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NONLOCAL CAHN-HILLIARD EQUATION

By

Jianlong Han

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ABSTRACT

NONLOCAL CAHN-HILLIARD EQUATION

By

Jianlong Han

The thesis includes three parts. In the first part, we study a nonlocal Cahn-Hilliard equation with no flux boundary condition and prove the existence, uniqueness and continuous dependence on initial data of the solution to this equation. We also apply a nonlinear *Poincaré* inequality to show the existence of an absorbing set in each constant mass affine space. In the second part, we study the existence, uniqueness and continuous dependence on initial data of the solution to a nonlocal Cahn-Hilliard equation with homogeneous Dirichlet boundary conditions on a bounded domain. Under a nondegeneracy assumption the solutions are classical but when this is relaxed, the equation is satisfied in a weak sense. Also we prove that there exists a global attractor in some metric space. In the third part, we establish the existence, uniqueness and continuous dependence on initial values for classical solutions to the Cauchy problem of a nonlocal Cahn-Hilliard equation. We also prove that under certain conditions, there exists a discontinuous steady state solution for this equation in a bounded domain.

To my mother

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CHAPTER 1

Introduction

An interesting phenomenon is observed when a molten binary alloy is rapidly cooled to a lower temperature. We find that the sample becomes inhomogeneous very quickly, decomposing into a very fine-grained structure - two concentration phases, one rich in one component and one rich in the other component. As time passes, the fine-grained structure becomes more coarse with larger particles growing and smaller particles tending to dissolve. The sudden appearance of a fine grained structure is called **spinodal decomposition**. The coarsening process is called **Ostwald Ripening**.

In 1953, materials scientists John Cahn and John Hilliard derived the following equation:

$$u_t = -\Delta(\varepsilon^2 \Delta u - F'(u)) \quad \text{for} \quad \mathrm{x} \in \Omega \subset \mathbb{R}^n \quad \text{and} \quad t > 0, \tag{1.1}$$

with the natural boundary conditions

$$\frac{\partial u}{\partial n} = 0 \quad \text{and} \quad \frac{\partial}{\partial n}(\Delta u - \lambda F'(u)) = 0 \quad \text{on} \quad \partial\Omega.$$

They conjectured that (sometimes called the Principle of Spinodal Decomposition), "Most solutions to the Cahn-Hilliard equation that start with initial data near a fixed constant in the spinodal region exhibit fine-grained decomposition." Since the conjecture agrees with the outcome of physical experiments, the Cahn-Hilliard equation has been accepted as a meaningful model of the dynamics of phase separation in binary alloys.

To derive the Cahn-Hilliard equation as a model for the evolution of a concentration field in a binary alloy, we take the point of view that microstructure changes in such a way as to decrease the total free energy of the sample, consistent with the second law of thermodynamics. Not only should the free energy decrease but it should do so as quickly as possible.

The Helmholtz free energy of a state is

$$E = H - TS,$$

where H is the total interaction energy, T is the absolute temperature, and S is the total entropy of mixing. Using accepted definitions of H and S, for a scaled concentration field u at fixed subcritical temperature, one can derive the expression

$$E(u) = \frac{C}{4}\int\int J(x-y)(u(x)-u(y))^2 dxdy + \int F(u(x))dx, \tag{1.2}$$

where C is a positive constant, F is a double well function, having local minima at ± 1, and the interaction kernel J is assumed to be integrable with positive integral and with $J(-x) = J(x)$ by the symmetry of interaction between sites.

The goal is to find dynamics for the field u which decreases $E(u)$ optimally, consistent with thermodynamic principles. This suggests the evolution law.

$$\gamma \frac{\partial u}{\partial t} = -\text{grad} E(u), \tag{1.3}$$

where $\gamma > 0$ is called the relaxation coefficient since it determines the rate at which u approaches equilibrium.

In the evolution equation, for each t, $u(t)$ is a function of position, that is, $u(t)$ lies in some space of functions X defined on a spatial domain, and so $\frac{\partial u}{\partial t} \in X$ for all $t > 0$. On the other hand, E: $X \to \mathbb{R}$ is a nonlinear functional and grad E(u) is therefore the linear functional on X defined by

$$< \mathrm{grad} E(u), v >= \frac{d}{dh} E(u+hv)|_{h=0}, \tag{1.4}$$

where $<,>$ is the duality pairing. In order to have (1.3) make sense, X must be a Hilbert space. First we consider the case $X = L^2(\mathbb{R}^n)$. In this case, we get

$$\mathrm{grad}_{L^2} E(u) = C[(\int J(z)dz)u - J * u] + F'(u),$$

where $*$ is convolution. On a bounded domain Ω, this is given by $J * u \equiv \int_\Omega J(x - y)u(y)dy$. Taking $\int J = 1$, $C = 1$ and denoting $f = F'$, equation (1.3) becomes

$$\gamma \frac{\partial u}{\partial t} = J * u - u - f(u). \tag{1.5}$$

We call this the Nonlocal Allen-Cahn equation. In (1.2), if we make the approximation

$$u(x) - u(y) \simeq \nabla u(x) \cdot (x - y),$$

and assume J is isotropic, equation (1.3) leads to

$$\frac{\partial u}{\partial t} = d \Delta u - f(u), \tag{1.6}$$

the appropriate boundary conditions being

$$\frac{\partial u}{\partial n} = 0.$$

This is called the Allen-Cahn equation.

Note that both versions of the Allen-Cahn equation do not preserve the average value of u. This violates conservation of species if we are modeling phase change in binary alloys. One way to correct this is to select a new metric space with respect to which we take the gradient of the energy. So following Fife [23], we consider a new Hilbert space H_0^{-1}, where H^{-1} is the dual of the Sobolev space H^1 and the subscript zero refers to mean value zero. Recall that, if $f \in L^2$ and $\int f = 0$, there is a unique ϕ such that

$$-\Delta \phi = f, \; \frac{\partial \phi}{\partial n} = 0, \text{ and} \int \phi = 0.$$

We use the notation $\phi = (-\Delta_0)^{-1} f$. Since $(-\Delta_0)^{-1}$ is a positive self adjoint operator, fractional powers of it are well defined. The space H_0^{-1} is the completion of the space of smooth functions of mean value zero in the norm

$$||u||_{H_0^{-1}} = ||(-\Delta_0)^{-\frac{1}{2}} u||_{L^2}.$$

The inner product is given by

$$< u, v >_{H_0^{-1}} = < (-\Delta_0)^{-\frac{1}{2}} u, (-\Delta_0)^{-\frac{1}{2}} v >_{L^2}$$

for u, v belonging to H_0^{-1}. If $u \in H_0^{-1}$ and $v \in L^2$, then

$$< u, v >_{H_0^{-1}} = < (-\Delta_0)^{-1} u, v >_{L^2} .$$

This means that the representative of grad E(u) in H_0^{-1} is $(-\Delta)(\text{grad}_{L^2} E(u))$ and instead of the Nonlocal Allen-Cahn equation, we have the following Nonlocal Cahn-Hilliard equation

$$\frac{\partial u}{\partial t} = \Delta((\int_\Omega J(x-y)dyu(x) - \int_\Omega J(x-y)u(y)dy + f(u)). \tag{1.7}$$

with natural boundary condition

$$\frac{\partial(\int_\Omega J(x-y)dyu(x) - \int_\Omega J(x-y)u(y)dy + f(u))}{\partial n} = 0.$$

Integrating the equation over Ω, using the Divergence Theorem and the boundary condition we have

$$\frac{d}{dt}\int_\Omega u = 0,$$

so species are conserved. Also, to see that energy decreases along trajectories, note that

$$\begin{aligned}
\frac{dE(u)}{dt} =& 2\int\int 2J(x-y)(u(x)-u(y))(u_t(x)-u_t(y))dxdy + \int F'(u)u_t dx \\
=& 2\int\int J(x-y)u(x)u_t(x)dxdy - 2\int\int J(x-y)u(x)u_t(y)dxdy - \\
& 2\int\int J(x-y)u(y)u_t(x)dxdy + 2\int\int J(x-y)u(y)u_t(y)dxdy + \int f(u)u_t dx.
\end{aligned} \tag{1.8}$$

For the case of a bounded domain Ω, using the symmetry of J, if we write $a(x) = \int_\Omega J(x-y)dy$, $J * u(x) = \int_\Omega J(x-y)u(y)dy$, and $k(u) = a(x)u - J * u(x) + f(u)$, then we have

$$
\begin{aligned}
\frac{dE(u)}{dt} &= \int (a(x)u(x) - J * u(x) + f(u))u_t dx \\
&= \int k(u)\Delta k(u) dx \\
&= -\int |\nabla k(u)|^2 dx \\
&\leq 0.
\end{aligned}
\tag{1.9}
$$

Again using a first order approximation for $u(x) - u(y)$, the local equation corresponding to (1.7) is

$$
\frac{\partial u}{\partial t} = -\Delta(d\Delta u - f(u)) \quad \text{for} \quad \mathrm{x} \in \Omega \subset \mathbb{R}^n \quad \text{and} \quad t > 0, \tag{1.10}
$$

with the natural boundary conditions

$$
\frac{\partial u}{\partial n} = 0 \quad \text{and} \quad \frac{\partial}{\partial n}(\Delta u - f(u))) = 0 \quad \text{on} \quad \partial\Omega.
$$

This is the Cahn-Hilliard equation (1.1).

Equations (1.5), (1.6), (1.7), and (1.10) are important in the study of materials science in modeling certain phenomena such as spinodal decomposition, Ostwald ripening, and grain boundary motion. Equations (1.7) and (1.10) share some common features, for example, the mass is conserved and the energy is decreased. There is a lot of work on equation (1.10), see for example [2], [3], [4], [5], [8], [9], [10], [11], [18], [19], [21], [35], [39] and references contained in those articles.

However, for equation (1.7), there are very few results. To the best of our knowledge, the only results related to equation (1.7) were given in [26] by H. Gajewski and K. Zacharias and in [29] by G.Giacomin and J. Lebowitz .

In [26], H. Gajewski and K. Zacharias considered the equation

$$\begin{aligned} &\frac{\partial}{\partial t}(f'^{-1}(v-w)) - \nabla\cdot(\mu v) = 0,\\ &\frac{\partial(\mu v)}{\partial n}|_{\partial\Omega} = 0, \\ &u(x,0) = u_0(x), \end{aligned} \tag{1.11}$$

where $\omega(x) = \int_\Omega \kappa(|x-y|)(1-2u(y))dy$, $v = f'(u)+\omega$, and $f(u) = u\log u + (1-u)\log(1-u)$.

The mobility μ has the form $\mu = \dfrac{a(x,\nabla v)}{f''(u)}$, where a satisfies:

$$(a(x,s_1)s_1 - a(x,s_2)s_2)(s_1-s_2) \geq \alpha_0|s_1-s_2|^2,\ \ s_1,\ s_2 \in \mathbb{R}_+,$$

$$(a(x,s_1)s_1 - a(x,s_2)s_2)(s_1-s_2) \leq \frac{\alpha_1}{3}|s_1-s_2|.$$

They proved the existence and uniqueness of a solution to equation (1.11).

In [29], G. Giacomin and J. Lebowitz considered the equation

$$\partial_t\rho = \nabla\cdot[\sigma(\rho)\nabla(\frac{\delta F_0(\rho)}{\delta\rho})]$$

on $\mathbb{T}^d$, the torus $\mathbb{R}^d\, mod\, \mathbb{Z}^d$, where $\dfrac{\delta F_0(\rho)}{\delta\rho}$ is the L_2 gradient of F_0. Here, σ is a function from $[0,1]$ taking nonnegative values and such that $\sigma(0)=\sigma(1)=0$, and

$$F_0(\rho) = \int_{\mathbb{T}^d} f_c(\rho(r))dr + \frac{1}{4}\int\int_{\mathbb{T}^d\times\mathbb{T}^d} J(r-r')((\rho(r)-\rho(r'))^2 dr dr'.$$

The function f_c has a double well structure, symmetric about $\frac{1}{2}$, with the minimum at values ρ^+ and $\rho^- < \rho^+$.

Denote $g(\rho) = f_c(\rho) + \frac{\bar{J}(0)}{2}(\rho-\frac{1}{2})^2$, where $\bar{J}(0) = \int_{\mathbb{T}^d} J(r)dr$. In [29], it is assumed that:

(1) There exists a constant $c > 0$ such that

$$\frac{1}{c} \leq D(\rho) \equiv \sigma(\rho)g''(\rho) \leq c$$

for all $\rho \in (0,1)$.

(2) Both g and σ are symmetric with respect to $\frac{1}{2}$.

(3) $J \in C^2(\mathbb{T}^d)$, $J \geq 0$ and $J(r)$ depends only on $|r|$.

With the above assumptions, G. Giacomin and J. Lebowitz indicated how one might prove the existence and uniqueness of solutions.

In chapter 2, we study equation (1.7) with no flux boundary condition and prove the existence, uniqueness and continuous dependence on initial data of the solution to this equation. We also apply a nonlinear *Poincaré* inequality to show the existence of an absorbing set in each constant mass affine space. In Chapter 3, we study the existence, uniqueness and continuous dependence on initial data of the solution to equation (1.7) with Dirichlet boundary conditions on a bounded domain. Under a nondegeneracy assumption the solutions are classical but when this is relaxed, the equation is satisfied in a weak sense. Also we prove that there exists a global attractor in some metric space. In chapter 4, we establish the existence, uniqueness and continuous dependence on initial values for classical solutions to the Cauchy problem of equation (1.7). We also prove that under certain conditions, there exists a discontinuous steady state solution for equation (1.7).

CHAPTER 2

The Neumann boundary problem for a nonlocal Cahn-Hilliard equation

2.1 Existence and uniqueness

Consider the integro-differential boundary value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta\left(\displaystyle\int_\Omega J(x-y)dyu(x) - \int_\Omega J(x-y)u(y)dy + f(u)\right) \text{ in } \quad \Omega,\, t > 0, \\ \dfrac{\partial\left(\int_\Omega J(x-y)dyu(x) - \int_\Omega J(x-y)u(y)dy + f(u)\right)}{\partial n} = 0 \text{ on } \quad \partial\Omega,\, t > 0, \\ u(x,0) = u_0(x). \end{cases} \tag{2.1}$$

In order to prove the existence of a classical solution to (2.1), we need the initial data to satisfy the boundary condition. So we assume $u_0(x) \in C^{2+\beta, \frac{2+\beta}{2}}(\bar{\Omega})$ for some $\beta > 0$, and $u_0(x)$ satisfies the compatibility condition:

$$\frac{\partial\left(\int_\Omega J(x-y)dyu_0(x) - \int_\Omega J(x-y)u_0(y)dy + f(u_0)\right)}{\partial n} = 0 \quad \text{on} \quad \partial\Omega. \tag{2.2}$$

Rewrite (2.1) as

$$\begin{cases} \dfrac{\partial u}{\partial t} = a(x,u)\Delta u + b(x,u,\nabla u) \text{ in } \quad \Omega,\, t>0, \\ a(x,u)\dfrac{\partial u}{\partial n} + \dfrac{\partial a(x)}{\partial n}u(x) - \displaystyle\int_\Omega \dfrac{\partial J(x-y)}{\partial n}u(y)dy = 0 \text{ on } \quad \partial\Omega,\, t>0, \\ u(x,0) = u_0(x), \end{cases} \tag{2.3}$$

where

$$\begin{aligned} a(x,u) &= a(x) + f'(u), \\ a(x) &= \int_\Omega J(x-y)dy, \\ b(x,u,\nabla u) &= 2\nabla a \cdot \nabla u + f''(u)|\nabla u|^2 + u\Delta a - (\Delta J) * u. \end{aligned}$$

We assume the following conditions:

(A_1) $a(x) \in C^{2+\beta}(\bar{\Omega})$, $f \in C^{2+\beta}(\mathbb{R})$.

(A_2) There exist $c_1 > 0$, $c_2 > 0$, and $r > 0$ such that

$$a(x,u) = a(x) + f'(u) \geq c_1 + c_2|u|^{2r}.$$

(A_3) $\partial\Omega$ is of class $C^{2+\beta}$.

Note that (A_2) implies

$$F(u) = \int_0^u f(s)ds \geq c_3|u|^{2r+2} - c_4 \tag{2.4}$$

for some positive constants c_3 and c_4.

For any $T > 0$, denote $Q_T = \Omega \times (0,T)$. We first establish an a priori bound for solutions of (2.1).

Theorem 2.1.1 *If $u(x,t) \in C^{2,1}(\bar{Q}_T)$ is a solution of equation (2.1), then*

$$\max_{\bar{Q}_T} |u(x,t)| \leq \bar{C}(u_0) \tag{2.5}$$

for some constant $\bar{C}(u_0)$.

In order to prove the theorem, we need the following lemma.

Lemma 2.1.2 *If $u(x,t) \in C^{2,1}(\bar{Q}_T)$ is a solution of equation (2.1), then there is a constant $C(u_0)$ such that*

$$\sup_{0\le t\le T} ||u(\cdot,t)||_q \le C(u_0) \tag{2.6}$$

for any $q \le 2r+2$.

PROOF. Let

$$E(u) = \frac{1}{4}\int\int J(x-y)(u(x)-u(y))^2 dxdy + \int F(u(x))dx. \tag{2.7}$$

It follows from (1.9) that

$$\frac{dE(u)}{dt} \le 0.$$

Therefore $E(u) \le E(u_0)$, i.e.,

$$\begin{aligned}&\frac{1}{4}\int\int J(x-y)(u(x)-u(y))^2 dxdy + \int F(u(x))dx\\ &\quad \le \frac{1}{4}\int\int J(x-y)(u_0(x)-u_0(y))^2 dxdy + \int F(u_0(x))dx.\end{aligned}$$

From condition (A_1), (2.4), and Young's inequality, we obtain

$$\int_\Omega |u|^{2r+2}dx \le C(u_0).$$

Since this is true for any $t > 0$, we have

$$\sup_{0\le t\le T}\int_\Omega |u|^{2r+2}dx \le C(u_0),$$

where $C(u_0)$ does not depend on T.

Since Ω is bounded, it follows that

$$\sup_{0\le t\le T} ||u||_q \le C(u_0)$$

for any $q \le 2r+2$.

We will prove the theorem with an iteration argument.

PROOF. For $p > 1$, multiply equation (2.1) by $u|u|^{p-1}$ and integrate over Ω, to obtain

$$\begin{aligned} \int u|u|^{p-1}u_t dx = &-\int a(x,u)\nabla u \cdot \nabla(u|u|^{p-1}(x))dx \\ &-\int\int \nabla J(x-y)u(x)\nabla(u|u|^{p-1}(x))dydx \\ &+\int\int \nabla J(x-y)u(y)\nabla(u|u|^{p-1}(x))dydx. \end{aligned} \tag{2.8}$$

Since

$$\int_\Omega a(x,u)\nabla u \cdot \nabla(u|u|^{p-1})dx = p\int_\Omega a(x,u)|u|^{p-1}|\nabla u|^2 dx \tag{2.9}$$

and

$$|\nabla|u|^{\frac{p+1}{2}}|^2 = \frac{(p+1)^2}{4}|u|^{p-1}|\nabla u|^2, \tag{2.10}$$

with condition (A_2), we have

$$\begin{aligned} \int_\Omega a(x,u)\nabla u \cdot \nabla(u|u|^{p-1})dx \geq &\frac{4pc_1}{(p+1)^2}\int_\Omega |\nabla|u|^{\frac{p+1}{2}}|^2 dx \\ &+\frac{4pc_2}{(p+2r+1)^2}\int_\Omega |\nabla|u|^{\frac{p+2r+1}{2}}|^2 dx. \end{aligned} \tag{2.11}$$

This yields

$$\begin{aligned} \frac{1}{p+1}\frac{d}{dt}\int_\Omega |u|^{p+1}dx + \frac{4pc_1}{(p+1)^2}\int_\Omega |\nabla|u|^{\frac{p+1}{2}}|^2 dx \\ \leq -\int\int \nabla J(x-y)u(x)\nabla(u|u|^{p-1}(x))dydx \\ +\int\int \nabla J(x-y)u(y)\nabla(u|u|^{p-1}(x))dydx. \end{aligned} \tag{2.12}$$

From Cauchy-Schwartz and Young's inequalities, together with

$$\nabla(u|u|^{p-1}) = p|u|^{p-1}\nabla u, \tag{2.13}$$

we have

$$
\begin{aligned}
&-\int\int \nabla J(x-y)u(x)\nabla(u|u|^{p-1}(x))dydx \\
&\le M_1 p\int_\Omega ||u|^p\,\nabla u(x)| \\
&\le M_1 p\int_\Omega |u|^{\frac{p-1}{2}}|\nabla u(x)|\,|u|^{\frac{p+1}{2}}dx \\
&\le \frac{c_1 p}{(p+1)^2}\int_\Omega |\nabla|u|^{\frac{p+1}{2}}|^2 dx + M_2 p\int_\Omega |u|^{p+1}dx,
\end{aligned}
\tag{2.14}
$$

for some positive constant M_2 which does not depend on p, and $M_1 = \sup\int|\nabla J(x-y)|dy$. Also we have

$$
\begin{aligned}
&\int\int|\nabla J(x-y)||u(y)|\nabla(u|u|^{p-1}(x))dydx \\
&\quad = p\int\int|\nabla J(x-y)||u(y)||u(x)|^{p-1}|\nabla u(x)|dxdy \\
&\quad \le p\int|u(x)|^{\frac{p-1}{2}}|\nabla u(x)||u(x)|^{\frac{p-1}{2}}\int|\nabla J(x-y)||u(y)|dydx \\
&\quad \le \epsilon p\int|u(x)|^{p-1}|\nabla u(x)|^2dx + c(\epsilon)p\int|u(x)|^{p-1}[\int|\nabla J(x-y)||u(y)|dy]^2 \\
&\quad \le \epsilon p\int|u(x)|^{p-1}|\nabla u(x)|^2dx \\
&\quad + c(\epsilon)p[\int|u(x)|^{p+1}dx]^{\frac{p-1}{p+1}}(\int[\int|\nabla J(x-y)||u(y)|dy]^{2\cdot\frac{p+1}{2}}dx)^{\frac{2}{p+1}} \\
&\quad \le \epsilon p\int|u(x)|^{p-1}|\nabla u(x)|^2dx \\
&\quad + c(\epsilon)p[\int|u(x)|^{p+1}dx]^{\frac{p-1}{p+1}}(\int|u(y)|^{p+1}dy)^{\frac{2}{p+1}}M_1^2 \\
&\quad \le \epsilon p\int|u(x)|^{p-1}|\nabla u(x)|^2dx + c(\epsilon)pM_1^2\int|u(x)|^{p+1}dx \\
&\quad \le \frac{c_1 p}{(p+1)^2}\int_\Omega|\nabla|u|^{\frac{p+1}{2}}|^2dx + M_3 p\int_\Omega|u|^{p+1}dx
\end{aligned}
\tag{2.15}
$$

for some constant M_3 which does not depend on p. Inequalities (2.12)-(2.15) imply

$$
\frac{d}{dt}\int_\Omega|u|^{p+1}dx + \frac{2pc_1}{(p+1)}\int_\Omega|\nabla|u|^{\frac{p+1}{2}}|^2dx \le C\cdot(p+1)^2\int_\Omega|u|^{p+1}dx. \tag{2.16}
$$

Now we need the following Gagliardo-Nirenberg inequality,

$$||D^j v||_{L^s} \le C_1 ||D^m v||^a_{L^r} ||v||^{1-a}_{L^q} + C_2 ||v||_{L^q}, \tag{2.17}$$

where

$$\frac{j}{m} \le a \le 1, \ \frac{1}{s} = \frac{j}{n} + a(\frac{1}{r} - \frac{m}{n}) + (1-a)\frac{1}{q}. \tag{2.18}$$

In (2.17), set $s = 2$, $j = 0$, $r = 2$, $m = 1$, to get

$$||v||_2^2 \le C_1 ||Dv||_2^{2a} ||v||_q^{2(1-a)} + C_2 ||v||_q^2. \tag{2.19}$$

Let $v = |u|^{\frac{\mu_k + 1}{2}}$, $\mu_k = 2^k$, $q = \dfrac{2(\mu_{k-1}+1)}{\mu_k + 1}$, and

$$a = \frac{n(2-q)}{n(2-q)+2q} = \frac{n}{n+2+2^{2-k}}. \tag{2.20}$$

Using Young's inequality this yields

$$\int_\Omega |u|^{\mu_k + 1} dx \le \epsilon \int_\Omega |\nabla |u|^{\frac{\mu_k+1}{2}}|^2 dx + c\epsilon^{-\frac{a}{1-a}} (\int_\Omega |u|^{\mu_{k-1}+1} dx)^{\frac{\mu_k+1}{\mu_{k-1}+1}}. \tag{2.21}$$

If we set $p = \mu_k$ in (2.16) and plug (2.21) into (2.16), we obtain

$$\begin{aligned} &\frac{d}{dt}\int_\Omega |u|^{\mu_k + 1} dx + \frac{2c_1\mu_k}{\mu_k+1}\int_\Omega |\nabla |u|^{\frac{\mu_k+1}{2}}|^2 dx \\ &\le C(\mu_k+1)^2(\epsilon \int_\Omega |\nabla |u|^{\frac{\mu_k+1}{2}}|^2 dx + c\epsilon^{-\frac{a}{1-a}} (\int_\Omega |u|^{\mu_{k-1}+1} dx)^{\frac{\mu_k+1}{\mu_{k-1}+1}}). \end{aligned} \tag{2.22}$$

Choosing $\epsilon = \dfrac{1}{C(\mu_k+1)^2} \cdot \dfrac{c_1\mu_k}{\mu_k+1}$, we have

$$\frac{d}{dt}\int_\Omega |u|^{\mu_k + 1} dx + C_1(k)\int_\Omega |\nabla |u|^{\frac{\mu_k+1}{2}}|^2 dx \le C_2(k)(\int_\Omega |u|^{\mu_{k-1}+1} dx)^{\frac{\mu_k+1}{\mu_{k-1}+1}}, \tag{2.23}$$

where $C_1(k) = \frac{c_1\mu_k}{\mu_k+1}$, $C_2(k) = C^{\frac{1}{1-a}} \cdot c \cdot (\frac{c_1\mu_k}{\mu_k+1})^{-\frac{a}{1-a}} \cdot (\mu_k+1)^{\frac{2}{1-a}}$.

Choosing $\epsilon = 1$ in (2.21), this and (2.23) also imply

$$\frac{d}{dt}\int_\Omega |u|^{\mu_k+1}dx + C_1(k)\int_\Omega |u|^{\mu_k+1}dx \le C_4(k)(\int_\Omega |u|^{\mu_{k-1}+1}dx)^{\frac{\mu_k+1}{\mu_{k-1}+1}}$$

where $C_4(k) = C_2(k) + c$.

By Gronwall's inequality, we have

$$\begin{aligned}\int_\Omega |u|^{\mu_k+1}dx &\le \int_\Omega |u_0|^{\mu_k+1}dx + \frac{C_4(k)}{C_1(k)}(\sup_{t\ge0}\int_\Omega |u|^{\mu_{k-1}+1}dx)^{\frac{\mu_k+1}{\mu_{k-1}+1}} \\ &\le \delta(k)\max\{M_0^{\mu_k+1}|\Omega|, (\sup_{t\ge0}\int_\Omega |u|^{\mu_{k-1}+1}dx)^{\frac{\mu_k+1}{\mu_{k-1}+1}}\},\end{aligned} \tag{2.24}$$

where $\delta(k) = c(1+\mu_k)^\alpha$, $\alpha = \frac{2}{1-a}$, and $M_0 = \sup_{x\in\Omega}|u_0|$. This implies

$$\begin{aligned}\int_\Omega |u|^{\mu_k+1}dx &\le \delta(k)\max\{M_0^{\mu_k+1}|\Omega|, (\sup_{t\ge0}\int_\Omega |u|^{\mu_{k-1}+1}dx)^{\frac{\mu_k+1}{\mu_{k-1}+1}}\} \\ &\le \prod_{i=0}^{k}(|\Omega|\delta(k-i))^{\frac{\mu_k+1}{\mu_{k-i}+1}}\max\{M_0^{\mu_k+1}, (\sup_{t\ge0}\int_\Omega |u|^2dx)^{\frac{\mu_k+1}{2}}\}.\end{aligned} \tag{2.25}$$

Since $\frac{\mu_k+1}{\mu_{k-i}+1} < 2^i$, we have

$$\begin{aligned}&\delta(k)\delta(k-1)^{\frac{\mu_k+1}{\mu_{k-1}+1}}\delta(k-2)^{\frac{\mu_k+1}{\mu_{k-2}+1}}\cdots\delta(1)^{\frac{\mu_k+1}{2}} \\ &\le c^{1+2+\cdots 2^{k-1}}\cdot(2^\alpha)^{k+(k-1)2+\cdots+(k-i)2^i+\cdots+2^{k-1}} \\ &\le c^{2^k-1}(2^\alpha)^{-k+2^{k+1}-2}\end{aligned} \tag{2.26}$$

and

$$|\Omega|\cdot|\Omega|^{\frac{\mu_k+1}{\mu_{k-1}+1}}\cdots|\Omega|^{\frac{\mu_k+1}{2}} \le |\Omega|^{2^k+1}. \tag{2.27}$$

Estimates (2.25)-(2.27) and Lemma 2.1.2 imply

$$\left(\int_\Omega |u|^{\mu_k+1}dx\right)^{\frac{1}{\mu_k+1}} \le C|\Omega|2^{2\alpha}\max\{M_0, \sup_{t\ge 0}\left(\int_\Omega |u|^2 dx\right)^{\frac{1}{2}}\} \le \bar{C}(u_0) \tag{2.28}$$

where $\bar{C}(u_0)$ does not depend on k. Since this is true for any k, letting $k \to \infty$ in (2.28), we have

$$||u||_\infty \le \bar{C}(u_0),$$

and therefore,

$$\sup_{0\le t\le T} ||u||_\infty \le \bar{C}(u_0). \tag{2.29}$$

Since $u \in C(\bar{Q}_T)$, it follows that

$$\max_{\bar{Q}_T} |u(x,t)| \le \bar{C}(u_0)$$

Remark 2.1.3 In (2.29), since $\bar{C}(u_0)$ does not depend on T, we also obtain a global bound for u whenever there is global existence of a classical solution.

Since $\max_{\bar{Q}_T}|u| \le M$, after a slight modification of the proof of Theorem 7.2 in Chapter V in [30], using the equivalent form (2.3) we have

Theorem 2.1.4 *For any solution $u \in C^{2,1}(\bar{Q}_T)$ of equation (2.1) having $\max_{Q_T}|u| \le C$, one has the estimates*

$$\max_{\bar{Q}_T} |\nabla u| \le K_1, \ |u|_{Q_T}^{(1+\delta)} \le K_2, \tag{2.30}$$

where constants K_1, K_2, and δ depend only on C, $||u_0||_{C^2(\bar{\Omega})}$ and Ω, $|\cdot|_{Q_T}^{(1+\delta)}$ is the Hölder norm given in [30].

In (2.3), setting $v(x,t) = u(x,t) - u_0(x)$, we obtain the equivalent form

$$\begin{cases} \dfrac{\partial v}{\partial t} = \tilde{a}(x,v,u_0)\Delta v + \tilde{b}(x,v,\nabla v,u_0) & \text{in} \quad \Omega,\ t>0, \\ \tilde{a}(x,v,u_0)\dfrac{\partial v}{\partial n} + \tilde{\psi}(x,v,u_0) = 0, & \text{on} \quad \partial\Omega,\ t>0, \\ v(x,0) = 0, & \end{cases} \tag{2.31}$$

where

$$\tilde{a}(x, v, u_0) = a(x, v + u_0),$$

$$\tilde{b}(x, v, \nabla v, u_0) = a(x, v + u_0)\Delta u_0 + b(x, v + u_0, \nabla(v + u_0)),$$

and

$$\tilde{\psi}(x, v, u_0) = \frac{\partial a(x)}{\partial n}(v(x,t) + u_0(x)) + \tilde{a}(x, v, u_0)\frac{\partial u_0}{\partial n} - \int_\Omega \frac{\partial J(x-y)}{\partial n}(v(y,t) + u_0(y))dy.$$

Since (2.2) implies $\tilde{\psi}(x, 0, u_0) = 0$, the compatibility condition for (2.31) is also satisfied.

Denote

$$Lv = \frac{\partial v}{\partial t} - \tilde{a}(x, v, u_0)\Delta v - \tilde{b}(x, v, \nabla v, u_0),$$

and

$$L_0 v = \frac{\partial v}{\partial t} - c_1 \Delta v,$$

where c_1 is the constant in condition (A_2).

Consider the following family of problems:

$$\begin{cases} \lambda Lv + (1-\lambda)L_0 v = 0 & \text{in } Q_T, \\ \lambda(\tilde{a}(x, v, u_0)\frac{\partial v}{\partial n} + \tilde{\psi}(x, v, u_0)) + (1-\lambda)(c_1(\frac{\partial v}{\partial n})) = 0 & \text{on } \partial\Omega \times [0, T], \\ v(x, 0) = 0. \end{cases} \tag{2.32}$$

Lemma 2.1.5 *If $v(x, t, \lambda) \in C^{2,1}(\bar{Q}_T)$ is a solution of (2.32), then*

$$\max_{\bar{Q}_T} |v(x, t, \lambda)| \leq K, \tag{2.33}$$

where K does not depend on λ.

PROOF. Since $\lambda\tilde{a}(x,v,u_0) + (1-\lambda)c_1 \geq \lambda c_1 + (1-\lambda)c_1 = c_1 > 0$, the terms in (2.32) also satisfy $(A_1) - (A_2)$ and so (2.33) follows from Theorem 2.1.1.

Consequently one may also conclude from Lemma 2.1.5 and Theorem 2.1.4 that:

Lemma 2.1.6 *If $v(x,t,\lambda) \in C^{2,1}(\bar{Q}_T)$ is a solution of equation (2.32), then*

$$\max_{\bar{Q}_T} |\nabla v(x,t,\lambda)| \leq K_1, \quad |v(x,t,\lambda)|_{Q_T}^{(1+\delta)} \leq K_2, \tag{2.34}$$

where constants K_1, K_2, and δ do not depend on λ.

We will use the following abstract result(see [30]):

Theorem 2.1.7 *(Leray-Schauder Fixed Point Theorem) Consider a transformation*

$$y = T(x,\lambda)$$

where x, y belong to a Banach space X and $0 \leq \lambda \leq 1$.

Assume:

(a) For any fixed λ, $T(\cdot,\lambda)$ is continuous on X.

(b) For x in bounded sets of X, $T(x,\lambda)$ is uniformly continuous in λ on [0,1].

(c) For any fixed λ, $T(\cdot,\lambda)$ is a compact transformation, i.e., it maps bounded subsets of X into precompact subsets of X.

(d) There exists a constant K such that every possible solution x of $x - T(x,\lambda) = 0$ with $\lambda \in [0,1]$ satisfies : $||x|| \leq K$

(e) The equation $x - T(x,0) = 0$ has a unique solution in X.

Then there exists a solution of the equation $x - T(x,1) = 0$.

Define a Banach space

$$X = \{v(x,t) \in C^{1+\beta, \frac{1+\beta}{2}}(\bar{Q}_T) : v(x,0) = 0\}$$

with the usual Hölder norm.

For any function $w \in X$ satisfying conditions $\max_{\bar{Q}_T}|w| \leq M$ and $\max_{\bar{Q}_T}|\nabla w| \leq M_1$, we consider the following linear problem

$$\begin{cases} v_t - (\lambda\tilde{a}(x,w,u_0) + (1-\lambda)c_1)\Delta v + \lambda\tilde{b}(x,w,\nabla w,u_0) = 0 & \text{in } Q_T, \\ \lambda(\tilde{a}(x,w,u_0)\dfrac{\partial v}{\partial n} + \tilde{\psi}(x,w,u_0)) + (1-\lambda)c_1\dfrac{\partial v}{\partial n} = 0 & \text{on } \quad \partial\Omega \times [0,T], \\ v(x,0) = 0. \end{cases} \tag{2.35}$$

It is clear that there exists a unique solution $v(x,t,\lambda) \in C^{2+\beta, \frac{2+\beta}{2}}(\bar{Q}_T)$ of (2.35).

Define $T(w,\lambda)$ by

$$T(w,\lambda) = v(x,t,\lambda).$$

Lemma 2.1.8 *For w being in a bounded set of X, $T(w,\lambda)$ is uniformly continuous in λ.*

PROOF. Let $w \in X$ with $||w||_X \leq M$ and let $v_1 = T(w,\lambda_1)$, $v_2 = T(w,\lambda_2)$, and $v = v_1 - v_2$. We have:

$$\begin{cases} v_t - (\lambda_1\tilde{a}(x,w,u_0) + (1-\lambda_1)c_1)\Delta v = (\lambda_1 - \lambda_2)h(x,w,v_2), \\ (\lambda_1\tilde{a}(x,w,u_0) + (1-\lambda_1)c_1)\dfrac{\partial v}{\partial n} = (\lambda_1 - \lambda_2)g(x,w,v_2), \\ v(x,0) = 0, \end{cases} \tag{2.36}$$

where

$$h(x,w,v_2) = (\tilde{a}(x,w,u_0) - c_1)\Delta v_2 - \tilde{b}(x,w,\nabla w,u_0),$$

and

$$g(x,w,v_2) = c_1\frac{\partial v_2}{\partial n} - \tilde{a}(x,w,u_0)\frac{\partial v_2}{\partial n}.$$

Since $|w|_X \leq M$ and $\lambda_2\tilde{a}(x,w,u_0) + (1-\lambda_2)c_1 \geq c_1 > 0$, from (2.35) we have $||v_2(x,t,\lambda_2)||_{C^{2,1}(\bar{Q}_T)} \leq N$ for some constant N independent of λ_2. Therefore

$$\max|h(x,w,v_2)| \leq N_1, \quad \max|g(x,w,v_2)| \leq N_2$$

for constants N_1 and N_2 that do not depend on λ_2. Note also that $\lambda\tilde{a}(x,w,u_0)+(1-\lambda)c_1 \geq c_1 > 0$ for all $\lambda \in [0,1]$. It then follows from linear parabolic theory that the solution of equation (2.36) will approach zero in X as $|\lambda_1-\lambda_2| \to 0$.

Similarly, one can see that for any fixed λ, $T(x,\lambda)$ is continuous in X. Furthermore, since $C^{2+\beta,\frac{2+\beta}{2}}(\bar{Q}_T) \hookrightarrow C^{1+\beta,\frac{1+\beta}{2}}(\bar{Q}_T)$ is compact, we see that $T(w,\lambda)$ is a compact transformation.

These observations, Lemma 2.1.5—Lemma 2.1.8 and the Leray-Schauder Theorem imply the existence of a solution $v(x,t)$ of (2.31), and therefore:

Theorem 2.1.9 *Let $\beta > 0$. For $u_0 \in C^{2+\beta}(\bar{\Omega})$ satisfying the boundary condition (2.2), there exists a solution u to (2.1) with $u \in C^{2+\beta,\frac{2+\beta}{2}}(\bar{Q}_T)$.*

We complete our goal of establishing well-posedness with the following:

Theorem 2.1.10 *(Uniqueness and continuous dependence on initial data)*

If $u_1(x,t)$ and $u_2(x,t)$ are two solutions corresponding initial data $u_{10}(x)$ and $u_{20}(x)$ of equation (2.1), then

$$\sup_{0\leq t\leq T}\int_\Omega |u_1-u_2|dx \leq C\int_\Omega |u_{10}-u_{20}|dx, \tag{2.37}$$

where C only depends on T.

PROOF. For any $\tau \in (0,T)$, $\theta \in C^{2,1}(\bar{Q}_\tau)$ with $\frac{\partial\theta}{\partial n}=0$ on $\partial\Omega\times(0,\tau)$, we have

$$\begin{aligned}\int_\Omega u_i(x,\tau)\theta(x,\tau)dx &= \int_\Omega u_i(x,0)\theta(x,0)dx + \int_0^\tau\int_\Omega (u_i\theta_t + B(x,u_i)\Delta\theta)dxdt \\ &\quad + \int_0^\tau\int_\Omega \theta\Delta J * u_i dxdt + \int_0^\tau\int_{\partial\Omega}\theta\frac{\partial J}{\partial n} * u_i dxdt,\end{aligned} \tag{2.38}$$

where $B(x,u)=a(x)u+f(u)$. Hence,

$$\begin{aligned}\int_\Omega (u_1-u_2)\theta(x,\tau)dx &= \int_\Omega (u_{10}-u_{20})\theta(x,0)dx \\ &\quad + \int_0^\tau\int_\Omega (u_1-u_2)(\theta_t+H\Delta\theta)dxdt + \int_0^\tau\int_\Omega \theta\Delta J*(u_1-u_2)dxdt \\ &\quad + \int_0^\tau\int_{\partial\Omega}\theta\frac{\partial J}{\partial n}*(u_1-u_2)dxdt,\end{aligned} \tag{2.39}$$

where

$$H(x,t)=\begin{cases}\dfrac{B(x,u_1)-B(x,u_2)}{u_1-u_2} & \text{for} \quad u_1\neq u_2\\ \dfrac{\partial B(x,u_1)}{\partial u} & \text{for} \quad u_1=u_2.\end{cases}$$

Let θ be the solution to the final value problem

$$\begin{cases}\dfrac{\partial\theta}{\partial t}=-H(x,t)\triangle\theta+\beta\theta & \text{in} \quad \Omega,\ 0\le t\le\tau,\\ \dfrac{\partial\theta}{\partial n}=0 & \text{on} \quad \partial\Omega,\ 0\le t\le\tau,\\ \theta(x,\tau)=h(x), & \end{cases} \tag{2.40}$$

where $h(x)\in C_0^\infty(\Omega)$, $0\le h\le 1$ and $\beta>0$ is a constant.

By the comparison theorem, we have

$$0\le\theta\le e^{\beta(t-\tau)}.$$

Therefore, from (2.39) we have

$$\begin{aligned}&\int_\Omega(u_1-u_2)hdx\\ &=\int_\Omega(u_{10}-u_{20})\theta(x,0)dx+\int_0^\tau\int_\Omega(u_1-u_2)\beta\theta dxdt\\ &+\int_0^\tau\int_\Omega\theta\triangle J*(u_1-u_2)dxdt+\int_0^\tau\int_{\partial\Omega}\theta\frac{\partial J}{\partial n}*(u_1-u_2)dxdt.\end{aligned} \tag{2.41}$$

Hence,

$$\begin{aligned}&\int_\Omega(u_1-u_2)hdx\\ &\le\int_\Omega|u_{10}-u_{20}|e^{-\beta\tau}dx+\int_0^\tau\int_\Omega|u_1-u_2|\beta e^{\beta(t-\tau)}dxdt\\ &+C_1\int_0^\tau\int_\Omega|u_1-u_2|e^{\beta(t-\tau)}dxdt+C_2\int_0^\tau\int_\Omega|u_1-u_2|e^{\beta(t-\tau)}dxdt.\end{aligned} \tag{2.42}$$

Letting $\beta\to 0$ and $h\to\operatorname{sign}(u_1-u_2)^+$ in (2.42), we have

$$\int_\Omega(u_1-u_2)^+dx\le\int_\Omega|u_{10}-u_{20}|dx+C_3\int_0^\tau\int_\Omega|u_1-u_2|dxdt. \tag{2.43}$$

Interchanging u_1 and u_2 gives

$$\int_\Omega|u_1-u_2|dx\le\int_\Omega|u_{10}-u_{20}|dx+C_3\int_0^\tau\int_\Omega|u_1-u_2|dxdt. \tag{2.44}$$

By Gronwall's inequality, (2.44) yields

$$\int_{\Omega} |u_1 - u_2| dx \leq C(T) \int_{\Omega} |u_{10} - u_{20}| dx. \tag{2.45}$$

Remark 2.1.11 If $u_0(x) \in L^{\infty}(\Omega)$, we can consider weak solutions as follows:

Define

$$X = \{f(x) \in C_0^{\infty}(\Omega) |\ g(x) = \int_{\Omega} J(x-y) f(y) dy,\ \frac{\partial g}{\partial n} | \partial\Omega = 0\}$$

and let

$$B = \text{Closure of X in the L}^2 \text{ norm.}$$

Definition 2.1.12 *A weak solution of (2.1) is a function* $u \in C([0,T], L^2(\Omega)) \cap L^{\infty}(Q_T) \cap L^2([0,T], H^1(\Omega))$, *with* $u_t \in L^2([0,T], H^{-1}(\Omega))$ *and* $\nabla h(x,u) \in L^2((0,T), L^2(\Omega))$ *such that*

$$\begin{aligned} < u_t(x,t), \psi(x) > &+ \int_{\Omega} \nabla h(x,u) \cdot \nabla \psi(x) dx \\ &- \int_{\Omega} (\nabla J * u(\cdot, s)) \cdot \nabla \psi(x) dx = 0 \end{aligned} \tag{2.46}$$

for all $\psi \in H^1(\Omega)$ *and a.e. time* $0 \leq t \leq T$, *where* $h(x,u) = a(x)u + f(u)$, $a(x) = \int_{\Omega} J(x-y) dy$, *and*

$$u(x,0) = u_0(x). \tag{2.47}$$

Theorem 2.1.13 *If* $(A_1) - (A_3)$ *are satisfied and* $u_0 \in L^{\infty}(\Omega) \cap B$, *then there exists a unique weak solution* u *of (2.1)*

Essentials of the proof: Since $u_0 \in L^{\infty}(\Omega) \cap B$, there exists a sequence $u_0^{(k)} \in X$ such that

$$\begin{aligned} &||u_0^{(k)} - u_0||_{L^2} \to 0, \\ &||u_0^{(k)}||_{\infty} < C, \end{aligned} \tag{2.48}$$

where C does not depend on k. Consider equation (2.1) with initial data $u_0^{(k)}$. There exists a unique classical solution $u^{(k)}$. By the energy estimate and other a priori bounds, one can find a subsequence and a weak limit u such that

$$\begin{aligned}
&u^{(k)} \rightharpoonup u \text{ in } \quad L^2((0,T), H^1(\Omega)), \\
&u^{(k)} \to u \text{ in } \quad L^2((0,T), L^2(\Omega)), \\
&||u||_\infty \leq C, \\
&h(x, u^{(k)}) \rightharpoonup h(x,u) \text{ in } \quad L^2((0,T), H^1(\Omega)), \\
&h(x, u^{(k)}) \to h(x,u) \text{ in } \quad L^2((0,T), L^2(\Omega)), \\
&u_t^{(k)} \rightharpoonup u_t \text{ in } \quad L^2((0,T), H^{-1}(\Omega)),
\end{aligned} \tag{2.49}$$

and u satisfies equation (2.46).

2.2 Long term behavior in the L^p norm

First, we establish a nonlinear version of the *Poincaré* inequality.

Proposition 2.2.1 *Let $\Omega \subset \mathbb{R}^n$ be smooth and bounded. For $p > 1$, there is a constant $C(\Omega, p)$ such that for all $u \in W^{1,2p}(\Omega)$ with $\int_\Omega u = 0$,*

$$\int_\Omega |u|^{2p} dx \leq C(\Omega, p) \int_\Omega |\nabla |u|^p|^2 dx. \tag{2.50}$$

PROOF. If (2.50) is not true, there exists a sequence $\{u_k\} \subset W^{1,2p}(\Omega)$ such that

$$\int_\Omega u_k = 0, \ \int_\Omega |u_k|^{2p} dx > k \int_\Omega |\nabla |u_k|^p|^2 dx. \tag{2.51}$$

If $w_k = \dfrac{u_k}{||u_k||_{2p}}$, then it follows that

$$\int_\Omega w_k = 0, \ \int_\Omega |w_k|^{2p} dx = 1, \ \int_\Omega |\nabla |w_k|^p|^2 dx < \frac{1}{k}. \tag{2.52}$$

Therefore, there exists a subsequence (still denoted by $\{|w_k|^p\}$) and $w \in H^1(\Omega)$ such that

$$|w_k|^p \rightharpoonup w \text{ in } H^1 \text{ and } |w_k|^p \to w \text{ in } L^2. \tag{2.53}$$

Since $\int_\Omega |\nabla |w_k|^p|^2 dx \le \frac{1}{k}$, for any $\varphi \in C_0^\infty(\Omega)$, we have

$$\int_\Omega \frac{\partial |w_k|^p}{\partial x_i} \varphi dx \to 0 \tag{2.54}$$

for i=1,$\cdots$,n. Therefore,

$$\int_\Omega \frac{\partial w}{\partial x_i} \varphi dx = 0 \tag{2.55}$$

for i=1,$\cdots$,n and $\varphi \in C_0^\infty(\Omega)$. So $\nabla w = 0$ a.e in Ω, and w is constant in Ω.

By taking a subsequence, (2.52) and (2.53) yield

$$w = (\frac{1}{|\Omega|})^{\frac{1}{2}}, \text{ and } |w_k|^p \to (\frac{1}{|\Omega|})^{\frac{1}{2}} \ a.e \text{ in } \Omega. \tag{2.56}$$

So, we have

$$|w_k| \to (\frac{1}{|\Omega|})^{\frac{1}{2p}} \ a.e \text{ in } \Omega. \tag{2.57}$$

Since $\int w_k = 0$, there exists a unique solution φ_k to

$$\begin{cases} -\Delta\varphi = w_k & \text{in } \Omega, \\ \frac{\partial \varphi}{\partial n} = 0 & \text{on } \partial\Omega, \\ \int_\Omega \varphi dx = 0. \end{cases} \tag{2.58}$$

From (2.58), we obtain

$$\int |\nabla \varphi_k|^2 = \int w_k \varphi_k \le ||w_k||_{L^2} ||\varphi_k||_{L^2}. \tag{2.59}$$

Since $\int_\Omega \varphi_k dx = 0$, by *Poincaré*'s inequality, $||\varphi_k||_{L^2} \le c||\nabla\varphi_k||_{L^2}$, therefore (2.58) and (2.59) imply

$$||\nabla \varphi_k||_{L^2} \le c||w_k||_{L^2} \tag{2.60}$$

and

$$\int_\Omega \nabla(|w_k|^{p-1} w_k) \nabla \varphi_k dx = \int_\Omega |w_k|^{p+1} dx. \tag{2.61}$$

Since

$$\nabla(|w_k|^{p-1}w_k) = p|w_k|^{p-1}\nabla w_k, \tag{2.62}$$

we have

$$|\nabla(|w_k|^{p-1}w_k)| = p|w_k|^{p-1}|\nabla w_k| = |\nabla|w_k|^p|. \tag{2.63}$$

Hence, from (2.61), we have

$$\begin{aligned}\int_\Omega |w_k|^{p+1}dx &= \int_\Omega \nabla(|w_k|^{p-1}w_k)\nabla\varphi_k dx \\ &\leq \int_\Omega |\nabla|w_k|^p||\nabla\varphi_k|dx \\ &\leq ||\nabla|w_k|^p||_{L^2}||\nabla\varphi_k||_{L^2} \\ &\to 0\end{aligned} \tag{2.64}$$

as $k \to \infty$, by (2.52) and (2.60). Hence, along a subsequence,

$$|w_k|^{p+1} \to 0 \ a.e \text{ in } \ \Omega,$$

i.e,

$$|w_k| \to 0 \ a.e \text{ in } \ \Omega. \tag{2.65}$$

This contradicts (2.57).

Remark 2.2.2 In [6], the same result was established independently by Alikakos and Rostamian, which was brought to my attention by Professor Alikakos.

The following lemma may be found in [38]

Lemma 2.2.3 *(Uniform Gronwall inequality) Let y be a positive absolutely continuous function on $(0,\infty)$ which satisfies*

$$y' + \nu y^p \leq \delta$$

with $p > 1, \nu > 0, \delta \geq 0$. Then, for $t \geq 0$, we have

$$y(t) \leq \left(\frac{\delta}{\nu}\right)^{\frac{1}{p}} + (\nu(p-1)t)^{\frac{-1}{p-1}}. \tag{2.66}$$

We use this to prove the following:

Proposition 2.2.4 *Let $\alpha_0 < (\frac{c_1}{c_2})^{\frac{1}{2r}}$, where c_1 and c_2 are the constants in assumption (A_2), and let $\bar{u}_0 = \frac{1}{|\Omega|}\int_\Omega u_0 dx$. If $u(x)$ is a solution of (2.1), and $|\bar{u}_0| \leq \alpha_0$, then for any $q > 1$, we have*

$$\int_\Omega |u - \bar{u}_0|^{q+1} dx < C_1 + (\frac{C_2 rt}{q+1})^{-\frac{q+1}{2r}} \tag{2.67}$$

where C_1 depends on α_0 and q, and C_2 depends on q.

PROOF. Let $u = v + \bar{u}_0$ in equation (2.1). Since

$$\frac{\partial v}{\partial t} = \frac{\partial u}{\partial t},$$
$$\nabla v = \nabla u,$$

and

$$\nabla a(x)v - (\nabla J) * v = \nabla a(x)u - (\nabla J) * u,$$

(2.1) becomes

$$\frac{\partial v}{\partial t} = \nabla \cdot (a(x,u)\nabla v + v\nabla a(x) - (\nabla J) * v), \tag{2.68}$$

with the boundary condition

$$(a(x,u)\nabla v + v\nabla a(x) - (\nabla J) * v) \cdot n = 0 \text{ on } \partial\Omega.$$

Multiplying equation (2.68) by $|v|^{q-1}v$, integrating by parts, and using Hölder's and Young's inequalities gives

$$\begin{aligned}\frac{1}{q+1}\frac{d}{dt}\int_\Omega |v|^{q+1} dx + q\int_\Omega a(x,u)|v|^{q-1}|\nabla v|^2 dx \\ \leq \epsilon(q)\int_\Omega |\nabla |v|^{\frac{q+1}{2}}|^2 dx + M(\epsilon(q))\int_\Omega |v|^{q+1} dx.\end{aligned} \tag{2.69}$$

Since $|v|^{2r} \le 2^{2r-1}(|u|^{2r} + |\bar{u}_0|^{2r})$, with condition (A_2) we have

$$a(x,u) > c_1 - c_2|\bar{u}_0|^{2r} + \frac{c_2}{2^{2r}}|v|^{2r}. \tag{2.70}$$

It follows from (2.69) and (2.70) that

$$\begin{aligned}
\frac{1}{q+1}\frac{d}{dt}\int_\Omega |v|^{q+1}dx &+ \frac{4q(c_1 - c_2|\bar{u}_0|^{2r})}{(q+1)^2}\int_\Omega |\nabla|v|^{\frac{q+1}{2}}|^2 dx \\
&+ \frac{4qc_2}{2^{2r}(q+2r+1)^2}\int_\Omega |\nabla|v|^{\frac{q+2r+1}{2}}|^2 dx \\
&\le \epsilon(q)\int_\Omega |\nabla|v|^{\frac{q+1}{2}}|^2 dx + M(\epsilon(q))\int_\Omega |v|^{q+1}dx.
\end{aligned} \tag{2.71}$$

Choosing $\alpha_0 < (\frac{c_1}{c_2})^{\frac{1}{2r}}$, $\epsilon(q) = \dfrac{4q(c_1 - c_2\alpha_0^{2r})}{(q+1)^2}$ in (2.71), for $|\bar{u}_0| \le \alpha_0$, we obtain

$$\begin{aligned}
\frac{1}{q+1}\frac{d}{dt}\int_\Omega |v|^{q+1}dx &+ \frac{4qc_2}{2^{2r}(q+2r+1)^2}\int_\Omega |\nabla|v|^{\frac{q+2r+1}{2}}|^2 dx \\
&\le M(\epsilon(q),\alpha_0)\int_\Omega |v|^{q+1}dx.
\end{aligned} \tag{2.72}$$

Since $\int v dx = 0$, Proposition 2.2.1 implies

$$\int_\Omega |v|^{q+2r+1}dx \le C\int_\Omega |\nabla|v|^{\frac{q+2r+1}{2}}|^2 dx. \tag{2.73}$$

It follows from (2.72)-(2.73) and *Hölder's* and Young's inequalities that

$$\frac{d}{dt}\int_\Omega |v|^{q+1}dx + C_3(q)\left(\int_\Omega |v|^{q+1}dx\right)^{\frac{q+2r+1}{q+1}} \le C_4(q,\alpha_0) \tag{2.74}$$

for constants $C_3(q)$ and $C_4(q,\alpha_0)$. By Lemma 2.2.3, we have

$$\int_\Omega |v|^{q+1}dx < \left(\frac{C_3(q)}{C_4(q,\alpha_0)}\right)^{\frac{q+1}{q+2r+1}} + \left(\frac{C_3(q)t2r}{q+1}\right)^{-\frac{q+1}{2r}}. \tag{2.75}$$

From Proposition 2.2.4, we have

Theorem 2.2.5 *Assume $\alpha_0 > 0$ is given in Proposition 2.2.4, $q > 1$, and*

$$\mu > \left(\frac{C_3(q)}{C_4(q,\alpha_0)}\right)^{\frac{1}{q+2r+1}} + \alpha_0|\Omega|^{\frac{1}{q+1}}.$$

Then for any solution of (2.1) with $\frac{1}{|\Omega|}|\int u_0 dx| = |\bar{u}_0| \le \alpha_0$, *there exists a time* $t_0(\alpha_0, q) \ge 0$ *such that*

$$||u||_{q+1} < \mu, \ for\ all\ t > t_0(\alpha_0, q). \tag{2.76}$$

Remark 2.2.6 Applying Proposition 2.2.1 to the standard Cahn-Hilliard equation, we can prove that there exists an absorbing set in each constant mass affine L^2 space directly as follows:

Let u satisfy

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta(-d\Delta u + f(u)) & \text{in} \quad \Omega,\ t > 0 \\ \dfrac{\partial u}{\partial n} = 0, \quad \dfrac{\partial \Delta u}{\partial n} = 0 & \text{on} \quad \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \end{cases} \tag{2.77}$$

where

$$f(u) = \sum_{j=1}^{2p-1} a_j u^j, \quad a_{2p-1} > 0, \quad p \in \mathbb{N}, \ p \ge 2.$$

For simplicity, assume $|\Omega| = 1$, let $\int u_0(x)dx = \bar{u}_0$, and $v = u - \bar{u}_0$.

Multiply equation (2.77) by v and integrate over Ω to get

$$\frac{d\int |v|^2 dx}{dt} + d\int |\Delta v|^2 + C\int |\nabla |v|^p|^2 \le C_1 \int |v|^2 + k(\bar{u}_0),$$

where $k(\bar{u}_0)$ depends only on $\bar{u}_0$.

Since $\int v = 0$, by Proposition 2.2.1, we have

$$\int |v|^{2p} \le C \int |\nabla |v|^p|^2.$$

By *Hölder's* and Young's inequalities we have

$$\frac{d\int |v|^2 dx}{dt} + C_2\Big(\int |v|^2\Big)^p \le k(\alpha).$$

So by Gronwall's inequality, there exists an absorbing set in the affine space $H_{\bar{u}_0} = \{u \in L^2, \frac{1}{|\Omega|}\int u = \bar{u}_0\}$.

2.3 Long term behavior in the H^1 norm

In Section 2.2, we considered the long term behavior of the solution in the L^p norm for any given $p \geq 1$. In particular, there exists a "local absorbing set" in the sense that if $|\int u_0|$ is not too large, the solution enters a fixed bounded set in the affine space $\bar{u}_0 + L^p$ in finite time (note that $\bar{u}_0 = \frac{1}{|\Omega|}\int_\Omega u_0$ is conserved by the evolution). In this section we consider the long term behavior of the solution in the H^1 norm. In this case, we do not need any restriction on $|\int u_0|$.

Note that (A_2) implies

$$f(u)u \geq c_5|u|^{2r+2} - c_6 \text{ for some constants } c_5 \text{ and } c_6.$$

We make additional assumptions on the nonlinearity,

(A_4) $|f(u)| \leq c_7|u|^{2r+1} + c_8$,

(A_5) $F(u) = \int_0^u f(s)ds \leq c_9|u|^{2r+2} + c_{10}$, and $c_5 > c_9$.

Remark 2.3.1 (A_2), (A_4), and (A_5) hold for $f(u) = c|u|^{2r}u+$ lower terms.

Denote $\bar{\psi} = \frac{1}{|\Omega|}\int_\Omega \psi dx$ and write $\varphi = \psi - \bar{\psi}$.

For $\varphi \in L^2(\Omega)$, satisfying $\bar{\varphi} = 0$, we consider the following equation:

$$\begin{cases} -\Delta\theta = \varphi \\ \frac{\partial\theta}{\partial n}|_{\partial\Omega} = 0 \\ \int_\Omega \theta = 0 \end{cases} \tag{2.78}$$

The equation (2.78) has a unique solution $\theta := (-\Delta_0)^{-1}(\varphi)$. Denote $||\varphi||_{-1} = (\int_\Omega \varphi(-\Delta_0)^{-1}(\varphi)dx)^{\frac{1}{2}}$. This is a continuous norm on $L^2(\Omega)$.

Since $\bar{u} = \bar{u}_0$ is constant, we may write the equation as

$$\frac{\partial(u - \bar{u})}{\partial t} = \Delta K(u), \tag{2.79}$$

where $K(u) = \int_\Omega J(x-y)dyu(x) - \int_\Omega J(x-y)u(y)dy + f(u)$. Applying the operator $(-\Delta_0)^{-1}$ to both sides of equation (2.79), we obtain

$$\frac{d(-\Delta_0)^{-1}(u-\bar{u})}{dt} + K(u) = 0. \tag{2.80}$$

Taking the scalar product with $u - \bar{u}$ in $L^2(\Omega)$, we have

$$\frac{1}{2}\frac{d}{dt}||u-\bar{u}||_{-1}^2 + (K(u), u-\bar{u}) = 0. \tag{2.81}$$

From condition (A_2)–(A_5), we have

$$\begin{aligned}
&(K(u), u-\bar{u})\\
&= \int (\int J(x-y)dyu(x) - \int J(x-y)u(y)dy + f(u))(u(x)-\bar{u})dx\\
&= \int\int J(x-y)u^2(x)dydx - \int\int J(x-y)u(y)u(x)dydx\\
&+ \int f(u)u(x)dx - \bar{u}\int f(u)dx\\
&= \frac{1}{2}\int\int J(x-y)(u(x)-u(y))^2dxdy + \int uf(u)dx - \bar{u}\int f(u)dx\\
&\geq \frac{1}{2}\int\int J(x-y)(u(x)-u(y))^2dxdy + \int (c_5|u|^{2r+2} - c_6)dx\\
&- |\bar{u}|\int (c_7|u|^{2r+1} + c_8)dx\\
&\geq \frac{1}{2}\int\int J(x-y)(u(x)-u(y))^2dxdy + c_5\int |u|^{2r+2}dx\\
&- \epsilon\int |u|^{2r+2}dx - c(\bar{u},\epsilon)
\end{aligned} \tag{2.82}$$

for any $\epsilon > 0$. Choosing $\epsilon = c_5 - c_9$, we have

$$\begin{aligned}
&(K(u), u-\bar{u})\\
&\geq \frac{1}{2}\int\int J(x-y)(u(x)-u(y))^2dxdy + c_9\int |u|^{2r+2}dx - c(\bar{u})\\
&\geq \frac{1}{4}\int\int J(x-y)(u(x)-u(y))^2dxdy + \int F(u)dx - c(\bar{u})\\
&\geq E(u) - c(\bar{u})\\
&= E(u) - c(\bar{u}_0).
\end{aligned} \tag{2.83}$$

Also from (2.83), we have

$$(K(u), u - \bar{u}) \geq c \int |u|^{2r+2} dx - c(\bar{u}_0) \tag{2.84}$$

for some positive constants c and $c(\bar{u}_0)$.

Since $||.||_{-1}$ is a continuous norm on $L^2(\Omega)$, we have

$$||u - \bar{u}||_{-1} \leq C||u - \bar{u}||_2. \tag{2.85}$$

Therefore,

$$\begin{aligned} ||u - \bar{u}_0||_{-1} &\leq C||u - \bar{u}_0||_2 \\ &\leq C||u - \bar{u}_0||_{2r+2} \\ &\leq C||u||_{2r+2} + C(\bar{u}_0) \end{aligned} \tag{2.86}$$

for some positive constants C and $C(\bar{u}_0)$. From (2.81), (2.84), and (2.86), it follows that

$$\frac{d}{dt}||u - \bar{u}_0||_{-1}^2 + C||u - \bar{u}_0||_{-1}^{2r+2} \leq C(\bar{u}_0). \tag{2.87}$$

By Lemma 2.2.3, we obtain

$$||u - \bar{u}_0||_{-1}^2 \leq \left(\frac{C(\bar{u}_0)}{C}\right)^{\frac{1}{r+1}} + (C(r)t)^{\frac{-1}{r}}. \tag{2.88}$$

Thus, we have proved:

Theorem 2.3.2 *There exists $M(\bar{u}_0)$ such that for any $\rho > M(\bar{u}_0)^{\frac{1}{2r+2}}$, there exists a time t_0 such that*

$$||u - \bar{u}_0||_{-1} \leq \rho, \quad \forall\, t \geq t_0. \tag{2.89}$$

From (2.81) and (2.82), we also obtain

$$\frac{1}{2}\frac{d}{dt}||u - \bar{u}_0||_{-1}^2 + E(u) \leq c(\bar{u}_0). \tag{2.90}$$

Integrating from t to $t+1$, then (2.89) implies

$$\int_t^{t+1} E(u(s))ds \le c^*(\bar{u}_0) \equiv c(\bar{u}_0) + \frac{\rho^2}{2} \tag{2.91}$$

for $t \ge t_0$. Since $E(u(t))$ is decreasing, (2.91) implies

$$E(u(t)) \le c^*(\bar{u}_0) \tag{2.92}$$

for $t \ge t_0 + 1$.

Since, from (2.4),

$$\begin{aligned} E(u(t)) &\ge \frac{1}{4}\int\int J(x-y)(u(x)-u(y))^2 dxdy + \int F(u)dx \\ &\ge c_3 \int |u|^{2r+2} - c_4, \end{aligned} \tag{2.93}$$

inequalities (2.92) and (2.93) yield

$$\int |u|^{2r+2} \le c_*(\bar{u}_0) \tag{2.94}$$

for $t \ge t_0$.

Corollary 2.3.3 *There exists $c_*(\bar{u}_0) > M(\bar{u}_0)^{\frac{1}{2r+2}}$ such that for any $\rho > c_*(\bar{u}_0)$, there exists a time t_0^* such that*

$$\int |u|^{r+1} \le c_*(\bar{u}_0) \ for\ t \ge t_0^*. \tag{2.95}$$

Next we estimate $||\nabla u||_2$.

Denote $h(x,u) = a(x)u + f(u)$. Multiplying (2.1) by $h(x,u)$ and integrating over Ω, we have

$$\int h(x,u)u_t + \int |\nabla h(x,u)|^2 = \int \nabla J * u \cdot \nabla h(x,u). \tag{2.96}$$

Since

$$h(x,u)u_t = (a(x)u + f(u))u_t = \frac{\partial}{\partial t}[\frac{1}{2}a(x)u^2 + F(u)], \tag{2.97}$$

and

$$\int \nabla J * u \cdot \nabla h(x,u) \leq c||u||_2^2 + \frac{1}{2}||\nabla h(x,u)||_2^2, \tag{2.98}$$

equation (2.96) yields

$$\frac{d}{dt}\int [\frac{1}{2}a(x)u^2 + F(u)] + \frac{1}{2}\int |\nabla h(x,u)|^2 \leq c||u||_2^2. \tag{2.99}$$

Integrate (2.99) from t to $t+1$, and use assumption (A_2) and Corollary 2.3.3, to obtain

$$\int_t^{t+1}\int |\nabla h(x,u)|^2 \leq c \tag{2.100}$$

for some constant c and all $t \geq t_0^*$.

Multiply (2.1) by $h(x,u)_t$ and integrate on Ω to obtain

$$\int h(x,u)_t u_t + \int \nabla h(x,u) \cdot \nabla h(x,u)_t = \int \nabla J * u \cdot \nabla h(x,u)_t. \tag{2.101}$$

Since

$$\begin{aligned} & h(x,u)_t u_t = a(x)u_t^2 + f'(u)u_t^2 \geq c_1 u_t^2, \\ & \int \nabla h(x,u) \cdot \nabla h(x,u)_t = \frac{1}{2}\frac{d}{dt}\int |\nabla h(x,u)|^2, \\ & \text{and} \\ & \int \nabla J * u \cdot \nabla h(x,u)_t = \frac{d}{dt}\int \nabla J * u \cdot \nabla h(x,u) - \int \nabla J * u_t \cdot \nabla h(x,u), \end{aligned} \tag{2.102}$$

we have

$$\begin{aligned} c_1 \int |u_t|^2 + \frac{1}{2}\frac{d}{dt}\int |\nabla h(x,u)|^2 \\ \leq \frac{d}{dt}\int \nabla J * u \cdot \nabla h(x,u) - \int \nabla J * u_t \cdot \nabla h(x,u). \end{aligned} \tag{2.103}$$

Estimate (2.103) with the Cauchy-Schwartz, and Young's inequalities imply

$$\frac{d}{dt}\int |\nabla h(x,u)|^2 \leq \frac{d}{dt}\int 2\nabla J * u \cdot \nabla h(x,u) + \gamma \int |\nabla h(x,u)|^2 \tag{2.104}$$

for some constant $\gamma > 0$.

For $t < s < t+1$, multiplying (2.104) by $e^{\gamma(t-s)}$, we have

$$\frac{d}{ds}[e^{\gamma(t-s)} \int |\nabla h(x,u)|^2] \le e^{\gamma(t-s)} \frac{d}{ds} \int 2\nabla J * u \cdot \nabla h(x,u). \tag{2.105}$$

Integrating (2.105) between s and $t+1$, we obtain

$$\begin{aligned} &e^{-\gamma} \int_\Omega |\nabla h(x,u(x,t+1))|^2 - e^{\gamma(t-s)} \int |\nabla h(x,u(x,s))|^2 \\ &\le \int_s^{t+1} e^{\gamma(t-\mu)} \frac{d}{d\mu} \int_\Omega 2\nabla J * u(\cdot,\mu) \cdot \nabla h(x,u(x,\mu)) dx d\mu. \end{aligned} \tag{2.106}$$

Write

$$\begin{aligned} &\int_s^{t+1} e^{\gamma(t-\mu)} \frac{d}{d\mu} \int_\Omega 2\nabla J * u(\cdot,\mu) \cdot \nabla h(x,u(x,\mu)) dx d\mu \\ &= e^{\gamma(t-\mu)} \int_\Omega 2\nabla J * u(\cdot,\mu) \cdot \nabla h(x,u(x,\mu)) dx |_s^{t+1} \\ &\quad - \int_s^{t+1} (-\gamma) e^{\gamma(t-\mu)} \int_\Omega 2\nabla J * u(\cdot,\mu) \cdot \nabla h(x,u(x,\mu)) dx d\mu \\ &= I_1 + I_2. \end{aligned} \tag{2.107}$$

Also,

$$\begin{aligned} I_1 &= e^{\gamma(t-\mu)} \int_\Omega 2\nabla J * u(\cdot,\mu) \cdot \nabla h(x,u(x,\mu)) dx |_s^{t+1} \\ &= e^{-\gamma} \int_\Omega 2\nabla J * u(\cdot,t+1) \cdot \nabla h(x,u(x,t+1)) \\ &\quad - e^{\gamma(t-s)} \int_\Omega 2\nabla J * u(\cdot,s) \cdot \nabla h(x,u(x,s)) dx. \end{aligned} \tag{2.108}$$

Using the Cauchy-Schwartz and Young's inequalities, this is bounded above by

$$\begin{aligned} &\frac{e^{-\gamma}}{2} \int_\Omega |\nabla h(x,u(x,t+1))|^2 + C \int_\Omega |u(x,t+1)|^2 \\ &\qquad + \int_\Omega |\nabla h(x,u(x,s))|^2 + C \int |u(x,s)|^2 \end{aligned}$$

for some constant C. Furthermore,

$$\begin{aligned} I_2 &= \int_s^{t+1} \gamma e^{\gamma(t-\mu)} \int_\Omega 2\nabla J * u(\cdot,\mu) \cdot \nabla h(x,u(x,\mu)) dx d\mu \\ &\le C \int_s^{t+1} [\int_\Omega |\nabla h(x,u(x,\mu))|^2 + \int_\Omega |u(x,\mu)|^2] d\mu. \end{aligned} \tag{2.109}$$

Estimate (2.106) becomes

$$\begin{aligned}
&e^{-\gamma}\int_\Omega |\nabla h(x,u(x,t+1))|^2 - e^{\gamma(t-s)}\int_\Omega |\nabla h(x,u(x,s))|^2 \leq \\
&\frac{e^{-\gamma}}{2}\int_\Omega |\nabla h(x,u(x,t+1))|^2 + C\int_\Omega |u(x,t+1)|^2 + \int |\nabla h(x,u(x,s))|^2 \qquad (2.110)\\
&+C\int_\Omega |u(x,s)|^2 + C\int_s^{t+1}[\int_\Omega |\nabla h(x,u(x,\mu))|^2 + \int_\Omega |u(x,\mu)|^2]d\mu.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{e^{-\gamma}}{2}\int_\Omega |\nabla h(x,u(x,t+1))|^2\\
&\leq e^{\gamma(t-s)}\int_\Omega |\nabla h(x,u(x,s))|^2 + C\int_\Omega |u(x,t+1)|^2 + \int_\Omega |\nabla h(x,u(x,s))|^2 \qquad (2.111)\\
&+C\int_\Omega |u(x,s)|^2 + C\int_s^{t+1}[\int_\Omega |\nabla h(x,u(x,\mu))|^2 + \int_\Omega |u(x,\mu)|^2]d\mu.
\end{aligned}$$

Integrating (2.111) from t to $t+1$ with respect to s, we have

$$\begin{aligned}
&\frac{e^{-\gamma}}{2}\int_\Omega |\nabla h(x,u(x,t+1))|^2 dx\\
&\leq \int_t^{t+1} e^{\gamma(t-s)}\int_\Omega |\nabla h(x,u(x,s))|^2 dxds + C\int_t^{t+1}\int_\Omega |u(x,t+1)|^2 dxds\\
&+\int_t^{t+1}\int_\Omega |\nabla h(x,u(x,s))|^2 dxds + C\int_t^{t+1}\int_\Omega |u(x,s)|^2 dxds\\
&+C\int_t^{t+1}\int_s^{t+1}[\int_\Omega |\nabla h(x,u(x,\mu))|^2 dx + \int_\Omega |u(x,\mu)|^2 dx]d\mu ds \qquad (2.112)\\
&\leq \int_t^{t+1}\int_\Omega |\nabla h(x,u(x,s))|^2 dxds + C\int_\Omega |u(x,t+1)|^2 dx\\
&+\int_t^{t+1}\int_\Omega |\nabla h(x,u(x,s))|^2 dxds + C\int_t^{t+1}\int_\Omega |u(x,s)|^2 dxds\\
&+C\int_t^{t+1}(\mu-t)[\int_\Omega |\nabla h(x,u(x,\mu))|^2 dx + \int_\Omega |u(x,\mu)|^2 dx]d\mu.
\end{aligned}$$

By (2.95) and (2.100), estimate (2.112) yields

$$\int_\Omega |\nabla h(x,u(x,t+1))|^2 dx \leq C(\bar{u}_0) \qquad (2.113)$$

for $t \geq t_0(\bar{u}_0)$ and some $C(\bar{u}_0) > 0$.

Since

$$\nabla h(x,u(x,t+1)) = (a(x) + f'(u(t+1)))\nabla u(x,t+1) - u(x,t+1)\nabla a(x), \qquad (2.114)$$

we have

$$\int_\Omega |\nabla h(x,u(x,t+1))|^2 \geq \frac{1}{2}\int_\Omega |(a(x)+f'(u(t+1)))|^2|\nabla u(x,t+1)|^2$$
$$-\int_\Omega |u(x,t+1)\nabla a(x)|^2 \tag{2.115}$$
$$\geq \int_\Omega \frac{1}{2}c_1^2|\nabla u(x,t+1)|^2 - D(\bar{u}_0)$$

for $t \geq t_0(\bar{u}_0)$ and some constant $D(\bar{u}_0)$.

Estimates (2.113) and (2.115) imply

$$\int_\Omega |\nabla u(x,t+1)|^2 \leq G(\bar{u}_0), \tag{2.116}$$

for $t \geq t_0^*(\bar{u}_0)$ and $G(\bar{u}_0) > 0$. Thus, we have

Theorem 2.3.4 *There exists a time $t_0^*(\bar{u}_0)$ such that*

$$||u||_{H^1} \leq c(\bar{u}_0) \ for\ t \geq t_0^*(\bar{u}_0). \tag{2.117}$$

Remark 2.3.5 [38] gives a similar result for the Cahn-Hilliard equation.

Also we have the following theorem

Theorem 2.3.6 *If u is a solution of (2.1), and $Q(u) = (\int_\Omega J(x-y)dy)u(x) - J * u(x) + f(u(x))$, then there exist a sequence $\{t_k\}$ and u^* such that*

$$u(t_k) \to u^* \text{ weakly in } H^1,$$
$$Q(u(t_k)) \to Q(u^*) \text{ weakly in } H^1, \tag{2.118}$$

and $Q(u^*)$ is a constant, i.e. u^* is a steady state solution of (2.1).

PROOF. If u is a solution of (2.1), from (1.9), we have

$$\frac{dE(u)}{dt} = -\int |\nabla Q(u)|^2 dx. \tag{2.119}$$

This implies

$$\int_0^T \int_\Omega |\nabla Q(u)|^2 dxdt = E(u(0)) - E(u(T)). \tag{2.120}$$

Recall that

$$E(u) = \frac{1}{4}\int\int J(x-y)(u(x)-u(y))^2 dxdy + \int F(u(x))dx.$$

Using (2.4), we have

$$-E(u(T)) \leq C, \tag{2.121}$$

where C does not depend on T.

(2.120)-(2.121) imply

$$\int_0^\infty \int_\Omega |\nabla Q(u)|^2 dxdt \leq \bar{C}, \tag{2.122}$$

for some positive number $\bar{C}$. So there exists a sequence $\{t_k\}$ with $t_k \in [k, k+1]$, such that

$$\int_\Omega |\nabla Q(u(t_k))|^2 dx \to 0. \tag{2.123}$$

From (2.29), Remark 2.1.3, and (2.117), we have

$$\begin{aligned} &||u(t_k)||_\infty \leq C_1, \\ &||u(t_k)||_{H^1} \leq C_2. \end{aligned} \tag{2.124}$$

Observations (2.123) and (2.124) imply that there exists a subsequence of $\{t_k\}$ (still denoted by $\{t_k\}$) such that

$$\begin{aligned} &u(t_k) \to u^* \text{ weakly in } H^1, \\ &u(t_k) \to u^* \text{ strongly in } L^2, \\ &Q(u(t_k)) \to v \text{ weakly in } H^1, \\ &\nabla v = 0 \; a.e \text{ in } \Omega. \end{aligned} \tag{2.125}$$

Since $||u(t_k)||_\infty \leq C_1$ and $||u^*||_\infty \leq C_1$, we have

$$||f(u(t_k)) - f(u^*)||_{L^2} \leq C||u(t_k) - u^*||_{L^2}. \tag{2.126}$$

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for some constant C.

(2.125) and (2.126) imply

$$\begin{aligned} &v = Q(u^*) \ a.e \text{ in } \Omega, \\ &v = \text{constant}, \\ &\int_\Omega u^* dx = \int_\Omega u_0 dx. \end{aligned} \tag{2.127}$$

So u^* is a steady state solution of (2.1).

2.4 Applications to other nonlocal problems

The method for the nonlocal Cahn-Hilliard equation can also be applied to other nonlocal problems. For example, we consider the following integrodifferential equation that may be related to interacting particle systems with Kawasaki dynamics (see [18], [31], [32], [33], [34]):

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta(u - tanh(\beta J * u)) \text{ in } \quad \Omega,\, t > 0, \\ \dfrac{\partial(u - tanh(\beta J * u))}{\partial n} = 0 \text{ on } \quad \partial\Omega,\, t > 0, \\ u(x, 0) = u_0(x), \end{cases} \tag{2.128}$$

where β is a constant and J is a smooth function.

Note that the average of u, $\bar{u}$ is constant in time.

If $u(x,t) \in C^{2+\alpha, \frac{2+\alpha}{2}}(\bar{Q}_T)$ is a solution of (2.128), multiplying equation (2.128) by u and integrating by parts, we have

$$\frac{1}{2}\frac{d\int |u|^2}{\partial t} + \int |\nabla u|^2 = \int \nabla(tanh(\beta J * u))\nabla u. \tag{2.129}$$

Since

$$\nabla(tanh(\beta J * u)) = \frac{4\beta \nabla J * u}{(e^{-\beta J * u} + e^{\beta J * u})^2}$$

and

$$(e^{-\beta J * u} + e^{\beta J * u})^2 \geq 4,$$

we have

$$\int |\nabla(tanh(\beta J * u))|^2 \leq \int |\beta|^2 |\nabla J * u|^2 \leq C(\Omega, J, \beta) \int |u|^2. \tag{2.130}$$

The Cauchy-Schwartz inequality and (2.129)-(2.130) imply

$$\frac{1}{2}\frac{d\int |u|^2}{\partial t} + \frac{1}{2}\int |\nabla u|^2 \leq C(\Omega, J, \beta) \int |u|^2. \tag{2.131}$$

By Gronwall's lemma, we obtain

$$\int |u|^2 \leq c(T, u_0). \tag{2.132}$$

A similar argument to that in the proof of Theorem 2.1.1 yields

$$\sup_{Q_T} |u| \leq C(u_0, T). \tag{2.133}$$

The analogues of Theorems 2.1.4-2.1.10 yield

Theorem 2.4.1 *If $a(x) = \int J(x-y)dy \in C^{2+\alpha}(\bar{\Omega})$, $\partial\Omega$ is of class $C^{2+\alpha}$ for some $\alpha > 0$, and $u_0(x) \in C^{2+\alpha}(\bar{\Omega})$ satisfies the compatibility condition, then there exists a unique solution $u(x,t) \in C^{2+\alpha, \frac{2+\alpha}{2}}(\bar{Q}_T)$ to (2.128).*

For the long term behavior of the solution, we consider

$$\frac{du}{dt} = \Delta K(u), \tag{2.134}$$

where $K(u) = u - tanh(\beta J * u)$.

Apply the operator $(-\Delta_0)^{-1}$ to both sides of (2.134), where $(-\Delta_0)^{-1}$ is defined in (2.78). We obtain

$$\frac{d(-\Delta_0)^{-1}(u - \bar{u}_0)}{dt} + K(u) = 0. \tag{2.135}$$

Taking the scalar product with $u - \bar{u}_0$ in $L^2(\Omega)$, we have

$$\frac{1}{2}\frac{d}{dt}||u - \bar{u}_0||_{-1}^2 + (K(u), u - \bar{u}_0) = 0. \tag{2.136}$$

Note that

$$\begin{aligned}(K(u), u - \bar{u}_0) &= (u - tanh(\beta J * u), u - \bar{u}_0)\\ &= (u - \bar{u}_0 + \bar{u}_0, u - \bar{u}_0) - (tanh(\beta J * u), u - \bar{u}_0)\\ &\geq \int |u - \bar{u}_0|^2 - |\bar{u}_0| \int |u - \bar{u}_0| - \int |u - \bar{u}_0|\\ &\geq \frac{1}{2}\int |u - \bar{u}_0|^2 - B(\bar{u}_0)\end{aligned} \tag{2.137}$$

for some constant $B(\bar{u}_0)$.

Continuity of the embedding gives

$$||u - \bar{u}_0||_{-1} \leq C||u - \bar{u}_0||_2. \tag{2.138}$$

Equation (2.136) yields

$$\frac{d}{dt}||u - \bar{u}_0||_{-1}^2 + C||u - \bar{u}_0||_{-1}^2 \leq 2B(\bar{u}_0). \tag{2.139}$$

Gronwall's inequality implies

$$||u - \bar{u}_0||_{-1}^2 \leq \frac{2B(\bar{u}_0)}{C} + Ke^{-Ct}. \tag{2.140}$$

So, there exists $C(\bar{u}_0) > \frac{2B(\bar{u}_0)}{C}$ and $t_0 := t_0(\bar{u}_0)$ such that for $t \geq t_0$

$$||u - \bar{u}_0||_{-1}^2 \leq C(\bar{u}_0), \tag{2.141}$$

and so there exists an absorbing set in the H_0^{-1} norm.

For $t > t_0$, integrating (2.136) from t to $t+1$ gives

$$\begin{aligned}&||u(\cdot, t+1) - \bar{u}_0||_{-1}^2 - ||u(\cdot, t) - \bar{u}_0||_{-1}^2\\ &\qquad + \int_t^{t+1} \int_\Omega |u(x,s) - \bar{u}_0|^2 dx ds \leq B(\bar{u}_0).\end{aligned} \tag{2.142}$$

Inequalities (2.141)-(2.142) imply

$$\int_t^{t+1} \int_\Omega |u(x,s) - \bar{u}_0|^2 dxds \le C(\bar{u}_0). \tag{2.143}$$

for some constant $C(\bar{u}_0)$. This yields

$$\int_t^{t+1} \int_\Omega |u(x,s)|^2 dxds \le \bar{C}(\bar{u}_0). \tag{2.144}$$

So there is a $t_1 \in [t, t+1]$, where $\int_\Omega |u(x,t_1)|^2 dx \le \bar{C}(\bar{u}_0)$. From (2.131) we have

$$\begin{aligned} \int_\Omega |u(x,t+1)|^2 dx &\le \int_\Omega |u(x,t_1)|^2 dx + C(\Omega, J, \beta) \int_t^{t+1} \int_\Omega |u(x,s)|^2 dxds \\ &\le C_1(\bar{u}_0) \end{aligned} \tag{2.145}$$

for $t > t_0$.

By (2.132) and (2.145), we have

$$\sup_{t \ge t_0} \int_\Omega |u(x,t)|^2 dx \le C(\bar{u}_0). \tag{2.146}$$

By (2.131) and (2.146), using a similar argument to that in the proof of Theorem 2.1.1, we have

$$\sup_{t \ge 0} ||u||_\infty \le C(u_0). \tag{2.147}$$

Next we estimate $||\nabla u||_2$

Integrating (2.131) from t to $t+1$ with $t \ge t_0$, we have

$$||u(\cdot, t+1)||_2^2 - ||u(\cdot, t)||_2^2 + \int_t^{t+1} \int_\Omega |\nabla u|^2 dx \le C \int_t^{t+1} ||u(\cdot, s)||_2^2. \tag{2.148}$$

Inequalities (2.145) and (2.148) yield

$$\int_t^{t+1} \int_\Omega |\nabla u|^2 dx \le C(\bar{u}_0). \tag{2.149}$$

Multiplying (2.128) by u_t and integrating over Ω, we obtain

$$\int (u_t)^2 + \int \nabla u \cdot \nabla u_t = \int \nabla(tanh(\beta J * u)) \cdot \nabla u_t. \tag{2.150}$$

Note that

$$\int \nabla(tanh(\beta J * u)) \cdot \nabla u_t = \frac{d}{dt}\int \nabla(tanh(\beta J * u)) \cdot \nabla u \\ - \int \nabla(tanh(\beta J * u))_t \cdot \nabla u, \tag{2.151}$$

$$\int \nabla u \cdot \nabla u_t = \frac{1}{2}\frac{d}{dt}\int |\nabla u|^2, \tag{2.152}$$

and

$$\begin{aligned} &|\nabla(tanh(\beta J * u))_t| = \\ &\left|\frac{(4\beta \nabla J * u_t)(e^{-\beta J * u} + e^{\beta J * u}) - 8\beta^2 J * u_t(e^{\beta J * u} - e^{-\beta J * u})(\nabla J) * u}{(e^{-\beta J * u} + e^{\beta J * u})^3}\right| \\ &\leq C||u_t||_2. \end{aligned} \tag{2.153}$$

It follows from (2.150)-(2.153) that

$$\begin{aligned} &\int (u_t)^2 + \frac{1}{2}\frac{d}{dt}\int |\nabla u|^2 \\ &= \frac{d}{dt}\int \nabla(tanh(\beta J * u)) \cdot \nabla u - \int \nabla(tanh(\beta J * u))_t \cdot \nabla u \\ &\leq \frac{d}{dt}\int \nabla(tanh(\beta J * u)) \cdot \nabla u + \int C||u_t||_2|\nabla u| \\ &\leq \frac{d}{dt}\int \nabla(tanh(\beta J * u)) \cdot \nabla u + \frac{1}{2}||u_t||_2^2 + C\int |\nabla u|^2. \end{aligned} \tag{2.154}$$

Therefore,

$$\frac{d}{dt}\int |\nabla u|^2 \leq \frac{d}{dt}\int 2\nabla(tanh(\beta J * u)) \cdot \nabla u + C\int |\nabla u|^2, \tag{2.155}$$

where C depends on $\bar{u}_0$, J and Ω.

For $t < s < t+1$, multiplying (2.155) by $e^{C(t-s)}$, we have

$$\begin{aligned} &\frac{d}{ds}[\int |\nabla u(x,s)|^2 e^{C(t-s)} dx] \\ &\qquad \leq e^{C(t-s)}\frac{d}{ds}[\int 2\nabla(tanh(\beta J * u(\cdot,s))) \cdot \nabla u(x,s) dx]. \end{aligned} \tag{2.156}$$

Integrating (2.156) between s and $t+1$, we obtain

$$\begin{aligned}
&\int |\nabla u(x,t+1)|^2 e^{-C} dx - \int |\nabla u(x,s)|^2 e^{C(t-s)} dx \\
&\leq \int_s^{t+1} e^{C(t-\mu)} \frac{d}{d\mu} [\int 2\nabla(tanh(\beta J * u(\cdot,\mu))) \cdot \nabla u(x,\mu) dx] d\mu \qquad (2.157)\\
&= I_1.
\end{aligned}$$

We compute

$$\begin{aligned}
I_1 &= [e^{C(t-\mu)} \int 2\nabla(tanh(\beta J * u(\cdot,\mu))) \cdot \nabla u(x,\mu) dx]|_s^{t+1} \\
&+ \int_s^{t+1} Ce^{C(t-\mu)} \int 2\nabla(tanh(\beta J * u(\cdot,\mu))) \cdot \nabla u(x,\mu) dx d\mu \\
&= e^{-C} \int 2\nabla(tanh(\beta J * u(\cdot,t+1))) \cdot \nabla u(x,t+1) dx \\
&- e^{C(t-s)} \int 2\nabla tanh \beta J * u(\cdot,s) \cdot \nabla u(x,s) dx \qquad (2.158)\\
&+ \int_s^{t+1} Ce^{C(t-\mu)} \int 2\nabla(tanh(\beta J * u(\cdot,\mu))) \cdot \nabla u(x,\mu) dx d\mu \\
&\equiv P_1 + P_2 + P_3.
\end{aligned}$$

First,

$$\begin{aligned}
P_1 &= e^{-C} \int 2\nabla(tanh(\beta J * u(\cdot,t+1))) \cdot \nabla u(x,t+1) dx \\
&\leq \frac{e^{-C}}{2} \int |\nabla u(x,t+1)|^2 dx + C \int |u(x,t+1)|^2 dx
\end{aligned} \qquad (2.159)$$

for some constant C. Also since $t < s < t+1$, $e^{C(t-s)} \leq 1$, and we have

$$P_2 \leq \int |\nabla u(x,s)|^2 dx + C \int |u(x,s)|^2 dx, \qquad (2.160)$$

and

$$P_3 \leq \int_s^{t+1} [\int |\nabla u(x,\mu)|^2 dx + C \int |u(x,\mu)|^2 dx] d\mu. \qquad (2.161)$$

Estimate (2.157) becomes

$$\begin{aligned}
&\int |\nabla u(x,t+1)|^2 e^{-C} dx - \int |\nabla u(x,s)|^2 e^{C(t-s)} dx \\
&\leq \frac{e^{-C}}{2} \int |\nabla u(x,t+1)|^2 dx + C \int |u(x,t+1)|^2 dx + \int |\nabla u(x,s)|^2 dx \qquad (2.162)\\
&+ C \int |u(x,s)|^2 dx + \int_s^{t+1} [\int |\nabla u(x,\mu)|^2 dx + C \int |u(x,\mu)|^2 dx] d\mu.
\end{aligned}$$

This yields

$$
\begin{aligned}
&\int |\nabla u(x,t+1)|^2 dx \\
&\leq C[\int |\nabla u(x,s)|^2 dx + \int |u(x,t+1)|^2 dx + \int |u(x,s)|^2 dx \\
&+ \int_t^{t+1} \int |\nabla u(x,\mu)|^2 dx d\mu + \int_t^{t+1} \int |u(x,\mu)|^2 dx d\mu].
\end{aligned}
\tag{2.163}
$$

Integrating (2.163) from t to $t+1$ with respect to s, we obtain

$$
\begin{aligned}
\int |\nabla u(x,t+1)|^2 dx &\leq C[\int_t^{t+1} \int |\nabla u(x,s)|^2 dx ds \\
&+ \int_t^{t+1} \int |u(x,t+1)|^2 dx ds + \int_t^{t+1} \int |u(x,s)|^2 dx ds \\
&+ \int_t^{t+1} \int |\nabla u(x,\mu)|^2 dx d\mu + \int_t^{t+1} \int |u(x,\mu)|^2 dx d\mu] \\
&= C[\int_t^{t+1} \int |\nabla u(x,s)|^2 dx ds + \int |u(x,t+1)|^2 dx ds \\
&+ \int_t^{t+1} \int |u(x,s)|^2 dx ds].
\end{aligned}
\tag{2.164}
$$

Therefore, estimates (2.144), (2.145), (2.149), and (2.164) imply

$$
\int |\nabla u(x,t+1)|^2 dx \leq C(\bar{u}_0) \tag{2.165}
$$

for $t \geq t_0(\bar{u}_0)$ and for some constant $C(\bar{u}_0)$.

This means that there exists an "absorbing set" in the affine space $H_{\bar{u}_0}$ relative to the H^1 norm.

CHAPTER 3

The Dirichlet boundary problem for a nonlocal Cahn-Hilliard equation

3.1 Existence, uniqueness and continuous dependence on initial data for classical solutions

In this part, we study the following integrodifferential equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta\left(\int_\Omega J(x-y)dyu(x) - \int_\Omega J(x-y)u(y)dy + f(u)\right) & \text{in} \quad Q_T, \\ u = 0 & \text{on} \quad S_T, \\ u(x,0) = u_0(x), \end{cases} \tag{3.1}$$

where as before $Q_T = \Omega \times (0,T)$, $S_T = \partial\Omega \times (0,T)$, $\Omega \subset \mathbb{R}^n$ is a bounded domain, $J(-x) = J(x)$ for $x \in \mathbb{R}^n$, and f is bistable. We do not assume that J is nonnegative but its integral is assumed to be positive.

Rewrite (3.1) as

$$u_t = (p(x) + f'(u))\Delta u + 2\nabla p(x) \cdot \nabla u + f''(u)\nabla u \cdot \nabla u + u\Delta p - (\Delta J) * u, \tag{3.2}$$

where

$$p(x) \equiv \int_\Omega J(x-y)dy, \text{ and } (\triangle J) * u \equiv \int_\Omega \triangle J(x-y)u(y)dy.$$

We make the following assumptions

(B_1) $J \in C^{2+\gamma}(\mathbb{R}^n)$, $f \in C^{2+\gamma}(\mathbb{R})$ for some $\gamma > 0$,

(B_2) There exists $c_1 > 0$ such that $a(x,u) \equiv p(x) + f'(u) \geq c_1$,

(B_3) $\partial\Omega$ is of class $C^{2+\gamma}$.

We first establish an a priori bound for the solution of (3.2).

Remark 3.1.1 *Note that bistability of f is not important for our results. However, the nonlinearity cannot have a slope that is too negative, whereas there is no such restriction for the local Cahn-Hilliard equation. This is not just a technicality since the equation has no solution with $p(x) + f'(u) < 0$.*

Proposition 3.1.2 *Assume $(B_1) - (B_3)$. If $u(x,t) \in C(\bar{Q}_T) \cap C^{2,1}(Q_T)$ is a solution of (3.2), then*

$$\max_{\bar{Q}_T} |u| \leq C(\Omega, T, u_0) \tag{3.3}$$

for some positive constant $C(\Omega, T, u_0)$.

PROOF. Set $u(x,t) = ve^{\sigma t}$, where σ is to be determined. Then $\nabla u = e^{\sigma t}\nabla v$, $\triangle u = e^{\sigma t}\triangle v$, and (3.2) becomes

$$\begin{aligned} e^{\sigma t}v_t + ve^{\sigma t}\sigma = (p(x) + f'(u))e^{\sigma t}\triangle v + 2\nabla p(x) \cdot \nabla v e^{\sigma t} \\ + f''(u)\nabla v \cdot \nabla v e^{2\sigma t} + \triangle p v e^{\sigma t} - (\triangle J) * v e^{\sigma t}. \end{aligned} \tag{3.4}$$

Multiplying (3.4) by v and using $v\triangle v = \frac{1}{2}\triangle v^2 - |\nabla v|^2$, we obtain

$$\begin{aligned} \frac{1}{2}(v^2)_t + v^2\sigma = \frac{1}{2}(p(x) + f'(u))\triangle v^2 - (p(x) + f'(u))|\nabla v|^2 + \nabla p(x) \cdot \nabla v^2 \\ + \frac{1}{2}f''(u)\nabla v \cdot \nabla v^2 e^{\sigma t} + \triangle p v^2 - v(\triangle J) * v. \end{aligned} \tag{3.5}$$

If there exists $(P_0, t_0) \in \bar{Q}_T$ with $t_0 > 0$ such that $v^2(P_0, t_0) = \max v^2$, then $\Delta v^2(P_0, t_0) \le 0$, $\nabla v^2(P_0, t_0) = 0$, $(v^2)_t(P_0, t_0) \ge 0$, and (3.5) yields

$$(\sigma - \Delta p)v^2(P_0, t_0) \le -\int_\Omega \Delta J(P_0 - y)v(y, t_0)dyv(P_0, t_0). \tag{3.6}$$

Choose σ large enough such that $\sigma - \max(\Delta p) > \delta > 0$, we have

$$\max|v| \le \frac{M}{\delta}\int_\Omega |v(y, t_0)|dy \le M_1 e^{-\sigma t_0}\int_\Omega |u(y, t_0)|dy \le M_2 e^{-\sigma t_0}||u||_2 \tag{3.7}$$

for some positive constants M, M_1 and M_2.

On the other hand, multiplying (3.1) by u and integrating over Ω, it follows from Hölder's and Young's inequalities and condition (B_2) that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx \le K\int_\Omega u^2 dx, \tag{3.8}$$

where K depends only on J and Ω. This yields

$$\int_\Omega u^2 dx \le C(T)\int_\Omega u_0^2 dx. \tag{3.9}$$

It follows from (3.7) and (3.9) that

$$|v(P_0, t_0)| \le C_1||u_0||_2. \tag{3.10}$$

Therefore,

$$\max|v| \le \max\{C_1||u_0||_2, \max|u_0|\}. \tag{3.11}$$

Since $\max|u| \le e^{\sigma T}\max|v|$, (3.3) follows from (3.11).

If $u(x,t)$ is a solution of (3.1), $\max_{\bar{Q}_T}|u| \le C$, after a slight modification of Theorem 7.2 in Chapter V in [30], we have

Theorem 3.1.3 *For any solution $u(x,t) \in C^{2+\gamma, 1+\frac{\gamma}{2}}(\bar{Q}_T)$ of equation (3.1) having $\max_{\bar{Q}_T}|u| \le C$ one has the estimates*

$$\max_{\bar{Q}_T}|\nabla u| \le K_1, \quad |u|_{\bar{Q}_T}^{(1+\gamma)} \le K_2, \tag{3.12}$$

where constants K_1, K_2 and γ depend only on C, J, c_1, u_0, and the boundary of Ω ($|\cdot|_{Q_T}^{(1+\gamma)}$ is a Hölder norm defined in [30]).

In order to prove the existence of a solution, we use Schaefer's fixed point theorem from [22] or [25].

Theorem 3.1.4 *(Schaefer's Fixed point Theorem). Suppose X is a Banach space, and*

$$A : X \to X$$

is a continuous and compact mapping. Assume further that the set

$$\{u \in X | u = \mu A[u] \, for \; some \; 0 \le \mu \le 1\}$$

is bounded. Then A has a fixed point.

The a priori bounds established above will be used in conjunction with this to prove

Theorem 3.1.5 *Suppose conditions $(B_1) - (B_3)$ hold, $u_0(x) \in C^{2+\gamma}(\bar{\Omega})$ and $u_0|_{\partial\Omega} = 0$. Then there exists a solution $u(x,t) \in C^{2+\gamma, 1+\frac{\gamma}{2}}(\bar{Q}_T)$ of equation* (3.1).

PROOF. Let $v = u - u_0$, then (3.2) becomes

$$\begin{cases} v_t = \tilde{a}(x,v)\Delta v + \tilde{b}(x,v,\nabla v), \\ v|_{\partial\Omega} = -u_0, \\ v(0,x) = 0, \end{cases} \tag{3.13}$$

where

$$\tilde{a}(x,v) = a(x, v + u_0),$$
$$\tilde{b}(x,v,\nabla v) = a(x, v + u_0)\Delta u_0 + b(x, v + u_0, \nabla(v + u_0)),$$

and

$$b(x,u,\nabla u) = 2\nabla p \cdot \nabla u + f''(u)|\nabla u|^2 + u\Delta p - (\Delta J) * u.$$

Define $X = \{w \in C^{1+\gamma, \frac{1+\gamma}{2}}(\bar{Q}_T) | \, w(0,x) = 0\}$.

For any $w \in X$, consider the following linear equation

$$\begin{cases} v_t = \tilde{a}(x,w)\Delta v + \tilde{b}(x,w,\nabla w), \\ v|_{\partial\Omega} = -u_0(x), \\ v(0,x) = 0. \end{cases} \tag{3.14}$$

Since $w \in X$, $\tilde{a}(x,w)$ and $\tilde{b}(x,w,\nabla w)$ belong to $C^{\gamma, \frac{\gamma}{2}}(\bar{Q}_T)$, and so there exists a unique solution $v \in C^{2+\gamma, 1+\frac{\gamma}{2}}(\bar{Q}_T)$ of equation (3.14).

Define an operator $A : X \to X$ such that

$$v = A[w].$$

Claim 1: A is continuous from X to X.

In fact, if $v_1 = A[w_1]$, $v_2 = A[w_2]$, and $q = v_1 - v_2$, then q satisfies

$$\begin{cases} q_t = \tilde{a}(x,w_1)\Delta q + k(x,w_1,w_2), \\ q|_{\partial\Omega} = 0, \\ q(0,x) = 0, \end{cases} \tag{3.15}$$

where

$$k(x,w_1,w_2) = (\tilde{a}(x,w_1) - \tilde{a}(x,w_2))\Delta v_2 + \tilde{b}(x,w_1,\nabla w_1) \\ -\tilde{b}(x,w_2,\nabla w_2).$$

Fixing w_2, as w_1 approaches w_2 in X, we have

$$(\tilde{a}(x,w_1) - \tilde{a}(x,w_2))\Delta v_2 \to 0 \text{ in } C^{\gamma, \frac{\gamma}{2}}(\bar{Q}_T),$$

$$\tilde{b}(x,w_1,\nabla w_1) - \tilde{b}(x,w_2,\nabla w_2) \to 0 \text{ in } C^{\gamma, \frac{\gamma}{2}}(\bar{Q}_T).$$

Therefore, $k(x,w_1,w_2) \to 0$ in $C^{\gamma, \frac{\gamma}{2}}(\bar{Q}_T)$.

This implies $q \to 0$ in X as $w_1 \to w_2$, so A is continuous in X.

Claim 2: If $w = \mu A[w]$, there exists a uniform bound C such that

$$||w||_X \leq C.$$

In fact, if $\mu = 0$, then $w = 0$. If $0 < \mu \leq 1$, since $A[w] = \frac{1}{\mu}w$, $\frac{1}{\mu}w$ is a solution of (3.14), we have

$$\begin{cases} w_t = \tilde{a}(x,w)\Delta w + \mu\tilde{b}(x,w,\nabla w), \\ w|_{\partial\Omega} = -\mu u_0(x), \\ w(0,x) = 0. \end{cases} \tag{3.16}$$

Proposition 3.1.2 and Theorem 3.1.3 imply

$$||w||_{C^{1+\gamma, \frac{1+\gamma}{2}}} \leq C,$$

where C does not depend on μ.

Finally, the compactness of A follows from the fact that $C^{2+\gamma, \frac{2+\gamma}{2}}(\bar{Q}_T) \hookrightarrow C^{1+\gamma, \frac{1+\gamma}{2}}(\bar{Q}_T)$ is compact. This completes the proof.

We will prove the uniqueness and continuous dependence on initial values of the solution in the next section.

3.2 Existence, uniqueness and continuous dependence on initial data for generalized solutions

In section 3.1, under the assumption (B_2), equation (3.1) is a nondegenerate parabolic equation. In this section, we consider the degenerate case. Consider the following equation with $u_0 \in L^\infty(\Omega)$

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta(h(x,u)) - \displaystyle\int_\Omega u(y)\Delta J(x-y)dy & \text{in} \quad Q_T, \\ u = 0 & \text{on} \quad S_T, \\ u(x,0) = u_0(x), \end{cases} \tag{3.17}$$

where

$$h(x,u) = p(x)u(x) + f(u)$$

with $p(x)$ defined in section 2. Instead of nondegeneracy condition (B_2), we assume:

(B_2') For every fixed x, $h(x,0) = 0$, and $\dfrac{\partial h(x,u)}{\partial u} \geq d_1|u|^{r_1}$ for some positive constants r_1 and d_1.

Definition 3.2.1 *A generalized solution of (3.17) is a function* $u \in C([0,T] : L^1(\Omega)) \cap L^\infty(Q_T)$ *such that*

$$\begin{aligned}\int_\Omega u(x,t)\psi(x,t)dx - \int\int_{Q_t} u(x,t)\psi_s(x,s)dxds &= \int\int_{Q_t} h(x,u)\Delta\psi(x,s)dxds \\ &- \int\int_{Q_t} (\Delta J * u(\cdot,s))\psi(x,s)dxds + \int_\Omega u(x,0)\psi(x,0)dx\end{aligned} \tag{3.18}$$

for all $\psi \in C^{2,1}(\bar{Q}_T)$ *such that* $\psi(x,t) = 0$ *for* $x \in \partial\Omega$ *and* $0 \leq t \leq T$, *and*

$$u(x,0) = u_0(x). \tag{3.19}$$

We first prove the uniqueness.

Proposition 3.2.2 *Let* u_1, u_2 *be two solutions of equation (3.17) with initial data* u_{10}, $u_{20} \in L^\infty(\Omega)$, *then*

$$||u_1(\tau) - u_2(\tau)||_{L^1(\Omega)} \leq C(T)||u_{10} - u_{20}||_{L^1(\Omega)}$$

for each $\tau \in (0,T)$, *and some constant* $C(T)$.

PROOF. For any $\tau \in (0,T)$, and $\psi \in C^{2,1}(\bar{Q}_\tau)$ with $\psi|_{\partial\Omega} = 0$ for $0 < t < \tau$, after multiplying (3.17) by ψ and integrating over $\Omega \times (0,\tau)$, we have

$$\begin{aligned}\int_\Omega u_i(x,\tau)\psi(x,\tau)dx = \int_\Omega u_i(x,0)\psi(x,0)dx &+ \int_0^\tau \int_\Omega (u_i\psi_t + h(x,u_i)\Delta\psi)dxdt \\ &+ \int_0^\tau \int_\Omega (\Delta J * u_i)\psi dxdt.\end{aligned} \tag{3.20}$$

Setting $z = u_1 - u_2$ and $z_0 = u_{10} - u_{20}$, equation (3.20) gives

$$\begin{aligned}\int_\Omega z(x,\tau)\psi(x,\tau)dx &= \int_\Omega z_0(x)\psi(x,0)dx \\ &+ \int_0^\tau \int_\Omega z(\psi_t + b(x,t)\Delta\psi)dxdt + \int_0^\tau \int_\Omega (\Delta J * z)\psi dxdt,\end{aligned} \tag{3.21}$$

where

$$b(x,t) = \begin{cases} \dfrac{h(x,u_1) - h(x,u_2)}{u_1 - u_2} & \text{for} \quad u_1 \neq u_2, \\ h_u(x,u_1) & \text{for} \quad u_1 = u_2. \end{cases}$$

Following the idea in [7], we consider the problem:

$$\begin{cases} \dfrac{\partial\psi}{\partial t} = -b\Delta\psi + \nu\psi & \text{in} \quad \Omega, \ 0 < t < \tau, \\ \psi = 0 & \text{on} \quad \partial\Omega, \ 0 < t < \tau, \\ \psi(x,\tau) = g(x), \end{cases} \tag{3.22}$$

where $g(x) \in C_0^\infty(\Omega)$, $0 \leq g \leq 1$, and $\nu > 0$ is constant.

Since b just belongs to $L^\infty(Q_T)$ and may be equal to zero, we perturb to get a nondegenerate equation, by setting $b_n = \rho_n * b + \dfrac{1}{n}$, where ρ_n is a mollifier in $\mathbb{R}^n$, and $\int_0^\tau \int_\Omega (\rho_n * b - b)^2 dxdt \leq \dfrac{1}{n^2}$. Consider

$$\begin{cases} \dfrac{\partial\psi}{\partial t} = -b_n\Delta\psi + \nu\psi & \text{in} \quad \Omega, \ 0 < t < \tau, \\ \psi = 0 & \text{on} \quad \partial\Omega, \ 0 < t < \tau, \\ \psi(x,\tau) = g(x). \end{cases} \tag{3.23}$$

Since $b_n \geq \dfrac{1}{n}$, the equation is a nondegenerate parabolic equation, and so there exists a solution $\psi_n \in C^{2,1}(\bar{Q}_\tau)$.

Lemma 3.2.3 *The solution of (3.23) has the following properties*

$$\begin{aligned} &(i) \ \ 0 \leq \psi_n \leq e^{\nu(t-\tau)}, \\ &(ii) \ \ \int_0^\tau \int_\Omega b_n |\Delta(\psi_n)|^2 dxdt \leq C, \\ &(iii) \ \ \sup_{0 \leq t \leq \tau} \int_\Omega |\nabla\psi_n|^2 dx \leq C, \end{aligned}$$

where the constant C depends only on g.

PROOF. Since $1 \geq g \geq 0$, by the comparison principle, $e^{\nu(t-\tau)} \geq \psi_n \geq 0$, this proves (i).

For (ii) and (iii), multiplying equation (3.23) by $\triangle\psi_n$ and integrating over $\Omega \times (t,\tau)$, we have

$$\begin{aligned}\frac{1}{2}\int_\Omega |\nabla\psi_n(x,t)|^2 dx + \int_t^\tau \int_\Omega b_n|\triangle\psi_n|^2 dxds \\ + \nu \int_t^\tau \int_\Omega |\nabla\psi_n|^2 dxds = \frac{1}{2}\int_\Omega |\nabla\psi_n(x,\tau)|^2 dx.\end{aligned} \tag{3.24}$$

Since $\psi_n(x,\tau) = g(x)$, $\nabla\psi_n(x,\tau) = \nabla g$, we have

$$\sup_{0\leq t\leq\tau} \int_\Omega |\nabla\psi_n|^2 dx \leq C(g) \tag{3.25}$$

and

$$\int_t^\tau \int_\Omega b_n|\triangle\psi_n|^2 dxdt \leq C(g).$$

Therefore,

$$\int_0^\tau \int_\Omega b_n|\triangle\psi_n|^2 dxdt \leq C.$$

Replacing ψ by ψ_n in (3.21), and using (3.23) we obtain

$$\begin{aligned}\int_\Omega z(x,\tau)g(x)dx - \int_0^\tau \int_\Omega z(b-b_n)\triangle\psi_n dxdt \\ = \int_\Omega z(x,0)\psi_n(0)dx + \int\int_{Q_\tau} (\triangle J * z + \nu z)\psi_n dxdt.\end{aligned} \tag{3.26}$$

Since

$$\begin{aligned}&\int_0^\tau \int_\Omega z(b-b_n)\triangle\psi_n dxdt \\ &\leq C(\int_0^\tau \int_\Omega \frac{(b-b_n)^2}{b_n} dxdt)^{\frac{1}{2}}(\int_0^\tau \int_\Omega b_n|\triangle\psi_n|^2 dxdt)^{\frac{1}{2}} \\ &\leq \frac{C}{\sqrt{n}} \to 0,\end{aligned}$$

equation (3.26) implies

$$\begin{aligned}&\int_\Omega z(x,\tau)g(x)dx \\ &\leq \int_\Omega |z(x,0)|e^{\nu(t-\tau)}dx + \int\int_{Q_\tau} |\triangle J * z + \nu z|e^{\nu(t-\tau)}dxdt.\end{aligned} \tag{3.27}$$

Letting $\nu \to 0$ and $g(x) \to \operatorname{sign} z^+(x,\tau)$ in (3.27), we have

$$\int_\Omega (u_1 - u_2)^+ dx \le \int_\Omega |u_{10} - u_{20}| dx + \int\int_{Q_\tau} |\triangle J * z| dx dt. \tag{3.28}$$

Interchanging u_1 and u_2 yields

$$\int_\Omega |u_2 - u_1| dx \le \int_\Omega |u_{20} - u_{10}| dx + C \int\int_{Q_\tau} |u_2 - u_1| dx dt. \tag{3.29}$$

(3.29) and Gronwall's inequality imply the conclusion.

Remark 3.2.4 *Since every classical solution is also a weak solution, this also proves the uniqueness and continuous dependence on initial values for classical solutions.*

To prove the existence of a solution to (3.18), we consider the regularized problem and take $u_0 \in C^{2+\gamma}(\bar\Omega)$ for some $\gamma > 0$, with $u_0|_{\partial\Omega} = 0$.

$$\begin{cases} \dfrac{\partial u}{\partial t} = \triangle(h^\epsilon(x,u)) - \displaystyle\int_\Omega \triangle J(x-y) u(y) dy & \text{in} \quad Q_T, \\ u = 0 & \text{on} \quad S_T, \\ u(x,0) = u_0(x), \end{cases} \tag{3.30}$$

where

$$h^\epsilon(x,u) = p(x)u(x) + f(u) + \epsilon u.$$

By Theorem 3.1.4, there exists a classical solution $u_\epsilon(x,t) \in C^{2+\gamma, \frac{2+\gamma}{2}}(\bar Q_T)$. These solutions are uniformly bounded:

Lemma 3.2.5 *There exists a constant C, independent of ϵ, such that*

$$\max_{\bar Q_T} |u_\epsilon(x,t)| \le C \tag{3.31}$$

for all $0 < \epsilon \le 1$.

PROOF. Multiplying equation (3.30) by u and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\int u^2 + \epsilon\int |\nabla u|^2 + \int (p(x) + f'(u))|\nabla u|^2 = \frac{1}{2}\int u^2 \triangle p(x) - \int u\triangle J * u. \tag{3.32}$$

Using Hölder's and Young's inequalities, we obtain

$$\max_{0\le t\le T}\int u^2 \le C_1,$$

where C_1 does not depend on ϵ. A similar argument to that in the proof of Proposition 3.1.2 yields (3.31).

Now we can prove the existence of a generalized solution.

We need the following lemma.

Lemma 3.2.6 *If $r > 0$ is a constant, u, $v \in \mathbb{R}$, then we have*

$$|(|u|^r u - |v|^r v)| \ge \frac{1}{2^r}|u - v|^{r+1}. \tag{3.33}$$

Multiplying equation (3.30) by $\dfrac{\partial h^\epsilon(x,u)}{\partial t}$ and integrating over Ω, we have

$$\int \frac{\partial h^\epsilon(x,u)}{\partial t}u_t + \frac{d}{dt}\int |\nabla h^\epsilon(x,u)|^2 = \int \frac{\partial h^\epsilon(x,u)}{\partial t}(-\triangle J * u). \tag{3.34}$$

Since u is uniformly bounded, and $\dfrac{\partial h^\epsilon(x,u)}{\partial t} = (\epsilon + p(x) + f'(u))u_t$, we have

$$\begin{aligned}\int \frac{\partial h^\epsilon(x,u)}{\partial t}(-\triangle J * u) &\le C\int (\epsilon + p(x) + f'(u))|u_t| \\ &\le \int \sqrt{(\epsilon + p(x) + f'(u))}|u_t|\sqrt{(\epsilon + p(x) + f'(u))} \\ &\le \frac{1}{2}\int (\epsilon + p(x) + f'(u))|u_t|^2 + C\int (\epsilon + p(x) + f'(u)),\end{aligned} \tag{3.35}$$

where C does not depend on ϵ.

Equation (3.34) and inequality (3.35) imply

$$\frac{1}{2}\int_0^t \int \frac{\partial h^\epsilon(x,u)}{\partial t}u_t + \int |\nabla h^\epsilon(x,u)|^2 \le \int |\nabla h^\epsilon(x,u_0)|^2 + C. \tag{3.36}$$

Note that the first term is positive from the expression for $\frac{\partial h^\epsilon}{\partial t}$. Therefore,

$$\sup_{0\le t\le T}\int |\nabla h^\epsilon(x,u)|^2 \le C, \tag{3.37}$$

and

$$\int_0^T\int \frac{\partial h^\epsilon(x,u)}{\partial t}u_t \le C, \tag{3.38}$$

where C does not depend on ϵ.

Since u is uniformly bounded, we also have

$$\begin{aligned}\int_0^T\int \left|\frac{\partial h^\epsilon(x,u)}{\partial t}\right|^2 &= \int_0^T\int (\epsilon+p(x)+f'(u))^2|u_t|^2\\ &\le C\int_0^T\int \frac{\partial h^\epsilon(x,u)}{\partial t}u_t \\ &\le C_1,\end{aligned} \tag{3.39}$$

where C_1 does not depend on ϵ.

We show the dependence on ϵ by writing u_ϵ. We have

$$\begin{aligned}\left|\left|\frac{\partial h^\epsilon(x,u_\epsilon)}{\partial t}\right|\right|_{L^2((0,T),L^2(\Omega))} &\le C,\\ \max_{0\le t\le T}||\nabla h^\epsilon(x,u_\epsilon)||_{L^2(\Omega)} &\le C,\\ \max_{Q_T}|h^\epsilon(x,u_\epsilon)| &\le C,\end{aligned} \tag{3.40}$$

where C does not depend on ϵ.

Also from condition (B_1) and (3.40), we obtain

$$\begin{aligned}\left|\left|\frac{\partial(|u_\epsilon|^{r_1}u_\epsilon)}{\partial t}\right|\right|_{L^2((0,T),L^2(\Omega))} &\le C,\\ \max_{0\le t\le T}||\nabla(|u_\epsilon|^{r_1}u_\epsilon)||_{L^2(\Omega)} &\le C,\\ \max_{\bar{Q}_T}|u_\epsilon|^{r_1+1} &\le C.\end{aligned} \tag{3.41}$$

It follows from (3.40) that $h^\epsilon(x,u_\epsilon)$ is equicontinuous from $[0,T]$ into $L^2(\Omega)$ with values in a bounded subset of $H^1(\Omega)$. Since $H^1(\Omega)\hookrightarrow L^2(\Omega)$ is compact, by

Arzela-Ascoli's lemma, there is a sequence $\epsilon_n \to 0$ such that $h^{\epsilon_n}(x, u_{\epsilon_n}) \to v$ in $C([0,T], L^2(\Omega))$.

Also from (3.41), we know that $|u_\epsilon|^{r_1} u_\epsilon$ is equicontinuous from $[0,T]$ into $L^2(\Omega)$ with values in a bounded subset of $H^1(\Omega)$. By Arzela-Ascoli's lemma, there exists $\epsilon_n \to 0$ such that $|u_n|^{r_1} u_n \to v_1$ in $C([0,T], L^2(\Omega))$, where $u_n = u_{\epsilon_n}$.

By Lemma 3.2.6, we have

$$\int_\Omega |u_n - u_m|^{r_1+1} dx \le 2^r \int_\Omega ||u_n|^{r_1} u_n - |u_m|^{r_1} u_m| dx. \tag{3.42}$$

Therefore, $\{u_n\}$ is a Cauchy sequence in $C([0,T], L^{r_1+1}(\Omega))$ and there exists u such that $u_n \to u$ in $C([0,T], L^{r_1+1}(\Omega))$.

By Lemma 3.2.5, we can also conclude that $u \in L^\infty(Q_T)$. Since f is differentiable, we have $h^{\epsilon_n}(x, u_{\epsilon_n}) \to h(x,u)$ in $C([0,T], L^{r_1+1}(\Omega))$.

Letting $\epsilon_n \to 0$, we see that u satisfies equation (3.18), and u is a generalized solution with initial data $u_0 \in C_0^{2+\gamma}(\bar{\Omega})$.

For $u_0 \in L^\infty(\Omega)$, choose $u_{0n} \in C_0^{2+\gamma}(\bar{\Omega})$ such that

$$||u_{0n} - u_0||_{L^1(\Omega)} \to 0 \ as \ n \to \infty. \tag{3.43}$$

By Proposition 3.2.2, we have

$$\sup_{0 \le t \le T} ||u_m(t) - u_n(t)||_{L^1(\Omega)} \le C ||u_{0m} - u_{0n}||_{L^1(\Omega)}, \tag{3.44}$$

where C does not depend on m, n. Furthermore, there is a constant C_1, depending only on $||u_0||_{L^\infty}$, such that $||u_j||_{L^\infty} \le C_1$.

By (3.43) and (3.44), there exists $u \in C([0,T], L^1(\Omega))$ such that $u_m(t) \to u$ in $C([0,T], L^1(\Omega))$, clearly u is a generalized solution.

We have proved:

Theorem 3.2.7 *For any $T > 0$ and $u_0 \in L^\infty(\Omega)$, if conditions (B_1), (B_2'), and (B_3) are satisfied, then there exists a unique function $u \in C([0,T], L^1(\Omega)) \cap L^\infty(Q_T)$ which satisfies equation (3.18).*

3.3 Long term behavior in the H^1 norm

In this section, we prove that there exists a continuous semigroup associated with equation (3.1). Then we consider the boundedness in time of the solution.

Definition 3.3.1 *A weak solution of (3.1) is a function*

$u \in C([0,T], L^2(\Omega)) \cap L^\infty([0,T], L^\infty(\Omega)) \cap L^2([0,T], H_0^1(\Omega)), u_t \in L^2([0,T], H^{-1}(\Omega)),$

$h(x,u) \in L^2((0,T), H^1(\Omega))$ *such that*

$$\begin{aligned} < u_t(x,t), \psi(x) > &+ \int_\Omega \nabla h(x,u) \cdot \nabla \psi(x) dx \\ &= \int_\Omega (\Delta J * u(\cdot, s)) \psi(x) dx \end{aligned} \tag{3.45}$$

for all $\psi \in H_0^1(\Omega)$ *and a.e. time* $0 \leq t \leq T$*, where* $h(x,u) = p(x)u + f(u)$*, and*

$$u(x,0) = u_0(x). \tag{3.46}$$

Theorem 3.3.2 *If* $(B_1) - (B_3)$ *are satisfied and* $u_0 \in L^\infty(\Omega)$*, then there exists a unique solution* u *of (3.45).*

PROOF. Since $u_0 \in L^\infty(\Omega)$, there exists a sequence $u_0^{(k)} \in C^{2+\gamma}(\bar{\Omega})$ with $\gamma > 0$ such that

$$\begin{aligned} &||u_0^{(k)} - u_0||_{L^2} \to 0, \\ &||u_0^{(k)}||_\infty < C, \end{aligned} \tag{3.47}$$

where C does not depend on k.

We consider the following problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta(h(x,u) - J * u) & \text{in} \quad Q_T, \\ u = 0 & \text{on} \quad S_T, \\ u(x,0) = u_0^{(k)}(x). \end{cases} \tag{3.48}$$

By Theorem 3.1.4, there exists a classical solution $u^{(k)} \in C^{2+\gamma, \frac{2+\gamma}{2}}(\bar{Q}_T)$, and

$$\max_{\bar{Q}_T} |u^{(k)}(x,t)| \leq \bar{C}, \tag{3.49}$$

where $\bar{C}$ does not depend on k.

Multiplying equation (3.48) by $u^{(k)}$ and integrating over Ω, we have

$$\frac{d\int_\Omega |u^{(k)}|^2 dx}{dt} + \int_\Omega \nabla h(x, u^{(k)}) \cdot \nabla u^{(k)} dx = \int_\Omega (\Delta J * u^{(k)}) u^{(k)} dx. \tag{3.50}$$

Since $\nabla h(x, u^{(k)}) \cdot \nabla u^{(k)} \geq c_1 |\nabla u^{(k)}|^2 + u \nabla p \cdot \nabla u^{(k)}$, where c_1 is defined in condition (B_2), from equation (3.50), we also have

$$\sup_{0 \leq t \leq T} ||u^{(k)}||_{L^2} \leq C_1(T), \tag{3.51}$$

$$\int_0^T \int_\Omega |\nabla u^{(k)}|^2 dx dt \leq C_2(T), \tag{3.52}$$

where $C_1(T)$, $C_2(T)$ do not depend on k.

Since by (3.49), $u^{(k)}$ is uniformly bounded, from inequality (3.52), we have

$$\begin{aligned} \int_0^T \int_\Omega |\nabla h(x, u^{(k)})|^2 dx dt &= \int_0^T \int_\Omega |u^{(k)} \nabla p + (p(x) + f'(u^{(k)})) \nabla u^{(k)}|^2 dx dt \\ &\leq C_3(T) \end{aligned} \tag{3.53}$$

for some positive constant $C_3(T)$ which does not depend on k.

From equality (3.53) and equation (3.48), we also have

$$||u_t^{(k)}||_{L^2((0,T), H^{-1}(\Omega))} \leq C_4(T), \tag{3.54}$$

where $C_4(T)$ does not depend on k.

Inequalities (3.51)-(3.54) imply that there exist subsequence of $\{u^k\}$ (still denoted by $\{u^k\}$) and v, u, g such that

$$\begin{aligned} h(x, u^{(k)}) &\rightharpoonup v \ \text{ in } \ L^2((0,T), H^1(\Omega)), \\ u^{(k)} &\rightharpoonup u \ \text{ in } \ L^2((0,T), H^1(\Omega)), \\ u_t^{(k)} &\rightharpoonup g \ \text{ in } \ L^2((0,T), H^{-1}(\Omega)). \end{aligned} \tag{3.55}$$

Since $||u^{(k)}||_{L^\infty} \leq \bar{C}$ and $u^{(k)} \to u$ *in* $L^2((0,T), L^2(\Omega))$, we have

$$\begin{aligned}
&||u||_{L^\infty} \leq \bar{C}, \\
&h(x, u^{(k)}) \to h(x, u) \text{ in } L^2((0,T), L^2(\Omega)), \\
&g = u_t, \\
&v = h(x, u).
\end{aligned} \tag{3.56}$$

This implies that u is a weak solution of (3.45). Uniqueness follows from Proposition 3.2.2.

Corollary 3.3.3 *If* $u_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$, *and if* $u \in C([0,T], L^2(\Omega)) \cap L^\infty([0,T], L^\infty(\Omega)) \cap L^2([0,T], H_0^1(\Omega))$, *with* $u_t \in L^2([0,T], H^{-1}(\Omega))$, *satisfies equation (3.45), then* $u \in C([0,T], L^2(\Omega)) \cap L^\infty([0,T], L^\infty(\Omega)) \cap L^\infty([0,T], H_0^1(\Omega))$, *and* $u_t \in L^2([0,T], L^2(\Omega))$. *Furthermore, if* $\dim \Omega = 1$, *we also have* $u \in L^2([0,T], H^2(\Omega))$, *and* $u \in C([0,T], H^1(\Omega))$.

PROOF. Since $u_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$, we may assume $||u_0^{(k)}||_{H^1} \leq C$ for some constant C which does not depend on k in (3.47). Multiplying equation (3.48) by $\dfrac{\partial h(x, u^{(k)})}{\partial t}$ and integrating over Ω, we have

$$\int \frac{\partial h(x, u^{(k)})}{\partial t} u_t + \frac{1}{2} \frac{d}{dt} \int |\nabla h(x, u^{(k)})|^2 = \int \frac{\partial h(x, u^{(k)})}{\partial t} (-\Delta J * u^{(k)}). \tag{3.57}$$

A similar argument to that in the proof of (3.34)-(3.38) in section 3.2 shows

$$\sup_{0 \leq t \leq T} \int |\nabla h(x, u^{(k)})|^2 \leq C, \tag{3.58}$$

and

$$\int_0^T \int \frac{\partial h(x, u^{(k)})}{\partial t} u_t^{(k)} \leq C, \tag{3.59}$$

where C does not depend on k.

Condition (B_2') and (3.58)-(3.59) imply

$$\sup_{0\le t\le T}\int |\nabla u^{(k)}|^2 \le C_1, \tag{3.60}$$

and

$$\int_0^T \int |u_t^{(k)}|^2 \le C_1, \tag{3.61}$$

where C_1 does not depend on k.

Therefore, by passing to limits as a subsequence of $k \to \infty$, we deduce
$u \in L^\infty([0,T], H_0^1(\Omega))$, $u_t \in L^2([0,T], L^2(\Omega))$, and
$h(x,u) \in L^\infty([0,T], H_0^1(\Omega))$.

If $\dim\Omega = 1$, multiplying (3.48) by $-\Delta u^{(k)}$ and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\int |\nabla u^{(k)}|^2 dx + \int \Delta h(x,u^{(k)})\Delta u^{(k)} dx = \int \Delta J * u^{(k)} \Delta u^{(k)} dx. \tag{3.62}$$

Since

$$\Delta h(x,u^{(k)}) = u^{(k)}\Delta p + 2\nabla p\cdot\nabla u^{(k)} + f''(u^{(k)})|\nabla u^{(k)}|^2 + (p+f'(u^{(k)}))\Delta u^{(k)}, \tag{3.63}$$

by Hölder's and Young's inequalities, and using (3.49) and (3.60), we have

$$\int \Delta J * u^{(k)}\Delta u^{(k)} dx \le \epsilon\int |\Delta u^{(k)}|^2 dx + C(\epsilon), \tag{3.64}$$

and

$$\begin{aligned}
\int \Delta h(x,u^{(k)})\Delta u^{(k)} dx &\ge \int (p+f'(u^{(k)}))|\Delta u^{(k)}|^2 dx - C(\epsilon)\int |\Delta p u^{(k)}|^2 dx \\
&\quad - \epsilon\int |\Delta u^{(k)}|^2 dx - C(\epsilon)\int |\nabla p\cdot\nabla u^{(k)}|^2 dx \\
&\quad - \epsilon\int |\Delta u^{(k)}|^2 dx - C(\epsilon)\int |f''(u^{(k)})|^2|\nabla u^{(k)}|^4 dx \\
&\quad - \epsilon\int |\Delta u^{(k)}|^2 dx \\
&\ge (c_1 - 3\epsilon)\int |\Delta u^{(k)}|^2 dx - C(\epsilon) - C(\epsilon)\int |\nabla u^{(k)}|^4 dx.
\end{aligned} \tag{3.65}$$

In order to estimate $\int |\nabla u^{(k)}|^4 dx$, we need the following Gagliardo-Nirenberg inequality,

$$||D^j v||_{L^s} \leq C_1 ||D^m v||_{L^r}^a ||v||_{L^q}^{1-a} + C_2 ||v||_{L^q}, \tag{3.66}$$

where

$$\frac{j}{m} \leq a \leq 1, \ \frac{1}{s} = \frac{j}{n} + a(\frac{1}{r} - \frac{m}{n}) + (1-a)\frac{1}{q}. \tag{3.67}$$

In (3.66), set $s = 4$, $j = 0$, $r = 2$, $m = 1$, $n = 1$, $r = 2$, $a = \frac{1}{4}$, $q = 2$ to get

$$||v||_4 \leq C_1 ||Dv||_2^{\frac{1}{4}} ||v||_2^{\frac{3}{4}} + C_2 ||v||_2. \tag{3.68}$$

Let $v = \nabla u^{(k)}$, then (3.68) and (3.60) give

$$||\nabla u^{(k)}||_4 \leq C_1 ||\Delta u^{(k)}||_2^{\frac{1}{4}} ||\nabla u^{(k)}||_2^{\frac{3}{4}} + C_2 ||\nabla u^{(k)}||_2 \leq C ||\Delta u^{(k)}||_2^{\frac{1}{4}} + C. \tag{3.69}$$

This and Young's inequality imply

$$||\nabla u^{(k)}||_4^4 \leq \epsilon_1 ||\Delta u^{(k)}||_2^2 + C(\epsilon_1). \tag{3.70}$$

Inequalities (3.65) and (3.70) imply that

$$\int \Delta h(x, u^{(k)}) \Delta u^{(k)} dx \geq (c_1 - 3\epsilon - C\epsilon_1) \int |\Delta u^{(k)}|^2 dx - C(\epsilon, \epsilon_1), \tag{3.71}$$

where constant $C(\epsilon, \epsilon_1)$ does not depend on k.

Equation (3.62), inequalities (3.64) and (3.71) imply

$$\frac{1}{2}\frac{d}{dt} \int |\nabla u^{(k)}|^2 dx + (c_1 - 4\epsilon - C\epsilon_1) \int |\Delta u^{(k)}|^2 dx \leq C(\epsilon, \epsilon_1). \tag{3.72}$$

Choose ϵ and ϵ_1 small enough such that $c_1 - 4\epsilon - C\epsilon_1 \geq \frac{c_1}{2}$, and integrate over (0,T) to obtain

$$\frac{1}{2} \int |\nabla u^{(k)}(T)|^2 dx - \frac{1}{2} \int |\nabla u^{(k)}(0)|^2 dx + \frac{c_1}{2} \int_0^T \int |\Delta u^{(k)}|^2 dx \leq C. \tag{3.73}$$

Therefore, there exists a subsequence such that $u_k \rightharpoonup u$ in $L^2((0,T),H^2(\Omega))$. Since $u \in L^2((0,T),H^2(\Omega))$ and $u_t \in L^2((0,T),L^2(\Omega))$, we also have $u \in C([0,T],H^1(\Omega))$. This completes the proof.

In order to prove the existence of an absorbing set, instead of (B_2), we assume

$(\bar{B}_2)$ There exist positive constants c_1, c_2 and r such that $a(x,u) \geq c_2|u|^r + c_1$.

Also, we assume

(B_4) There exist positive constants c_3 and c_4 such that $a(x,u) \leq c_3|u|^r + c_4$.

First we study long term behavior in the L^p norm.

We need the following version of Gronwall's lemma (see Temam [38]):

Lemma 3.3.4 *(Uniform Gronwall inequality) Let y be a positive absolutely continuous function on $(0,\infty)$ which satisfies*

$$y' + \nu y^p \leq \delta$$

with $p > 1, \nu > 0, \delta \geq 0$. Then, for $t \geq 0$, we have

$$y(t) \leq \left(\frac{\delta}{\nu}\right)^{\frac{1}{p}} + (\nu(p-1)t)^{\frac{-1}{p-1}}. \tag{3.74}$$

With this we can establish

Proposition 3.3.5 *If u is a solution of (3.1), then for $p \geq 1$, we have*

$$\int_\Omega |u|^{p+1}dx < \left(\frac{d_1(p)}{d_2(p)}\right)^{\frac{p+1}{p+r+1}} + \left(\frac{d_2(p)rt}{p+1}\right)^{-\frac{p+1}{r}} \tag{3.75}$$

where $d_1(p)$ and $d_2(p)$ are constants which do not depend on the initial data.

PROOF. Multiplying equation (3.1) by $u|u|^{p-1}$ and integrating over Ω, we obtain

$$\begin{aligned} \int u|u|^{p-1}u_t dx = &-\int a(x,u)\nabla u\cdot\nabla(u|u|^{p-1})dx \\ &+\int\int \triangle J(x-y)u(x)u(x)|u|^{p-1}dydx \\ &-\int\int \triangle J(x-y)u(y)u(x)|u|^{p-1}dydx \\ &+\int\int \nabla J(x-y)\cdot\nabla u(x)u(x)|u|^{p-1}dydx. \end{aligned} \tag{3.76}$$

Since

$$\int_\Omega u|u|^{p-1}u_t dx = \frac{1}{p+1}\frac{d}{dt}\int_\Omega |u|^{p+1}dx, \tag{3.77}$$

$$\int_\Omega a(x,u)\nabla u\cdot\nabla(u|u|^{p-1})dx = p\int_\Omega a(x,u)|u|^{p-1}|\nabla u|^2 dx, \tag{3.78}$$

$$|\nabla|u|^{\frac{p+1}{2}}|^2 = \frac{(p+1)^2}{4}|u|^{p-1}|\nabla u|^2, \tag{3.79}$$

and

$$|\nabla|u|^{\frac{p+r+1}{2}}|^2 = \frac{(p+r+1)^2}{4}|u|^{p+r-1}|\nabla u|^2, \tag{3.80}$$

from $(\bar{B}_2)$ we have

$$\begin{aligned} \int_\Omega a(x,u)\nabla u\cdot\nabla(u|u|^{p-1})dx \geq &\frac{4pc_1}{(p+1)^2}\int_\Omega |\nabla|u|^{\frac{p+1}{2}}|^2 dx \\ &+\frac{4pc_2}{(p+r+1)^2}\int_\Omega |\nabla|u|^{\frac{p+r+1}{2}}|^2 dx. \end{aligned} \tag{3.81}$$

Equations (3.76)-(3.80) and inequality (3.81) yield

$$\begin{aligned} &\frac{1}{p+1}\frac{d}{dt}\int_\Omega |u|^{p+1}dx + \frac{4pc_1}{(p+1)^2}\int_\Omega |\nabla|u|^{\frac{p+1}{2}}|^2 dx + \frac{4pc_2}{(p+r+1)^2}\int_\Omega |\nabla|u|^{\frac{p+r+1}{2}}|^2 dx \\ &\leq \int_\Omega\int_\Omega \triangle J(x-y)|u|^{p+1}dydx - \int_\Omega\int_\Omega \triangle J(x-y)u(y)u(x)|u|^{p-1}dydx \\ &+\int_\Omega\int_\Omega \nabla J(x-y)\cdot\nabla u(x)u(x)|u|^{p-1}dydx. \end{aligned} \tag{3.82}$$

By Hölder's and Young's inequalities, we have

$$\begin{aligned}
&\int_\Omega\int_\Omega u(x)|u|^{p-1}\nabla J(x-y)\cdot\nabla u(x)dydx \\
&\leq M\int_\Omega (|u|^{\frac{p-1}{2}}|\nabla u(x)|)|u|^{\frac{p+1}{2}}dx \\
&\leq \epsilon\int_\Omega |u|^{p-1}|\nabla u(x)|^2dx + C(\epsilon)\int_\Omega |u|^{p+1}dx \\
&= \frac{4\epsilon}{(p+1)^2}\int_\Omega |\nabla|u|^{\frac{p+1}{2}}|^2dx + C(\epsilon)\int_\Omega |u|^{p+1}dx.
\end{aligned} \tag{3.83}$$

Also,

$$\begin{aligned}
&\int\int |\Delta J(x-y)||u(y)||u(x)|^p dydx \\
&\leq (\int |u|^{1+p}dx)^{\frac{p}{1+p}}(\int(\int |\Delta J(x-y)||u(y)|dy)^{1+p}dx)^{\frac{1}{1+p}} \\
&\leq C\int_\Omega |u|^{p+1}dx,
\end{aligned} \tag{3.84}$$

and

$$\int_\Omega\int_\Omega |\Delta J(x-y)||u(x)|^{p+1}dydx \leq C\int_\Omega |u|^{p+1}dx. \tag{3.85}$$

Inequality (3.82) and estimates (3.83)-(3.85) imply

$$\begin{aligned}
&\frac{1}{p+1}\frac{d}{dt}\int_\Omega |u|^{p+1}dx + \frac{4pc_1}{(p+1)^2}\int_\Omega |\nabla|u|^{\frac{p+1}{2}}|^2dx + \frac{4pc_2}{(p+r+1)^2}\int_\Omega |\nabla|u|^{\frac{p+r+1}{2}}|^2dx \\
&\leq \frac{4\epsilon}{(p+1)^2}\int_\Omega |\nabla|u|^{\frac{p+1}{2}}|^2dx + C(\epsilon)\int_\Omega |u|^{p+1}dx.
\end{aligned} \tag{3.86}$$

Let $\epsilon = \frac{pc_1}{2}$ in (3.86), then we obtain

$$\begin{aligned}
&\frac{1}{p+1}\frac{d}{dt}\int_\Omega |u|^{p+1}dx + \frac{2pc_1}{(p+1)^2}\int_\Omega |\nabla|u|^{\frac{p+1}{2}}|^2dx + \frac{4pc_2}{(p+r+1)^2}\int_\Omega |\nabla|u|^{\frac{p+r+1}{2}}|^2dx \\
&\leq C\int_\Omega |u|^{p+1}dx.
\end{aligned} \tag{3.87}$$

Therefore,

$$\frac{d}{dt}\int_{\Omega}|u|^{p+1}dx + d_1(p)\int_{\Omega}|\nabla|u|^{\frac{p+r+1}{2}}|^2dx \leq d_2(p)\int_{\Omega}|u|^{p+1}dx \tag{3.88}$$

for some constants $d_1(p)$ and $d_2(p)$.

Set $v = |u|^{\frac{p+r+1}{2}}$ and $\gamma = \frac{2(p+1)}{p+r+1}$, so that (3.88) becomes

$$\frac{d}{dt}\int_{\Omega}|v|^{\gamma}dx + d_1(p)\int_{\Omega}|\nabla v|^2dx \leq d_2(p)\int_{\Omega}|v|^{\gamma}dx. \tag{3.89}$$

By *Poincaré's* inequality, $\int_{\Omega}|v|^2dx \leq C\int_{\Omega}|\nabla v|^2dx$, we have

$$\frac{d}{dt}\int_{\Omega}|v|^{\gamma}dx + d_1(p)\int_{\Omega}|v|^2dx \leq d_2(p)\int_{\Omega}|v|^{\gamma}dx, \tag{3.90}$$

where $d_1(p)$ and $d_2(p)$ have been redefined.

Since $\gamma < 2$, it follows from *Hölder's* and Young's inequalities that

$$\frac{d}{dt}\int_{\Omega}|v|^{\gamma}dx + d_2(p)(\int_{\Omega}|v|^{\gamma}dx)^{\frac{2}{\gamma}} \leq d_2(p). \tag{3.91}$$

where $d_1(p)$ and $d_2(p)$ have been redefined.

The conclusion follows from Lemma 3.3.4.

Using a similar argument to that in the proof of Theorem 2.1.1 in Chapter 2, we obtain

Proposition 3.3.6 *If $u_0 \in L^{\infty}(\Omega)$, then*

$$\sup_{t\geq 0}||u||_{\infty} \leq C(u_0). \tag{3.92}$$

Next we need to estimate $||\nabla u||_2$.

Theorem 3.3.7 *Assume that u is a solution of (3.1) and conditions (B_1), $(\bar{B}_2)$, (B_3) and (B_4) are satisfied. There exists $t_0 > 0$ such that if $t \geq t_0$ then*

$$\sup_{t\geq t_0}||\nabla u||_2 < C, \tag{3.93}$$

where constant C does not depend on initial data.

PROOF. Multiplying (3.1) by $a(x,u)\frac{\partial u}{\partial t}$, and integrating over Ω, we obtain

$$\begin{aligned}\int_\Omega a(x,u)|\frac{\partial u}{\partial t}|^2 dx = &\int_\Omega \nabla\cdot(a(x,u)\nabla u)a(x,u)\frac{\partial u}{\partial t}dx+\\ &\int_\Omega\int_\Omega \nabla J(x-y)\cdot\nabla u(x)a(x,u)\frac{\partial u}{\partial t}dxdy+\\ &\int_\Omega\int_\Omega(\Delta J)(x-y)u(x)a(x,u)\frac{\partial u}{\partial t}dxdy-\\ &\int_\Omega\int_\Omega(\Delta J)(x-y)u(y)a(x,u)\frac{\partial u}{\partial t}dxdy.\end{aligned}\tag{3.94}$$

Since

$$\begin{aligned}\int_\Omega \nabla\cdot(a(x,u)\nabla u)a(x,u)\frac{\partial u}{\partial t}dx &= -\int_\Omega a(x,u)\nabla u\cdot\nabla(a(x,u)\frac{\partial u}{\partial t})dx\\ &= -\int_\Omega(a(x,u)\nabla p(x)\cdot\nabla u\frac{\partial u}{\partial t}dx-\\ &\quad a(x,u)f''(u)|\nabla u|^2\frac{\partial u}{\partial t}-a^2(x,u)\nabla u\frac{\partial\nabla u}{\partial t})dx,\end{aligned}\tag{3.95}$$

and

$$\frac{1}{2}\frac{d}{dt}\int_\Omega a^2(x,u)|\nabla u|^2dx = \int_\Omega(a(x,u)f''(u)|\nabla u|^2\frac{\partial u}{\partial t}+a^2(x,u)\nabla u\frac{\partial\nabla u}{\partial t})dx,\tag{3.96}$$

this yields

$$\begin{aligned}\int_\Omega\nabla\cdot(a(x,u)\nabla u)a(x,u)\frac{\partial u}{\partial t}dx = &-\frac{1}{2}\frac{d}{dt}\int_\Omega a^2(x,u)|\nabla u|^2dx\\ &-\int_\Omega(a(x,u)\nabla p(x)\cdot\nabla u\frac{\partial u}{\partial t}dx.\end{aligned}\tag{3.97}$$

It follows from (3.94) and (3.97) that

$$\begin{aligned}\int_\Omega a(x,u)|\frac{\partial u}{\partial t}|^2dx = &-\frac{1}{2}\frac{d}{dt}\int_\Omega a^2(x,u)|\nabla u|^2dx\\ &-\int_\Omega(a(x,u)\nabla p(x)\cdot\nabla u\frac{\partial u}{\partial t}dx\\ &+\int_\Omega\int_\Omega\nabla J(x-y)\cdot\nabla u(x)a(x,u)\frac{\partial u}{\partial t}dxdy\\ &+\int_\Omega\int_\Omega(\Delta J)(x-y)u(x)a(x,u)\frac{\partial u}{\partial t}dxdy\\ &-\int_\Omega\int_\Omega(\Delta J)(x-y)u(y)a(x,u)\frac{\partial u}{\partial t}dxdy.\end{aligned}\tag{3.98}$$

Note that

$$\int_\Omega a(x,u)\nabla p(x)\cdot\nabla u\frac{\partial u}{\partial t}dx \le \epsilon\int_\Omega a(x,u)|\frac{\partial u}{\partial t}|^2dx + C(\epsilon)\int_\Omega a(x,u)|\nabla u|^2dx, \quad (3.99)$$

$$\int_\Omega\int_\Omega \nabla J(x-y)\cdot\nabla u(x)a(x,u)\frac{\partial u}{\partial t}dxdy \le \epsilon\int_\Omega a(x,u)|\frac{\partial u}{\partial t}|^2dx + C(\epsilon)\int_\Omega a(x,u)|\nabla u|^2dx, \quad (3.100)$$

$$\int_\Omega\int_\Omega (\Delta J)(x-y)u(x)a(x,u)\frac{\partial u}{\partial t}dxdy \le \epsilon\int_\Omega a(x,u)|\frac{\partial u}{\partial t}|^2dx + C(\epsilon)\int_\Omega a(x,u)u^2dx, \quad (3.101)$$

and

$$\begin{aligned}\int_\Omega\int_\Omega (\Delta J)(x-y)u(y)a(x,u)\frac{\partial u}{\partial t}dxdy &\le \epsilon\int_\Omega a(x,u)|\frac{\partial u}{\partial t}|^2dx \\ &\quad + C(\epsilon)\int_\Omega a(x,u)(\int_\Omega (\Delta J)(x-y)u(y)dy)^2dx.\end{aligned} \quad (3.102)$$

Choosing $\epsilon = \frac{1}{4}$, it follows from (3.98)-(3.102) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega a^2(x,u)|\nabla u|^2dx &\le C(\int_\Omega a(x,u)|\nabla u|^2dx \\ &\quad + \int_\Omega a(x,u)u^2dx + \int_\Omega a(x,u)(\int_\Omega \Delta J(x-y)u(y)dy)^2dx).\end{aligned} \quad (3.103)$$

From condition (B_4), we have

$$\begin{aligned}\int_\Omega a(x,u)|\nabla u|^2dx &\le \int_\Omega (c_3|u|^r + c_4)|\nabla u|^2dx \\ &= \int_\Omega \left(\frac{2}{r+2}\right)^2 c_3|\nabla|u|^{\frac{r+2}{2}}|^2dx + \int_\Omega c_4|\nabla u|^2dx,\end{aligned} \quad (3.104)$$

$$\begin{aligned}\int_\Omega a(x,u)u^2dx &\le \int_\Omega (c_3|u|^r + c_4)u^2dx \\ &= \int_\Omega c_3|u|^{r+2}dx + \int_\Omega c_4u^2dx,\end{aligned} \quad (3.105)$$

and

$$\int_\Omega a(x,u)\Big(\int_\Omega (\Delta J)(x-y)u(y)dy\Big)^2 dx \le \int_\Omega (c_3|u|^r + c_4)\Big(\int_\Omega (\Delta J)(x-y)u(y)dy\Big)^2 dx$$
$$\le C\int_\Omega u(y)^2 dy \int_\Omega (c_3|u|^r + c_4)dx. \tag{3.106}$$

Note that Proposition 3.3.5 implies

$$\int_\Omega c_3|u(x,t)|^{r+2}dx + \int_\Omega c_4 u(x,t)^2 dx \le C \tag{3.107}$$

for $t \ge t_0$ and for some constant C which does not depend on initial data and

$$\int_\Omega u(y)^2 dy \int_\Omega (c_3|u|^r + c_4)dx \le C \tag{3.108}$$

for $t \ge t_0$ and for some constant C which does not depend on initial data.

Also, inequality (3.87) and Proposition 3.3.5 yield

$$\int_t^{t+1}\int_\Omega \left(\frac{2}{r+2}\right)^2 |\nabla |u|^{\frac{r+2}{2}}|^2 dx + \int_t^{t+1}\int_\Omega c_4|\nabla u|^2 dx \le C \tag{3.109}$$

for $t \ge t_0$ and for some constant C which does not depend on initial data,

and

$$\int_t^{t+1}\int_\Omega a^2(x,u)|\nabla u|^2 dxds \le \int_t^{t+1}\int_\Omega (c_3|u|^r + c_4)^2|\nabla u|^2 dxds$$
$$\le \int_t^{t+1}\int_\Omega c_5(|\nabla|u|^{r+1}|^2 + |\nabla u|^2)dxds \tag{3.110}$$
$$\le C$$

for $t \ge t_0$ and for some constant C which does not depend on initial data.

It follows from (3.103)-(3.108) that

$$\frac{d}{dt}\int_\Omega a^2(x,u)|\nabla u|^2 dx \le C_1\Big(\int_\Omega |\nabla|u|^{\frac{r+2}{2}}|^2 dx + \int_\Omega |\nabla u|^2 dx + C_2\Big) \tag{3.111}$$

for some constants C_1 and C_2 which do not depend on initial data and for $t \ge t_0$.

For $t_0 < t < s < t+1$, integrating inequality (3.111) between s and $t+1$, we obtain

$$\int_\Omega a^2(x,u(x,t+1))|\nabla u(x,t+1)|^2dx - \int_\Omega a^2(x,u(x,s))|\nabla u(x,s)|^2dx$$
$$\leq C_1(\int_s^{t+1}[\int_\Omega |\nabla|u(x,\mu)|^{\frac{r+2}{2}}|^2dx + \int_\Omega |\nabla u(x,\mu)|^2dx + C_2]d\mu). \tag{3.112}$$

Integrating (3.112) from t to $t+1$ with respect to s, we have

$$\int_\Omega a^2(x,u(x,t+1))|\nabla u(x,t+1)|^2dx \leq \int_t^{t+1}\int_\Omega a^2(x,u(x,s))|\nabla u(x,s)|^2dxds$$
$$+ C_1(\int_t^{t+1}\int_s^{t+1}[\int_\Omega |\nabla|u(x,\mu)|^{\frac{r+2}{2}}|^2dx + \int_\Omega |\nabla u(x,\mu)|^2dx + C_2]d\mu ds). \tag{3.113}$$

By (3.109), (3.110), (3.112) and Fubini Theorem, we obtain

$$\int_\Omega a^2(x,u(x,t+1))|\nabla u(x,t+1)|^2dx \leq C \tag{3.114}$$

for some constant C and for $t \geq t_0$.

Condition (B_3) and (3.114) yield

$$\int_\Omega |\nabla u(x,t+1)|^2dx \leq C \tag{3.115}$$

for $t \geq t_0$ and for some constant C which does not depend on initial data.

3.4 Existence of a global attractor

In this section, we prove that there exists a global attractor for weak solutions in some metric space for $n=1$.

Let H be a metric space, $S(t)$ $(t \geq 0)$ be a family of operators, which map H into itself and enjoy the usual semigroup properties

$$\begin{cases} S(t+s) = S(t)\cdot S(s) & \forall s,\, t \geq 0, \\ S(0) = I \text{ (Identity in } H), \end{cases} \tag{3.116}$$

$$S(t) \text{ is a continuous operator from } H \text{ into itself for all } t \geq 0. \tag{3.117}$$

The following lemma may be found in [38].

Lemma 3.4.1 *Assume that H is a metric space and that the operators $S(t)$ satisfy (3.116) and (3.117) and the following condition:*

For every bounded set $\mathcal{B}$ there exists t_0 which may depend on $\mathcal{B}$ such that $\cup_{t \geq t_0} S(t)\mathcal{B}$ is relatively compact in H.

Assume that there exists an open set $\mathcal{U}$ and a bounded subset $\mathcal{B}$ of $\mathcal{U}$ such that $\mathcal{B}$ is absorbing in $\mathcal{U}$. Then the ω-limit set of $\mathcal{B}$, $\mathcal{A} = \omega(\mathcal{B})$, is a compact attractor which attracts the bounded sets of $\mathcal{U}$ (for the inclusion relation). Furthermore, if H is a Banach space and $\mathcal{U}$ is convex and connected, then $\mathcal{A}$ is connected too.

Let $X = L^\infty(\Omega)$ with the metric from $L^1(\Omega)$. From Theorem 3.3.2 and Corollary 3.3.3, there exists a semigroup $S(t)$ associated with equation (3.1).

From Proposition 3.3.6, we have that $S(t)$ maps $L^\infty(\Omega)$ to $L^\infty(\Omega)$.

Since for $n = 1$, $H^1(\Omega) \hookrightarrow C^\mu(\bar{\Omega})$ is compact, from Theorem 3.3.7, we see that There exists an absorbing set in $H^1 \cap X$.

Therefore, by Lemma 3.4.1, we have the following theorem.

Theorem 3.4.2 *For $n = 1$, if conditions (B_1), $(\bar{B}_2)$, (B_3), and (B_4) are satisfied, then the semigroup associated with (3.1) possesses an attractor $\mathcal{A} \subset H^1(\Omega) \cap X$ which is maximal and compact.*

CHAPTER 4

The Cauchy problem and steady state solutions for a nonlocal Cahn-Hilliard equation

4.1 The Cauchy problem for a nonlocal Cahn-Hilliard equation

We consider the following equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta(\varphi(u) - J * u) \text{ in } \quad \mathbb{R}^n \times (0,T), \\ u(x,0) = u_0(x), \end{cases} \tag{4.1}$$

where $\varphi(u) = u + f(u)$, f is bistable (e.g. $f(u) = u(u^2 - 1)$), $*$ is convolution.

For $T > 0$, let $Q_T = \mathbb{R}^n \times (0,T)$. We make the following assumptions:

(C_1) $f \in C^{2+\beta}(\mathbb{R})$ and $\varphi'(u) \geq c$ for some positive constants c and β,

(C_2) $J \in C^{2+\beta}(\mathbb{R}^n)$, $\Delta J \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, and $\int_{\mathbb{R}^n} J = 1$.

First, we prove the uniqueness and continuous dependence of solutions on initial data. We have

Proposition 4.1.1 *Let u_i $(i = 1, 2)$ be two solutions of equation (4.1) with initial data u_{i0} $(i = 1, 2)$. If conditions (C_1) and (C_2) are satisfied, if $u_i \in C([0,T], L^1(\mathbb{R}^n)) \cap L^\infty(Q_T)$, and if $u_{i0} \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ $(i = 1, 2)$, then*

$$\sup_{0 \le t \le T} \int_{\mathbb{R}^n} |u_1 - u_2| dx \le C(T) \int_{\mathbb{R}^n} |u_{10} - u_{20}| dx \tag{4.2}$$

for some constant $C(T)$.

PROOF. For any $\tau \in (0, T)$, and $\psi \in C^{2,1}(Q_\tau)$, with $\psi = 0$ for $|x|$ large enough, after multiplying (4.1) by ψ, and integrating over $[0, \tau] \times \mathbb{R}^n$, we have

$$\int_{\mathbb{R}^n} u_i(x,\tau)\psi(x,\tau)dx = \int_{\mathbb{R}^n} u_i(x,0)\psi(x,0)dx + \int_0^\tau \int_{\mathbb{R}^n} (u_i\psi_t + \varphi(u_i)\Delta\psi)dxdt - \int_0^\tau \int_{\mathbb{R}^n} \psi\Delta J * u_i dxdt. \tag{4.3}$$

Set $z = u_1 - u_2$, $z_0 = u_{10} - u_{20}$, then (4.3) gives

$$\int_{\mathbb{R}^n} z(x,\tau)\psi(x,\tau)dx = \int_{\mathbb{R}^n} z_0(x)\psi(x,0)dx + \int_0^\tau \int_{\mathbb{R}^n} z(x,t)(\psi_t + b(x,t)\Delta\psi)dxdt - \int_0^\tau \int_{\mathbb{R}^n} \psi\Delta J * z(x,t)dxdt, \tag{4.4}$$

where

$$b(x,t) = \begin{cases} \dfrac{\varphi(u_1) - \varphi(u_2)}{u_1 - u_2} & \text{for} \quad u_1 \neq u_2, \\ \varphi'(u_1) & \text{for} \quad u_1 = u_2. \end{cases} \tag{4.5}$$

Let $g(x) \in C_0^\infty(\mathbb{R}^n)$ have compact support, $0 \le g(x) \le 1$, and take $\lambda > 0$. We will choose ψ, above, to satisfy certain conditions. First, consider the following final value problem on a large ball $B_R(0)$

$$\begin{cases} \dfrac{\partial\psi}{\partial t} = -b(x,t)\Delta\psi + \lambda\psi & \text{for} \quad |x| < R,\ 0 < t < \tau, \\ \psi = 0 & \text{on} \quad |x| = R,\ 0 < t < \tau, \\ \psi(x,\tau) = g(x) & \qquad |x| \le R. \end{cases} \tag{4.6}$$

There exists a unique solution of (4.6) $\psi \in C^{2,1}(B_R(0) \times (0,\tau))$ which satisfies the following properties:

$$0 \le \psi \le e^{\lambda(t-\tau)}, \tag{4.7}$$

$$\int_0^\tau \int_{B_R(0)} b(x,t)|\Delta\psi|^2 dxdt \le C, \tag{4.8}$$

$$\sup_{0\le t\le \tau} \int_{B_R(0)} |\nabla\psi|^2 dx \le C, \tag{4.9}$$

where the constant C only depends on g.

In order to extend ψ to be zero outside of $B_R(0)$, we define $\xi_R \in C_0^\infty(\mathbb{R}^n)$ such that

$$\begin{cases} 0 \le \xi_R \le 1, \\ \xi_R = 1 \quad if \ |x| < R-1, \\ \xi_R = 0 \quad if \ |x| > R - \dfrac{1}{2}, \\ |\nabla\xi_R(x)|, \ |\Delta\xi_R(x)| \le C \end{cases} \tag{4.10}$$

for some constant C which does not depend on R.

Let $\gamma = \xi_R\psi$, where ψ satisfies (4.6) in $B_R(0)$ and is zero outside. Using γ instead of ψ in (4.4), we have

$$\begin{aligned} &\int_{\mathbb{R}^n} z(x,\tau)g\xi_R dx - \int_{\mathbb{R}^n} \xi_R(x)z_0(x)\psi(x,0)dx + \int\int_{Q_\tau} (\Delta J * z - \lambda z)\xi_R\psi dxdt \\ &= \int\int_{Q_\tau} b(x,t)z(x,t)(2\nabla\xi_R \cdot \nabla\psi + \psi\Delta\xi_R)dxdt \\ &\equiv G(z,R). \end{aligned} \tag{4.11}$$

Since u_1 and u_2 belong to $L^\infty(Q_T)$, and since b is positive, from estimates (4.7)-(4.9) and (4.10), we have

$$\begin{aligned} |G(z,R)| &\le \int_0^\tau \int_{B_R\setminus B_{R-1}} (b|u_1 - u_2|((2|\nabla\xi_R||\nabla\psi| + |\psi||\Delta\xi_R|)) \\ &\le C\int_0^\tau \int_{B_R\setminus B_{R-1}} b(|u_1| + |u_2|)(|\nabla\psi| + 1)dxdt \\ &\le C\int_0^\tau \int_{B_R\setminus B_{R-1}} (|u_1| + |u_2|)dxdt. \end{aligned} \tag{4.12}$$

Since u_1 and u_2 belong to $L^1(Q_T)$, letting $R \to \infty$ we have $G(z,R) \to 0$.

This implies

$$\int_{\mathbb{R}^n} z(x,\tau)g(x)dx \leq \int_{\mathbb{R}^n} |z_0(x)|e^{-\lambda\tau}dx + \int_0^\tau \int_{\mathbb{R}^n} (|\triangle J * z - \lambda z|e^{\lambda(t-\tau)}dxdt. \quad (4.13)$$

Letting $\lambda \to 0$ and $g(x) \to \operatorname{sign} z^+(x,\tau)$, we obtain

$$\int_{\mathbb{R}^n} (u_1 - u_2)^+ dx \leq \int_{\mathbb{R}^n} |u_{10} - u_{20}|dx + C\int_0^\tau \int_{\mathbb{R}^n} |u_1 - u_2|dxdt. \quad (4.14)$$

Interchanging u_1 and u_2 yields

$$\int_{\mathbb{R}^n} |u_1 - u_2|dx \leq \int_{\mathbb{R}^n} |u_{10} - u_{20}|dx + C\int_0^\tau \int_{\mathbb{R}^n} |u_1 - u_2|dxdt. \quad (4.15)$$

Inequality (4.2) follows from (4.15) and Gronwall's inequality.

Next we prove the existence of a solution to equation (4.1).

Theorem 4.1.2 *For any $T > 0$, if $u_0(x) \in C_0^{2+\beta}(\mathbb{R}^n)$, and if φ and J satisfy assumptions (C_1) and (C_2), then there exists a unique solution of (4.1) which belongs to $C^{2+\beta, \frac{2+\beta}{2}}(Q_T) \cap L^1(Q_T) \cap L^\infty(Q_T)$.*

PROOF. Since $u_0(x) = 0$ for $|x|$ large enough, we consider

$$\begin{cases} \dfrac{\partial u}{\partial t} = \triangle(\varphi(u) - J * u) & \text{in} \quad B_R(0) \times (0,T), \\ u(x,t) = 0 & \text{on} \quad \partial B_R(0) \times (0,T), \\ u(x,0) = u_0(x). \end{cases} \quad (4.16)$$

From Theorem 3.1.5 in Chapter 3, there exists a unique solution of (4.16) $u(x,t) \in C^{2+\beta, \frac{2+\beta}{2}}(B_R(0) \times (0,T))$.

Let $u(x,t) = ve^t$ in (4.16), then we have

$$e^t v_t + ve^t = \varphi'(u)e^t \triangle v + \varphi''(u)|\nabla v|^2 e^{2t} - e^t \triangle J * v. \quad (4.17)$$

Multiplying (4.17) by v and using $v\Delta v = \frac{1}{2}\Delta v^2 - |\nabla v|^2$, we obtain

$$\frac{1}{2}(v^2)_t + v^2 = \frac{1}{2}\varphi'(u)\Delta v^2 + \frac{1}{2}\varphi''(u)\nabla v \cdot \nabla v^2 e^t - \varphi'(u)|\nabla v|^2 - v\Delta(J * v). \quad (4.18)$$

If there exists $(P_0, t_0) \in B_R(0) \times (0, T]$ such that $v^2(P_0, t_0) = \max v^2$, then $\Delta v^2(P_0, t_0) \leq 0$, $\nabla v^2(P_0, t_0) = 0$, $\nabla v(P_0, t_0) = 0$, $v_t^2(P_0, t_0) \geq 0$, and (4.18) yields

$$v^2(P_0, t_0) \leq -\int_{B_R} \Delta J(P_0 - y)v(y, t_0)dy v(P_0, t_0). \quad (4.19)$$

This yields

$$\max |v| \leq M \int_{B_R} |v(y, t_0)|dy \quad (4.20)$$

for some constant M which does not depend on R.

Since $u = 0$ is also a solution of (4.16) with initial data $u_0 = 0$, by Proposition 3.2.2 in Chapter 3, we have

$$\int_{B_R} |u(x, t) - 0|dx \leq C(T) \int_{B_R} |u_0 - 0|dx \quad (4.21)$$

for some constant $C(T)$ which does not depend on R.

Inequalities (4.20) and (4.21) imply

$$\max |v| \leq C(T) \int_{B_R} |u_0|dx. \quad (4.22)$$

Since $u_0 \in L^1(\mathbb{R}^n)$, we have

$$\max |v| \leq B(T) \quad (4.23)$$

for some constant $B(T)$ which does not depend on R.

This yields

$$\max |u| \leq B(T)e^T \quad (4.24)$$

for some constant $B(T)$ which does not depend on R.

Now we have proved the solution of (4.16) is uniformly bounded, i.e.,

$$\max_{B_R\times[0,T]} |u(x,t)| \leq C$$

for any $R > 0$, where C does not depend on R.

A similar argument to that in the proof of Theorem 3.1.3 in Chapter 3 yields

$$||u_R||_{2+\beta} \leq C(K,T) \tag{4.25}$$

for any $R > K \equiv$ constant, where u_R is a solution of (4.16) in $B_R \times (0,T)$ and $C(K,T)$ is a constant which does not depend on R ($||\cdot||_{2+\beta}$ is a *Hölder* norm defined in [30]).

By employing the usual diagonal process, we can choose a sequence $\{R_i\}$ such that u_{R_i}, Du_{R_i}, and $D^2u_{R_i}$ converge to u, Du, and D^2u pointwise, and u satisfies equation (4.1). From (4.21) and (4.24), we also have $u \in L^1(Q_T) \cap L^\infty(Q_T)$.

Uniqueness follows from Proposition 4.1.1.

4.2 Steady state solutions for a nonlocal Cahn-Hilliard equation

In this section, we consider the following equation:

$$\begin{cases} \displaystyle\int_\Omega J(x-y)dyu(x) - \int_\Omega J(x-y)u(y)dy + f(u) = C \ \text{ in } \Omega, \\ \displaystyle\int_\Omega u(x)dx = 0, \end{cases} \tag{4.26}$$

where Ω is a bounded domain, C is a constant. The case when $\Omega = \mathbb{R}$ or $\mathbb{R}^n$ has been treated by others (see [10], [12], [16], [17] and references therein).

Proposition 4.2.1 *Suppose $\Omega \subset \mathbb{R}^n$ is a closed and bounded set, $J(x) \geq 0$ and is continuous on $\mathbb{R}^n$, supp $\supset B_\delta(0)$ for some positive constant δ, and f is nondecreasing. Then the only continuous solution of equation (4.26) is zero.*

PROOF. Without loss of generality, we assume that $f(0) = 0$. If $f(0) \neq 0$, we may use $f(u) - f(0)$ instead of $f(u)$ in (4.26).

Case 1: $C \leq 0$ in equation (4.26).

If the conclusion is not true, since $\int u dx = 0$, and u is continuous on Ω, there exists $P_0 \in \Omega$ such that $u(P_0) = \max u(x) > 0$.

Let $A = \{y \in \Omega | u(y) = \max u(x)\}$.

We claim: There exist $P_0 \in \partial A$ and $r > 0$ such that $K := (\Omega \setminus A) \cap B_r(P_0)$ has positive measure. If this is not true, we have $meas(\Omega \setminus A) = 0$. This and $u(x) = \max u$ on A imply $\int_\Omega u = \int_A u > 0$. This contradicts $\int_\Omega u = 0$.

Since $SuppJ \supset B_\delta(0)$ implies $SuppJ(P_0 - \cdot) \supset B_\delta(P_0)$, choosing $r_1 = \min\{\delta, r\}$ gives

$$meas(K \cap B_{r_1}(P_0)) > 0, \tag{4.27}$$

$$J(P_0 - y) > 0 \text{ on } K \cap B_{r_1}(P_0), \tag{4.28}$$

and

$$u(P_0) - u(y) > 0 \text{ on } K \cap B_{r_1}(P_0). \tag{4.29}$$

Inequalities (4.27)-(4.29) imply

$$\int_\Omega J(P_0 - y)(u(P_0) - u(y))dy \geq \int_{K \cap B_{r_1}(P_0)} J(P_0 - y)(u(P_0) - u(y))dy > 0. \tag{4.30}$$

This and $f(u(P_0)) \geq 0$ imply

$$\int_\Omega J(P_0 - y)u(P_0)dy - \int_\Omega J(P_0 - y)u(y)dy + f(u(P_0)) > 0, \tag{4.31}$$

contradicting (4.26).

Case 2: $C > 0$ in (4.26).

In this case, taking P_0 such that $u(P_0) = \min u < 0$ leads to a contradiction in a similar way.

If $f'(u)$ changes sign, we make the following assumptions:

(E_1) $\Omega = (-1,1)$ if $\dim \Omega = 1$, $\Omega = (-1,1) \times \Omega'$ if $\dim \Omega > 1$.

(E_2) $J(x) = J(|x|)$, $J(x) \geq 0$, and

$$M \geq \sup_{x \in \Omega} \int_\Omega J(x-y)dy \geq \inf_{x \in \Omega} \int_\Omega J(x-y)dy \geq m > 0$$

for positive constants M and m.

(E_3) $f \in C^1(\mathbb{R})$, f is odd, $f(1) = 0$, there exist $\delta > 0$ and $a \in (0,1)$ such that $f'(x) \geq \delta$ on $[a, \infty)$, and $f(-a) \geq (1+a)M$.

(E_4) $C = 0$ in (4.26).

Remark 4.2.2 Condition (E_3) implies that $f(-1) = 0$, $f'(u) \geq \delta$ on $(-\infty, -a]$, and $-f(a) \geq (1+a)M$.

Let $j(x) = \int_\Omega J(x-y)dy$. From (E_2), we have

$$m \leq j(x) \leq M. \tag{4.32}$$

Dividing equation (4.26) by $j(x)$, we consider

$$\begin{cases} u(x) - \dfrac{1}{j(x)} \displaystyle\int_\Omega J(x-y)u(y)dy + \dfrac{f(u(x))}{j(x)} = 0, \\ \displaystyle\int_\Omega u(x)dx = 0. \end{cases} \tag{4.33}$$

Theorem 4.2.3 *If assumptions $(E_1)-(E_4)$ are satisfied, then there exists a solution of equation (4.33) such that*

$$u(x) \begin{cases} \geq a \textit{ for } x \in M_1 \equiv (0,1) \times \Omega', \\ \leq -a \textit{ for } x \in M_2 \equiv (-1,0) \times \Omega' . \end{cases} \tag{4.34}$$

Moreover, we have

$$-1 \leq u(x) \leq 1. \tag{4.35}$$

PROOF. Following the idea in [10], let

$$B = \{u \in L^\infty(\Omega) | \, u(-x_1, x') = -u(x_1, x'), \; u(x) \in [a, 1] \text{ for } x \in M_1\}.$$

The definition of B implies that $u(x) \in [-1, -a]$ for $x \in M_2$.

Define

$$Tu(x) = u(x) + h[\frac{1}{j(x)} \int_\Omega J(x-y)u(y)dy - u(x) - \frac{1}{j(x)} f(u(x))].$$

We want to show $T : B \to B$ is a contraction map if h is small enough.

In fact, since $j(x) = \int_\Omega J(x-y)dy$, with assumption (E_2), we have $j(-x_1, x') = j(x_1, x')$. And if $u(x) \in B$, we have:

$$\begin{aligned}
&T(u(-x_1, x')) = u(-x_1, x') + \frac{h}{j(-x_1, x')} \int_{-1}^{1} \int_{\Omega'} J(-x_1 - y_1, x' - y')u(y_1, y')dy_1 dy' \\
&- hu(-x_1, x') + \frac{h}{j(-x_1, x')} f(u(-x_1, x')) \\
&= -u(x_1, x') - \frac{h}{j(x_1, x')} \int_{-1}^{1} \int_{\Omega'} J(-x_1 + z_1, x' - y')u(z_1, y')dz_1 dy' \\
&+ hu(x_1, x') - \frac{h}{j(x_1, x')} f(u(x_1, x')) \\
&= -u(x_1, x') - \frac{h}{j(x_1, x')} \int_{-1}^{1} \int_{\Omega'} J(x_1 - z_1, x' - y')u(z_1, y')dz_1 dy' \\
&+ hu(x_1, x') - \frac{h}{j(x_1, x')} f(u(x_1, x')) \\
&= -(u(x_1, x') + \frac{h}{j(x_1, x')} \int_{-1}^{1} \int_{\Omega'} J(x_1 - z_1, x' - y')u(z_1, y')dz_1 dy' \\
&- hu(x_1, x') + \frac{h}{j(x_1, x')} f(u(x_1, x'))) \\
&= -T(u(x_1, x')).
\end{aligned} \tag{4.36}$$

Choose h small enough such that

$$h \frac{1}{j(x)} f'(u) < 1 - h \tag{4.37}$$

for $u \in [-1, -a] \cup [a, 1]$ and $x \in \Omega$.

This implies that $u - h[u + \frac{1}{j(x)} f(u)]$ is increasing in u on $[a, 1]$. Since $u(y) \geq a$ for $y \in M_1$, and $u(y) \geq -1$ for $y \in M_2$, we have for $x \in M_1$

$$\begin{aligned}
Tu(x) &= h\frac{1}{j(x)}\int_\Omega J(x-y)u(y)dy + u - h[u + \frac{1}{j(x)} f(u)] \\
&\geq h\frac{1}{j(x)}\int_\Omega J(x-y)u(y)dy + a - ha - h\frac{1}{j(x)} f(a) \\
&= h\frac{1}{j(x)}\int_{M_1} J(x-y)u(y)dy + h\frac{1}{j(x)}\int_{M_2} J(x-y)u(y)dy + a - ha - h\frac{1}{j(x)} f(a) \\
&\geq ha\frac{1}{j(x)}\int_{M_1} J(x-y)dy - h\frac{1}{j(x)}\int_{M_2} J(x-y)dy + a - ha - h\frac{1}{j(x)} f(a) \\
&= a - ha\frac{1}{j(x)}\int_{M_2} J(x-y)dy - h\frac{1}{j(x)}\int_{M_2} J(x-y)dy - \frac{h}{j(x)} f(a) \\
&\geq a - \frac{h}{j(x)}[(1+a)\int_{M_2} J(x-y)dy + f(a)] \\
&\geq a
\end{aligned} \tag{4.38}$$

by (E_3).

Also,

$$\begin{aligned}
Tu(x) &= h\frac{1}{j(x)}\int_\Omega J(x-y)u(y)dy + u - h[u + \frac{1}{j(x)} f(u)] \\
&\leq h\frac{1}{j(x)}\int_\Omega J(x-y)u(y)dy + 1 - h - h\frac{1}{j(x)} f(1) \\
&\leq 1
\end{aligned} \tag{4.39}$$

for $x \in M_1$.

Estimates (4.36)-(4.39) imply that T maps B to B.

For $u, v \in B$, choosing h small enough so that $0 < 1 - h(1 + \delta\frac{1}{M}) < 1$, we have

$$
\begin{aligned}
||Tu - Tv||_\infty =& ||(u-v) + \frac{h}{j(x)}\int_\Omega J(x-y)(u(y)-v(y))dy \\
& - h(u(x)-v(x)) - \frac{h}{j(x)}(f(u)-f(v))||_\infty \\
=& ||(1 - h - \frac{hf'(\theta u + (1-\theta)v)}{j(x)})(u-v) + \frac{h}{j(x)}\int_\Omega J(x-y)(u(y)-v(y))dy||_\infty \\
\le& (1 - h(1 + \delta\frac{1}{M}))||u-v||_\infty + h||(u-v)||_\infty \\
\le& (1 - h\delta\frac{1}{M})||u-v||_\infty,
\end{aligned}
\tag{4.40}
$$

where $\theta(x) \in (0,1)$ for all $x \in \Omega$. Here we used (E_3) and the fact that for any $x \in \Omega$ either $u(x)$, $v(x) \ge a$ or $u(x)$, $v(x) \le -a$.

Therefore, T is a contraction map from B to B and so there exists a unique point $u \in B$ such that $Tu = u$. Estimates (4.35) follows from the definition of B.

Remark 4.2.4 If we just consider the solution to

$$
\int_\Omega J(x-y)u(x)dy - \int_\Omega J(x-y)u(y)dy + f(u) = 0 \quad \text{in } \Omega \tag{4.41}
$$

without the condition $\int_\Omega u dx = 0$, then the conditions that f is odd and $J(x) = J(|x|)$ are not necessary. In this case, we can use a similar method to that in [10] to prove the existence of a discontinuous solution under conditions (E_2), $(E_3)'$, and (E_4), where

$(E_3)'$: $f \in C^1(R)$, $f(-1) = f(c) = f(1) = 0$ for $c \in (-1,1)$, there exist $\delta > 0$, $a \in (0,1)$, $b \in (-1,0)$ such that $f'(x) \ge \delta$ on $[a,\infty) \cup (-\infty, b)$, $f(a) \le -(1+a)M$, and $f(b) \ge (1+b)M$, where M is defined in (E_2).

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