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# Pattern Formation in Some Nonlinear Systems

By

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# ABSTRACT

**Pattern Formation in Some Nonlinear Systems**

By

*Peng Feng*

We study the pattern formation arising in some nonlinear systems. The problems come from material science and biological science. We study the instability in the corresponding mathematical models and explain the pattern formation. In particular, one mathematical model we studied is an elliptic equation and the other is a degenerate parabolic system. For the elliptic equation, we study the radial symmetric solutions and the bifurcation into nonradial ones. For the parabolic system, we study the existence of traveling wave solutions and the instability of the flat front.

To my family.

# ACKNOWLEDGMENTS

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I would like to express my gratitude to all those who gave me the possibility to finish this thesis. Especially I am obliged to Professor P. Bates, B.S. Yan, P. Lamm, M. Fraizier, M.X. Tang. I would also like to thank those for all their help and valuable conversations: Professor Emmanuele DiBenedetto of Vanderbilt University, Professor Yi Li of University of Iowa, Professor A. Friedman of Ohio State University.

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# Introduction

The diversity of the natural shapes that surround us has a profound impact on the quality of our lives. For this reason alone, it is not surprising that the origins of these shapes have been the subject of serious study since antiquity. It has long been believed that a quantitative study of natural forms will help us understanding their origins and behaviors. Many physicists and mathematicians have devoted to developing general approaches toward the quantitative description of the complex patterns that are characteristic of most natural phenomena.

Among these patterns, there are relatively simple ones that can arise in many simple and complex systems. For example, spiral and helix patterns are often found in sea shells. One of the most studied examples of spiral pattern-formation in reaction-diffusion systems is the Belousov-Zhabotinsky reaction. Labyrinthine patterns are also considered to be a relatively simple pattern. These maze-like patterns can be found in the well known Rayleigh-Béard convection experiments, in which a thin layer of fluid is heated from below. Another phenomenon that has been extensively studied is the viscous fingering in the well-known Hele-Shaw experiment in which a less viscous fluid is injected into a more viscous fluid in a Hele-Shaw cell.

D'Archy Thompson was one of the first to attempt a mathematical description of pattern forming process in his work "On growth and form" from 1917. Cross and Hohenberg provide an excellent technical introduction into the mathematical concepts of pattern formation. Below, we briefly describe the basic principles and

mathematical tools.

Pattern formation implies a change in symmetry. Spontaneous symmetry breaking is particularly evident in fluids, which are structureless in equilibrium but exhibit a surprisingly variety of patterns under nonequilibrium conditions.

In this thesis, we first present a mathematical model that describes the deflection of an elastic membrane in an electromechanical system. The membrane is supported by an annular boundary. The deflection of the membrane satisfies the following parabolic equation

$$\frac{\partial u}{\partial t} = \Delta u + \frac{\lambda}{(1-u)^2} \quad \text{in} \quad \Omega; \quad u = 0 \quad \text{on} \quad \partial\Omega; \quad u(x, y, 0) = 0. \tag{1}$$

Here $\Omega$ is an annulus.

In this dissertation, we show that the equilibrium solution does not necessarily inherit the radial symmetry of the supporting boundary. Instead, nonradial solution may bifurcate from the radial ones. We also study the exact multiplicity of such radial solutions. We also address the finite time touch down behavior in such equation. These issues will be presented in Chapter 1.

Pattern forming process can often be found in biological system. Recently, the pattern formed by a bacterial colony growing on a thin agar plate has attracted many mathematicians and biologists in an attempt to explain the intriguing patterns. A particular interesting pattern is shown in Fig. 1.

Reaction-diffusion system has been often used as continuous model to describe patterns and waves arising in far from equilibrium states. In spite that the equations look so simple, numerics has revealed that the equations generate complex but regulated spatio-temporal patterns. On the the typical system is the following

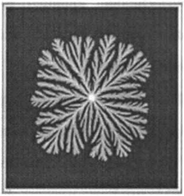


Figure 1. A typical pattern in bacterial colony

reaction-diffusion system for two components $u$ and $v$:

$$u_t = d_u \Delta u + uv^m - au - bu^n \tag{2}$$

$$v_t = d_v \Delta v - uv^m - a(1-v) \tag{3}$$

where $a$ and $b$ are positive constants. In particular, when $m = 2$ and $n = 1$, it is called Gray-Scott equations. For $m = 2$, it is known that the system generates diverse complex patterns in high dimensions depending on the values of $m$ and $n$. One typical phenomenon is the occurrence of spot patterns which are generated through self-replicating process.

On the other hand, when $a = 0$, (2)-(3) reduce to the following closed system

$$u_t = d_u \Delta u + uv^m - bu^n, \tag{4}$$

$$v_t = d_v \Delta v - uv^m. \tag{5}$$

In a bounded domain with the Neumann boundary conditions, it is proved that any solution $(u, v)$ becomes spatially homogeneous asymptotically, that is, there occurs no pattern formation at all. Therefore, it has long been believed that such systems are not so interesting from pattern formation viewpoints. Recently, Mimura et al found that such system generate complex patterns in transient process, although it generates no pattern asymptotically.

While diffusion alone tends to create uniform states, in 1952, Alan Turing suggested that diffusion coupled with chemical reaction may lead to spatial patterns in chemical composition. He speculated that such mechanisms would be sufficient to explain patterns such as zebra stripes.

Several similar systems have been proposed by different authors to describe the growth of bacterial colony on a thin plate. The systems have the following general form

$$\frac{\partial b}{\partial t} = \nabla \cdot \{D_b(b, n)\nabla b\} + g(b, n), \tag{6}$$

$$\frac{\partial n}{\partial t} = D_n \nabla^2 n - f(b, n), \tag{7}$$

where $D_b$ and $D_n$ are the diffusion coefficients of the bacteria and the nutrient, respectively. Here $b$ and $n$ represent the density of the bacteria and nutrient. The numerics showed that the system captures many patterns successfully.

Another purpose of this dissertation is to present a method to understand how patterns can arise in such a system. We study the traveling wave solutions to this system and the instability of the wave front. The results will be presented in Chapter 2.

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# CHAPTER 1

# Pattern Formation in A Mechanical System

## 1.1 Introduction and Summary of result

In 1968, at the age of 82, in the context of investigating fundamental questions in electrohydrodynamics, Geoffrey Ingram Taylor [*] one of the great physicists of the twentieth century, studied the electrostatic deflection of elastic membranes. Taylor's device is a soap film as the membrane material. He then applied a fixed high voltage potential difference between two supported circular membrane. He showed experimentally that the two membranes snap together and touch at a critical voltage. At smaller voltage, even though the membranes remained separate, they either became unstable or failed to exist. This instability is now known to researchers in microelectromechanical system (MEMS) and nanoelectromechanical systems (NEMS)

[*]Although many researchers are familiar with one or another of Taylors contributions, few seem to be aware of the incredible breadth of his scientific publications and their relevance to important research questions today. The same person who is commonly remembered as the namesake for several basic fluid flow instabilities (TaylorCouette, RayleighTaylor, and SaffmanTaylor) also was the first to show experimentally that a diffraction pattern produced by shining light on a needle does not change when the intensity of light is decreased. And these topics are only the beginning. Taylor made fundamental contributions to turbulence, championing the need for developing a statistical theory, and performing the first measurements of the effective diffusivity and viscosity of the atmosphere.

fields as the pull-in instability.

The use of electrostatic forces to provide actuation is now a method of central importance in MEMS and is of growing importance in NEMS. Here, we study the electrostatic deflection of an annular elastic membrane. We investigate whether electrostatically deflected membranes always inherit the radial symmetry of the membrane's domain.

In the recent work of Pelesko, Bernstein and McCuan [19], they derived a mathematical model for the deflection of the membrane at the equilibrium state. The model is essentially a semilinear elliptic equation with the corresponding boundary conditions. This model incorporated the voltage as a bifurcation parameter. They showed that asymmetric solutions exist through numerical investigation. A bifurcation diagram was obtained. They conjectured that there are an infinite number of branches intersecting the upper radially symmetric solution branch. However, they were unable to obtain the complete bifurcation diagram numerically where the equation is very close to become singular.

Motivated by their work, we shall study this problem theoretically and we will obtain the complete bifurcation diagram and prove their conjecture that there are indeed infinitely many symmetry breaking point at the upper branch of the radial solutions.

The problem we will study is the following semilinear elliptic equation with Dirichlet boundary condition:

$$-\Delta u = \frac{\lambda}{(1-u)^2} \quad \text{in} \quad \Omega, \tag{1.1}$$

$$u = 0 \quad \text{on} \quad \partial\Omega \tag{1.2}$$

where $\Omega = \{x \in R^2 : \epsilon_1 < |x| < 1\}$ is an annulus in $R^2$, $u$ is the displacement of the membrane. $\lambda$ serves as a bifurcation parameter and its exact meaning will be shown in the next section.

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We investigate the exact number of positive radial solutions and non-radially symmetric bifurcation of the above problem. The exact number of positive radial solutions may be 2, 1, or 0 depending on the value of $\lambda$. The upper branch of radial solutions has non-radially symmetric bifurcation at infinitely many $\lambda_N \in (0, \lambda^*)$. The proof of multiplicity result relies on the characterization of the shape of the time-map and shooting method. The proof of bifurcation result relies on a well known bifurcation theorem by Kielhöfer [11].

In fact, we can prove the following results:

**Theorem 1.1** *There exists a $\lambda^*$ such that the problem has no positive radial solution for $\lambda > \lambda^*$, one radial solution for $\lambda = \lambda^*$ and exactly two radial solutions for $0 < \lambda < \lambda^*$.*

**Theorem 1.2** *There exists infinitely many $\lambda_k \in (0, \lambda^*)$ such that the upper branch of radially symmetric solutions has a non-radially symmetric bifurcation at each $\lambda_k$, $k = 1, 2, \ldots$.*

The chapter is organized as follows. In section 1.2, we shall show the full derivation of the model proposed in [19]. In section 1.3, we show the existence results for small $\lambda$. In section 1.4, we obtain the multiplicity results and prove Theorem 1.1. In section 1.5, we study the radial symmetry breaking problem and prove Theorem 1.2. In section 1.6, we study the finite time touch down of the corresponding parabolic equation.

## 1.2 Formulation of the model

We model the device shown in Figure 1.1, which consists of an annular elastic membrane suspended above a rigid plate. The membrane is supported along the inner and outer boundaries. A voltage difference is applied across the device in order to

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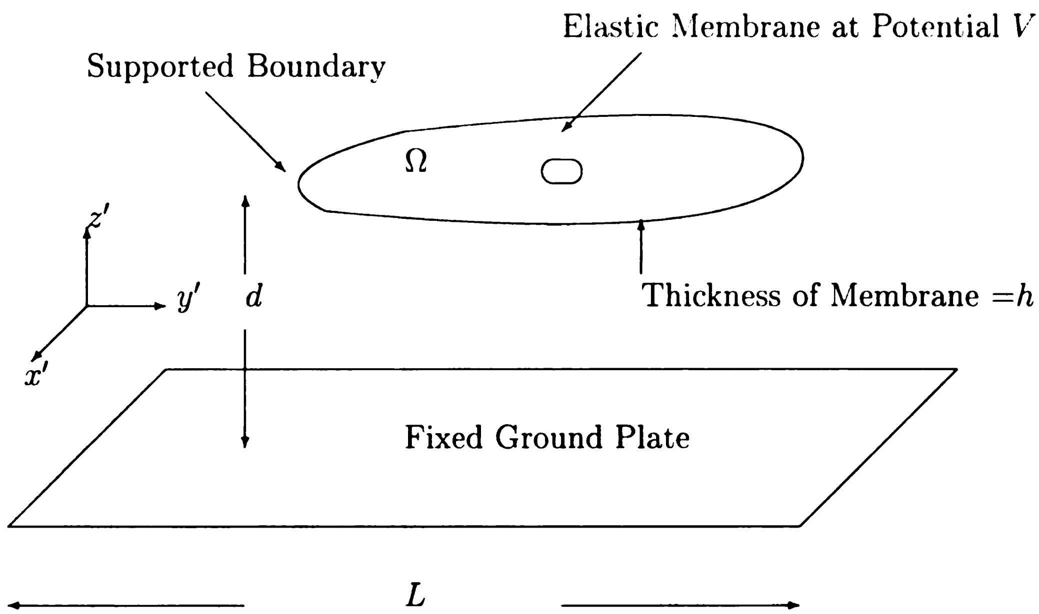


Figure 1.1. A basic electrostatically actuated elastic membrane. The prime coordinates indicate they have not yet been scaled.

cause deflection of the membrane. In particular, the upper surface of the membrane is held at potential $V$, while the ground plate is held at zero potential.

We shall note the fact that most MEMS devices are of small aspect ratio, and use thin components, *i.e.*, $h/d \ll 1$ in Figure 1.1. We derive an approximate solution.

We assume the electrostatic potential $\phi$ satisfies Laplace's equation everywhere away from the membrane and the plate:

$$\Delta\phi = 0. \tag{1.3}$$

It also satisfies appropriate boundary conditions on the membrane, that is,

$$\phi = V \quad \text{on} \quad \text{elastic} \quad \text{plate} \tag{1.4}$$

and

$$\phi = 0 \quad \text{on} \quad \text{ground} \quad \text{plate}. \tag{1.5}$$

We model the elastic plate using the plate equation. In particular, the deflection $u'$ of the membrane satisfies

$$\rho h \frac{\partial^2 u'}{\partial^2 t'} + a\frac{\partial u'}{\partial t'} - \mu \nabla_\perp^2 u' + D\nabla_\perp^4 u' = -\frac{\epsilon_0}{2} \mid \nabla \phi \mid^2 . \tag{1.6}$$

Here $\rho$ is the density of the membrane, $h$ is the thickness, $\mu$ is the tension in the membrane, $D$ is the flexural rigidity, and $\epsilon_0$ is the permittivity of free space. $\nabla_\perp$ represents the differentiation with respect to $x'$ and $y'$. The standard plate equation has been modified in two ways. First, a damping term has been added. The parameter $a$ is the damping constant. Second, we have assumed $a$ is proportional to velocity. We shall rescale the system and rewrite in dimensionless form. We rescale the electrostatic potential with the applied voltage, time with a damping timescale of the system, the $x'$ and $y'$ with a characteristic length of the device, and $z'$ and $u'$ with the size of the gap between the ground plate and the elastic membrane. We define

$$u = \frac{u'}{d}, \quad \phi = \frac{\phi}{V}, \quad x = \frac{x'}{L}, \quad y = \frac{y'}{L}, \quad z = \frac{z'}{d}, \quad t = \frac{\mu t'}{aL^2}. \tag{1.7}$$

In dimensionless form, we have

$$\begin{aligned}
\epsilon^2 \left( \frac{\partial^2 \phi}{\partial^2 x^2} + \frac{\partial^2 \phi}{\partial^2 y^2} \right) + \frac{\partial^2 \phi}{\partial^2 z^2} &= 0, && (1.8)\\
\phi &= 0 \quad \text{on} \quad \text{ground} \quad \text{plate}, && (1.9)\\
\phi &= 1 \quad \text{on} \quad \text{membrane}, && (1.10)\\
\frac{1}{\alpha^2}\frac{\partial^2 u}{\partial^2 t} + \frac{\partial u}{\partial t} - \nabla_\perp^2 u + \delta \nabla_\perp^4 u &= -\lambda \big[ \epsilon^2 \mid \nabla_\perp \phi \mid^2 + (\frac{\partial \phi}{\partial z})^2 \big]. && (1.11)
\end{aligned}$$

Here $\phi$ is a dimensionless potential scaled with respect to voltage $V$, $x$ and $y$ are scaled with respect to the length of the ground pate $L$, $z$ is scaled with respect to the gap size $d$. We assume the displacement of the membrane $u$ satisfies

$$\Delta u = \lambda \left[ \delta^2 \left(\frac{\partial^2 \phi}{\partial^2 x^2} + \frac{\partial^2 \phi}{\partial^2 y^2}\right) + \frac{\partial^2 \phi}{\partial^2 z^2} \right], \tag{1.12}$$

$$u = 0 \quad \text{on} \quad \text{boundary}. \tag{1.13}$$

Here $\alpha = \frac{aL}{\sqrt{\rho h \mu}}$ is the inverse of the quality factor for the system. $\delta = \frac{D}{L^2 \mu}$ measures the relative importance of tension and rigidity. $\epsilon = \frac{d}{L}$ is the aspect ratio of the system. $\lambda = \epsilon_0 V^2 L^2 / 2\mu d^3$, where $T$ is the tension in the membrane and $\epsilon_0$ is the permittivity of free space. Note that $\lambda$ is a dimensionless number which characterizes the relative strengths of electrostatic and mechanical forces in the system. As $\lambda$ is proportional to the applied voltage, it serves as a convenient bifurcation parameter.

Assuming $d \ll L$, that is $\epsilon \ll 1$. Physically, this means that the lateral dimension of the device are large compared to the gap between the membrane and the ground plate. For many MEMS systems this is an excellent approximation. We exploit the small-aspect ratio by setting $\epsilon$ goes to zero in equation (1.8). This reduces the electrostatic problem to

$$\frac{\partial^2 \phi}{\partial z^2} = 0, \tag{1.14}$$

which we may solve to find the approximate potential,

$$\phi \approx Az + B.$$

We are primarily concerned with the field between the plates and hence apply the boundary condition on $\phi$ which is

$$\phi(x, y, u, t) = 1,$$

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and

$$\phi(x, y, 0, t) = 0.$$

Hence

$$\phi \approx \frac{z}{u}.$$

Therefore, by sending $\epsilon$ goes to zero and use this approximate potential in equation (1.11), we find

$$\frac{1}{\alpha^2}\frac{\partial^2 u}{\partial^2 t} + \frac{\partial u}{\partial t} - \nabla_{\perp}^2 u + \delta \nabla_{\perp}^4 u = -\frac{\lambda}{u^2}. \tag{1.15}$$

We shall focus on the equilibrium state deflection and set the time derivatives in equation (1.15) to zero. We further simplify equation (1.15) by assuming that our elastic membrane has no rigidity, *i.e.*, $\delta = 0$ in (1.15). For convenience, we change variable $u \mapsto 1 - u$. The result is the following semi-linear elliptic equation for the displacement $u$:

$$-\Delta u = \frac{\lambda}{(1-u)^2} \quad \text{in} \quad \Omega, \tag{1.16}$$

$$u = 0 \quad \text{on} \quad \partial\Omega. \tag{1.17}$$

## 1.3 Existence

In this section, we shall study the following semilinear elliptic equation with Dirichlet boundary condition.

$$-\Delta u = \frac{\lambda}{(1-u)^2} \quad \text{on} \quad \Omega, \tag{1.18}$$

$$u = 0 \quad \text{on} \quad \partial\Omega. \tag{1.19}$$

First we show that there is no solution to our problem when $\lambda$ is sufficiently large. That is, the membrane fails to exist when the voltage is sufficiently large. We prove

**Theorem 1.3** *There exists a $\lambda^*$ such that when $\lambda > \lambda^*$ there is no solution to equa-*

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*tion (1.1) and equation (1.2).*

**Proof.** Let $\lambda_1$ be the lowest eigenvalue of

$$-\Delta u = \lambda u \quad \text{on} \quad \Omega, \tag{1.20}$$

$$u = 0 \quad \text{on} \quad \partial\Omega, \tag{1.21}$$

with $u_1$ is the corresponding eigenfunction which can be chosen strictly positive on $\Omega$.

Multiplying equation (1.1) by $u_1$ and integrating gives

$$\int_\Omega -u\Delta u_1 = \lambda \int_\Omega \frac{u_1}{(1-u)^2}.$$

Or,

$$\lambda_1 \int_\Omega uu_1 = \lambda \int_\Omega \frac{u_1}{(1-u)^2}.$$

Since $\frac{1}{(1-u)^2} \geq \frac{27}{4}u$, we have

$$\lambda_1 \int_\Omega uu_1 = \lambda \int_\Omega \frac{u_1}{(1-u)^2} \geq \frac{27}{4} \int_\Omega u_1 u.$$

Hence, $\lambda \leq \frac{\lambda_1}{27}$.

**Remark.** Any smooth solution $u$ must be nonnegative by the maximum principle.

Next we shall obtain the existence result for small $\lambda$. We have the following theorem.

**Theorem 1.4** *There exists a solution to equation (1.1) and equation (1.2) for some small $\lambda$.*

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To prove this theorem, we should apply the well-known method of upper and lower solutions [18]. Let us recall the following definition:

**Definition 1.1** *A function $\bar{u} \in C^2(\Omega)$ is called an upper solution of equation (1.1) and equation (1.2) if it satisfies the inequalities*

$$-\Delta \bar{u} \geq \frac{\lambda}{(1-\bar{u})^2} \quad \text{on} \quad \Omega, \tag{1.22}$$

$$\bar{u} \geq 0 \quad \text{on} \quad \partial\Omega. \tag{1.23}$$

*Similarly, $\underline{u}$ is called a lower solution if it satisfies all the reversed inequalities.*

The following two lemmas provide us with a proper choice of lower and upper solutions.

**Lemma 1.1** *Any constant $c < 0$ is a lower solution.*

**Lemma 1.2** $\underline{u} = \frac{1}{3}v_1$ *is an upper solution when $\lambda \leq \frac{4}{27}\alpha_1 m$. Here $\alpha_1$ and $v_1$ is the first eigenvalue and eigenfunction for the following problem:*

$$-\Delta v = \alpha v \quad \text{on} \quad \Omega', \tag{1.24}$$

$$v = 0 \quad \text{on} \quad \partial\Omega', \tag{1.25}$$

*where $\Omega'$ is a proper domain with smooth boundary which contains $\Omega$ and has been chosen such that $m \leq v_1 \leq 1$ on $\Omega$.*

**Proof.** It is sufficient to show that

$$-\Delta \bar{u} \geq \frac{\lambda}{(1-\bar{u})^2} \quad \text{on} \quad \Omega.$$

In fact,

$$-\Delta \bar{u} = -\frac{1}{3}\Delta v_1 = \frac{1}{3}\alpha_1 v_1 \geq \frac{1}{3}\alpha_1 m \geq \frac{\lambda}{3}\cdot\frac{27}{4} \geq \frac{\lambda}{(1-1/3v_1)^2} = \frac{\lambda}{(1-\bar{u})^2}.$$

This completes the proof.

Using Lemma 1.1 and 1.2 and the well-known theorem that a solution exists between an ordered pair of upper solution and lower solution, we obtain the existence result.

## 1.4 Multiplicity

In this section we are concerned with the multiplicity of positive radial solutions. Consider the following equations:

$$\Delta u + f(u) = 0 \quad \text{in} \quad \Omega, \tag{1.26}$$

$$u = 0 \quad \text{on} \quad \partial\Omega, \tag{1.27}$$

where $\Omega$ is either a radial or annular domain in $R^n$ and $f$ is a strictly convex $C^2$ function on $[0,\infty)$. According to the well-known result of Gidas, Ni and Nirenberg [7] every positive solution of the above equations is radially symmetric if the domain is a ball, *i.e.*, it's only a function of $r = |x|$. The number of positive solutions has been widely studied for different types of $f$ on a general bounded domain using variational and topological methods, see *e.g.*, [1, 2]. The problem is both fundamental and often difficult.

During the last decade, there has been tremendous progress in studying these problems when $\Omega$ is a ball or entire $R^n$, see, *e.g.*, [16, 17]. If the domain is a ball, then ODE techniques can be applied to get more information on the number of solutions [23]. Similarly, if the domain is a ball, we can also derive the exact multiplicity of radial solutions [8, 12, 13, 14, 25].

A radial solution has the form $u = u(r)$ where $r = |x| = (x_1^2 + \ldots + x_n^2)^{1/2}$. First note for $i = 1, \ldots, n$,

$$\frac{\partial r}{\partial x_i} = \frac{1}{2}(x_1^2 + \ldots + x_n^2)^{1/2} 2x_i = \frac{x_i}{r}.$$

We thus have

$$u_{x_i} = u'(r)\frac{x_i}{r}, \quad u_{x_i x_i} = u''(r)\frac{x_i^2}{r^2} + u'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right).$$

Consequently,

$$\Delta u = u''(r) + \frac{n-1}{r}u'(r).$$

Hence the radial solution of equation (1.1) and equation (1.2) satisfies the following equations

$$u''(r) + \frac{1}{r}u'(r) + \frac{\lambda}{(1-u)^2} = 0 \quad \text{in} \quad (\epsilon_1, 1), \tag{1.28}$$

$$u(\epsilon_1) = u(1) = 0. \tag{1.29}$$

Let $s = -\ln r, w(s) = u(r)$, then $w(s)$ satisfies

$$w'' + \lambda e^{-2s}\frac{1}{(1-w)^2} = 0 \quad \text{in} \quad (0, -\ln \epsilon_1), \tag{1.30}$$

$$w(0) = w(-\ln \epsilon_1) = 0. \tag{1.31}$$

Henceforth, we shall consider the following initial value problem

$$u''(r) + \lambda e^{-2r}\frac{1}{(1-u(r))^2} = 0 \quad \text{in} \quad (0, -\ln \epsilon_1), \tag{1.32}$$

$$u(0) = 0 \quad \text{and} \quad u'(0) = p. \tag{1.33}$$

**Definition 1.2** *Let $u(\cdot) = u(\cdot, p, \lambda)$ be the solution of equations (1.32) and (1.33) and we define the time-map associated to the above initial value problem to be the following function $R$:*

$$R(p, \lambda) = min\{R > 0 : u(R, p, \lambda) = 0\}.$$

We shall prove in the next lemma that $R(p, \lambda)$ is well defined for all $p$. By the boundary condition, $u$ has exactly one critical point, at which it takes the maximum value. We shall denote this critical point by $\tau(p, \lambda)$. Hence

$$u'(r) > 0 \quad \text{for} \quad r \in (0, \tau(p, \lambda)) \quad \text{and} \quad u'(r) < 0 \quad \text{for} \quad r \in (\tau(p, \lambda), R(p, \lambda)).$$

Also note that $u$ satisfies the following integral equation

$$u(r) = pr + \lambda \int_0^r (s - r)e^{-2s} \frac{1}{(1 - u(s))^2} ds.$$

To prove our multiplicity result, we need to establish several useful lemmas.

**Lemma 1.3** *$R(p, \lambda)$ is well defined.*

**Proof.** First we claim that it is indeed well defined for $p$ sufficiently small or large. Suppose otherwise that $\lim_{r \to +\infty} u'(r) = 0$. Multiplying equation (1.32) by $u'$ and integrating gives

$$\int_0^r u''(s)u'(s)ds = -\lambda \int_0^r \frac{e^{-2s}}{(1 - u(s))^2} u'(s)ds.$$

Hence

$$\frac{1}{2}u'(r)^2 - \frac{1}{2}p^2 = \lambda - \lambda \frac{e^{-2r}}{1 - u(r)} - 2\lambda \int_0^r \frac{e^{-2s}}{1 - u(s)} ds.$$

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Let $h(r) = \int_0^r \frac{e^{-2s}}{1-u(s)}ds$, then we have

$$\lambda h'(r) + 2\lambda h(r) - \frac{1}{2}p^2 + \frac{1}{2}u'(r)^2 - \lambda = 0.$$

When $r$ is sufficiently large,

$$\begin{aligned} h(r) &= \int_0^r \frac{e^{-2s}}{1-u(s)}ds \\ &= -\frac{1}{\lambda}\int_0^r u''(s)(1-u(s))ds \\ &= \frac{1}{\lambda}\left[-(1-u)u' + p - \int_0^r u'^2(s)ds\right] \\ &\leq \frac{p}{\lambda}. \end{aligned} \tag{1.34}$$

Hence for sufficiently large $r$,

$$\begin{aligned} \lambda h'(r) &= -2\lambda h(r) + \frac{1}{2}p^2 - \frac{1}{2}u'(r)^2 + \lambda \\ &\geq -2p + \frac{1}{2}p^2 - \frac{1}{2}u'(r)^2 + \lambda \\ &\geq c > 0 \end{aligned} \tag{1.35}$$

for some constant $c$ and $p$ either sufficiently large or small.

Therefore,

$$\frac{e^{-2r}}{1-u(r)} \geq \frac{c}{\lambda} > 0$$

for sufficiently large $r$. It follows that

$$\lim_{r\to+\infty} u(r) = 1$$

for $p$ sufficiently large or small. Applying L'Hopital's rule we have

$$\begin{aligned} \lim_{r\to+\infty} h'(r) &= \lim_{r\to+\infty} \frac{e^{-2r}}{1-u(r)} = \lim_{r\to+\infty} \frac{2e^{-2r}}{u'(r)} = \lim_{r\to+\infty} \frac{-4e^{-2r}}{u''(r)} \\ &= \lim_{r\to+\infty} 4e^{-2r}\frac{1-u(r)}{e^{-2r}} = \lim_{r\to+\infty} 4(1-u(r)) = 0. \end{aligned} \tag{1.36}$$

This is a contradiction to the previous conclusion that $h'(r) \geq c > 0$. Hence $R(p, \lambda)$ is well defined for $p$ sufficiently large and small. By continuous dependence on parameters, $R(p, \lambda)$ is well defined for all $p$. This completes the proof.

**Lemma 1.4**

$$\lim_{p\to 0+} R(p,\lambda) = \lim_{p\to 0+} \tau(p,\lambda) = 0.$$

**Proof.** Suppose otherwise, there exists a $\lambda > 0$, $\epsilon > 0$ and a sequence $p_k \to 0+$ such that

$$R_k \equiv R(p_k, \lambda) \geq \epsilon.$$

Since

$$\begin{aligned} u(r, p_k) &= p_k r + \lambda \int_0^r (s-r)e^{-2s}\frac{1}{(1-u(s))^2}ds \\ &\leq p_k r + \lambda \int_0^r (s-r)e^{-2s}ds \\ &= p_k r + \lambda\left[-\frac{1}{2}e^{-2s}(s-r)|_0^r + \int_0^r \frac{1}{2}e^{-2s}ds\right] \\ &= p_k r + \lambda\left[-\frac{r}{2} - \frac{e^{-2r}}{4} + \frac{1}{4}\right] \\ &< p_k r - \frac{\lambda r^2}{4}, \end{aligned} \tag{1.37}$$

Thus $R_k < \frac{4p_k}{\lambda}$.

Hence,

$$\begin{aligned} p_k R_k &= -\lambda \int_0^{R_k} (s - R_k) e^{-2s} \frac{1}{(1-u(s))^2} ds \\ &\geq \lambda \int_0^{\epsilon} (\epsilon - s) e^{-2s} ds \\ &> 0. \end{aligned} \tag{1.38}$$

This is a contradiction. Hence $\lim_{p\to 0+} R(p,\lambda) = 0$. It follows that $\lim_{p\to 0+} \tau(p,\lambda) = 0$. This completes the proof.

**Lemma 1.5**

$$\lim_{p\to+\infty} R(p,\lambda) = \lim_{p\to+\infty} \tau(p,\lambda) = 0.$$

**Proof.** Suppose $\lim_{p\to\infty} \tau(p,\lambda) \neq 0$, then there exists a $\tau_0 > 0$ and a sequence $p_k \to +\infty$ with $u_k(r) \equiv u(r, p_k, \lambda) > 0$ and $u_k'(r) > 0$ in $(0, \tau_0)$.
Let $\bar{\tau} = \tau_0/2$, we claim

$$\lim_{k\to+\infty} sup u_k(\bar{\tau}) = 1.$$

Otherwise, there exists $\epsilon > 0$ such that $0 < u_k(\bar{\tau}) \leq 1 - \epsilon$. It follows that

$$\begin{aligned} u_k(\bar{\tau}) &= p_k\bar{\tau} + \lambda \int_0^{\bar{\tau}} (r - \bar{\tau}) e^{-2r} \frac{1}{(1-u(r))^2} dr \\ &\geq p_k\bar{\tau} + \frac{\lambda}{\epsilon^2} \int_0^{\bar{\tau}} (r - \bar{\tau}) e^{-2r} dr \end{aligned} \tag{1.39}$$

which is impossible since $p_k \to +\infty$. Hence choosing a subsequence if necessary, we may assume

$$\lim_{k\to+\infty} u_k(\bar{\tau}) = 1.$$

Note that $u_k$ satisfies

$$u''(r) + \frac{\lambda e^{-2r}}{u_k(1-u_k)^2} u(r) = 0 \quad \text{in} \quad (\bar{\tau}, \tau_0).$$

Let

$$M_k = inf\{\frac{1}{u_k(1-u_k)^2} : r \in (\bar{\tau}, \tau_0)\},$$

then

$$\lim_{k\to+\infty} M_k = \infty.$$

Note that $\lambda e^{-2r} \geq \lambda e^{-2\tau_0}$ in $(\bar{\tau}, \tau_0)$. Let $v_k$ solves

$$v''(r) + \lambda e^{-2\tau_0} M_k v(r) = 0 \quad \text{in} \quad (\bar{\tau}, \tau_0).$$

It follows that $v_k$ has at least two zeros in $(\bar{\tau}, \tau_0)$ when $k$ is sufficiently large. By Sturm Comparison Principle, $u_k$ has at least one zero in $(\bar{\tau}, \tau_0)$. But this is impossible. Hence

$$\lim_{p\to+\infty} \tau(p, \lambda) = 0.$$

Finally, we prove $\lim_{p\to+\infty} R(p, \lambda) = 0$. Otherwise, there exists a point $r_0 > 0$ and a sequence $p_k \to +\infty$ with

$$u_k(r) > 0 \quad \text{and} \quad u_k'(r) \leq 0 \quad \text{in} \quad (\tau_k, r_0)$$

where $u_k \equiv u(r, p_k, \lambda)$ and $\tau_k \equiv \tau(p_k, \lambda)$.

Let $\bar{r} = \frac{r_0}{2}$, in view of previous lemma that $\lim_{p\to+\infty} \tau(p, \lambda) = 0$, we may assume $\bar{r} > \tau_k$ for any $k$. We claim that

$$\limsup_{k\to+\infty} u_k(\bar{r}) < 1.$$

Otherwise, by Sturm Comparison Principle again, $u_k$ has zeros in $(\tau_k, \bar{r})$ when $k$ is sufficiently large which is impossible since $\tau_k \to 0$ as $k \to +\infty$.

Note that

$$u'(r) = -\int_{\tau_k}^{r} \frac{\lambda e^{-2s}}{(1-u(s))^2} ds,$$

and

$$\left(\frac{1}{2}u'^2 + \frac{\lambda e^{-2r}}{1-u(r)}\right)' = -\frac{2\lambda e^{-2r}}{1-u(r)}. \tag{1.40}$$

Integrate equation (1.40) on $(\tau_k, \bar{r})$, we have

$$\frac{1}{2}u'(\bar{r})^2 = -\frac{\lambda e^{-2\bar{r}}}{1-u(\bar{r})} + \lambda\frac{e^{-2\tau_k}}{1-u(\tau_k)} - \int_{\tau_k}^{r} \frac{2\lambda e^{-2s}}{1-u(s)} ds.$$

On the other hand, we have

$$\begin{aligned}
\frac{1}{2}u'(r)^2 + \int_{\tau_k}^{r} \frac{2\lambda e^{-2s}}{1-u(s)} ds &\leq \frac{1}{2}u'(r)^2 + \int_{\tau_k}^{\bar{r}} \frac{2\lambda e^{-2s}}{(1-u(s))^2} ds \\
&\leq \frac{1}{2}u'(\bar{r})^2 + 2|u'(\bar{r})|.
\end{aligned} \tag{1.41}$$

Hence

$$-\frac{\lambda e^{-2\bar{r}}}{1-u(\bar{r})} + \lambda\frac{e^{-2\tau_k}}{1-u(\tau_k)} \leq \frac{1}{2}u'(\bar{r})^2 + 2|u'(\bar{r})|. \tag{1.42}$$

Integrate equation (1.40) on $(0, \tau_k)$, we have

$$\frac{\lambda e^{-2\tau_k}}{1-u(\tau_k)} + \int_0^{\tau_k} \frac{2\lambda e^{-2s}}{1-u(s)} ds = \frac{1}{2}p_k^2 + \lambda.$$

Therefore,

$$\frac{\lambda e^{-2\tau_k}}{1-u(\tau_k)} \geq \frac{1}{2}(\frac{1}{2}p_k^2 + \lambda). \tag{1.43}$$

Combining equation (1.42) and (1.43), we have

$$u'(\bar{r}) \to -\infty.$$

Thus for $r > \bar{r}$, we have

$$u_k(r_0) < u_k(\bar{r}) + u_k'(\bar{r})(r_0 - \bar{r}) \to -\infty,$$

a contradiction to $u_k(r_0) > 0$. This completes the proof.

**Lemma 1.6** *Define $\tilde{R} = \tilde{R}(\lambda) = sup\{R(p,\lambda), p > 0\}$. Then $\tilde{R}(\lambda)$ is strictly decreasing.*

**Proof.** Let $0 < \lambda_1 < \lambda_2$ and $u_2$ is a solution at $\lambda_2$ on $(0, \tilde{R}(\lambda_2))$. Let $v(s) = cu_2(r)$ with $r = s/c$ where $c$ is some constant greater but close to 1. It's easy to see $v(0) = 0$ and $v(\tilde{R}(\lambda_2) + \epsilon) = 0$ for $\epsilon = (c-1)\tilde{R}(\lambda_2)$.

Note that

$$\begin{aligned} v'' + \lambda_1 \frac{e^{-2s}}{(1-v(s))^2} &= \frac{1}{c}u_2''(r) + \lambda_1 \frac{e^{-2s}}{(1-cu_2(r))^2} \\ &= -\frac{1}{c}\left(\lambda_2 \frac{e^{-2r}}{(1-u_2(r))^2} - \lambda_1 \frac{e^{-2s}}{(1-cu_2(r))^2}\right) \le 0 \quad (1.44) \end{aligned}$$

When $c$ is sufficient close to 1. Hence $v$ is a lower solution for

$$v''(r) + \lambda_1 \frac{e^{-2r}}{(1-v)^2} = 0,$$

$$v(0) = 0, \quad v(\tilde{R}(\lambda_2) + \epsilon) = 0.$$

Hence $\tilde{R}(\lambda_1) \ge \tilde{R}(\lambda_2) + \epsilon$. Hence $\tilde{R}(\lambda)$ is strictly decreasing. This completes the proof.

**Lemma 1.7** $\lim_{\lambda \to 0+} \tilde{R}(\lambda) = +\infty, \quad \lim_{\lambda \to +\infty} \tilde{R}(\lambda) = 0$

**Proof.** Suppose $\lim_{\lambda \to 0+} \tilde{R}(\lambda) \ne +\infty$, then there exists a number $R^* > 0$ such that a sequence $\lambda_k \to 0+$ with $\lim_{k \to +\infty} \tilde{R}(\lambda_k) = \lim_{k \to +\infty} \tilde{R}(\lambda_k, p_k) = R^*$. Let us write

$u_k(r) = u(r, \lambda_k, p_k)$, then

$$\begin{aligned} 0 = u_k(R^*) &= p_k R^* + \lambda_k \int_0^{R^*} (s - R^*) e^{-2s} \frac{1}{(1 - u_k(s))^2} ds \\ &\geq p_k R^* + \frac{\lambda_k}{\epsilon^2} \int_0^{R^*} (s - R^*) e^{-2s} ds. \end{aligned} \tag{1.45}$$

Hence $p_k \to 0+$ or $R^* = 0$. But this contradicts the facts that $\lim_{p \to 0+} R(p, \lambda) = 0$ and $R(\lambda)$ is strictly decreasing. This completes the proof.

Finally for any given $\lambda$, we study the shape of $R(p)$. Note that $R(p)$ is determined by the implicit equation

$$u(R(p), p) = 0. \tag{1.46}$$

Differentiating equation (1.46) we get the following equations for the derivatives of R:

$$u_r(R(p), p) R'(p) + u_p(R(p), p) = 0, \tag{1.47}$$

$$u_{rr}(R(p), p) R'(p)^2 + 2u_{rp}(R(p), p) R'(p) + u_r(R(p), p) R''(p) + u_{pp}(R(p), p) = 0. \tag{1.48}$$

We write $h(r, p) = u_p(r, p)$, $z(r, p) = u_{pp}(r, p)$ and $v(r, p) = u_r(r, p)$, then equation (1.47) can be written as

$$v(R(p), p) R'(p) + h(R(p), p) = 0. \tag{1.49}$$

If $R'(p) = 0$, we conclude from equation (1.48) that

$$v(R(p), p) R''(p) + z(R(p), p) = 0. \tag{1.50}$$

Applying equation (1.50), we may prove the following important Lemma.

**Lemma 1.8** *For a given $\lambda$, if $R'(p) = 0$, then $R''(p) < 0$.*

**Proof.** Note that $h(r,p)$ satisfies the following initial problem

$$h'' + \frac{2\lambda e^{-2r}}{(1-u)^3} h(r,p) = 0, \tag{1.51}$$

$$h(0,p) = 0, \quad h'(0,p) = 1. \tag{1.52}$$

If $R'(p) = 0$, then equation (1.47) gives us $h(R(p),p) = 0$.

We claim that $h(r,p) > 0$ on $(0, R(p))$. Otherwise let $h(\xi(p),p) = 0$ and $h > 0$ on $(0, \xi(p))$. Note that $v$ satisfies the following

$$v'' + \frac{2\lambda e^{-2r}}{(1-u)^3} v - \frac{2\lambda e^{-2r}}{(1-u)^2} = 0, \tag{1.53}$$

$$v(0,p) = p, \quad v'(0,p) = -\lambda. \tag{1.54}$$

Recall that $v(\tau(p),p) = 0$. If $\xi(p) \geq \tau(p)$, then $v < 0$ on $(\xi(p), R(p))$. By Sturm Comparison Theorem, $v$ should have a zero on $(\xi(p), R(p))$ since $h(R(p),p) = 0$. This is impossible.

If $\xi(p) < \tau(p)$, then $v < 0$ on $(\tau(p), R(p))$. Since $0 = v(\tau(p),p) > h(\tau(p),p)$, by Sturm Second Comparison Theorem, $v > h$ on $(\tau(p), R(p))$ which is impossible since $h$ has to cross over $v$ and reaches zero at $R(p)$.

Next we claim $z(R(p),p) < 0$. Note that

$$z'' + \frac{2\lambda e^{-2r}}{(1-u)^3} z + \frac{6\lambda e^{-2r}}{(1-u)^4} h^2 = 0, \tag{1.55}$$

$$z(0,p) = 0, \quad z'(0,p) = 0. \tag{1.56}$$

We claim $z$ is negative in some neighborhood of 0. Otherwise by observing equation (1.55), we have $z'' < 0$. It follows that $z' < 0$ in the neighborhood of 0 since $z'(0,p) = 0$. This contradicts the assumption.

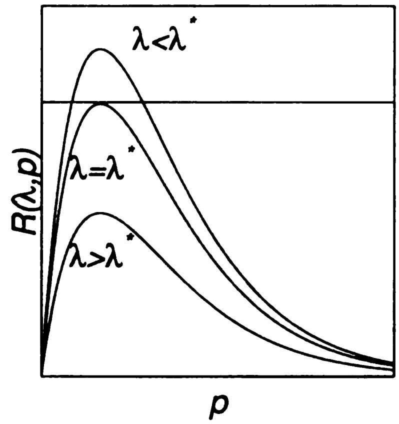


Figure 1.2. Timemap Diagram

Next we claim $z < 0$ in $(0, R(p)]$. Otherwise, let $z(r_1, p) = 0$ with $z < 0$ in $(0, r_1)$. Comparing equation (1.51) and equation (1.55), it follows that $h$ must have a zero in $(0, r_1)$ which contradicts our previous statement. Hence $z(R(p), p) < 0$ and it follows from equation (1.50) that $R''(p) < 0$.

We are now in position to prove **Theorem 1.1**.

**Proof.** In view of the above lemmas, we may obtain the timemap diagram as shown in Figure 1.2. From which we can easily conclude the theorem. In fact, for any given $\epsilon_1 > 0$, $\exists\ \lambda^*$ such that $\tilde{R}(\lambda^*) = -\ln \epsilon_1$ and there is a unique $q$ such that $R(\lambda^*, p) = -\ln \epsilon_1$, thus there exists a unique radial solution at $\lambda = \lambda^*$. For $\lambda < \lambda^*$, we can find $p_1, p_2$ such that $R(\lambda, p_1) = R(\lambda, p_2) = -\ln \epsilon_1$. The problem has two radial solutions in this case. For $\lambda > \lambda^*$, since $\tilde{R}(\lambda) < -\ln \epsilon_1$, there is no radial solution. This result is shown in Figure 1.3.

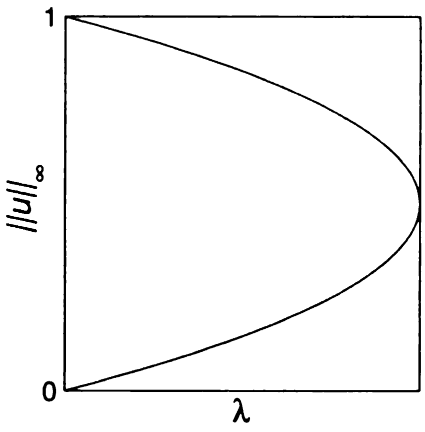


Figure 1.3. Bifurcation Diagram

## 1.5 Symmetry breaking

In previous section, we studied the multiplicity of radial solutions. Our purpose in this section is to study how radial symmetry can be broken, that is, to describe the bifurcations of these radial solutions to non-radial solutions. The problem of non-radial bifurcation from radial solutions on balls were studied by Dancer [4] and Smoller and Wasserman [22, 24], on an annulus by Lin [15] and others.

Let us first introduce a few basic concepts and the celebrated result by Crandall and Rabinowitz [3]. We will also illustrate the theorem by an example. Then we shall introduce a very useful bifurcation theorem due to Kielhöfer [11] which is what we applied in this thesis. For a summary on bifurcation theory, see for example [9].

We shall consider two real Banach Spaces, $U$ and $V$, as well as a nonlinear abstract operator

$$F: \quad R \times U \to V$$

of the form

$$F(\lambda, u) = L(\lambda)u + R(\lambda, u)$$

and the associated nonlinear abstract equation

$$F(\lambda, u) = 0$$

where the following assumptions are assumed to be satisfied:

- There exists $\lambda_0 \in R$ and $a, b \in R, \quad a < \lambda_0 < b$, such that $L(\lambda)$ is a linear operator from $U$ to $V$ for all $\lambda \in (a, b)$. Moreover, $\exists\ r \geq 2$ such that the map $\lambda \rightarrow L(\lambda)$ is of class $C^r$ and $L(\lambda_0)$ is a Fredholm operator of index zero.($R[L(\lambda_0)]$ is a closed subspace of $V$ and $dimN[L(\lambda_0)] = codimR[L(\lambda_0)] < \infty$.
- R is an operator of class $C^r$ such that $R(\lambda, 0) = 0$ and $D_u R(\lambda, 0) = 0$ for each $\lambda \in (a, b)$.

**Definition 1.3** *$(\lambda_0, 0)$ is a bifurcation point from the curve of $(\lambda, 0)$ if there exists a sequence $(\lambda_n, u_n) \in (a, b) \times (U \backslash \{0\})$ such that $\lim(\lambda_n, u_n) = (\lambda_0, 0)$ and $F(\lambda_n, u_n) = 0$.*

**Definition 1.4** *$\lambda_0$ is a nonlinear eigenvalue of $L(\lambda)$ if $(\lambda_0, 0)$ is a bifurcation point from the curve $(\lambda, 0)$ and $R(\lambda, u)$ satisfies the second assumption.*

In other word, $\lambda_0$ is a nonlinear eigenvalue of $L(\lambda)$ if the fact that bifurcation occurs is exclusively based on the linear part.

**Definition 1.5** *zero as a simple eigenvalue of $L(\lambda_0)$ if $N[L(\lambda_0)] \oplus R[L(\lambda_0)] = V$. For this, we need to assume $U \subset V$.*

**Definition 1.6** *$\lambda_0$ is an eigenvalue of the pair $(L_0, L_1)$ if zero is an eigenvalue of $L_0 - \lambda_0 L_1$ where $L_0 := L(\lambda_0)$ and $L_1 := \frac{dL}{d\lambda}(\lambda_0)$.*

Crandall and Rabinowitz found that if $F(\lambda, u)$ is of class $C^2$ and

$$dimN[L_0] = 1,$$

$$L_1(N[L_0]) \oplus R[L_0] = V,$$

then $(\lambda_0, 0)$ is a bifurcation point from $(\lambda, 0)$. Moreover, under these assumptions, the set of solutions bifurcating from $(\lambda, 0)$ at $\lambda = \lambda_0$ consists of a curve of class $C^1$. The second condition is usually referred to as the Crandall-Rabinowitz transversality condition, or nondegeneracy condition. Due to the huge number of applications of the result, it has become one of the most celebrated in nonlinear functional analysis. We shall state this theorem more rigorously as following

**Theorem 1.5** *(**Crandall-Rabinowitz**[3]) Suppose $F(\lambda, u)$ is of class $C^r$ for some $r \geq 2$ and zero is a simple eigenvalue of $(L_0, L_1)$. Let $Y \subset U$ be a subspace such that*

$$N[L_0] \oplus Y = U.$$

*Then, there exists $\epsilon > 0$ and two mappings of class $C^{r-1}$*

$$\lambda : (-\epsilon, \epsilon) \to R, \quad y : (-\epsilon, \epsilon) \to Y$$

*such that*

$$\lambda(0) = \lambda_0, \quad y(0) = 0$$

*and for each $s \in (-\epsilon, \epsilon)$*

$$F(\lambda(s), u(s)) = 0$$

$$u(s) := s(\phi_0 + y(s)).$$

*Moreover, there exists $\rho > 0$ such that if $F(\lambda, u) = 0$ and $(\lambda, u) \in B_\rho(\lambda_0, 0)$, then either $u = 0$ or $(\lambda, u) = (\lambda(s), u(s))$ for some $s \in (-\epsilon, \epsilon)$. Here $B_\rho(\lambda_0, 0)$ is a ball centered at $(\lambda_0, 0)$ with radius $\rho$. Furthermore, if $F$ is assumed to be real analytic, then so are $\lambda(s)$ and $u(s)$.*

Note that $(\lambda'(0), \phi_0)$ is the tangent vector to the curve $(\lambda(s), u(s))$ at $(\lambda_0, 0)$; it is usually called the *bifurcation direction.*

We shall illustrate the application of Crandall-Rabinowitz theorem by the following example.

**Example 1.1** *Consider the nonlinear boundary value problem*

$$-u''(x) = \lambda u(x)[1 + h(u(x))], \quad x \in (0,1), \tag{1.57}$$

$$u(0) = u(1) = 0, \tag{1.58}$$

*where $\lambda$ is a real parameter and $h$ is a function of class $C^3$ such that*

$$h(u) = h_2 u^2 + o(u^2)$$

*as $u \to 0$ with $h_2 \neq 0$.*

*Note that $u \equiv 0$ is a solution of eq. (1.57) for any $\lambda \in R$. Our goal is to apply Crandall-Rabinowitz theorem to show the existence of nonzero solution of eq. (1.57) with small amplitude and arbitrary nodal behaviour. To put it under the abstract framework introduced earlier, we consider the Banach spaces*

$$U = C_0^2([0,1]), \quad V = C([0,1])$$

*then the operator $F$ defined by*

$$F(\lambda, u) = u'' + \lambda u[1 + h(u)], \quad (\lambda, u) \in R \times U$$

*makes sense and is of class $C^3$ if $||u||_\infty$ is sufficiently small. We also have*

$$L(\lambda)u := u'' + \lambda u, \quad R(\lambda, u) := \lambda u h(u).$$

*It is easy to see that the eigenvalues of* $L(\lambda)$ *are*

$$\lambda_n = n^2\pi^2$$

*where* $n \in N$, $n \geq 1$. *Moreover, for each* $n \geq 1$,

$$N[L(\lambda_n)] = span[\sin(n\pi x)]$$

*and due to Fredholm alternative,*

$$R[L(\lambda_n)] = \left\{ v \in V : \int_0^1 v(x)\sin(n\pi x)dx = 0 \right\}.$$

*Thus, for any integer* $n \geq 1$,

$$sin(n\pi \cdot x) \notin R[(L\lambda_n)]$$

*and hence zero is a simple eigenvalue of* $L(\lambda_n)$. *Therefore, applying Crandall-Rabinowitz theorem, we have the following result*

***Corollary 1.1*** *Let* $n \geq 1$ *be an integer and consider*

$$Y_n := \left\{ u \in U : \int_0^1 u(x)\sin(n\pi x)dx = 0 \right\}.$$

*Then there exist* $\epsilon > 0$ *and two mappings:*

$$\Lambda_n : (-\epsilon, \epsilon) \to R \quad y_n : (-\epsilon, \epsilon) \to Y_n$$

*of class* $C^2$ *such that*

$$\Lambda_n(0) = \lambda_n, \quad y_n(0) = 0$$

*and for each* $s \in (-\epsilon, \epsilon)$

$$F(\Lambda_n(s), s[\sin(n\pi\cdot) + y_n(s)]) = 0.$$

*Moreover, if* $(\lambda, u)$ *is sufficiently close to* $(\lambda_n, 0)$ *and* $F(\lambda, u) = 0$*, then either* $u = 0$ *or there exists* $s \in (-\epsilon, \epsilon)$ *for which*

$$(\lambda, u) = (\Lambda_n(s), s[\sin(n\pi\cdot) + y_n(x)]).$$

*Therefore, for any integer* $n \geq 1$*, eq. (1.57) has a curve of solutions* $(\Lambda_n(s), u_n(s))$ *emanating from* $(\lambda, 0)$ *at* $\lambda = \lambda_n$ *where* $u_n(s) := s[\sin(n\pi\cdot) + y_n(x)]$.

Let $a(\lambda)$ denote the classical eigenvalue of the family $L(\lambda)$ perturbed from the zero eigenvalue of $L(\lambda_0)$:

$$L(\lambda)\phi(\lambda) = a(\lambda)\phi(\lambda),$$

$$0 \neq \phi(\lambda) \in D(L),$$

$$a(\lambda_0) = 0, \quad \phi(\lambda_0) = \phi_0,$$

where $\phi_0$ spans $N(L(\lambda_0))$. Then the following result holds

**Proposition 1.1** *If zero is a simple eigenvalue of* $L(\lambda_0)$*, then* $a'(\lambda_0) \neq 0$ *if and only if zero is a simple eigenvalue of the pair* $(L_0, L_1)$*,i.e.,* $L_1(N[L_0]) \oplus R[L_0] = V$.

As we recall Crandall's theorem which states that if zero is a simple eigenvalue of the pair $(L_0, L_1)$, then $(\lambda_0, 0)$ is a bifurcation point. In other word, $(\lambda_0, 0)$ is a bifurcation point if $a'(\lambda_0) \neq 0$. However, the meaning of nondegeneracy condition is not so clear, in [3] it is a technical condition which was just needed to apply the implicit function theorem to a modified equation where the trivial solution was eliminated.

The following theorem deals with degenerate eigenvalue and is due to Kielhöfer [11].

**Theorem 1.6** *Assume $U \subset V$ and zero is a simple eigenvalue of $L(\lambda_0)$. Then $\lambda_0$ is a nonlinear eigenvalue of $L(\lambda)$ if and only if $a(\lambda)$ changes sign as $\lambda$ crosses $\lambda_0$.*

With the aid of this result, we now study the symmetry breaking problem. We shall consider the linearized problem about a given radial solution $u$:

$$\Delta w + \frac{2\lambda}{(1-u)^3} w = 0.$$

We may write $w$ in the spherical harmonic decomposition form:

$$w = \sum_{N=0}^{\infty} a_N(r)\Phi_N(\theta),$$

and $a_N$ satisfies the equation:

$$a_N'' + \frac{1}{r}a_N' + \left[\frac{2\lambda}{(1-u)^3} - \frac{N^2}{r^2}\right] a_N = 0$$

together with the boundary conditions $a_N(1) = 0 = a_N(\epsilon_1)$.

If the above equation admits a nonzero solution $a_N \neq 0$ for some $N \geq 1$, then radial symmetry breaks. We consider the following eigenvalue problem:

$$a_N'' + \frac{1}{r}a_N' + \left[\frac{2\lambda}{(1-u)^3} - \frac{N^2}{r^2}\right] a_N = -\mu_{N,k} a_N.$$

Let $U = C_0^2(\epsilon_1, 1)$ and $V = C(\epsilon_1, 1)$.

We have the following lemma.

**Lemma 1.9** *If $u$ is a radial solution on the upper branch, then for arbitrary positive integer $N$, $\mu_{N,1}(\lambda) < 0$ for $\lambda$ sufficiently close to zero.*

**Proof.** It is well known that the eigenvalue $\mu_{N,1}(\lambda)$ can be characterized by

$$\mu_{N,1} = \inf \left\{ \frac{Q(\phi)}{\int_{\epsilon_1}^{1} r\phi^2 dr}, \quad \phi \in C_0^2([\epsilon_1, 1]) \right\},$$

where

$$Q(\phi) = \int_{\epsilon_1}^{1} r\left(\phi'^2 - \frac{2\lambda}{(1-u)^3}\phi^2 + N^2 r^{-2}\phi^2\right) dr.$$

If $u$ is a positive radial solution, then

$$\int_\Omega |\nabla u|^2 = \lambda \int_\Omega \frac{u}{(1-u)^2}.$$

Since $u$ is a solution on the upper branch, $||u||_\infty \to 1$ as $\lambda \to 0+$. Note that for arbitrary $p > 0$, there exists $\alpha > 0$ such that

$$\frac{2u}{1-u} \geq p \quad \text{for} \quad u \geq 1 - \alpha.$$

Let $\Omega_1 = \{x \in \Omega : u \geq 1 - \alpha\}$, $\Omega_2 = \{x \in \Omega : u < 1 - \alpha\}$.

Hence

$$\begin{aligned}
2\pi Q(u) &= \lambda \int_\Omega \left(\frac{1}{(1-u)^2} - \frac{2u}{(1-u)^3}\right) u + N^2 \int_\Omega \frac{u^2}{r^2} \\
&= \int_\Omega |\nabla u|^2 - \lambda \int_\Omega \frac{2u}{(1-u)} \cdot \frac{1}{(1-u)^2} u + N^2 \int_\Omega \frac{u^2}{r^2} \\
&\leq (1-p) \int_\Omega |\nabla u|^2 + \frac{N^2}{\epsilon_1^2} \int_\Omega u^2 - \int_{\Omega_2} \left[p - \frac{2u}{(1-u)}\right] \cdot \frac{\lambda}{(1-u)^2} u \\
&\leq (1 - p + \frac{N^2}{\epsilon_1^2 \nu_1}) \int_\Omega |\nabla u|^2 - M
\end{aligned} \tag{1.59}$$

for some constant $M > 0$ which is independent of $\lambda$.

Hence for any given $N > 0$, $\mu_{N,1} < 0$ since $p > 0$ can always be chosen to be sufficiently large.

**Remark.** It's easy to see that if $u$ is an upper branch solution, then

$$(\int_\Omega |\nabla u|^2)^{\frac{1}{2}} \geq \sqrt{\frac{2\pi\epsilon_1}{1-\epsilon_1}}$$

as $\lambda \to 0$. In fact,

$$\begin{aligned} u(r) &= \int_{\epsilon_1}^{r} u'(s)ds \leq (1-\epsilon_1)^{\frac{1}{2}}(\int_{\epsilon_1}^{1}(u'(s))^2 ds)^{\frac{1}{2}} \\ &\leq (1-\epsilon_1)^{\frac{1}{2}}\frac{1}{\sqrt{2\pi\epsilon_1}}(\int_\Omega |\nabla u|^2)^{\frac{1}{2}} \end{aligned} \tag{1.60}$$

**Lemma 1.10** $\mu_{0,1}(\lambda^*) = 0$.

**Proof.** Let $\mu_1(\lambda)$ be the principal eigenvalue of

$$\psi'' + \frac{1}{r}\psi' = -\mu\frac{2}{(1-u)^3}\psi, \ x \in (\epsilon_1, 1),$$

$$\psi(\epsilon_1) = \psi(1) = 0.$$

Clearly,

$$\mu_1(\lambda) = \inf\left\{\int_{\epsilon_1}^{1} r\psi'^2 / \int_{\epsilon_1}^{1}\frac{2r}{(1-u)^3}\psi^2, \quad \psi \in C_0^2([\epsilon_1, 1])\right\}.$$

On the other hand, note that

$$\mu_{N,1} = \inf\left\{Q(\phi)/\int_{\epsilon_1}^{1} r\phi^2 dr, \quad \phi \in C_0^2([\epsilon_1, 1])\right\},$$

where

$$Q(\phi) = \int_{\epsilon_1}^{1} r\left(\phi'^2 - \frac{2\lambda}{(1-u)^3}\phi^2\right)dr.$$

Thus

$$\mu_{0,1} \geq \inf\left\{[\mu_1(\lambda) - \lambda]\cdot\frac{1}{Q(\phi)}\int_{\epsilon_1}^{1}\frac{2r}{(1-u)^3}\phi^2 dr\right\}.$$

By a more general result by Keller [10] which states that $\mu_1(\lambda^*) = \lambda^*$, it follows that $\mu_{0,1}(\lambda^*) = 0$.

**Definition 1.7** $\lambda_0$ *is called a non-radial bifurcation point with mode* $k$ *if* $\mu_{k,1}(\lambda_0) = 0$ *where* $k$ *is a positive integer.*

We now apply the above result to prove the symmetry breaking result.

**Proof of Theorem 1.2.** Since $\mu_{0,1}(\lambda^*) = 0$, it follows that $\mu_{N,1}(\lambda^*) > 0$ for $N \geq 1$. By Lemma 1.9, for any $N \geq 1$ there exists $\lambda_N \in (0, \lambda^*)$ such that $\mu_{N,1}(\lambda_N) = 0$ and $\mu(\lambda)$ changes sign as $\lambda$ crosses $\lambda_N$. See Figure 1.4. Hence by Theorem 1.6, there is a bifurcation at $\lambda_N$ where the radial symmetry breaks. The proof is completed.

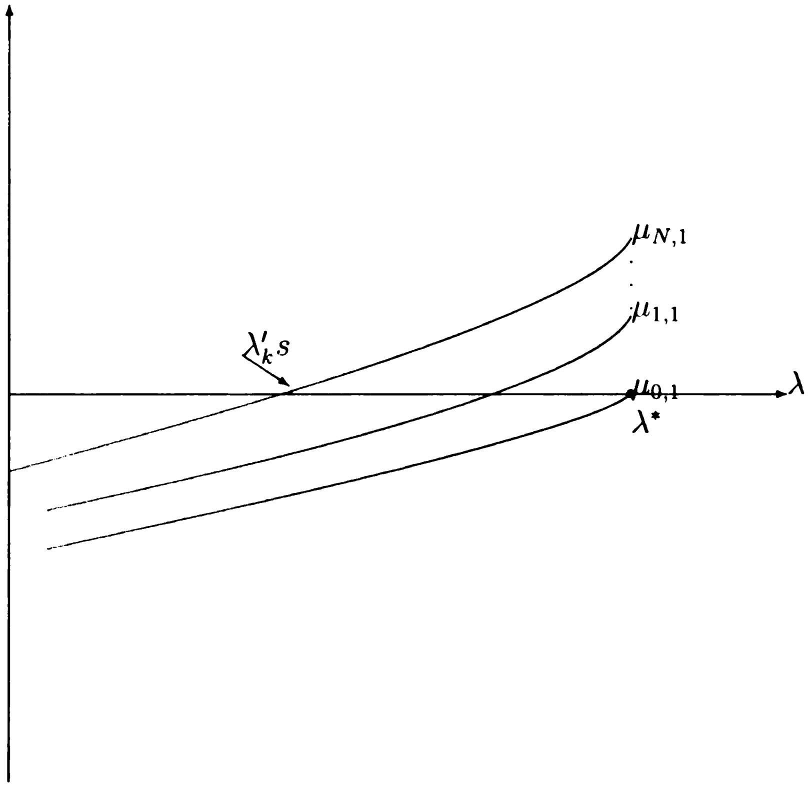


Figure 1.4. The change of sign as $\mu_{k,1}$ crosses $\lambda$ axis at $\lambda_k$

## 1.6 Finite Time Touchdown and Touchdown Profile

In this section, we modify a recent result by Flores et al. [6] and show that above a critical voltage, the membrane will touch the plate in finite time. These are apparently more important issue in the actual design of a MEMS device. For example, how to increase the stable operating range by increasing the critical voltage (pull-in voltage) $V_*$. This increase in the stable operating range may be important for the design of microresonators. Another example is how to decrease the time for touchdown, thereby increasing the speed of switch.

Now we consider the following time-dependent deflection of the membrane

$u(x, y, t)$ which satisfies

$$\frac{\partial u}{\partial t} = \Delta u + \frac{\lambda}{(1-u)^2} \quad \text{in} \quad \Omega; \quad u = 0 \quad \text{on} \quad \partial\Omega; \quad u(x, y, 0) = 0. \tag{1.61}$$

The solution $u$ of (1.61) is said to touchdown at finite time if the maximum value of $u$ reaches 1 at some finite time $t = T_* < \infty$. At such time, the membrane touches the fixed ground plate. We prove the following theorem.

**Theorem 1.7** *Let $\lambda_1 > 0$ and $u_1$ be the smallest eigenvalue and the corresponding eigenfunction of the Dirichlet eigenvalue problem*

$$-\Delta u = \lambda u \quad \text{on} \quad \Omega; \quad u = 0 \quad \text{on} \quad \partial\Omega. \tag{1.62}$$

*If $\lambda > \bar{\lambda} \equiv \frac{4\lambda_1}{27}$, then the solution $u$ of* (1.61) *reachese -1 at finite time.*

**Proof.** This proof is based on a key result in [6].

Without loss of generality we assume that $u_1 > 0$ in $\Omega$, and we normalize $u_1$ so that $\int_\Omega u_1 = 1$. Multiplying (1.61) by $u_1$ and integrating over $\Omega$, we have

$$\frac{d}{dt}\int_\Omega u_1 u = \int_\Omega u_1 \Delta u + \lambda \int_\Omega \frac{u_1}{(1-u)^2}.$$

Using Green's Theorem, we obtain

$$\frac{d}{dt}\int_\Omega u_1 u = -\lambda_1 \int_\Omega u_1 u + \lambda \int_\Omega \frac{u_1}{(1-u)^2}.$$

Introducing an energy-type variable $E(t)$ by $E(t) = \int_\Omega u_1 u$, we have

$$\frac{dE}{dt} + \lambda_1 E = \lambda \int_\Omega \frac{u_1}{(1-u)^2}. \tag{1.63}$$

Applying Jensen's inequality on the right-hand side of (1.63), we obtain

$$\frac{dE}{dt} + \lambda_1 E \geq \frac{\lambda}{(1-E)^2}.$$

On the other hand, we have

$$E(0) = 0, \quad E(t) \leq \sup_\Omega u \int_\Omega u_1 = \sup_\Omega u.$$

Now we let $F(t)$ solves

$$\frac{dF}{dt} + \lambda_1 F = \frac{\lambda}{(1-F)^2}, \quad F(0) = 0. \tag{1.64}$$

It follows by a standard comparison principle that $E(T) \geq F(t)$ on the domains of existence. Therefore, we conclude

$$F(t) \leq E(t) \leq \sup_\Omega u.$$

On the other hand, we may determine $t$ in terms of $F$ by separating variables in (1.64). And we have that when

$$T = \int_0^1 \left[\lambda_1 s - \frac{\lambda}{(1-s)^2}\right]^{-1} ds$$

$F = 1$.

$T$ is finite if the integral is finite. A simple calculation shows that the integrand is finite if $\lambda > \bar{\lambda} \equiv \frac{4\lambda_1}{27}$. Thus if $t = T_*$ where $T_* < T$, $u$ reaches 1.

# CHAPTER 2

# Pattern Formation in A Biological System

## 2.1 Description of Biological Experiment

Bacteria grown on thin agar plate may develop colonies of various spatial patterns, depending on both bacterial species and environmental conditions. We briefly describe an experimental study done by Ohgiwari (1992). They used a bacterial strain of *Bacillus subtilis.* The bacteria was point-inoculated at the center of an agar plate containing peptone as a nutrient in a plastic petri dish with a diameter of 88mm.

The swimming of the bacteria is a random walk type of movement and can be done only in a fluid with low viscosity. To produce such fluid the bacteria cooperatively secrete lubricant in which they can swim.

In order to move, reproduce and perform other metabolic activities, the bacteria consume nutrients which are given in limited supply. The growth of a colony is thus limited by the diffusion of nutrient toward the colony—-the bacterial reproduction rate is limited by the level of nutrient available for the cells. If the nutrient is deficient for a long enough time, the bacteria begin the process of sporulation. They stop normal activities and change into a spore. The sporulating bacteria may emit a wide range of materials, some of which are unique to sporulating bacteria. These emitted chemicals might be used to signal other bacteria about the condition at the location of spores.

Even though the initial seed of bacteria is radially symmetric, the colonies may develop various morphological patterns in response to environmental conditions. Observations of such patterns occurred as early as 1938. It was first reported by Fujikawa and Matsushita [32] that the colony may exhibit branching patterns similar to the type known from the study of fractal formation in the process of diffusion-limited aggregation (DLA). When the agar medium becomes softer, the colony turns to show a dense-branching morphology (DBM) with a smooth circular envelope. When both the nutrient concentration and the softness of the agar are high, the colony grows almost homogeneously into a radial shape. Other types of patterns such as concentric rings and Eden-like pattern may also appear.

## 2.2 General Mathematical Model

To explain each characteristic colony pattern, various models have been developed. For example, a diffusion limited aggregation model for DLA pattern proposed by Fujikawa and Matsushita [33], a communicating walkers model for DLA and DBM patterns by Ben-Jacob et al. [26, 27]. In a recent paper, Kawasaki et al. [39] developed a simple reaction diffusion model which closely captured all five different colony patterns. Other reaction diffusion models can be found in, for example, Kitsunezaki et al. [40], Matsushita et al. [43], Lacasta et al. [41].

All of the above models contains the population density $b(x, t)$ of the bacterial cells at time $t$ and spatial location $x$ and the concentration of the nutrient $n(x, t)$.

The models can be generalized into the form

$$\frac{\partial b}{\partial t} = \nabla \cdot \{D_b(b,n)\nabla b\} + g(b,n), \tag{2.1}$$
$$\frac{\partial n}{\partial t} = D_n \nabla^2 n - f(b,n), \tag{2.2}$$

where $D_b$ and $D_n$ are the diffusion coefficients of the bacteria and the nutrient, respectively. It is usually assumed that the diffusion coefficient $D_n$ is a constant, while the diffusion coefficient of the bacteria $D_b$ depends on both bacterial density and the nutrient concentration.

In [39], they chose $D_b(b,n) = \sigma nb$, $g(b,n) = nb$, $f(b,n) = \kappa nb$.

In [40], $D_b(b,n) = \sigma b^k$, $g(b,n) = nb - \mu b$, $f(b,n) = bn$. Note that the first term in $g(b,n)$ represents the rate at which the nutrient is consumed, the second term represents bacteria becoming stationary.

The initial concentration of bacteria and nutrient is set to be

$$b(x,0) = b_0(x),$$

$$n(x,0) = n_0,$$

where $n_0$ is a positive constant since the nutrient is initially uniformly distributed, and $b_0(x)$ is the initial bacteria density which is a function with compact support.

A typical numerical simulation is shown in the next figure.

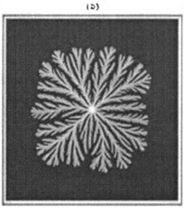

Figure 2.1. A typical numerical simulation

## 2.3 Existence and Large Time Behavior of Weak Solution

We consider the following nonlinear diffusion system:

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \Delta(u^k) + u^m v - a(u,v)u^n \quad \text{in} \quad Q_T := \Omega \times (0,T), &(2.3)\\
\frac{\partial v}{\partial t} &= d\Delta(v^l) - u^m v \quad \text{in} \quad Q_T, &(2.4)
\end{aligned}
$$

where $k, l, m$ and $n$ are positive integers, $d$ is a positive constant and $a(u,v)$ is a strictly positive function of $u$ and $v$. $\Omega$ is a smooth domain of $R^N$ and $T > 0$. The following boundary and initial conditions are assumed

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{2.5}$$

$$(u(x,0), v(x,0)) = (u_0(x), v_0(x)) \quad \text{for all } x \in \Omega. \tag{2.6}$$

In the context of bacteria growth, $u$ is the bacteria density and $v$ is the nutrient density.

When $m = n = 1$ and $k = l = 1$ and $a(u, v)$ is a constant, the model is the well-known Gray-Scott model in chemistry. A special case is the scalar equation

$$u_t = \Delta(u^k), \tag{2.7}$$

which is called the porous medium equation for $k > 1$. To understand some basic behavior of the solution, we shall first present this equation as an example. Specifically, we look for a "similarity" solution of the porous medium equation and observe its property.

**Example 2.1** *Let us look for a solution $u$ with the form*

$$u(x,t) = \frac{1}{t^\alpha} w(\frac{x}{t^\beta}), \quad x \in R^n \tag{2.8}$$

*where the constants $\alpha$, $\beta$ and the function $w : R^n \to R$ are to be determined.*

*We insert (2.8) into (2.7) and obtain*

$$\alpha t^{-(\alpha+1)} w(y) + \beta t^{-(\alpha+1)} y \cdot Dw(y) + t^{-(\alpha k + 2\beta)} \Delta(w^k)(y) = 0 \tag{2.9}$$

*for $y = t^{-\beta} x$. To put (2.9) into an expression involving only variable $y$, we require*

$$\alpha + 1 = \alpha k + 2\beta.$$

*Then (2.9) reduces to*

$$\alpha w + \beta y \cdot Dw + \Delta(w^k) = 0. \tag{2.10}$$

*Let us further assume that $w$ is in fact radial, that is, $w(y) = \omega(|y|)$. Then (2.10)* *becomes*

$$\alpha\omega + \beta r\omega' + (\omega^k)'' + \frac{n-1}{r}(\omega^k)' = 0, \tag{2.11}$$

*where* $r = |y|$, $' = \frac{d}{dr}$. *Now if we set* $\alpha = n\beta$, *we have*

$$(r^{n-1}(\omega^k)')' + \beta(r^n\omega)' = 0.$$

*Thus*

$$r^{n-1}(\omega^k)' + \beta r^n\omega = a$$

*for some constant* $a$. *Assuming* $\lim_{r\to\infty}\omega = \lim_{r\to\infty}\omega' = 0$, *we conclude* $a$ *is in fact 0. Hence*

$$(\omega^{k-1})' = -\frac{k-1}{k}\beta r.$$

*Consequently,*

$$\omega^{k-1} = b - \frac{k-1}{2k}\beta r^2,$$

*where* $b$ *is a constant; and so*

$$\omega = \left(b - \frac{k-1}{2k}\beta r^2\right)_+^{(k-1)^{-1}},$$

*where we took the positive part of the right hand side to ensure* $\omega \geq 0$. *Recalling* $w(y) = \omega(r)$ *and the rescaling, we obtain*

$$u(x,t) = \frac{1}{t^\alpha}\left(b - \frac{k-1}{2k}\beta\frac{|x|^2}{t^{2\beta}}\right)_+^{(k-1)^{-1}},$$

*where*

$$\alpha = \frac{n}{n(k-1)+2}, \quad \beta = \frac{1}{n(k-1)+2}.$$

*This solution is called Barenblatt's solution to the porous medium equation.*

*Based upon this special solution, one can easily observe that it has compact support* *for each time $t > 0$. This is a general feature for nonnegative weak solutions of the porous medium equation with compactly supported initial data. The porous medium equation becomes degenerate wherever $u = 0$, and this type of parabolic equation is generally called degenerate parabolic equation. So the set $u > 0$ moves with a finite propagation speed which is a general feature of degenerate parabolic equation. Thus it is often regarded as a better model of diffusive spreading than the linear heat equation (which predicts infinite propagation speed).*

*One may also observe that for $k = 2$, the first derivative of the solution is discontinuous but finite at the boundary of the support, and for $k > 2$, the first derivative diverges at the boundary of the support. This is also a general feature of degenerate parabolic equation, that is, the solution may not be classical. Hence one shall seek solutions in a weaker sense.*

**Definition 2.1** *We say $(u, v)$ is a weak solution on $[0, T]$ if it satisfies:*

*(i) $u,\ v \in C(\bar{Q}_T)$ and $u,\ v \geq 0$,*

*(ii) For all $\phi \in C^{2,1}(\bar{Q}_T)$ such that $\dfrac{\partial \phi}{\partial \nu} = 0$ on $\partial\Omega \times [0, T]$, we have for all $t \in [0, T]$:*

$$\begin{aligned} \int_\Omega u(t)\phi(t) &= \int_\Omega u_0\phi(0) + \int_0^t \int_\Omega (u^k \Delta\phi + u\phi_t + (u^m v - a(u, v)u^n)\phi), && (2.12)\\ \int_\Omega v(t)\phi(t) &= \int_\Omega v_0\phi(0) + \int_0^t \int_\Omega (dv^l \Delta\phi + u\phi_t - u^m v\phi). && (2.13) \end{aligned}$$

In a very recent study by Mimura et al. [30], they studied the existence and large time behavior of such weak solutions. The proof is very technical and tedious, we refer interested readers to their paper. Here we shall state their main results:

Suppose that

1) $u_0, v_0 \in C(\bar{\Omega})$ and $0 \leq u_0, v_0 \leq M$ for some constant $M > 0$. and

2) $a(u,v)$ is strictly positive and locally Lipschitz or $a = 0$, and

$$1 \leq m < \begin{cases} k + 2/N, & \text{if } N \geq 3, \\ k + 1, & \text{if } N = 1, 2. \end{cases} \tag{2.14}$$

In the case $a \neq 0$, the problem admits a unique weak solution $(u, v)$ satisfying

$$0 \leq u(x,t) \leq C_0, \text{ and } 0 \leq v(x,t) \leq M \text{ for all } (x,t) \in \Omega \times (0,T),$$

for some constant $C_0 > 0$.

There exists a constant $v^\infty$ such that

$$\lim_{t \to \infty} (u(t), v(t)) = (0, v^\infty) \text{ uniformly in } \bar{\Omega}.$$

Furthermore, if $1 \leq m < n$, $v^\infty = 0$, while if $1 \leq n \leq m$, then $v^\infty > 0$. Especially if $m = n$, then $v^\infty \leq a(0, v^\infty)$.

For the case $a = 0$, they obtained similar result and

$$\lim_{t \to \infty} (u(t), v(t)) = (< u_0 + v_0 >, 0) \text{ uniformly in } \bar{\Omega}.$$

## 2.4 Main result and methods

As we indicated earlier, bacteria grown on the surface of thin agar plates often develop colonies of various spatial patterns, such as fractal morphogenesis, dense-branching pattern. Recently Kawasaki et al. [39] proposed a degenerate parabolic system with cross diffusion that captures the qualitative features of the growth patterns. The model is

$$
\begin{aligned}
\frac{\partial b}{\partial t} &= D_b \nabla \cdot \{nb\nabla b\} + nb, && (2.15)\\
\frac{\partial n}{\partial t} &= D_n \nabla^2 n - nb && (2.16)
\end{aligned}
$$

with initial conditions

$$
b(x, y, 0) = b_0, \quad n(x, y, 0) = n_0.
$$

Here $b$ is the bacteria density and $n$ is the nutrient density.

Recently, Maini et al. [45] considered the one-dimensional version of the model and a special case $D_n = 0$. They studied the existence and uniqueness of traveling wave solutions. The found that such solutions exists only for speeds greater than some threshold speed and the wave with the minimum speed has a sharp profile. For speeds greater than this minimum speed the waves are smooth. By considering the special case $D_n = 0$, the authors were able to reduce the problem to a phase-space analysis in $R^2$.

In this thesis, we consider a more general model and we will not assume $D_n = 0$. The model we will study takes the form

$$
\begin{aligned}
\frac{\partial b}{\partial t} &= D_b \nabla \cdot \{n^p b^k \nabla b\} + n^q b^l, && (2.17)\\
\frac{\partial n}{\partial t} &= D_n \nabla^2 n - n^q b^l. && (2.18)
\end{aligned}
$$

To our knowledge, there is little theory on traveling wave solutions on such coupled degenerate diffusion system with cross-diffusion.

Our method is based on Schauder-fixed point theorem. By fixing $n$ in a properly chosen space $V$, we investigate the existence of traveling wave solution $b$. For such a traveling wave $b$, we can find a traveling wave solution $\tilde{n}$ which lies in a compact subset of $V$. In this way, we may define a continuous mapping $T: V \to V$ with $Tn = \tilde{n}$. Invoking Schauder-fixed point theorem, we conclude the existence of a traveling wave solution pair $(b, n)$.

A key part of the thesis is to investigate the existence of finite traveling wave solution for a degenerate cross-diffusion equation. Our method is inspired by a very recent study of a similar problem by Malaguti [42]. We will transform the existence of traveling wave solution $b$ to the solvability of a first-order singular boundary value problem which can be done by typical shooting and comparison argument. For more information on singular ODE, see [28, 34, 38, 42, 44].

Our main goal is to prove the following theorem.

**Theorem 2.1** *For $k = 1$, $q \geq 1$, $p = 0$, $l > 1$, there exists a constant velocity $v_*$ such that the system admits a traveling wave solution $(b(\xi), n(\xi))$ where $b$ is a monotone finite traveling wave solution and $n$ is a monotone classical traveling wave solution. Here $\xi = x - vt$ is the usual wave coordinate and by finite traveling wave we mean*

$$\xi^* = \sup\{\xi : \ b(\xi) > 0\} < \infty.$$

**Remark.** Even though our result is very restricted on the values of $k$ and $p$, most part of our analysis in this thesis works through without such restrictions. We shall keep them until the occasion arises when we need to impose such restrictions. It is our future concern to extend our result to more general cases.

In section 2.5, we simulate the 1D problem and oberseve the long time behavior of the solutions. In section 2.6 we will derive some properties of traveling wave solutions. In section 2.7 we investigate the existence of finite traveling wave $b$ for a given $n$. For this part, the result is general and we place no restriction on $k$. In section 2.8, we derive the equivalent problem and study the existence of a traveling wave $n$ for the determined finite traveling wave solution $b$. In this part we shall require that $k = 1$ in order to derive the equivalent problem. In section 2.8 we apply Schauder fixed point theorem to deduce the existence of traveling wave solutions $(b, n)$ where $b$ is of finite type. In section 2.10, we study the instability of the flat front and explain the fingering pattern in the bacterial colony.

## 2.5 Numerical Simulation to 1D Problem

In this section we consider the one dimensional problem on $[0, 1]$

$$\frac{\partial b}{\partial t} = \frac{\partial}{\partial x}(nb\frac{\partial b}{\partial x}) + nb, \tag{2.19}$$

$$\frac{\partial n}{\partial t} = \frac{\partial^2 n}{\partial x^2} - nb, \tag{2.20}$$

with Neumann boundary condition and initial data is chosen to be

$$b(x, 0) = -(x - 0.25)(x - 0.75), \ \ n(x, 0) = 1.$$

We numerically simulate the problem on time interval $[0, 0.2]$ and we have the results shown in Figure 2.2 and 2.3. The numerical results show $b$ stays compact supported.

The long time behavior can be seen when we calculate the density functions up to $t = 9$, the results are shown in Figure 2.4 and 2.5. We can see that $b$ tends to a uniform state and $n$ tends to 0 asymptotically.

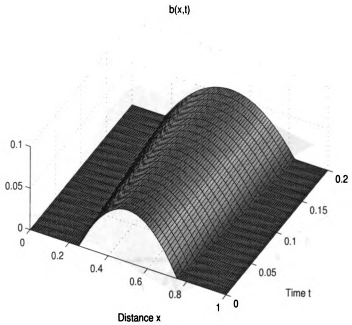


Figure 2.2. Bacteria Density

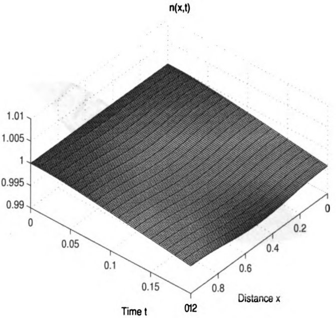


Figure 2.3. Nutrient Density

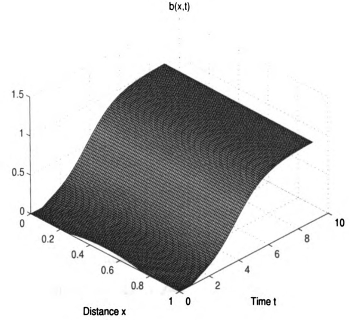


Figure 2.4. Bacteria Density

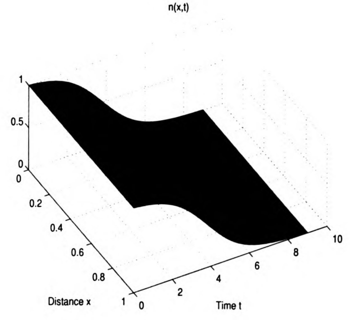


Figure 2.5. Nutrient Density

## 2.6 Travelling Wave Solution

We consider the system

$$\frac{\partial b}{\partial t} = D\frac{\partial}{\partial x}(n^p b^k \frac{\partial b}{\partial x}) + n^q b^l, \tag{2.21}$$

$$\frac{\partial n}{\partial t} = \frac{\partial^2 n}{\partial x^2} - n^q b^l, \tag{2.22}$$

where $(x,t) \in R \times R^+$ and $D$ is the rescaled non-dimensionalised diffusion coefficient of $b$.

Let us denote the spatially uniform steady states by $(b,n) = (b_s, 0), (0, n_s)$, where $n_s$ and $b_s$ are some constants. We may assume that initially the nutrient is uniformly distributed in the plate and there is no bacteria seed. Hence a proper initial steady state is given by

$$b = 0, \quad n = 1, \quad \text{for} \quad \text{all} \quad -\infty < x < \infty.$$

We shall also assume that

$$\frac{\partial b}{\partial x}, \frac{\partial n}{\partial x} \to 0 \text{ as } x \to \pm\infty, \quad \text{for} \quad \text{all} \quad t > 0. \tag{2.23}$$

Let $\xi = x - vt$ be the wave coordinates in which $b$ and $n$ solve

$$D(n^p b^k b')' + vb' + n^q b^l = 0 \tag{2.24}$$

$$n'' + vn' - n^q b^l = 0 \tag{2.25}$$

where $'$ denotes the derivative with respect to wave coordinate $\xi$. Equations (2.24)-(2.25) are to be solved subject to the following boundary conditions ahead of the wave

$$b \to 0, n \to 1 \text{ as } \xi \to +\infty, \tag{2.26}$$

and behind the wave

$$b \to b_s, n \to n_s \text{ as } \xi \to -\infty. \tag{2.27}$$

### 2.6.1 Properties of the traveling wave solutions.

*Property 1.* $n_s = 0$, $b_s = 1$.

**Proof.** Integrating on both sides of equation (2.24) from $-\infty$ to $+\infty$ gives

$$-vb_s + \int_{-\infty}^{+\infty} n^q b^l = 0 \tag{2.28}$$

from which we conclude that $b_s > 0$ since $v > 0$. But $n_s b_s = 0$, hence $n_s = 0$. To conclude that $b_s = 1$ we integrate Equation (2.24) from $-\infty$ to $\xi$, we have

$$n^p b^k b' + v(b - b_s) + \int_{-\infty}^{\xi} n^q b^l = 0. \tag{2.29}$$

On the other hand, integrating equation (2.25) from $-\infty$ to $\xi$, we have

$$\int_{-\infty}^{\xi} n^q b^l = n'(\xi) + vn(\xi). \tag{2.30}$$

Substituting expression (2.30) into equation (2.29) and passing $\xi$ to $+\infty$, we have

$$-vb_s + v = 0. \tag{2.31}$$

Therefore $b_s \equiv 1$.

*Property 2.* If $n$, $b$ are the traveling wave solutions, then $n$ is monotone increasing and $b$ is monotone decreasing if $0 < b < 1$.

**Proof.** The monotonicity of $n$ follows directly from equation (2.25). In fact,

$$n'' + vn' = n^q b^l \geq 0, \tag{2.32}$$

or equivalently

$$(n'e^{v\xi})' \geq 0. \tag{2.33}$$

Integrating this inequality from $-\infty$ to $\xi$, we obtain $n'e^{v\xi} > 0$. Hence $n' > 0$ for all $\xi$.

For a traveling wave solution $b(\xi)$, we prove that

$$b'(\xi) < 0 \text{ for } 0 < b(\xi) < 1.$$

We first suppose that $b'(\xi_0) = 0$ for some $\xi_0$, then from equation (2.24) we have

$$Dn^p b^k b'' + D(n^p b^k)'b' + vb' + n^q b^l = 0. \tag{2.34}$$

Hence

$$b''(\xi_0) = -\frac{n(\xi_0)^q b(\xi_0)^l}{Dn(\xi_0)^p b(\xi_0)^k} < 0. \tag{2.35}$$

Therefore we may define

$$\xi_* = \inf\{\xi : b'(s) > 0 \quad \text{for all} \quad s \in (\xi, \xi_0).\}. \tag{2.36}$$

Therefore $\xi_* > -\infty \Rightarrow b'(\xi_*) = 0$ since $0 < b < 1 = b(-\infty) = 1$. Furthermore, $b'(\xi_*) = 0$. On the other hand, from equation (2.24) we conclude that $n^p b^k b'$ is positive and strictly decreasing in $(\xi_*, \xi_0]$, in contradiction with $b'(\xi_*) = 0 = b'(\xi_0)$.

Thus if $b'(\xi) > 0$ in some interval $(\xi_1, \xi_2)$ with $b'(\xi_1) = 0$, we necessarily have $b(\xi_1) = 0$ which again in contradiction since $n^p b^k b'$ is positive and decreasing in

$(\xi_1, \xi_2)$. Therefore we conclude that $b'(\xi) < 0$ when $0 < b < 1$.

## 2.7 Existence of $b$ for a given $n$

Let

$$\xi_1 = \inf\{\xi : b(\xi) < 1\} \in R \cup \{-\infty\};$$

$$\xi_2 = \sup\{\xi : b(\xi) > 0\} \in R \cup \{+\infty\}.$$

Since $b$ is strictly decreasing on $(\xi_1, \xi_2)$, it follows that the inverse of $b(\xi)$, denoted by $\xi = \xi(b)$ is well defined on $(0,1)$ and takes value in $(\xi_1, \xi_2)$. Therefore we may define

$$n(b) = n(\xi(b)) \tag{2.37}$$

and

$$u(b) = Dn^p(b)b^k b'(\xi(b)) < 0 \quad \text{for} \quad \text{all} \quad b \in (0,1). \tag{2.38}$$

In fact, $\xi_1 = -\infty$. We state this in the following lemma.

**Lemma 2.1** $\xi_1 = -\infty$.

**Proof.** The proof is similar to the proof of property 2 in previous section.

Let $n(b) \in V$ where $V$ is the closed convex set of the Banach space $C^0([0,1])$ defined by

$$V := \{n(b) \in C^0[0,1],\ 0 \le n \le 1,\ \limsup_{b \to 1^-} \frac{n(b)}{1-b} \le L,\ n(b) \ge 1/L(1-b)\}$$

where $L$ is a sufficiently large constant and will be chosen later. We shall see later in the thesis the reason why we define such a space.

Differentiating both sides of equation (2.38) with respect to $\xi$ we have

$$u'(b)b'(\xi) = Dn^p b^k b'' + (Dn^p b^k)'b' = -vb' - n^q b^l. \tag{2.39}$$

Therefore $u$ as a function of $b$ satisfies

$$u'(b) = -v - \frac{Dn^{p+q}(b)b^{k+l}}{u}, \quad b \in (0,1). \tag{2.40}$$

Our next step is to derive the boundary condition at 0 and 1.

**Lemma 2.2** $u(0^+) = u(1^-) = 0$.

**Proof.** If $\xi_2 = +\infty$, *i.e.*, $b$ is a classical traveling wave, it follows that

$$u(0^+) = \lim_{\xi \to +\infty} Dn^p(b)b^k b'(\xi) = 0.$$

If $\xi_2 < +\infty$, *i.e.*, $b$ is a finite traveling wave solution, we may assume that it vanishes at $\xi = 0$. We expect that as $\xi \to 0$

$$b(\xi) \sim A(-\xi)^\alpha. \tag{2.41}$$

Substitute this expression into

$$Dn^p b^k b'' + D(n^p b^k)'b' + vb' + n^q b^l = 0$$

where the derivative is with respect to $\xi$, we obtain

$$Dn^p A^{k+1}[\alpha(\alpha-1) + k\alpha^2](-\xi)^{(k+1)\alpha-2} + Dpn^{p-1}n'(-A^{k+1}\alpha)(-\xi)^{(k+1)\alpha-1}$$
$$-vA\alpha(-\xi)^{\alpha-1} + n^q A^l(-\xi)^{\alpha l} = 0. \tag{2.42}$$

Equating the dominated terms at $\xi = 0$ we have

$$(k+1)\alpha - 2 = \alpha - 1,$$

$$Dn^{*p}A^{k+1}[\alpha(\alpha-1) + k\alpha^2] - vA\alpha = 0$$

where $n^* = \lim_{b\to 0} n(b) > 0$.

Hence we have

$$\alpha = 1/k, \tag{2.43}$$

$$A = \frac{kv}{Dn^{*p}}. \tag{2.44}$$

From the definition of $u$, we have

$$\begin{aligned} u(0^+) &= \lim_{b\to 0^+} Dn^p(b)b^k b'(\xi) \\ &= \lim_{\xi\to 0^+} Dn^p A^k(-\xi)\cdot(-A\alpha)(-\xi)^{1/k-1} \\ &= 0. \end{aligned} \tag{2.45}$$

$u(1^-) = 0$ follows from the definition.

Therefore $u$ solves the following singular boundary value problem:

$$u'(b) = -v - \frac{Dn^{p+q}(b)b^{k+l}}{u} \quad \text{for} \quad b \in (0,1), \tag{2.46}$$

subject to

$$u(0^+) = u(1^-) = 0. \tag{2.47}$$

We now consider the solvability of this singular problem.

### 2.7.1 First-order singular boundary value problem

Given $n(b) \in V$, we consider the following existence problem:

**Problem $P_1$.** Find a pair $(v, u)$ with $v > 0$ such that

$$u'(b) = -v - \frac{Dn^{p+q}(b)b^{k+l}}{u} \quad \text{for} \quad b \in (0,1), \tag{2.48}$$

$$u(0^+) = u(1^-) = 0, \tag{2.49}$$

$$u < 0 \text{ in } (0,1). \tag{2.50}$$

### 2.7.2 Existence of a critical velocity

**In this section we show that $P_1$ is solvable with a unique negative solution $u = u(b)$ if and only if $v \geq v^*$ for some positive $v^*$ which depends on the choice of $n(b)$.**

**We first prove the following lemma**

**Lemma 2.3** *If there exists $\phi \in C^1(0,1)$ such that*

$$\phi'(b) > -v - \frac{Dn^{p+q}(b)b^{k+l}}{\phi(b)} \quad \text{for} \quad b \in (0,1), \tag{2.51}$$

*such that $\phi(0^+) = 0$ and $\phi(b) < 0$ for $b \in (0,1)$. Then $P_1$ is solvable and the solution $0 > u(b) > \phi(b)$.*

**Proof.** The proof is based on shooting argument and some comparison techniques.

First, for a fixed constant $b_0 \in (0,1)$, note $\phi(b_0) < 0$, let $\alpha \in [\phi(b_0), 0)$, we consider the following initial value problem:

$$u' = -v - \frac{Db^{k+l}n^{p+q}(b)}{u}, \quad 0 < b < 1,$$

$$u(b_0) = \alpha \geq \phi(b_0),$$

and let $u$ represent the unique solution of it. We first claim that $u(b) < 0$ on $(0, b_0)$. Suppose otherwise there is a $b^* \in (0, b_0)$ satisfying $u(b) < 0$ for all $b \in (b^*, b_0]$ and

$$\lim_{b \to b^{*}+} u(b) = 0.$$

Since

$$\lim_{(b,u)\to(b^*,0)} (-vu - Db^{k+l}n^{p+q}(b)) = -D(b^*)^{k+l}n^{p+q}(b^*) < 0,$$

hence we can find a constant $c > 0$ such that

$$u' = \frac{-vu - Db^{k+l}n^{p+q}(b)}{u} > \frac{-c}{u(b)} > 0$$

for all $b^* < b < b^* + c$, which would imply that $u > 0$ for $b^* < b < b^* + c$, clearly a contradiction.

Next we assert if $\alpha > \phi(b_0)$, then

$$\phi(b) < u(b) < 0 \quad \text{for} \quad 0 < b \leq b_0.$$

Since $\phi(b_0) < \alpha = u(b_0)$, we may define

$$\underline{b} = \inf\{b : \phi(s) < u(s) \quad \text{for} \quad s \in (b, b_0)\}$$

Suppose $\underline{b} > 0$, then $u(\underline{b}) = \phi(\underline{b})$ and

$$\begin{aligned} u'(\underline{b}) &= -v - \frac{D\underline{b}^{k+l}n^{p+q}(\underline{b})}{u(\underline{b})} \\ &= -v - \frac{D\underline{b}^{k+l}n^{p+q}(\underline{b})}{\phi(\underline{b})} \\ &< \phi'(\underline{b}), \end{aligned}$$

which is clearly absurd.

In view of the arguments above, we have proved that $u$ is well defined in $(0, b_0)$. Now we may define $(0, b_\alpha)$ to be the maximal existence interval for solution $u$. The aim is to show that $b_\alpha = 1$ for some $\alpha \in [\phi(b_0), 0)$. Let $u_1$ and $u_2$ be two distinct solutions corresponding to initial value $\alpha_1$ and $\alpha_2$ respectively. Suppose for definiteness that

$$\alpha_1 < \alpha_2,$$

then

$$u_1 < u_2 \quad \text{in} \quad (0, min\{b_{\alpha_1}, b_{\alpha_2}\}),$$

hence

$$b_{\alpha_1} \geq b_{\alpha_2}.$$

We now claim that if the $\mid \alpha \mid$ is sufficiently small, then

$$b_\alpha < 1 \quad \text{and} \quad u(b_\alpha^-) = 0.$$

The trick is to construct an upper solution and use a similar comparison argument as in the earlier part of the proof of this lemma. Since

$$\lim_{(b,u)\to(b_0,0)} -vu - Db^{k+l}n^{p+q}(b) < 0,$$

there exists a sufficiently small constant $M > 0$ and $\lambda < 2M$ such that

$$-v - \frac{Db^{k+l}n^{p+q}(b)}{u} > -\frac{M}{u},$$

for all $-\lambda < u < 0$ and $b_0 \leq b < b_0 + \lambda$. Let $\alpha > -\lambda$ and define

$$\psi = -\sqrt{\alpha^2 - 2M(b - b_0)}$$

for $b_0 \leq b \leq b_0 + \frac{\alpha^2}{2M}$ which solves the following initial problem:

$$\psi' = -\frac{M}{\psi},$$

$$\psi(b_0) = \alpha.$$

We have

$$b_0 + \frac{\alpha^2}{2M} < b_0 + \lambda < 1.$$

Moreover, $u'(b_0) > 0$, hence $u(b) > u(b_0) = \alpha > -\lambda$ in a right neighborhood of $b_0$. Put

$$I = [b_0, min\{b_0 + \frac{\alpha^2}{2M}, b_\alpha\}],$$

we deduce that

$$u(b) > -\lambda \quad \text{for} \quad \text{all} \quad b \in I.$$

Apply a similar comparison argument as before to conclude $u(b) > \psi(b)$ for all $b \in (b_0, b_\alpha) \cap (b_0, b_0 + \frac{\alpha^2}{2M})$. Since $\phi(b_0 + \frac{\alpha^2}{2M}) = 0$, we have

$$b_\alpha \leq b_0 + \frac{\alpha^2}{2M} < 1.$$

Now we let $\alpha^* = inf\{\alpha \in (\phi(b_0), 0) : b_\alpha < 1\}$, then $b_{\alpha^*} = 1$, therefore the corresponding solution $u$ is defined and negative on $(0,1)$ and $u > \phi$ in $(0,1)$ and $u(0^+) = 0$. This completes the proof.

We are now in position to prove the following solvability result for Problem $P_1$.

**Theorem 2.2** *There exists $v^* > 0$, such that for all $v > v^*$, Problem $P_1$ has a unique* *negative solution.*

**Proof.** We first show $P_1$ is solvable for $v$ sufficiently large. To this aim, we let

$$c = sup_{s\in(0,1)} \frac{Dn^{p+q}(s)s^{k+l}}{s}, \tag{2.52}$$

which is well defined. Let

$$\phi(b) = -\sqrt{c}b.$$

Then, for $v > 2\sqrt{c}$, we have

$$\begin{aligned} -v - \frac{Db^{k+l}n^{p+q}(b)}{\phi(b)} &< -2\sqrt{c} + \frac{Db^{k+l}n^{p+q}(b)}{\sqrt{c}b} \\ &\leq -2\sqrt{c} + \sqrt{c} = \phi'(b) \end{aligned} \tag{2.53}$$

for all $b \in (0, 1)$.

Hence $\phi(b)$ satisfies condition of Lemma 2.3. Therefore $P_1$ is solvable for every $v > 2\sqrt{c}$. We now show that $P_1$ is not solvable for $v = 0$. Otherwise, let $u$ solves

$$u' = -\frac{Db^{k+l}n^{p+q}(b)}{u}$$

defined on some interval $(a, 1)$ with $0 < a < 1$ and $u(b) < 0$ for all $b \in (a, 1)$. Integrate the equation above in $[b, \tilde{b}]$ with $a < b < \tilde{b} < 1$ we obtain

$$u^2(\tilde{b}) = u^2(b) - 2\int_b^{\tilde{b}} Ds^{k+l}n^{p+q}(s)ds. \tag{2.54}$$

Therefore, if $u(1^-) = 0$, we have

$$u(b) = -\sqrt{2\int_b^1 Ds^{k+l}n^{p+q}(s)ds} \tag{2.55}$$

which implies $u(0^+) < 0$, a contradiction.

We now let

$$v^* = \inf\{v : P_1 \quad is \quad solvable\}$$

which is well defined and $v^* > 0$ based on the observation above.

We prove that for every $v > v^*$, $P_1$ is solvable. Given $v > v^*$, take $\bar{v}$ such that $P_1$ is solvable with $\bar{v} < v$ and the unique solution $\bar{u}$ for $\bar{v}$. Since

$$\bar{u}' = -\bar{v} - \frac{Db^{k+l}n^{p+q}(b)}{\bar{u}} > -v - \frac{Db^{k+l}n^{p+q}(b)}{\bar{u}},$$

hence $\bar{u}$ satisfies condition of Lemma 2.3. Therefore, we conclude the solvability for $v$.

Finally we prove that $P_1$ admits at most one solution. Suppose for contradiction that $u_1$ and $u_2$ are two distinct solutions of $P_1$. For definiteness, we assume

$$u_1(b_0) > u_2(b_0) \quad \text{for} \quad b_0 \in (0,1),$$

it follows that

$$\begin{aligned} u_1'(b_0) - u_2'(b_0) &= -\frac{Db_0^{k+l}n^{p+q}(b_0)}{u_1(b_0)} + \frac{Db_0^{k+l}n^{p+q}(b_0)}{u_2(b_0)} \\ &> 0. \end{aligned} \tag{2.56}$$

Hence if $u_1(b_0) > u_2(b_0)$, then $u_1'(b_0) > u_2'(b_0)$. Therefore, it is impossible that

$$u_1(1^-) = u_2(1^-) = 0.$$

This completes the proof.

### 2.7.3 Finite traveling wave at $v^*$

The main goal of this section is to show that $b$ is a finite traveling wave at the minimum wave speed. We will also show that $b$ is a classical traveling wave if $v > v^*$. The idea is to characterize the type of $b$ by the value of $u'(0^+)$.

**Lemma 2.4** *$u'(0^+) = 0$ or $u'(0^+) = -v$.*

**Proof.** We first show that if $u'(b^*) = 0$ for some sufficiently small $b^*$ then $u''(b^*) > 0$. To this aim, we write

$$g(b) = Db^{k+l}n^{p+q}(b),$$

it follows that

$$u''(b) = \frac{g(b)u'(b)}{u^2} - \frac{g'(b)}{u}.$$

Note that

$$g'(b) = Db^{k+l-1}n^{p+q-1}[(k+1)n(b) + (p+q)bn'(b)] > 0$$

for $b$ sufficiently small. Hence if there exists $b^*$ sufficiently small such that $u'(b^*) = 0$, then $u''(b^*) = -\dfrac{g'(b^*)}{u} > 0$.

Therefore we conclude that there exists $0 < \bar{b} < b^*$ such that

$$u''(b) > 0 \quad \text{or} \quad u''(b) < 0 \quad \text{on} \quad (0, \bar{b}).$$

*Case 1:* $u''(b) > 0$ on $(0, \bar{b})$. In this case $u'(0^+) < 0$ exists and it follows from

$$(u' + v) \cdot \frac{u}{b} = -Db^{k+l-1}n^{p+q}(b)$$

that $u'(0^+) = -v$.

*Case 2:* $u''(b) < 0$ on $(0, \bar{b})$. In this case, $-v < u'(0^+) \leq 0$, similarly we obtain $u'(0^+) = 0$.

**Lemma 2.5** *$b$ is a finite traveling wave if and only if $u'(0^+) = -v$.*

**Proof.** If $b$ is finite traveling wave, then from Lemma 2.2, at $\xi = 0$, we have

$$b(\xi) = \frac{kv}{Dn^{*p}}(-\xi)^{1/k}. \tag{2.57}$$

Therefore by equation (2.39) we have

$$\begin{aligned} u'(0^+) &= \lim_{b\to 0^+}(-v - \frac{n^q b^l}{b'}) \\ &= -v + \lim_{\xi\to 0^+} n^q(\frac{kv}{Dn^p})^{l-1}k(-\xi)^{l/k-1/k+1} \\ &= -v. \end{aligned} \tag{2.58}$$

On the other hand, if $u'(0^+) = -v$, we have

$$\begin{aligned} \lim_{\xi\to\xi_2^-} b'(\xi) &= \lim_{b\to 0^+} \frac{u(b)}{b}\cdot\frac{b}{Db^k n^p} \\ &= -\lim_{b\to 0^+}\frac{v}{Db^{k-1}n^p}, \end{aligned} \tag{2.59}$$

which is $-\dfrac{v}{Dn^{*p}}$ if $k = 1$ and $-\infty$ if $k > 1$. This implies that $b$ is a finite traveling wave solution.

**Lemma 2.6** *For $v$ sufficiently large, $b(\xi) > 0$ for all $\xi \in R$. In other word, $b(\xi)$ is a classical traveling wave solution.*

**Proof.** Note that for $v$ sufficiently large, there exists $\lambda > 0$ such that

$$-v + \frac{D}{\lambda} \le -(k+l)\lambda. \tag{2.60}$$

We claim that

$$u(b) \ge -\lambda b^{k+l}, \tag{2.61}$$

for all $b \in [0,1]$. Suppose $u(\bar{b}) < -\lambda \bar{b}^{k+l}$ for some $\bar{b} \in (0,1)$, then

$$\begin{aligned} u'(\bar{b}) &= -v - \frac{D\bar{b}^{k+l} n^{p+q}(\bar{b})}{u(\bar{b})} \\ &< -v + \frac{D\bar{b}^{k+l} n^{p+q}(\bar{b})}{\lambda \bar{b}^{k+l}} \\ &< -v + \frac{D}{\lambda} \\ &\leq -\lambda(k+l) \leq -\lambda(k+l)\bar{b}^{k+l-1}. \end{aligned} \tag{2.62}$$

Hence

$$u(b) + \lambda b^{k+l} < u(\bar{b}) + \lambda \bar{b}^{k+l} < 0 \quad \text{for all } b > \bar{b} \tag{2.63}$$

which contradicts $u(1^-) = 0$.

Now for any $\xi_0 \in (-\infty, \xi_2)$, let $b(\xi_0) = b_0$, we have

$$\begin{aligned} \xi_0 - \xi_2 &= \int_0^{b_0} \xi'(b) db \\ &= \int_0^{b_0} \frac{Dn^p(b) b^k}{u(b)} db \\ &\leq -\frac{1}{\lambda} \int_0^{b_0} \frac{Dn^p(b)}{b^l} db = -\infty. \end{aligned} \tag{2.64}$$

Hence $\xi_2 = +\infty$ which implies that $b$ is a classical traveling wave solution. Note here we applied the fact that $l \geq 1$.

Now our goal is to show that when $v = v^*$, $b$ is a finite traveling wave solution. In view of Lemma 2.5, we shall show that when $v = v^*$, the solution of

$$u' = -v - \frac{Db^{k+l} n^{p+q}(b)}{u},$$

$$u(0^+) = u(1^-) = 0,$$

satisfies $u'(0^+) = -v^*$.

The trick is to construct a converging sequence of solutions which satisfy this property. To show this, we define a continuous function $g_n(b)$ on $[0,1]$ as:

$$g_n(b) = 0 \quad \text{on} \quad \left[0, \frac{1}{n+2}\right], \qquad 0 \le g_n \le Db^{k+l}n^{p+q}(b) \quad \text{for} \quad b \in \left[\frac{1}{n+2}, \frac{1}{n+1}\right]$$
$$\text{and } g_n(b) = Db^{k+l}n^{p+q}(b) \quad \text{on} \quad \left[\frac{1}{n+1}, 1\right].$$

It holds that

$$g_n \le g_{n+1}.$$

We consider the following problem:

*To find a function $u_n : (0,1) \to R^-$ and $v_n > 0$ such that*

$$u_n' = -v_n - \frac{g_n(b)}{u_n},$$

$$u_n(0^+) = u_n(1^-) = 0.$$

Let us emphasize here that $v_n$ is also an unknown of the problem. We shall prove the following theorem.

**Theorem 2.3** *There exists a unique solution $u_n : (0,1) \to R^-$ and $v_n$ of the problem. $u$ is of class $C^1$.*

**Proof.** The problem is equivalent to

$$u_n' = -v_n - \frac{g_n}{u_n} \quad \text{on} \quad \left[\frac{1}{\text{n}+2}, 1\right], \tag{2.65}$$

$$u_n = -v_n b \quad \text{on} \quad \left[0, \frac{1}{n+2}\right], \tag{2.66}$$

$$u_n'\left(\frac{1}{n+2}\right) = -v_n, \tag{2.67}$$

$$u_n(1^-) = 0. \tag{2.68}$$

Note that the solution of this equivalent problem is of $C^1$ since $u'(\frac{1}{n+2}) = -v_n$. We apply the shooting argument and comparison techniques on $[\frac{1}{n+2}, 1]$ similar as in Section 3 to the problem:

$$\begin{aligned} u_n' &= -v_n - \frac{g_n}{u_n} \quad \text{on} \quad [\frac{1}{n+2}, 1], && (2.69) \\ u_n(\frac{1}{n+2}) &= -\frac{v_n}{n+2}, && (2.70) \\ u_n'(\frac{1}{n+2}) &= -v_n. && (2.71) \end{aligned}$$

We can show that there exists a unique $v_n$ such that $u_n(1^-) = 0$.

We now examine several properties of $u_n$. Some of them will be applied in the proof of the next lemma. The proof is all the similar shooting and comparison argument. We shall briefly show the proof of the second one.

**Lemma 2.7** *$u_n$ satisfies:*

*p1* $u_n(b) = -v_n b$ on $[0, \frac{1}{n+2}]$.

*p2* $u_n(b) \geq -v^* b$ for all $b \in [0, 1]$.

*p3* $u_n \geq u_{n+1}$ *for all* $b \in [0, 1]$.

**Proof.** *p2* Since $u_n(b) = -v_n(b) > -v^* b$ on $(0, \frac{1}{n+2}]$, hence if $\exists\, \bar{b}$ such that $u_n(\bar{b}) < -v^* b$ for some $n$, $\bar{b}$ must be in $(\frac{1}{n+2}, 1]$. Furthermore, $\exists\, \tilde{b} \in (\frac{1}{n+2}, \bar{b})$ such that $u_n(\tilde{b}) = -v^* \tilde{b}$ and $u_n'(\tilde{b}) < -v^*$. However,

$$u_n'(\tilde{b}) = -v_n - \frac{g_n(\tilde{b})}{u_n(\tilde{b})} > -v^* - \frac{g_n(\tilde{b})}{-v^* \tilde{b}} > -v^*,$$

which is a contradiction.

Now we will prove that $v_n$ determined in the theorem above has the following properties.

**Lemma 2.8** *$\{v_n\}$ is monotone increasing and* $\lim_{n \to +\infty} v_n = v^*$.

**Proof.** First we show that $\{v_n\}$ is monotone increasing. Suppose $\exists v_n > v_{n+1}$ for some $n$. Then,

$$u_n(\frac{1}{n+2}) = \frac{-v_n}{n+2} < \frac{-v_{n+1}}{n+2} < u_{n+1}(\frac{1}{n+2}).$$

We also have

$$u_n'(\frac{1}{n+2}) = -v_n < -v_{n+1} - \frac{g_{n+1}(\frac{1}{n+2})}{u_{n+1}(\frac{1}{n+2})} = u_{n+1}'(\frac{1}{n+2})$$

since

$$g_{n+1}(\frac{1}{n+2}) > 0.$$

Hence $u_n(b) < u_{n+1}(b)$ in a right neighborhood of $\frac{1}{n+2}$, therefore in this neighborhood

$$u_n'(b) = -v_n - \frac{g_n}{u_n} < -v_{n+1} - \frac{g_{n+1}}{u_{n+1}} = u_{n+1}'(b).$$

Hence

$$u_n(b) - u_{n+1}(b) < u_n(\frac{1}{n+2}) - u_{n+1}(\frac{1}{n+2}) < 0$$

for all $b \in [\frac{1}{n+2}, 1]$.

This is a contradiction with $u_n(1^-) = u_{n+1}(1^-) = 0$. Hence

$$v_n \leq v_{n+1}.$$

We now prove that $v_n \leq v^*$. Suppose $\exists\ v_n > v^*$, then

$$u_n(\frac{1}{n+2}) = \frac{-v_n}{n+2} < \frac{-v^*}{n+2} < u(\frac{1}{n+2}),$$

where we have applied the fact that $u' > -v^*$ for all $b \in (0, 1)$.

We also have

$$u_n'(\frac{1}{n+2}) = -v_n < -v^* < u'(\frac{1}{n+2}).$$

Hence $u_n(b) < u(b) < 0$ is a right neighborhood of $\frac{1}{n+2}$. For every $b$ in this neighborhood, we have

$$u_n'(b) = -v_n - \frac{g_n}{u_n} < -v^* - \frac{g_n}{u} \leq -v^* - \frac{Db^{k+l}n^{p+q}(b)}{u} = u'(b).$$

Hence we conclude that

$$u_n(b) < u(b) \quad \text{for} \quad \text{all} \quad b \in \left[\frac{1}{n+2}, 1\right]$$

which contradicts with $u_n(1^-) = u(1^-) = 0$.

Finally, we show $\lim_{n\to+\infty} v_n = v^*$. Let

$$\bar{v} = \limsup_{n\to+\infty} v_n \leq v^*.$$

Note that we may define $u(b) = \lim_{n\to+\infty} u_n(b)$, we have

$$\begin{aligned} u(b) - u(\frac{1}{n+1}) &= \lim_{n\to+\infty} \left(u_n(b) - u_n(\frac{1}{n+1})\right) \\ &= \lim_{n\to+\infty} \int_{\frac{1}{n+1}}^{b} u_n' \\ &= \lim_{n\to+\infty} \int_{\frac{1}{n+1}}^{b} -v_n - \frac{g_n}{u_n} \\ &= \int_0^b -\bar{v} - \frac{Db^{k+l}n^{p+q}(b)}{u}. \end{aligned} \tag{2.72}$$

Hence $u$ solves

$$u' = -\bar{v} - \frac{Db^{k+l}n^{p+q}(b)}{u} \quad \text{on} \quad (0,1),$$

Moreover, we have $u(0^+) = 0$ and $u(1^-) = 0$.

By the definition of $v^*$ we conclude that $\bar{v} \geq v^*$. Hence $\bar{v} \equiv v^*$. Note that

$$u(b) = \lim_{n \to +\infty} u_n(b) \geq -v^* b \quad \text{for} \quad b \in [0, 1).$$

Applying monotone convergence theorem to $\{u_n\}$ we can show that $u$ solves

$$u' = -v^* - \frac{D b^{k+l} n^{p+q}(b)}{u} \quad \text{in} \quad (0, 1),$$

$$u(0^+) = 0, \quad u(1^-) = 0.$$

Hence

$$-v^* \leq \frac{u}{b} \leq \frac{u_n}{b} = -v_n \quad \text{for} \quad b \in \left(0, \frac{1}{n+2}\right].$$

Therefore

$$u'(0) = -v^*.$$

This shows that the traveling wave at minimum wave speed $v^*$ is of finite type.

We conclude this section with a few properties of the negative solution $u$.

**Lemma 2.9** *Let $u = u(b)$ be a negative solution of Problem $P_1$. Then, there exists $u'(1^-) = 0$.*

**Proof.** Let $u(b)$ be a solution of $P_1$. Let $M := \limsup_{b \to 1^-} \frac{u(b)}{b-1}$ and $m := \liminf_{b \to 1^-} \frac{u(b)}{b-1} \geq 0$. There are three possible cases: either $m = M > 0$ or $M > m \geq 0$ and $M = m = 0$. We show the first two are impossible.

In fact, if $m = M > 0$, then

$$\lim_{b \to 1^-} u'(b) = -v - \lim_{b \to 1^-} \frac{D n^{p+q} b^{k+l}}{b-1} \frac{b-1}{u(b)} = -v < 0,$$

a contradiction. Here we applied the fact that $\limsup_{b \to 1^-} \frac{n(b)}{1-b} \leq L$.

If $M > m \geq 0$. Then for any given $m_0 \in (m, M)$ there exists a sequence $\{b_n\}$ which converges to 1 such that

$$\frac{u(b_n)}{b_n - 1} = m_0 \textit{ and } \left(\frac{u(b)}{b-1}\right)'_{b=b_n} \leq 0.$$

Hence,

$$\lim_{n\to+\infty} u'(b_n) = \lim_{n\to+\infty} -v - \frac{1}{m_0}\frac{Dn^{p+q}(b_n)b_n^{k+l}}{b_n - 1} = -v < 0,$$

therefore we may assume $u'(b_n) < 0$ for every $n$. On the other hand,

$$\left(\frac{u(b)}{b-1}\right)'_{b=b_n} = \frac{1}{b_n - 1}\left[u'(b_n) - \frac{u(b_n)}{b_n - 1}\right] = \frac{1}{b_n - 1}[u'(b_n) - m_0] > 0.$$

A contradiction to previous statement that $\left(\frac{u(b)}{b-1}\right)'_{b=b_n} \leq 0$. Therefore, $M = m = 0$, *i.e.*, there exists $u'(1^-) = 0$.

Using this lemma, we can further show that

**Lemma 2.10** *For the solution $u$ of Problem $P_1$, there exists $C_1 < 0$, $C_2 < 0$ such that*

$$C_2(1-b)^{p+q} \leq u(b) \leq C_1(1-b)^{p+q} \text{ for } b \text{ sufficiently close to } 1.$$

**Proof.** We only need to show that it is impossible to find negative constants $\tilde{C}_1, \tilde{C}_2$ such that $u(b) \geq \tilde{C}_1(1-b)^{p+q+\gamma}$ or $u(b) \leq \tilde{C}_2(1-b)^{p+q-\gamma}$ for any $\gamma > 0$ as $b \to 1^-$.

Otherwise, we either have

$$\begin{aligned}
u'(b) &= -v - \frac{Dn^{p+q}b^{k+l}}{u(b)} \\
&\geq -v - \frac{Dn^{p+q}b^{k+l}}{\tilde{C}_1(1-b)^{p+q+\gamma}} \\
&\geq -v - \frac{D(1/L)^{p+q}(1-b)^{p+q}b^{k+l}}{\tilde{C}_1(1-b)^{p+q+\gamma}} \\
&> 0
\end{aligned}$$

for $b$ sufficiently close to 1. A contradiction to Lemma 2.9.

Or we have

$$\begin{aligned} u'(b) &\le -v - \frac{DL^{p+q}(1-b)^{p+q}b^{k+l}}{\tilde{C}_1(1-b)^{p+q-\gamma}} \\ &\to -v \end{aligned}$$

as $b \to 1^-$. A contradiction to Lemma 2.9 again.

In fact, we can improve this result in the next lemma.

**Lemma 2.11** *There exists $C_2 < 0$ such that $u(b) \ge C_2(1-b)^{p+q}$ for all $b \in [0,1]$.*

**Proof.** Suppose that there exists $\bar{b} \in [0,1]$ such that $u(\bar{b}) < C_2(1-\bar{b})^{p+q}$. Then we have

$$\begin{aligned} u'(\bar{b}) &= -v - \frac{Dn^{p+q}(\bar{b})\bar{b}^{k+l}}{u(\bar{b})} \\ &\le -v - \frac{DL^{p+q}(1-\bar{b})^{p+q}\bar{b}^{k+l}}{C_2(1-\bar{b})^{p+q}} \\ &< -C_2(p+q)(1-\bar{b})^{p+q-1} \end{aligned}$$

for $C_2$ sufficiently large so that

$$-v - \frac{DL^{p+q}\bar{b}^{k+l}}{C_2} < -C_2(p+q)(1-\bar{b})^{p+q-1}.$$

Thus

$$u(b) - C_2(1-b)^{p+q} < u(\bar{b}) - C_2(1-\bar{b})^{p+q} < 0$$

for every $b > \bar{b}$. A contradiction to $u(1^-) = 0$. This ends the proof. This result is illustrated in Figure 2.6.

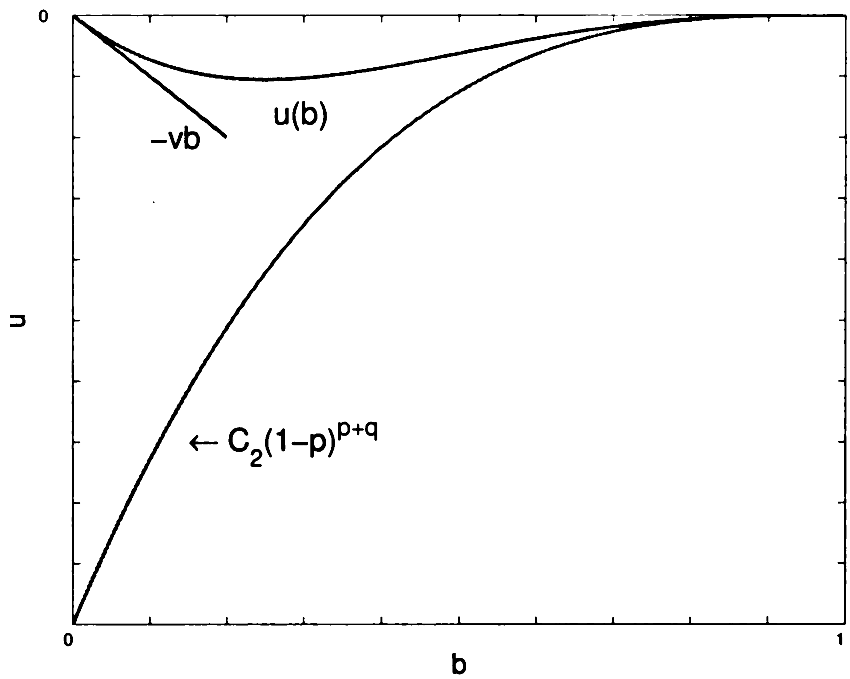


Figure 2.6. Sketch of u

## 2.8 The Equivalent Problem

Now that we have shown that for any $n \in V$, there exists a $v^*$ which depends on the choice of $n$ such that $b$ is a finite traveling wave, we may assume that

$$b(\xi) = 0 \quad \text{for} \quad \xi \geq 0. \tag{2.73}$$

We shall simplify the problem by reducing it to a system in the interval $\xi < 0$ only. Note that $n$ satisfies

$$n'' + vn' = 0 \quad \text{for} \quad \xi \geq 0. \tag{2.74}$$

We have the following lemma:

**Lemma 2.12** *The problem*

$$n'' + vn' = 0 \quad \text{in} \quad (0, +\infty), \tag{2.75}$$

$$n(0) = n^* \quad \text{with} \quad 0 < n^* < 1, \tag{2.76}$$

$$n'(0) = v(1 - n^*), \tag{2.77}$$

*has a unique bounded solution that satisfies* $\lim_{\xi \to +\infty} n(\xi) = 1$.

**Proof.** We consider the following ODE system:

$$n' = p,$$

$$p' = -vp.$$

A phase plane analysis shows that every trajectory can intersect the $n - axis$ at most once. Hence $p'$ changes sign at most once, and consequently $n(+\infty)$ exists. Let $n(+\infty) = c$, a direct integration shows that

$$\int_0^{+\infty} n'' = -\int_0^{+\infty} vn',$$

hence

$$-v(1 - n^*) = -vc + vn^*.$$

Therefore

$$\lim_{\xi \to +\infty} n(\xi) = c = 1.$$

In view of this lemma, we may reformulate the problem into:

**Problem** $P_2$. *Find* $(v, b, n)$ *with* $v > 0$ *such that*

$$D(n^p b^k)'' + vb' + n^q b^l = 0, \quad \xi < 0, \tag{2.78}$$

$$n'' + vn' - n^q b^l = 0, \quad \xi < 0, \tag{2.79}$$

$$n'(0) = v(1 - n(0)), \quad n(-\infty) = 0, \tag{2.80}$$

$$b(-\infty) = 1, \quad b(0) = 0. \tag{2.81}$$

We have seen that as $\xi$ varies from $-\infty$ to 0, $b = b(\xi)$ decreases monotonicity from 1 to 0. We can therefore define

$$\xi = \xi(b) \quad \text{as the inverse function of} \quad b(\xi)$$

where $b$ varies from 0 to 1 and $\xi$ takes value in $(-\infty, 0)$.

As before we define

$$u(b) = Dn^p b^k b'(\xi(b)) < 0 \quad \text{for all} \quad b \in (0,1) \tag{2.82}$$

and

$$n(b) = n(\xi(b)). \tag{2.83}$$

Since

$$\frac{d}{d\xi} = b'(\xi) \cdot \frac{d}{db} = \frac{u(b)}{Dn^p b^k} \cdot \frac{d}{db},$$

we can transform problem $P_2$ into the following equivalent problem:

**Problem** $P_3$. *Find* $(v, u, n)$ *solves*

$$u' = -v - \frac{Db^{k+l} n^{p+q}(b)}{u}, \tag{2.84}$$

$$\frac{u}{Db^k n^p}\left(\frac{u}{Db^k n^p} n'\right)' + v \frac{u}{Db^k n^p} n' - n^q b^l = 0, \tag{2.85}$$

$$u(0^+) = u(1^-) = 0, \tag{2.86}$$

$$n'(0^+) = v(1-n(0)) \cdot \lim_{b\to 0^+} \frac{Db^k n^p}{u(b)} = -(1-n(0))Dn(0)^p (k=1), \quad n(1^-) = 0, \tag{2.87}$$

$$u < 0 \quad \text{in} \quad (0,1) \tag{2.88}$$

*we shall also need*

$$\int_0^1 \frac{Db^k n^p}{u(b)} db = -\infty. \tag{2.89}$$

**Remark.** The first equality in equation (2.87) is equivalent to the first equality of equation (2.80) only for $k = 1$. Therefore from here we shall take $k = 1$.

## 2.9 A fixed point

Give $n(b) \in V$, let $(v, u)$ be the unique solution of Problem $P_1$ such that $u'(0^+) = -v$, *i.e.*, the corresponding $b$ is a finite traveling wave. Consider the following:

**Problem $P_4$.** *Find $\tilde{n}(b)$ such that*

$$\left(\frac{u}{Dbn^p}\tilde{n}'\right)' + v\tilde{n}' - \frac{Dn^p\tilde{n}^q b^{l+1}}{u} = 0 \text{ in } (0,1), \tag{2.90}$$

$$\tilde{n}' + D(1-\tilde{n})n^p = 0 \text{ at } b = 0, \tag{2.91}$$

$$\tilde{n}(1) = 0. \tag{2.92}$$

We shall show that Problem $P_4$ admits a unique solution that is also in $V$. To prove this fact, we begin with the following local existence result.

**Lemma 2.13**

$$\left(\frac{u}{Db^k n^p}\tilde{n}'\right)' + v\tilde{n}' - \frac{Dn^p\tilde{n}^q b^{l+k}}{u} = 0 \text{ in } (0,1), \tag{2.93}$$

$$\tilde{n}(0) = n_0, \quad 0 < n_0 < 1, \tag{2.94}$$

$$\tilde{n}'(0) = D(n_0 - 1)n^{*p}, \tag{2.95}$$

$$\tilde{n}(b) > 0 \text{ for b} > 0 \text{ and close to } 0. \tag{2.96}$$

*has a local solution. Here* $n^* = \lim_{b\to 0} n(b)$.

**Proof.** Take

$$E = \{\tilde{n}(b)|\tilde{n}(b) \in C([0, b_0]), 0 \le \tilde{n} \le n_0 \text{ for } 0 \le b \le b_0,$$

$$\text{and some small } b_0 > 0 \text{ to be determined later}\}.$$

Then $E$ is a closed convex set in $C([0, b_0])$. Before proceeding it is helpful to rewrite equation (2.93) as an equivalent integral equation. Assuming that $n$ is a smooth solution on the interval $(0, b_0]$ with $b_0 \le 1$, we integrate once to get

$$\frac{u}{Dbn^p}\tilde{n}' - \lim_{s\to 0^+}\frac{u(s)}{Dsn^p(s)}\tilde{n}' + v\tilde{n}(b) - vn_0 = \int_0^b \frac{Dn^p\tilde{n}^q s^{1+l}}{u(s)}. \tag{2.97}$$

Since

$$\lim_{s\to 0^+}\frac{u(s)}{Dsn^p(s)}\tilde{n}' = \frac{-v}{Dn^{*p}}D(n_0 - 1)n^{*p},$$

we have

$$\frac{u}{Dbn^p}\tilde{n}' + v\tilde{n}(b) - v = \int_0^b \frac{Dn^p\tilde{n}^q s^{1+l}}{u(s)}. \tag{2.98}$$

A second integration yields

$$\tilde{n}(b) = n_0 + \int_0^b \frac{Dsn^p(s)}{u(s)}v(1 - \tilde{n}(s))ds + \int_0^b \frac{Dsn^p}{u(s)}\int_0^s \frac{Dn^p\tilde{n}^q\tau^{1+l}}{u(\tau)}d\tau. \tag{2.99}$$

For small $b_0$ and $0 \le b \le b_0$, we have

$$0 \le \tilde{n} \le n_0 < +\infty,$$

$$\tilde{n}'(b) = \frac{Dbn^p(b)}{u(b)} v(1-\tilde{n}(b)) + \frac{Dbn^p}{u(b)} \int_0^b \frac{Dn^p \tilde{n}^q \tau^{1+l}}{u(\tau)} d\tau \leq C < +\infty.$$

This shows the operator defined above is a compact operator. Therefore it has a fixed point in $E$, *i.e.*, it has a local solution.

To establish the existence and uniqueness of solution to $P_4$, we shall also need to following lemmas:

**Lemma 2.14** *If $\tilde{n}(0) = n_0$ is sufficiently close to 1, then $\tilde{n}(1) \geq 0$. If $\tilde{n}(0) = n_0$ is sufficiently close to 0, then $\tilde{n}(b) = 0$ for some $0 < b < 1$.*

**Proof.** We prove this Lemma by contradictions. We start with the following equation:

$$\tilde{n}' + \frac{vDbn^p}{u} \tilde{n} = \left[ v + \int_0^b \frac{Dn^p \tilde{n}^q s^{l+1}}{u(s)} \right] \frac{Dbn^p}{u}. \tag{2.100}$$

Let $h(b) = \frac{Dbn^p}{u}$ which is clearly nonpositive. The above equation can be rewritten as

$$(e^{\int_0^b vh(s)ds} \tilde{n})' = e^{\int_0^b vh(s)ds} h(b) \left[ v + \int_0^b h(s) \tilde{n}^q s^l ds \right]. \tag{2.101}$$

Let $H(b) = e^{\int_0^b vh(s)ds}$, and $J(b) = v + \int_0^b h(s)\tilde{n}^q s^l ds$. Integrating both sides, we have

$$\begin{aligned}
H(b)\tilde{n} - \tilde{n}(0) &= \int_0^b \frac{1}{v} H'(s) J(s) ds \\
&= \frac{1}{v} H(b)J(b) - \frac{1}{v} H(0)J(0) - \int_0^b \frac{1}{v} H(s)h(s)\tilde{n}^q(s) s^l ds \\
&= \frac{1}{v} H(b)J(b) - 1 - \int_0^b \frac{1}{v} H(s)h(s)\tilde{n}^q(s) s^l ds.
\end{aligned} \tag{2.102}$$

Suppose that $n_0$ is sufficiently close to 1 but $\tilde{n}(b) = 0$ for some $0 < b < 1$. Then the left side of the above equation is sufficiently close to -1. But since $H(b) > 0$, $J(b) = \frac{u}{Dbn^p}\tilde{n}' + v\tilde{n}(b) > 0$ and $\int_0^b \frac{1}{v} H(s)h(s)\tilde{n}^q(s)s^l ds < 0$. Thus the equality above is impossible. Hence we conclude that $\tilde{n}(1) \geq 0$ for $n_0$ sufficiently close to 1.

To prove the second statement, we note that

$$u''(0) = \lim_{b\to 0} \frac{u' + v}{b} = \lim_{b\to 0} -\frac{Db^l n^{p+q}}{u} = 0$$

for $l > 1$.

Since

$$\begin{aligned}
\frac{u}{Dbn^p}\tilde{n}'' &= -\left[\left(\frac{u}{Dbn^p}\right)' + v\right]\tilde{n}' + \frac{Dn^p\tilde{n}^q b^{1+l}}{u} \\
&= -\left[\frac{u'}{Dbn^p} - \frac{u}{Db^2n^p} - \frac{upn^{p-1}n'}{Dbn^{2p}} + v\right]\tilde{n}' + \frac{Dn^p\tilde{n}^q b^{1+l}}{u}, \qquad (2.103)
\end{aligned}$$

and

$$\lim_{b\to 0} \frac{u'}{Dbn^p} - \frac{u}{Db^2n^p} = -\frac{u''(0)}{2Dn^{*p}} = 0.$$

Thus if $p = 0$, $l > 1$, we have $\tilde{n}''(0) < 0$.

Note that there exists $\delta > 0$, such that

$$\left(v + \frac{u'}{Db} - \frac{u}{Db^2}\right) > \frac{v}{2} \quad \text{and} \quad \frac{v}{2} < |\frac{u}{b}| < 2v \quad \text{on} \quad [0, \delta].$$

We claim that $\tilde{n}'' < 0$ on$\{b \in [0, \delta] : \tilde{n} > 0\}$.

If not, $\exists\ \delta_1 \in (0, \delta)$ such that

$$\tilde{n}''(\delta_1) = 0, \quad \tilde{n}'(b) < 0 \quad \text{on} \quad (0, \delta_1).$$

Thus if we let

$$I(b) = \frac{u}{Db}\tilde{n}'' = -\left[\frac{u'}{Db} - \frac{u}{Db^2} + v\right]\tilde{n}' + \frac{D\tilde{n}^q b^{1+l}}{u}.$$

We have $I(\delta_1) = 0$.

On the other hand,

$$I > -\frac{v}{2}\tilde{n}'(0) - \frac{D\tilde{n}^q(0)}{v/2}\delta^l > 0 \quad \text{for} \quad \tilde{n}(0) << 1.$$

This ends the proof of the lemma.

**Remark.** Note in the proof of this lemma, we added the restriction on $p$ and $l$ such that $p = 0$, $l > 1$. We expect the result to be true for more general $p$ and $l$. For the rest of this thesis, keep in mind that $p = 0$ even though our argument as follows does not necessarily need this condition and thus we shall still keep this notation.

**Lemma 2.15** *Let $\tilde{n}_1$, $\tilde{n}_2$ be the solutions corresponding two different initial value $n_{10}$ and $n_{20}$ with $n_{10} > n_{20}$, then $n_1 > n_2$ on $[0, min\{T(n_{10}), T(n_{20})\}]$ where $T(n_{i0})$ represent the corresponding maximal existence interval in the sense that $\tilde{n}(T) = 0$.*

**Proof.** Suppose otherwise there is a $b_0 \in [0, min\{T(n_{10}), T(n_{20})\}]$ such that $\tilde{n}_1(b_0) = \tilde{n}_2(b_0)$ and $\tilde{n}_1 > \tilde{n}_2$ on $[0, b_0)$, then $\tilde{n}_1'(b_0) < \tilde{n}_2'(b_0)$. We should have

$$\frac{u(b_0)}{Db_0 n^p(b_0)}\tilde{n}_1'(b_0) + v\tilde{n}_1(b_0) - v > \frac{u(b_0)}{Db_0 n^p(b_0)}\tilde{n}_2'(b_0) + v\tilde{n}_2(b_0) - v, \tag{2.104}$$

while

$$\int_0^{b_0} \frac{Dn^p\tilde{n}_1^q s^{1+l}}{u(s)} < \int_0^{b_0} \frac{Dn^p\tilde{n}_2^q s^{1+l}}{u(s)}, \tag{2.105}$$

which is clearly a contradiction to (2.98).

In view of the above Lemmas, we have established the following theorem

**Theorem 2.4** *Problem $P_2$ admits a unique solution. The solution $\tilde{n}$ satisfies the following property:*

$$0 \le \tilde{n} \le 1,$$

$$\tilde{n} \ge \frac{1}{L}(1 - b),$$

$$\limsup_{\beta\to 1^-} \frac{\tilde{n}}{1-b} \le L,$$

$$-M \le \tilde{n}' \le 0.$$

**Proof.** Since equation (2.90) is only degenerate at $b = 1$. The boundedness of $\tilde{n}'$ on $[0, 1)$ follows from standard ODE theory. We shall study the boundedness at $b = 1$. To this purpose, we let $n(b) \sim (1-b)^\alpha$ with $\alpha \ge 1$ as $b \to 1$. Let $\tilde{n}(b) \sim (1-b)^\beta$ as $b \to 1$. In view of equation (2.90), we have

$$u \sim -\frac{Dn^{p+q}}{v} \; as \; b \to 1.$$

Thus as $b \to 1$, $\tilde{n}$ satisfies

$$(\frac{n^q}{-v}\tilde{n}')' + v\tilde{n}' + v\frac{\tilde{n}^q}{n^q} = 0. \tag{2.106}$$

Suppose that $\beta < 1$, matching the leading singular terms in the equation above, we have

$$\beta - 1 = q(\beta - \alpha),$$

or

$$\beta = \frac{\alpha q - 1}{q - 1}.$$

This is clearly impossible since $\alpha \ge 1$. Thus we conclude $\beta \ge 1$, or equivalently, $\tilde{n}'(1)$ exits. In fact, by matching the coefficients, one can show that

$$\limsup_{b\to 1^-} \frac{\tilde{n}}{1-b} \le L.$$

Let $W = \frac{u}{Dbn^p}\tilde{n}'$ and differentiate (2.85), we get

$$(\frac{u}{Dbn^p}W')' + vW' - q\tilde{n}^{q-1}\tilde{n}'b^l - l\tilde{n}^q b^{l-1} = 0. \tag{2.107}$$

Also

$$W(0) > 0, \quad W(1) = 0.$$

The maximum principle yields $W > 0$ in $(0,1)$, *i.e.*, $\tilde{n}' < 0$ in $(0,1)$. Note that to deduce $W(1) = 0$, we have applied the fact that $\tilde{n}'(1)$ is bounded.

By Lemma 2.14, we have the bound on $\tilde{n}(0)$. The uniqueness follows from Lemma 2.15. This ends the proof of this theorem.

We shall now combine Lemma 2.15 with Theorem 2.4. For every $n \in V$, we define $(v, u)$ to be the solution of $P_1$ and $\tilde{n}$ by Theorem 2.4, and introduce the mapping $T$ by

$$Tn = \tilde{n}.$$

Clearly, $T$ maps $V$ into itself, and its image lies in a compact subset of $V$ (since $-M \leq \tilde{n}' \leq 0$). By the uniqueness part in Lemma 2.15, it also follows that $T$ is continuous. Invoking the Schauder fixed point theorem, we conclude that there exists at least one fixed point for $T$. We shall denote it by $(v^*, u, n)$. To show that the corresponding $(v^*, b, n)$ is a solution to problem $P_2$, we shall only need to show equality (2.89).

In fact, we do have the following conclusion.

**Lemma 2.16** $\displaystyle\int_0^1 \frac{Db^k n^p}{u(b)} db = -\infty.$

**Proof.** We shall start with the following facts. There exists a constant $L > 0$ such that $n(b) \geq 1/L(1-b)$ and a constant $C_2 < 0$ such that $u(b) \geq C_2(1-b)^{p+q}$ where $(u, n)$ is the pair in the fixed point stated above. The first fact is trivial. The second

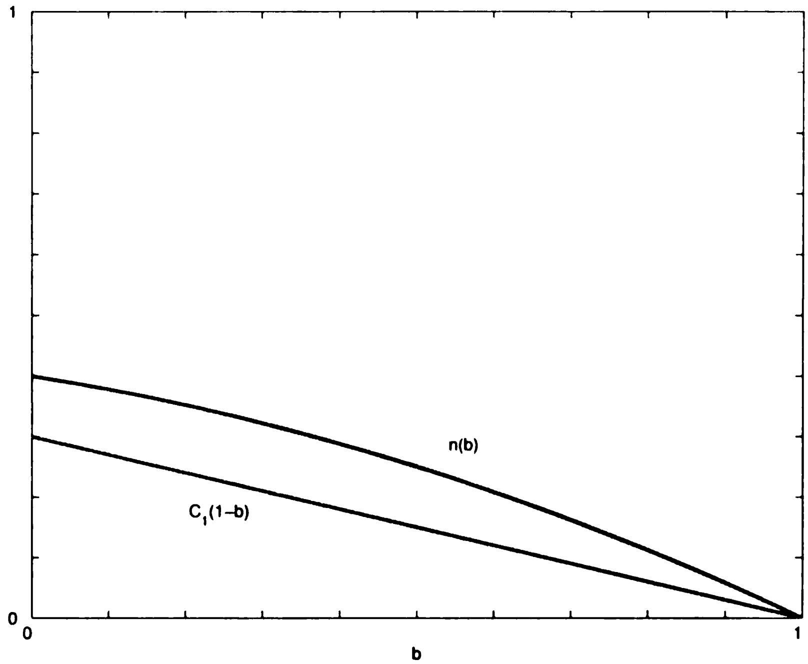


Figure 2.7. Sketch of n

fact follows from Lemma 2.11. Thus

$$\int_0^1 \frac{Db^k n^p}{u(b)} db \le \int_0^1 \frac{Db^k (1/L)^p (1-b)^p}{C_2(1-b)^{p+q}} = -\infty.$$

**Remark.** Note that we kept $k$ and $p$ here and it is true for general $k$ and $p$ as long as $q \ge 1$.

## 2.10 Instability of Flat Front

In this section, we carry out the linear stability analysis of the traveling wave front. A similar analysis on a nondegenerate reaction diffusion system can be found in the papers by Horváth et al.[36, 37]. They introduced a small spatial perturbation of the traveling wave solution. Upon linearization of the system, they obtained a dispersion relation. Using numerical method, they obtained a detailed diagram of the dispersion relation and explained the instability phenomenon. Because of degeneracy at the flat wave front, we not only perturb the wave front but also the bacteria and nutrient function at the front.

The system we will analyze takes the form

$$\frac{\partial b}{\partial t} = D\Delta b^{k+1} + nb, \tag{2.108}$$

$$\frac{\partial n}{\partial t} = \Delta n - nb. \tag{2.109}$$

For the convenience of analysis, we introduce the perturbed coordinate:

$$\eta = \xi + \epsilon exp(\lambda t)\cos(qy)$$

in terms of which the bacteria and nutrient density can be expressed as

$$b(\eta, y, t) = b_0(\eta) + \epsilon b_1(\eta) exp(\lambda t)\cos(qy),$$

$$n(\eta, y, t) = n_0(\eta) + \epsilon n_1(\eta) exp(\lambda t)\cos(qy).$$

Here $\xi$ is the traveling wave coordinate, $b_0$ and $n_0$ are the traveling wave solutions.

$$\begin{aligned} n_t &= (n_0)_\eta(-v_0 + \epsilon\lambda exp(\lambda t)\cos(qy)) + \epsilon(n_1)_\eta(-v_0 + \epsilon\lambda exp(\lambda t)\cos(qy))exp(\lambda t) \\ &\quad \cos(qy) + \epsilon n_1 \lambda exp(\lambda t)\cos(qy), \\ n_{xx} &= (n_0)_{\eta\eta} + \epsilon(n_1)_{\eta\eta} exp(\lambda t)\cos(qy), \end{aligned}$$

$$
\begin{aligned}
n_y &= (n_0)_\eta(-\epsilon q)exp(\lambda t)\sin(qy) + \epsilon(n_1)_\eta(-\epsilon q)exp(\lambda t)\sin(qy)exp(\lambda t)\cos(qy) + \\
&\quad \epsilon n_1 exp(\lambda t)(-q)\sin(qy), \\
n_{yy} &= (n_0)_\eta(-\epsilon q^2)exp(\lambda t)\cos(qy) + (n_0)_{\eta\eta}(-\epsilon q)^2 exp(2\lambda t)(\sin qy)^2 \\
&\quad +\epsilon n_1 exp(\lambda t)(-q^2)\cos(qy) + O(\epsilon^2).
\end{aligned}
$$

Substituting the above expressions and Equalizing the $\epsilon$ terms in equation (2.109), we obtain

$$
(n_0)_\eta\lambda - v_0(n_1)_\eta + n_1\lambda = (n_1)_{\eta\eta} - q^2(n_0)_\eta - q^2 n_1 - n_0 b_1 - n_1 b_0. \qquad (2.110)
$$

Similarly we calculate

$$
\begin{aligned}
b_t &= (b_0)_\eta(-v_0 + \epsilon\lambda exp(\lambda t)\cos(qy) + \epsilon(b_1)_\eta \\
&\quad (-v_0 + \epsilon\lambda exp(\lambda t)\cos(qy))exp(\lambda t)\cos(qy) + \epsilon b_1\lambda exp(\lambda t)\cos(qy) \\
(b^{k+1})_x &= (k+1)b^k((b_0)_\eta + \epsilon(b_1)_\eta exp(\lambda t)\cos(qy)) \\
(b^{k+1})_{xx} &= (k+1)kb^{k-1}((b_0)_\eta + \epsilon(b_1)_\eta exp(\lambda t)\cos(qy))^2 \\
&\quad +(k+1)b^k((b_0)_{\eta\eta} + \epsilon(b_1)_{\eta\eta} exp(\lambda t)\cos(qy) \\
(b^{k+1})_y &= (k+1)b^k[(b_0)_\eta(-\epsilon q)exp(\lambda t)\sin(qy) + \epsilon(b_1)_\eta(-\epsilon q)exp(\lambda t)\sin(qy)exp(\lambda t) \\
&\quad \cos(qy) + \epsilon b_1(-q)exp(\lambda t)\sin(qy)] \\
(b^{k+1})_{yy} &= (k+1)b^k[(b_0)_\eta(-\epsilon q)exp(\lambda t)\sin(qy) + \epsilon(b_1)_\eta(-\epsilon q)exp(\lambda t)\sin(qy)exp(\lambda t) \\
&\quad \cos(qy) + \epsilon b_1(-q)exp(\lambda t)\sin(qy)]^2 + (k+1)b^k[(b_0)_{\eta\eta}(\epsilon q)^2\exp(2\lambda t)\sin^2(qy) \\
&\quad +(b_0)_\eta(-\epsilon q^2)\exp(\lambda t)\cos(qy) + \epsilon b_1\exp(\lambda t)(-q^2)\cos(qy) + O(\epsilon^2)] \qquad (2.111)
\end{aligned}
$$

Substituting the above expressions into equation (2.108) and equalizing the $\epsilon$

terms, we have

$$
\begin{aligned}
(b_0)_\eta\lambda - v_0(b_1)_\eta + b_1\lambda &= (k+1)k(k-1)b_0^{k-2}b_1(b_0)_\eta^2 + 2(k+1)kb_0^{k-1}(b_0)_\eta(b_1)_\eta \\
&\quad + (k+1)kb_0^{k-1}b_1(b_0)_{\eta\eta} + (k+1)(b_0)^k(b_1)_{\eta\eta} \\
&\quad + (k+1)(b_0)^k[-q^2(b_0)_\eta - q^2b_1] + n_1b_0 + n_0b_1. \qquad (2.112)
\end{aligned}
$$

Using equations (2.110) and (2.112) we can linearize the original system (2.108)-(2.109) into

$$
\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}\begin{pmatrix} b_1 \\ n_1 \end{pmatrix} = \begin{pmatrix} \lambda + Df'q^2 & 0 \\ 0 & \lambda + q^2 \end{pmatrix}\begin{pmatrix} b_1 + (b_0)_\eta \\ n_1 + (n_0)_\eta \end{pmatrix} \qquad (2.113)
$$

*where*

$$
\begin{aligned}
L_{11} &= D\frac{\partial^2}{\partial\eta^2}(f') + v_0\frac{\partial}{\partial\eta} + n_0 \\
&= Df'\frac{\partial^2}{\partial\eta^2} + (2Df''\frac{\partial b_0}{\partial\eta} + v_0)\frac{\partial}{\partial\eta} \\
&\quad + Df'''(\frac{\partial b_0}{\partial\eta})^2 + Df''\frac{\partial^2 b_0}{\partial\eta^2} + n_0, \qquad (2.114) \\
L_{12} &= b_0, \qquad (2.115) \\
L_{21} &= -n_0, \qquad (2.116) \\
L_{22} &= \frac{\partial^2}{\partial\eta^2} + v_0\frac{\partial}{\partial\eta} - b_0. \qquad (2.117)
\end{aligned}
$$

Here $f = b_0^{k+1}$ and the differentiation is with respect to $b_0$.

Similarly, if we have the following system:

$$
\begin{cases} b_t = D\nabla\cdot(n\nabla b^{k+1}) + nb \\ n_t = \Delta n - nb \end{cases} \qquad (2.118)
$$

After some tedious calculation, we arrive at the following linearized system:

$$\begin{pmatrix} \bar{L}_{11} & \bar{L}_{12} \\ \bar{L}_{21} & \bar{L}_{22} \end{pmatrix} \begin{pmatrix} b_1 \\ n_1 \end{pmatrix} = \begin{pmatrix} \lambda + Dn_0 f' q^2 & 0 \\ 0 & \lambda + q^2 \end{pmatrix} \begin{pmatrix} b_1 + (b_0)_\eta \\ n_1 + (n_0)_\eta \end{pmatrix} \tag{2.119}$$

$$\begin{aligned} \bar{L}_{11} &= Dn_0 \frac{\partial^2}{\partial \eta^2}(f') + v_0 \frac{\partial}{\partial \eta} + n_0 \\ &= Dn_0 f' \frac{\partial^2}{\partial \eta^2} + (2Dn_0 f'' \frac{\partial b_0}{\partial \eta} + v_0) \frac{\partial}{\partial \eta} \\ &\quad + Dn_0 f''' (\frac{\partial b_0}{\partial \eta})^2 + Dn_0 f'' \frac{\partial^2 b_0}{\partial \eta^2} + n_0 \\ &\quad + Dn_0 f'(n_0)_\eta \frac{\partial}{\partial \eta} + Dn_0 f''(n_0)_\eta (b_0)_\eta, \end{aligned} \tag{2.120}$$

$$\bar{L}_{12} = b_0 + Df'(b_0)_\eta \frac{\partial}{\partial \eta} + Df''((b_0)_\eta)^2 + Df'(b_0)_{\eta\eta}, \tag{2.121}$$

$$\bar{L}_{21} = -n_0, \tag{2.122}$$

$$\bar{L}_{22} = \frac{\partial^2}{\partial \eta^2} + v_0 \frac{\partial}{\partial \eta} - b_0. \tag{2.123}$$

Now if we consider the regular perturbation, similarly we obtain the following linearized system:

$$\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} \bar{b}_1 \\ \bar{n}_1 \end{pmatrix} = \begin{pmatrix} \lambda + f' q^2 & 0 \\ 0 & \lambda + q^2 \end{pmatrix} \begin{pmatrix} \bar{b}_1 \\ \bar{n}_1 \end{pmatrix} \tag{2.124}$$

This implies that

$$\bar{b}_1 = b_1 + (b_0)_\eta, \quad \bar{n}_1 = n_1 + (n_0)_\eta.$$

As we've seen before, $b_0$ is singular at the front, the above expression indicates that $\frac{\bar{b}_1}{b_0}$ *is not* a small perturbation. That's why we can't use the usual perturbation method.

For this eigenvalue problem, we consider two separate regions. For $\eta > 0$, we have

$$b_1 = 0$$

$$\frac{\partial^2 n_1}{\partial \eta^2} + v_0 \frac{\partial n_1}{\partial \eta} - (\lambda + q^2) n_1 = (\lambda + q^2) \frac{\partial n_0}{\partial \eta}$$

which can be solved. In fact,

$$n_1 = (1 - c_0) v_0 exp(-v_0 \eta) + d_0 exp(-w \eta).$$

with $w = \frac{v_0 - \sqrt{v_0^2 + 4(\lambda + q^2)}}{2}$ and $d_0$ is to be determined using the boundary condition at $\eta = 0$. Since $n_1$ and its derivative are continuous at $\eta = 0$, we have

$$n_1(0) = (1 - c_0) v_0 + d_0,$$

and

$$\partial_\eta n_1 = -(1 - c_0) v_0^2 - d_0 w.$$

Since we require that $b_1 / b_0$ is bounded, we may assume that at $\eta = 0$ $b_1$ vanishes as

$$b_1 = B(-\eta)^\beta.$$

By a straightforward calculation we obtain

$$B = -\frac{\lambda}{k v_0} (\frac{k v_0}{D})^{1/k}$$

and

$$\beta = \frac{1}{k.}$$

Hence

$$\frac{b_1}{b_0} \rightarrow \frac{\lambda}{kv} \quad \text{for} \quad \eta \rightarrow 0$$

which is finite.

The boundary conditions at $\eta \rightarrow -\infty$ are given by

$$b_1 \rightarrow 0, \quad \partial_\eta b_1 \rightarrow 0,$$

$$n_1 \rightarrow 0, \quad \partial_\eta n_1 \rightarrow 0.$$

Note $\lambda$ is small when $q$ is small. Also $n_1 = 0$ and $b_1 = 0$ for $q = 0$ and so for small $q$, $b_1$ and $n_1$ are both of order $q^2$ too. This implies that in equation (2.113) for $q$ small, the terms on the right hand side involving $b_1$ and $n_1$ are of order $q^4$. Hence to order $q^2$, we have

$$\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} b_1 \\ n_1 \end{pmatrix} = \begin{pmatrix} \lambda + Df'q^2 & 0 \\ 0 & \lambda + q^2 \end{pmatrix} \begin{pmatrix} (b_0)_\eta \\ (n_0)_\eta \end{pmatrix} \tag{2.125}$$

Let $L^*$ be the adjoint operator of $L$ which is obtained from

$$\int_{-\infty}^{\infty} (\psi_1, \psi_2) L \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \int_{-\infty}^{\infty} (\phi_1, \phi_2) L^* \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{2.126}$$

Let $\psi_1$ and $\psi_2$ be the eigenvector of

$$L^* = \begin{pmatrix} Df'\frac{\partial^2}{\partial\eta^2} - v_0\frac{\partial}{\partial\eta} + n_0 & -n_0 \\ b_0 & \frac{\partial^2}{\partial\eta^2} - v_0\frac{\partial}{\partial\eta} - b_0 \end{pmatrix} \tag{2.127}$$

Hence by the definition of $L^*$, we have

$$\int_{-\infty}^{\infty}(\psi_1,\psi_2)L\begin{pmatrix} b_1 \\ n_1 \end{pmatrix} = \int_{-\infty}^{\infty}(b_1,b_2)L^*\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{2.128}$$

Hence

$$\int_{-\infty}^{\infty}(\psi_1,\psi_2)L\begin{pmatrix} \lambda + Df'q^2 & 0 \\ 0 & \lambda + q^2 \end{pmatrix}\begin{pmatrix} \frac{\partial b_0}{\partial \eta} \\ \frac{\partial n_0}{\partial \eta} \end{pmatrix} = 0 \tag{2.129}$$

from which we obtain the following relation

$$\lambda \int_{-\infty}^{+\infty}(\psi_1\frac{\partial b_0}{\partial \eta} + \psi_2\frac{\partial n_0}{\partial \eta}) = -q^2\int_{-\infty}^{+\infty}(D\psi_1 f'\cdot\frac{\partial b_0}{\partial \eta} + \psi_2\frac{\partial n_0}{\partial \eta}) \tag{2.130}$$

So when $q$ is small, we have

$$\frac{d\lambda}{dq^2}|_{q=0} = -\frac{I_1}{I_2} \tag{2.131}$$

with

$$I_1 = \int_{-\infty}^{+\infty}(D\psi_1 f'\cdot\frac{\partial b_0}{\partial \eta} + \psi_2\frac{\partial n_0}{\partial \eta}) \tag{2.132}$$

and

$$I_2 = \int_{-\infty}^{+\infty}(\psi_1\frac{\partial b_0}{\partial \eta} + \psi_2\frac{\partial n_0}{\partial \eta}). \tag{2.133}$$

From this dispersion relation, one may be able to conclude that for fix $k$, there exists a threshold $D_k$ such that the flat interface is unstable for small $q$ if $D < D_k$ and it is stable for all $q$ if $D > D_k$. A possible dispersion diagram is as shown in Figure 2.8.

This can be explained in the following way. We consider a perturbed flat front moving from left to right as sketched. At the tip of bacteria finger that penetrate

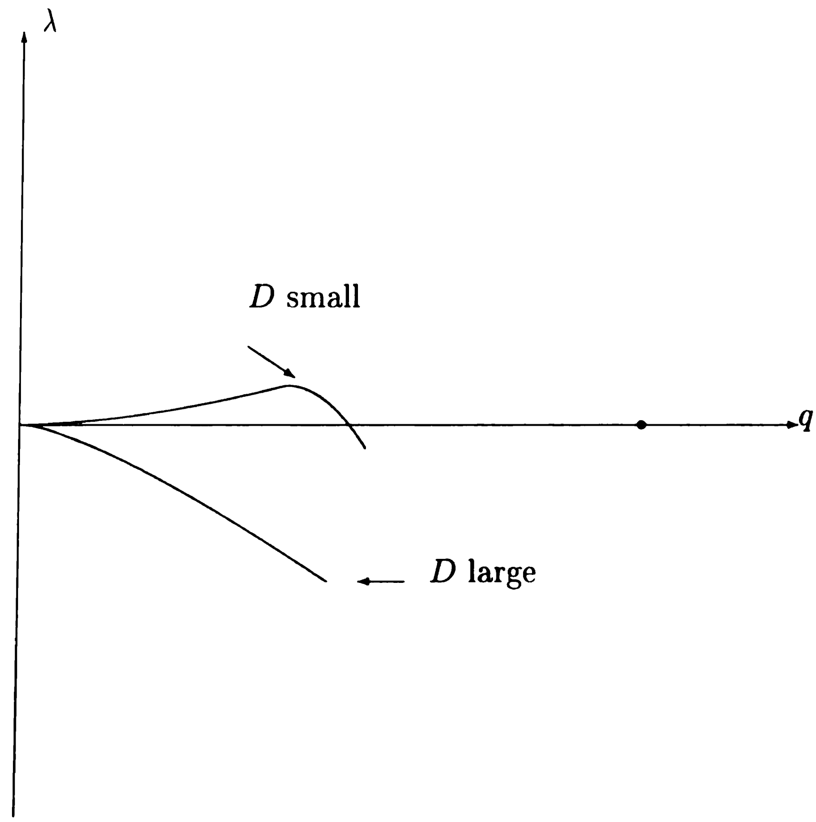


Figure 2.8. An illustration of the dispersion relation for different $D$ values

into the nutrient region, the nutrient gradients are compressed and the nutrient diffusion is enhanced. The "feeding" of the interface from the nutrient is hence enhanced there, and this tends to make such fingers grow larger. On the other hand, on the back of such fingers, the bacterial diffusion is reduced, this tends to reduce the finger from growing, and hence stabilize the interface perturbation. The relative strength of the two effects is determined by $D$. When $D > D_k$, the stabilizing effect takes over, when $D < D_k$, the destabilizing effect wins over.

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