This is to certify that the
dissertation entitled

Local Regularization for the Autoconvolution Problem

presented by

Zhewei Dai

has been accepted towards fulfillment
of the requirements for the

Ph.D. degree in Applied Mathematics

Patricia K Lamm

Major Professor's Signature

7/13/05

Date

PLACE IN RETURN BOX to remove this checkout from your record.
TO AVOID FINES return on or before date due.
MAY BE RECALLED with earlier due date if requested.

DATE DUE	DATE DUE	DATE DUE

Local Regularization for the Autoconvolution Problem

By

Zhewei Dai

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

2005

ABSTRACT

Local Regularization for the Autoconvolution Problem

By

Zhewei Dai

We develop a local regularization theory for the nonlinear autoconvolution problem. Unlike the classic regularization techniques such as Tikhonov regularization, this theory provides regularization methods that preserve the causal nature of the autoconvolution problem, allowing for fast sequential numerical solution. We prove the convergence of the regularized solutions to the true solution as the noise level in the data shrinks to zero, with a certain convergence rate. We propose several regularization methods and provide theoretic basis for their convergence. Our numerical results confirm effectiveness of the methods, suggesting superiority of our methods over the existing ones, especially in recovering sharp features in the solution.

To Michael

ACKNOWLEDGMENTS

First and foremost, I would like to thank my advisor and mentor, Professor Patricia Lamm. Thank you for your constant encouragement, your generosity with your time, your tireless proofreading and commenting on this thesis and your contagious enthusiasm about mathematical research. The great deal I have learned from you are all very invaluable, and I simply could not have asked for a better advisor.

I would also like to thank other members of my committee, Professor Michael Frazier, Chang Yi Wang, Baisheng Yan and Zhengfang Zhou. They all have been my teachers at some point of my graduate life, and have impressed me with their dedication to mathematical research and education.

I could not have made this far without the love and support of my family and friends. While there are too many to name, I would like to express my deepest appreciation for my parents, for raising me into what I am today, for always trusting me in every endeavor I pursue in life.

Finally, thanks to Michael Hamlin, my husband and best friend, who has been through this all. Thank you for helping with the numerical work of this thesis, thank you for your love, your patience, your confidence in me and in us.

TABLE OF CONTENTS

Introduction

Linear and nonlinear Volterra integral equations arise in various applications, for example, in recovering a space curve from its curvature and torsion, in the theory of industrial inventory problems, and in nuclear reactor kinetics [23].

In this paper, we will study the solution of a nonlinear Volterra problem. We consider the autoconvolution problem of finding $x \in L_2(0,T)$ solving

$$G(x)(t) = f(t), \quad \text{a.e. t} \in [0, \text{T}], \tag{1}$$

where G is the nonlinear Volterra operator given by

$$G(x)(t) = \int_0^t x(t-s)\, x(s)\, ds, \quad \text{a.e. t} \in [0, \text{T}], \tag{2}$$

and $f \in Range(G) \subseteq L_2(0,T)$.

Before turning to this problem, we first give some background information on the linear counterpart to this problem. Let us consider a *linear* first-kind Volterra integral equation of the form

$$\mathcal{A}u = f, \tag{3}$$

where $\mathcal{A}$ is a bounded linear operator on $L_2(0,T)$ defined by

$$\mathcal{A}u(t) = \int_0^t k(t-s)u(s)\, ds, \quad \text{a.e. t} \in [0, \text{T}], \tag{4}$$

with the kernel function $k \in L_2(0,T)$. Here, f is in the range of $\mathcal{A}$ and our objective is to find $\bar{u} \in L_2(0,T)$ that satisfies equation (4).

A classic example of such equation is the Inverse Heat Conduction Problem (IHCP). If we apply heat at the end of a semi-infinite bar where we call location $x = 0$, and measure the temperature $f(t)$ as a function of time t somewhere away from the heat source at location $x = 1$, then the problem of recovering the temperature $u(t)$ at the heat source is to solve equation (4), with the kernel given by

$$k(t) = \frac{1}{2\sqrt{\pi}\, t^{3/2}} \exp(-\frac{1}{4t}). \tag{5}$$

Unfortunately, such problems are ill-posed because solutions do not have a continuous dependence on data: very small errors in the measured data can lead to large errors in the solution. Since the available data in practice always contain uncertainty, regularization methods should be employed to stabilize the problem.

One of the most well-known regularization methods is Tikhonov regularization. Instead of solving for u that satisfies $\mathcal{A}u = f^\delta$, we solve the following constrained minimization problem

$$\min \; ||\mathcal{A}u - f^\delta||^2 + \alpha||Lu||^2, \tag{6}$$

where f^δ is the measured noisy data, and $\alpha > 0$ is the regularization parameter. Here L is a closed linear operator often picked as either the identity operator or the derivative operator. The role of L then is to penalize highly oscillatory solutions, thus stabilize the problem.

The Tikhonov theory states that there is a choice of $\alpha = \alpha(\delta)$ such that as the noise level $\delta \to 0$,

- $\alpha(\delta) \to 0$,
- and the corresponding Tikhonov solution u_δ^α to (6) converges to the true $\bar{u}$.

Classical regularization methods such as Tikhonov regularization have inherent disadvantages in solving Volterra problems. Volterra problems are causal in the sense that the solution u at a given time t does not affect the data f on the interval $[0, t)$. Therefore, it makes sense to use future data f on the interval $[t, T]$ only in reconstructing $u(t)$. Tikhonov regularization however replaces the original causal problem by a "full domain" problem. It uses all values of data in the whole domain $[0, T]$ in reconstructing the solution u at any given time t. This becomes apparent when we consider the necessary condition for the minimization problem (6):

$$(\mathcal{A}^{\star}\mathcal{A} + \alpha L^{\star}L)u = \mathcal{A}^{\star}f^{\delta},$$

where $\mathcal{A}^{\star}\mathcal{A}$ is a non-causal operator even with a Volterra operator $\mathcal{A}$.

In the late 1960's, J. V. Beck developed a regularization scheme for the discretized IHCP which retains the causal nature of the problem [2]. The numerical implementation of the Beck method is also more efficient than those of classical regularization methods, because of its sequential nature. It was not until mid-1990's that P. K. Lamm established the theoretical basis for the convergence of the sequential local regularization method, and Beck's approach was generalized to a wide class of linear first-kind Volterra problems [13][14][15]. While Beck's method was an approach developed to handle a finite dimensional problem, the current theory of local regularization methods can be placed in both finite and infinite dimensional settings.

To motivate the sequential local regularization method for linear Volterra problems, we let $\bar{R} > 0$ be a small fixed constant and assume that equation (4) holds on an extended interval $[0, T + R]$ for $0 < R < \bar{R}$. Then $\bar{u}(t)$ solves

$$\int_0^{t+\rho} k(t + \rho - s)\, u(s)\, ds = f(t + \rho), \qquad t \in [0, T],\ \rho \in [0, R].$$

Split the integral at t, then change the variable of integration, we get

$$\int_0^t k(t+\rho-s)\,u(s)\,ds + \int_0^\rho k(\rho-s)\,u(t+s)\,ds = f(t+\rho) \quad t\in[0,T], \rho\in[0,R].$$

We integrate both sides of the equation with respect to a suitable Borel measure $\eta = \eta_R(\rho)$ (which we will clarify later) on $[0,R]$, then

$$\int_0^t \int_0^R k(t+\rho-s)\,d\eta_R(\rho)\,u(s)\,ds + \int_0^R \int_0^\rho k(\rho-s)\,u(t+s)\,ds\,d\eta_R(\rho) \\ = \int_0^R f(t+\rho)\,d\eta_R(\rho), \quad t\in[0,T]. \quad (7)$$

Note that we still have an equation that $\bar{u}$ satisfies exactly.

In reality, we often only have access to some perturbed $f^\delta \in C[0, T+\bar{R}]$ such that

$$\|f^\delta - f\|_\infty \le \delta \qquad \text{for some} \quad \delta > 0. \quad (8)$$

Some regularization method needs to be employed.

We motivate the regularization method by considering what would happen if, for fixed t, we momentarily held u constant on a small local interval $[t, t+R]$. This motivates us to replace $u(t+s)$ by $u(t)$ in the second term of equation (7). Here, the length R of the local interval becomes the regularization parameter. We then obtain the regularized equation in u valid for all $t\in[0,T]$,

$$\alpha_R\,u(t) + \int_0^t \tilde{k}_R(t-s)\,u(s)\,ds = \tilde{f}_R^\delta(t), \quad (9)$$

where

$$\tilde{k}_R(t) = \int_0^R k(t+\rho)\, d\eta_R(\rho), \tag{10}$$

$$\tilde{f}_R^\delta(t) = \int_0^R f^\delta(t+\rho)\, d\eta_R(\rho), \tag{11}$$

$$\alpha_R = \int_0^R \int_0^\rho k(\rho - s)\, ds\, d\eta_R(\rho). \tag{12}$$

The existing theory for the local regularization of linear Volterra problems requires the assumption that the convolution kernel k in equation (4) is ν-smoothing, i.e.,

$$k \in C^\nu[0,T] \quad \text{such that} \quad k(0) = k'(0) = \cdots = k^{\nu-2}(0) = 0 \quad \text{and} \quad k^{\nu-1}(0) \neq 0.$$

It is well-known that the degree of ill-posedness of problem (4) is characterized by the the degree of smoothness of the kernel k and the behavior of k at 0: the ill-posedness increases as ν increases. For example, the Inverse Heat Conduction Problem is severely ill-posed since the heat kernel (5) is infinitely smoothing, i.e., $k^{(p)}(0) = 0$ for $p = 0, 1, \cdots$.

Let the Borel measure $\eta_R(\rho)$ on $[0, R]$ satisfy the following three conditions:

1. For $i = 0, 1, \ldots, \nu$, there is some $\mu \in \mathbb{R}$ and $c_i = c_i(\mu) \in \mathbb{R}$ independent of R such that

$$\int_0^R \rho^i\, d\eta_R(\rho) = R^{i+\mu}\left(c_i + \mathcal{O}(R)\right), \quad \text{as} \quad R \to 0, \tag{13}$$

   with $c_\nu \neq 0$.

2. The parameters c_i, $i = 0, 1, \ldots, \nu$, satisfy the condition that all roots of the polynomial $p_\nu(\lambda)$ defined by

$$p_\nu(\lambda) = \frac{c_\nu}{\nu\,!}\,\lambda^\nu + \frac{c_{\nu-1}}{(\nu-1)\,!}\,\lambda^{\nu-1} + \ldots + \frac{c_1}{1\,!}\,\lambda + \frac{c_0}{0\,!}$$

have negative real parts.

3. There exists a $\tilde{C} \geq 0$ independent of R such that

$$\left| \int_0^R g(\rho)\, d\eta_R(\rho) \right| \leq \tilde{C}\, \|g\|_\infty R^\mu,$$

for all $g \in C[0, R]$ and all $R > 0$ sufficiently small.

It is worth noting that the Borel measures η_R satisfying these conditions are not necessarily positive. Therefore, a signed Borel measure is allowed. It was shown in [18] that under the above three conditions on $\eta_R(\rho)$, we have $\alpha_R \neq 0$ for all $R > 0$ sufficiently small and all ν-smoothing k, where $\nu = 1, 2, 3, \ldots$. Therefore (9) is a well-posed second-kind Volterra equation, with solutions depending continuously on data f^δ. We summarize the convergence theory loosely in the following Theorem, for a more precise version and information about convergence rate, we refer to [18].

Theorem 0.1 *Let $\bar{u}$ denote the solution of equation (4) given 'true' data $f \in C[0, T + \bar{R}]$. Assume k is ν-smoothing and that η_R is a family of signed Borel measures satisfying hypotheses (1)-(3) for all $R \in [0, \bar{R}]$. Then for all $t \in [0, T]$ and $f^\delta \in C[0, T + \bar{R}]$ satisfying (8), there is a choice of $R = R(\delta)$ such that $R(\delta) \to 0$ as $\delta \to 0$ and*

$$u_R^\delta(t) \to \bar{u}(t) \quad as \quad \delta \to 0,$$

where u_R^δ is the solution to the regularized equation (9).

Even though the full convergence of the sequential local regularization method for linear first-kind Volterra equations is obtained by allowing *signed* Borel measures, we would like to point out that convergence was also obtained using *positive* Borel measures for ν-smoothing Volterra problems, where $\nu = 1, 2, 3, 4$. There is to date no convergence theory for positive Borel measures except in those cases. For details, see [13], [15] and [25].

Different variations of the local regularization methods for linear ill-posed problems were developed over the last decade [3][4][5][14][16][17][20][21][22]. For example, in [3] and [4], the motivation for the regularization equation is to consider using a polynomial function to fit the data on the local interval instead of a constant function. In [16], [21] and [22], a variable regularization parameter $R(t)$ is used instead of a single constant R. This technique allows for more or less smoothing at different parts of the domain, and it enforces another advantage of local regularization over Tikhonov regularization in the ability of recovering sharp features of the solution. The methods were even extended to Fredholm (non-Volterra) problems in [5] and [17], where the solution is obtained by iteratively solving many small localized problems.

While the theory for the local regularization methods of linear Volterra problems is rather complete, the nonlinear theory is largely absent. It was only recently that local regularization theory was extended to the nonlinear Hammerstein equation:

$$\int_0^t k(t,s)g(u(s),s)ds = f(t) \quad \text{for } t \in [0,T].$$

Notice the Hammerstein equation composes the desired solution u with another arbitrary function g. Based on the linear convergence theory, the key issue here becomes how we can stably recover u from inverting the function g. We have shown in [19] that local regularization of this problem is successful under certain conditions on this external function g. It is not surprising that the proof utilizes the linear theory.

In this paper, we develop a local regularization theory for the nonlinear autoconvolution problem of finding $x \in L_2(0,1)$ solving equation (1) with G given by (2). We will use the underlying ideas of local regularization to formulate the regularization equation for the autoconvolution problem. However, due to its nonlinearity, we expect the convergence theory to be fundamentally different from what is in the linear case.

CHAPTER 1

Properties of the Autoconvolution Equation

Autoconvolution has been of interest to scientists for decades because of its applications in various fields. It arises in stochastics where the density function of a continuous random variable V is reconstructed after observing the density function of the random variable $S = V_1 + V_2$, where V_1 and V_2 are identically and independently distributed random variables of V. For more details, see [10]. Another application of autoconvolution occurs in spectroscopy. Baumeister presents in [1] a reference list of physically motivated papers concerning this class of problems. He also discusses in detail the mathematical model of deconvolution of "appearance potential"(AP) spectra to investigate electronic properties of solids in their surface region. In this context, the density of unoccupied states in the surface region of a solid is recovered from the measured AP-spectrum data.

In this chapter, we will summarize the properties of the autoconvolution equation. For details of these results, we refer to [7] [8] [9] [10]. We will also mention some existing regularization methods towards the end the chapter.

1.1 Continuity and Compactness

In the linear case, the inverse problem (3) is ill-posed when the operator $\mathcal{A}$ is compact. As we will show in this section, the autoconvolution operator is continuous in $L_2(0,1)$, but fails to be compact in $L_2(0,1)$.

Recall the autoconvolution operator $G : L_2(0,1) \to L_2(0,1)$ defined as

$$G(x)(t) = \int_0^t x(t-s)\,x(s)\,ds = f(t), \quad \text{a.e. } t \in [0,1]. \tag{1.1}$$

More generally, $G : D(G) \subseteq B_1 \to B_2$ where B_1 and B_2 are Banach spaces containing real functions on $[0,1]$, with properties to be specified below.

Proposition 1.1 *[10] If $x \in L_2(0,1)$, then $G(x) \in C[0,1]$ and $[G(x)](0) = 0$. Moreover, $G : L_2(0,1) \to C[0,1]$ is a continuous nonlinear integral operator with*

$$\|G(x)\|_{L_2(0,1)} \le \|G(x)\|_{C[0,1]} \le \|x\|^2_{L_2(0,1)}.$$

Note that the continuity of $G : D(G) \subseteq B_1 \to B_2$ remains true if B_1 has a stronger norm than $L_2(0,1)$ or if the norm of B_2 is weaker compared to that of $C[0,1]$.

Definition 1.1 *We call a linear or nonlinear operator $A : D(A) \subseteq B_1 \to B_2$* compact *if the range $R_A(S) \equiv \{y \in B_2, y = A(x), x \in S\}$ is a relatively compact subset of B_2 whenever S is a B_1-bounded subset of $D(A)$.*

Proposition 1.2 *[10] The autoconvolution operator $G : L_2(0,1) \to L_2(0,1)$ is not a compact operator. On the other hand, the Fréchet derivative $G'(x) : L_2(0,1) \to L_2(0,1)$ defined by*

$$G'(x)(h)(t) = 2\int_0^t x(t-s)h(s)\,ds, \qquad t \in [0,1], \quad h \in L_2(0,1),$$

is a compact bounded linear operator for all $x \in L_2(0,1)$.

As an example of noncompactness of G, we consider an infinite sequence

$$x_n(t) = \sin(nt), \quad 0 \leq t \leq 1, \, n = 1, 2, \ldots,$$

then,

$$y_n(t) = [G(x_n)](t) = -\frac{t\cos(nt)}{2} + \frac{\sin(nt)}{2n},$$

for $0 \leq t \leq 1$ and $n = 1, 2, \cdots$. Note that $x_0(t) = 0$ and $y_0(t) = [G(x_0)](t) = 0$ for $t \in [0, 1]$. Evidently, the sequence $\{x_n\}$ is bounded in $L_2(0,1)$, but we don't have strong convergence of $y_n \to y_0 = 0$ since $\|y_n\|_{L_2(0,1)} \to \frac{\sqrt{6}}{12} \neq 0$. Therefore, the autoconvolution operator G doesn't take every bounded subset in $L_2(0,1)$ into a relatively compact subset in $L_2(0,1)$.

Even though the operator G is not compact in the general setting where $G : L_2(0,1) \to L_2(0,1)$, compactness of G can be obtained by restricting its domain $D(G)$ to a relatively compact subset of $L_2(0,1)$ due to the continuity of G. For example, if $D(G)$ only contains equibounded monotone functions, then $G : D(G) \to L_2(0,1)$ becomes a compact operator.

1.2 Weak Closedness and Injectivity

In the following, we restrict the domain of the autoconvolution problem to

$$D(G) \equiv \{x \in L_2(0,1), \, x(t) \geq 0 \text{ for } a.e. \; t \in [0,1]\}. \tag{1.2}$$

To apply the classical Tikhonov regularization theory to nonlinear inverse problems[6], it is required that the nonlinear operator be weakly closed. We can ensure the weak-closedness of the autoconvolution operator G if using the restricted domain $D(G)$ as defined in (1.2).

Definition 1.2 *A linear or nonlinear operator $A : D(A) \subseteq B_1 \to B_2$ (where B_1 and B_2 are Hilbert spaces) is* weakly sequentially closed *if, for any sequence $\{x_n\}_{n=1}^{\infty} \subset D(A)$, weak convergence $x_n \rightharpoonup x_0$ in B_1 and $A(x_n) \rightharpoonup y_0$ imply $x_0 \in D(A)$ and $A(x_0) = y_0$.*

Proposition 1.3 *The autoconvolution operator $G : D(G) \subset L_2(0,1) \to L_2(0,1)$ is weakly continuous, $D(G)$ is weakly closed in $L_2(0,1)$, therefore G is weakly sequentially closed.*

We can also prove the injectivity of the autoconvolution operator using the Titchmarsh's Lemma.

Lemma 1.1 *For $f, g \in L_2(0,1)$, let there exist a value ν $(0 < \nu < 1)$ such that*

$$\int_0^t f(t-s)\, g(s)\, ds = 0 \quad (0 \le t \le \nu).$$

Then there exist numbers $\alpha, \beta \in [0,1]$ with $\alpha + \beta \ge \nu$, $f(t) = 0$ a.e. in $t \in [0, \alpha]$ and $g(t) = 0$ a.e. in $t \in [0, \beta]$

Then it follows:

Theorem 1.1 *If we define for any $x \in L_2(0,1)$,*

$$\varepsilon\,(x) \equiv \sup\{\, 0 \le \varepsilon \le 1 : \, x(t) = 0 \text{ a.e. on } [0, \varepsilon]\},$$

then the autoconvolution equation (1.1) subject to the domain (1.2) has a unique solution if and only if $f(t) \ge 0$ a.e. $t \in [0,1]$ and $\varepsilon(f) = 0$. If $x^\star$ is the uniquely determined solution, then it fulfills the condition $\varepsilon(x^\star) = 0$.

1.3 Local Ill-posedness

Let us first consider a general operator equation

$$F(x) = y, \tag{1.3}$$

where the operator $F : D(F) \subseteq X \to Y$ maps between Hilbert spaces X and Y. We denote the norms in X and Y by $\|\cdot\|_X$ and $\|\cdot\|_Y$. If F is nonlinear, we will focus our attention to a solution point $x^\star \in D(F)$ of equation (1.3) and a family of closed balls centered at $x^\star$ with radius r, i.e., $\mathcal{B}(x^\star, r) \equiv \{x \in X,\ \|x - x^\star\|_X \leq r\}$.

Definition 1.3 *We call the equation (1.3) locally ill-posed in $x^\star$ if, for arbitrary small $r > 0$, there is an infinite sequence $\{x_n\} \in D(F) \cap \mathcal{B}(x^\star, r)$ with*

$$\|F(x_n) - F(x^\star)\|_Y \to 0, \quad \textit{but} \quad \|x_n - x^\star\|_X \nrightarrow 0 \quad \textit{as} \quad n \to \infty.$$

Otherwise the equation is called locally well-posed in $x^\star$.

In linear inverse problems, ill-posedness typically occurs when the operator F is compact; in particular, if F is linear and compact, then the problem is unstable if and only if the range of F is infinitely dimensional. However, ill-posedness can also occur when F fails to be compact, and the autoconvolution problem represents a nonlinear example of this case.

Proposition 1.4 *For the $D(G)$ defined in (1.2), the inverse autoconvolution operator G^{-1} is discontinuous at every point $y = G(x) \in L_2(0,1)$, $x \in D(G) \subset L_2(0,1)$, i.e., the autoconvolution equation is locally ill-posed at every point $x \in D(G) \subset L_2(0,1)$.*

Various degrees of ill-posedness for the autoconvolution equation are discussed in [8] and [10]. In short, we expect a correlation between the degree of ill-posedness and boththe smoothness of the solution x and the behavior of x at 0. It is not surprising that the same kind of dependence exists in the linear problems except that the role of the kernel function k is now taken over by the solution x: the global smoothness of k is replaced by the local smoothness of x. Furthermore, the main difficulty of the autoconvolution problem is associated with values of the solution x with small t, particularly at $x(0)$.

Various regularization methods have been studied for the autoconvolution equation. We can utilize Tikhonov regularization theory for nonlinear inverse problems since the autoconvolution operator $G : D(G) \subseteq L_2(0,1) \to L_2(0,1)$ is continuous and weakly closed, and G has a compact Fréchet derivative at any $x \in L_2(0,1)$ that satisfies the assumptions to guarantee stability in the Tikhonov theory [6] [10]. However, the drawbacks of Tikhonov regularization (e.g., loss of causality of the Volterra problem) still exist here, and because of the nonlinearity of the problem, the numerical implementation becomes even more expensive.

The ill-posed autoconvolution problem (1.1) can be changed into a well-posed one by imposing appropriate a prior restrictions $x \in Q$. If Q is a relatively compact subset of $L_2(0,1)$ and G is injective on Q, then the inverse operator G^{-1} exists and is continuous by Tikhonov lemma. In [9], the domain of the autoconvolution equation is restricted to a subset B of $D(G)$ (as defined in (1.2)), where B contains solutions x that are uniformly bounded below and above by positive constants, and with a prescribed upper bound c for the total variation. Here, c is the regularization parameter.

The most recent approach of which we are aware, to regularize the autoconvolution problem, was studied by Janno in [12]. Lavrent'ev regularization method was applied to the autoconvolution equation and convergence was obtained. The advantage of Lavrent'ev method is that it preserves the causal nature of Volterra problems and therefore leads to a fast sequential method. The major drawback for the method is that it requires and depends on an initial guess of the true solution. When the initial guess is far away from the true solution, the method appears less able to recover the true solution.

In the next chapter, we develop a local regularization method for the autoconvolution equation. A convergence theorem is proved, and we provide an effective regularization method which preserves the casual nature of the problem for the most part without having to introduce an initial guess.

CHAPTER 2

Convergence of Local Regularization

2.1 Formulation of the Regularized Equation

Recall that we are considering the problem of finding $x \in L_2(0,1)$ satisfying the autoconvolution equation

$$G(x)(t) = f(t), \quad \text{a.e. t} \in [0,1], \tag{2.1}$$

where G is the nonlinear Volterra operator given by

$$G(x)(t) = \int_0^t x(t-s)\, x(s)\, ds \quad \text{a.e. t} \in [0,1], \tag{2.2}$$

and $f \in Range(G) \subseteq L_2(0,1)$. Note that without loss of generality, we have assumed that $T = 1$.

Suppose $0 < \bar{x} \in C^1[0,1]$ is the true solution of equation (2.1). As in the local regularization approach described in the Introduction for the linear Volterra problems, we extend equation (2.1) slightly into the future, i.e., let $\bar{R} > 0$ be a small fixed constant such that $\bar{R} \ll 1$ and assume that equation (2.1) holds on an extended interval $[0, 1+R]$ for $0 < R < \bar{R}$. Then $\bar{x}(t)$ solves

$$\int_0^{t+\rho} x(t+\rho-s)\,x(s)\,ds = f(t+\rho), \qquad t \in [0,1],\ \rho \in [0,R].$$

Split the integral at ρ and t, then change the variable of integration, we get

$$2\int_0^{\rho} x(t+\rho-s)\,x(s)\,ds + \int_{\rho}^{t} x(t+\rho-s)\,x(s)\,ds = f(t+\rho), \quad t \in [0,1],\ \rho \in [0,R]. \quad (2.3)$$

In order to consolidate the local future information introduced by the variable ρ, we integrate both sides of the equation (2.3) with respect to a suitable Borel measure $\eta = \eta(\rho) > 0$ (which we will clarify later) on $[0, R]$, then

$$2\int_0^R \int_0^{\rho} x(t+\rho-s)\,x(s)\,ds\,d\eta(\rho) + \int_0^R \int_{\rho}^{t} x(t+\rho-s)\,x(s)\,ds\,d\eta(\rho)$$
$$= \int_0^R f(t+\rho)\,d\eta(\rho), \qquad t \in [0,1]. \quad (2.4)$$

Note that $\bar{x}$ still satisfies equation (2.4) exactly.

In reality, instead of having the exact data f, we always only have access to some approximation $f^\delta \in C[0, 1+\bar{R}]$, such that

$$\|f^\delta - f\|_\infty \le \delta \qquad \text{for some} \quad \delta > 0. \quad (2.5)$$

So, instead of solving $Gx = f$, we need to solve $Gx = f^\delta$. As discussed in Chapter 1, this latter problem is locally ill-posed in $L_2(0,1)$ in the sense that $\delta \to 0$ does not guarantee the convergence of the solution to $\bar{x}$, i.e., the solution of the equation does not depend continuously on the right hand side. Thus a regularization method is needed.

As in the linear case, we motivate the regularization method by momentarily holding x constant on a small local interval $[t, t+R]$. Therefore, we replace $x(t+\rho)$ by $x(t)$ for $\rho \in [0, R]$ in the first term of equation (2.4). Here, the length of this local interval R serves as the regularization parameter. We then obtain the regularization equation

$$\alpha_R(x)x + F_R(x) = f_R^\delta, \tag{2.6}$$

where for $t \in [0,1]$,

$$\begin{aligned}
\alpha_R(x) &\equiv 2\int_0^R \int_0^\rho x(s)\, ds\, d\eta(\rho), \\
F_R(x)(t) &\equiv \int_0^R \int_\rho^t x(t+\rho-s)\, x(s)\, ds\, d\eta(\rho), \\
f_R^\delta(t) &\equiv \int_0^R f^\delta(t+\rho)\, d\eta(\rho).
\end{aligned} \tag{2.7}$$

We will first study the class of the Borel measures we use. If $\eta(\rho) > 0$ is a continuous Borel measure on $[0, R]$ of the form

$$\int_0^R g(\rho)\, d\eta(\rho) = \int_0^R g(\rho)\omega(\rho)\, d\rho,$$

where $\omega(\rho) \in L_\infty[0, R]$, such that $0 < \omega_1 \le \omega(\rho) \le \omega_2 < \infty, \rho \in [0, R]$, for constants $\omega_1, \omega_2 > 0$, then for any real numbers m and n,

$$\frac{\int_0^R \rho^m\, d\eta(\rho)}{\int_0^R \rho^n\, d\eta(\rho)} = \frac{\int_0^R \rho^m \omega(\rho)\, d\rho}{\int_0^R \rho^n \omega(\rho)\, d\rho} \le \frac{\omega_2 \int_0^R \rho^m\, d\rho}{\omega_1 \int_0^R \rho^n\, d\rho} = \frac{\omega_2(n+1)}{\omega_1(m+1)} R^{m-n}.$$

If $\eta(\rho) > 0$ is a discrete Borel measure on $[0, R]$ of the form

$$\int_0^R g(\rho)\, d\eta(\rho) = \sum_{i=1}^k g(\rho_i)\, \tilde{\omega}_i,$$

where for $i = 1, \cdots, k$, $0 < \bar{\omega}_1 \le \tilde{\omega}_i \le \bar{\omega}_2 < \infty, \rho_i = C_i R$ for $C_i \in [0, 1]$ and there exists at least some C_i for $1 \le i \le k$ such that $C_i \neq 0$, then

$$\frac{\int_0^R \rho^m \, d\eta(\rho)}{\int_0^R \rho^n \, d\eta(\rho)} = \frac{\sum_{i=1}^{k} (C_i R)^m \, \widetilde{\omega}_i}{\sum_{i=1}^{k} (C_i R)^n \, \widetilde{\omega}_i} \le \frac{\bar{\omega}_2 k}{\bar{\omega}_1 \min\limits_{\substack{i=1,\cdots,k \\ C_i \neq 0}} C_i^n} R^{m-n}.$$

Therefore, a reasonable general assumption for our generic Borel measure η is that for any real numbers m and n, there exists a constant $C(m, n, \eta) > 0$, such that

$$\frac{\int_0^R \rho^m \, d\eta(\rho)}{\int_0^R \rho^n \, d\eta(\rho)} \le C(m, n, \eta) R^{m-n}. \tag{2.8}$$

Our main convergence result is stated below. It follows as a corollary to Theorem 2.2 of the next section.

Theorem 2.1 *Assume the perturbed data f^δ satisfies $|f(t) - f^\delta(t)| \le \delta$ for $t \in [0, 1 + \bar{R}]$, and that the Borel measure $\eta > 0$ satisfies (2.8). Then there exists $\bar{C} > 0$ and $k_1 > 0$ independent of R such that if the true solution $\bar{x} \in C^1[0,1]$ of the autoconvolution equation is positive and satisfies*

$$\bar{x}(0) > \bar{C} \, \|\bar{x}'\|_\infty,$$

then for $R = R(\delta) > 0$ selected satisfying

$$R(\delta) \to 0 \quad \textit{and} \quad \frac{\delta}{R^2(\delta)} \le k_1$$

as $\delta \to 0$, it follows that the solution $x^\delta_{R(\delta)}$ of the regularization equation (2.6) asso-

ciated with data f^δ satisfies

$$\|x^\delta_{R(\delta)} - \bar{x}\|_{L_2(0,1)} = \mathcal{O}(\delta^{1/2})$$

as $\delta \to 0$.

The proof of this result will take the remainder of this section and involves introducing some additional spaces and norms. To begin, we rewrite equation (2.4) using similar notation as in equation (2.6), then $\bar{x}$ satisfies

$$\alpha_R(\bar{x})\bar{x} + F_R(\bar{x}) = f_R + \epsilon_R, \tag{2.9}$$

where for $t \in [0,1]$,

$$\begin{aligned}
\alpha_R(\bar{x}) &= 2\int_0^R \int_0^\rho \bar{x}(s)\, ds\, d\eta(\rho), \\
F_R(\bar{x})(t) &= \int_0^R \int_\rho^t \bar{x}(t+\rho-s)\bar{x}(s)\, ds\, d\eta(\rho), \\
f_R(t) &\equiv \int_0^R f(t+\rho)\, d\eta(\rho), \\
\epsilon_R &\equiv 2\int_0^R \int_0^\rho (\bar{x}(t) - \bar{x}(t+\rho-s))\, \bar{x}(s)\, ds\, d\eta(\rho).
\end{aligned} \tag{2.10}$$

Let us define a new R-dependent topology on $L_2(0,1)$:

$$\begin{aligned}
\langle x, y\rangle_{\sigma,R} &\equiv \frac{1}{C(R)}\langle x, y\rangle_{L_2^\sigma(0,R)} + \langle x, y\rangle_{L_2^\sigma(0,1)} \\
&\equiv \frac{1}{C(R)}\int_0^R e^{-2\sigma t}x(t)\, y(t)\, dt + \int_0^1 e^{-2\sigma t}x(t)\, y(t)\, dt,
\end{aligned}$$

where $\sigma > 0$ and $C(R) > 0$, with conditions on σ and $C(R)$ to be specified later.

Note that the weighted inner product

$$\langle x, y\rangle_{L_2^\sigma(0,T)} = \int_0^T e^{-2\sigma t} x(t)\, y(t)\, dt \qquad \text{for} \quad T > 0$$

induces a weighted norm

$$\|x\|_{L_2^\sigma(0,T)} = \left(\int_0^T e^{-2\sigma t} x^2(t)\, dt\right)^{1/2},$$

which is equivalent to the L_2 norm since

$$e^{-\sigma T}\|\cdot\|_{L_2(0,T)} \le \|\cdot\|_{L_2^\sigma(0,T)} \le \|\cdot\|_{L_2(0,T)}$$

for $T > 0$.

Throughout this chapter, we will couple the $L_2(0,1)$ space with the newly defined R-dependent topology and denote it as $L_2^{\sigma,R}(0,1)$. We then denote a closed ball centered at x_0 with radius r under this new topology by

$$\mathcal{B}(x_0, r) = \{z \in L_2^{\sigma,R}(0,1), \|z - x_0\|_{\sigma,R} \le r\}, \tag{2.11}$$

where

$$\|\cdot\|_{\sigma,R} = \left\{\frac{1}{\mathcal{C}(R)}\|\cdot\|^2_{L_2^\sigma(0,R)} + \|\cdot\|^2_{L_2^\sigma(0,1)}\right\}^{1/2}.$$

Note that if $x \in L_2^{\sigma,R}(0,1)$, then $x \in L_2^\sigma(0,1)$ and $x|_{[0,R]} \in L_2^\sigma(0,R)$ and thus the norms $\|\cdot\|_{L_2^\sigma(0,1)}$ and $\|\cdot\|_{L_2^\sigma(0,R)}$ still have meaning for such x.

Lemma 2.1 *The operator $F_R : L_2^{\sigma,R}(0,1) \to L_2^{\sigma,R}(0,1)$ as defined in (2.7) is Fréchet differentiable and F_R' is uniformly Lipschitz in $L_2^{\sigma,R}(0,1)$, i.e., there exists some constant $k > 0$, such that for $\sigma \ge 1/2$ and $R \le 1$,*

$$\|F_R'(x_1) - F_R'(x_2)\| \le k\|x_1 - x_2\|_{\sigma,R}$$

for $x_1, x_2 \in L_2^{\sigma,R}(0,1)$*, where* $\|\cdot\|$ *denotes the usual* $\mathcal{L}(L_2^{\sigma,R}(0,1))$ *operator norm.*

Proof: For any $x, h \in L_2^{\sigma,R}(0,1)$ and $t \in [0,1]$, we calculate

$$\begin{aligned}
& F_R(x+h)(t) - F_R(x)(t) \\
&= \int_0^R \int_\rho^t h(t+\rho-s)\, x(s)\, ds\, d\eta(\rho) + \int_0^R \int_\rho^t x(t+\rho-s)\, h(s)\, ds\, d\eta(\rho) \\
&\quad + \int_0^R \int_\rho^t h(t+\rho-s)\, h(s)\, ds\, d\eta(\rho) \\
&= 2\int_0^R \int_\rho^t x(t+\rho-s)\, h(s)\, ds\, d\eta(\rho) + \int_0^R \int_\rho^t h(t+\rho-s)\, h(s)\, ds\, d\eta(\rho).
\end{aligned}$$

For fixed $t \in [0,1]$, $\rho \in [0,R]$, let $\tau = t + \rho - s$, then

$$\begin{aligned}
\left| \int_\rho^t h(t+\rho-s)\, h(s)\, ds \right| &\le \left(\int_\rho^t h^2(t+\rho-s)\, ds \right)^{1/2} \left(\int_\rho^t h^2(s)\, ds \right)^{1/2} \\
&\le \left(\int_\rho^t e^{2\sigma\tau} e^{-2\sigma\tau} h^2(\tau)\, d\tau \right)^{1/2} \left(\int_\rho^t e^{2\sigma s} e^{-2\sigma s} h^2(s)\, ds \right)^{1/2} \\
&\le e^{2\sigma}\, \|h\|^2_{L_2^\sigma(0,1)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left| \int_0^R \int_\rho^t h(t+\rho-s)\, h(s)\, ds\, d\eta(\rho) \right| &\le \int_0^R \left| \int_\rho^t h(t+\rho-s)\, h(s)\, ds \right| d\eta(\rho) \\
&\le e^{2\sigma}\, \|h\|^2_{L_2^\sigma(0,1)} \int_0^R d\eta(\rho),
\end{aligned}$$

and

$$\begin{aligned}
\left\| \int_0^R \int_\rho^{\cdot} h(\cdot+\rho-s)\, h(s)\, ds\, d\eta(\rho) \right\|_{L_2^\sigma(0,1)} &\le e^{2\sigma}\, \|h\|^2_{L_2^\sigma(0,1)} \int_0^R d\eta(\rho)\, \|1\|_{L_2^\sigma(0,1)} \\
&\le e^{2\sigma}\, \|h\|^2_{L_2^\sigma(0,1)} \int_0^R d\eta(\rho) \qquad (2.12)
\end{aligned}$$

for $\sigma \geq 1/2$. Here we have used the fact that

$$\|1\|_{L_2^\sigma(0,1)} = \left(\int_0^1 e^{-2\sigma t}\, dt\right)^{1/2} = \left(\frac{1-e^{-2\sigma}}{2\sigma}\right)^{1/2} \leq 1$$

for $\sigma \geq 1/2$.

Similarly, we can show for $\sigma \geq 1/2$,

$$\left\| \int_0^R \int_\rho^{\cdot} h(\cdot + \rho - s)\, h(s)\, ds\, d\eta(\rho) \right\|_{L_2^\sigma(0,R)} \leq e^{2\sigma} \|h\|^2_{L_2^\sigma(0,R)} \int_0^R d\eta(\rho). \tag{2.13}$$

Combining inequalities (2.12) and (2.13), we get

$$\begin{aligned}
&\left\| \int_0^R \int_\rho^{\cdot} h(\cdot + \rho - s)\, h(s)\, ds\, d\eta(\rho) \right\|^2_{\sigma,R} \\
&= \left\| \int_0^R \int_\rho^{\cdot} h(\cdot + \rho - s)\, h(s)\, ds\, d\eta(\rho) \right\|^2_{L_2^\sigma(0,1)} \\
&\qquad + \frac{1}{C(R)} \left\| \int_0^R \int_\rho^{\cdot} h(\cdot + \rho - s)\, h(s)\, ds\, d\eta(\rho) \right\|^2_{L_2^\sigma(0,R)} \\
&\leq \left[e^{2\sigma} \|h\|^2_{L_2^\sigma(0,1)} \int_0^R d\eta(\rho) \right]^2 + \frac{1}{C(R)} \left[e^{2\sigma} \|h\|^2_{L_2^\sigma(0,R)} \int_0^R d\eta(\rho) \right]^2 \\
&\leq \left[e^{2\sigma} \int_0^R d\eta(\rho) \right]^2 \|h\|^2_{L_2^\sigma(0,1)} \left[\|h\|^2_{L_2^\sigma(0,1)} + \frac{1}{C(R)} \|h\|^2_{L_2^\sigma(0,R)} \right] \\
&\leq \left[e^{2\sigma} \int_0^R d\eta(\rho) \right]^2 \|h\|^2_{\sigma,R} \|h\|^2_{\sigma,R} .
\end{aligned}$$

Therefore,

$$\left\| \int_0^R \int_\rho^{\cdot} h(\cdot + \rho - s)\, h(s)\, ds\, d\eta(\rho) \right\|_{\sigma,R} \leq e^{2\sigma} \int_0^R d\eta(\rho)\ \|h\|^2_{\sigma,R} .$$

Hence, F_R is Fréchet differentiable at $x \in L_2^{\sigma,R}(0,1)$ with $F_R'(x) \in \mathcal{L}(L_2^{\sigma,R}(0,1))$ given by

$$F_R'(x)(h)(t) = 2 \int_0^R \int_\rho^t x(t + \rho - s)\, h(s)\, ds\, d\eta(\rho) \tag{2.14}$$

for any $h \in L_2^{\sigma,R}(0,1)$ and $t \in [0,1]$.

To show the Lipschitz condition, we will first consider $\|F_R'(x_1) - F_R'(x_2)\|_{L_2^\sigma(0,1)}$, for $x_1, x_2 \in L_2^{\sigma,R}(0,1)$. For fixed $t \in [0,1], \rho \in [0,R]$, let $\tau = t + \rho - s$, we have

$$
\begin{aligned}
&\left| \int_\rho^t (x_1(t+\rho-s) - x_2(t+\rho-s))\, h(s)\, ds \right| \\
&\leq \left(\int_\rho^t (x_1(t+\rho-s) - x_2(t+\rho-s))^2 ds \right)^{1/2} \left(\int_\rho^t h^2(s)\, ds \right)^{1/2} \\
&= \left(\int_\rho^t e^{2\sigma\tau} e^{-2\sigma\tau} (x_1(\tau) - x_2(\tau))^2\, d\tau \right)^{1/2} \left(\int_\rho^t e^{2\sigma s} e^{-2\sigma s} h^2(s)\, ds \right)^{1/2} \\
&\leq e^{2\sigma} \|x_1 - x_2\|_{L_2^\sigma(0,1)} \|h\|_{L_2^\sigma(0,1)},
\end{aligned}
$$

therefore,

$$
\begin{aligned}
&\left| \int_0^R \int_\rho^t (x_1(t+\rho-s) - x_2(t+\rho-s))\, h(s)\, ds\, d\eta(\rho) \right| \\
&\leq \int_0^R \left| \int_\rho^t (x_1(t+\rho-s) - x_2(t+\rho-s))\, h(s)\, ds \right| d\eta(\rho) \\
&\leq e^{2\sigma} \|x_1 - x_2\|_{L_2^\sigma(0,1)} \|h\|_{L_2^\sigma(0,1)} \int_0^R d\eta(\rho).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\|F_R'(x_1)(h) - F_R'(x_2)(h)\|_{L_2^\sigma(0,1)} \\
&= \left\| 2 \int_0^R \int_\rho^{\cdot} (x_1(\cdot+\rho-s) - x_2(\cdot+\rho-s))\, h(s)\, ds\, d\eta(\rho) \right\|_{L_2^\sigma(0,1)} \\
&\leq 2e^{2\sigma} \|x_1 - x_2\|_{L_2^\sigma(0,1)} \|h\|_{L_2^\sigma(0,1)} \int_0^R d\eta(\rho)\, \|1\|_{L_2^\sigma(0,1)} \\
&\leq 2e^{2\sigma} \|x_1 - x_2\|_{L_2^\sigma(0,1)} \|h\|_{L_2^\sigma(0,1)} \int_0^R d\eta(\rho)
\end{aligned}
$$

for $\sigma \geq 1/2$.

Similarly, we can get

$$\begin{aligned}
&\|F'_R(x_1)(h) - F'_R(x_2)(h)\|_{L_2^\sigma(0,R)} \\
&\leq 2e^{2\sigma}\|x_1 - x_2\|_{L_2^\sigma(0,R)}\|h\|_{L_2^\sigma(0,R)} \int_0^R d\eta(\rho) \\
&\leq 2e^{2\sigma}\|x_1 - x_2\|_{L_2^\sigma(0,1)}\|h\|_{L_2^\sigma(0,R)} \int_0^R d\eta(\rho)
\end{aligned}$$

for $\sigma \geq 1/2$.

Therefore, for any $h \in L_2^{\sigma,R}(0,1)$,

$$\begin{aligned}
&\|F'_R(x_1)(h) - F'_R(x_2)(h)\|_{\sigma,R}^2 \\
= \;& \|F'_R(x_1)(h) - F'_R(x_2)(h)\|_{L_2^\sigma(0,1)}^2 + \frac{1}{C(R)}\|F'_R(x_1)(h) - F'_R(x_2)(h)\|_{L_2^\sigma(0,1)}^2 \\
\leq \;& \left[2\,e^{2\sigma}\int_0^R d\eta(\rho)\,\|x_1 - x_2\|_{L_2^\sigma(0,1)}\right]^2 \left[\|h\|_{L_2^\sigma(0,1)}^2 + \frac{1}{C(R)}\|h\|_{L_2^\sigma(0,R)}^2\right] \\
\leq \;& \left[2\,e^{2\sigma}\int_0^R d\eta(\rho)\,\|x_1 - x_2\|_{\sigma,R}\right]^2 \|h\|_{\sigma,R}^2.
\end{aligned}$$

Hence,

$$\|F'_R(x_1) - F'_R(x_2)\| \leq 2\,e^{2\sigma}\int_0^R d\eta(\rho)\,\|x_1 - x_2\|_{\sigma,R}$$

for any $x_1, x_2 \in L_2^{\sigma,R}(0,1)$, i.e., F'_R is uniformly Lipschitz in $L_2^{\sigma,R}(0,1)$ with Lipschitz constant $k = 2e^{2\sigma}\int_0^R d\eta(\rho)$. □

The following Lemma follows immediately from Lemma 2.1.

Lemma 2.2 *Let $v, v_1, v_2 \in \mathcal{B}(0,r) \subseteq L_2^{\sigma,R}(0,1)$ and $x \in L_2^{\sigma,R}(0,1)$, then the remainder*

$$\mathcal{R}_R(x,v) \equiv F_R(x+v) - F_R(x) - F'_R(x)v \tag{2.15}$$

of the Fréchet derivative $F'_R(x)$ satisfies

$$\|\mathcal{R}_R(x,v)\|_{\sigma,R} \le e^{2\sigma} \int_0^R d\eta(\rho)\, \|v\|^2_{\sigma,R}, \tag{2.16}$$

$$\|\mathcal{R}_R(x,v_1) - \mathcal{R}_R(x,v_1)\|_{\sigma,R} \le 2\, e^{2\sigma} \int_0^R d\eta(\rho)\, \max\{\|v_1\|_{\sigma,R}, \|v_2\|_{\sigma,R}\}\, \|v_1 - v_2\|_{\sigma,R}. \tag{2.17}$$

Proof: We can write

$$\mathcal{R}_R(x,v) = \int_0^1 (F'_R(x+tv) - F'_R(x))\, v\, dt, \tag{2.18}$$

then

$$\begin{aligned}
\|\mathcal{R}_R(x,v)\|_{\sigma,R} &\le \int_0^1 \|(F'_R(x+tv) - F'_R(x))\, v\|_{\sigma,R}\, dt \\
&\le \int_0^1 (\|F'_R(x+tv) - F'_R(x)\|\, \|v\|_{\sigma,R})\ dt \\
&\le \int_0^1 \left(2\, e^{2\sigma} \int_0^R d\eta(\rho)\, \|tv\|_{\sigma,R}\, \|v\|_{\sigma,R}\right) dt \\
&\le 2\, e^{2\sigma} \int_0^R d\eta(\rho)\, \|v\|^2_{\sigma,R} \int_0^1 t\, dt \\
&= e^{2\sigma} \int_0^R d\eta(\rho)\, \|v\|^2_{\sigma,R}.
\end{aligned}$$

Similarly, we can write

$$\mathcal{R}_R(x,v_1) - \mathcal{R}_R(x,v_2) = \int_0^1 (F'_R(x + tv_1 + (1-t)v_2) - F'_R(x))(v_1 - v_2)\, dt,$$

then,

$$
\begin{aligned}
&\|\mathcal{R}_R(x, v_1) - \mathcal{R}_R(x, v_2)\|_{\sigma,R} \\
&\leq \int_0^1 \|(F'_R(x + tv_1 + (1-t)v_2) - F'_R(x))(v_1 - v_2)\|_{\sigma,R}\, dt \\
&\leq \int_0^1 (\|F'_R(x + tv_1 + (1-t)v_2) - F'_R(x)\|\, \|v_1 - v_2\|_{\sigma,R})\, dt \\
&\leq \int_0^1 \left(2\, e^{2\sigma} \int_0^R d\eta(\rho)\, \|tv_1 + (1-t)v_2\|_{\sigma,R}\, \|v_1 - v_2\|_{\sigma,R}\right) dt \\
&\leq 2\, e^{2\sigma} \int_0^R d\eta(\rho)\, \|v_1 - v_2\|_{\sigma,R} \max\{\|v_1\|_{\sigma,R}, \|v_1\|_{\sigma,R}\} \int_0^1 1\, dt \\
&= 2\, e^{2\sigma} \int_0^R d\eta(\rho)\, \|v_1 - v_2\|_{\sigma,R} \max\{\|v_1\|_{\sigma,R}, \|v_1\|_{\sigma,R}\}.
\end{aligned}
$$

□

For $h \in L_2^{\sigma,R}(0,1)$, $t \in [0,1]$, we define

$$
B_R(\bar{x})(h)(t) \equiv 2 \int_0^R \int_0^t \bar{x}(t + \rho - s)\, h(s)\, ds\, d\eta(\rho), \tag{2.19}
$$

$$
D_R(\bar{x})(h)(t) \equiv 2 \int_0^R \int_0^\rho \bar{x}(t + \rho - s)\, h(s)\, ds\, d\eta(\rho), \tag{2.20}
$$

then

$$
F'_R(\bar{x})(h)(t) = B_R(\bar{x})(h)(t) - D_R(\bar{x})(h)(t). \tag{2.21}
$$

Note that both $B_R(\bar{x})$ and $D_R(\bar{x})$ are bounded linear operators in $L_2^{\sigma,R}(0,1)$.

Expand $F_R(x)$ in equation (2.6) using the Fréchet derivative $F'_R(\bar{x})$, we get

$$
\alpha_R(x)x + F_R(\bar{x}) + F'_R(\bar{x})(x - \bar{x}) + \mathcal{R}_R(\bar{x}, x - \bar{x}) = f_R^\delta. \tag{2.22}
$$

After combining equation (2.9), (2.21) with equation (2.22) and some simple algebra, we obtain

$$(\alpha_R(\bar{x})I + B_R(\bar{x}))(x - \bar{x}) = f_R^\delta - f_R - \epsilon_R - \mathcal{R}_R(\bar{x}, x - \bar{x})$$
$$+ D_R(\bar{x})(x - \bar{x}) + (\alpha_R(\bar{x}) - \alpha_R(x))x, \quad (2.23)$$

where I is the identity operator on $L_2^{\sigma,R}(0,1)$.

Let us further denote for $v \in L_2^{\sigma,R}(0,1)$,

$$E_R(\bar{x}, v) = D_R(\bar{x})(v) - \alpha_R(v)\,\bar{x}, \quad (2.24)$$

where we note that $v \to E_R(\bar{x}, v)$ is linear. Then equation(2.23) becomes

$$(\alpha_R(\bar{x})I + B_R(\bar{x}))(x - \bar{x}) = f_R^\delta - f_R - \epsilon_R - \mathcal{R}_R(\bar{x}, x - \bar{x})$$
$$+ E_R(\bar{x}, x - \bar{x}) + (\alpha_R(x) - \alpha_R(\bar{x}))(\bar{x} - x). \quad (2.25)$$

In order to be able to invert $(\alpha_R(\bar{x})I + B_R(\bar{x})) \in \mathcal{L}(L_2^{\sigma,R}(0,1))$, we need the following lemma.

Lemma 2.3 *If $\bar{x}(t) > 0$ for $t \in [0, 1 + \bar{R}]$ and $\bar{x} \in W^{2,\infty}(0, 1 + \bar{R})$, then for positive Borel measure $\eta(\rho)$, there exists $\sigma_0 > 0$ independent of $R > 0$, such that the operator $B_R(\bar{x})$ is accretive in $L_2^{\sigma,R}(0,1)$ for $\sigma \geq \sigma_0$; i.e.,*

$$\langle B_R(\bar{x})v, v\rangle_{\sigma,R} \geq 0 \qquad \text{for any} \quad v \in L_2^{\sigma,R}(0,1).$$

Proof: Recall that for $h \in L_2^{\sigma,R}(0,1)$, $t \in [0,1]$,

$$\begin{aligned} B_R(\bar{x})(h)(t) &= 2\int_0^R \int_0^t \bar{x}(t+\rho-s)\,h(s)\,ds\,d\eta(\rho) \\ &= \int_0^t \left(2\int_0^R \bar{x}(t+\rho-s)\,d\eta(\rho)\right) h(s)\,ds. \end{aligned}$$

Let us denote for $t \in [0,1]$,

$$a_R(t) \equiv 2\int_0^R \bar{x}(t+\rho)\,d\eta(\rho),$$

then

$$B_R(\bar{x})(h)(t) = \int_0^t a_R(t-s)h(s)\,ds. \tag{2.26}$$

We will first show $B_R(\bar{x})$ is accretive in $L_2^\sigma(0,1)$, i.e.,

$$\begin{aligned} &\int_0^1 e^{-2\sigma t}\cdot\left(\int_0^t a_R(t-s)v(s)ds\right) v(t)\,dt \\ &= \int_0^1\int_0^t e^{-\sigma(t-s)}\,a_R(t-s)e^{-\sigma s}\,v(s)\,ds\,e^{-\sigma t}v(t)\,dt \geq 0 \end{aligned}$$

for any $v \in L_2^\sigma(0,1)$. This is equivalent to the following condition

$$\int_0^1\int_0^t e^{-\sigma(t-s)}\,a_R(t-s)\,v(s)\,ds\,v(t)\,dt \geq 0$$

for any $v \in L_2^\sigma(0,1)$, i.e., the operator $B_R^\sigma(\bar{x})$ is accretive under $\langle\cdot,\cdot\rangle_{\sigma=0}$ (the regular L_2 norm) where $B_R^\sigma(\bar{x})$ is defined as in (2.26) except that the kernel $a_R(t)$ for B_R is replaced by $a_R[\sigma](t) = e^{-\sigma t}a_R(t)$. In what follows, we will show accretivity of $B_R^\sigma(\bar{x})$ in $L_2(0,1)$ for all σ sufficiently large.

Let us define

$$A_0 = \min_{t\in[0,1+\bar{R}]} \bar{x}(t) > 0, \qquad A_1 = \|\bar{x}'\|_{L^\infty[0,1+\bar{R}]}, \qquad A_2 = \|\bar{x}''\|_{L^\infty[0,1+\bar{R}]},$$

then

$$\min_{t\in[0,1]} a_R(t) \geq 2\,A_0 \int_0^R d\eta(\rho),$$
$$\|a_R'\|_{L^\infty[0,1]} \leq 2\,A_1 \int_0^R d\eta(\rho),$$
$$\|a_R''\|_{L^\infty[0,1]} \leq 2\,A_2 \int_0^R d\eta(\rho).$$

Consider $a_R[\sigma](t) = e^{-\sigma t} a_R(t)$, we have for a.e. $t \in [0,1]$,

$$a_R[\sigma](t) \geq e^{-\sigma t}\left(2\,A_0 \int_0^R d\eta(\rho)\right),$$
$$a_R'[\sigma](t) = e^{-\sigma t}(-\sigma a_R(t) + a_R'(t)) \leq e^{-\sigma t}\,2\,(-\sigma A_0 + A_1)\int_0^R d\eta(\rho),$$
$$a_R''[\sigma](t) = e^{-\sigma t}(\sigma^2 a_R(t) - 2\sigma a_R'(t) + a_R''(t)) \geq e^{-\sigma t}\,2\,(\sigma^2 A_0 - 2\sigma A_1 - A_2)\int_0^R d\eta(\rho).$$

Since $\int_0^R d\eta(\rho) > 0$, we can take

$$\sigma_0 \geq \frac{A_1 + \sqrt{A_1^2 + A_0 A_2}}{A_0}, \qquad \text{(independent of } R\text{)}$$

then for $\sigma \geq \sigma_0$, we have

$$a_R[\sigma](t) \geq 0, \qquad a_R[\sigma]'(t) \leq 0, \qquad a_R[\sigma]''(t) \geq 0$$

for a.e. $t \in [0,1]$.

Therefore, the kernel $a_R[\sigma]$ is nonnegative, nonincreasing and convex. According to Lemma 2 of [12], for $\sigma \geq \sigma_0$, we have $B_R^\sigma(\bar{x})$ accretive on $L_2(0,1)$, from which it follows that $B_R(\bar{x})$ is accretive in $L_2^\sigma(0,1)$. Similarly, we can verify that $B_R(\bar{x})$ is accretive in $L_2^\sigma(0,R)$ for $\sigma \geq \sigma_0$, where the σ_0 is the same as in $L_2^\sigma(0,1)$ case.

Hence, for any $v \in L_2^{\sigma,R}(0,1), \sigma \geq \sigma_0$,

$$\begin{aligned}\langle B_R(\bar{x})v, v\rangle_{\sigma,R} &= \frac{1}{\mathcal{C}(R)}\langle B_R(\bar{x})v, v\rangle_{L_2^\sigma(0,R)} + \langle B_R(\bar{x})v, v\rangle_{L_2^\sigma(0,1)} \\ &\geq \frac{1}{\mathcal{C}(R)}0 + 0 = 0,\end{aligned}$$

we then have $B_R(\bar{x})$ accretive with respect to $\langle\cdot,\cdot\rangle_{\sigma,R}$ for $\sigma \geq \sigma_0$. □

Consequently, $(\alpha_R(\bar{x})I + B_R(\bar{x}))^{-1} \in \mathcal{L}(L_2^{\sigma,R}(0,1))$ and we have the following estimates (see [24]):

$$\|(\alpha_R(\bar{x})I + B_R(\bar{x}))^{-1}\| \leq \frac{1}{\alpha_R(\bar{x})}$$

$$\|(\alpha_R(\bar{x})I + B_R(\bar{x}))^{-1}B_R(\bar{x})\| \leq 1$$

for $\sigma \geq \sigma_0$, where $\sigma_0 > 0$ is independent of R.

We are now ready to formularize our regularized equation as

$$x = H_R\, x, \tag{2.27}$$

where

$$\begin{aligned}H_R\, x = (\alpha_R(\bar{x})I + B_R(\bar{x}))^{-1}[f_R^\delta - f_R - \epsilon_R - \mathcal{R}_R(\bar{x}, x - \bar{x}) \\ + E_R(\bar{x}, x - \bar{x}) + (\alpha_R(x) - \alpha_R(\bar{x}))(\bar{x} - x)] + \bar{x}.\end{aligned} \tag{2.28}$$

2.2 Convergence of the Regularized Equation

In this section, we are going to prove that the regularized equation (2.27) has a unique solution, and this solution converges to the true solution $\bar{x}$ as the noise level in the data $\delta \to 0$. In the following lemmas, we bound the right hand side of equation (2.28) term by term.

Lemma 2.4 *If* $|f^\delta(t) - f(t)| \le \delta$ *for* $t \in [0, 1+R]$ *and* $\sigma \ge 1/2$, *then*

$$\|f_R^\delta - f_R\|_{\sigma,R} \le \delta \int_0^R d\eta(\rho) \sqrt{1 + \frac{R + \mathcal{O}(R^2)}{\mathcal{C}(R)}}. \tag{2.29}$$

Proof: Since

$$\|f_R^\delta - f_R\|_{\sigma,R}^2 = \frac{1}{\mathcal{C}(R)} \|f_R^\delta - f_R\|_{L_2^\sigma(0,R)}^2 + \|f_R^\delta - f_R\|_{L_2^\sigma(0,1)}^2,$$

we first consider

$$\begin{aligned}
\|f_R^\delta - f_R\|_{L_2^\sigma(0,R)} &\le \left\| \int_0^R |f^\delta(\cdot + \rho) - f(\cdot + \rho))| \, d\eta(\rho) \right\|_{L_2^\sigma(0,R)} \\
&\le \left\| \int_0^R \delta \, d\eta(\rho) \right\|_{L_2^\sigma(0,R)} \le \delta \int_0^R d\eta(\rho) \, \|1\|_{L_2^\sigma(0,R)} \\
&\le \delta \int_0^R d\eta(\rho) \left(\int_0^R e^{-2\sigma t} \, dt \right)^{1/2} = \delta \int_0^R d\eta(\rho) \sqrt{\frac{1 - e^{-2\sigma R}}{2\sigma}}.
\end{aligned}$$

Taylor expansion of $e^{-2\sigma R}$ around $R = 0$ gives

$$e^{-2\sigma R} = 1 - 2\sigma R + \mathcal{O}(R^2).$$

Thus,

$$\|f_R^\delta - f_R\|_{L_2^\sigma(0,R)} \le \delta \int_0^R d\eta(\rho) \sqrt{R + \mathcal{O}(R^2)}.$$

Similarly, we consider

$$\|f_R^\delta - f_R\|_{L_2^\sigma(0,1)} \le \delta \int_0^R d\eta(\rho) \left(\int_0^1 e^{-2\sigma t}\, dt \right)^{1/2}$$
$$= \delta \int_0^R d\eta(\rho) \sqrt{\frac{1-e^{-2\sigma}}{2\sigma}} \le \delta \int_0^R d\eta(\rho)$$

for $\sigma \ge 1/2$. The lemma then follows. □

Lemma 2.5 *If $\bar{x}(t) \in C^1[0,1]$ and $\sigma \ge 1/2$, then ϵ_R defined in (2.10) satisfies*

$$\|\epsilon_R\|_{\sigma,R} \le \left[\frac{2\sqrt{2}}{\sqrt{3}} \|\bar{x}'\|_\infty \bar{x}(0) \int_0^R \rho^2\, d\eta(\rho) + \frac{2\sqrt{2}}{3} \|\bar{x}'\|_\infty^2 \int_0^R \rho^3\, d\eta(\rho) \right] \sqrt{1 + \frac{R + \mathcal{O}(R^2)}{\mathcal{C}(R)}}. \tag{2.30}$$

Proof: For fixed $t \in [0,1], \rho \in [0,R]$, we have

$$\left| \int_0^\rho (\bar{x}(t) - \bar{x}(t+\rho-s))\, \bar{x}(s)\, ds \right|$$
$$\le \left(\int_0^\rho (\bar{x}(t) - \bar{x}(t+\rho-s))^2 ds \right)^{1/2} \left(\int_0^\rho \bar{x}^2(s)\, ds \right)^{1/2}.$$

Since $\bar{x}(t) \in C^1[0,1]$, there exist $\xi_1, \xi_2 \in [0,1]$, such that for $0 < s < \rho < R$,

$$\bar{x}(t+\rho-s) - \bar{x}(t) = \bar{x}'(\xi_1)(\rho - s), \qquad \bar{x}(s) = \bar{x}(0) + \bar{x}'(\xi_2)s,$$

then

$$
\begin{aligned}
&\left|\int_0^\rho (\bar{x}(t) - \bar{x}(t+\rho-s))\,\bar{x}(s)\,ds\right| \\
&\leq \left(\int_0^\rho (\bar{x}'(\xi_1)\,(\rho-s))^2\,ds\right)^{1/2} \left(\int_0^\rho (\bar{x}(0) + \bar{x}'(\xi_2)\,s)^2\,ds\right)^{1/2} \\
&\leq \|\bar{x}'\|_\infty \left(\int_0^\rho (\rho-s)^2\,ds\right)^{1/2} \left(2\int_0^\rho \bar{x}(0)^2\,ds + 2\int_0^\rho (\bar{x}'(\xi_2)\,s)^2\,ds\right)^{1/2} \\
&\leq \|\bar{x}'\|_\infty \frac{\rho^{3/2}}{\sqrt{3}}\sqrt{2}\left(\bar{x}(0)^2\,\rho + \|\bar{x}'\|_\infty^2 \frac{\rho^3}{3}\right)^{1/2}.
\end{aligned}
$$

Now consider

$$
\begin{aligned}
\|\epsilon_R\|_{L_2^\sigma(0,1)} &= \left\|2\int_0^R \int_0^\rho (\bar{x}(\cdot) - \bar{x}(\cdot+\rho-s))\,\bar{x}(s)\,ds\,d\eta(\rho)\right\|_{L_2^\sigma(0,1)} \\
&\leq \left\|2\int_0^R \left|\int_0^\rho (\bar{x}(\cdot) - \bar{x}(\cdot+\rho-s))\,\bar{x}(s)\,ds\right|\,d\eta(\rho)\right\|_{L_2^\sigma(0,1)} \\
&\leq 2\int_0^R \|\bar{x}'\|_\infty \sqrt{\frac{2}{3}}\,\rho^{3/2}\left(\bar{x}(0)^2\,\rho + \|\bar{x}'\|_\infty^2 \frac{\rho^3}{3}\right)^{1/2} d\eta(\rho)\,\|1\|_{L_2^\sigma(0,1)} \\
&\leq 2\int_0^R \|\bar{x}'\|_\infty \sqrt{\frac{2}{3}}\,\rho^{3/2}\left(\bar{x}(0)\,\rho^{1/2} + \|\bar{x}'\|_\infty \frac{\rho^{3/2}}{\sqrt{3}}\right) d\eta(\rho)\,\|1\|_{L_2^\sigma(0,1)} \\
&\leq \frac{2\sqrt{2}}{\sqrt{3}}\|\bar{x}'\|_\infty\,\bar{x}(0)\int_0^R \rho^2\,d\eta(\rho)\,\|1\|_{L_2^\sigma(0,1)} + \frac{2\sqrt{2}}{3}\|\bar{x}'\|_\infty^2 \int_0^R \rho^3\,d\eta(\rho)\,\|1\|_{L_2^\sigma(0,1)}.
\end{aligned}
$$

Similarly,we have

$$
\begin{aligned}
\|\epsilon_R\|_{L_2^\sigma(0,R)} \leq \frac{2\sqrt{2}}{\sqrt{3}}\|\bar{x}'\|_\infty\,\bar{x}(0)\int_0^R \rho^2\,d\eta(\rho)\,\|1\|_{L_2^\sigma(0,R)} \\
+ \frac{2\sqrt{2}}{3}\|\bar{x}'\|_\infty^2 \int_0^R \rho^3\,d\eta(\rho)\,\|1\|_{L_2^\sigma(0,R)}.
\end{aligned}
$$

As seen in the proof of the previous lemma,

$$
\|1\|_{L_2^\sigma(0,1)} = \sqrt{\frac{1-e^{2\sigma}}{2\sigma}} \leq 1 \qquad \text{for} \quad \sigma \geq 1/2,
$$

and

$$\|1\|_{L_2^\sigma(0,R)} = \sqrt{R + \mathcal{O}(R^2)}.$$

Therefore, the lemma follows. $\square$

Lemma 2.6 *If $\bar{x}(t) \in C^1[0,1]$ and for $\sigma \geq 1/2$, then $E_R(\bar{x}, x-\bar{x})$ defined in (2.24) satisfies*

$$\|E_R(\bar{x}, x-\bar{x})\|_{\sigma,R} \leq \frac{2}{\sqrt{3}} \|\bar{x}'\|_\infty e^{\sigma R} \|x-\bar{x}\|_{\sigma,R} \int_0^R \rho^{3/2}\, d\eta(\rho) \sqrt{1 + \frac{R + \mathcal{O}(R^2)}{\mathcal{C}(R)}}. \quad (2.31)$$

Proof: For fixed $t \in [0,1], \rho \in [0,R]$, we have

$$\begin{aligned}
|\, E_R(\bar{x}, x-\bar{x})(t)| &= |D(\bar{x})(x-\bar{x})(t) - (\alpha_R(x) - \alpha_R(\bar{x}))\bar{x}(t)| \\
&\leq 2 \int_0^R \int_0^\rho |\bar{x}(t+\rho-s) - \bar{x}(t)|\; |x(s) - \bar{x}(s)|\; ds\, d\eta(\rho) \\
&\leq 2\, \|\bar{x}'\|_\infty \int_0^R \int_0^\rho (\rho - s)\; |(x(s) - \bar{x}(s))|\; ds\, d\eta(\rho).
\end{aligned}$$

Consider, for fixed $\rho > 0$,

$$\begin{aligned}
&\int_0^\rho (\rho - s)\, |(x(s) - \bar{x}(s))|\, ds \\
&\leq \left(\int_0^\rho (\rho - s)^2\, ds \right)^{1/2} \left(\int_0^\rho e^{2\sigma s} e^{-2\sigma s} (x(s) - \bar{x}(s))^2\, ds \right)^{1/2} \\
&\leq \frac{\rho^{3/2}}{\sqrt{3}} e^{\sigma\rho}\, \|x - \bar{x}\|_{L_2^\sigma(0,1)}.
\end{aligned}$$

Therefore, for fixed t and ρ,

$$|E_R(\bar{x}, x-\bar{x})(t)| \leq \frac{2}{\sqrt{3}} \|\bar{x}'\|_\infty e^{\sigma R}\, \|x - \bar{x}\|_{L_2^\sigma(0,1)} \int_0^R \rho^{3/2}\, d\eta(\rho).$$

It is now not hard to see that

$$\|E_R(\bar{x}, x-\bar{x})\|_{L_2^\sigma(0,1)} \le \frac{2}{\sqrt{3}} \|\bar{x}'\|_\infty e^{\sigma R} \|x-\bar{x}\|_{L_2^\sigma(0,1)} \int_0^R \rho^{3/2}\, d\eta(\rho)\, \|1\|_{L_2^\sigma(0,1)},$$

and

$$\|E_R(\bar{x}, x-\bar{x})\|_{L_2^\sigma(0,R)} \le \frac{2}{\sqrt{3}} \|\bar{x}'\|_\infty e^{\sigma R} \|x-\bar{x}\|_{L_2^\sigma(0,1)} \int_0^R \rho^{3/2}\, d\eta(\rho)\, \|1\|_{L_2^\sigma(0,R)}.$$

Combine the above two inequalities with the fact that

$$\|x-\bar{x}\|_{L_2^\sigma(0,1)} \le \|x-\bar{x}\|_{\sigma,R},$$

we obtain the lemma. □

Lemma 2.7 *Assume the Borel measure $\eta > 0$ satisfies (2.8), if further $\bar{x}(t) \in C^1[0,1]$, then for $R > 0$ sufficiently small,*

$$\frac{1}{\alpha_R(\bar{x})} \le \frac{1}{\bar{x}(0) \int_0^R \rho\, d\eta(\rho)}. \tag{2.32}$$

Proof: We can write for some $\xi(s) \in [0, R]$,

$$\begin{aligned}
\alpha_R(\bar{x}) &= 2\int_0^R \int_0^\rho \bar{x}(s)\, ds\, d\eta(\rho) \\
&= 2\int_0^R \int_0^\rho (\bar{x}(0) + s\,\bar{x}'(\xi(s)))\, ds\, d\eta(\rho) \\
&= 2\,\bar{x}(0) \int_0^R \rho\, d\eta(\rho)\ [1 + g(R)],
\end{aligned}$$

where

$$g(R) = \frac{1}{\bar{x}(0)\int_0^R \rho\, d\eta(\rho)} \int_0^R \int_0^\rho s\, \bar{x}'(\xi(s))\, ds\, d\eta(\rho).$$

Since

$$|g(R)| \leq \frac{1}{\bar{x}(0)\int_0^R \rho\, d\eta(\rho)}\, \|\bar{x}'\|_\infty \int_0^R \frac{\rho^2}{2}\, d\eta(\rho) \leq \frac{\|\bar{x}'\|_\infty\, C(2,1,\eta)}{2\,\bar{x}(0)} R,$$

then

$$g(R) = \mathcal{O}(R) \to 0 \quad \text{as} \quad R \to 0.$$

Thus,

$$\begin{aligned}
\frac{1}{\alpha_R(\bar{x})} &= \frac{1}{2\,\bar{x}(0)\int_0^R \rho\, d\eta(\rho)} \cdot \frac{1}{1+g(R)} \\
&= \frac{1}{2\,\bar{x}(0)\int_0^R \rho\, d\eta(\rho)} \cdot [1 + g(R) + \mathcal{O}(R^2)],
\end{aligned}$$

where $|g(R) + \mathcal{O}(R^2)| \leq CR$ for R sufficiently small and some constant C. Therefore,

$$\frac{1}{\alpha_R(\bar{x})} \leq \frac{1}{2\,\bar{x}(0)\int_0^R \rho\, d\eta(\rho)} \cdot (1 + \mathcal{O}(R)) \leq \frac{1}{\bar{x}(0)\int_0^R \rho\, d\eta(\rho)}$$

for $R > 0$ sufficiently small. □

Lemma 2.8 *For $x, \bar{x} \in L_2^{\sigma,R}(0,1)$, we have*

$$|\alpha_R(x) - \alpha_R(\bar{x})| \leq 2\, e^{\sigma R} \int_0^R \rho^{1/2}\, d\eta(\rho) \cdot \sqrt{\mathcal{C}(R)} \cdot \|x - \bar{x}\|_{\sigma,R}. \tag{2.33}$$

Proof: For fixed $\rho \in [0, R]$, we have

$$\left|\int_0^\rho (x(s) - \bar{x}(s))\, ds\right| \le \left(\int_0^\rho (x(s) - \bar{x}(s))^2\, ds\right)^{1/2} \cdot \left(\int_0^\rho 1\, ds\right)^{1/2}$$
$$\le e^{\sigma R} \|x - \bar{x}\|_{L_2^\sigma(0,R)} \cdot \rho^{1/2}.$$

Therefore,

$$\begin{aligned}
|\alpha_R(x) - \alpha_R(\bar{x})| &= \left|2 \int_0^R \int_0^\rho (x(s) - \bar{x}(s))\, ds\, d\eta(\rho)\right| \\
&\le 2 \int_0^R \left|\int_0^\rho (x(s) - \bar{x}(s))\, ds\right| d\eta(\rho) \\
&\le 2 \int_0^R e^{\sigma R} \|x - \bar{x}\|_{L_2^\sigma(0,R)} \cdot \rho^{1/2} d\eta(\rho) \\
&= 2\, e^{\sigma R}\, \|x - \bar{x}\|_{L_2^\sigma(0,R)} \int_0^R \rho^{1/2} d\eta(\rho). \\
&\le 2\, e^{\sigma R} \int_0^R \rho^{1/2}\, d\eta(\rho) \cdot \sqrt{C(R)} \cdot \|x - \bar{x}\|_{\sigma,R}.
\end{aligned}$$

□

We are now ready to state our main convergence theorem.

Theorem 2.2 *Assume the autoconvolution problem (2.4) has a positive solution* $\bar{x} \in C^1[0, 1 + R]$ *satisfying*

$$\bar{x}(0) > 9\, b^2\, e^{2\sigma} \cdot \|\bar{x}'\|_\infty \tag{2.34}$$

for $\sigma = \max\left\{\sigma_0, \frac{1}{2}\right\}$ *and* $b = \max\{C(0, 1, \eta), C(2, 1, \eta)\}$. *Assume further that the Borel measure* $\eta(\rho) > 0$ *satisfies (2.8). Then there exist constants* $k_1 > 0$, $k_2 \neq 0$, *and* $\hat{C} > 0$, *all independent of* R, *such that if*

$$|f^\delta(t) - f(t)| \le \delta \le k_1 R^2, \qquad t \in [0, 1 + R], \tag{2.35}$$

then the regularized equation (2.27) has a unique solution x_R^δ satisfying

$$\frac{1}{k_2^2 R}\|x_R^\delta - \bar{x}\|^2_{L_2^g(0,R)} + \|x_R^\delta - \bar{x}\|^2_{L_2^g(0,1)} \leq \hat{C}^2 R^2.$$

Proof: We will apply the contraction mapping principle to the regularized equation (2.27) in the ball $\mathcal{B}(\bar{x}, \hat{C}R)$.

Let $\mathcal{C}(R) = k_2^2\, R$, we have

$$\sqrt{1 + \frac{R + \mathcal{O}(R^2)}{\mathcal{C}(R)}} = \sqrt{1 + \frac{R + \mathcal{O}(R^2)}{k_2^2\, R}} = \sqrt{\frac{k_2^2 + 1}{k_2^2}} \cdot \sqrt{1 + \mathcal{O}(R)}.$$

Let $k_3 = \sqrt{\dfrac{k_2^2 + 1}{k_2^2}}$, then $k_3 > 1$ and

$$\sqrt{1 + \frac{R + \mathcal{O}(R^2)}{\mathcal{C}(R)}} = k_3\,(1 + \mathcal{O}(R)) = k_3\, + \mathcal{O}(R).$$

Combining the results of the previous lemmas in this chapter, we have

$$
\begin{aligned}
&\|H_R x - \bar{x}\|_{\sigma,R} \\
&\leq \frac{1}{\alpha(\bar{x})}\|f_R^\delta - f_R\|_{\sigma,R} + \frac{1}{\alpha(\bar{x})}\|\epsilon_R\|_{\sigma,R} + \frac{1}{\alpha(\bar{x})}\|\mathcal{R}_R(\bar{x}, x-\bar{x})\|_{\sigma,R} \\
&\quad + \frac{1}{\alpha(\bar{x})}\|E_R(\bar{x}, x-\bar{x})\|_{\sigma,R} + \frac{|\alpha_R(x) - \alpha_R(\bar{x})|}{\alpha_R(\bar{x})}\|x-\bar{x}\|_{\sigma,R} \\
&\leq \frac{\delta}{\bar{x}(0)} C(0,1,\eta) R^{-1}(k_3 + \mathcal{O}(R)) \\
&\quad + \left(\frac{2\sqrt{2}}{\sqrt{3}}\|\bar{x}'\|_\infty C(2,1,\eta)\, R + \frac{2\sqrt{2}\,\|\bar{x}'\|_\infty^2}{3\bar{x}(0)} C(3,1,\eta)\, R^2\right)\cdot (k_3 + \mathcal{O}(R)) \\
&\quad + \frac{e^{2\sigma}}{\bar{x}(0)} C(0,1,\eta)\, R^{-1}\,\|x-\bar{x}\|_{\sigma,R}^2 \\
&\quad + \frac{2\,\|\bar{x}'\|_\infty\, e^{\sigma R}}{\sqrt{3}\,\bar{x}(0)} C(\frac{3}{2},1,\eta)\, R^{1/2}\,(k_3 + \mathcal{O}(R))\,\|x-\bar{x}\|_{\sigma,R} \\
&\quad + \frac{2\,e^{\sigma R}\sqrt{C(R)}}{\bar{x}(0)} C(\frac{1}{2},1,\eta)\, R^{-1/2}\,\|x-\bar{x}\|_{\sigma,R}^2
\end{aligned}
$$

for $\sigma \geq \max\{\sigma_0, \frac{1}{2}\}$.

Since $\|x-\bar{x}\|_{\sigma,R} \leq \hat{C}R$ and assumptions (2.35), we have

$$
\begin{aligned}
&\|H_R x - \bar{x}\|_{\sigma,R} \\
&\leq \frac{k_1(k_3 + \mathcal{O}(R))}{\bar{x}(0)} C(0,1,\eta)\, R + \frac{2\sqrt{2}}{\sqrt{3}}\|\bar{x}'\|_\infty C(2,1,\eta)(k_3 + \mathcal{O}(R))R \\
&\quad + \frac{2\sqrt{2}\,\|\bar{x}'\|_\infty^2}{3\bar{x}(0)} C(3,1,\eta)\,(k_3 + \mathcal{O}(R))\, R^2 + \frac{e^{2\sigma}}{\bar{x}(0)} C(0,1,\eta)\,\hat{C}^2\, R \\
&\quad + \frac{2\,\|\bar{x}'\|_\infty e^{\sigma R}}{\sqrt{3}\,\bar{x}(0)} C(\frac{3}{2},1,\eta)\,(k_3 + \mathcal{O}(R))\,\hat{C}\, R^{\frac{3}{2}} + \frac{2\,e^{\sigma R}\,k_2\, C(\frac{1}{2},1,\eta)}{\bar{x}(0)}\hat{C}^2\, R^2.
\end{aligned}
$$

Therefore, for sufficiently small R, to have $\|H_R x - \bar{x}\|_{\sigma,R} \leq \hat{C}R$ for some $\hat{C} > 0$, a sufficient condition is

$$
\frac{1}{\bar{x}(0)}\, b\, k_1\, k_3 + \frac{2\sqrt{2}}{\sqrt{3}}\|\bar{x}'\|_\infty\, b\, k_3 + \frac{e^{2\sigma}}{\bar{x}(0)}\, b\,\hat{C}^2 < \hat{C}. \tag{2.36}
$$

Let us pick k_1 such that

$$k_1 = \left(2 - \frac{2\sqrt{2}}{\sqrt{3}}\right) \|\bar{x}'\|_\infty \, \bar{x}(0),$$

equation (2.36) then becomes

$$L(\hat{C}) \equiv \frac{e^{2\sigma}\, b}{\bar{x}(0)} \hat{C}^2 - \hat{C} + 2\, b\, k_3 \, \|\bar{x}'\|_\infty < 0. \tag{2.37}$$

It is not hard to see $L(\hat{C}) = 0$ has two distinct positive solutions by assumption (2.34) when $k_3 \in (1, 9/8]$. Let us denote these two solutions by $\hat{C}_1$ and $\hat{C}_2$ such that $0 < \hat{C}_1 < \hat{C}_2$. Then for $\hat{C}$ satisfying $\hat{C}_1 < \hat{C} < \hat{C}_2$, we have $L(\hat{C}) < 0$, thus $\|H_R x - \bar{x}\|_{\sigma,R} \leq \hat{C}R$ for R sufficiently small.

To further demonstrate that H_R is a contraction on $\mathcal{B}(\bar{x}, \hat{C}R)$, we let $x_1, x_2 \in \mathcal{B}(\bar{x}, \hat{C}R)$, then

$$\begin{aligned}
&\|H_R\, x_1 - H_R\, x_2\|_{\sigma,R} \\
&= \|(\alpha_R(\bar{x})I + B_R(\bar{x}))^{-1}\{\mathcal{R}_R(\bar{x}, x_2 - \bar{x}) - \mathcal{R}_R(\bar{x}, x_1 - \bar{x}) + E_R(\bar{x}, x_1 - x_2) \\
&\qquad - [(\alpha_R(x_1) - \alpha_R(\bar{x}))(x_1 - \bar{x}) - (\alpha_R(x_2) - \alpha_R(\bar{x}))(x_2 - \bar{x})]\}\|_{\sigma,R} \\
&\leq \frac{1}{\alpha_R(\bar{x})}\|\mathcal{R}_R(\bar{x}, x_2 - \bar{x}) - \mathcal{R}_R(\bar{x}, x_1 - \bar{x})\|_{\sigma,R} + \frac{1}{\alpha_R(\bar{x})}\|E_R(\bar{x}, x_1 - x_2)\|_{\sigma,R} \\
&\quad + \frac{1}{\alpha_R(\bar{x})}\|(\alpha_R(x_1) - \alpha_R(\bar{x}))(x_1 - \bar{x}) - (\alpha_R(x_2) - \alpha_R(\bar{x}))(x_2 - \bar{x})\|_{\sigma,R}.
\end{aligned}$$

Since

$$
\begin{aligned}
&\frac{1}{\alpha_R(\bar{x})}\|(\alpha_R(x_1)-\alpha_R(\bar{x}))(x_1-\bar{x})-(\alpha_R(x_2)-\alpha_R(\bar{x}))(x_2-\bar{x})\|_{\sigma,R} \\
&= \frac{1}{\alpha_R(\bar{x})}\|(\alpha_R(x_1)-\alpha_R(x_2))(x_1-\bar{x})+(\alpha_R(x_2)-\alpha_R(\bar{x}))(x_1-x_2)\|_{\sigma,R} \\
&\le \frac{|\alpha_R(x_1)-\alpha_R(x_2)|}{\alpha(\bar{x})}\|x_1-\bar{x}\|_{\sigma,R}+\frac{|\alpha_R(x_2)-\alpha_R(\bar{x})|}{\alpha(\bar{x})}\|x_1-x_2\|_{\sigma,R} \\
&\le \frac{2\,e^{\sigma R}\int_0^R \rho^{\frac{1}{2}}\,d\eta(\rho)\cdot\sqrt{C(R)}}{\bar{x}(0)\int_0^R \rho\,d\eta(\rho)}\cdot\{\|x_1-x_2\|_{\sigma,R}\|x_1-\bar{x}\|_{\sigma,R}+\|x_2-\bar{x}\|_{\sigma,R}\,\|x_1-x_2\|_{\sigma,R}\} \\
&\le \frac{4\,e^{\sigma R}\,k_2\,C(\frac{1}{2},1,\eta)\hat{C}R}{\bar{x}(0)}\|x_1-x_2\|_{\sigma,R},
\end{aligned}
$$

then

$$
\begin{aligned}
\|H_Rx_1-H_Rx_2\|_{\sigma,R} \le{}& \left[\frac{2\,e^{2\sigma}\,C(0,1,\eta)}{\bar{x}(0)}\,\hat{C}\right]\|x_1-x_2\|_{\sigma,R} \\
&+\left[\frac{2\,\|\bar{x}'\|_\infty e^{\sigma R}}{\sqrt{3}\,\bar{x}(0)}\,C(\frac{3}{2},1,\eta)\,(k_3+\mathcal{O}(R))\,R^{\frac{1}{2}}+\frac{4\,e^{\sigma R}\,k_2\,C(\frac{1}{2},1,\eta)\hat{C}R}{\bar{x}(0)}\right]\|x_1-x_2\|_{\sigma,R}.
\end{aligned}
$$

Thus,

$$
\hat{C}<\frac{\bar{x}(0)}{2\,e^{2\sigma}\,b}\Longrightarrow\hat{C}<\frac{\bar{x}(0)}{2\,e^{2\sigma}\,C(0,1,\eta)}, \tag{2.38}
$$

and for $R>0$ sufficiently small this leads to

$$
\|H_R\,x_1-H_R\,x_2\|_{\sigma,R}\le q\cdot\|x_1-x_2\|_{\sigma,R}
$$

for any $x_1,x_2\in\mathcal{B}(\bar{x},\hat{C}R)$ with some $q<1$.

Notice further that $\dfrac{\hat{C}_1+\hat{C}_2}{2}=\dfrac{\bar{x}(0)}{2\,e^{2\sigma}\,b}$, therefore our regularized equation (2.27) has a unique solution in $\mathcal{B}(\bar{x},\hat{C}R)$ for $\hat{C}$ that satisfies $\hat{C}_1<\hat{C}<\dfrac{\bar{x}(0)}{2\,e^{2\sigma}\,b}$. □

Remark 2.2.1 *The usual convergence result is stated using the regularization parameter as a function of the noise level δ in the the data. In this case, the results of*

Theorem 2.2 indicate that we need $R = R(\delta)$ so that

$$R(\delta) \to 0 \quad and \quad \frac{\delta}{R^2(\delta)} \le k_1$$

as $\delta \to 0$.

Remark 2.2.2 *The local convergence rate obtained by Theorem 2.2 is $\mathcal{O}(\delta^{1/2})$, i.e.,*

$$\|x_R^\delta - \bar{x}\|_{\sigma,R} \sim \mathcal{O}(\delta^{1/2}) \quad as \ \delta \to 0.$$

2.3 Alternate Methods on $[0, R]$

As we will show in the next chapter, if we use a piecewise function to approximate the solution x, and recover x using the standard collocation on the interval $[0, 1]$ of the regularization equation (2.6), we have a sequential numerical method for recovering $x(t)$ for $R < t \le 1$. Unfortunately, for the x values on the interval $0 \le t \le R$, we have to solve a nonlinear system of equations which can be numerically expensive. This disadvantage motivates us to look for cheaper alternatives to recover x on the interval $0 \le t \le R$. In this section, we will propose some alternative methods and give theoretical basis for them.

In what follows we will show that for any function x_R sufficiently close to $\bar{x}$ on $[0, R]$, we may find a unique $\hat{x}_R^\delta \in L_2(0, 1)$ for which

$$\hat{x}_R^\delta(t) = x_R(t), \quad t \in [0, R],$$

and such that $\hat{x}_R^\delta$ satisfies equation (2.6) on the restricted interval $(R, 1]$. That is, $\hat{x}_R^\delta \in L_2(0, 1)$ satisfies

$$\alpha_R(x)\, x(t) + F_R(x)(t) = f_R^\delta(t), \quad t \in (R, 1]. \tag{2.39}$$

Further, we will see that under suitable conditions on the true solution $\bar{x}$ and the choice of $R = R(\delta)$, the function $\hat{x}_R^\delta$ is a good approximation of $\bar{x}$ for δ small. The advantage to this new approach is that we are free to find easier ways of determining an approximation x_R to $\bar{x}$ on $[0, R]$ than that obtained by solving equation (2.6) on the interval $[0, R]$.

Theorem 2.3 *Assume the autoconvolution problem (2.4) has a positive solution $\bar{x} \in C^1[0, 1+R]$ satisfying*

$$\bar{x}(0) > 9\, b^2\, e^{2\sigma} \cdot \|\bar{x}'\|_\infty \tag{2.40}$$

for $\sigma = \max\left\{\sigma_0, \frac{1}{2}\right\}$ and $b = \max\{C(0,1,\eta), C(2,1,\eta)\}$. Assume further that the Borel measure $\eta(\rho) > 0$ satisfies (2.8), and that $\{x_R\}_{R\in(0,\bar{R}]}$ is any family of $L_\infty(0,1)$ functions satisfying

$$\sup_{t\in[0,R]} |x_R(t) - \bar{x}(t)| \le \tilde{C} R^p, \tag{2.41}$$

for some $\tilde{C} > 0$ and $p > 1$. Then there exist constants $k_1 > 0$, $k_2 > 0$, and $\hat{C} > 0$, all independent of R, such that if

$$|f^\delta(t) - f(t)| \le \delta \le k_1 R^2, \qquad t \in [0, 1+R], \tag{2.42}$$

there exists an unique $\hat{x}_R^\delta \in L_2(0,1)$ with $\hat{x}_R^\delta = x_R(t)$, $t \in (0, R]$, such that $\hat{x}_R^\delta$ satisfies equation (2.6) for $t \in (R, 1]$ and for which

$$\frac{1}{k_2^2 R}\|\hat{x}_R^\delta - \bar{x}\|^2_{L_2^2(0,R)} + \|\hat{x}_R^\delta - \bar{x}\|^2_{L_2^2(0,1)} \le \hat{C}^2 R^2.$$

Proof: To begin, let $\hat{C} < \dfrac{\bar{x}(0)}{2\,e^{2\sigma}\,b}$ and define a new ball around $\bar{x}$ by

$$\hat{\mathcal{B}}_R(\bar{x}) \equiv \{x \in L_2^{\sigma,R}(0,1) : x|_{[0,R]} = x_R|_{[0,R]}, \|x - \bar{x}\|_{\sigma,R} \le \hat{C}R\}. \tag{2.43}$$

In what follows we will use the fact that $\hat{\mathcal{B}}_R(\bar{x}) \subseteq \mathcal{B}(\bar{x}, \hat{C}R)$, where $\mathcal{B}(\bar{x}, \hat{C}R)$ was fundamental to the proof of Theorem 2.2. We claim that the ball $\hat{\mathcal{B}}_R(\bar{x})$ is not empty. Indeed, if $\hat{x}_R \in L_2^{\sigma,R}(0,1)$ is defined as

$$\hat{x}_R = \begin{cases} x_R(t) & \text{if } t \in [0,R] \\ \bar{x}(t) & \text{if } t \in (R,1], \end{cases}$$

then, for $\mathcal{C}(R) = k_2^2 R$,

$$\begin{aligned} \|\hat{x}_R - \bar{x}\|_{\sigma,R}^2 &= \frac{1}{\mathcal{C}(R)}\|x_R - \bar{x}\|_{L_2^\sigma(0,R)}^2 + \|\hat{x}_R - \bar{x}\|_{L_2^\sigma(0,1)}^2 \\ &\le \frac{1}{\mathcal{C}(R)}\tilde{C}^2 R^{2p} \int_0^R e^{-2\sigma t}\,dt + \tilde{C}^2 R^{2p} \int_0^R e^{-2\sigma t}\,dt \\ &\le \tilde{C}^2 R^{2p}\left(1 + \frac{1}{k_2^2 R}\right) R \\ &= \frac{\tilde{C}^2 R^{2p}}{k_2^2}(1 + k_2^2 R) \\ &\le \hat{C}^2 R^2 \end{aligned} \tag{2.44}$$

for R sufficiently small and $p > 1$, i.e., $\hat{x}_R \in \hat{\mathcal{B}}_R(\bar{x})$.

Let us further define an new operator $\bar{H}_R$ such that

$$\bar{H}_R(x)(t) = \begin{cases} x(t) & \text{if } t \in [0,R] \\ H_R x(t) & \text{if } t \in (R,1]. \end{cases}$$

We will show that the operator $\bar{H}_R(x)$ has a fixed point in the ball $\hat{\mathcal{B}}_R(\bar{x})$ by contraction mapping theorem.

We let $x_1, x_2 \in \hat{\mathcal{B}}_R(\bar{x})$, then

$$
\begin{aligned}
&\|\bar{H}_R(x_1) - \bar{H}_R(x_2)\|^2_{\sigma,R} \\
&= \frac{1}{C(R)}\|\bar{H}_R(x_1) - \bar{H}_R(x_2)\|^2_{L_2^\sigma(0,R)} + \|\bar{H}_R(x_1) - \bar{H}_R(x_2)\|^2_{L_2^\sigma(0,1)} \\
&= 0 + \int_0^1 e^{-2\sigma t}\,(\bar{H}_R(x_1)(t) - \bar{H}_R(x_2)(t))^2\,dt \\
&= \int_R^1 e^{-2\sigma t}\,(\bar{H}_R(x_1)(t) - \bar{H}_R(x_2)(t))^2\,dt \\
&\le \|H_R(x_1) - H_R(x_2)\|^2_{L_2^\sigma(0,1)} \\
&\le \|H_R(x_1) - H_R(x_2)\|^2_{\sigma,R}.
\end{aligned}
$$

Therefore the same condition (2.38) as in Theorem 2.2 is needed for $\bar{H}_R$ to be a contraction in the ball $\hat{\mathcal{B}}_R(\bar{x})$.

We now consider $\|\bar{H}_R(x) - \bar{x}\|_{\sigma,R}$ for $x \in \hat{\mathcal{B}}_R(\bar{x})$. Note that

$$
\|\bar{H}_R(x) - \bar{x}\|^2_{\sigma,R} = \frac{1}{C(R)}\|\bar{H}_R(x) - \bar{x}\|^2_{L_2^\sigma(0,R)} + \|\bar{H}_R(x) - \bar{x}\|^2_{L_2^\sigma(0,1)},
$$

where

$$
\begin{aligned}
\|\bar{H}_R(x) - \bar{x}\|^2_{L_2^\sigma(0,R)} &= \int_0^R e^{-2\sigma R}\,(x_R(t) - \bar{x}(t))^2\,dt \\
&\le \tilde{C}^2\,R^{2p}\int_0^R e^{-2\sigma t}\,dt \\
&= \tilde{C}^2\,R^{2p}\,\frac{1 - e^{-2\sigma R}}{2\sigma},
\end{aligned}
$$

and

$$
\begin{aligned}
&\|\bar{H}_R(x) - \bar{x}\|^2_{L_2^\sigma[0,1]} \\
&= \int_0^R e^{-2\sigma R} \left(\bar{H}_R(x)(t) - \bar{x}(t)\right)^2 dt + \int_R^1 e^{-2\sigma R} \left(\bar{H}_R(x)(t) - \bar{x}(t)\right)^2 dt \\
&= \int_0^R e^{-2\sigma R} \left(x_R(t) - \bar{x}(t)\right)^2 dt + \int_R^1 e^{-2\sigma R} \left(H_R(x)(t) - \bar{x}(t)\right)^2 dt \\
&\leq \tilde{C}^2 R^{2p} \frac{1 - e^{-2\sigma R}}{2\sigma} + \|H_R(x) - \bar{x}\|^2_{L_2^\sigma(0,1)}.
\end{aligned}
$$

We therefore have

$$
\begin{aligned}
\|\bar{H}_R(x) - \bar{x}\|^2_{\sigma,R} &\leq \|H_R(x) - \bar{x}\|^2_{L_2^\sigma(0,1)} + \tilde{C}^2 R^{2p} \frac{1 - e^{-2\sigma R}}{2\sigma} \left(1 + \frac{1}{C(R)}\right) \\
&\leq \|H_R(x) - \bar{x}\|^2_{\sigma,R} + \tilde{C}^2 R^{2p} \frac{1 - e^{-2\sigma R}}{2\sigma} \left(1 + \frac{1}{C(R)}\right),
\end{aligned}
$$

or,

$$
\|\bar{H}_R(x) - \bar{x}\|_{\sigma,R} \leq \|H_R(x) - \bar{x}\|_{\sigma,R} + \tilde{C} R^p \sqrt{\frac{1 - e^{-2\sigma R}}{2\sigma} \left(1 + \frac{1}{C(R)}\right)}.
$$

If we let $M = \dfrac{2}{k_2}$, then

$$
\begin{aligned}
\sqrt{\frac{1 - e^{-2\sigma R}}{2\sigma} \left(1 + \frac{1}{C(R)}\right)} &= \sqrt{[R + \mathcal{O}(R^2)] \left(1 + \frac{1}{k_2^2 R}\right)} \\
&= \sqrt{\frac{1}{k_2^2} + \mathcal{O}(R)} \leq M
\end{aligned}
$$

for R sufficiently small. Thus

$$
\|\bar{H}_R(x) - \bar{x}\|_{\sigma,R} \leq \|H_R(x) - \bar{x}\|_{\sigma,R} + \tilde{C} M R^p. \tag{2.45}
$$

Notice that if $p > 1$, the corollary now follows from the proof of Theorem 2.2 under the exact same conditions. $\square$

This theorem suggests that as long as we have a higher than $\mathcal{O}(\mathcal{R})$ approximation for $\bar{x}$ on the interval $[0, R]$ and still use equation (2.6) to recover a new solution on the interval $(R, 1]$, convergence is guaranteed under the same conditions of Theorem 2.2. For example, if we have access to $\bar{x}(0)$ and $\bar{x}'(0)$, then the function

$$x_R(t) = \bar{x}(0) + \bar{x}'(0)t \tag{2.46}$$

for $t \in [0, R]$ satisfies (2.41) for $p = 2$. Therefore, it is theoretically justified that we can utilize this linear approximation in recovering the solution on the interval $[0, R]$.

Unfortunately, we do not always have access to $\bar{x}'(0)$, or to $\bar{x}(0)$ for that matter, despite the fact that a full convergence theory for other prominent regularization methods cannot be established unless the value of $\bar{x}(0)$ is actually made an explicit part of these methods. Indeed, convergence rates for Tikhonov regularization [6] and the Levrent'ev regularization method [12] (another method preserving the causality of the Volterra problem) cannot be obtained unless an auxiliary function x_0 is used in an essential way in these methods; here, x_0 is assumed to satisfy

$$x_0 = \bar{x} + G'(\bar{x})w$$

for suitable w. From the form of $G'(\bar{x})$ we see that $x_0(0) = \bar{x}(0)$, so that $\bar{x}(0)$ actually must be used as part of these methods. Thus, if we too make the assumption that $\bar{x}(0)$ is known, then by letting

$$x_R(t) = \bar{x}(0) \tag{2.47}$$

for $t \in [0, R]$, we only have an $\mathcal{O}(R)$ approximation of $\bar{x}(t)$ for $t \in [0, R]$. We will show in the following theorem that the convergence can still be obtained in the case of $p = 1$ under a slightly tighter condition on the true solution $\bar{x}$ and a restriction on

$\tilde{C}$ in (2.41).

Theorem 2.4 *Assume the autoconvolution problem (2.4) has a positive solution* $\bar{x} \in C^1[0, 1+R]$ *satisfying*

$$\bar{x}(0) > 13\, b^2\, e^{2\sigma} \cdot \|\bar{x}'\|_\infty \tag{2.48}$$

for $\sigma = \max\left\{\sigma_0, \frac{1}{2}\right\}$ *and* $b = \max\{C(0,1,\eta), C(2,1,\eta)\}$. *Assume further that the Borel measure* $\eta(\rho) > 0$ *satisfies (2.8), and that* $\{x_R\}_{R \in (0,\bar{R}]}$ *is any family of* $L_\infty(0,1)$ *functions satisfying*

$$\sup_{t \in [0,R]} |x_R(t) - \bar{x}(t)| \le \tilde{C} R^p, \tag{2.49}$$

where $p = 1$ *and some* $\tilde{C} > 0$. *Then there exist constants* $k_1 > 0$, $k_2 > 0$, *and* $\hat{C} > 0$, *all independent of* R, *such that if for* $\forall\ \mu > 0$ *fixed,*

$$\tilde{C} \le \frac{\hat{C} k_2}{1+\mu}. \tag{2.50}$$

and

$$|f^\delta(t) - f(t)| \le \delta \le k_1 R^2, \qquad t \in [0, 1+R], \tag{2.51}$$

then there exists an unique $\hat{x}_R^\delta \in L_2(0,1)$ *with* $\hat{x}_R^\delta = x_R(t)$, $t \in (0, R]$, *such that* $\hat{x}_R^\delta$ *satisfies equation (2.6) for* $t \in (R, 1]$ *and for which*

$$\frac{1}{k_2^2 R} \|\hat{x}_R^\delta - \bar{x}\|^2_{L_2^2(0,R)} + \|\hat{x}_R^\delta - \bar{x}\|^2_{L_2^2(0,1)} \le \hat{C}^2 R^2.$$

We are not going to repeat the proof since it follows the same procedure as the proof of Corollary 2.3. The following changes need to be made to adapt the smaller p ($p = 1$).

- To guarantee the ball $\hat{\mathcal{B}}_R(\bar{x})$ is nonempty, we need the extra condition (2.50). This can be easily seen from inequality (2.44).

- We cannot immediately use Theorem 2.2 after (2.45) is obtained. Instead, we argue that for sufficiently small R, a sufficient condition for $\|\bar{H}_R x - \bar{x}\|_{\sigma,R} \leq \hat{C}R$ is that

$$M\tilde{C} + \frac{1}{\bar{x}(0)}\, b\, k_1\, k_3 + \frac{2\sqrt{2}}{\sqrt{3}}\|\bar{x}'\|_\infty\, b\, k_3 + \frac{e^{2\sigma}}{\bar{x}(0)}\, b\, \hat{C}^2 < \hat{C}. \tag{2.52}$$

As in the proof of Theorem 2.2, we may still pick k_1 such that

$$k_1 = \left(2 - \frac{2\sqrt{2}}{\sqrt{3}}\right) \|\bar{x}'\|_\infty\, \bar{x}(0),$$

and it is also convenient to add a second condition on $\tilde{C}$, namely

$$\tilde{C} \leq \frac{b\, k_3\, \|\bar{x}'\|_\infty}{M} \leq \frac{b\, k_2\, k_3\, \|\bar{x}'\|_\infty}{2}. \tag{2.53}$$

Then inequality (2.52) is true if

$$\hat{L}(\hat{C}) \equiv \frac{e^{2\sigma}\, b}{\bar{x}(0)}\, \hat{C}^2 - \hat{C} + 3\, b\, k_3\, \|\bar{x}'\|_\infty < 0. \tag{2.54}$$

It is not hard to see $\hat{L}(\hat{C}) = 0$ has two distinct positive solutions by assumption (2.48) when $k_3 \in (1, 13/12]$. Let us denote these two solutions by $\hat{\hat{C}}_1$ and $\hat{\hat{C}}_2$ such that $0 < \hat{\hat{C}}_1 < \hat{\hat{C}}_2$. Then for $\hat{C}$ satisfying $\hat{\hat{C}}_1 < \hat{C} < \hat{\hat{C}}_2$, we have $\hat{L}(\hat{C}) < 0$, thus $\|H_R x - \bar{x}\|_{\sigma,R} \leq \hat{C}R$ for R sufficiently small.

- Notice that we have imposed two conditions on $\tilde{C}$ so far, and we need to show that they are compatible, i.e., if $\tilde{C}$ satisfies (2.53), it also satisfies (2.50). This implies that we need to show

$$\frac{b\,k_2\,k_3\,\|\bar{x}'\|_\infty}{2} \le \frac{\hat{C}k_2}{1+\mu},$$

or

$$\hat{C} \ge \frac{b\,k_3\,\|\bar{x}'\|_\infty(1+\mu)}{2}. \tag{2.55}$$

We note that the $\hat{C}$'s allowed by the corollary satisfy

$$\hat{\hat{C}}_1 < \hat{C} < \frac{\hat{\hat{C}}_1 + \hat{\hat{C}}_2}{2} = \frac{\bar{x}(0)}{2\,e^{2\sigma}b}, \tag{2.56}$$

where $0 < \hat{\hat{C}}_1 < \hat{\hat{C}}_2$ are the two solutions of $\hat{L}(C) = 0$. Combining (2.55) and (2.56), to ensure the existence of $\hat{C}$, all we need is that

$$\frac{b\,k_3\,\|\bar{x}'\|_\infty(1+\mu)}{2} < \frac{\bar{x}(0)}{2\,e^{2\sigma}b},$$

which is equivalent to

$$\bar{x}(0) > (1+\mu)\,b^2\,e^{2\sigma}\,k_3\,\|\bar{x}'\|_\infty. \tag{2.57}$$

We point out further that μ could be taken small when R is sufficiently small, therefore the condition (2.57) is ensured by the assumption (2.48) of the corollary.

Remark 2.3.1 *The results of Corollary 2.3 and Corollary 2.4 indicate that we need* $R = R(\delta)$ *so that*

$$R(\delta) \to 0 \quad \textit{and} \quad \frac{\delta}{R^2(\delta)} \le k_1$$

as $\delta \to 0$ *to obtain convergence.*

Remark 2.3.2 *The local convergence rates obtained by Corollary 2.3 and Corollary*

2.4 are $\mathcal{O}(\delta^{1/2})$, the same as Theorem 2.2; i.e.,

$$\|\hat{x}_R^\delta - \bar{x}\|_{\sigma,R} \sim \mathcal{O}(\delta^{1/2}) \quad \textit{as } \delta \to 0.$$

As mentioned earlier, Theorem 2.3 and 2.4 provide us the freedom of finding easier ways of determining the approximation x_R to $\bar{x}$ on $[0, R]$ than that obtained by solving equation (2.6) on the interval $[0, R]$. We have seen so far two possible x_R's we can use: a constant function on $[0, R]$ as defined in (2.47) or a linear function on $[0, R]$ as defined in (2.46), both of which require the value $\bar{x}(0)$, which is not always available. We also notice a waste of data when using those two x_R's since the collected data values $f^\delta(t)$ for $t \in [0, R]$ are not utilized at all in recovering the solution.

After investigating the numerical solution of the autoconvolution equation (2.1) (with f replaced by f^δ) after a standard collocation on the interval $[0, 1]$, we notice that the calculated solution recovers $\bar{x}$ quite well for small t, even though it does substantially worse as t increases. Note that no special regularization method is used other than changing an infinite-dimensional problem into a finite dimensional problem in the course of discretization. The increasingly worse solution on the interval $[0, 1]$ is not surprising since the error in the earlier part of the solution can propagate through the rest of the interval and due to the ill-posedness of the problem. But it does not prevent us from hoping that if we only solve the unregularized equation (2.1) on the small interval $[0, R]$, this solution x_R will be close to $\bar{x}$ on the interval $[0, R]$ for R sufficiently small.

We first discretize the autoconvolution equation

$$G(x)(t) = \int_0^t x(t-s)x(s)\,ds = f^\delta(t), \quad t \in [0, R], \tag{2.58}$$

with $|f^\delta(t) - f(t)| \le \delta$ for $t \in [0, R]$. Let $K = K(R) \ge 1$ be an integer (we will specify later), we partition the interval $[0, R]$ into K equal-length subintervals; i.e., let

$$\Delta t = R/K,$$

$$t_i = i\Delta t, \quad i = 0, 1, \cdots, K.$$

For $i = 2, 3, \cdots, K$, let $\chi_i(t)$ be the indicator function on the interval $(t_{i-1}, t_i]$. Let $\chi_1(t)$ be the indicator function on the interval $[t_0, t_1]$. We further denote an approximation space of piecewise constant functions on $[0, R]$ as $\mathcal{S}_K = \text{span}\{\chi_i, i = 1, 2, \cdots, K\}$.

A standard discretization of equation (2.58) involves finding $x_R \in \mathcal{S}_K$, i.e., x_R of the form

$$x_R(t) = \sum_{l=1}^{K} x_l \, \chi_l(t), \tag{2.59}$$

where the constants $x_l \in \mathbb{R}$, $l = 1, 2, \cdots, K$, are determined by requiring x to solve the collocation equations

$$G(x)(t_i) = f^\delta(t_i), \quad i = 1, 2, \cdots, K,$$

i.e.,

$$\int_0^{t_i} x(t_i - s)x(s)ds = f^\delta(t_i), \quad i = 1, 2, \cdots, K. \tag{2.60}$$

Therefore,

$$\sum_{\gamma=1}^{i} \int_{t_{\gamma-1}}^{t_\gamma} \left[\sum_{l=1}^{K} x_l \, \chi_l(t_i - s) \right] \left[\sum_{p=1}^{K} x_p \, \chi_p(s) \right] ds = f^\delta(t_i), \quad i = 1, 2, \cdots, K.$$

Note that for $s \in (t_{\gamma-1}, t_\gamma]$, $\chi_l(t_i - s) = 1$ iff $t_i - t_\gamma = t_{l-1}$, i.e., $l = i - \gamma + 1$; and for

$s \in (t_{\gamma-1}, t_\gamma]$, $\chi_p(s) = 1$ iff $p = \gamma$. Thus the collocation equations become

$$\sum_{\gamma=1}^{i} \int_{t_{\gamma-1}}^{t_\gamma} x_{i-\gamma+1}\, x_\gamma\, ds = f^\delta(t_i), \quad i = 1, 2, \cdots, K,$$

i.e.,

$$\sum_{\gamma=1}^{i} x_{i-\gamma+1}\, x_\gamma = \frac{f^\delta(t_i)}{\Delta t}, \quad i = 1, 2, \cdots, K. \tag{2.61}$$

The collocation equations (2.61) allow us to explicitly solve for x_i, $i = 1, 2, \cdots, K$, provided that $f^\delta(t_1) > 0$. Namely,

$$x_1 = \sqrt{\frac{f^\delta(t_1)}{\Delta t}}, \tag{2.62}$$

and if $x_1, \cdots, x_{i-1}$ have been found already, x_i is determined by

$$x_i = \frac{\frac{f^\delta(t_i)}{\Delta t} - (x_{i-1}x_2 + \cdots + x_2 x_{i-1})}{2x_1}, \tag{2.63}$$

therefore, $x_2, x_3, \cdots, x_K$ can be found uniquely and sequentially.

So far, we have shown that there is a unique solution x_R to the discrete autoconvolution equation (2.60), and $x_R(t) = \sum_{l=1}^{K} x_l\, \chi_l(t)$ for $t \in [0, R]$ with constants x_i's specifically given by (2.62) and (2.63). We will show in the following claim that this x_R is a close approximation of $\bar{x}$ on the interval $[0, R]$ for R sufficiently small.

Corollary 2.1 *Assume the autoconvolution problem (2.1) has a positive solution* $\bar{x} \in C^1[0,1]$. *Let* $x_R = \sum_{l=1}^{K} x_l\, \chi_l(t)$ *be the unique solution of the discrete autoconvolution equation (2.60) on the interval [0,R], where the constants* x_i, $i = 1, 2, \cdots, K$, *are specified in (2.62) and (2.63) and* $K = K(R) \geq 1$ *is an integer. Then if there exists a constant* $\tilde{M} > 0$, *such that* $K = K(R) \leq \tilde{M}$ *uniformly in* R, *and*

$$|f^\delta(t) - f(t)| = |\delta(t)| \leq \delta \leq k_1 R^2, \qquad t \in [0, 1], \tag{2.64}$$

convergence of $x_R(t)$ to the true solution $\bar{x}(t)$ for $t \in [0, R]$ occurs at the collocation points as $R \to 0$, i.e.,

$$|x_R(t_i) - \bar{x}(t_i)| \sim \mathcal{O}(R), \quad \textit{for } i = 1, 2, \cdots, K, \tag{2.65}$$

for R sufficiently small. Further, we have a constant $\tilde{C}$ depending on $\bar{x}$ but independent of R such that

$$|x_R(t) - \bar{x}(t)| \leq \tilde{C} R \tag{2.66}$$

for R sufficiently small and all $t \in [0, R]$. Thus if $\bar{x}$ is such that $\tilde{C}$ satisfies (2.50), we obtain the conclusions of Theorem 2.4 if we use the family $\{x_R\}_{R \in (0, \bar{R}]}$ where x_R is defined by (2.59), (2.62) and (2.63).

Proof: The true solution $\bar{x}$ solves the autoconvolution equation (2.1) for any $t \in [0, 1]$, $\bar{x}$ then solves more specifically the following collocation equations.

$$\int_0^{t_i} \bar{x}(t_i - s)\bar{x}(s)\, ds = f(t_i), \quad i = 1, 2, \cdots, K,$$

or

$$\sum_{\gamma=1}^{i} \int_{t_{\gamma-1}}^{t_\gamma} \bar{x}(t_i - s)\bar{x}(s)\, ds = f(t_i), \quad i = 1, 2, \cdots, K.$$

Since $\bar{x} \in C^1[0, 1]$, then for $0 \leq t_{\gamma-1} \leq s \leq t_\gamma \leq R$, there exist $\zeta_{i,1}(s), \zeta_{i,1}(s) \in [0, R]$ such that

$$\bar{x}(t_i - s) = \bar{x}(t_{i-\gamma+1}) + (t_i - s - t_{i-\gamma+1})\, \bar{x}'(\zeta_{i,1}(s)) = \bar{x}(t_{i-\gamma+1}) + \mathcal{O}(\Delta t),$$

and

$$\bar{x}(s) = \bar{x}(t_\gamma) + (s - t_\gamma)\bar{x}'(\zeta_{i,2}(s)) = \bar{x}(t_\gamma) + \mathcal{O}(\Delta t).$$

Therefore, we have

$$\sum_{\gamma=1}^{i}\int_{t_{\gamma-1}}^{t_\gamma}(\bar{x}(t_{i-\gamma+1})\,\bar{x}(t_\gamma)+\mathcal{O}(\Delta t))\,ds=f(t_i),\quad i=1,2,\cdots,K,$$

i.e.,

$$\Delta t\sum_{\gamma=1}^{i}(\bar{x}(t_{i-\gamma+1})\,\bar{x}(t_\gamma)+\mathcal{O}(\Delta t))=f(t_i),\quad i=1,2,\cdots,K,$$

or

$$\sum_{\gamma=1}^{i}\bar{x}(t_{i-\gamma+1})\,\bar{x}(t_\gamma)=\frac{f(t_i)}{\Delta t}+g_i(\Delta t),\tag{2.67}$$

where $|g_i(\Delta t)|\le\bar{k}\Delta t$ for $i=1,2,\cdots,K$ and $\bar{k}>0$ constant independent of K,R,δ. The collocation equations (2.67) allow us to solve for $\bar{x}(t_i)$ for $i=1,2,\cdots,K$. To simplify the notation, we will denote $\bar{x}(t_i)$ as $\bar{x}_i$ from now on. Then we have

$$\bar{x}_1=\sqrt{\frac{f(t_1)}{\Delta t}+g_1(\Delta t)},\tag{2.68}$$

and if $\bar{x}_1,\cdots,\bar{x}_{i-1}$ have been found already, $\bar{x}_i$ is determined by

$$\bar{x}_i=\frac{\frac{f(t_i)}{\Delta t}+g_i(\Delta t)-(\bar{x}_{i-1}\bar{x}_2+\cdots+\bar{x}_2\bar{x}_{i-1})}{2\bar{x}_1},\tag{2.69}$$

therefore, $\bar{x}_2,\bar{x}_3,\cdots,\bar{x}_K$ can be found using equation (2.69).

We will now prove by induction that

$$|x_i-\bar{x}_i|\sim\mathcal{O}(R),\quad\text{for } i=1,2,\cdots,K,$$

for R sufficiently small.

For $i=1$, we have

$$x_1^2-\bar{x}_1^2=\frac{\delta_1}{\Delta t}-g_1(\Delta t),$$

therefore

$$x_1 - \bar{x}_1 = \left(\frac{\delta_1}{\Delta t} - g_1(\Delta t)\right) \cdot \frac{1}{x_1 + \bar{x}_1},$$

where $\delta_i = f^\delta(t_i) - f(t_i)$ for $i = 1, 2, \cdots, N$. Notice that

$$\begin{aligned} x_1 + \bar{x}_1 &= \sqrt{\frac{f(t_1) + \delta_1}{\Delta t}} + \sqrt{\frac{f(t_1) + \Delta t g(\Delta t)}{\Delta t}} \\ &= \frac{1}{\sqrt{\Delta t}} \left(\sqrt{f(t_1) + \delta_1} + \sqrt{f(t_1) + \Delta t\, g(\Delta t)}\right), \end{aligned}$$

since $|\delta_i| \leq \delta \leq k_1 R^2$ for $i = 1, 2, \cdots, N$, we have

$$\begin{aligned} \sqrt{f(t_1) + \delta_1} &= \sqrt{f(t_1)}\,(1 + \mathcal{O}(R)) \\ \sqrt{f(t_1) + \Delta t\, g(\Delta t)} &= \sqrt{f(t_1)}\,(1 + \mathcal{O}(R)). \end{aligned}$$

Therefore, for $\Delta t = R/K$ and R sufficiently small, we have

$$\begin{aligned} |x_1 - \bar{x}_1| &\leq \frac{\delta + \Delta t g_1(\Delta t)}{\sqrt{\Delta t}} \cdot \frac{1}{2\sqrt{f(t_1)}}(1 + \mathcal{O}(R)) \\ &\leq (k_1\, K^{1/2}\, R^{3/2} + \bar{k}\, K^{-3/2}\, R^{3/2}) \cdot \frac{1}{2\sqrt{f(t_1)}}(1 + \mathcal{O}(R)), \end{aligned}$$

obviously, $1 \leq K = K(R) \leq \tilde{M}$ guarantees $|x_1 - \bar{x}_1|$ to be at least $\mathcal{O}(R)$. Let there exist a constant q_1 such that $x_1 - \bar{x}_1 = q_1 R$.

Assume that for $1 \leq j \leq i-1$ and R sufficiently small, $|x_j - \bar{x}_j| = C_j(R)R$ for some constant $C_j(R) > 0$ uniformly bounded independent of R, then there exist a constant $q_2 = q_2(i, R) > 0$ such that

$$|(x_{i-1}x_2 + \cdots + x_2 x_{i-1}) - (\bar{x}_{i-1}\bar{x}_2 + \cdots + \bar{x}_2\bar{x}_{i-1})| \leq q_2 R,$$

where $q_2(i, R)$ depends on C_j's for $j = 1, \cdots, i-1$ and $q_2(i, R)$ uniformly bounded

independent of R. Let us denote

$$\Delta_i \equiv f^\delta(t_i) - (x_{i-1}x_2 + \cdots + x_2 x_{i-1}),$$
$$\hat{\Delta}_i \equiv \frac{f(t_i)}{\Delta t} + g_i(\Delta t) - (\bar{x}_{i-1}\bar{x}_2 + \cdots + \bar{x}_2\bar{x}_{i-1}),$$

then

$$\begin{aligned} |x_i - \bar{x}_i| &= \left|\frac{\Delta_i}{2x_1} - \frac{\hat{\Delta}_i}{2\bar{x}_1}\right| = \left|\frac{\bar{x}_1\Delta_i - x_1\hat{\Delta}_i}{2x_1\bar{x}_1}\right| \\ &= \left|\frac{\bar{x}_1\Delta_i - (\bar{x}_1 + q_1 R)\hat{\Delta}_i}{2x_1\bar{x}_1}\right| \le \left|\frac{\Delta_i - \hat{\Delta}_i}{2x_1}\right| + \left|\frac{q_1 R\hat{\Delta}_i}{2x_1\bar{x}_1}\right|. \end{aligned}$$

Note that

$$\begin{aligned} |\Delta_i - \hat{\Delta}_i| &= \left|\frac{\delta_i}{\Delta t} - g_i(\Delta t) - [(x_{i-1}x_2 + \cdots + x_2x_{i-1}) - (\bar{x}_{i-1}\bar{x}_2 + \cdots + \bar{x}_2\bar{x}_{i-1})]\right| \\ &\le \frac{k_1 R^2}{R/K} + \bar{k}\frac{R}{K} + q_2 R \\ &\le (k_1 K + \frac{\bar{k}}{K} + q_2)R, \end{aligned}$$

and $\hat{\Delta}_i = \bar{x}_i \cdot 2\bar{x}_1$ is a positive constant, then for $1 \le K = K(R) \le \tilde{M}$ uniformly in R, we have

$$|x_i - \bar{x}_i| \sim \mathcal{O}(R)$$

for R sufficiently small. Therefore, (2.65) is true.

It follows immediately that for $t \in [0, R]$,

$$\left|\sum_{l=1}^{k} x_l \chi_l(t) - \bar{x}(t)\right| \sim \mathcal{O}(R)$$

for R sufficiently small, since for $t_{l-1} \le t \le t_l$, $\bar{x}(t) = \bar{x}(t_l) + \mathcal{O}(R)$ for R sufficiently small. □

We have shown in Corollary 2.1 that the unique solution $x_R(t) = \sum_{l=1}^{k} x_l \chi_l(t)$ to the discrete autoconvolution equation (2.60) for $t \in [0, R]$ is an $\mathcal{O}(R)$ approximation of $\bar{x}(t)$ on the interval $[0, R]$. By Theorem 2.4, this could provide us another alternative for constructing a convergent regularized solution $\hat{x}_R^\delta$. Namely, we solve the discrete autoconvolution equation (2.60) for $t \in [0, R]$, and then the regularized equation (2.6) for $t \in [R, 1]$. This approach allows us to fully utilize the measured data ($f^\delta(t)$ for all $t \in [0, 1]$) and the measured data only (no $\bar{x}$ value required) in recovering the solution. Note that the size of $\tilde{C}$ depends on the true solution $\bar{x}$, so that the verification of the condition (2.50) depends on the problem. However, as we will show in the next chapter, the numerical results suggest the success of this approach.

CHAPTER 3

Discretization and Numerical Results

To solve the autoconvolution equation (2.1) on the interval $[0, 1]$ numerically without any special regularization, the same kind of discretization approach as described in Section 2.3 can be utilized on the interval [0,1]. Due to the causal nature of the problem, the discretized version of equation (2.1) can be sequentially, thus efficiently solved. However, the inherent instability of the problem remains prominent in the solution. In this chapter, we will consider a discretized version of the regularized equation (2.6), which leads to a stable method on the interval $[0, 1]$ that is sequential for the majority of the interval. We will then summarize the approaches motivated by Theorem 2.3 and 2.4, all of which are sequential on the interval $[0, 1]$. We also include the numerical results for all discussed methods.

3.1 Discretization of the Regularized Equation

Recall the regularized equation (2.6) of finding $x \in L_2(0,1)$ satisfying

$$2\int_0^R \int_0^\rho x(s)\,ds\,d\eta(\rho)x(t) + \int_0^R \int_\rho^t x(t+\rho-s)\,x(s)\,ds\,d\eta(\rho) = \int_0^R f^\delta(t+\rho)\,d\eta(\rho), \tag{3.1}$$

for $t \in [0,1]$. In this section, we will discretize this regularized equation through standard collocation on the interval $[0,1]$. For simplicity, we will assume that

$$\int_0^R g(\rho)\,d\eta(\rho) = \int_0^R g(\rho)\,d\rho. \tag{3.2}$$

Note that the simple Borel measure $\eta(\rho)$ used in (3.2) satisfies the general condition as required in (2.8).

Let $N = 1, 2, 3, \cdots$, We partition the interval $[0,1]$ into N equal-length subintervals; i.e., let

$$\Delta t = 1/N,$$
$$t_i = i\Delta t, \quad i = 0, 1, \cdots, N.$$

For $i = 2, 3, \cdots, N$, let $\chi_i(t)$ be the indicator function on the interval $(t_{i-1}, t_i]$. Let $\chi_1(t)$ be the indicator function on the interval $[t_0, t_1]$. We further denote an approximation space of piecewise constant functions on $[0,1]$ as $\mathcal{S}_N = \text{span}\{\,\chi_i,\, i = 1, 2, \cdots, N\}$.

A standard discretization of equation (3.1) involves finding $x \in \mathcal{S}_N$, i.e., x of the form

$$x(t) = \sum_{p=1}^N c_p\,\chi_p(t), \tag{3.3}$$

where the constants $c_p \in \mathbb{R}$, $p = 1, 2, \cdots, N$, are determined by requiring x to solve the collocation equations

$$2\int_0^R \int_0^\rho x(s)\,ds\,d\rho\, x(t_i) + \int_0^R \int_\rho^{t_i} x(t_i+\rho-s)\,x(s)\,ds\,d\rho = \int_0^R f^\delta(t_i+\rho)\,d\rho \quad (3.4)$$

for $i = 1, 2, \cdots, N$. We will further assume $R = r\Delta t$, where $r \in \{1, 2, \cdots, N\}$ is fixed, and we will use the apporximation

$$\int_0^R g(\rho)\,d\rho \approx \Delta t \cdot \sum_{j=1}^{r} g(t_{j-1}).$$

We will now study equation (3.4) term by term for fixed $i = 1, 2, \cdots, N$. First, we consider the right hand side of equation (3.4), and we have

$$\int_0^R f^\delta(t_i+\rho)\,d\rho = \Delta t\,[f^\delta(t_i) + f^\delta(t_{i+1}) + \cdots + f^\delta(t_{i+r-1})] = \Delta t \cdot \sum_{q=0}^{r-1} f^\delta(t_{i+q}).$$

The first term on the left hand side of equation (3.4) becomes

$$\begin{aligned}
&2\int_0^R \int_0^\rho x(s)\,ds\,d\rho \cdot x(t_i)\\
&= 2\Delta t \left(\int_0^0 x(s)\,ds + \int_0^{t_1} x(s)\,ds + \cdots + \int_0^{t_{r-1}} x(s)\,ds\right) \cdot c_i\\
&= 2\Delta t\ [0 + c_1\Delta t + (c_1+c_2)\Delta t + \cdots + (c_1+c_2+\cdots+c_{r-1})\Delta t] \cdot c_i\\
&= 2\,(\Delta t)^2 \cdot \left(\sum_{l=1}^{r-1}\sum_{m=1}^{l} c_m\right) \cdot c_i,
\end{aligned}$$

where we notice that the coefficient of c_i involves only $c_1, c_2, \cdots, c_{r-1}$. The second term of the left hand side is more complicated, and, depending on the value of i in relation to r, we come to different forms for the term. In general, for any fixed $i = 1, 2, \cdots, N$, we have

$$\int_0^R \int_\rho^{t_i} x(t_i+\rho-s)\,x(s)\,ds\,d\rho = \Delta t \sum_{j=0}^{r-1} \int_{t_j}^{t_i} x(t_{i+j}-s)x(s)\,ds. \qquad (3.5)$$

Further investigation of equation (3.5) leads us to the following cases.

- If $i < r - 1$, then

$$\begin{aligned}
&\int_0^R \int_\rho^{t_i} x(t_i + \rho - s)\, x(s)\, ds\, d\rho \\
&= (\Delta t)^2 \{[(c_1 c_i + \cdots + c_i c_1) + (c_2 c_i + \cdots + c_i c_2) + \cdots + c_i c_i] \\
&\quad - [c_{i+1} c_{i+1} + (c_{i+1} c_{i+2} + c_{i+2} c_{i+1}) + \cdots + (c_{i+1} c_{r-1} + \cdots + c_{r-1} c_{i+1})]\} \\
&= (\Delta t)^2 \cdot \left[\sum_{m=1}^{i} \sum_{l=m}^{i} c_l\, c_{i+m-l} - \sum_{m=i+1}^{r-1} \sum_{l=i+1}^{m} c_l\, c_{i+1+m-l} \right],
\end{aligned}$$

which is nonlinear in c_i and involves values beyond c_i; more specifically, it involves all values of $c_1, c_2, \cdots, c_{r-1}$.

- If $i = r - 1$, then

$$\begin{aligned}
&\int_0^R \int_\rho^{t_i} x(t_i + \rho - s)\, x(s)\, ds\, d\rho \\
&= (\Delta t)^2 [(c_1 c_i + \cdots + c_i c_1) + (c_2 c_i + \cdots + c_i c_2) + \cdots + c_i c_i] \\
&= (\Delta t)^2 \cdot \sum_{m=1}^{i} \sum_{l=m}^{i} c_l\, c_{i+m-l},
\end{aligned}$$

which is also nonlinear in c_i and involves all values of $c_1, c_2, \cdots, c_{r-1}$.

- If $i \geq r$, then

$$\begin{aligned}
&\int_0^R \int_\rho^{t_i} x(t_i + \rho - s)\, x(s)\, ds\, d\rho \\
&= (\Delta t)^2 \cdot [0 + (c_1 c_i + \cdots + c_i c_1) + (c_2 c_i + \cdots + c_i c_2) + \cdots + (c_r c_i + \cdots + c_i c_r)] \\
&= (\Delta t)^2 \cdot \sum_{m=1}^{r} \sum_{l=m}^{i} c_l\, c_{i+m-l}, \qquad (3.6)
\end{aligned}$$

With this result, we notice that specifically for $i = r$, (3.6) is quadratic in c_r once $c_1, c_2, \cdots, c_{r-1}$ are found. Even better, for $i > r$, (3.6) is linear in c_i once

$c_1, c_2, \cdots, c_r$ are found.

Therefore, to determine the values for c_i where $i = 1, 2, \cdots, N$, we solve the following N equations.

- For $i = 1, 2, \cdots, r-2$, we have the first $r-2$ equations.

$$\left(2\,(\Delta t)^2 \cdot \sum_{l=1}^{r-1} \sum_{m=1}^{l} c_m\right) \cdot c_i + (\Delta t)^2 \cdot \left[\sum_{m=1}^{i} \sum_{l=m}^{i} c_l\, c_{i+m-l} - \sum_{m=i+1}^{r-1} \sum_{l=i+1}^{m} c_l\, c_{i+1+m-l}\right] = \Delta t \cdot \sum_{q=0}^{r+1} f^{\delta}(t_{i+q}).$$

Note that all values of $c_1, c_2, \cdots, c_{r-1}$ are involved in any of these $r-2$ equations.

- For $i = r-1$, we have the $(r-1)$-th equation.

$$\left(2\,(\Delta t)^2 \cdot \sum_{l=1}^{r-1} \sum_{m=1}^{l} c_m\right) \cdot c_i + (\Delta t)^2 \cdot \sum_{m=1}^{i} \sum_{l=m}^{i} c_l\, c_{i+m-l} = \Delta t \cdot \sum_{q=0}^{r+1} f^{\delta}(t_{i+q}).$$

Note that all values of $c_1, c_2, \cdots, c_{r-1}$ are also involved in this $(r-1)$-th equation. Since we now have $r-1$ equations with $r-1$ unknowns, we are ready to solve $c_1, c_2, \cdots, c_{r-1}$. We would expect this procedure numerically expensive since we are basically solving a system of nonlinear equations for $c_1, c_2, \cdots, c_{r-1}$.

- Once $c_1, c_2, \cdots, c_{r-1}$ are found, we can solve for $c_r, \cdots, c_N$ sequentially by

$$\left(2\,(\Delta t)^2 \cdot \sum_{l=1}^{r-1} \sum_{m=1}^{l} c_m\right) \cdot c_i + (\Delta t)^2 \cdot \sum_{m=1}^{r} \sum_{l=m}^{i} c_l\, c_{i+m-l} = \Delta t \cdot \sum_{q=0}^{r+1} f^{\delta}(t_{i+q})$$

for $i = r, r+1, \cdots, N$. As noted before, we still have to solve a nonlinear equation for c_r since the r-th equation is quadratic in c_r. However, once $c_1, c_2, \cdots, c_r$ have already been determined, the remaining $N-r$ equations can be sequentially solved quickly since the ith equation is linear in c_i for $i > r$.

We have described so far a collocation scheme on the local regularization equation (2.6) for finding a step function that satisfies the equation (2.6) at N discrete points. This method (hereafter referred to as Method 1) leads to a nonlinear system of equations in recovering $x(t)$ for $t \in [0, R]$, even though it is sequential and linear in recovering $x(t)$ for $t \in (R, 1]$. Several cheaper and effective alternative methods to recover $x(t)$ on the interval [0,R] are presented below; their convergence was guaranteed by the two theorems in Section 2.3. While using equation (2.6) to recover $x(t)$ on the interval $(R, 1]$, these methods obtain $x(t)$ on the interval $[0, R]$ much more efficiently.

- Method 2: let $x(t) = \bar{x}(0)$ for $t \in [0, R]$.
- Method 3: let $x(t) = \bar{x}(0) + \bar{x}'(0)t$ for $t \in [0, R]$.
- Method 4: let $x(t)$ be the unique solution to the discretized autoconvolution equation (2.60) on the interval $[0, R]$, where no special regularization is used. For details, we refer to Section 2.3.

3.2 Numerical Results

The examples in this section provide evidence of the effectiveness of the local regularization methods on the autoconovolution equation. The local regularization methods are also superior to the existing methods in that they don't necessarily require the value of $\bar{x}(0)$ and they maintain the causal nature of the problem at least for the majority of the domain. In order for easy comparison with the results of existing methods [9] [12] , we try to recover a continuous $\bar{x}$ and a discontinuous one. We select our true solution $\bar{x}$ ahead of time, then generate the data function f by integration, $f(t) = \int_0^t \bar{x}(t-s)\bar{x}(s)\,ds$, for $t \in [0, 1]$. The perturbed data f^δ was then produced by adding uniformly distributed noise from the interval $[-\delta f(t), \delta f(t)]$ to the discrete values of $f(t)$ for $t = t_i$, where $i = 1, 2, \cdots, N$. In each of the examples, we show the recovered solution against the true solution $\bar{x}$, first without any regularization, and then with regularization using Methods 1, 2, 3 and 4. In all figures presented below, the true solution $\bar{x}$ is plotted as a dashed line, while the solid line expresses the approximate solution computed according to the specified method.

3.2.1 Example with continuous $\bar{x}$ function

The true solution in this example is a continuous function

$$\bar{x}(t) = 1 - 3(t - 1/2)^2, \quad 0 \le t \le 1,$$

with the true data then given by

$$f(t) = \frac{3}{10}t^5 - \frac{3}{2}t^4 + t^3 + \frac{3}{4}t^2 + \frac{1}{16}t, \quad 0 \le t \le 1.$$

For Figures 3.1–3.5, we use relative noise level $\delta = 10^{-3}$, $N = 100$ and $r = 4$ for the regularized problems. All sample solutions shown use exactly the same noisy data. In Figure 3.1, we show the solution without any special regularization. As we can see, the recovered solution contains strong oscillation, especially for t large in the domain $[0, 1]$. Figures 3.2–3.5 show the regularized solution using Methods 1–4 as introduced in the last section. All four methods are effective in producing stable solutions that are very close to the true solution $\bar{x}$. It is worth noting in Figure 3.3 that, even after the poor start in approximating $\bar{x}$ (as dictated by the method), the solution was able to come back on track.

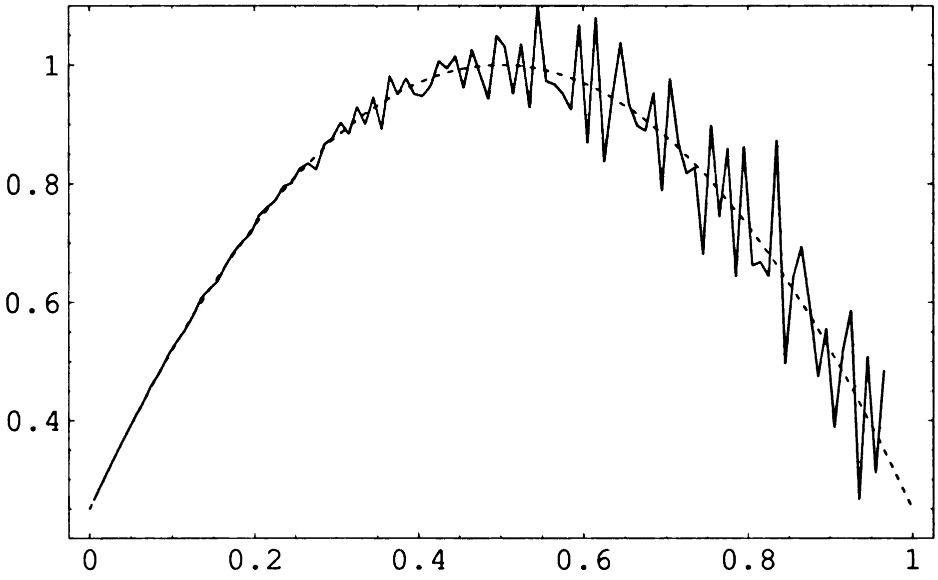

Figure 3.1. Solution without regularization, $\delta = 10^{-3}$, $N = 100$

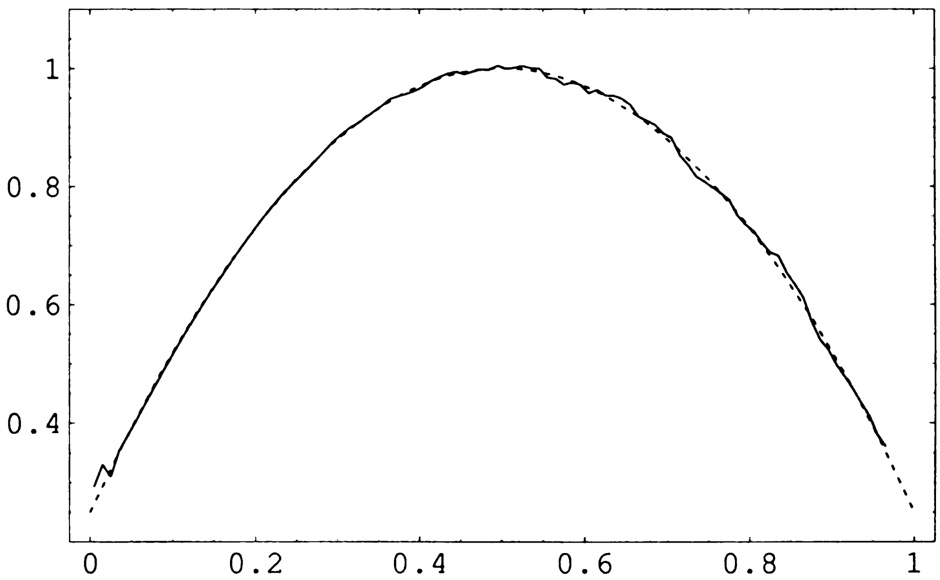

Figure 3.2. Solution obtained by Method 1, $\delta = 10^{-3}$, $N = 100$, $r = 4$

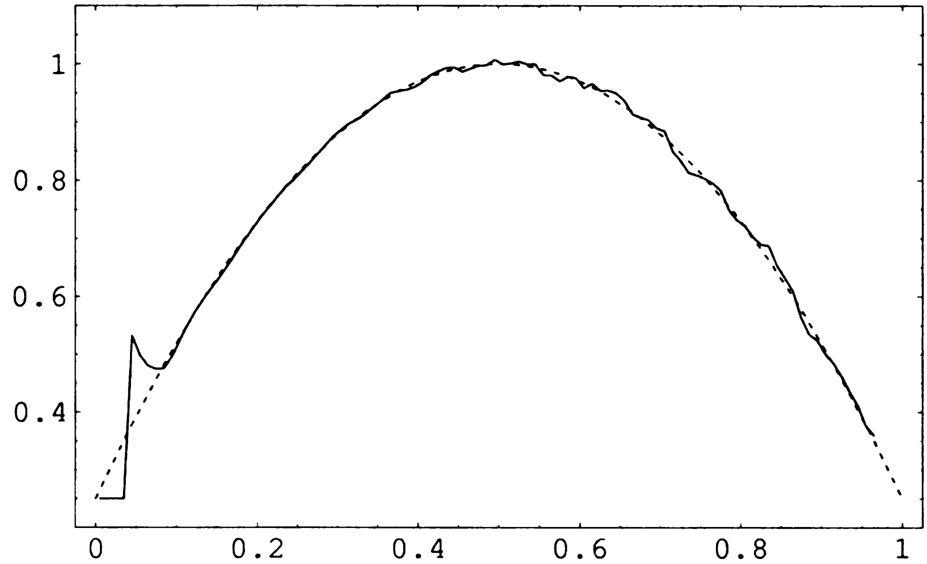

Figure 3.3. Solution obtained by Method 2, $\delta = 10^{-3}$, $N = 100$, $r = 4$

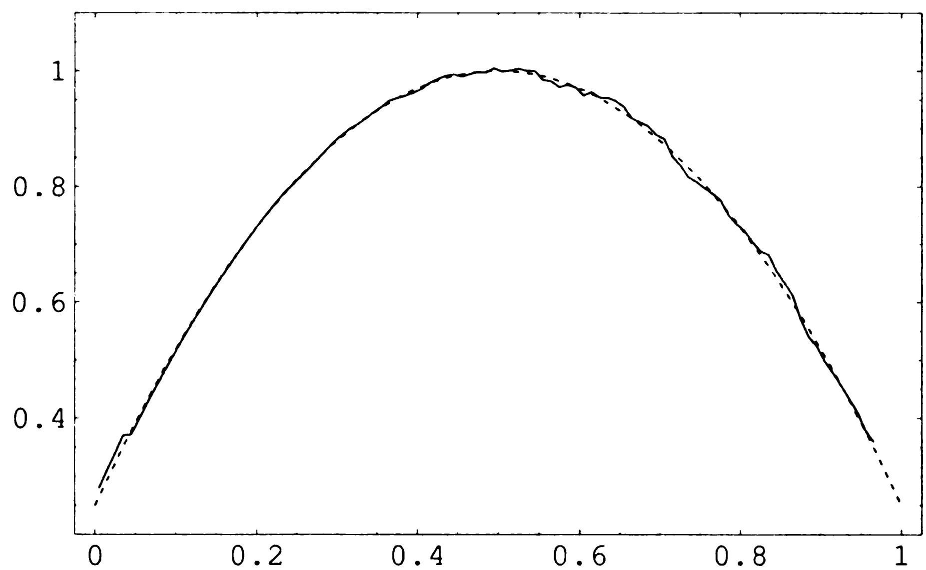

Figure 3.4. Solution obtained by Method 3, $\delta = 10^{-3}$, $N = 100$, $r = 4$

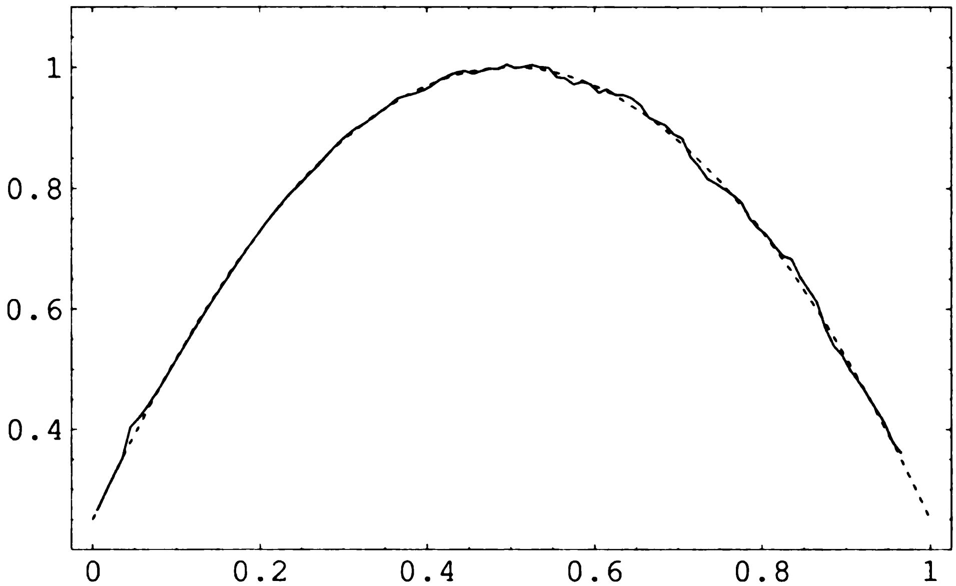

Figure 3.5. Solution obtained by Method 4, $\delta = 10^{-3}$, $N = 100$, $r = 4$

For Figures 3.6–3.10, we use a bigger relative noise level $\delta = 10^{-2}$, $N = 100$. In Figure 3.6, we show the solution without any special regularization. Figures 3.7–3.10 show the regularized solution using Methods 1-4. We use $r = 4$ for Method 1 and $r = 6$ for Methods 2–4. We can observe the same effectiveness in the regularization methods as in the case of noise level $\delta = 10^{-3}$.

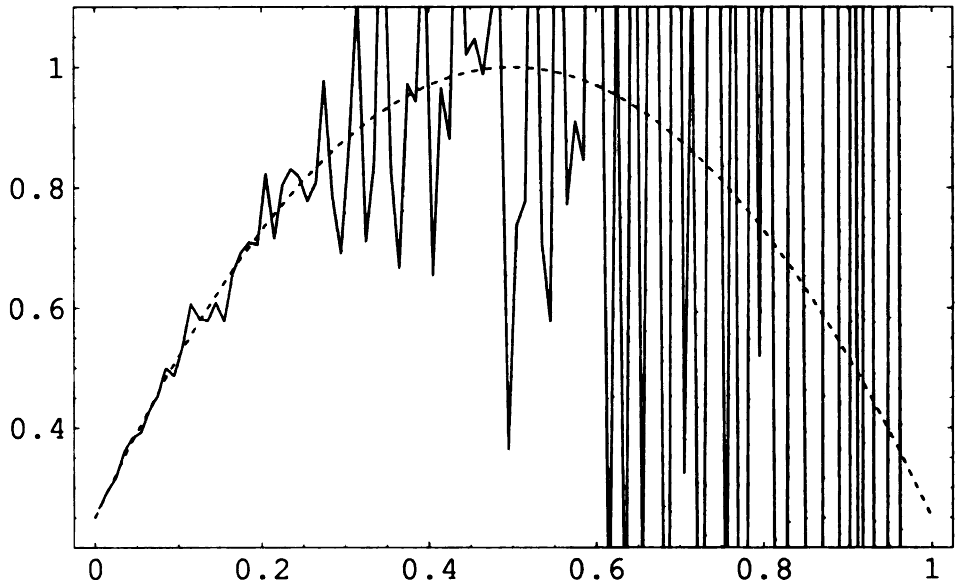

Figure 3.6. Solution without regularization, $\delta = 10^{-2}$, $N = 100$

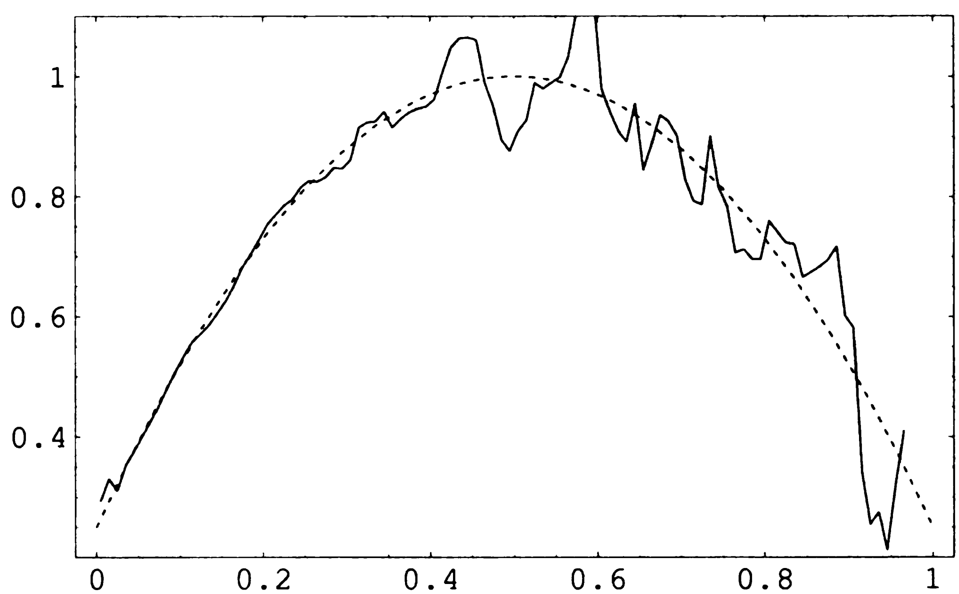

Figure 3.7. Solution obtained by Method 1, $\delta = 10^{-2}$, $N = 100$, $r = 4$

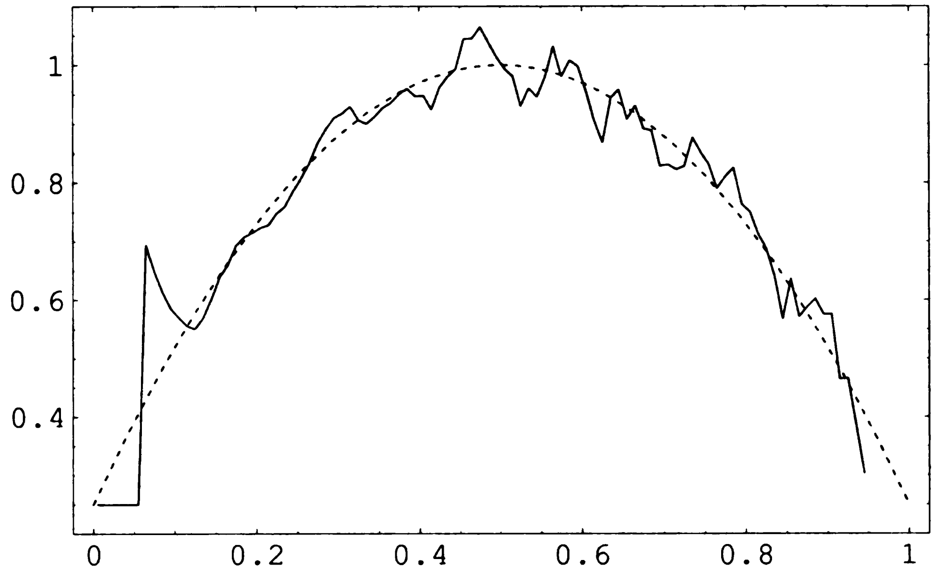

Figure 3.8. Solution obtained by Method 2, $\delta = 10^{-2}$, $N = 100$, $r = 6$

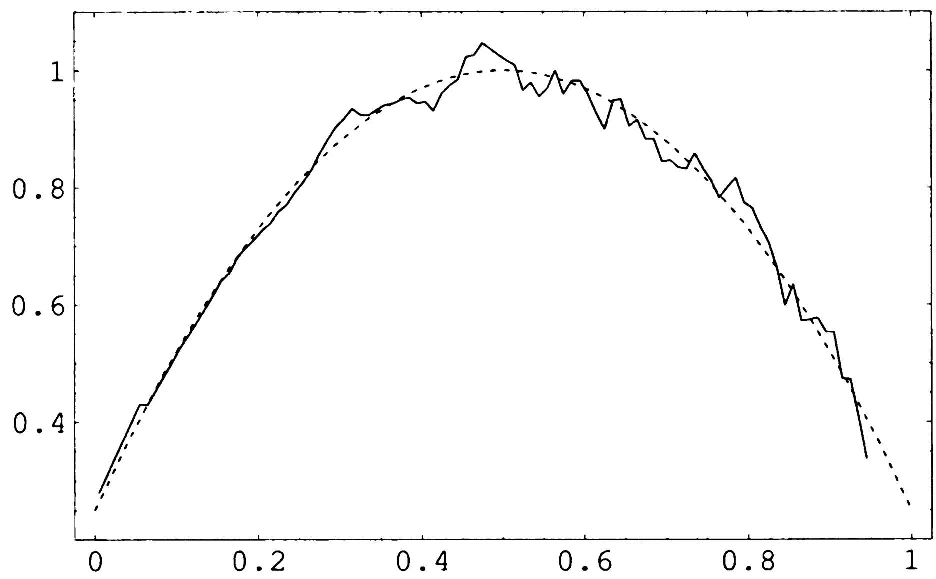

Figure 3.9. Solution obtained by Method 3, $\delta = 10^{-2}$, $N = 100$, $r = 6$

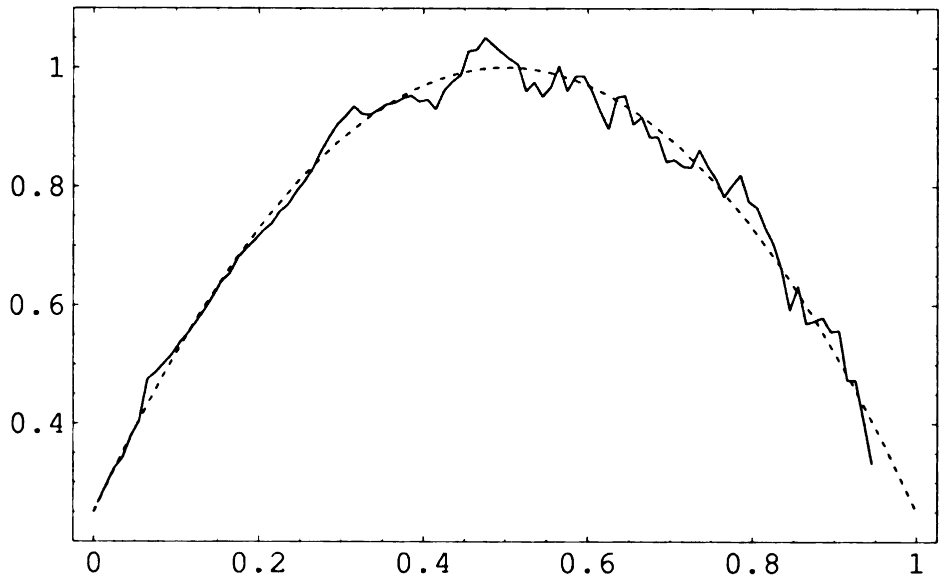

Figure 3.10. Solution obtained by Method 4, $\delta = 10^{-2}$, $N = 100$, $r = 6$

3.2.2 Example with discontinuous $\bar{x}$ function

In this example, we will use a step function as our true solution

$$\bar{x}(t) = \begin{cases} 0.5 & \text{if } t \in [0, 0.5] \\ 0.25 & \text{if } t \in (0.5, 0.8] \\ 0.75 & \text{if } t \in (0.8, 1], \end{cases}$$

the true data f is then given by

$$f(t) = \begin{cases} 0.25t & \text{if } t \in [0, 0.5] \\ 0.125 & \text{if } t \in (0.5, 0.8] \\ 0.5t - 0.275 & \text{if } t \in (0.8, 1]. \end{cases}$$

For all the figures shown in this example, we use relative noise level $\delta = 10^{-3}$, $N = 200$ and $r = 4$ for the regularized problem. Again, all sample solutions shown use the same noisy data. In Figure 3.11, we show the solution without any special regularization. As we can see, the recovered solution contains increasingly strong oscillation as t increases in the domain $[0, 1]$. Figures 3.12-3.15 show the regularized solution using Methods 1-4. All four methods effectively produce stable solutions, where the location of jumps are determined quite precisely. We note in this example that all four regularization methods produce nearly identical solutions. The reason is that our four methods differ only in how they handle the interval $[0, R]$, and our discontinuous $\bar{x}$ is constant on that interval, and all methods are reasonably good at reconstructing the constant. These results are much better than that were presented in [9] [12]. It is worth noting that the true solution $\bar{x}$ here does not satisfy the assumptions of Theorem 2.2. We point out that the assumptions in Theorem 2.2 are only sufficient conditions for convergence, and convergence may also exist for the $\bar{x}$'s that don't satisfy those assumptions.

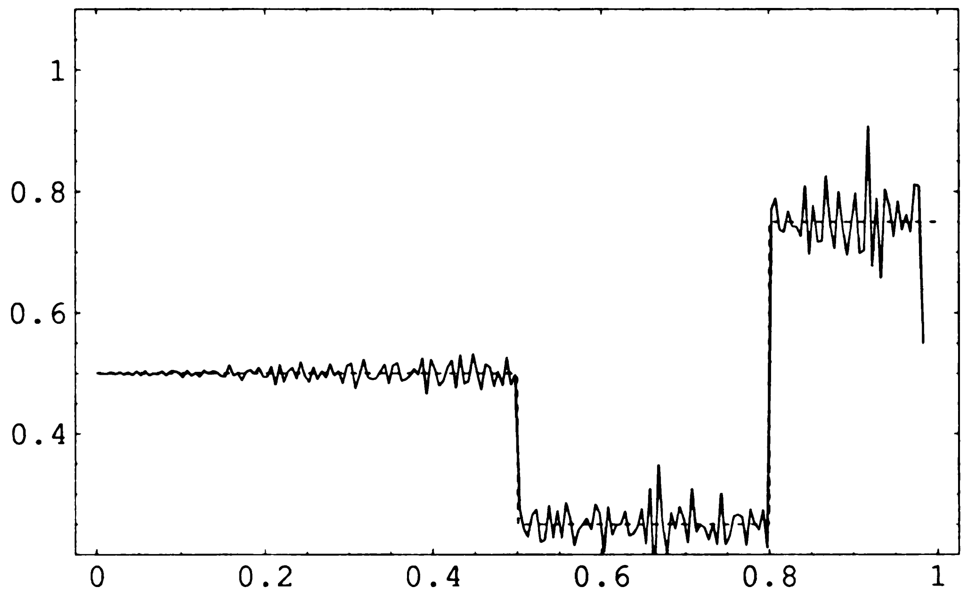

Figure 3.11. Solution without regularization, $\delta = 10^{-3}$, $N = 200$

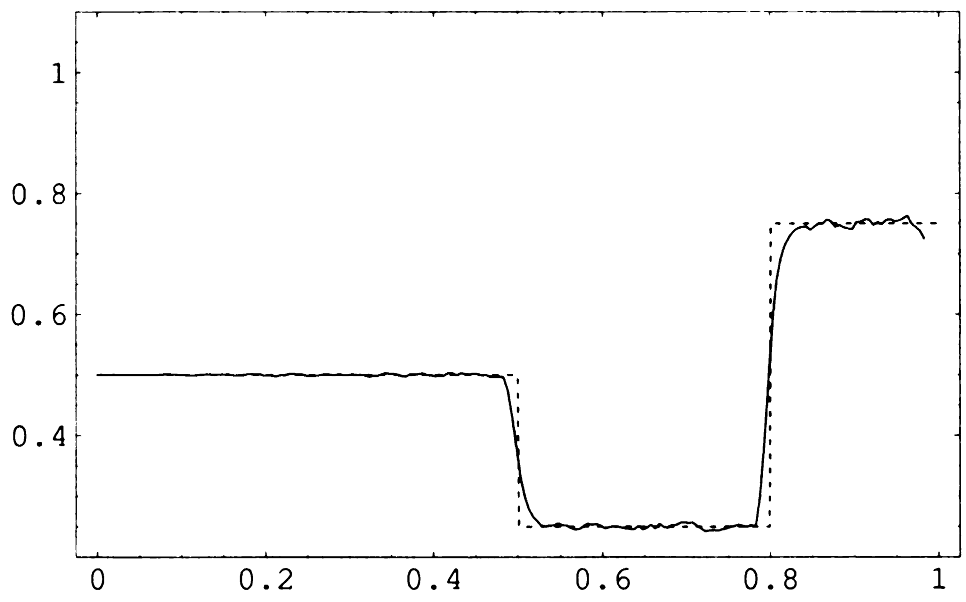

Figure 3.12. Solution obtained by Method 1, $\delta = 10^{-3}$, $N = 200$, $r = 4$

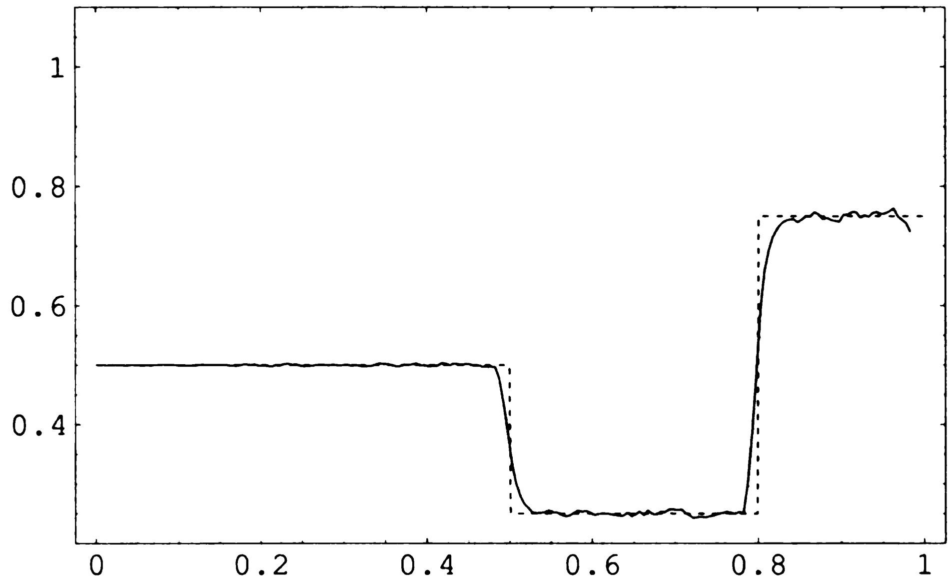

Figure 3.13. Solution obtained by Method 2, $\delta = 10^{-3}$, $N = 200$, $r = 4$

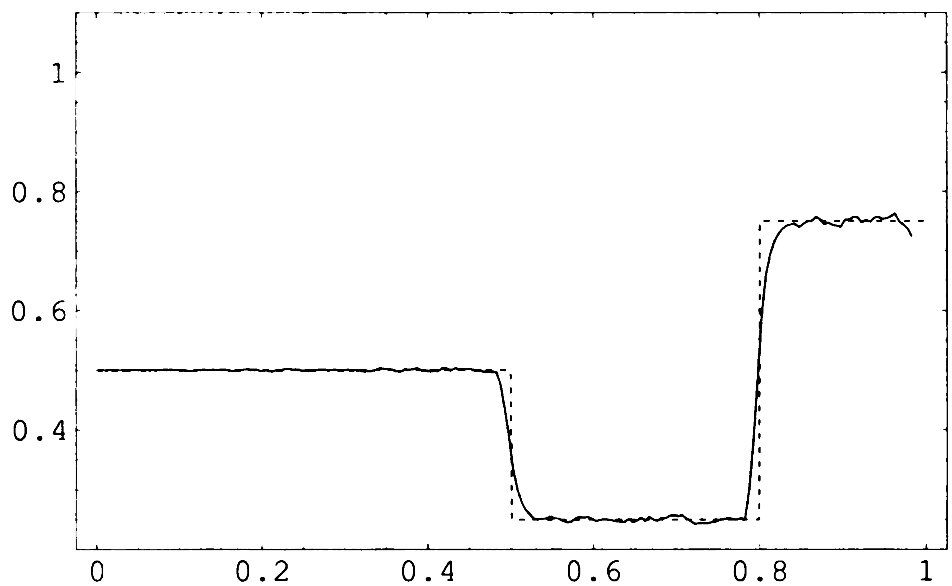

Figure 3.14. Solution obtained by Method 3, $\delta = 10^{-3}$, $N = 200$, $r = 4$

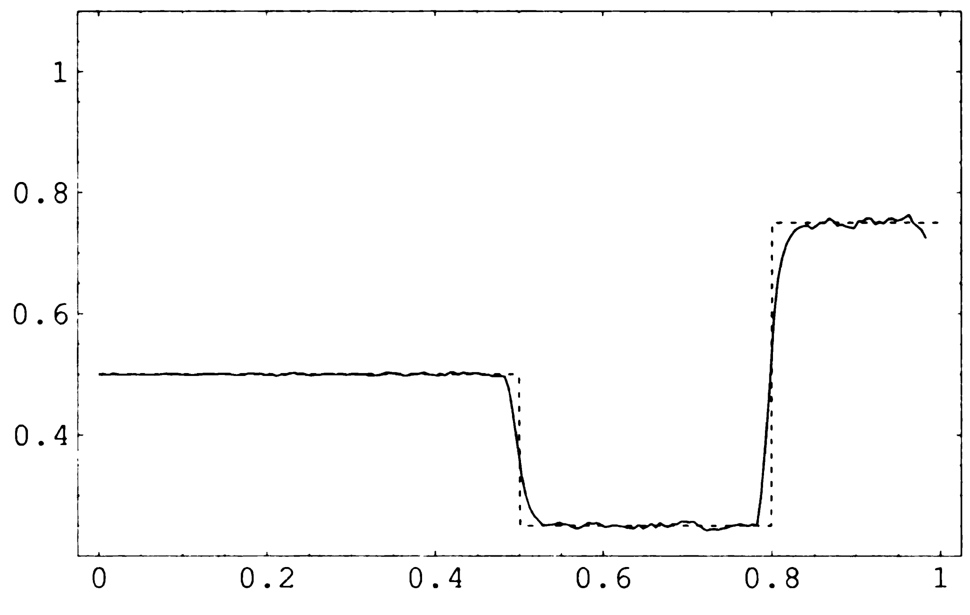

Figure 3.15. Solution obtained by Method 4, $\delta = 10^{-3}$, $N = 200$, $r = 4$

CHAPTER 4

Summary and Future Work

In this paper, we developed a local regularization theory for the autoconvolution equation, a nonlinear Volterra problem. Several local regularization methods have been established, all of which provide stable solutions for the autoconvolution problem. The methods also preserve the causal nature of the autoconvolution equation, allowing for numerically fast sequential solution.

Using the underlying idea of local regularization, we formulated the local regularized equation for the autoconvolution problem. We proved the convergence of the solution produced by this regularized equation to the true solution of the autoconvolution equation as the noise level in the data shrinks to zero. The convergence rate we obtained is the same as that of the classic Tikhonov regularization and Lavrent'ev regularization.

Simple collocation of the regularized equation leads to a sequential method over the majority of the domain, i.e., on the interval $(R, 1]$; and a nonlinear system of equations for $t \in [0, R]$. This motivates us to look for other alternatives in seeking the solution on the interval $[0, R]$, while still maintaining the same regularized equation on the interval $(R, 1]$. We proved two theorems, which provided the convergence of two alternative methods. However, both of those two methods require some knowledge of the true solution $\bar{x}(t)$ at $t = 0$. In practice, we do not always have access to the true solution. A fourth method was then presented, where we simply solve the unregularized discrete autoconvolution equation on the interval $[0, R]$. We demonstrated also the convergence of this method which result from using this piecewise constant function on $[0, R]$.

Finally, we have shown numerical results which provide evidence that the local regularization methods developed in this work are superior to the other existing regularization methods, especially in capturing sharp features in the solution. In fact, the numerical results confirm the effectiveness of these local regularization methods, even in cases not completely falling under the assumptions of the general theory we developed here. It is our hope that, through further study, we will be able to weaken the conditions imposed on the true solution $\bar{x}$ in Theorem 2.2 and Corollary 2.3 and 2.4.

One of the most commonly asked question with regard to the local regularization methods is how one picks the regularization parameter r. It is currently an open question for linear problems, and we are also seeking answers in the case of the nonlinear autoconvolution equation.

The Discrepancy Principle is one of the most successful criteria in determining the regularization parameter α in the Tikhonov regularization. To summarize the Discrepancy Principle, we assume that the perturbed data f^δ has an absolute noise level $\hat{\delta}$, i.e., $||f^\delta - f|| \leq \hat{\delta}$. The Tikhonov theory states that for every choice of $\alpha > 0$, the Tikhonov problem (6) has a unique solution u_α^δ, and that the discrepancy

$$\delta_d \equiv ||\mathcal{A}u_\alpha^\delta - f^\delta||$$

is monotone in α. The discrepancy principle picks α such that

$$\delta_d = \tau\hat{\delta}$$

with $\tau \geq 1$ some constant. In practice, τ is often picked as $\sqrt{2}$.

In the following, we use the discrepancy principle in picking our best r for a given relative noise level δ and fixed discretization parameter N. Method 4 is used for all numerical experiments that follow. We first investigate the example of the continuous $\bar{x}$ as presented in Section 3.2.1. Note that exactly the same noisy data is used for results within the same table. In Table 4.1– 4.3, we present a comparison of the values of discrepancy δ_d and the absolute data noise $\hat{\delta}$ for different values of r at various relative noise levels. Unlike the Tikhonov regularization, the discrepancy δ_d is not exactly a monotone increasing function of the regularization parameter r. Therefore, we predict a good r is such that the discrepancy δ_d first exceeds the absolute data noise $\hat{\delta}$. The Discrepancy Principle then suggests $r = 13$ for $\delta = 10^{-2}$, $r = 9$ for $\delta = 5 \times 10^{-3}$, and $r = 5$ for $\delta = 10^{-3}$, as highlighted in the tables. It is a satisfying observation that the suggested r decreases as the relative noise level in the data decreases, since, naturally, less regularization is needed for less noisy data. To further demonstrate the Discrepancy Principle does work in this case, we included in Figure 4.1 the numerical result using the predicted $r = 13$ at relative noise level $\delta = .01$. We can see a significant improvement in recovering the true $\bar{x}$ than that in Figure 3.10, where $r = 6$ was used. In fact, the relative root-mean-square (rms) error of the reconstructed x is 0.0258251 with $r = 13$, which is just about half of the rms error (0.0502236) using $r = 6$, when the same set of noisy data is used.

r	$\hat{\delta}$	δ_d
8	.0249610	.0188627
9	.0236252	.0182476
10	.0231554	.0194804
11	.0226741	.0197377
12	.0226101	.0211548
• 13	.0219453	.0228909
14	.0219144	.0182378

Table 4.1. Relative noise level $\delta = 10^{-2}$, $N = 100$, continuous $\bar{x}$

r	$\hat{\delta}$	δ_d
7	.0130590	.0112185
8	.0124805	.0117506
• 9	.0118126	.0123196
10	.0115777	.0140569

Table 4.2. Relative noise level $\delta = 5 \times 10^{-3}$, $N = 100$, continuous $\bar{x}$

r	$\hat{\delta}$	δ_d
4	.00270788	.00265035
• 5	.00269173	.00371632
6	.00262241	.00508314
7	.00261181	.00663777
8	.00249610	.00821011

Table 4.3. Relative noise level $\delta = 10^{-3}$, $N = 100$, continuous $\bar{x}$

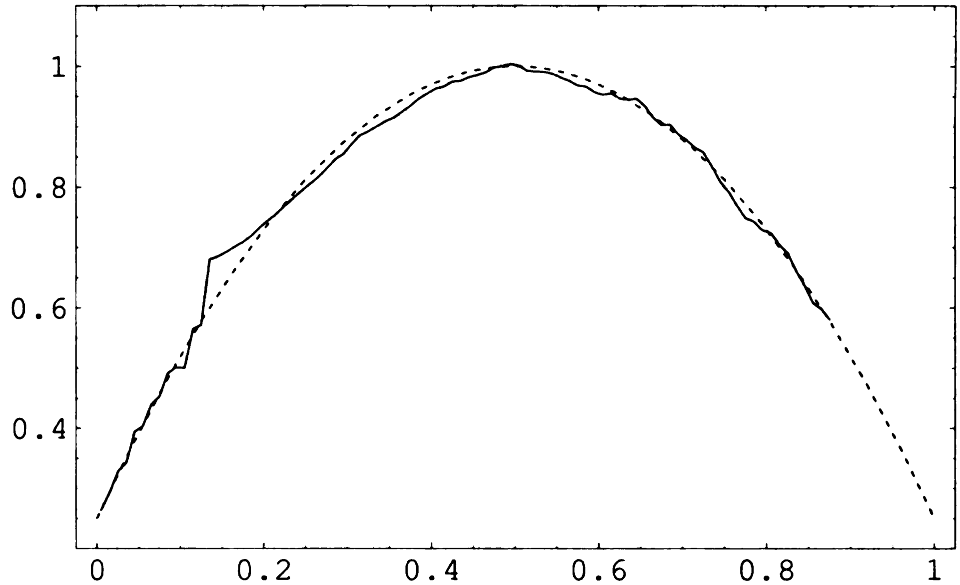

Figure 4.1. Solution obtained by Method 4, $\delta = 10^{-2}$, $N = 100$, $r = 13$

We conduct a similar analysis in the case of a discontinuous $\bar{x}$, using the $\bar{x}$ presented in Section 3.2.2, and the effectiveness of the Discrepancy Principle is further confirmed. As highlighted in the tables, we can see the best r's predicted are $r = 9$ for $\delta = 10^{-2}$, $r = 6$ for $\delta = 5 \times 10^{-3}$ and $r = 3$ for $\delta = 10^{-3}$. In Figure 4.2, we show the reconstructed solution using the predicted $r = 3$ at $\delta = 10^{-3}$. While Figure 4.2 looks quite similar to Figure 3.15 where $r = 4$ was used, we can quantitively conclude $r = 3$ is a better choice since the relative rms error is 0.072, which is slightly better than the rms error (0.086) in the case of $r = 4$ when the same set of noisy data is used.

r	$\hat{\delta}$	δ_d
3	0.013285	0.00754975
4	0.0131227	0.00831738
5	0.0127907	0.00890927
6	0.0126117	0.00966738
7	0.012555	0.010586
8	0.0125535	0.0117132
• 9	0.0122756	0.0128592
10	0.0120137	0.0142196

Table 4.4. Relative noise level $\delta = 10^{-2}$, $N = 200$, discontinuous $\bar{x}$

r	$\hat{\delta}$	δ_d
3	0.00664252	0.0040549
4	0.00656133	0.00482411
5	0.00639536	0.00568239
• 6	0.00630583	0.00675673
7	0.00627752	0.00801038

Table 4.5. Relative noise level $\delta = 5 \times 10^{-3}$, $N = 200$, discontinuous $\bar{x}$

r	$\hat{\delta}$	δ_d
2	0.00133155	0.00113062
• 3	0.0013285	0.00202835
4	0.00131227	0.00308485

Table 4.6. Relative noise level $\delta = 10^{-3}$, $N = 200$, discontinuous $\bar{x}$

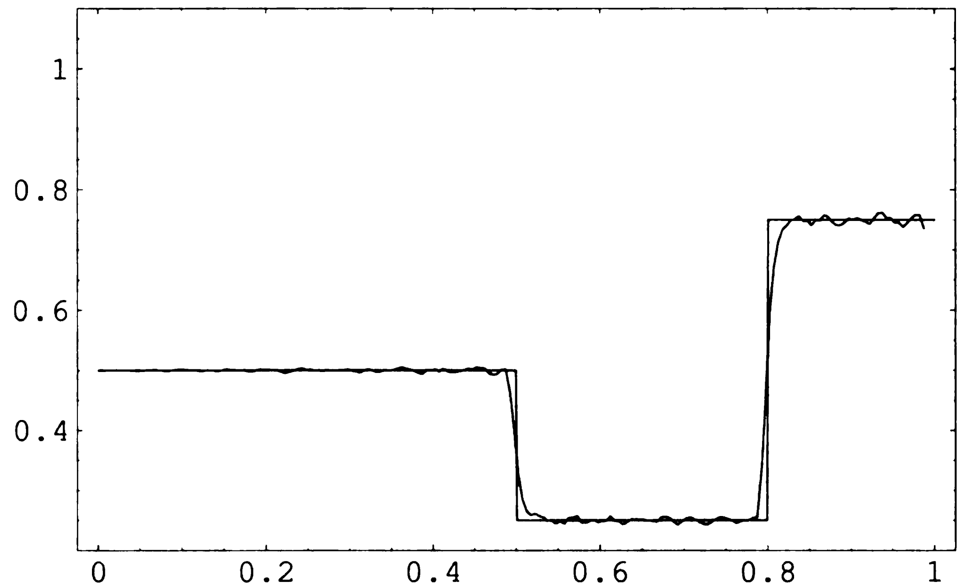

Figure 4.2. Solution obtained by Method 4, $\delta = 10^{-3}$, $N = 200$, $r = 3$

We also conducted some numerical experiments to shed light on the question of picking r for various values of N. In Figure 4.3, we have used Method 4 to reconstruct the continuous function used in Section 3.2.1 with the relative noise level $\delta = 10^{-3}$. The relative rms error of the reconstructed x is plotted as a function of N, for several values of r. As expected, as N increases, the ideal choice of r increases as well. For a given choice of r, generally, as N increases, the rms error decreases until an optimum N, and then increases again. Each value of r thus has a "sweet spot," where the rms error is lower than for any other value of r. In this figure, for this function, we see that $r = 2$ is optimal for $25 \lesssim N \lesssim 40$, while $r = 3$ is optimal for $45 \lesssim N \lesssim 60$, and so on. We plan to study this question further.

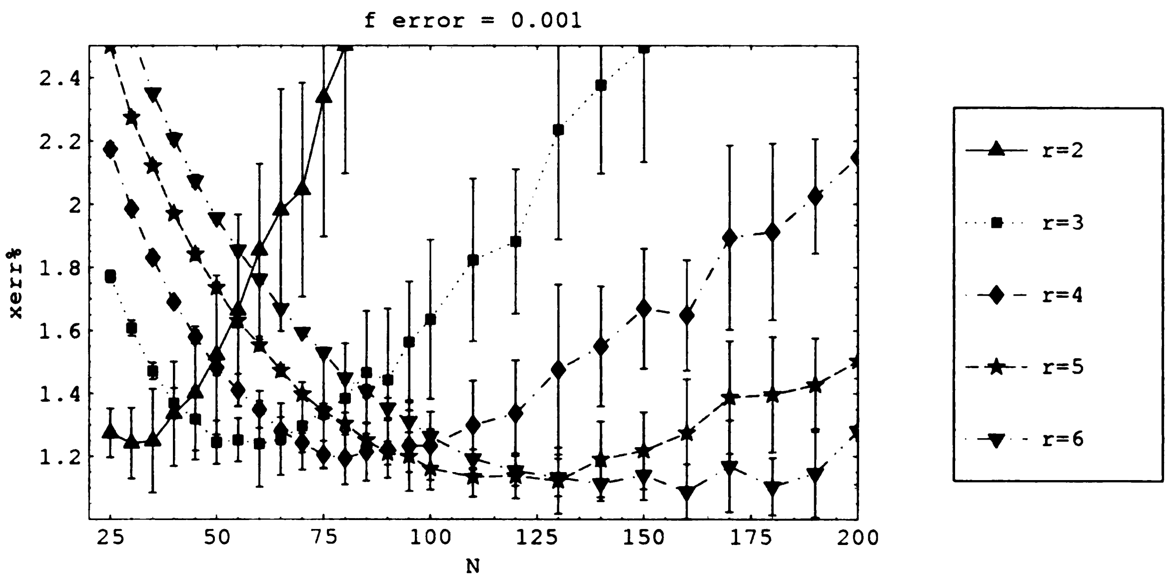

Figure 4.3. Relative error of numerical solution, for various values of N and r

Even though this study is the first time in our knowledge that local regularization is extended to real nonlinear Volterra problems, it is applied to a very specific nonlinear Volterra problem, the autoconvolution equation. We would like to extend the nonlinear theory to a more general class of nonlinear Volterra problems.

BIBLIOGRAPHY

BIBLIOGRAPHY

[1] J. Baumeister, Deconvolution of appearance potential spectra, *In: R.Kleinmann, et al.(eds): Direct and inverse boundary value problems. Proc. Conf. Oberwolfach. Lang, Frankfurt am Main* (1991) 1-13.

[2] J.V. Beck, B. Blackwell, and Jr. C.R.St. Clair, Inverse Heat Conduction, Wiley-Interscience, (1985).

[3] A. C. Cinzori, Continuous future polynomial regularization of 1-smoothing Volterra problems, *Inverse Problems* **20** (2004) 1791-1806.

[4] A. C. Cinzori and P. K. Lamm, Future polynomial regularization of ill-posed Volterra equations, *SIAM J.Numer.Anal* **37** (2000) 949-979.

[5] C. Cui, Local regularization methods for n-dimensional first-kind integral equations, Ph.D. Thesis, *Michigan State University*, 2005

[6] H. W. Engl, K. Kunisch, A. Neubauer Convergence rates for Tikhonov regularization of nonlinear ill-posed problems, *Inverse Problems* **5** (1989) 524-540.

[7] G. Fleischer, B. Hofmann, On inversion rates for the autoconvolution equation, *Inverse Problems* **12** (1996) 419-435.

[8] G. Fleischer, B. Hofmann, The local degree of ill-posedness and the autoconvolution equation, *Nonlinear Analysis, Theory, Methods & Application.* Vol.30, No.6, (1997) 3323-3332.

[9] G. Fleischer, R. Gorenflo, B. Hofmann, On the autoconvolution equation and total variation constraints, *Z. Angew. Math. Mech* **79** (1999) 149-159.

[10] R. Gorenflo, B. Hofmann, On autoconvolution and regularization, *Inverse Problems* **10** (1994) 353-373.

[11] J. Janno, On a regularization method for the autoconvolution equation, *Z. Angew. Math. Mech* **77** (1997) 393-394.

[12] J. Janno, Lavrent'ev regularization of ill-posed problems containing nonlinear near-to-monotone operators with application to autoconvolution equation, *Inverse Problems* **16** (2000) 333-348.

[13] P. K. Lamm, Approximation of ill-posed Volterra problems via predictor-corrector regularization methods, *SIAM J. Appl. Math.* **195** (1995) 469-494.

[14] P. K. Lamm, Future-sequential regularization methods for ill-posed Volterra equations: applications to the inverse heat conduction problem, *J. Math. Anal. Appl.* **56** (1996) 524-541.

[15] P. K. Lamm, Regularized inversion of finitely smoothing Volterra operators: predictor-corrector regularization methods, *Inverse Problems* **13** (1997) 375-402.

[16] P. K. Lamm, Variable-smoothing regularization methods for inverse problems, *Theory and Practice of Control and Systems* ed A.Conte and A.M.Perdon (Singapore: World Scientific) (1999).

[17] P. K. Lamm, Variable-smoothing local regularization methods for first-kind integral equations, *Inverse Problems* **19** (2003) 195-216.

[18] P. K. Lamm, Full convergence of sequential local regularization methods for Volterra inverse problems, *Inverse Problems* **21** (2005) 785-803.

[19] P. K. Lamm and Z. Dai, On local regularization methods for linear Volterra problems and nonlinear equations of Hammerstein type, *Preprint.*

[20] P. K. Lamm and L. Eldén, Numerical solution of first-kind Volterra equations by sequential Tikhonov regularization, *SIAM J. Numer. Anal.* **34** (1997)1432-1450.

[21] P. K. Lamm and T. Scofield, Sequential predictor-corrector methods for the variable regularization of Volterra inverse problems, *Inverse Problems* **16** (2000) 373-399.

[22] P. K. Lamm and T. Scofield, Local regularization methods for the stabilization for ill-posed Volterra problems, *Numer. Funct. Anal. Opt* **23** (2001) 913-940.

[23] R. Miller, Nonlinear Volterra integral equations, *W.A.Benjamin, Inc* (1971).

[24] H. Tanabe, Equations of evolution, London:Pitman, (1979).

[25] W. Ring and J. Prix, Sequential predictor-corrector regularization methods and their limitations, *Inverse Problem* **16** (2000) 619-634.