \leq t_0$$

(see Figure 2.1). Let x_1 and x_2 denote the intersection points of $t_\pm$ with the x-axis.

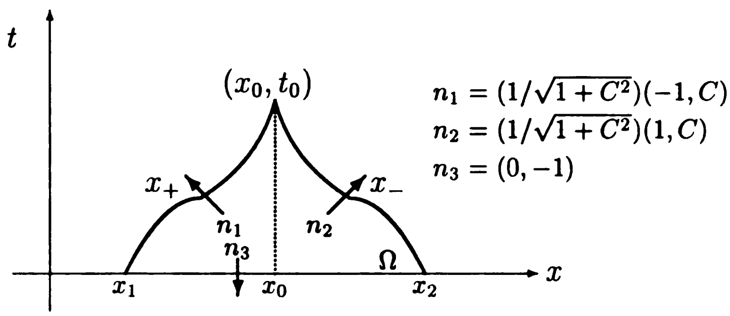

Figure 2.2. Outer normal vectors

For the initial data in (2.6), We can estimate the total energy

$$
\begin{aligned}
E(u) &= \int_{-\infty}^{\infty} u_t^2 + C^2 u_x^2 dx \qquad (2.10)\\
&= \int_{-\infty}^{\infty} 2C^2 \phi'(\frac{x}{\epsilon})^2 dx \\
&\leq 2C_1^2 \int_{-\infty}^{\infty} \phi'(\frac{x}{\epsilon})^2 dx \\
&= 2C_1^2 \epsilon \int_{-\infty}^{\infty} \phi'(y)^2 dy \\
&= 2C_1^2 \epsilon \|\phi'\|_{L2}^2 \\
&= M\epsilon
\end{aligned}
$$

for some constant M, where C_1 is an upper bound for $C(u)$.

Now, we will consider the divergence theorem with the vector $\vec{F} = (u_t^2 - C^2 u_x^2, u_t^2 + C^2 u_x^2)$ and the characteristic region, Ω in Figure2.1.

$$
\int_{\partial\Omega} \vec{F} \cdot \vec{n} ds = \int\int_{\Omega} div\vec{F} dxdt. \qquad (2.11)
$$

The left hand side of (2.11) is calculated with outer normal vector n_1, n_2, and n_3 (See Figure 2.2). So, that is

$$\int_{x+} \frac{1}{\sqrt{1+C^2}}(-u_t^2 + C^2u_x^2 + Cu_t^2 + C^3u_x^2)ds \tag{2.12}$$
$$+ \int_{x-} \frac{1}{\sqrt{1+C^2}}(u_t^2 - C^2u_x^2 + Cu_t^2 + C^3u_x^2)ds$$
$$- \int_{x1}^{x2} (u_t^2 + C^2u_x^2)dx.$$

The right hand side of (2.11) is

$$\int\int_\Omega (u_t^2 - C^2u_x^2)_x + (u_t^2 + C^2u_x^2)_t dxdt. \tag{2.13}$$

The first term in (2.13) turns to

$$\begin{aligned} & \int\int_\Omega (u_t^2 - C^2u_x^2)_x dxdt \\ = & \int_0^{t_0}\int_{x+(t)}^{x-(t)} (u_t^2 - C^2u_x^2)_x dxdt \\ = & \int_{x-} \frac{1}{\sqrt{1+C^2}}(u_t^2 - C^2u_x^2)ds - \int_{x+} \frac{1}{\sqrt{1+C^2}}(u_t^2 - C^2u_x^2)ds. \end{aligned}$$

And the second term in (2.13) turns to

$$\begin{aligned} & \int\int_\Omega 2u_tu_{tt} + 2(C(u)u_x)_tC(u)u_xdxdt \\ = & \int\int_\Omega 2u_tu_{tt} + 2(C(u)u_t)_xC(u)u_xdxdt \\ = & \int\int_\Omega 2u_tu_{tt} - 2C(u)u_t(C(u)u_x)_xdxdt \\ + & \int_{x-} \frac{2C^2(u)}{\sqrt{1+C^2}}u_xu_tds - \int_{x+} \frac{2C^2(u)}{\sqrt{1+C^2}}u_xu_tds \\ = & \int\int_\Omega 2u_t(u_{tt} - C(u)(C(u)u_x)_x)dxdt \\ + & \int_{x-} \frac{2C^2(u)}{\sqrt{1+C^2}}u_xu_tds - \int_{x+} \frac{2C^2(u)}{\sqrt{1+C^2}}u_xu_tds \\ = & \int_{x-} \frac{2C^2(u)}{\sqrt{1+C^2}}u_xu_tds - \int_{x+} \frac{2C^2(u)}{\sqrt{1+C^2}}u_xu_tds. \end{aligned}$$

Hence, (2.13) is equal to

$$\int_{x_-} \frac{1}{\sqrt{1+C^2}}(u_t^2 - C^2u_x^2)ds - \int_{x_+} \frac{1}{\sqrt{1+C^2}}(u_t^2 - C^2u_x^2)ds \tag{2.14}$$
$$+ \int_{x_-} \frac{2C^2(u)}{\sqrt{1+C^2}}u_xu_tds - \int_{x_+} \frac{2C^2(u)}{\sqrt{1+C^2}}u_xu_tds.$$

Combining (2.11), (2.12), and (2.14), we have

$$\int_{x_-} \frac{C}{\sqrt{1+C^2}}(u_t^2 + C^2u_x^2)ds + \int_{x_+} \frac{C}{\sqrt{1+C^2}}(u_t^2 + C^2u_x^2)ds \tag{2.15}$$
$$- \int_{x_-} \frac{2C^2(u)}{\sqrt{1+C^2}}u_xu_tds + \int_{x_+} \frac{2C^2(u)}{\sqrt{1+C^2}}u_xu_tds$$
$$= \int_{x1}^{x2} (u_t^2 + C^2u_x^2)ds.$$

By (2.15) and total energy (2.10), we have

$$\int_{x_-} \frac{C}{\sqrt{1+C^2}}(u_t^2 + C^2u_x^2 - 2Cu_xu_t)ds$$
$$+ \int_{x_+} \frac{C}{\sqrt{1+C^2}}(u_t^2 + C^2u_x^2 + 2Cu_xu_t)ds \le M\epsilon,$$
$$\int_{x_-} \frac{C}{\sqrt{1+C^2}}(u_t - Cu_x)^2ds + \int_{x_+} \frac{C}{\sqrt{1+C^2}}(u_t + Cu_x)^2ds \le M\epsilon,$$

$$\frac{C_0}{\sqrt{1+C_1^2}}\int_{x_+}(u_t + Cu_x)^2ds \le \int_{x_+} \frac{C}{\sqrt{1+C^2}}(u_t + Cu_x)^2ds \le M\epsilon,$$
$$\int_{x_+}(u_t + Cu_x)^2ds \le \frac{M\sqrt{1+C_1^2}}{C_0}\epsilon = K\epsilon. \tag{2.16}$$

Similarly, one can obtain

$$\int_{x_-}(u_t + Cu_x)^2ds \le K\epsilon.$$

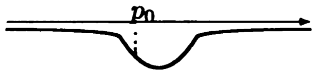

Figure 2.3. $\phi(x)$

Using the energy estimate, we can see the deviation of u from u_0 as follows:

$$\begin{aligned} |u(t,x)-u_0| &= |u(t,x_+(t))-u(x,0)+u(x,0)-u_0| \qquad (2.17)\\ &\le |u(t,x_+(t))-u(x,0)|+|u(x,0)-u_0| \\ &\le \int_0^t \frac{d}{dt}u(\tau,x_+(\tau))d\tau+\epsilon \\ &= \int_0^t (Cu_x+u_t)d\tau+\epsilon \\ &\le (\int_0^t (u_t+Cu_x)^2 d\tau)^{1/2}\sqrt{t}+\epsilon \\ &\le K\sqrt{\epsilon t}+\epsilon \end{aligned}$$

for some constant K.

We choose $\delta > 0$ small enough that $C'(u)$ does not change sign in the interval $[u_0-\delta, u_0]$. Actually

$$0 < C'(u) \le C_1' \text{ for } u \in (u_0-\delta, u_0) \qquad (2.18)$$

for some positive constant C_1' since $C''(u_0) < 0$ and $C'(u_0) = 0$. We also choose a point $p_0 \in \mathbb{R}$ such that $\phi'(p_0) < 0$, and sufficiently small number $\epsilon_0, \sigma > 0$ such that the term on the right hand side of (2.17) satisfies $K\sqrt{\epsilon t}+\epsilon \le \delta$ for $0 < \epsilon \le \epsilon_0, 0 \le t \le \sigma/\epsilon$.

Now we have the following lemma for the sign of R.

Lemma 2.3 *Let* $-1 < \phi < 0$ *and* $u_t(0,x) = -C(u(0,x))u_x(0,x)$. *Then* $R(t,x) \le 0$ *when* $0 < t < \min\{T, \sigma/\epsilon\}$.

Proof. From (2.8), the derivative of R along the minus characteristic line $\frac{dx}{dt} = -C(u)$ is

$$\frac{dR}{dt} = \frac{C'}{4C}(R^2 - S^2).$$

With zero initial data for R, i.e., $R(0) = 0$ from the choice of $u_t(0,x)$, and

$$\frac{dR}{dt}|_{t=0} = \frac{C'(u_0 + \epsilon\phi(x/\epsilon))}{4C(u_0 + \epsilon\phi(x/\epsilon))}(-S^2) \le 0. \tag{2.19}$$

In (2.19), the equality can only hold when $S|_{t=0} = 0$.

That means, by initial data in (2.6), $S|_{t=0} = -2C(u(0,x))\phi'(\frac{x}{\epsilon}) = 0$, i.e. $\phi'(\frac{x}{\epsilon}) = 0$.

At that point

$$\frac{d^2R}{dt^2}|_{t=0} = \frac{d}{dt}(\frac{C'}{4C})(R^2 - S^2) + (\frac{C'}{4C})(2R\frac{dR}{dt} - 2S\frac{dS}{dt}) = 0,$$

and

$$\begin{aligned} \frac{d^3R}{dt^3}|_{t=0} &= \frac{d^2}{dt^2}(\frac{C'}{4C})(R^2 - S^2) + 2\frac{d}{dt}(\frac{C'}{4C})(2R\frac{dR}{dt} - 2S\frac{dS}{dt}) \\ &+ (\frac{C'}{4C})(2(\frac{dR}{dt})^2 + 2R\frac{d^2R}{dt^2} - 2(\frac{dS}{dt})^2 - 2S\frac{d^2S}{dt^2}) \\ &= -(\frac{C'}{2C})(\frac{dS}{dt})^2 \le 0. \end{aligned} \tag{2.20}$$

And in (2.20), the equality cannot hold, because of the calculation of the derivative of S along the minus characteristic line.

$$\begin{aligned} \frac{dS}{dt} &= S_t - CS_x \\ &= u_{tt} - C'(u)u_tu_x - 2C(u)u_{xt} + C(u)C'(u)u_x^2 + C^2(u)u_{xx} \\ &= 2C^2(u)u_{xx} + 2C(u)C'(u)u_x^2 - C'(u)u_tu_x - 2C(u)u_{xt}. \end{aligned}$$

Using the initial data in (2.6), we have

$$\begin{aligned}\frac{dS}{dt}|_{t=0} &= 2(C(u(0,x)))^2\frac{1}{\epsilon}\phi''(\frac{x}{\epsilon}) + 3C(u(0,x))C'(u(0,x))(\phi'(\frac{x}{\epsilon}))^2 \\ &+ 2(C(u(0,x)))^2\frac{1}{\epsilon}\phi''(\frac{x}{\epsilon}) \\ &= \frac{4(C(u(0,x)))^2}{\epsilon}\phi''(\frac{x}{\epsilon}) \neq 0.\end{aligned}$$

So we can conclude that $\frac{d^3R}{dt^3}|_{t=0} < 0$, so finally we can say that $\frac{dR}{dt}|_{t=0} < 0$, when t is very small, R is a decreasing function at $t = 0$.

Now we consider the initial value problem

$$\frac{dR}{dt} = \frac{C'}{4C}(R^2 - S^2) \leq \frac{C_1'}{4C_0}R^2, \quad R(0) \leq 0.$$

$R \leq 0$ follows from that the only solution of the initial value problem

$$\frac{dR}{dt} = \frac{C_1'}{4C_0}R^2, \ R(0) = 0$$

is the zero solution, and a standard comparison theorem in ODE (ordinary differential equation).

We proved that $R(t,x) \leq 0$ along the minus characteristic line when $0 < t < \min\{T, \sigma/\epsilon\}$. □

Lemma 2.4 *R is bounded. More exactly, we show that there is a constant $k > 0$ such that*

$$-k\epsilon \leq R(t,x) \leq 0 \ \textit{for} \ 0 < \epsilon \leq \epsilon_0, \ \ 0 \leq t \leq \sigma/\epsilon.$$

Proof. By Lemma 2.3, $R \leq 0$ for $0 < \epsilon \leq \epsilon_0,\ \ 0 \leq t \leq \sigma/\epsilon$.

Estimating $\frac{dR}{dt}$ from below gives

$$\frac{dR}{dt}(t, x_-(t)) = \frac{C'}{4c}(R^2 - S^2) \geq -\frac{C_1'}{4C_0}S^2.$$

Therefore, integrating along the minus characteristics we get

$$\begin{aligned} R(t_0, x_0) &\geq -\frac{C_1'}{4C_0}\int_0^{t_0} S^2(t, x_-(t))dt \\ &= -\frac{C_1'}{4C_0}\int_{x_0}^{x_2} \frac{S^2(t_-(x), x)}{C(u(t_-(x), x))}dx \\ &\geq -\frac{C_1'}{4C_0^2}E(u) = -k\epsilon. \end{aligned}$$

□

Lemma 2.5 *S becomes infinite for some $t \leq t^* = o(1/\epsilon)$.*

Proof. We integrate the equation for S in (2.7) along the plus characteristic $\frac{dx}{dt} = C(u)$ passing through the point $x = \epsilon p_0$ at $t = 0$.

Using Lemma 2.2, this gives

$$\frac{dS}{dt} = \frac{C'}{4C}(S^2 - R^2) \geq \frac{C'}{4C_1}(S^2 - k^2\epsilon^2) \tag{2.21}$$

for some constant k in the region $0 < \epsilon \leq \epsilon_0$ and $0 < t \leq \sigma\epsilon^{-1}$.

Now we note that $S(0, \epsilon p_0) = -2C(u(0, \epsilon p_0))\phi'(p_0)$.

And by Taylor's series (1.11) and $u(0, \epsilon p_0) = u_0 + \epsilon\phi(p_0)$

$$\begin{aligned} S(0, \epsilon p_0) &= -2(C(u_0) + \sum_{k\geq 2} a_i\epsilon^k\phi^k(p_0))\phi'(p_0) \\ &= -2(C(u_0) + o(\epsilon^2))\phi'(p_0) \\ &= 2C(u_0)(-\phi'(p_0)) + 2(o(\epsilon^2))(-\phi'(p_0)) > 0, \end{aligned}$$

where a_i are positive constant.

Choosing ϵ_0 smaller if necessary, we can assume that

$$\frac{dS}{dt}|_{t=0} \geq \frac{C'}{4C_1}[(2C(u_0)(-\phi'(p_0)))^2 - k^2\epsilon^2]. \tag{2.22}$$

And also by Taylor's series (1.11) and $u(0, \epsilon p_0) = u_0 + \epsilon\phi(p_0)$,

$$C'(u_0 + \epsilon\phi(p_0)) = b_1 C''(u_0)\epsilon\phi(p_0) + \sum_{k\geq 2} c_i \epsilon^k \phi^k(p_0)) = b_1 C''(u_0)\epsilon\phi(p_0) + o(\epsilon^2).$$

So (2.22) turn to

$$\frac{dS}{dt}|_{t=0} \geq \frac{b_1 C''(u_0)\epsilon\phi(p_0)}{4C_1}[(2C^2(u_0)(-\phi'(p_0)))^2 - k_1^2\epsilon^2]$$

for some constant b_1 and k_1.

And $C(u)$ has maximum at u_0 so $C''(u_0) < 0$ and $\phi(p_0) < 0$, so $b_1 C''(u_0)\phi(p_0) > 0$, finally

$$\frac{dS}{dt}|_{t=0} \geq \frac{\epsilon k_2}{4C_1}[(2C^2(u_0)(-\phi'(p_0)))^2 - k^2\epsilon^2] > 0.$$

with constant k_2 and small enough ϵ.

Then S is an increasing function of t along the plus characteristics with positive data, and the quadratic growth in the inequality (2.21) will drive S to infinity. To obtain an upper-bound for the singularity formation time, we integrate the ODE (ordinary differential equation)

$$\begin{aligned} \frac{dS}{dt} &= a^2\epsilon(S^2 - b^2\epsilon^2) \\ S(0) &= \sigma^2 > 0. \end{aligned}$$

Using the separation of variables and partial sum,

$$\begin{aligned}
\frac{1}{(S-b\epsilon)(S+b\epsilon)}dS &= a^2\epsilon dt \\
\frac{1}{2b\epsilon}\left(\frac{1}{S-b\epsilon}-\frac{1}{S+b\epsilon}\right)dS &= a^2\epsilon dt \\
\ln(S-b\epsilon)-\ln(S+b\epsilon) &= 2a^2b\epsilon^2 t + C \\
\frac{S-b\epsilon}{S+b\epsilon} &= C_e 2a^2b\epsilon^2 t
\end{aligned}$$

And using the initial data then the solution is

$$\frac{S-b\epsilon}{S+b\epsilon} =_e 2a^2b\epsilon^2 t\frac{\sigma^2-b\epsilon}{\sigma^2+b\epsilon}.$$

Since S tends to infinity, then $(S-b\epsilon)/(S+b\epsilon)$ goes to 1. From

$$1 =_e 2a^2b\epsilon^2 t\frac{\sigma^2-b\epsilon}{\sigma^2+b\epsilon}$$

and by Taylor series,

$$\begin{aligned}
2a^2b\epsilon^2 t &= \ln\left(\frac{\sigma^2+b\epsilon}{\sigma^2-b\epsilon}\right) = \ln\left(1+\frac{2b\epsilon}{\sigma^2-b\epsilon}\right) \\
&= \frac{2b\epsilon}{\sigma^2-b\epsilon}-\frac{1}{2}\left(\frac{2b\epsilon}{\sigma^2-b\epsilon}\right)^2+\cdots.
\end{aligned}$$

So this solution develops a singularity at time

$$t^* = \frac{1}{2a^2b\epsilon^2}\ln\left(\frac{\sigma^2+b\epsilon}{\sigma^2-b\epsilon}\right) = \frac{1}{2a^2b\epsilon^2}\left(\frac{2b\epsilon}{\sigma^2-b\epsilon}-\frac{1}{2}\left(\frac{2b\epsilon}{\sigma^2-b\epsilon}\right)^2+\cdots\right) \sim \frac{1}{a^2\sigma^2\epsilon}$$

.

Therefore S becomes infinite when $t = o(\frac{1}{\epsilon})$. Since R remains of order ϵ, the derivative u_x and u_t become infinite simultaneously when $t = o(\frac{1}{\epsilon})$, provide that a smooth solution for u exists up to that time. $\square$

And this is the complete proof of Theorem 2.2. $\square$

CHAPTER 3

The wave equation in 2 space dimension

In this chapter we study the equation in space dimension $n = 2$. Again, the nonlinear wave equation is

$$u_{tt} - C(u)\nabla \cdot (C(u)\nabla u) = 0. \tag{3.1}$$

3.1 The energy method

The equation (3.1) can be written to

$$u_{tt} - C^2(u)\Delta u + C(u)C'(u)|\nabla u|^2 = 0 \tag{3.2}$$

By the Taylor expansion with $u = u_0 + v(x,t)$, the equation (3.2) turns to

$$v_{tt} - C^2(u_0)\Delta v = \beta_1 v|\nabla v|^2 + \sum_{k\geq 2} \beta_{2,k} v^k |\nabla v|^2 + \sum_{k\geq 2} \beta_{3,k} v^k \Delta v. \tag{3.3}$$

In (3.3), space dimension $n = 2$ and $\alpha = 2$ as we stated in Theorem 1.1 of Chapter 1. And $\frac{n-1}{2}(1-\frac{2}{\alpha n})\alpha = \frac{1}{2} < 1$, Theorem 1.1 can not be directly used for global existence. It seems that the major difficult is from the first term on the right hand side of (3.3) because this term corresponds to $\alpha = 2$. In fact, it seems that nothing is known for the global existence of small solutions for even

$$\Box v = v|\nabla v|^2.$$

It is well known that any nontrivial solutions for

$$\Box v = |\nabla v|^2$$

blows up in finite time. The factor v before $|\nabla v|^2$ should help some. The question is whether it helps enough for the global existence. We have some preliminary estimates which indicate the blow up. But we have some gaps, and are not confidently enough to include them in this dissertation.

3.2 The characteristic method

Throughout this section, we will consider the nonlinear wave equations (3.1) using the method in Chapter 2, characteristic method. However, unlike energy methods, this approach only works for the functions with radial symmetry, essentially one space dimension.

For this, let $u(x,t) = \varphi(r,t), \quad 0 \geq r = |x|$. Then the equation (3.1) becomes

$$\varphi_{tt} - C(\varphi)(C(\varphi)\varphi_r)_r - \frac{C^2(\varphi)}{r}\varphi_r = 0. \tag{3.4}$$

Since $|\nabla u|^2 = ((x_1/r)\varphi'(r), (x_2/r)\varphi'(r)) \cdot ((x_1/r)\varphi'(r), (x_2/r)\varphi'(r)) = (\varphi'(r))^2$ and $\Delta u = \partial_1((x_1/r)\varphi'(r)) + \partial_2((x_2/r)\varphi'(r)) = \varphi''(r) + \frac{2}{r}\varphi'(r) - \frac{1}{r}\varphi'(r) = \varphi''(r) + \frac{1}{r}\varphi'(r)$.

In the sense of distributions in the space $W^{1,\infty}(\mathbb{R}^2_+)$, the energy estimate gives us

$$E(u) \equiv \int_0^\infty r\varphi_t^2 + rC^2(\varphi)\varphi_r^2 dr = \text{constant}.$$

To see this, multiplying $2r\varphi_t$ on the original differential equation (3.4) and integration give us

$$\int_0^\infty 2r\varphi_{tt}\varphi_t - 2rC(\varphi)(C(\varphi)\varphi_r)_r\varphi_t - 2C^2(\varphi)\varphi_r\varphi_t dr = 0.$$

Then integration by parts,

$$\int_0^\infty 2r\varphi_{tt}\varphi_t + 2(rC(\varphi)\varphi_t)_r C(\varphi)\varphi_r) - 2C^2(\varphi)\varphi_r\varphi_t dr = 0.$$

By product rule,

$$\int_0^\infty 2r\varphi_{tt}\varphi_t + 2rC'(\varphi)C(\varphi)\varphi_t\varphi_r^2 + 2rC^2(\varphi)\varphi_{tr}\varphi_r dr = 0.$$

We have

$$\frac{d}{dt}\int_0^\infty [r(\varphi_t^2) + rC^2(\varphi)\varphi_r^2]dr = 0,$$

which implies the conservation of energy.

Now we will consider the initial value problem,

$$\begin{cases} v_{tt} - C(v)(C(v)v_r)_r - (C^2(v)/r)v_r = 0, \\ v(0,r) = u_0 + \epsilon\phi(\frac{r}{\epsilon}), \quad v_t(0,r) = \psi(\frac{r}{\epsilon}), \end{cases} \tag{3.5}$$

where $\epsilon > 0$ is sufficiently small, $\phi \in C_0^\infty(\mathbb{R})$ with ϕ and ψ are even function since the initial data are also radial symmetric, and $C(v) \in C^2(\mathbb{R})$ satisfies the following:

(a) there exist positive constants $0 < C_0 < C_1 < \infty$ such that $C_0 \le C(v) \le C_1$ for all $u \in \mathbb{R}$,

(b) $C'(v_0) = 0$ and $C''(v_0) \neq 0$.

We will write the the non-linear wave equation (3.4) as a system of first order equations by introducing new dependent variables

$$\begin{aligned} R &= \sqrt{r}(v_t + C(v)v_r), \\ S &= \sqrt{r}(v_t - C(v)v_r). \end{aligned} \tag{3.6}$$

Then, for smooth solutions, equation (3.4) is equivalent to the following system for (R, v, S),

$$\begin{aligned} R_t - CR_r &= \frac{C'}{4C\sqrt{r}}(R^2 - S^2) - \frac{C}{2r}S, \\ v_t &= \frac{R+S}{2\sqrt{r}}, \\ S_t + CS_r &= \frac{C'}{4C\sqrt{r}}(S^2 - R^2) + \frac{C}{2r}S, \end{aligned} \tag{3.7}$$

with the constraint

$$v_r = \frac{R-S}{2C\sqrt{r}}. \tag{3.8}$$

The reasons we introduced the factor $\sqrt{r}$ in (3.6) are the following:

1. The resulting equation (3.7) for R and S along the characteristic lines look simpler, and will be easier to estimate later; and
2. R and S are in $L^2[0, \infty]$ from energy conservatives.

Given any point (t_0, r_0) in the upper half plane $t > 0$, let $t_\pm(x) = r_\pm(t)$ denote the plus and minus characteristics through (t_0, r_0), extended backward in time depending on the r_0:

$$\frac{dt_\pm(r)}{dr} = \pm\frac{1}{C(v)}, \quad t_\pm(r_0) = t_0 \;\; 0 \leq t \leq t_0$$

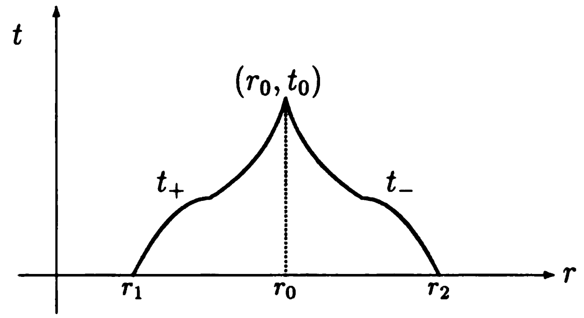

Figure 3.1. A characteristic region with far from origin

If r_0 is far from origin then see Figure 3.1. And if r_0 is close to origin then we will have to consider the reflection of the line on the minus region since the symmetry. Let r_1 and r_2 denote the intersection points of $t_\pm$ with the r-axis.

For the initial data in (3.5), We can estimate the total energy

$$\begin{aligned} E(u) &= \int_0^\infty r(v_t^2 + C^2 v_r^2)dr \\ &\le \int_0^\infty r\{\psi^2(\frac{r}{\epsilon}) + C^2(\phi'(\frac{r}{\epsilon}))^2\}dr \\ &\le \epsilon^2 \int_0^\infty y\{\psi^2(y) + C_1^2(\phi'(y))^2\}dy \\ &\le \epsilon^2 \int_0^\infty \psi^2(y) + C_1^2(\phi'(y))^2 dy \\ &\le M\epsilon^2 \end{aligned} \tag{3.9}$$

for some constant M, where C_1 is an upper bound for $C(v)$.

Like Chapter 2, we will use the divergence theorem for a vector field $\vec{F} = (ru_t^2 - rC^2u_r^2, ru_t^2 + rC^2u_r^2)$ in a region bounded by two characteristic lines. See Figure 3.1.

$$\int_{\partial\Omega} \vec{F} \cdot \vec{n} ds = \int\int_{\Omega} div\vec{F} dr dt. \tag{3.10}$$

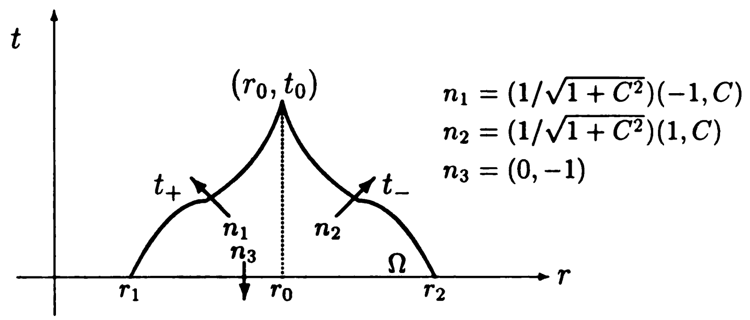

Figure 3.2. Outer normal vectors when far from origin

The left hand side of (3.10) is calculated with outer normal vector n_1, n_2, and n_3 (See Figure 3.2). That is,

$$\begin{aligned} &\int_{t_+} \frac{r}{\sqrt{1+C^2}}(-u_t^2 + C^2u_r^2 + Cu_t^2 + C^3u_r^2)ds \\ + &\int_{t_-} \frac{r}{\sqrt{1+C^2}}(u_t^2 - C^2u_r^2 + Cu_t^2 + C^3u_r^2)ds \\ - &\int_{t1}^{t2} r(u_t^2 + C^2u_r^2)dr. \end{aligned} \tag{3.11}$$

The right hand side of (3.10) is

$$\int\int_\Omega [r(u_t^2 - C^2u_r^2)]_r + [r(u_t^2 + C^2u_r^2)]_t drdt. \tag{3.12}$$

The first term in (3.12) turns to

$$\begin{aligned} &\int\int_\Omega [r(u_t^2 - C^2u_r^2)]_r drdt \\ = &\int_0^{t_0}\int_{t_+}^{t_-} [r(u_t^2 - C^2u_r^2)]_r drdt \\ = &\int_{t_-} \frac{r}{\sqrt{1+C^2}}(u_t^2 - C^2u_r^2)ds - \int_{t_+} \frac{r}{\sqrt{1+C^2}}(u_t^2 - C^2u_r^2)ds. \end{aligned}$$

And the second term in (3.12) turns to

$$
\begin{aligned}
& \int\int_{\Omega} r[2u_t u_{tt} + 2(C(u)u_r)_t C(u)u_r] drdt \\
= & \int\int_{\Omega} r[2u_t u_{tt} + 2(C(u)u_t)_r C(u)u_r] drdt \\
= & \int\int_{\Omega} r[2u_t u_{tt} - 2C(u)u_t(C(u)u_r)_r] - 2C^2(u)u_t u_r drdt \\
+ & \int_{t_-} \frac{2rC^2(u)}{\sqrt{1+C^2}} u_r u_t ds - \int_{t_+} \frac{2rC^2(u)}{\sqrt{1+C^2}} u_r u_t ds \\
= & \int\int_{\Omega} 2ru_t(u_{tt} - C(u)(C(u)u_r)_r) - \frac{C^2(u)u_r}{r} drdt \\
+ & \int_{t_-} \frac{2rC^2(u)}{\sqrt{1+C^2}} u_r u_t ds - \int_{t_+} \frac{2tC^2(u)}{\sqrt{1+C^2}} u_r u_t ds \\
= & \int_{t_-} \frac{2rC^2(u)}{\sqrt{1+C^2}} u_r u_t ds - \int_{t_+} \frac{2rC^2(u)}{\sqrt{1+C^2}} u_r u_t ds.
\end{aligned}
$$

Hence, (3.12) becomes

$$
\begin{aligned}
& \int_{t_-} \frac{r}{\sqrt{1+C^2}} (u_t^2 - C^2 u_r^2) ds - \int_{t_+} \frac{r}{\sqrt{1+C^2}} (u_t^2 - C^2 u_r^2) ds \\
+ & \int_{t_-} \frac{2rC^2(u)}{\sqrt{1+C^2}} u_r u_t ds - \int_{t_+} \frac{2rC^2(u)}{\sqrt{1+C^2}} u_r u_t ds.
\end{aligned}
\tag{3.13}
$$

Combine (3.10), (3.11), and (3.13), we have

$$
\begin{aligned}
& \int_{t_-} \frac{rC}{\sqrt{1+C^2}} (u_t^2 + C^2 u_r^2) ds + \int_{t_+} \frac{rC}{\sqrt{1+C^2}} (u_t^2 + C^2 u_r^2) ds \\
- & \int_{t_-} \frac{2rC^2(u)}{\sqrt{1+C^2}} u_r u_t ds + \int_{t_+} \frac{2rC^2(u)}{\sqrt{1+C^2}} u_r u_t ds \\
= & \int_{t1}^{t2} r(u_t^2 + C^2 u_r^2) ds.
\end{aligned}
\tag{3.14}
$$

By (3.14) and total energy (3.9), we have

$$\begin{aligned}
&\int_{t_-} \frac{rC}{\sqrt{1+C^2}}(u_t^2 + C^2u_r^2 - 2Cu_ru_t)ds \\
+ &\int_{t_+} \frac{rC}{\sqrt{1+C^2}}(u_t^2 + C^2u_r^2 + 2Cu_ru_t)ds \\
= &\int_{t1}^{t2} r(u_t^2 + C^2u_r^2)dr
\end{aligned}$$

which means that,

$$\int_{t_-} \frac{rC}{\sqrt{1+C^2}}(u_t - Cu_r)^2 ds + \int_{t_+} \frac{rC}{\sqrt{1+C^2}}(u_t + Cu_r)^2 ds \le M\epsilon^2.$$

Hence, we can estimate u along characteristics

$$\frac{C_0}{\sqrt{1+C_1^2}} \int_{t_+} r(u_t + Cu_r)^2 ds \le \int_{t_+} \frac{rC}{\sqrt{1+C^2}}(u_t + Cu_r)^2 ds \le M\epsilon^2,$$

$$\int_{t_+} r(u_t + Cu_r)^2 ds \le K\epsilon^2. \tag{3.15}$$

Similarly,

$$\int_{t_-} r(u_t - Cu_r)^2 ds \le K\epsilon^2. \tag{3.16}$$

These estimates are not enough to conclude that the solution is very close to u_0. But from the equation for R and S on characteristic curve, we can see that the derivative of R along the minus characteristic line $\frac{dx}{dt} = -C(v)$,

$$\begin{aligned}
\frac{dR}{dt}(t, r_-(t)) &= R_t - C(v)R_r \\
&= \frac{C'}{4C\sqrt{r}}R^2 - [\frac{C'}{4C\sqrt{r}}S^2 + \frac{C}{2r}S].
\end{aligned}$$

And the derivative of S along the plus characteristic line $\frac{dx}{dt} = C(v)$ is

$$\begin{aligned}\frac{dS}{dt}(t, r_+(t)) &= S_t + C(v)S_r \\ &= \frac{C'}{4C\sqrt{r}}S^2 + \frac{C}{2r}S - \frac{C'}{4C\sqrt{r}}R^2.\end{aligned}$$

The idea is to construct an initial data such that $R(0,r) = 0, \quad S(0,r) \geq 0$, and $C'(v(0,r)) \geq 0$. Our hope is to prove the following:

1. $R \leq 0$ for $t \geq 0$,
2. $S \geq 0$ for $t \geq 0$,
3. R^2 is suitably controlled; and
4. S blows up in finite time.

If we could establish 1, 2 and 3, then 4 might follow easily since the equations for S has the form

$$\frac{dS}{dt}(t, r_+(t)) \geq \frac{h(t)S^2}{\sqrt{t}}$$

Since $r \sim t$ when $t \to \infty$. This differential inequality will generate the singularity of S in finite time. 1 and 2 can be shown if R is suitably controlled. From

$$\frac{dR}{dt}(t, r_-(t)) \geq -(\frac{C'}{4C\sqrt{r_-(t)}}S^2 + \frac{C}{2r_-(t)}S),$$

one can see the possibility if the integral of the right hand side along the minus characteristic r_- is small.

Note that we already knew this

$$\int_0^{t_0} S^2(t, r_-(t))dt < M\epsilon^2,$$

we need a better estimate of S near $r = 0$. Our conjecture is that this is true, we will continue to study this case carefully.

We also would like to remark that one can easily derive an nonlinear wave equation of the form

$$u_{tt} - C_1(u)(C_1(u)u_x)_x - C_2(u)(C_2(u)u_y)_y = 0.$$

from the theory of liquid crystals. An interesting case is that $u = u_0$ is maximum of $C_1(u)$ and simultaneously u_0 is the minimum of $C_2(u)$. In this case, do we have global solution close to u_0? It seems that there is no answer to this simple question.

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