This is to certify that the
dissertation entitled

Local Regularization Methods For n-Dimensional First-Kind
Integral Equations

presented by

Changjun Cui

has been accepted towards fulfillment
of the requirements for the

Doctoral degree in Mathematics

Patricia K Lamm
Major Professor's Signature

5/6/05

Date

PLACE IN RETURN BOX to remove this checkout from your record.
TO AVOID FINES return on or before date due.
MAY BE RECALLED with earlier due date if requested.

DATE DUE	DATE DUE	DATE DUE

Local Regularization Methods for n-Dimensional First-Kind Integral Equations

By

Changjun Cui

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

2005

ABSTRACT

Local Regularization Methods for n-Dimensional First-Kind Integral Equations

By

Changjun Cui

We examine a new method for regularizing ill-posed first-kind (non-Volterra) integral equations on R^n. We analyze the method and prove that the regularized solutions converge to the true solution. We also develop a numerical algorithm based on this theory, and present evidence that this local regularization method is superior to classical regularization methods in that it can recover sharp features in the true solution. Our numerical examples also show that this algorithm has a faster speed of convergence compared to existing techniques.

ACKNOWLEDGMENTS

My foremost thank goes to my thesis adviser Professor Patricia Lamm.Without her insights, directions, suggestions, and encouragements, this dissertation would not have been possible. Especially, I thank her for her patience and strong support that carried me on through difficult times, while I was making progress on my research at a slower pace compared to other students.

I also want to thank all professors I have taken classes with at Michigan State University - several of them are also my thesis committee members. I am deeply grateful for the knowledge and research skills I learned during my graduate studies.

TABLE OF CONTENTS

1 Introduction **1**

2 Basic Definitions **9**

2.1 The Local Regularization Parameters $\bar{\mathbf{r}}$ and α 9

2.2 The Spaces X, $\mathcal{X}$ and $\mathcal{X}_{\mathbf{r}}$ 9

2.3 The Map $E_{\mathbf{r}} : \mathcal{X}_{\mathbf{r}} \mapsto \mathcal{X}$ 10

2.4 Construction of $F_{\mathbf{r}}$, $\bar{F}_{\mathbf{r}}$, $F^{\delta}_{\mathbf{r}}$, $\bar{F}^{\delta}_{\mathbf{r}}$, $U_{\mathbf{r}}$, $\bar{U}_{\mathbf{r}} \in \mathcal{X}_{\mathbf{r}}$ 11

2.5 Operators $A_{\mathbf{r}}$, $B_{\mathbf{r}}$, l, T, $T_{\mathbf{r}}$ and $C_{\mathbf{r}}$ 12

2.6 The Local Regularization Problem $\mathcal{P}^{\delta}_{\mathbf{r},\alpha}$ 17

3 Convergence **20**

4 Numerical Implementation **33**

4.1 The Discrete Local Regularization Problem 33

4.2 Numerical Examples 45

BIBLIOGRAPHY **78**

1 Introduction

Consider the equation

$$Au = f \tag{1.1}$$

where for $\Omega = [0,1]^n \subset R^n$, A is the bounded linear operator on $L^2(\Omega)$ given by

$$Au(\mathbf{x}) = \int_\Omega k(\mathbf{x}, \mathbf{s})u(\mathbf{s})d\mathbf{s}, \qquad \text{a.a.} \qquad x \in \Omega. \tag{1.2}$$

Here $k \in L^\infty(\Omega \times \Omega)$ satisfies

$$|k(\mathbf{x}, \mathbf{s}) - k(\mathbf{y}, \mathbf{s})| \leq L_k(\mathbf{s}) \, \|\mathbf{x} - \mathbf{y}\|^{\mu_k} \qquad \text{a.a.} \qquad \mathbf{x}, \mathbf{y}, \mathbf{s} \in \Omega, \tag{1.3}$$

for $\mu_k > 0$ and $L_k \in L^2(\Omega)$, where $\|\cdot\|$ is the Euclidean norm in R^n. We assume that equation (1.1) has solutions and $\bar{u} \in L^2(\Omega)$ is the (unique) minimum norm solution. Clearly $f \in L^\infty(\Omega)$, and we will assume $\bar{u} \in L^\infty(\Omega)$.

Equation (1.1) can be used as a mathematical model of an inverse problem, i.e., given data f which represents a desired or an observed effect in real world problems, determine the cause which is represented by a physically relevant solution (often a generalized solution) $\bar{u}$ of equation (1.1).

Mathematical inverse problems arise in many areas, ranging from physics to population problems. For example:

1. The Inverse Heat Conduction Problem (IHCP): consider an insulated semi-infinite bar coincident with the non-negative x-axis. An unknown heat source $u(t)$ is applied to the end of bar at $x = 0$, here t is the time variable. The temperature of the bar at $x = 1$ is measured as $f(t)$. Then the unknown heat source $u(t)$ is the solution of first-kind Volterra integral equation

$$\int_0^t k(t-s)u(s)ds = f(t), \qquad t \in [0,1], \tag{1.4}$$

with kernel function

$$k(t) = \frac{1}{2\sqrt{\pi}t^{3/2}} \cdot e^{-1/4t}, \tag{1.5}$$

2. Blurred Image Reconstruction: let $\bar{u}(\mathbf{x})$, $\mathbf{x} = (x_1, x_2)$, represent gray-level values of an image on $\Omega \subset R^2$, where $\Omega = [0,1] \times [0,1]$. The image blurring is governed by a blurring operator H: for $\beta > 0$,

$$Hu(\mathbf{x}) = \frac{\beta}{\pi} \cdot \int_\Omega e^{-\beta\|\mathbf{x}-\mathbf{y}\|^2} \cdot u(\mathbf{y}) \cdot d\mathbf{y}. \tag{1.6}$$

When $\beta \to \infty$, H becomes the identity operator and no blurring occurs. The image reconsruction is solving the first-kind integral equation $Hu = g$ where g is the observed image gray level. Notice this is a non-Volterra equation.

3. X-ray Tomography, or Radon inversion: this is a medical application in which we recover a human body's density f from X-ray measurements in a cross-section of the human body.

4. The time resolved fluorescence problem in physical chemistry: a substance to be analyzed is illuminated by a short-pulse and absorbs energy (*photons*) as some molecules switch over into an excited state. In order to determine the fluorescence lifetime distribution $a(t)$, which determines the probability density of a certain molecule excited at time $t_0 = 0$ to emit a photon at time t, we need to solve an inverse problem represented by a first-kind integral equation with observed fluorescence intensity as given data.

Most inverse problems are ill-posed in the sense that solution $\bar{u}$ does not depend continuously on data f. As a matter of fact, in (1.1), as long as k is a non-degenerate function on Ω, the range of A is not closed which means A^{-1} is unbounded or discontinuous. So an inverse problem is usually unstable. In a practical setting, this implies that whenever our given data is an imprecise, or noisy measurement (say f^δ), we can not rely upon the solution $\bar{u}^\delta$ obtained by solving equation (1.1) using the perturbed data f^δ in place of accurate data f. Unfortunately, the presence of a small data error is unavoidable due to measurement errors or computer round-off errors, and this leads to highly oscillatory solutions.

To overcome the ill-posedness of the inverse problems, different *regularization* methods have been developed. The regularization of an ill-posed problem gives a method of constructing $\bar{u}^\delta$ in a way that we have guaranteed convergence $\bar{u}^\delta \to \bar{u}$ as the noise level $\delta \to 0$. That is, for sufficiently small δ such as $\|f - f^\delta\| \leq \delta$, the regularized approximation $\bar{u}^\delta$ should be reasonably close to the true solution $\bar{u}$.

The most well-known and classical regularization method is Tikhonov regularization. Assume $K : H_1 \longrightarrow H_2$ is a compact linear operator, where H_1, H_2 are Hilbert spaces. The Singular System, or Singular Value Decomposition (SVD) of K is defined as the sequence $\{u_n, v_n; \lambda_n\}$ that satisfies the following conditions:

- $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$,
- $\lambda_n \longrightarrow 0$ as $n \longrightarrow \infty$,
- $\{u_n\}$ and $\{v_n\}$ are complete orthonormal sets in H_1, H_2 respectively,
- $Ku_n = \lambda_n v_n$ and $K^* v_n = \lambda_n u_n$ for $n = 1, 2, \cdots$.

If $g \in H_2$ satisfies $\sum_{n=1}^{\infty} \left(\frac{1}{\lambda_n} \cdot \langle g, v_n \rangle\right)^2 < \infty$, then one can show that the minimum norm solution $\bar{u}$ of $Ku = g$ is given by

$$\bar{u} = K^{\dagger} g = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \cdot \langle g, v_n \rangle \cdot u_n, \tag{1.7}$$

where $K^{\dagger}$ is the generalized inverse of K. When the true data g is replaced by noisy data g^{δ}, $K^{\dagger} g$ as defined in equation (1.7) would magnify the high frenquency components of the error by $1/\lambda_n \longrightarrow \infty$ as $n \longrightarrow \infty$. A classical regularization technique uses the construction

$$u_{\alpha}^{\delta} = \sum_{n=1}^{\infty} M_{\alpha}(\lambda_n) \cdot \langle g^{\delta}, v_n \rangle \cdot u_n, \tag{1.8}$$

to limit the effect of high frenquency errors. It can be shown that with suitably chosen M_{α} and $\alpha = \alpha(\delta)$, we can ensure that $u_{\alpha}^{\delta} \longrightarrow \bar{u}$ as $\delta \longrightarrow 0$. The scalar α is the regularization parameter.

Tikhonov regularization method is defined by

$$M_\alpha(\lambda) = \frac{\lambda}{\lambda^2 + \alpha}, \tag{1.9}$$

Historically, Tikhonov introduced his regularization method by the formulation

$$u_\alpha^\delta = \arg\min_u \{\|Ku - g^\delta\|^2 + \alpha \cdot \|u\|^2\}, \tag{1.10}$$

which is equivalent to the definition given by (1.9). Using (1.10), it's easier to see that the Tikhonov regularization parameter α serves to have a trade-off between accuracy of the model-fitting (i.e. the first term in (1.10)) while achieving stability through penalty term $\|u\|$. For Tikhonov regularization theory, see [5] and [6].

Tikhonov regularization has a very noticable disadvantage: it tends to produce an overly smooth solution. Sharp or discontinuous features in the true solution tend to be lost during Tikhonov regularization process.

Solutions with sharp or discontinuous features are typical in image recontruction and signal processing applications. More recently, several regularizations have been proposed and studied to avoid the over-smoothness during regularization process. One method, called bounded variation regularization, is to seek

$$u_\beta^\delta = \arg\min_u \{\|Ku - g^\delta\|^2 + \beta \cdot G(u)\}, \tag{1.11}$$

where the penalty term is the bounded variation seminorm

$$G(u) = \int_{\Omega} |\nabla u|(x)dx. \tag{1.12}$$

While the bounded variation regularizations can capture sharp features and discontinuities much better compared to classical Tikhonov regularization, it does bring in its own drawbacks. One drawback is that the penalty term $G(u)$ is not differentiable, which leads to nontrivial difficulties in practical implementation because standard optimization packages can not be used. As a result, numerical algorithms derived from bounded variation regularization methods are costly and have slow speed of convergence. For references in bounded variation regularization, see [1],[2], [3],[4], and [15].

Another regularization theory being studied is the method of local regularization. As the name suggests, the motivation of local regularization is to regularize locally by applying more fine-tuned local controls during the regularization process in order to recover as much features in the true solution as possible.

To illustrate the idea of local regularization, let's look at equation (1.1). We define Local Regularization Parameter $\mathbf{r} = (r_1, r_2, \cdots, r_n)$ where $r_i \in (0, \frac{1}{2})$ and consider

$$A\bar{u}(\mathbf{x} + \boldsymbol{\rho}) = f(\mathbf{x} + \boldsymbol{\rho}),$$

where

$$\boldsymbol{\rho} = (\rho_i)_{i=1}^n,\ \rho_i \in [-r_i, r_i] \text{ and } \mathbf{x} = (x_i)_{i=1}^n,\ x_i \in [r_i, 1-r_i]. \tag{1.13}$$

This can be rewritten as

$$\int_\Omega k(\mathbf{x}+\boldsymbol{\rho}, \mathbf{s})\bar{u}(\mathbf{s})d\mathbf{s} = f(\mathbf{x}+\boldsymbol{\rho}), \text{ for } \boldsymbol{\rho} \text{ and } \mathbf{x} \text{ satisfying (1.13).}$$

That is,

$$\int_{\Omega\setminus B(\mathbf{x},\mathbf{r})} k(\mathbf{x}+\boldsymbol{\rho}, \mathbf{s})\bar{u}(\mathbf{s})d\mathbf{s} + \int_{B(\mathbf{x},\mathbf{r})} k(\mathbf{x}+\boldsymbol{\rho}, \mathbf{s})\bar{u}(\mathbf{s})d\mathbf{s} = f(\mathbf{x}+\boldsymbol{\rho}), \tag{1.14}$$

where

$$B(\mathbf{x},\mathbf{r}) = \{\mathbf{y} = (y_i)_{i=1}^n \in R^n;\ \mid y_i - x_i \mid \leq r_i\}.$$

It is easy to see that $B(\mathbf{x},\mathbf{r}) \subset \Omega$.

Equation (1.14) implies

$$\int_{B(\mathbf{0},\mathbf{r})} k(\mathbf{x}+\boldsymbol{\rho}, \mathbf{x}+\mathbf{s})\bar{u}(\mathbf{x}+\mathbf{s})d\mathbf{s} + \int_{\Omega\setminus B(\mathbf{x},\mathbf{r})} k(\mathbf{x}+\boldsymbol{\rho}, \mathbf{s})\bar{u}(\mathbf{s})d\mathbf{s} = f(\mathbf{x}+\boldsymbol{\rho}). \tag{1.15}$$

The above equations hold true for $\boldsymbol{\rho}$ and $\mathbf{x}$ satisfying (1.13).

The first term on the left-hand side of equation (1.15) represents the action of A on the $\mathbf{x}$-dependent "local part" of $\bar{u}$, so that equation (1.15) suggests a decomposition of the operator A into "global" and "local" parts, for each $\mathbf{x} \in \Omega$. This splitting of A is the basis of the local regularization method and allows us to apply regularization only to the local part of A.

It is worth noting that, in addition to the advantage of reconstructing sharp features in the true solution, when applied to a Volterra equation such as

$$\int_0^t k(t,s)u(s)ds = f(t), \tag{1.16}$$

local regularization methods can retain the causal structure of the original Volterra problem, which would be destroyed if using classical Tikhonov regularization. For this type of local regularization theory and numerical results (called *sequential* local regularization), see [7],[8], [9], [10],[12], [13] and [14].

When solving a non-Volterra type inverse problem (which is the subject of this paper), an *iterative* local regularization procedure is used. Intuitively, the procedure works like this: an initial guess is made for the solution on the entire domain, then the solution is iteratively updated on small subdomains by solving a local regularization problem while assuming the solution off the small subdomain (i.e. the global part) is fixed at its previously set value. This method is first introduced in [11]. In this paper, we study the first-kind integral equation on R^n by using a different set of regularization parameters. The resulting numerical algorithm generates faster convergence speed compared to results presented in [11].

2 Basic Definitions

In this section, we will make clear the "local regularization" ideas presented in the Introduction section (equation 1.15) by defining all required spaces, operators and regulariztion parameters.

2.1 The Local Regularization Parameters $\bar{\mathbf{r}}$ and α

Our first regularization parameter is $\mathbf{r} = (r_i)_{i=1}^n$, where $r_i \in (0, \frac{1}{2})$.

We use the following notations in this paper:

$$\mathbf{1} - \mathbf{r} = (1 - r_i)_{i=1}^n,$$

$$(\mathbf{r}, \mathbf{1} - \mathbf{r}) = \{\mathbf{x} \in R^n; \quad r_i < x_i < 1 - r_i \text{ for } i = 1, 2, \cdots, n\},$$

$$(-\mathbf{r}, \mathbf{r}) = \{\mathbf{x} \in R^n; \quad -r_i < x_i < r_i \text{ for } i = 1, 2, \cdots, n\}.$$

The second regularization parameter is given by $\alpha \in \Lambda$, where

$$\Lambda \equiv \{\alpha \in L^\infty(\Omega); \quad \alpha_{\text{min}} \equiv \inf_{x \in \Omega} \alpha(\mathbf{x}) > 0\}.$$

2.2 The Spaces X, $\mathcal{X}$ and $\mathcal{X}_\mathbf{r}$

Let $\Delta = 1$. We use the following notations:

$$\mathrm{X} = L^2((-\Delta, \Delta)^n) \text{ with usual norm } |\cdot|_X \text{ and usual inner product } \langle\cdot,\cdot\rangle_X,$$

$\mathcal{X} \equiv L^2(\Omega; X)$ with norm

$$\| \tilde{\varphi} \|_{\mathcal{X}}^2 \equiv \int_\Omega | \tilde{\varphi}(\mathbf{x}) |_X^2 \, d\mathbf{x} = \int_\Omega \int_{(-\Delta,\Delta)^n} | \tilde{\varphi}(\mathbf{x})(\boldsymbol{\rho}) |^2 \, d\boldsymbol{\rho} \, d\mathbf{x} \text{ for } \tilde{\varphi} \in \mathcal{X},$$

and the associated inner product $\langle \cdot, \cdot \rangle_{\mathcal{X}}$.

We also define $\mathcal{X}_{\mathbf{r}} = L^2((\mathbf{r}, \mathbf{1} - \mathbf{r}); L^2(-\mathbf{r}, \mathbf{r}))$ with norm

$$\| \varphi \|_{\mathbf{r}}^2 \equiv \frac{1}{2^n r_1 r_2 \cdots r_n} \int_{[\mathbf{r},\mathbf{1}-\mathbf{r}]} \int_{[-\mathbf{r},\mathbf{r}]} |\varphi(\mathbf{x})(\boldsymbol{\rho})|^2 d\boldsymbol{\rho} \, d\mathbf{x}, \tag{2.1}$$

Given $\alpha \in \Lambda$, we can also define an equivalent norm on $\mathcal{X}_{\mathbf{r}}$:

$$\| \varphi \|_{\mathbf{r},\alpha}^2 \equiv \frac{1}{2^n r_1 r_2 \cdots r_n} \int_{[\mathbf{r},\mathbf{1}-\mathbf{r}]} \alpha(\mathbf{x}) \int_{[-\mathbf{r},\mathbf{r}]} |\varphi(\mathbf{x})(\boldsymbol{\rho})|^2 d\boldsymbol{\rho} \, d\mathbf{x},$$

2.3 The Map $E_{\mathbf{r}} : \mathcal{X}_{\mathbf{r}} \mapsto \mathcal{X}$

Define $E_{\mathbf{r}} : \mathcal{X}_{\mathbf{r}} \mapsto \mathcal{X}$ by

$$E_{\mathbf{r}}\varphi(\mathbf{x})(\boldsymbol{\rho}) \equiv \begin{cases} \varphi(\mathbf{x})(\rho_1 r_1, \rho_2 r_2, \cdots, \rho_n r_n), & x \in (\mathbf{r}, \mathbf{1} - \mathbf{r}) \text{ and } \rho \in (-\Delta, \Delta)^n, \\ \\ 0, & \text{otherwise}, \end{cases}$$

for $\boldsymbol{\rho} \in [-\Delta, \Delta]^n$ and $\mathbf{x} \in \Omega$.

We now show that

$$\| E_{\mathbf{r}}\varphi \|_{\mathcal{X}}^2 = 2^n \, \| \varphi \|_{\mathbf{r}}^2, \tag{2.2}$$

In fact,

$$
\begin{aligned}
\| E_{\mathbf{r}}\varphi \|_{\mathcal{X}}^2 &= \int_\Omega \|E_{\mathbf{r}}\varphi(\mathbf{x})\|_X^2 \, d\mathbf{x} \\
&= \int_\Omega \int_{[-\Delta,\Delta]^n} |E_{\mathbf{r}}\varphi(\mathbf{x})(\boldsymbol{\rho})|^2 d\boldsymbol{\rho}\, d\mathbf{x} \\
&= \int_{[\mathbf{r},\mathbf{1}-\mathbf{r}]} \int_{[-\Delta,\Delta]^n} |\varphi(\mathbf{x})(\rho_1 r_1, \rho_2 r_2, \cdots, \rho_n r_n)|^2 d\boldsymbol{\rho}\, d\mathbf{x} \\
&= \int_{[\mathbf{r},\mathbf{1}-\mathbf{r}]} \int_{[-\mathbf{r},\mathbf{r}]} |\varphi(\mathbf{x})(\boldsymbol{\eta})|^2 \cdot \frac{1}{r_1 r_2 \cdots r_n} \cdot d\boldsymbol{\eta}\, d\mathbf{x} \qquad (2.3) \\
&= 2^n \|\varphi\|_{\mathbf{r}}^2.
\end{aligned}
$$

In equation (2.3), we used $\boldsymbol{\eta} = (\rho_1 r_1, \rho_2 r_2, \cdots, \rho_n r_n)$.

2.4 Construction of $F_{\mathbf{r}}$, $\bar{F}_{\mathbf{r}}$, $F_{\mathbf{r}}^\delta$, $\bar{F}_{\mathbf{r}}^\delta$, $U_{\mathbf{r}}$, $\bar{U}_{\mathbf{r}} \in \mathcal{X}_{\mathbf{r}}$

We define

$$F_{\mathbf{r}}(\mathbf{x})(\boldsymbol{\rho}) = f(\mathbf{x}+\boldsymbol{\rho}), \qquad \text{a.a. } \boldsymbol{\rho} \in [-\mathbf{r},\mathbf{r}], \quad \mathbf{x} \in [\mathbf{r}, \mathbf{1}-\mathbf{r}],$$

and

$$\bar{F}_{\mathbf{r}}(\mathbf{x})(\boldsymbol{\rho}) = f(\mathbf{x}), \qquad \text{a.a. } \boldsymbol{\rho} \in [-\mathbf{r},\mathbf{r}], \quad \mathbf{x} \in [\mathbf{r}, \mathbf{1}-\mathbf{r}].$$

Similar definitions can be made for $F_{\mathbf{r}}^\delta$, $\bar{F}_{\mathbf{r}}^\delta$ (where f^δ replaces f) and for $U_{\mathbf{r}}, \bar{U}_{\mathbf{r}}$ (where $\bar{u}$ replaces f).

Apparently $U_{\mathbf{r}}$, $\bar{U}_{\mathbf{r}} \in \mathcal{X}_{\mathbf{r}}$ and we can show that

$$\|U_{\mathbf{r}}\| \leq \|\bar{u}\|_{\infty}, \quad \|\bar{U}_{\mathbf{r}}\| \leq \|\bar{u}\|_{\infty}. \tag{2.4}$$

In fact,

$$\|U_{\mathbf{r}}\|_{\mathbf{r}}^2 = \frac{1}{2^n r_1 r_2 \cdots r_n} \int_{[\mathbf{r},\mathbf{1}-\mathbf{r}]} \int_{[-\mathbf{r},\mathbf{r}]} |\bar{u}(\mathbf{x}+\boldsymbol{\rho})|^2 d\boldsymbol{\rho}\, d\mathbf{x}$$

$$\leq \|\bar{u}\|_{\infty}^2 \cdot 2^n r_1 r_2 \cdots r_n \cdot \frac{1}{2^n r_1 r_2 \cdots r_n} \cdot (1-2r_1)(1-2r_2)\cdots(1-2r_n)$$

$$\leq \|\bar{u}\|_{\infty}^2.$$

and similar statements can be made for $\bar{U}_{\mathbf{r}}$.

2.5 Operators $A_{\mathbf{r}}$, $B_{\mathbf{r}}$, l, T, $T_{\mathbf{r}}$ and $C_{\mathbf{r}}$

The operator $A_{\mathbf{r}} : \mathcal{X}_{\mathbf{r}} \mapsto \mathcal{X}_{\mathbf{r}}$ is given by

$$A_{\mathbf{r}}\varphi(\mathbf{x})(\boldsymbol{\rho}) = \int_{[-\mathbf{r},\mathbf{r}]} k(\mathbf{x}+\boldsymbol{\rho},\, \mathbf{x}+\mathbf{s})\varphi(\mathbf{x})(\mathbf{s})\, d\mathbf{s} \quad \text{a.a. } \rho \in [-\mathbf{r},\mathbf{r}],\ \mathbf{x} \in [\mathbf{r}, \mathbf{1}-\mathbf{r}]. \tag{2.5}$$

Then

$$
\begin{aligned}
\|A_{\mathbf{r}}\varphi\|_{\mathbf{r}}^2 &= \frac{1}{2^n r_1 r_2 \cdots r_n} \int_{[\mathbf{r},\mathbf{1}-\mathbf{r}]} \int_{[-\mathbf{r},\mathbf{r}]} |A_{\mathbf{r}}\varphi(\mathbf{x})(\boldsymbol{\rho})|^2 d\boldsymbol{\rho}\, d\mathbf{x} \\
&= \frac{1}{2^n r_1 r_2 \cdots r_n} \int_{[\mathbf{r},\mathbf{1}-\mathbf{r}]} \int_{[-\mathbf{r},\mathbf{r}]} \left| \int_{[-\mathbf{r},\mathbf{r}]} k(\mathbf{x}+\boldsymbol{\rho},\, \mathbf{x}+\mathbf{s})\varphi(\mathbf{x})(\mathbf{s}) d\mathbf{s} \right|^2 d\boldsymbol{\rho}\, d\mathbf{x} \\
&\le \frac{1}{2^n r_1 r_2 \cdots r_n} \int_{[\mathbf{r},\mathbf{1}-\mathbf{r}]} \int_{[-\mathbf{r},\mathbf{r}]} \left| \int_{[-\mathbf{r},\mathbf{r}]} k^2(\mathbf{x}+\boldsymbol{\rho}, \mathbf{x}+\mathbf{s}) d\mathbf{s} \right| \\
&\quad \cdot \left| \int_{[-\mathbf{r},\mathbf{r}]} |\varphi(\mathbf{x})(\mathbf{s})|^2 d\mathbf{s} \right| d\boldsymbol{\rho}\, d\mathbf{x} \\
&\le \frac{1}{2^n r_1 r_2 \cdots r_n} \int_{[\mathbf{r},\mathbf{1}-\mathbf{r}]} \int_{[-\mathbf{r},\mathbf{r}]} 2^n r_1 r_2 \cdots r_n \cdot \|k\|_\infty^2 \cdot \int_{[-\mathbf{r},\mathbf{r}]} |\varphi(\mathbf{x})(\mathbf{s})|^2 d\mathbf{s}\, d\boldsymbol{\rho}\, d\mathbf{x} \\
&= \|k\|_\infty^2 \cdot \int_{[\mathbf{r},\mathbf{1}-\mathbf{r}]} \int_{[-\mathbf{r},\mathbf{r}]} \int_{[-\mathbf{r},\mathbf{r}]} |\varphi(\mathbf{x})(\mathbf{s})|^2 d\mathbf{s}\, d\boldsymbol{\rho}\, d\mathbf{x} \\
&= 2^n r_1 r_2 \cdots r_n \cdot \|k\|_\infty^2 \cdot \int_{[\mathbf{r},\mathbf{1}-\mathbf{r}]} \int_{[-\mathbf{r},\mathbf{r}]} |\varphi(\mathbf{x})(\mathbf{s})|^2 d\mathbf{s}\, d\mathbf{x} \\
&= 2^n r_1 r_2 \cdots r_n \cdot \|k\|_\infty^2 \cdot 2^n r_1 r_2 \cdots r_n \cdot \|\varphi\|_{\mathbf{r}}^2.
\end{aligned}
$$

This shows that $A_{\mathbf{r}}$ is a bounded linear operator on $\mathcal{X}_{\mathbf{r}}$ with

$$\| A_{\mathbf{r}} \| \le 2^n r_1 r_2 \cdots r_n \ \| k \|_\infty . \tag{2.6}$$

We also define $B_{\mathbf{r}} : L^2(\Omega) \mapsto \mathcal{X}_{\mathbf{r}}$ by, for $\eta \in \mathrm{L}^2(\Omega)$ and a.a. $\mathbf{x} \in \Omega$,

$$B_{\mathbf{r}}\eta(\mathbf{x})(\boldsymbol{\rho}) = \int_{\Omega-[\mathbf{x}-\mathbf{r},\mathbf{x}+\mathbf{r}]} k(\mathbf{x}+\boldsymbol{\rho}, \mathbf{s})\eta(\mathbf{s})d\mathbf{s}, \qquad \text{a.a. } \boldsymbol{\rho} \in [-\mathbf{r},\mathbf{r}] \text{ and } \mathbf{x} \in [\mathbf{r}, \mathbf{1}-\mathbf{r}].$$

We can similarly show that $B_{\mathbf{r}}$ is a bounded linear operator from $L^2(\Omega)$ to $\mathcal{X}_{\mathbf{r}}$ with

$$\| B_{\mathbf{r}} \| \le \| k \|_\infty . \tag{2.7}$$

As a matter fact, we have

$$
\begin{aligned}
\| B_{\mathbf{r}}\eta \|_{\mathbf{r}}^2 &= \frac{1}{2^n r_1 r_2 \cdots r_n} \int_{[\mathbf{r},1-\mathbf{r}]} \int_{[-\mathbf{r},\mathbf{r}]} |B_{\mathbf{r}}\eta(\mathbf{x})(\boldsymbol{\rho})|^2 d\boldsymbol{\rho}\, d\mathbf{x} \\
&= \frac{1}{2^n r_1 r_2 \cdots r_n} \int_{[\mathbf{r},1-\mathbf{r}]} \int_{[-\mathbf{r},\mathbf{r}]} \left| \int_{\Omega - B(\mathbf{x},\mathbf{r})} k(\mathbf{x}+\boldsymbol{\rho},\mathbf{s})\eta(\mathbf{s}) d\mathbf{s} \right|^2 d\boldsymbol{\rho}\, d\mathbf{x} \\
&\le \frac{1}{2^n r_1 r_2 \cdots r_n} \int_{[\mathbf{r},1-\mathbf{r}]} \int_{[-\mathbf{r},\mathbf{r}]} \left| \int_{\Omega - B(\mathbf{x},\mathbf{r})} k^2(\mathbf{x}+\rho,\mathbf{s}) d\mathbf{s} \right| \\
&\quad \cdot \left| \int_{\Omega - B(\mathbf{x},\mathbf{r})} \eta^2(\mathbf{s}) d\mathbf{s} \right| d\boldsymbol{\rho}\, d\mathbf{x} \\
&\le \frac{1}{2^n r_1 r_2 \cdots r_n} \int_{[\mathbf{r},1-\mathbf{r}]} \int_{[-\mathbf{r},\mathbf{r}]} \|k\|_\infty^2 \cdot \left| \int_\Omega \eta^2(\mathbf{s}) d\mathbf{s} \right| d\boldsymbol{\rho}\, d\mathbf{x} \\
&= \frac{1}{2^n r_1 r_2 \cdots r_n} \cdot \|k\|_\infty^2 \int_{[\mathbf{r},1-\mathbf{r}]} \int_{[-\mathbf{r},\mathbf{r}]} \|\eta\|_{L^2(\Omega)}^2 \, d\boldsymbol{\rho}\, d\mathbf{x} \\
&\le \|k\|_\infty^2 \cdot \|\eta\|_{L^2(\Omega)}^2 .
\end{aligned}
$$

Let $l \in X^*$ be fixed and normalized so that $l(\mathbf{1}) = \mathbf{1}$, where $\mathbf{1} \in X$ is given by $\mathbf{1}(\rho) = \mathbf{1}$ for a.a. ρ. Let γ_l denote the unique nonzero element of X satisfying

$$l(v) = \langle v, \gamma_l \rangle_X, \; v \in X. \tag{2.8}$$

Notice that we must have $\int_{[-\Delta,\Delta]^n} \gamma_l(\rho)\, d\rho = 1$.

Define $T \in \mathcal{L}(\mathcal{X}, L^2(\Omega))$ by

$$T\tilde{\varphi}(\mathbf{x}) \equiv l(\tilde{\varphi}(\mathbf{x})), \text{ a.a. } \mathbf{x} \in \Omega. \tag{2.9}$$

for $\tilde{\varphi} \in \mathcal{X}$. We also define a bounded linear operator $T_{\mathbf{r}} : \mathcal{X}_{\mathbf{r}} \mapsto L^2(\Omega)$ via $T_{\mathbf{r}} \equiv T\, E_{\mathbf{r}}$.

As a matter fact, we have

$$
\begin{aligned}
\| B_{\mathbf{r}}\eta \|_{\mathbf{r}}^2 &= \frac{1}{2^n r_1 r_2 \cdots r_n} \int_{[\mathbf{r},\mathbf{1}-\mathbf{r}]} \int_{[-\mathbf{r},\mathbf{r}]} |B_{\mathbf{r}}\eta(\mathbf{x})(\boldsymbol{\rho})|^2 d\boldsymbol{\rho}\, d\mathbf{x} \\
&= \frac{1}{2^n r_1 r_2 \cdots r_n} \int_{[\mathbf{r},\mathbf{1}-\mathbf{r}]} \int_{[-\mathbf{r},\mathbf{r}]} \left| \int_{\Omega - B(\mathbf{x},\mathbf{r})} k(\mathbf{x}+\boldsymbol{\rho},\mathbf{s})\eta(\mathbf{s}) d\mathbf{s} \right|^2 d\boldsymbol{\rho}\, d\mathbf{x} \\
&\le \frac{1}{2^n r_1 r_2 \cdots r_n} \int_{[\mathbf{r},\mathbf{1}-\mathbf{r}]} \int_{[-\mathbf{r},\mathbf{r}]} \left| \int_{\Omega - B(\mathbf{x},\mathbf{r})} k^2(\mathbf{x}+\rho,\mathbf{s}) d\mathbf{s} \right| \\
&\quad \cdot \left| \int_{\Omega - B(\mathbf{x},\mathbf{r})} \eta^2(\mathbf{s}) d\mathbf{s} \right| d\boldsymbol{\rho}\, d\mathbf{x} \\
&\le \frac{1}{2^n r_1 r_2 \cdots r_n} \int_{[\mathbf{r},\mathbf{1}-\mathbf{r}]} \int_{[-\mathbf{r},\mathbf{r}]} \|k\|_\infty^2 \cdot \left| \int_{\Omega} \eta^2(\mathbf{s}) d\mathbf{s} \right| d\boldsymbol{\rho}\, d\mathbf{x} \\
&= \frac{1}{2^n r_1 r_2 \cdots r_n} \cdot \|k\|_\infty^2 \int_{[\mathbf{r},\mathbf{1}-\mathbf{r}]} \int_{[-\mathbf{r},\mathbf{r}]} \|\eta\|_{L^2(\Omega)}^2 \, d\boldsymbol{\rho}\, d\mathbf{x} \\
&\le \|k\|_\infty^2 \cdot \|\eta\|_{L^2(\Omega)}^2 .
\end{aligned}
$$

Let $l \in X^*$ be fixed and normalized so that $l(\mathbf{1}) = \mathbf{1}$, where $\mathbf{1} \in X$ is given by $\mathbf{1}(\rho) = \mathbf{1}$ for a.a. ρ. Let γ_l denote the unique nonzero element of X satisfying

$$
l(v) = \langle v,\, \gamma_l \rangle_X, \;\; v \in X. \tag{2.8}
$$

Notice that we must have $\int_{[-\Delta,\Delta]^n} \gamma_l(\rho)\, d\rho = 1$.

Define $T \in \mathcal{L}(\mathcal{X}, L^2(\Omega))$ by

$$
T\tilde{\varphi}(\mathbf{x}) \equiv l(\tilde{\varphi}(\mathbf{x})), \;\; \text{a.a. } \mathbf{x} \in \Omega. \tag{2.9}
$$

for $\tilde{\varphi} \in \mathcal{X}$. We also define a bounded linear operator $T_{\mathbf{r}} : \mathcal{X}_{\mathbf{r}} \mapsto L^2(\Omega)$ via $T_{\mathbf{r}} \equiv T\, E_{\mathbf{r}}$.

Then we have

$$T_{\mathbf{r}}\,\bar{U}_{\mathbf{r}}(\mathbf{x}) = \begin{cases} \bar{u}(\mathbf{x}), & \mathbf{x} \in (\mathbf{r}, \mathbf{1}-\mathbf{r}), \\ \\ 0, & \text{otherwise}\ , \end{cases}$$

for $\mathbf{x} \in \Omega$. This is because

$$E_{\mathbf{r}}\,\bar{U}_{\mathbf{r}}(\mathbf{x})(\boldsymbol{\rho}) = \begin{cases} \bar{u}(\mathbf{x}), & \mathbf{x} \in (\mathbf{r}, \mathbf{1}-\mathbf{r}), \boldsymbol{\rho} \in (-\Delta, \Delta)^n, \\ \\ 0, & \text{otherwise}\ . \end{cases}$$

So

$$T_{\mathbf{r}}\,\bar{U}_{\mathbf{r}}(\mathbf{x}) = \int_{[-\Delta,\Delta]^n} E_{\mathbf{r}}\,\bar{U}_{\mathbf{r}}(\mathbf{x})(\boldsymbol{\rho})\,\gamma_l(\boldsymbol{\rho})\,d\boldsymbol{\rho}$$

$$= \begin{cases} \int_{[-\Delta,\Delta]^n} \bar{u}(\mathbf{x})\,\gamma_l(\boldsymbol{\rho})\,d\boldsymbol{\rho}, & \mathbf{x} \in (\mathbf{r}, \mathbf{1}-\mathbf{r}), \\ \\ 0, & \text{otherwise}\ , \end{cases}$$

$$= \begin{cases} \bar{u}(\mathbf{x}), & \mathbf{x} \in (\mathbf{r}, \mathbf{1}-\mathbf{r}), \\ \\ 0, & \text{otherwise}\ . \end{cases}$$

Notice that

$$\|T_{\mathbf{r}}\bar{U}_{\mathbf{r}} - \bar{u}\|^2_{L^2(\Omega)} = \int_{\Omega} \left|T_{\mathbf{r}}\bar{U}_{\mathbf{r}}(\mathbf{x}) - \bar{u}(\mathbf{x})\right|^2 d\mathbf{x}$$

$$= \int_{\Omega\setminus(\mathbf{r},\mathbf{1}-\mathbf{r})} |\bar{u}(\mathbf{x})|^2 \, d\mathbf{x}$$

$$\leq (3^n - 1) \cdot \|\mathbf{r}\|_\infty \cdot \|\bar{u}\|^2_\infty, \tag{2.10}$$

where

$$\|\mathbf{r}\|_\infty = \max_{i=1,2,\cdots n} r_i,$$

we get that

$$\|T_{\mathbf{r}}\bar{U}_{\mathbf{r}} - \bar{u}\|_{L^2(\Omega)} = O(\sqrt{\|\mathbf{r}\|}).$$

We can also get

$$\|T_{\mathbf{r}}\varphi\|^2_{\mathcal{X}} \leq \|T\|^2 \cdot \|E_{\mathbf{r}}\varphi\|^2_{\mathcal{X}} = \|T\|^2 \cdot 2^n \cdot \|\varphi\|^2_r.$$

Hence

$$\|T_{\mathbf{r}}\| \leq \sqrt{2^n}\,\|T\|. \tag{2.11}$$

Finally we define operator $C_{\mathbf{r}} \equiv A_{\mathbf{r}} + B_{\mathbf{r}}T_{\mathbf{r}}$. Because $A_{\mathbf{r}} \in L(\mathcal{X}_{\mathbf{r}})$, $B_{\mathbf{r}} \in L(L_2(\Omega), \mathcal{X}_{\mathbf{r}})$ and $T_{\mathbf{r}} \in L(\mathcal{X}_{\mathbf{r}}, L_2(\Omega))$, we know that $C_{\mathbf{r}} \in L(\mathcal{X}_{\mathbf{r}})$.

2.6 The Local Regularization Problem $\mathcal{P}^{\delta}_{\mathbf{r},\alpha}$

Definition 2.1 Let $\mathbf{r}$ and α satisfy the conditions defined in section (2.1) and $\delta > 0$. Also assume that $f^{\delta} \in L^{\infty}(\Omega)$ is given satisfying $\| f - f^{\delta} \|_{\infty} < \delta$. The Problem $\mathcal{P}^{\delta}_{\mathbf{r},\alpha}$ is the problem of finding $\varphi^{\delta}_{\mathbf{r},\alpha}$ such that

$$\varphi^{\delta}_{\mathbf{r},\alpha} = \arg \min_{\varphi \in \mathcal{X}_{\mathbf{r}}} \{\| C_{\mathbf{r}}\varphi - F^{\delta}_{\mathbf{r}} \|^2_{\mathbf{r}} + \| \varphi \|^2_{\mathbf{r},\alpha}\}.$$

The following theorem follows from classical Tikhonov regularization theory.

Theorem 2.1 Let $\mathbf{r}, \alpha, \delta$ and f^{δ} satisfy the conditions stated in Definition 2.1. Then there exists a unique solution $\varphi^{\delta}_{\mathbf{r},\alpha}$ of Problem $\mathcal{P}^{\delta}_{\mathbf{r},\alpha}$. Both $\varphi^{\delta}_{\mathbf{r},\alpha} \in \mathcal{X}_{\mathbf{r}}$ and $\eta^{\delta}_{\mathbf{r},\alpha} \equiv T_{\mathbf{r}}\varphi^{\delta}_{\mathbf{r},\alpha} \in L^2(\Omega)$ depend continuously on $F^{\delta}_{\mathbf{r}} \in \mathcal{X}_{\mathbf{r}}$ and thus on data $f^{\delta} \in L^{\infty}(\Omega)$.

We first present the following lemma.

Lemma 2.1 Let $\mathbf{r}, \alpha, \delta$ and f^{δ} satisfy the conditions of Definition 2.1 and let $\varphi^{\delta}_{\mathbf{r},\alpha}$ denote the solution of Problem $\mathcal{P}^{\delta}_{\mathbf{r},\alpha}$. Then

$$\| C_{\mathbf{r}}\varphi^{\delta}_{\mathbf{r},\alpha} - F^{\delta}_{\mathbf{r}} \|^2_{\mathbf{r}} + \| \varphi^{\delta}_{\mathbf{r},\alpha} \|^2_{\mathbf{r},\alpha} \leq C \ \left[(r_1^2 r_2^2 \cdots r_n^2 \|k\|^2_{\infty} + \|\alpha\|_{\infty}) \|\bar{u}\|^2_{\infty} + \delta^2\right],$$

for some $C > 0$ independent of $\mathbf{r}, \alpha$ and δ.

Proof:

First we observe that

$$
\begin{aligned}
& \| A_{\mathbf{r}} U_{\mathbf{r}} + B_{\mathbf{r}} T_{\mathbf{r}} \bar{U}_{\mathbf{r}} - F_{\mathbf{r}} \|_{\mathbf{r}}^2 \\
= \;& \frac{1}{2^n r_1 r_2 \cdots r_n} \int_{[\mathbf{r}, 1-\mathbf{r}]} \int_{[-\mathbf{r}, \mathbf{r}]} \left| A_{\mathbf{r}} U_{\mathbf{r}}(\mathbf{x})(\boldsymbol{\rho}) + B_{\mathbf{r}} T_{\mathbf{r}} \bar{U}_{\mathbf{r}}(\mathbf{x})(\boldsymbol{\rho}) - F_{\mathbf{r}}(\mathbf{x})(\boldsymbol{\rho}) \right|^2 d\boldsymbol{\rho}\, d\mathbf{x} \\
= \;& \frac{1}{2^n r_1 r_2 \cdots r_n} \int_{[\mathbf{r}, 1-\mathbf{r}]} \int_{[-\mathbf{r}, \mathbf{r}]} \left| A_{\mathbf{r}} U_{\mathbf{r}}(\mathbf{x})(\boldsymbol{\rho}) + B_{\mathbf{r}} \bar{u}(\mathbf{x})(\boldsymbol{\rho}) - f(\mathbf{x} + \boldsymbol{\rho}) \right|^2 d\boldsymbol{\rho}\, d\mathbf{x} \\
= \;& 0,
\end{aligned}
$$

because (1.15) shows exactly that $f(\mathbf{x}+\boldsymbol{\rho}) = A_{\mathbf{r}} U_{\mathbf{r}}(\mathbf{x})(\boldsymbol{\rho}) + B_{\mathbf{r}} \bar{u}(\mathbf{x})(\boldsymbol{\rho})$. Hence

$$
\begin{aligned}
& \| C_{\mathbf{r}} \varphi_{\mathbf{r},\alpha}^{\delta} - F_{\mathbf{r}}^{\delta} \|_{\mathbf{r}}^2 + \| \varphi_{\mathbf{r},\alpha}^{\delta} \|_{\mathbf{r},\alpha}^2 \\
\leq \;& \| C_{\mathbf{r}} \bar{U}_{\mathbf{r}} - F_{\mathbf{r}}^{\delta} \|_{\mathbf{r}}^2 + \| \bar{U}_{\mathbf{r}} \|_{\mathbf{r},\alpha}^2 \\
\leq \;& 2(\| A_{\mathbf{r}}(\bar{U}_{\mathbf{r}} - U_{\mathbf{r}}) \|_{\mathbf{r}} + \| A_{\mathbf{r}} U_{\mathbf{r}} + B_{\mathbf{r}} T_{\mathbf{r}} \bar{U}_{\mathbf{r}} - F_{\mathbf{r}} \|_{\mathbf{r}})^2 + 2 \| F_{\mathbf{r}} - F_{\mathbf{r}}^{\delta} \|_{\mathbf{r}}^2 + \| \bar{U}_{\mathbf{r}} \|_{\mathbf{r},\alpha}^2 \\
= \;& 2 \| A_{\mathbf{r}}(\bar{U}_{\mathbf{r}} - U_{\mathbf{r}}) \|_{\mathbf{r}}^2 + 2 \| F_{\mathbf{r}} - F_{\mathbf{r}}^{\delta} \|_{\mathbf{r}}^2 + \| \bar{U}_{\mathbf{r}} \|_{\mathbf{r},\alpha}^2 .
\end{aligned}
$$

The conclusion of this Lemma then follows from the facts that

$$
\begin{aligned}
\|F_{\mathbf{r}} - F_{\mathbf{r}}^{\delta}\|_{\mathbf{r}}^2 &= \frac{1}{2^n r_1 r_2 \cdots r_n} \int_{[\mathbf{r}, 1-\mathbf{r}]} \int_{[-\mathbf{r}, \mathbf{r}]} |F_{\mathbf{r}}(\mathbf{x})(\boldsymbol{\rho}) - F_{\mathbf{r}}^{\delta}(\mathbf{x})(\boldsymbol{\rho})|^2 \, d\boldsymbol{\rho}\, d\mathbf{x} \\
&= \frac{1}{2^n r_1 r_2 \cdots r_n} \int_{[\mathbf{r}, 1-\mathbf{r}]} \int_{[-\mathbf{r}, \mathbf{r}]} |f(\mathbf{x}+\boldsymbol{\rho}) - f^{\delta}(\mathbf{x}+\boldsymbol{\rho})|^2 \, d\boldsymbol{\rho}\, d\mathbf{x} \\
&\leq \|f - f^{\delta}\|_{\infty}^2 \cdot \frac{1}{2^n r_1 r_2 \cdots r_n} \int_{[\mathbf{r}, 1-\mathbf{r}]} \int_{[-\mathbf{r}, \mathbf{r}]} d\boldsymbol{\rho}\, d\mathbf{x} \\
&\leq \|f - f^{\delta}\|_{\infty}^2,
\end{aligned}
$$

and (using (2.4) ,(2.6))

$$
\|\bar{U}_{\mathbf{r}}\|_{\mathbf{r},\alpha}^2 \leq \|\alpha\|_{\infty} \cdot \|\bar{U}_{\mathbf{r}}\|_{\mathbf{r}}^2 \leq \|\alpha\|_{\infty} \cdot \|\bar{u}\|_{\infty}^2,
$$

$$\|A_{\mathbf{r}}(\bar{U}_{\mathbf{r}} - U_{\mathbf{r}})\|_{\mathbf{r}}^2 \le \|A_{\mathbf{r}}\|^2 \cdot \|\bar{U}_{\mathbf{r}} - U_{\mathbf{r}}\|_{\mathbf{r}}^2 \le 2^{2n}\, r_1^2\, r_2^2 \cdots r_n^2 \cdot \|k\|_\infty^2 \cdot 4 \cdot \|\bar{u}\|_\infty^2.$$

3 Convergence

Our main convergence result is as follows:

Theorem 3.1 Let $\{\delta_k\}_{k=1}^{\infty} \subseteq R^+$ with $\delta_k \to 0$ as $k \to \infty$. Let $\{\mathbf{r}_k\}_{k=1}^{\infty}$ and $\{\alpha_k\}_{k=1}^{\infty} \subset \Lambda$ be given such that $\|\mathbf{r}_k\| \in (0, \frac{1}{2})$ and $\|\mathbf{r}_k\|, \| \alpha_k \|_\infty \to 0$ as $k \to \infty$. Assume further that there is $M > 0$ such that

(1) $\delta_k^2/\alpha_{k,\min} \to 0$,

(2) $\|\mathbf{r}_k\|^n/\delta_k \leq M$,

(3) $\| \alpha_k\|_\infty/\alpha_{k,\min} \to 1$,

as $k \to \infty$. For each $k = 1, 2, \cdots,$ let $f_{\delta_k} \in L^\infty(\Omega)$ be given with $\|f - f^{\delta_k}\| < \delta_k$, let $\varphi^{\delta_k}_{\mathbf{r}_k,\alpha_k} \in \mathcal{X}_{\mathbf{r}_k}$ denote the solution of Problem $\mathcal{P}^{\delta_k}_{\mathbf{r}_k,\alpha_k}$ associated with f^{δ_k}, and let

$$\eta_k \equiv T_{\mathbf{r}_k}\varphi^{\delta_k}_{\mathbf{r}_k,\alpha_k}. \tag{3.1}$$

Then

$$\eta_k \to \bar{u} \text{ in } L^2(\Omega),$$

as $k \to \infty$, where $\bar{u}$ is the solution of the original problem (1.1).

To simplify the notation, we will henceforth write $\varphi_k \equiv \varphi^{\delta_k}_{\mathbf{r}_k,\alpha_k}$, $\mathcal{P}_k \equiv \mathcal{P}^{\delta_k}_{\mathbf{r}_k,\alpha_k}$, $\mathcal{X}_k \equiv \mathcal{X}_{\mathbf{r}_k}$, $F_k \equiv F_{\mathbf{r}_k}$, $F_k^\delta \equiv F_{\mathbf{r}_k}^{\delta_k}$, $U_k \equiv U_{\mathbf{r}_k}$, $\bar{U}_k \equiv \bar{U}_{\mathbf{r}_k}$, $E_k \equiv E_{\mathbf{r}_k}$, $T_k \equiv T_{\mathbf{r}_k}$, $A_k \equiv A_{\mathbf{r}_k}$, and so on.

Lemma 3.1 Let $\{\delta_k\}_{k=1}^{\infty} \subseteq R^+$ and $\{\alpha_n\}_{n=1}^{\infty} \subseteq \Lambda$. Let $\{\mathbf{r}_k\}_{k=1}^{\infty}$ satisfy $\|\mathbf{r}_k\| \in (0, \frac{1}{2})$ and assume that there exists $M > 0$ such that

(1) $\delta_k^2/\alpha_{k,\min} \leq M$,

(2) $\|\mathbf{r}_k\|^n/\delta_k \leq M$,

(3) $\| \alpha_k \|_\infty / \alpha_{k,\min} \leq M$,

as $k \to \infty$. For each $k = 1, 2, \cdots$, let $f^{\delta_k} \in L^\infty(\Omega)$ be given with $\| f - f^{\delta_k} \|_\infty < \delta_k$, and let $\varphi_k \in \mathcal{X}_k$ denote the solution of Problem $\mathcal{P}_k$ associated with f^{δ_k}, with $\eta_k \equiv T_k\varphi_k \in L^2(\Omega)$.
Let $\tilde{\varphi}_k \equiv E_k\varphi_k \in \mathcal{X}$. Then there is $\tilde{\varphi} \in \mathcal{X}$ and a subsequence of $\{\tilde{\varphi}_k\}$ which converges weakly in $\mathcal{X}$ to $\tilde{\varphi}$. That is, relabelling the subsequential indices,

$$\tilde{\varphi}_k \rightharpoonup \tilde{\varphi} \text{ in } \mathcal{X} \text{ as } k \to \infty.$$

Further, $\eta \in L^2(\Omega)$ defined by

$$\eta \equiv T\tilde{\varphi}$$

is such that (using the same relabelling of indices as above)

$$\eta_k \rightharpoonup \eta \text{ in } L^2(\Omega) \text{ as } k \to \infty.$$

Proof. We note that

$$\begin{aligned}
\| E_k\varphi_k \|_{\mathcal{X}}^2 &= 2^n \cdot \| \varphi_k \|_{\mathbf{r}_k}^2 \\
&\le \frac{2^n}{\alpha_{k,\min}} \| \varphi_k \|_{\mathbf{r}_k,\alpha_k}^2 \\
&\le \frac{2^n}{\alpha_{k,\min}} (\| C_k\varphi_k - F_k^\delta \|_{\mathbf{r}_k}^2 + \| \varphi_k \|_{\mathbf{r}_k,\alpha_k}^2) \\
&\le \frac{2^n}{\alpha_{k,\min}} \cdot C \left[(r_{k1}^2 r_{k2}^2 \cdots r_{kn}^2 \|k\|_\infty^2 + \|\alpha_k\|_\infty) \|\bar{u}\|_\infty^2 + \delta_k^2 \right].
\end{aligned}$$

We used Lemma 2.1 in the last step. We may use assumptions (1) - (3) to obtain that

$$\|\tilde{\varphi}_k\|_{\mathcal{X}} = \|E_k\varphi_k\|_{\mathcal{X}}$$

is uniformly bounded for all $k = 1, 2, \cdots$. The remaining statements of the lemma follow from the fact that $\mathcal{X}$ is a Hilbert space, and the observation that $\eta_k \equiv T_k\varphi_k \equiv TE_k\varphi_k \equiv T\tilde{\varphi}_k$ for $k = 1, 2, \cdots$, where we recall that $T : \mathcal{X} \to L^2(\Omega)$ is a bounded linear operator.

Lemma 3.2 Assume $\{\delta_k\}, \{\mathbf{r}_k\}, \{\alpha_k\}$ and $\{f^{\delta_k}\}$ are given satisfying the conditions of Lemma 3.1, where we additionally assume that $\delta_k \to 0$, $\|\mathbf{r}_k\| \to 0$, and $\|\alpha_k\|_\infty \to 0$, as $k \to \infty$. Then for η and $\tilde{\varphi}$ given by Lemma 3.1, it follows that η is a solution of $Au = f$ and $\tilde{\varphi}$ is a solution of $\tilde{A}\psi = f$, for $\tilde{A} \in \mathcal{L}(\mathcal{X}, L^2(\Omega))$ defined by $\tilde{A} = AT$.

Proof. For $k = 1, 2, \cdots$, define $\bar{A}_k \in \mathcal{L}(L^2(\Omega), \mathcal{X}_k)$ via

$$\bar{A}_k u(\mathbf{x})(\boldsymbol{\rho}) = Au(\mathbf{x}), \qquad \text{a.a. } \rho \in [-\mathbf{r}_k, \mathbf{r}_k] \text{ and } \mathbf{x} \in [\mathbf{r}_k, \mathbf{1} - \mathbf{r}_k],$$

for A the original operator defined in (1.2) and $u \in L^2(\Omega)$. Then, for η given by Lemma 3.1, we have

$$\begin{aligned}
& \|\bar{A}_k\eta - \bar{F}_k\|^2_{\mathbf{r}_k} \\
= & \frac{1}{2^n r_{k1} r_{k2} \cdots r_{kn}} \int_{[\mathbf{r}_k, \mathbf{1}-\mathbf{r}_k]} \int_{[-\mathbf{r}_k, \mathbf{r}_k]} \left|\bar{A}_k\eta(\mathbf{x})(\boldsymbol{\rho}) - \bar{F}_k(\mathbf{x})(\boldsymbol{\rho})\right|^2 d\boldsymbol{\rho}\, d\mathbf{x} \\
= & \frac{1}{2^n r_{k1} r_{k2} \cdots r_{kn}} \int_{[\mathbf{r}_k, \mathbf{1}-\mathbf{r}_k]} \int_{[-\mathbf{r}_k, \mathbf{r}_k]} \left|A\eta(\mathbf{x}) - f(\mathbf{x})\right|^2 d\boldsymbol{\rho}\, d\mathbf{x} \\
= & \int_{[\mathbf{r}_k, \mathbf{1}-\mathbf{r}_k]} |A\eta(\mathbf{x}) - f(\mathbf{x})|^2 \, d\mathbf{x} \\
= & \int_{[\mathbf{r}_k, \mathbf{1}-\mathbf{r}_k]} |A\eta(\mathbf{x}) - f(\mathbf{x})|^2 \cdot \mathcal{X}_{[\mathbf{r}_k, \mathbf{1}-\mathbf{r}_k]}(\mathbf{x})\, d\mathbf{x} \\
\longrightarrow & \int_\Omega |A\eta(\mathbf{x}) - f(\mathbf{x})|^2 \, d\mathbf{x} = \|A\eta - f\|_{L^2(\Omega)} \quad \text{as } k \to \infty,
\end{aligned} \tag{3.2}$$

where $\mathcal{X}_{[\mathbf{r}_k, \mathbf{1}-\mathbf{r}_k]}(\cdot)$ is the characteristic function of the interval $[\mathbf{r}_k, \mathbf{1} - \mathbf{r}_k]$. Notice that in (3.2), the convergence is guaranteed because we can conclude $A\eta - f \in L^\infty(\Omega)$ from $f \in L^\infty(\Omega)$ and $A\eta \in L^2(\Omega)$.

So we only need to show that $\|\bar{A}_k\eta - \bar{F}_k\|_{\mathbf{r}_k} \to 0$ as $k \to \infty$.

To this purpose, we split $\|\bar{A}_k\eta - \bar{F}_k\|_{\mathbf{r}_k}$ and get

$$\|\bar{A}_k\eta - \bar{F}_k\|_{\mathbf{r}_k} \leq T_1^k + T_2^k + T_3^k + T_4^k + T_5^k,$$

where, using φ_k and η_k as defined in Lemma 3.1,

$$T_1^k = \|A_k\varphi_k + B_kT_k\varphi_k - F_k^\delta\|_{\mathbf{r}_k},$$

$$T_2^k = \|A_k\varphi_k\|_{\mathbf{r}_k},$$

$$T_3^k = \|\bar{A}_k\eta - B_kT_k\varphi_k\|_{\mathbf{r}_k},$$

$$T_4^k = \|F_k^\delta - F_k\|_{\mathbf{r}_k},$$

$$T_5^k = \|F_k - \bar{F}_k\|_{\mathbf{r}_k}.$$

In the remainder we will show that $T_i^k \to 0$ as $k \to \infty$ for $i = 1, 2, 3, 4, 5$, so we can conclude $| A\eta - f |= 0$ or η solves $Au = f$.

$T_1^k \to 0$: Use the fact that

$$(T_1^k)^2 = \|C_k\varphi_k - F_k^\delta\|_{\mathbf{r}_k}^2,$$

and the result of Lemma 2.1.

$T_2^k \to 0$: Use (2.2) and (2.6), we get

$$T_2^k \le \|A_k\| \cdot \|\varphi_k\|_{\mathbf{r}_k} \le 2^n\, r_{k1}\, r_{k2} \cdots r_{kn} \cdot \|k\|_\infty \cdot \frac{1}{\sqrt{2^n}} \cdot \|E_{\mathbf{r}_k}\varphi_k\|_{\mathcal{X}} \to 0 \quad \text{as } k \to \infty,$$

because $\|E_{\mathbf{r}_k}\varphi_k\|_{\mathcal{X}} = \|\tilde{\varphi}_k\|_{\mathcal{X}}$ is bounded (Lemma 3.1).

$T_3^k \to 0$: Notice that

$$\begin{aligned}
T_3^k &= \|\bar{A}_k \eta - B_k T_k \varphi_k\|_{\mathbf{r}_k} \\
&= \|\bar{A}_k \eta - B_k \eta_k\|_{\mathbf{r}_k} \\
&\leq \|\bar{A}_k(\eta - \eta_k)\|_{\mathbf{r}_k} + \|(\bar{A}_k - B_k)\eta_k\|_{\mathbf{r}_k}.
\end{aligned} \tag{3.3}$$

For the first term in (3.2), we have

$$\begin{aligned}
&\|\bar{A}_k(\eta - \eta_k)\|_{\mathbf{r}_k}^2 \\
= \; &\frac{1}{2^n r_{k1} r_{k2} \cdots r_{kn}} \int_{[\mathbf{r}_k, 1-\mathbf{r}_k]} \int_{[-\mathbf{r}_k, \mathbf{r}_k]} \left|\bar{A}_k(\eta - \eta_k)(\mathbf{x})(\boldsymbol{\rho})\right|^2 d\boldsymbol{\rho}\, d\mathbf{x} \\
= \; &\frac{1}{2^n r_{k1} r_{k2} \cdots r_{kn}} \int_{[\mathbf{r}_k, 1-\mathbf{r}_k]} \int_{[-\mathbf{r}_k, \mathbf{r}_k]} \left|A(\eta - \eta_k)(\mathbf{x})\right|^2 d\boldsymbol{\rho}\, d\mathbf{x} \\
= \; &\int_{[\mathbf{r}_k, 1-\mathbf{r}_k]} \left|A(\eta - \eta_k)(\mathbf{x})\right|^2 d\mathbf{x} \\
\leq \; &\|A(\eta - \eta_k)\|_{L^2(\Omega)}^2 \to 0,
\end{aligned}$$

as $k \to \infty$ (from the compactness of A on $L^2(\Omega)$ and Lemma 3.1).

$T_3^k \to 0$: Notice that

$$\begin{aligned}
T_3^k &= \|\bar{A}_k\eta - B_k T_k \varphi_k\|_{\mathbf{r}_k} \\
&= \|\bar{A}_k\eta - B_k \eta_k\|_{\mathbf{r}_k} \\
&\leq \|\bar{A}_k(\eta - \eta_k)\|_{\mathbf{r}_k} + \|(\bar{A}_k - B_k)\eta_k\|_{\mathbf{r}_k}.
\end{aligned} \tag{3.3}$$

For the first term in (3.2), we have

$$\begin{aligned}
&\|\bar{A}_k(\eta - \eta_k)\|_{\mathbf{r}_k}^2 \\
= \;& \frac{1}{2^n\, r_{k1}\, r_{k2} \cdots r_{kn}} \int_{[\mathbf{r}_k, \mathbf{1}-\mathbf{r}_k]} \int_{[-\mathbf{r}_k, \mathbf{r}_k]} \left|\bar{A}_k(\eta - \eta_k)(\mathbf{x})(\boldsymbol{\rho})\right|^2 d\boldsymbol{\rho}\, d\mathbf{x} \\
= \;& \frac{1}{2^n\, r_{k1}\, r_{k2} \cdots r_{kn}} \int_{[\mathbf{r}_k, \mathbf{1}-\mathbf{r}_k]} \int_{[-\mathbf{r}_k, \mathbf{r}_k]} \left|A(\eta - \eta_k)(\mathbf{x})\right|^2 d\boldsymbol{\rho}\, d\mathbf{x} \\
= \;& \int_{[\mathbf{r}_k, \mathbf{1}-\mathbf{r}_k]} \left|A(\eta - \eta_k)(\mathbf{x})\right|^2 d\mathbf{x} \\
\leq \;& \|A(\eta - \eta_k)\|_{L^2(\Omega)}^2 \to 0,
\end{aligned}$$

as $k \to \infty$ (from the compactness of A on $L^2(\Omega)$ and Lemma 3.1).

For the second term in (3.2), we have, for $\rho \in [-\mathbf{r}_k, \mathbf{r}_k]$ and $\mathbf{x} \in [\mathbf{r}_k, \mathbf{1} - \mathbf{r}_k]$,

$$
\begin{aligned}
& \left|(\bar{A}_k - B_k)\eta_k(\mathbf{x})(\boldsymbol{\rho})\right|^2 \\
= \ & \left|\bar{A}_k\eta_k(\mathbf{x})(\boldsymbol{\rho}) - B_k\eta_k(\mathbf{x})(\boldsymbol{\rho})\right|^2 \\
= \ & \left|A\eta_k(\mathbf{x})(\boldsymbol{\rho}) - \int_{\Omega - B(\mathbf{x},\mathbf{r}_k}) k(\mathbf{x}+\boldsymbol{\rho},\mathbf{s})\eta_k(\mathbf{s})\,d\mathbf{s}\right|^2 \\
= \ & \left|\int_{\Omega} k(\mathbf{x},\mathbf{s})\eta_k(\mathbf{s})\,d\mathbf{s} - \int_{\Omega - B(\mathbf{x},\mathbf{r}_k)} k(\mathbf{x}+\boldsymbol{\rho},\mathbf{s})\eta_k(\mathbf{s})\,d\mathbf{s}\right|^2 \\
\leq \ & 2\left|\int_{\Omega}(k(\mathbf{x},\mathbf{s}) - k(\mathbf{x}+\boldsymbol{\rho},\mathbf{s}))\,\eta_k(\mathbf{s})\,d\mathbf{s}\right|^2 + 2\left|\int_{B(\mathbf{x},\mathbf{r}_k)} k(\mathbf{x}+\boldsymbol{\rho},\mathbf{s})\,\eta_k(\mathbf{s})\,d\mathbf{s}\right|^2 \\
\leq \ & 2\int_{\Omega}\left|k(\mathbf{x},\mathbf{s}) - k(\mathbf{x}+\boldsymbol{\rho},\mathbf{s})\right|^2 d\mathbf{s} \int_{\Omega}\eta_k^2(\mathbf{s})\,d\mathbf{s} \\
& + 2\int_{B(\mathbf{x},\mathbf{r}_k)} k^2(\mathbf{x}+\boldsymbol{\rho},\mathbf{s})\,d\mathbf{s}\int_{B(\mathbf{x},\mathbf{r}_k)}\eta_k^2(\mathbf{s})\,d\mathbf{s} \\
\leq \ & 2\cdot\int_{\Omega}|L_k(\mathbf{s})|^2\cdot\|\rho\|^{2\mu_k}\,d\mathbf{s}\cdot\|\eta_k\|^2 + 2\cdot\|k\|_{\infty}^2\cdot 2^n\, r_{k1}\, r_{k2}\cdots r_{kn}\cdot\|\eta_k\|^2 \\
\leq \ & 2\ \|\ \eta_k\ \|^2\ (\|\rho\|^{2\mu_k}\ \|\ L_k\ \|^2 + \|k\|_{\infty}^2\cdot 2^n\, r_{k1}\, r_{k2}\cdots r_{kn}).
\end{aligned}
$$

thus

$$
\begin{aligned}
& \|(\bar{A}_k - B_k)\eta_k\|_{\mathbf{r}_k}^2 \\
= \ & \frac{1}{2^n\, r_{k1}\, r_{k2}\cdots r_{kn}}\int_{[\mathbf{r}_k,\mathbf{1}-\mathbf{r}_k]}\int_{[-\mathbf{r}_k,\mathbf{r}_k]}\left|(\bar{A}_k - B_k)\eta_k(\mathbf{x})(\boldsymbol{\rho})\right|^2 d\boldsymbol{\rho}\,d\mathbf{x} \\
\leq \ & \frac{1}{2^n r_{k1} r_{k2}\cdots r_{kn}}\cdot \\
& \int_{[\mathbf{r}_k,\mathbf{1}-\mathbf{r}_k]}\int_{[-\mathbf{r}_k,\mathbf{r}_k]} 2\ \|\ \eta_k\ \|^2\ (\|\rho\|^{2\mu_k}\ \|\ L_k\ \|^2 + \|k\|_{\infty}^2\cdot 2^n r_{k1} r_{k2}\cdots r_{kn})\,d\boldsymbol{\rho}\,d\mathbf{x} \\
\leq \ & \frac{\|\eta_k\|^2}{2^{n-1} r_{k1} r_{k2}\cdots r_{kn}}\cdot \\
& \int_{[\mathbf{r}_k,\mathbf{1}-\mathbf{r}_k]}\left(\|L_k\|^2\, 2^n r_{k1} r_{k2}\cdots r_{kn}\cdot\|\mathbf{r}_k\|^{2\mu_k} + \|k\|_{\infty}^2\cdot(2^n r_{k1} r_{k2}\cdots r_{kn})^2\right)d\mathbf{x} \\
\leq \ & \|\eta_k\|^2\left[2\cdot\|L_k\|^2\cdot\|\mathbf{r}_k\|^{2\mu_k} + 2^{n+1} r_{k1} r_{k2}\cdots r_{kn}\cdot\|k\|_{\infty}^2\right].
\end{aligned}
$$

This shows $\|(\bar{A}_k - B_k)\eta_k\|^2_{\mathbf{r}_k} \to 0$ as $k \to \infty$ since $\|\mathbf{r}_k\| \to 0$ as $k \to \infty$, and

$$\|\eta_k\| \;=\; \|T_k\tilde{\varphi}_k\| \;\le\; \|T_k\| \cdot \|\tilde{\varphi}_k\| \le \sqrt{2^n} \cdot \|T\| \cdot \|\tilde{\varphi}_k\| \quad \text{(using (2.11))}$$

is bounded (Lemma 3.1).

$T_4^k \to 0$: $\;(T_4^k)^2 \le \|f^{\delta_k} - f\|^2_\infty \le \delta_k^2$.

$T_5^k \to 0$:

$$\begin{aligned}
(T_5^k)^2 \;&=\; \frac{1}{2^n\, r_{k1}\, r_{k2} \cdots r_{kn}} \int_{[\mathbf{r}_k, 1-\mathbf{r}_k]} \int_{[-\mathbf{r}_k, \mathbf{r}_k]} \left|(\bar{F}_k(\mathbf{x})(\boldsymbol{\rho}) - F_k(\mathbf{x})(\rho))\right|^2 d\boldsymbol{\rho}\, d\mathbf{x} \\
&=\; \frac{1}{2^n\, r_{k1}\, r_{k2} \cdots r_{kn}} \int_{[\mathbf{r}_k, 1-\mathbf{r}_k]} \int_{[-\mathbf{r}_k, \mathbf{r}_k]} \left|(f(\mathbf{x} + \boldsymbol{\rho}) - f(\mathbf{x})\right|^2 d\boldsymbol{\rho}\, d\mathbf{x} \\
&=\; \frac{1}{2^n\, r_{k1}\, r_{k2} \cdots r_{kn}} \int_{[\mathbf{r}_k, 1-\mathbf{r}_k]} \int_{[-\mathbf{r}_k, \mathbf{r}_k]} \Big| \int_\Omega (k(\mathbf{x} + \boldsymbol{\rho}, \mathbf{s}) - k(\mathbf{x}, \mathbf{s}))\bar{u}(\mathbf{s})\, ds\Big|^2 d\boldsymbol{\rho}\, d\mathbf{x} \\
&\le\; \frac{1}{2^n\, r_{k1}\, r_{k2} \cdots r_{kn}} \int_{[\mathbf{r}_k, 1-\mathbf{r}_k]} \int_{[-\mathbf{r}_k, \mathbf{r}_k]} \Big(\int_\Omega \left|k(\mathbf{x} + \boldsymbol{\rho}, \mathbf{s}) - k(\mathbf{x}, \mathbf{s})\right|^2 ds\Big) \cdot \\
&\quad\;\; \Big(\int_\Omega |\bar{u}(\mathbf{s})|^2\, ds\Big)\, d\boldsymbol{\rho}\, d\mathbf{x} \\
&\le\; \frac{\|\bar{u}\|^2}{2^n\, r_{k1}\, r_{k2} \cdots r_{kn}} \int_{[\mathbf{r}_k, 1-\mathbf{r}_k]} \int_{[-\mathbf{r}_k, \mathbf{r}_k]} \int_\Omega \left|L_k(\mathbf{s})\right|^2 \cdot \|\rho\|^{2\mu_k}\, d\mathbf{s}\, d\boldsymbol{\rho}\, d\mathbf{x} \\
&\le\; \frac{\|L_k\|^2\, \|\bar{u}\|^2}{2^n\, r_{k1}\, r_{k2} \cdots r_{kn}} \int_{[\mathbf{r}_k, 1-\mathbf{r}_k]} \int_{[-\mathbf{r}_k, \mathbf{r}_k]} \|\rho\|^{2\mu_k}\, d\boldsymbol{\rho}\, d\mathbf{x} \\
&=\; \frac{\|L_k\|^2\, \|\bar{u}\|^2}{2^n\, r_{k1}\, r_{k2} \cdots r_{kn}} \int_{[\mathbf{r}_k, 1-\mathbf{r}_k]} \|\mathbf{r}_k\|^{2\mu_k} \cdot 2^n\, r_{k1}\, r_{k2} \cdots r_{kn}\, d\mathbf{x} \\
&\le\; \|L_k\|^2 \cdot \|\bar{u}\|^2 \cdot \|\mathbf{r}_k\|^{2\mu_k} \qquad \to 0,
\end{aligned}$$

as $k \to \infty$ because $\|\mathbf{r}_k\| \to 0$.

This completes the proof of Lemma 3.2.

Proof of Theorem 3.1 From Lemmas 3.1 and 3.2 we have that $\eta_k \rightharpoonup \eta$, for $\eta \in L^2(\Omega)$ a solution of $Au = f$, and $\tilde{\varphi}_n \equiv E_n\varphi_n \rightharpoonup \tilde{\varphi}$ for $\tilde{\varphi} \in \mathcal{X}$ a solution of $\tilde{A}\psi \equiv AT\psi = f$, $\psi \in \mathcal{X}$. In both cases the convergence is subsequential (the indices have been relabelled).

We will first show that $\tilde{\varphi} = \tilde{U}$ where $\tilde{U} \in \mathcal{X}$ is defined by

$$\tilde{U}(\mathbf{x})(\boldsymbol{\rho}) = \frac{\bar{u}(\mathbf{x})\,\gamma_l(\boldsymbol{\rho})}{|\,\gamma_l\,|_X^2}, \quad \text{a.a. } \rho \in [-\Delta, \Delta]^n,\ x \in \Omega. \tag{3.4}$$

From $T\tilde{U}(\mathbf{x}) = l(\bar{u}(\mathbf{x})\,\gamma_l(\cdot)/\,|\,\gamma_l\,|_X^2) = \bar{u}(\mathbf{x})$ for a.a. $x \in \Omega$, we know that $\tilde{U}$ solves the equation

$$\tilde{A}\psi = AT\psi = f.$$

In fact we will show that $\tilde{U} \in (\ker\tilde{A})^{\perp} \subseteq \mathcal{X}$, i.e. $\tilde{U}$ is the minimum norm solution of $\tilde{A}\psi = f$.

Indeed, let $\tilde{\psi} \in \ker\tilde{A}$, then

$$\begin{aligned}
\langle \tilde{U}, \tilde{\psi} \rangle_{\mathcal{X}} &= \int_{\Omega} |\tilde{U}(\mathbf{x}) \cdot \tilde{\psi}(\mathbf{x})|_X \, d\mathbf{x} \\
&= \int_{\Omega} \int_{[-\Delta,\Delta]^n} \tilde{U}(\mathbf{x})(\boldsymbol{\rho}) \, \tilde{\psi}(\mathbf{x})(\boldsymbol{\rho}) \, d\boldsymbol{\rho} \, d\mathbf{x} \\
&= \frac{1}{|\gamma_l|_X^2} \int_{\Omega} \bar{u}(\mathbf{x}) \int_{[-\Delta,\Delta]^n} \gamma_l(\boldsymbol{\rho}) \tilde{\psi}(\mathbf{x})(\boldsymbol{\rho}) \, d\boldsymbol{\rho} \, d\mathbf{x} \\
&= \frac{1}{|\gamma_l|_X^2} \int_{\Omega} \bar{u}(\mathbf{x}) \, l(\tilde{\psi}(\mathbf{x})) \, d\mathbf{x} \\
&= \frac{1}{|\gamma_l|_X^2} \langle \bar{u}, T\tilde{\psi} \rangle_{L^2(\Omega)} \\
&= 0,
\end{aligned}$$

because $\tilde{\psi} \in \ker\tilde{A}$ implies $T\tilde{\psi} \in \ker A$. This shows $\tilde{U} \in (\ker\tilde{A})^{\perp}$.

Next we show that $\|\tilde{\phi}\|_{\mathcal{X}} \le \|\tilde{U}\|_{\mathcal{X}}$. In fact, since $\tilde{\phi}_k = E_k \phi_k$, it follows that

$$\begin{aligned}
\|\tilde{\phi}_k\|_{\mathcal{X}}^2 &= 2^n \, \|\phi_k\|_{\mathbf{r}_k}^2 \\
&\le \frac{2^n}{\alpha_{k,\min}} [\, \|C_k \phi_k - F_k^{\delta}\|_{\mathbf{r}_k}^2 \, + \, \|\phi_k\|_{\mathbf{r}_k,\alpha_k}^2] \\
&\le \frac{2^n}{\alpha_{k,\min}} [\, \|C_k \tilde{V}_k - F_k^{\delta}\|_{\mathbf{r}_k}^2 \, + \, \|\tilde{V}_k\|_{\mathbf{r}_k,\alpha_k}^2],
\end{aligned} \tag{3.5}$$

where $\tilde{V}_k \in \mathcal{X}_k$ is defined as

$$\tilde{V}_k(\mathbf{x})(\boldsymbol{\rho}) = \tilde{U}(\mathbf{x})(\rho_1/r_{k1}, \rho_2/r_{k2}, \cdots, \rho_n/r_{kn}), \mathbf{x} \in (\mathbf{r}_k, \mathbf{1} - \mathbf{r}_k) \text{ and } \rho \in (-\mathbf{r}_k, \mathbf{r}_k).$$

Notice that by the definition of $E_{\mathbf{r}}$, we have

$$E_k \tilde{V}_k(\mathbf{x})(\rho) = \begin{cases} \tilde{U}(\mathbf{x})(\boldsymbol{\rho}), & \mathbf{x} \in (\mathbf{r}_k, \mathbf{1} - \mathbf{r}_k) \text{ and } \rho \in (-\Delta, \Delta)^n, \\ 0, & \text{otherwise .} \end{cases}$$

So

$$T_k \tilde{V}_k(\mathbf{x}) = \int_{[-\Delta,\Delta]^n} E_k \tilde{V}_k(\mathbf{x})(\boldsymbol{\rho}) \cdot \gamma_l(\boldsymbol{\rho})\, d\boldsymbol{\rho} = \begin{cases} \bar{u}(\mathbf{x}), & t \in (\mathbf{r}_k, \mathbf{1} - \mathbf{r}_k), \\ 0, & \text{otherwise .} \end{cases}$$

We also have

$$\begin{aligned} \|\tilde{V}_k\|^2_{\mathbf{r}_k} &= \frac{1}{2^n\, r_{k1}\, r_{k2} \cdots r_{kn}} \cdot \\ &\quad \int_{[\mathbf{r}_k, \mathbf{1}-\mathbf{r}_k]} \int_{[-\mathbf{r}_k, \mathbf{r}_k]} \bar{u}^2(\mathbf{x}) \cdot \gamma_l^2(\rho_1/r_{k1}, \rho_2/r_{k2}, \cdots, \rho_n/r_{kn}) \,/ |\gamma_l|^4_X \, d\boldsymbol{\rho}\, d\mathbf{x} \\ &= \frac{1}{2^n\, r_{k1}\, r_{k2} \cdots r_{kn}} \int_{[\mathbf{r}_k, \mathbf{1}-\mathbf{r}_k]} \bar{u}^2(\mathbf{x})\, d\mathbf{x} \int_{[-1,1]^n} \gamma_l^2(\eta) \,/ |\gamma_l|^4_X \cdot r_{k1} r_{k2} \cdots r_{kn}\, d\eta \\ &\leq \frac{1}{2^n |\gamma_l|^2_X} \int_{[\mathbf{r}_k, \mathbf{1}-\mathbf{r}_k]} \bar{u}^2(\mathbf{x})\, d\mathbf{x}. \end{aligned} \tag{3.6}$$

Now consider the first term in (3.5):

$$\begin{aligned} \|C_k \tilde{V}_k - F_k^\delta\|^2_{\mathbf{r}_k} &= \|A_k \tilde{V}_k + B_k T_k \tilde{V}_k - F_k^\delta\|^2_{\mathbf{r}_k} \\ &\leq 4\,\|A_k(\tilde{V}_k - U_k)\|^2_{\mathbf{r}_k} + 4\,\|A_k U_k + B_k T_k \tilde{V}_k - F_k\|^2_{\mathbf{r}_k} + 2\|F_k - F_k^\delta\|^2_{\mathbf{r}_k}, \end{aligned}$$

where $\|A_k U_k + B_k T_k \tilde{V}_k - F_k\|^2_{\mathbf{r}_k} = \|A_k U_k + B_k \bar{u} - F_k\|^2_{\mathbf{r}_k} = 0$ (using the proof of

Lemma 2.1 and the fact that $T_k\tilde{V}_k = \bar{u}$ for $x \in (\mathbf{r}_k, \mathbf{1}-\mathbf{r}_k)$, $\|F_k - F_k^\delta\|_{\mathbf{r}_k}^2 \le \delta_k^2$, and

$$\begin{aligned}\|A_k(\tilde{V}_k - U_k)\|_{\mathbf{r}_k}^2 &\le 2\|A_k\|^2\left(\|\tilde{V}_k\|_{\mathbf{r}_k}^2 + \|U_k\|_{\mathbf{r}_k}^2\right)\\ &\le Cr_{k1}^2 r_{k2}^2 \cdots r_{kn}^2,\end{aligned}$$

because of (2.4), (2.6) and the fact that $\|\tilde{V}_k\|_{\mathbf{r}_k}^2 \le \frac{1}{2|\gamma_l|_X^2} \cdot \|\bar{u}\|_{L^2(\Omega)}^2$ from (3.6). Therefore, for the first term in (3.5) we have

$$\|C_k\tilde{V}_k - F_k^\delta\|_{\mathbf{r}_k}^2 \le C \cdot (r_{k1}^2 r_{k2}^2 \cdots r_{kn}^2 + \delta_k^2).$$

For the second term in (3.5) we have

$$\|\tilde{V}_k\|_{\mathbf{r}_k,\alpha_k}^2 \le \|\alpha_k\|_\infty \cdot \|\tilde{V}_k\|_{\mathbf{r}_k}^2 = \frac{\|\alpha_k\|_\infty}{2^n|\gamma_l|_X^2}\int_{[\mathbf{r}_k,\mathbf{1}-\mathbf{r}_k]} \bar{u}^2(\mathbf{x})\,d\mathbf{x}.$$

Thus, continuing from (3.5), we get

$$\|\tilde{\phi}_k\|_{\mathcal{X}}^2 \le \frac{2^n}{\alpha_{k,\min}}\left[C(r_{k1}^2 r_{k2}^2 \cdots r_{kn}^2 + \delta_k^2) + \frac{\|\alpha_k\|_\infty}{2^n|\gamma_l|_X^2}\int_{[\mathbf{r}_k,\mathbf{1}-\mathbf{r}_k]} \bar{u}^2(\mathbf{x})\,d\mathbf{x}\right].$$

It follows that

$$\begin{aligned}\|\tilde{\phi}\|_{\mathcal{X}}^2 &\le \liminf \|\tilde{\phi}_k\|_{\mathcal{X}}^2 && (3.7)\\ &\le \limsup \frac{2^n}{\alpha_{k,\min}}\left[C(r_{k1}^2 r_{k2}^2 \cdots r_{kn}^2 + \delta_k^2) + \frac{\|\alpha_k\|_\infty}{2^n|\gamma_l|_X^2}\int_{[\mathbf{r}_k,\mathbf{1}-\mathbf{r}_k]} \bar{u}^2(\mathbf{x})\,d\mathbf{x}\right]\\ &= \|\bar{u}\|_{L^2(\Omega)}^2 / \mid \gamma_l \mid_X^2 && (3.8)\\ &= \|\tilde{U}\|_{\mathcal{X}}^2, && (3.9)\end{aligned}$$

under the assumptions of the theorem, so that

$$\|\tilde{\phi}\|_{\mathcal{X}} \leq \|\tilde{U}\|_{\mathcal{X}}.$$

By uniqueness of the minimum norm solution of $\tilde{A}\psi = f$, it follows that $\tilde{\phi} = \tilde{U}$.

All inequalities between (3.7) and (3.9) must then be equalities, so

$$\|\tilde{\phi}_k\|_{\mathcal{X}} \to \|\tilde{\phi}\|_{\mathcal{X}} \quad \text{as } k \to \infty.$$

This fact combined with the weak convergence of $\tilde{\phi}_k$ to $\tilde{\phi}$ in the Hilbert space $\mathcal{X}$ implies the strong convergence, thus

$$\eta_k = T\tilde{\phi}_k \to T\tilde{\phi} = T\tilde{U} = \bar{u}, \quad \text{as } k \to \infty.$$

so that $\eta = \bar{u}$.

Standard arguments can be used to extend the subsequential convergence we just proved to full sequential convergence. This completes the proof of Theorem 3.1.

4 Numerical Implementation

In this section we develop a discrete implementation of the local regularization method presented in the previous sections. We'll also study two numerical examples. The algorithms and examples presented here are the case of R^1. Numerical implementation in the case of R^n will be the subject of a future study.

It's also worth noting that the algorithm here is different from the results in [11], and as we'll see the new algorithm has better performance in the numerical examples.

4.1 The Discrete Local Regularization Problem

Let $\Delta x = \frac{1}{N}$ for fixed $N = 1, 2, \cdots$, and define

$$x_j = j \cdot \Delta x, \qquad j = 0, 1, 2, \cdots, N,$$

$$\rho_l = l \cdot \Delta x, \qquad l = -N, -N+1, \cdots, 0, 1, 2, \cdots, N,$$

$$\chi_l(x) = \begin{cases} 1, & x \in (\rho_{l-1}, \rho_l], \\ 0, & \text{Otherwise.} \end{cases}$$

The regularization parameter $\alpha(x)$ is replaced with a discrete approximation

$$\alpha(x) = \sum_{j=1}^{N} \alpha_j \cdot \chi_j(x) \qquad x \in [0, 1].$$

The other regularization parameter r is chosen as $r \in (0, \frac{1}{4})$ such that

$$r = \frac{i_r}{N},$$

where $1 \leq i_r < \frac{N}{4}$.

We also need a discrete approximation of the space $\mathcal{X}_r$. It is defined as

$$\mathcal{X}_r^N = \{\varphi : \varphi(x)(\rho) = \sum_{j=i_r}^{N-i_r-1} \sum_{l=-i_r}^{i_r-1} c_{jl} \cdot \chi_j(x) \cdot \chi_l(\rho)\},$$

with norms

$$\| \varphi \|_{N,r}^2 = \frac{1}{2^n r_1 r_2 \cdots r_n} \cdot \sum_{j=i_r}^{N-i_r-1} \sum_{l=-i_r}^{i_r-1} \left|\varphi(x_j)(\rho_l)\right|^2 \cdot (\Delta x)^2,$$

or

$$\| \varphi \|_{N,r,\alpha}^2 = \frac{1}{2^n r_1 r_2 \cdots r_n} \cdot \sum_{j=i_r}^{N-i_r-1} \sum_{l=-i_r}^{i_r-1} \left|\varphi(x_j)(\rho_l)\right|^2 \cdot \alpha(x_j) \cdot (\Delta x)^2.$$

Now we need to calculate the operators $A_{\mathbf{r}}, B_{\mathbf{r}}, T_{\mathbf{r}}$ and C_r.We use

$$\gamma_l(\rho) = \begin{cases} \frac{1}{c}, & \rho \in (-1, 0), \\ \\ 0, & \text{Otherwise}, \end{cases}$$

for $0 < c << 1$. This corresponds to

$$T_{\mathbf{r}}\varphi(x) \;=\; \begin{cases} \frac{1}{cr}\cdot\int_{-cr}^{0}\varphi(x)(\rho)\,d\rho, & x\in[r,1-r], \\ \\ 0, & \text{Otherwise,} \end{cases}$$

because for $x \in [r, 1-r]$,

$$T_{\mathbf{r}}\varphi(x) = TE_{\mathbf{r}}\varphi(x) = \int_{-1}^{1}\gamma_l(\rho)\cdot\varphi(x)(r\rho)\,d\rho = \frac{1}{c}\int_{-c}^{0}\varphi(x)(r\rho)\,d\rho = \frac{1}{cr}\int_{-cr}^{0}\varphi(x)(\eta)\,d\eta.$$

In the development of our algorithm, we choose

$$c = \Delta x. \tag{4.1}$$

Then for $\varphi \in \mathcal{X}_r^N$ and $x \in [r, 1-r]$, we have

$$\begin{aligned}
T_{\mathbf{r}}\varphi(x) &= \frac{1}{r\cdot\Delta x}\int_{-r\Delta x}^{0}\varphi(x)(\eta)\,d\eta \\
&= \frac{1}{r\cdot\Delta x}\int_{-r\Delta x}^{0}\sum_{j=i_r}^{N-i_r-1}\sum_{l=-i_r}^{i_r-1} c_{jl}\cdot\chi_j(x)\cdot\chi_l(\rho)\,d\rho \\
&= \frac{1}{r\cdot\Delta x}\cdot\sum_{j=i_r}^{N-i_r-1}\sum_{l=-i_r}^{i_r-1}\int_{-r\Delta x}^{0} c_{jl}\cdot\chi_j(x)\cdot\chi_l(\rho)\,d\rho \\
&= \frac{1}{r\cdot\Delta x}\cdot\sum_{j=i_r}^{N-i_r-1}\chi_j(x)\cdot\sum_{l=-i_r}^{i_r-1}\int_{-r\Delta x}^{0} c_{jl}\cdot\chi_l(\rho)\,d\rho \\
&= \frac{1}{r\cdot\Delta x}\cdot\sum_{j=i_r}^{N-i_r-1}\chi_j(x)\cdot\int_{-r\Delta x}^{0} c_{j0}\cdot\chi_0(\rho)\,d\rho \\
&= \frac{1}{r\cdot\Delta x}\cdot\sum_{j=i_r}^{N-i_r-1}\chi_j(x)\cdot c_{j0}\cdot r\Delta x \\
&= \sum_{j=i_r}^{N-i_r-1} c_{j0}\cdot\chi_j(x) \\
&= \varphi(x)(0). \qquad (4.2)
\end{aligned}$$

To calculate $A_{\mathbf{r}} : \mathcal{X}_r^N \to \mathcal{X}_r^N$, we fix j and l such that $i_r \le j \le N - i_r - 1$ and

$-i_r \leq l \leq i_r - 1$. Then

$$
\begin{aligned}
A_{\mathbf{r}}\varphi(x_j)(\rho_l) &= \int_{-\frac{i_r}{N}}^{\frac{i_r}{N}} k(x_j+\rho_l, x_j+\mathrm{s})\varphi(x_j)(\mathrm{s})\, ds \\
&= \int_{-\frac{i_r}{N}}^{\frac{i_r}{N}} k(x_j+\rho_l, x_j+\mathrm{s})\,\Big[\sum_{p=i_r}^{N-i_r-1}\sum_{q=-i_r}^{i_r-1} c_{pq}\cdot\chi_p(x_j)\cdot\chi_q(\mathrm{s})\Big]\, ds \\
&= \int_{-\frac{i_r}{N}}^{\frac{i_r}{N}} k(x_j+\rho_l, x_j+\mathrm{s})\,\Big[\sum_{q=-i_r}^{i_r-1} c_{jq}\chi_q(\mathrm{s})\Big]\, ds \\
&= \sum_{q=-i_r}^{i_r-1} \int_{-\frac{i_r}{N}}^{\frac{i_r}{N}} k(x_j+\rho_l, x_j+\mathrm{s})\cdot c_{jq}\cdot\chi_q(\mathrm{s})\, ds \\
&= \sum_{q=-i_r}^{i_r-1} \int_{\rho_{q-1}}^{\rho_q} k(x_j+\rho_l, x_j+\mathrm{s})\cdot c_{jq}\, ds \\
&= \sum_{q=-i_r}^{i_r-1} c_{jq}\cdot \int_{\rho_{q-1}}^{\rho_q} k(x_j+\rho_l, x_j+\mathrm{s})\, ds \\
&= \sum_{q=-i_r}^{i_r-1} c_{jq}\cdot \int_{\rho_{q-1}+x_j}^{\rho_q+x_j} k(t_{j+l}, v)\, dv \\
&= \sum_{q=-i_r}^{i_r-1} c_{jq}\cdot \Delta_{j+l,j+q}, \qquad (4.3)
\end{aligned}
$$

where $\Delta_{i,j}$ is defined as

$$
\Delta_{i,j} = \int_{\rho_{j-1}}^{\rho_j} k(t_i, s)\, ds. \qquad (4.4)
$$

For each j satisfying $i_r \leq j \leq N - i_r - 1$, we can define a matrix $\mathbf{A}_j$ and a vector

$\mathbf{c}_j$ as:

$$\begin{aligned}\mathbf{A}_j &= \left(\Delta_{j+l_1,j+l_2}\right)_{-i_r\le l_1,l_2\le i_r-1}\\ &= \begin{pmatrix}\Delta_{j-i_r,j-i_r} & \Delta_{j-i_r,j-i_r+1} & \cdots & \Delta_{j-i_r,j+i_r-1}\\ \Delta_{j-i_r+1,j-i_r} & \Delta_{j-i_r+1,j-i_r+1} & \cdots & \Delta_{j-i_r+1,j+i_r-1}\\ \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots\\ \Delta_{j+i_r-1,j-i_r} & \Delta_{j+i_r-1,j-i_r+1} & \cdots & \Delta_{j+i_r-1,j+i_r-1}\end{pmatrix}_{2\,i_r\times 2\,i_r}\end{aligned}$$

$$\mathbf{c}_j = \begin{pmatrix}c_{j,-i_r} & c_{j,-i_r+1} & \cdots & c_{j,i_r-1}\end{pmatrix}^T \in \mathbf{R}^{2\,i_r}.$$

Then we have, for $i_r \le j \le N-i_r-1$ and $-i_r \le l \le i_r-1$,

$$A_{\mathbf{r}}\varphi(x_j)(\rho_l) = \left(\mathbf{A}_j\cdot\mathbf{c}_j\right)_l. \tag{4.5}$$

Now we calculate $B_{\mathbf{r}}T_{\mathbf{r}} : \mathcal{X}_{\mathbf{r}}^N \to \mathcal{X}_{\mathbf{r}}^N$. According to the definition of $B_{\mathbf{r}}$ and (4.2), we have, for $i_r \le j \le N-i_r-1$ and $-i_r \le l \le i_r-1$,

$$\begin{aligned}B_{\mathbf{r}}T_{\mathbf{r}}\varphi(x_j)(\rho_l) &= \Big(\int_0^{x_j-\frac{i_r}{N}} + \int_{x_j+\frac{i_r}{N}}^1\Big)k(x_j+\rho_l,s)\cdot\Big(\sum_{u=i_r}^{N-i_r-1} c_{u0}\,\chi_u(s)\Big)\,ds\\ &= \sum_{u=i_r}^{N-i_r-1}\int_0^{x_j-t_{i_r}} k(x_j+\rho_l,s)\cdot c_{u0}\cdot\chi_u(s)\,ds\\ &\quad+\sum_{u=i_r}^{N-i_r-1}\int_{x_j+t_{i_r}}^1 k(x_j+\rho_l,s)\cdot c_{u0}\cdot\chi_u(s)\,ds. \end{aligned}\tag{4.6}$$

Case 1: If $i_r \leq j < 2\,i_r$, then $0 \leq j - i_r \leq i_r - 1$, hence $\chi_u(s) = 0$ for $s \in [0, x_{j-i_r}]$ and $i_r \leq u \leq N - i_r - 1$. Also from $j < 2\,i_r \leq N - 2\,i_r - 1$ we have $j + i_r < N - i_r - 1$. So from (4.6) we get

$$
\begin{aligned}
B_{\mathbf{r}}\, T_{\mathbf{r}} \varphi(x_j)(\rho_l) &= \sum_{u=i_r}^{N-i_r-1} \int_{x_j+t_{i_r}}^{1} k(x_j + \rho_l, s) \cdot c_{u\,0} \cdot \chi_u(\mathbf{s})\, ds \\
&= \sum_{u=i_r+j+1}^{N-i_r-1} \int_{t_{u-1}}^{t_u} k(x_j + \rho_l, s) \cdot c_{u\,0}\, ds \\
&= \sum_{u=i_r+j+1}^{N-i_r-1} c_{u\,0}\, \Delta_{j+l,u}. \qquad (4.7)
\end{aligned}
$$

Case 2: If $N - 2\,i_r - 1 \leq j \leq N - i_r - 1$, then $j + i_r > N - i_r - 1$. This implies that for $u = i_r, i_r + 1, \cdots, N - i_r - 1$ and $s \in [x_{j+i_r}, 1]$, we have $\chi_u(s) = 0$.So from (4.6) we get

$$
\begin{aligned}
B_{\mathbf{r}}\, T_{\mathbf{r}} \varphi(x_j)(\rho_l) &= \sum_{u=i_r}^{N-i_r-1} \int_{0}^{x_j-t_{i_r}} k(x_j + \rho_l, s) \cdot c_{u\,0} \cdot \chi_u(s)\, ds \\
&= \sum_{u=i_r}^{j-i_r} \int_{t_{u-1}}^{t_u} k(x_j + \rho_l, s) \cdot c_{u\,0}\, ds \\
&= \sum_{u=i_r}^{j-i_r} c_{u\,0}\, \Delta_{j+l,u}. \qquad (4.8)
\end{aligned}
$$

Case 3: If $2\,i_r \leq j < N - 2\,i_r - 1$, then from (4.6) we get

$$\begin{aligned}
B_{\mathbf{r}}\, T_{\mathbf{r}}\varphi(x_j)(\rho_l) &= \sum_{u=i_r}^{N-i_r-1} \int_0^{x_j - t_{i_r}} k(x_j + \rho_l, s) \cdot c_{u\,0} \cdot \chi_u(s)\, ds \\
&\quad + \sum_{u=i_r}^{N-i_r-1} \int_{x_j + t_{i_r}}^{1} k(x_j + \rho_l, s) \cdot c_{u\,0} \cdot \chi_u(s)\, ds \\
&= \sum_{u=i_r}^{j-i_r} \int_{t_{u-1}}^{t_u} k(x_j + \rho_l, s) \cdot c_{u\,0}\, ds + \sum_{u=i_r+j+1}^{N-i_r-1} \int_{t_{u-1}}^{t_u} k(x_j + \rho_l, s) \cdot c_{u\,0}\, ds \\
&= \Big(\sum_{u=i_r}^{j-i_r} + \sum_{u=i_r+j+1}^{N-i_r-1}\Big) c_{u\,0}\, \Delta_{j+l,u}. \qquad (4.9)
\end{aligned}$$

We can also define a matrix $\mathbf{B}_j$ and vector $\bar{\mathbf{c}}$, for each j, such that

$$B_{\mathbf{r}}\, T_{\mathbf{r}}\varphi(x_j)(\rho_l) = \left(\mathbf{B}_j \cdot \bar{\mathbf{c}}\right)_l \qquad (4.10)$$

The vector $\bar{\mathbf{c}}$ is defined as

$$\bar{\mathbf{c}} = \begin{pmatrix} c_{i_r,0} & c_{i_r+1,0} & \cdots & c_{N-i_r-1,0} \end{pmatrix}^T \in \mathbf{R}^{N-2\,i_r}$$

The definition of $\mathbf{B}_j$ is divided into three cases:

Case 1: $i_r \leq j < 2\,i_r$

$$\mathbf{B}_j \;=\; \left(\mathbf{0}_{2\,i_r \times (j+1)} \;\vdots\; \left(\Delta_{j+l,u}\right)_{2\,i_r \times (N-2\,i_r-j-1)}\right) \qquad (4.11)$$

or

$$\begin{pmatrix} 0 & \cdots & 0 & \Delta_{j-i_r,j+i_r+1} & \Delta_{j-i_r,j+i_r+2} & \cdots & \Delta_{j-i_r,N-i_r-1} \\ 0 & \cdots & 0 & \Delta_{j-i_r+1,j+i_r+1} & \Delta_{j-i_r+1,j+i_r+2} & \cdots & \Delta_{j-i_r+1,N-i_r-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \Delta_{j+i_r-1,j+i_r+1} & \Delta_{j+i_r-1,j+i_r+2} & \cdots & \Delta_{j+i_r-1,N-i_r-1} \end{pmatrix}_{2\,i_r \times (N-2\,i_r)}$$

Case 2: $N - 2\,i_r - 1 < j \leq N - i_r - 1$

$$\begin{aligned} \mathbf{B}_j &= \left(\left(\Delta_{j+l,u}\right)_{2\,i_r \times (j-2\,i_r+1)} \,\vdots\, \mathbf{0}_{2\,i_r \times (N-j-1)} \right) \\ &= \begin{pmatrix} \Delta_{j-i_r,i_r} & \Delta_{j-i_r,i_r+1} & \cdots & \Delta_{j-i_r,j-i_r} & 0 & \cdots & 0 \\ \Delta_{j-i_r+1,i_r} & \Delta_{j-i_r+1,i_r+1} & \cdots & \Delta_{j-i_r+1,j-i_r} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \Delta_{j+i_r-1,i_r} & \Delta_{j+i_r-1,i_r+1} & \cdots & \Delta_{j+i_r-1,j-i_r} & 0 & \cdots & 0 \end{pmatrix}_{2\,i_r \times (N-2\,i_r)} \end{aligned} \tag{4.12}$$

Case 3: $2\,i_r \leq j < N - 2\,i_r - 1$

$$\mathbf{B}_j = \left(\left(\Delta_{j+l,u}\right)_{2\,i_r \times (j-2\,i_r+1)} \,\vdots\, \mathbf{0}_{2\,i_r \times 2\,i_r} \,\vdots\, \left(\Delta_{j+l,u}\right)_{2\,i_r \times (N-2\,i_r-j-1)} \right) \tag{4.13}$$

or

$$\begin{pmatrix} \Delta_{j-i_r,i_r} & \cdots & \Delta_{j-i_r,j-i_r} & 0 & \cdots & 0 & \Delta_{j-i_r,j+i_r+1} & \cdots & \Delta_{j-i_r,N-i_r-1} \\ \Delta_{j-i_r+1,i_r} & \cdots & \Delta_{j-i_r+1,j-i_r} & 0 & \cdots & 0 & \Delta_{j-i_r+1,j+i_r+1} & \cdots & \Delta_{j-i_r+1,N-i_r-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ \Delta_{j+i_r-1,i_r} & \cdots & \Delta_{j+i_r-1,j-i_r} & 0 & \cdots & 0 & \Delta_{j+i_r-1,j+i_r+1} & \cdots & \Delta_{j+i_r-1,N-i_r-1} \end{pmatrix}$$

The last vector we need to calculate is $F_{\mathbf{r}}^{N,\delta}$ in the discrete case. By definition,

$$F_{\mathbf{r}}^{N,\delta}(x_j)(\rho_l) = f(x_j + \rho_l), \quad \text{for } i_r \le j \le N - i_r - 1, \ -i_r \le l \le i_r - 1.$$

This prompts us to define

$$\bar{\mathbf{f}}_j = \begin{pmatrix} f(x_j + \rho_{-i_r}) & f(x_j + \rho_{-i_r+1}) & \cdots & f(x_j + \rho_{i_r-1}) \end{pmatrix}^T \in \mathbf{R}^{2i_r}. \tag{4.14}$$

Finally we can define the discrete format of the local regularization problem $\mathcal{P}_{r,\alpha}^{\delta}$

given in Definition 2.1:

$$
\begin{aligned}
& \| A_{\mathbf{r}}\varphi + B_{\mathbf{r}} T_{\mathbf{r}}\varphi - F_r^{N,\delta} \|_{N,r}^2 + \| \varphi \|_{N,r,\alpha}^2 \\
= & \frac{1}{2r} \sum_{j=i_r}^{N-i_r-1} \sum_{l=-i_r}^{i_r-1} \left| A_{\mathbf{r}}\varphi(x_j)(\rho_l) + B_{\mathbf{r}} T_{\mathbf{r}}\varphi(x_j)(\rho_l) - F_{\mathbf{r}}^{N,\delta}(x_j)(\rho_l) \right|^2 \cdot (\Delta x)^2 \\
& + \frac{1}{2r} \sum_{j=i_r}^{N-i_r-1} \sum_{l=-i_r}^{i_r-1} \left| \varphi(x_j)(\rho_l) \right|^2 \cdot \alpha(x_j) \cdot (\Delta x)^2 \\
= & \frac{1}{2r} \sum_{j=i_r}^{N-i_r-1} \sum_{l=-i_r}^{i_r-1} \left| (\mathbf{A}_j \cdot \mathbf{c}_j)_l + (\mathbf{B}_j \cdot \bar{\mathbf{c}})_l - (\bar{\mathbf{f}}_j)_l \right|^2 \cdot (\Delta x)^2 \\
& + \frac{1}{2r} \sum_{j=i_r}^{N-i_r-1} \sum_{l=-i_r}^{i_r-1} (c_{jl})^2 \cdot \alpha(x_j) \cdot (\Delta x)^2 \\
= & \frac{1}{2r} \cdot (\Delta x)^2 \cdot \sum_{j=i_r}^{N-i_r-1} \sum_{l=-i_r}^{i_r-1} \left(\left| (\mathbf{A}_j \cdot \mathbf{c}_j + \mathbf{B}_j \cdot \bar{\mathbf{c}} - \bar{\mathbf{f}}_j)_l \right|^2 + (c_{jl})^2 \cdot \alpha(x_j) \right) \\
= & \frac{(\Delta x)^2}{2r} \cdot \sum_{j=i_r}^{N-i_r-1} H_j(\mathbf{c}_j; \bar{\mathbf{c}}),
\end{aligned} \tag{4.15}
$$

where $H_j(\mathbf{c}_j; \bar{\mathbf{c}})$ is defined as

$$
H_j(\mathbf{c}_j; \bar{\mathbf{c}}) = \sum_{l=-i_r}^{i_r-1} \left(\left| (\mathbf{A}_j \cdot \mathbf{c}_j + \mathbf{B}_j \cdot \bar{\mathbf{c}} - \bar{\mathbf{f}}_j)_l \right|^2 + (c_{jl})^2 \cdot \alpha(x_j) \right).
$$

In order to describe the relaxation type of minimization method we are going to use to solve our discrete regularization problem, it is necessary to introduce the following notations:

Fix m, where $i_r \le m \le N - i_r - 1$, define

$$
J_m(\mathbf{c}_m) = H_m(\mathbf{c}_m; \bar{\mathbf{c}}).
$$

For $j \neq m$, where $i_r \leq m \leq N - i_r - 1$, notice that $H_j(\mathbf{c}_j; \bar{\mathbf{c}})$ depends on $\mathbf{c}_m$ only through the component $c_{m\,0}$ in $\bar{\mathbf{c}}$. So it is valid to define that

$$\hat{J}_m(c_{m0}) = \sum_{\substack{j = i_r \\ j \neq m}}^{N-i_r-1} H_j(\mathbf{c}_j; \bar{\mathbf{c}}),$$

then

$$\frac{2r}{(\Delta x)^2} \left(\| \, A_{\mathbf{r}}\varphi + B_{\mathbf{r}} T_{\mathbf{r}}\varphi - F_r^{N,\delta} \, \|_{N,r}^2 \; + \; \| \, \varphi \, \|_{N,r,\alpha}^2 \right) = J_m(\mathbf{c}_m) + \hat{J}_m(c_{m\,0}).$$

Using notations $J_m(\mathbf{c}_m)$ and $\hat{J}_m(c_{m\,0})$, we introduce the following iterative relaxation-type minimization algorithm for the discrete regularization problem:

Local Regularization Algorithm 1

1. Initialize vectors $\mathbf{c}_1, \, \mathbf{c}_2, \, \cdots \, \mathbf{c}_N$.

2. Do for $m = i_r, \, i_r + 1, \, \cdots, \, N - i_r - 1$:

   (a) Holding the previously determined values of $\mathbf{c}_j, \; j \neq m$, find $\bar{\beta} \in \mathbf{R}^{2\,i_r}$ solving

   $$\min\{ J_m(\beta) + \hat{J}_m(\beta_0), \; \beta = \begin{pmatrix} \beta_{-i_r} & \beta_{-i_r+1} & \cdots & \beta_{i_r-1} \end{pmatrix} \in \mathbf{R}^{2\,i_r} \}.$$

   (b) Set $\mathbf{c}_m = \bar{\beta}$.

3. Go to step (2).

Local Regularization Algorithm 2

1. Initialize vectors $\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_N$.

2. Do for $m = i_r, i_r + 1, \cdots, N - i_r - 1$:

   (a) Holding the previously determined values of $\mathbf{c}_j$, $j \neq m$, find $\bar{\beta} \in \mathbf{R}^{2i_r}$ solving

$$\min\{J_m(\beta),\ \beta = \begin{pmatrix}\beta_{-i_r} & \beta_{-i_r+1} & \cdots & \beta_{i_r-1}\end{pmatrix} \in \mathbf{R}^{2i_r}\}.$$

   (b) Set $\mathbf{c}_m = \bar{\beta}$.

3. Go to step (2).

4.2 Numerical Examples

We will study two numerical examples of a one-dimensional image processing problem using the **Local Regularization Algorithm 2** presented above. The kernal function k is

$$k(x, s) = \frac{\gamma}{\pi}\exp(-\gamma(x - s)^2). \tag{4.16}$$

The corresponding operator A represents the blurring operator.

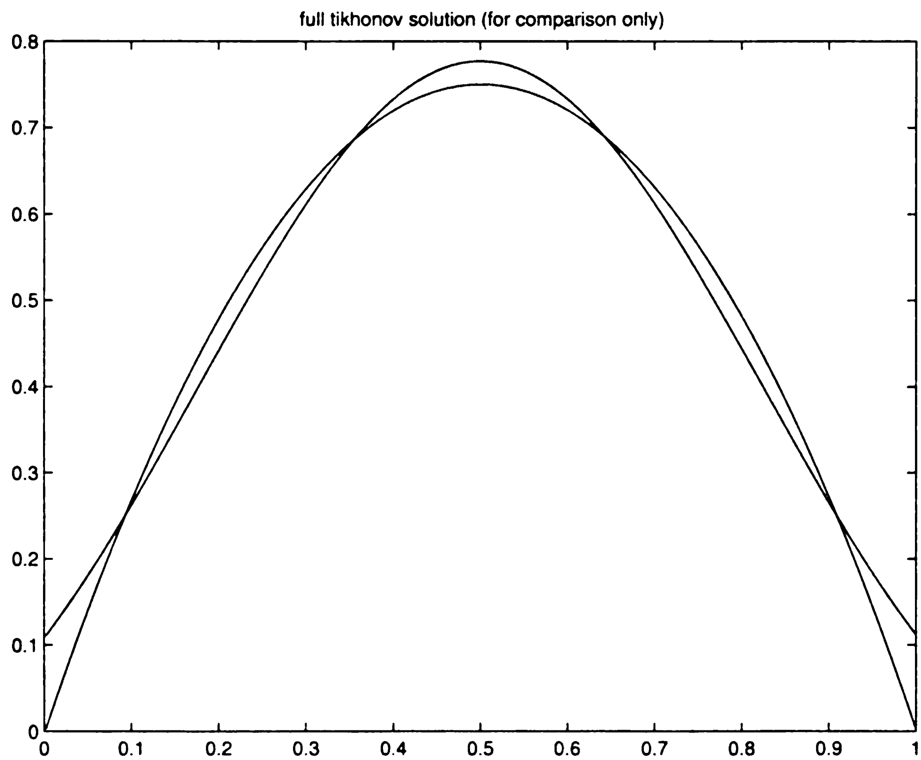
In our examples, we choose $\gamma = 5$. A true solution $\bar{u}$ was pre-selected and the noisy

data f^δ used in the regularization process is a random perturbation of $f = A\bar{u}$, where f^δ differs from f with 1% relative error.

All calculations are conducted using Matlab version 6.1.

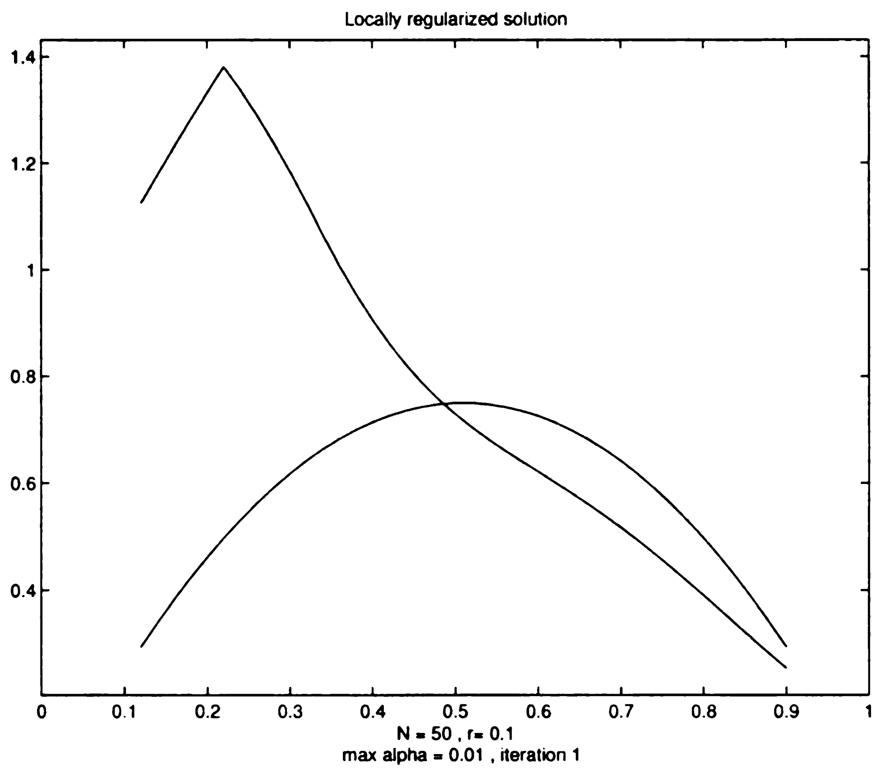
Example 4.1 In the first example, our true solution is $\bar{u}(x) = 3x\,(1-x)$.

The classical Tikhonov regularization generates the following result:

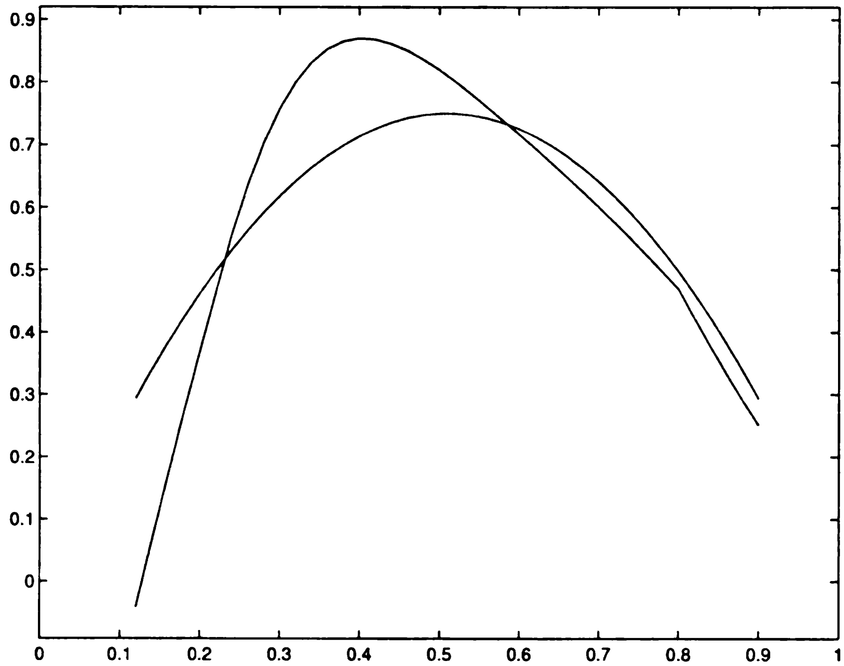


Results from various iterations of our local regularization algorithm are shown below - with different selections of regularization parameters α and r:

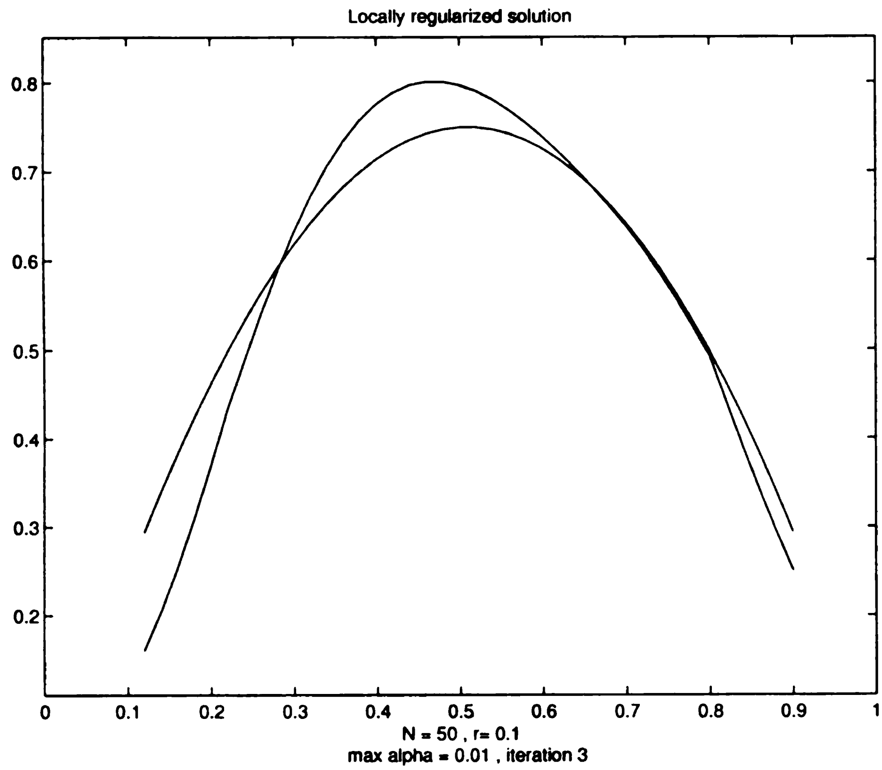
Local Regularization Results with $N = 50$, $\alpha = 0.01$ and $r = 0.1$:

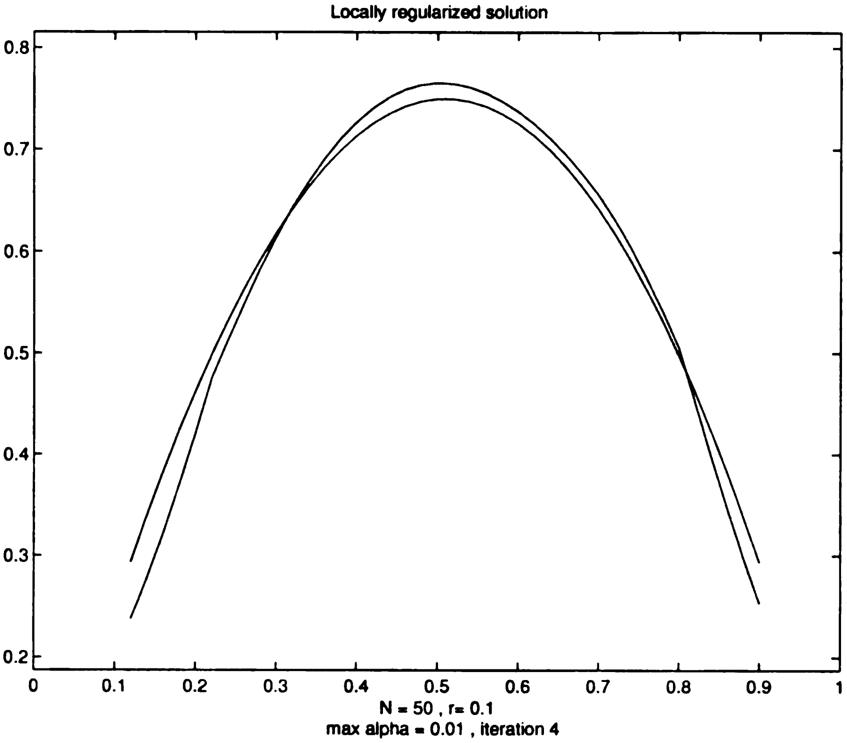


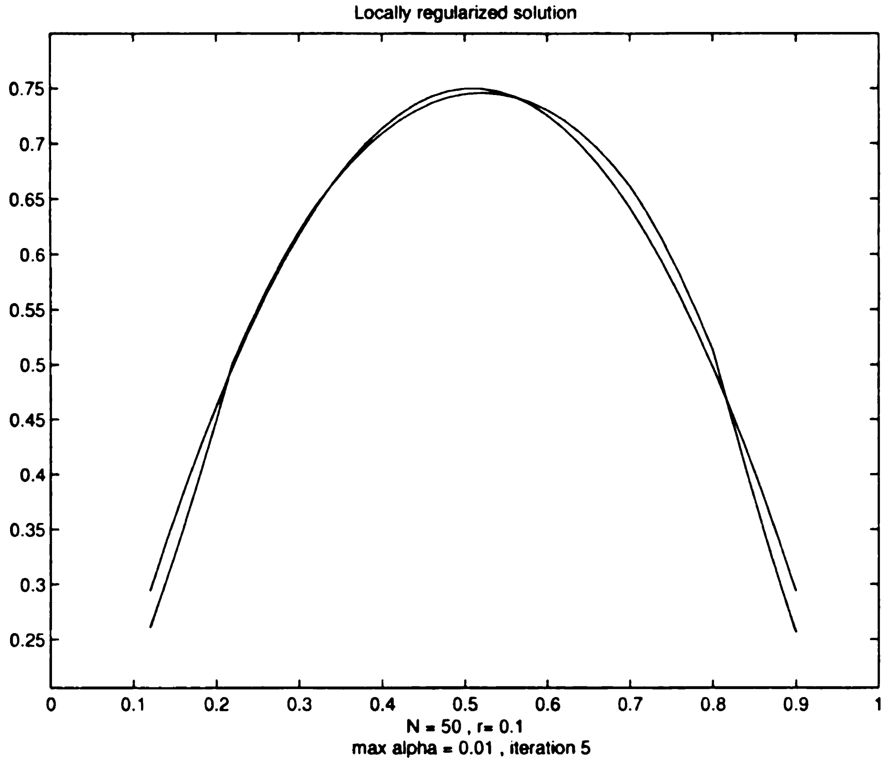
Locally regularized solution



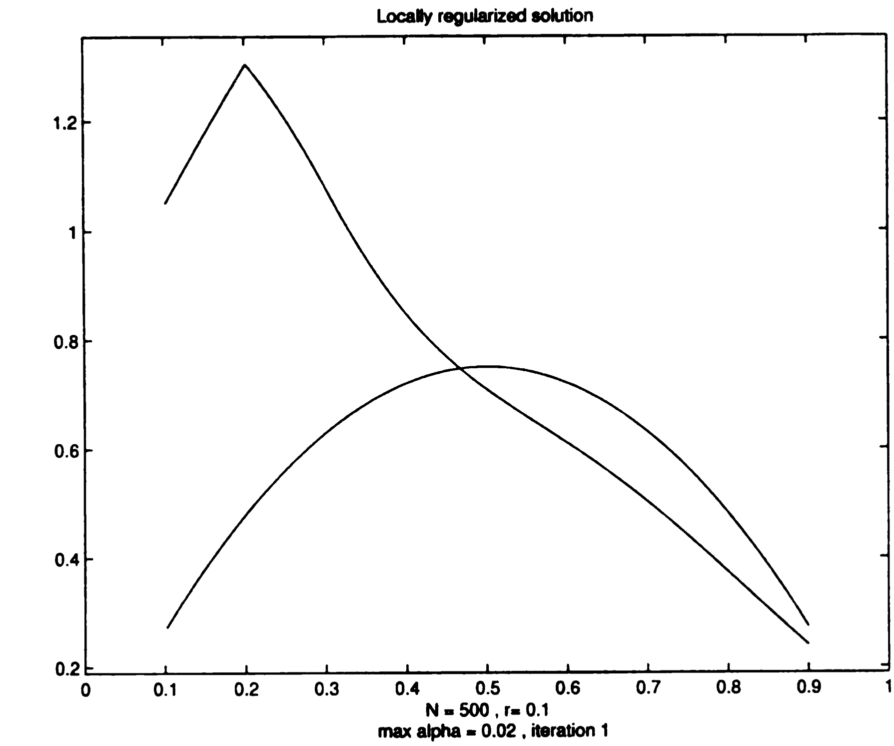
N = 50 , r= 0.1
max alpha = 0.01 , iteration 2

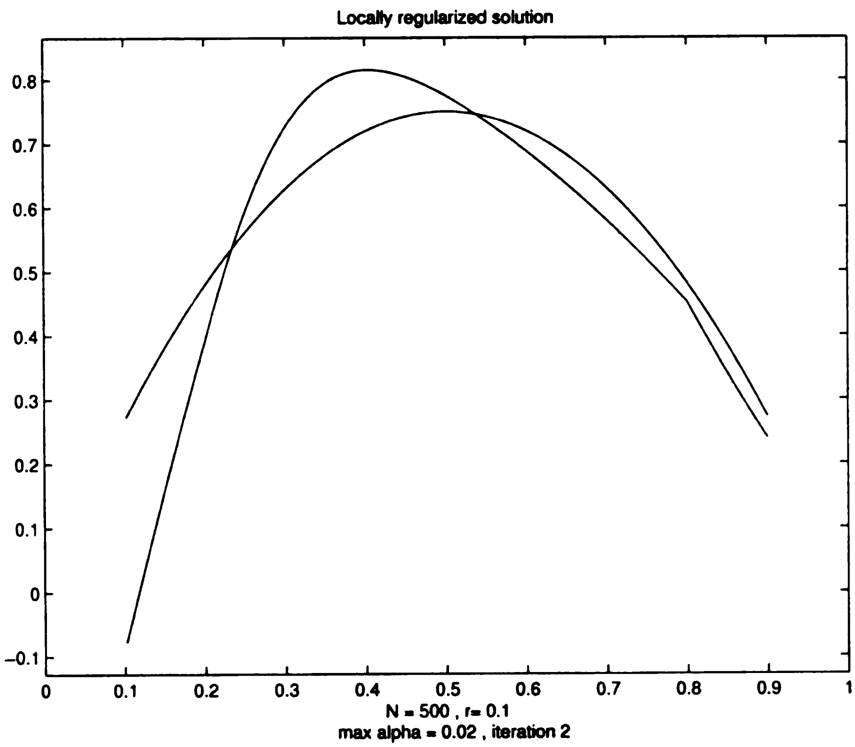

Locally regularized solution
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
N = 50 , r= 0.1
max alpha = 0.01 , iteration 3

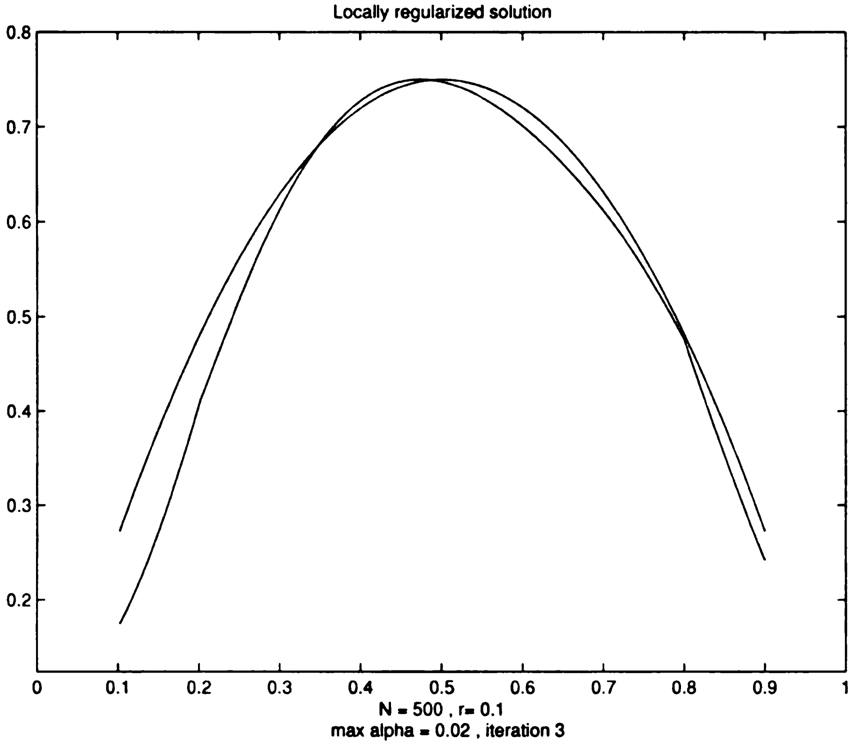

Locally regularized solution
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
N = 50 , r= 0.1
max alpha = 0.01 , iteration 4



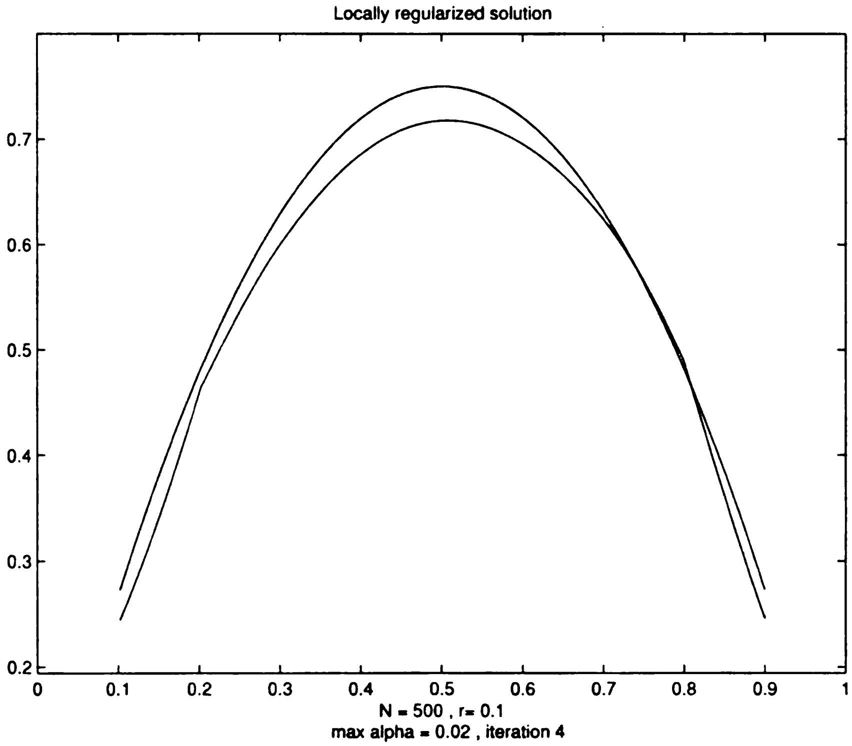
Local Regularization Results with $N = 500$, $\alpha = 0.02$ and $r = 0.1$:


Locally regularized solution
1.2
1
0.8
0.6
0.4
0.2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
N = 500 , r= 0.1
max alpha = 0.02 , iteration 1

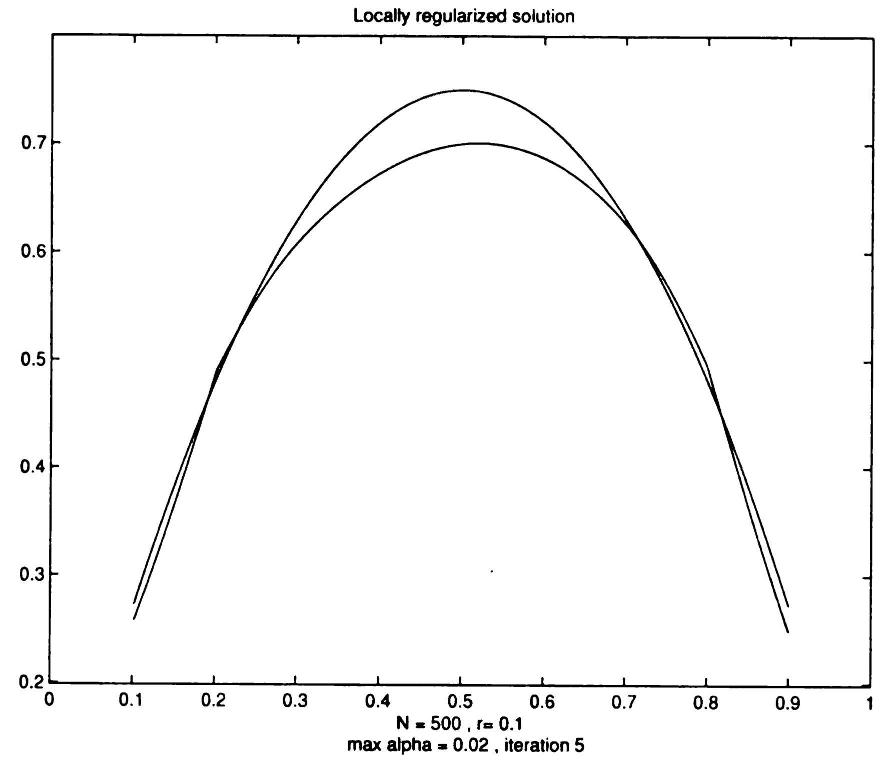

Locally regularized solution
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
−0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
N = 500 , r= 0.1
max alpha = 0.02 , iteration 2

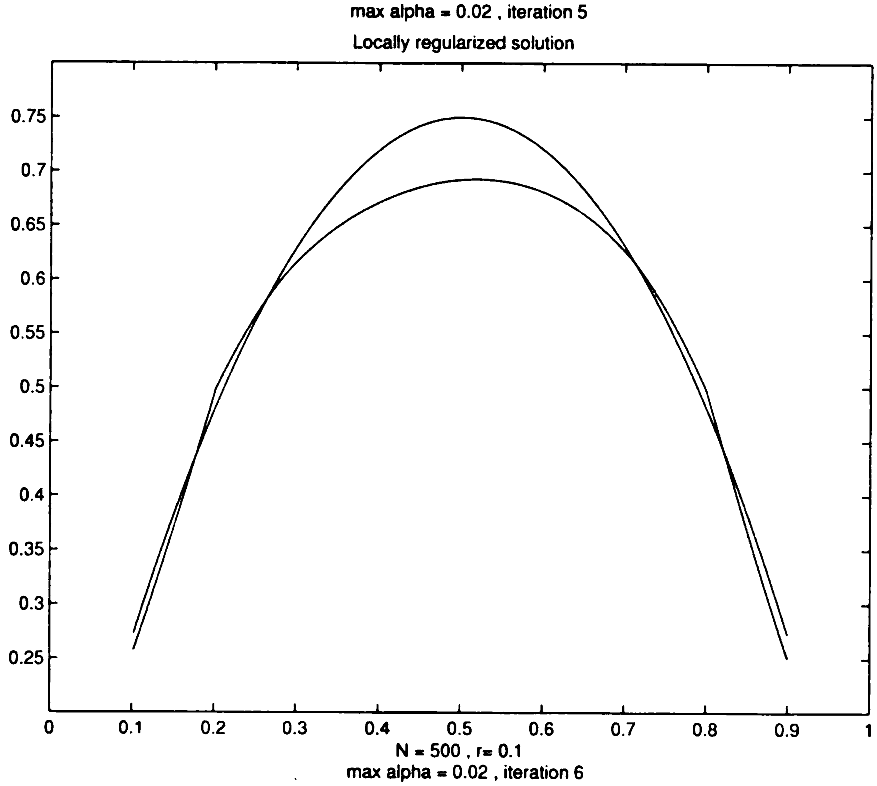


Locally regularized solution
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
N = 500 , r= 0.1
max alpha = 0.02 , iteration 3

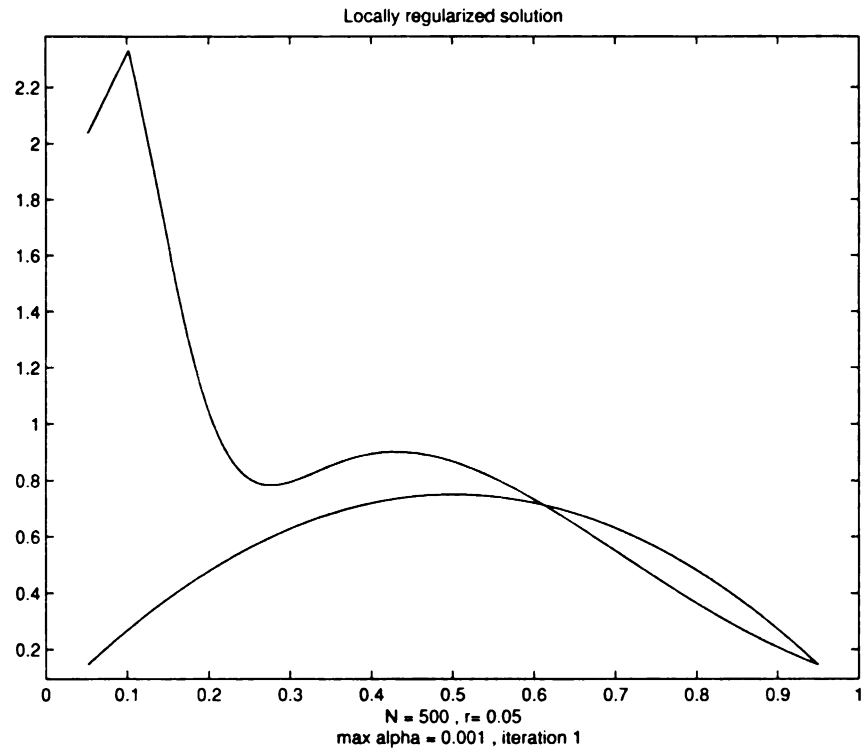


Locally regularized solution
0.7
0.6
0.5
0.4
0.3
0.2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
N = 500 , r= 0.1
max alpha = 0.02 , iteration 4

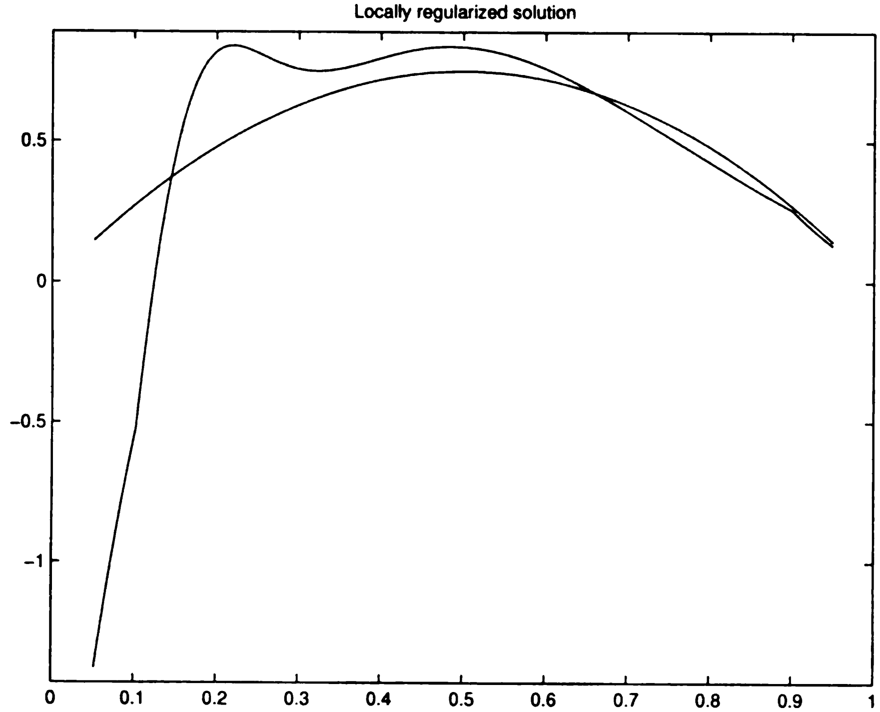

Locally regularized solution
0.7
0.6
0.5
0.4
0.3
0.2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
N = 500 , r= 0.1
max alpha = 0.02 , iteration 5


Locally regularized solution
0.75
0.7
0.65
0.6
0.55
0.5
0.45
0.4
0.35
0.3
0.25
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
N = 500 , r= 0.1
max alpha = 0.02 , iteration 6

Local Regularization Results with $N = 500$, $\alpha = 0.001$ and $r = 0.05$:

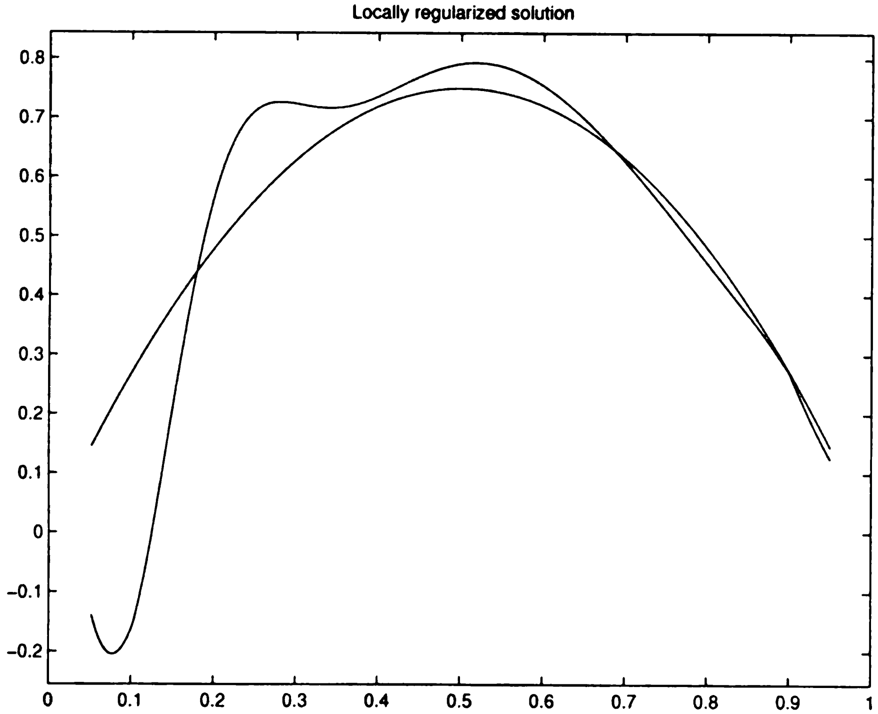


Locally regularized solution



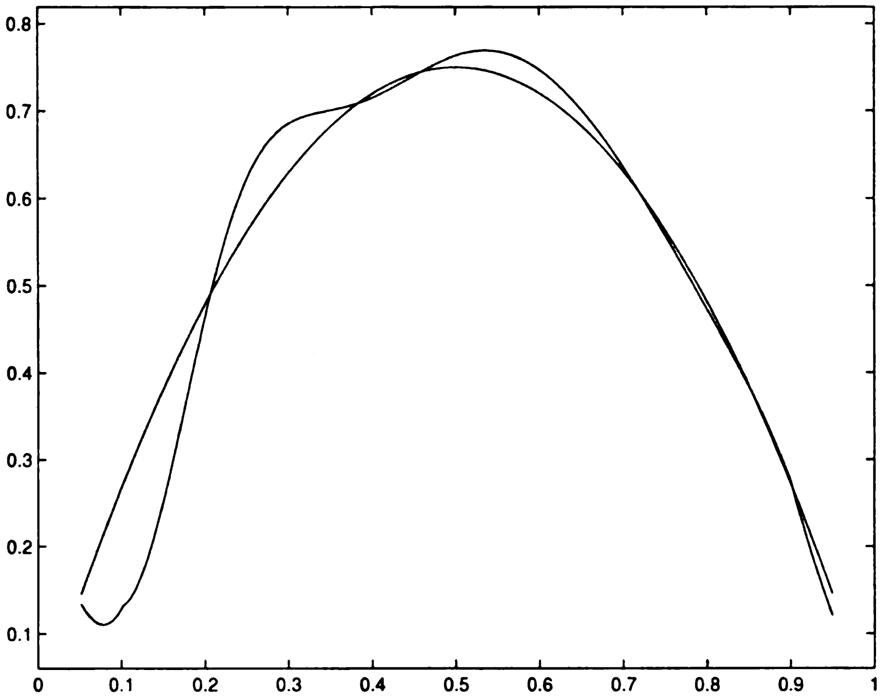
N = 500 , r= 0.05
max alpha = 0.001 , iteration 2

Locally regularized solution



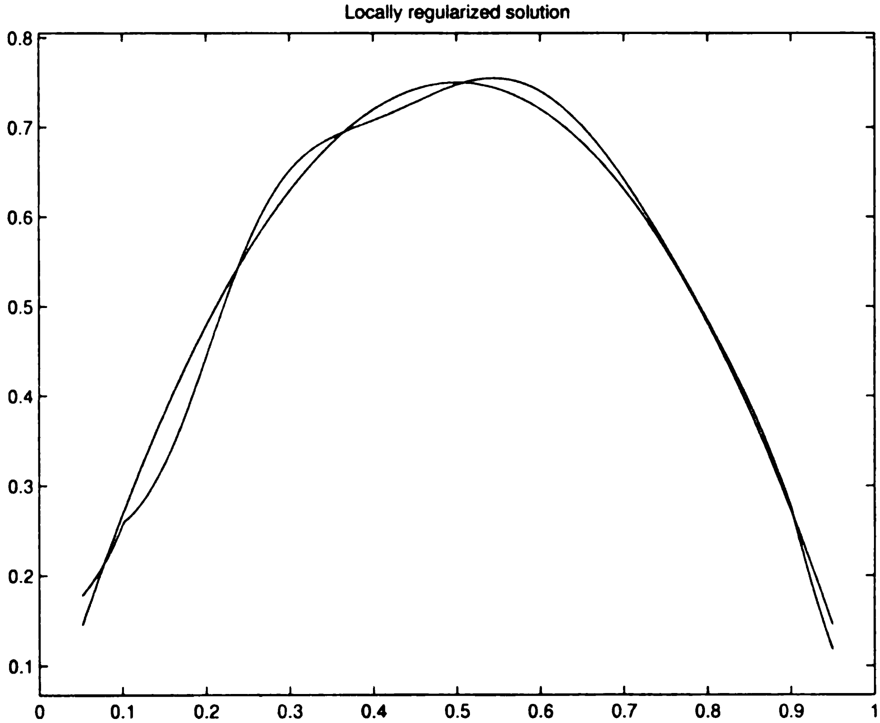
N = 500 , r= 0.05
max alpha = 0.001 , iteration 3

Locally regularized solution



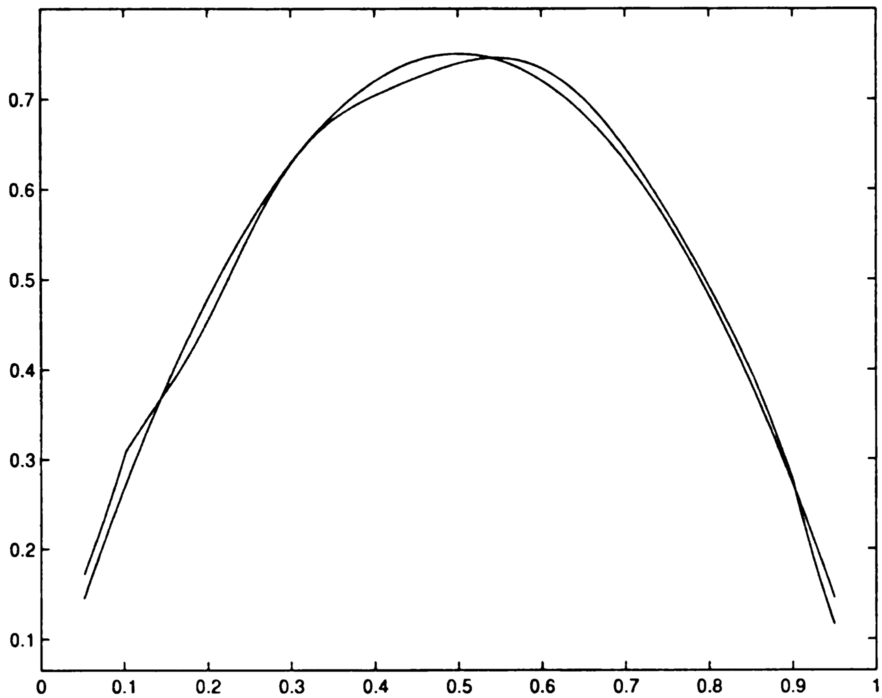
N = 500 , r= 0.05
max alpha = 0.001 , iteration 4

Locally regularized solution



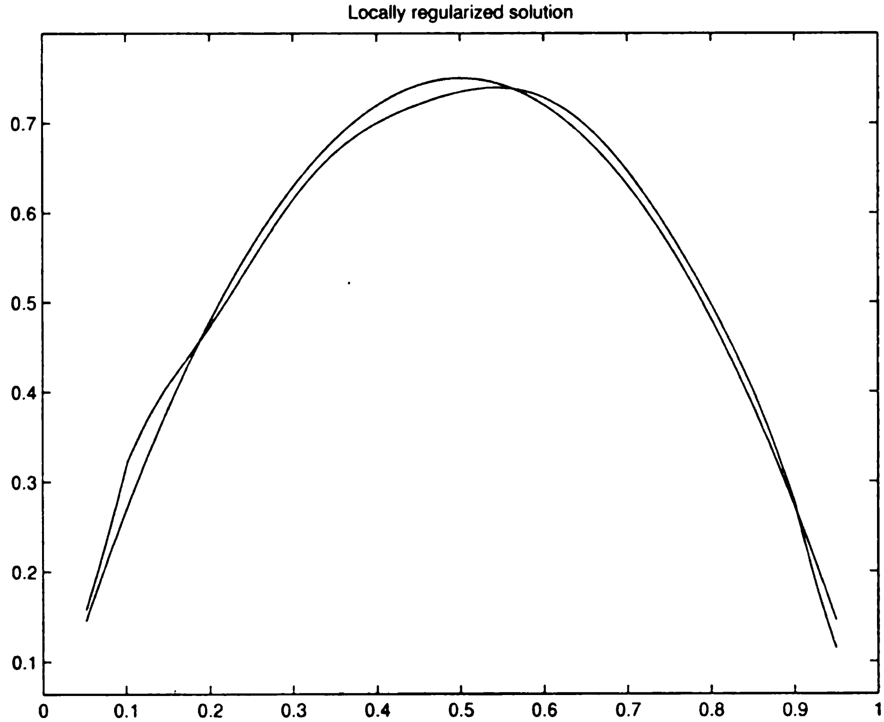
N = 500 , r= 0.05
max alpha = 0.001 , iteration 5

Locally regularized solution



N = 500 , r= 0.05
max alpha = 0.001 , iteration 6

Locally regularized solution



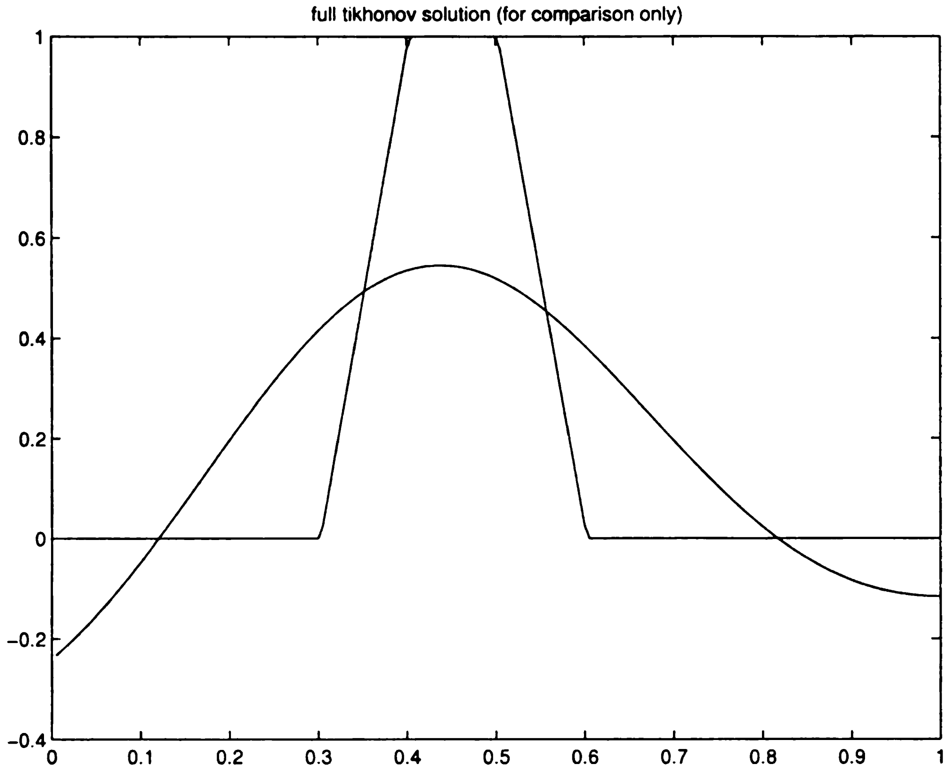
N = 500 , r= 0.05
max alpha = 0.001 , iteration 7

Example 4.2 In this example, our true solution is

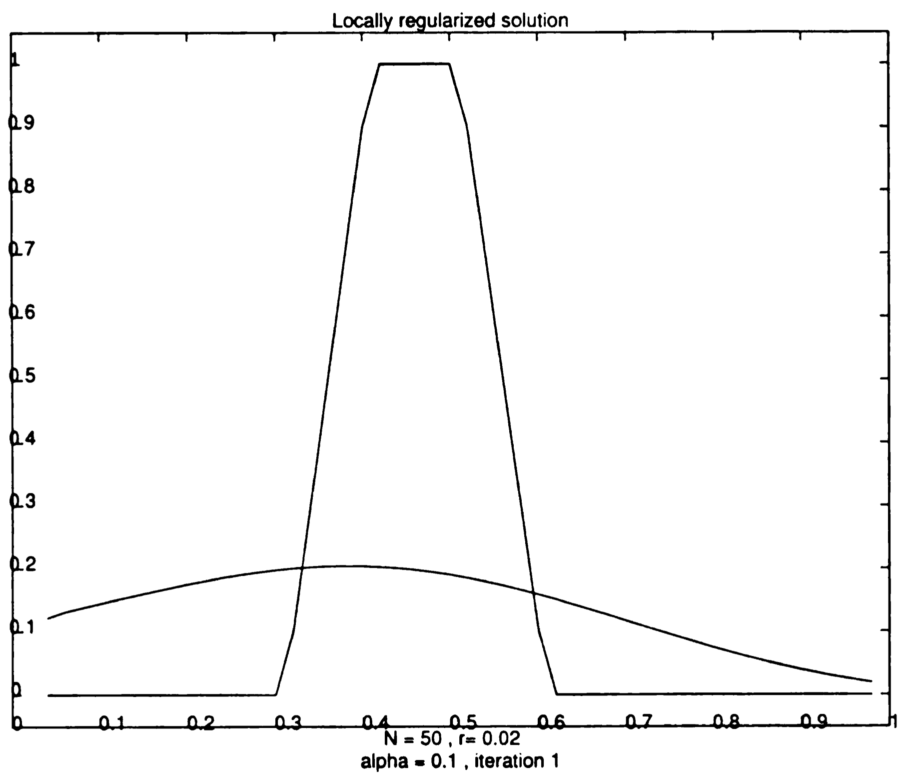
$$\bar{u}(x) = \begin{cases} 0, & x \in [0, 0.3] \\ 10\,(x - 0.3), & x \in [0.3, 0.4] \\ 1, & x \in [0.4, 0.5] \\ 10\,(0.6 - x), & x \in [0.5, 0.6] \\ 0, & x \in [0.6, 1] \end{cases}$$

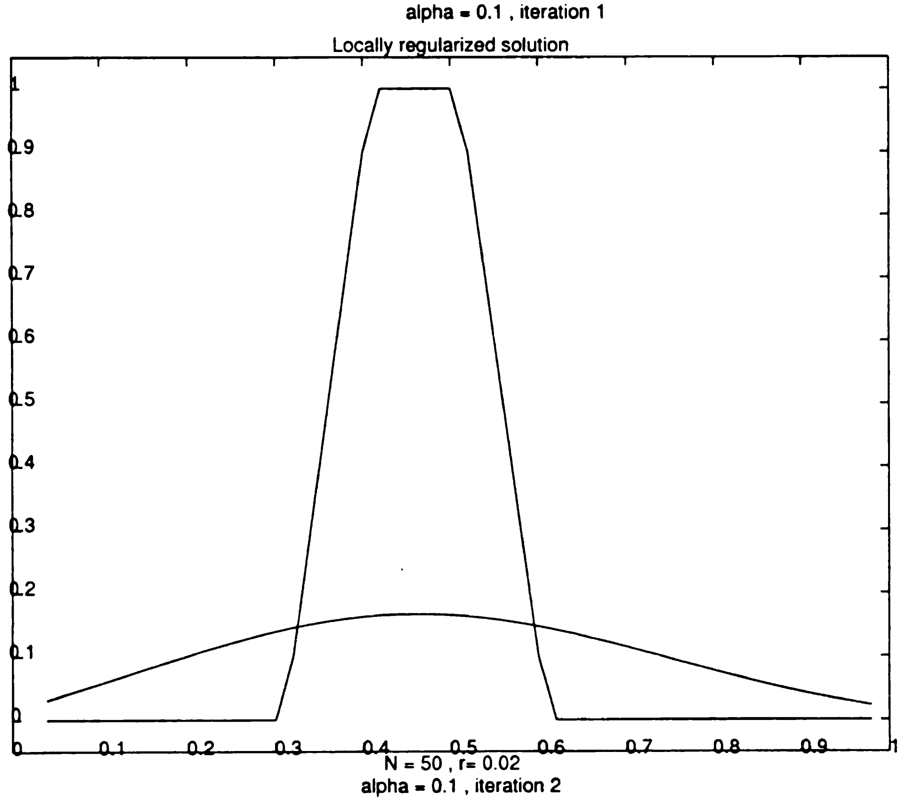
In this example we use $r = 0.02$. This time we try to use different choicess for α - both constant values and variable $\alpha(x)$.

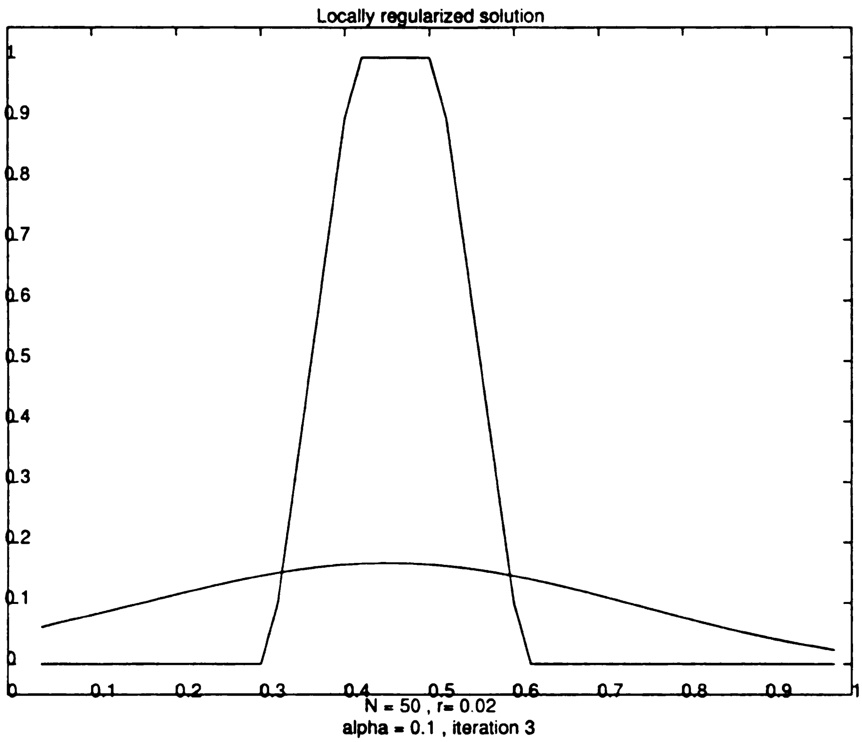
The classical Tikhonov regularization generates the following result:



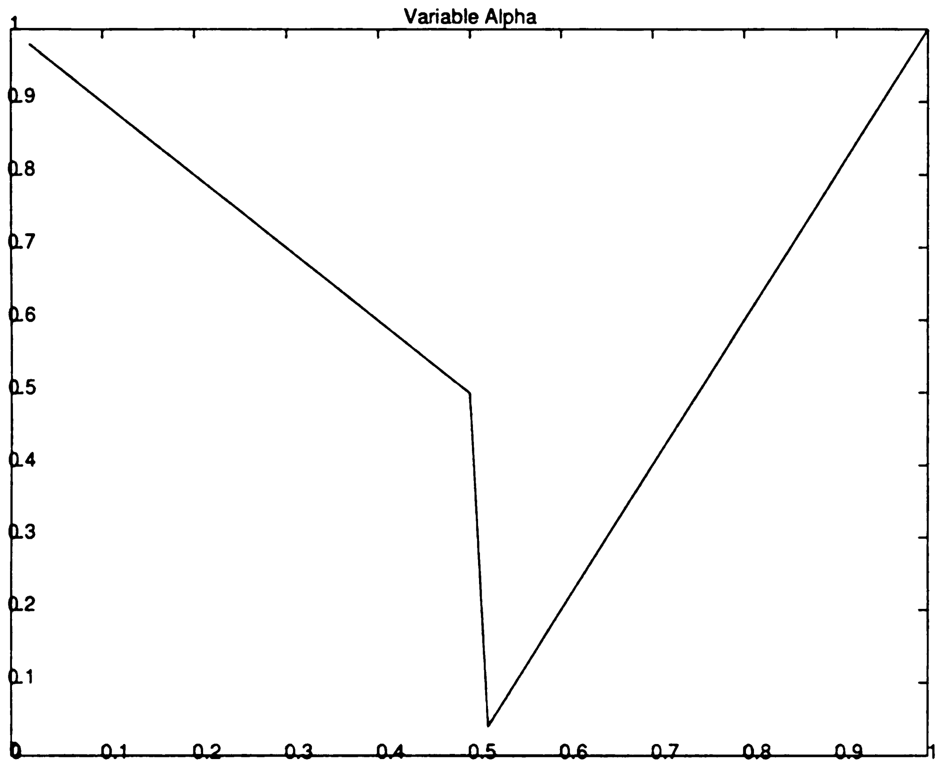
As we'll see in the next results, the performance of our local regularization algorithm is not so good when using a constant value α (in this case, $\alpha = 0.1$):


Locally regularized solution
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
N = 50 , r= 0.02
alpha = 0.1 , iteration 1

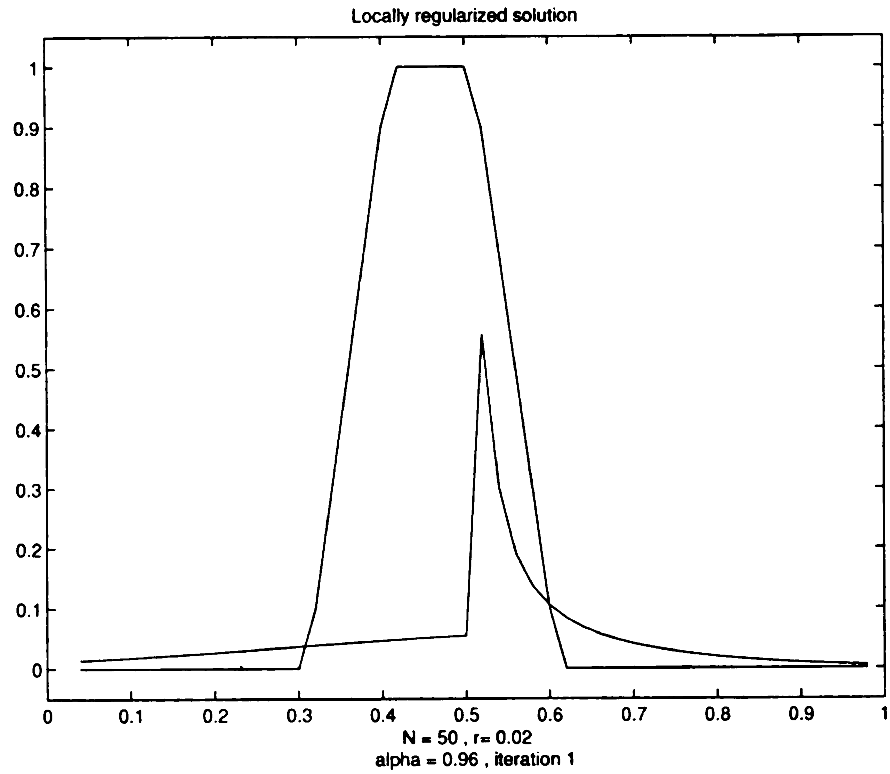

Locally regularized solution
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
N = 50 , r= 0.02
alpha = 0.1 , iteration 2

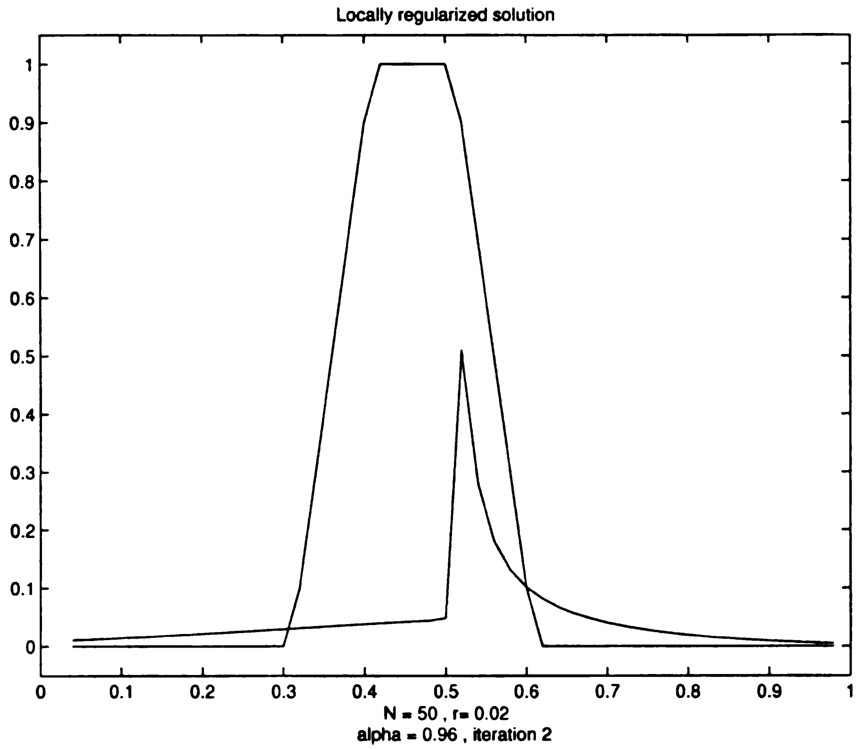


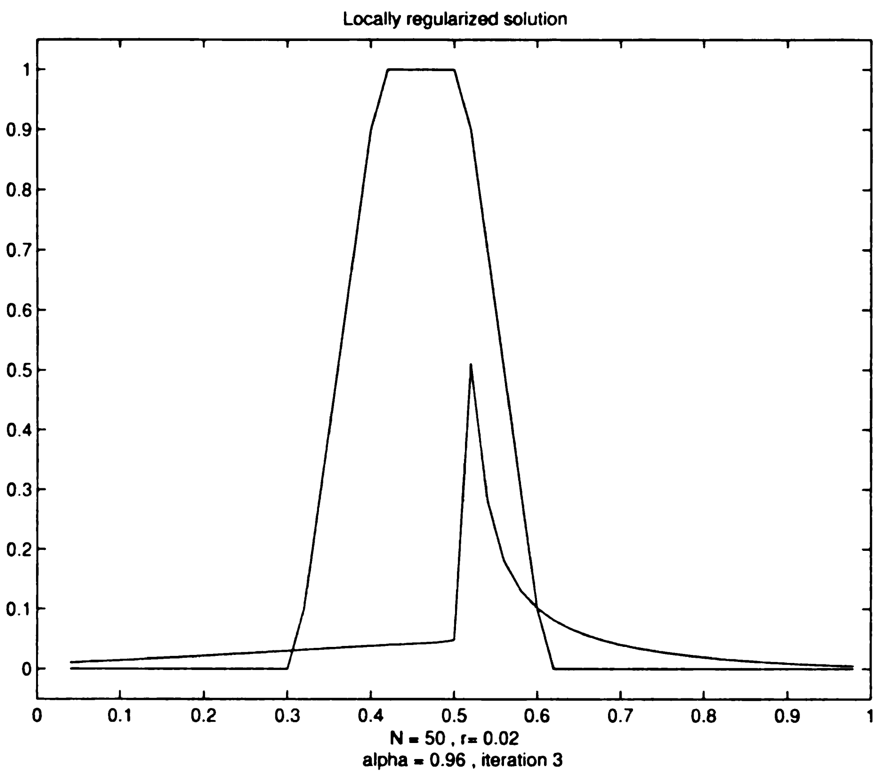
Our first choice of variable α regularization parameter is:



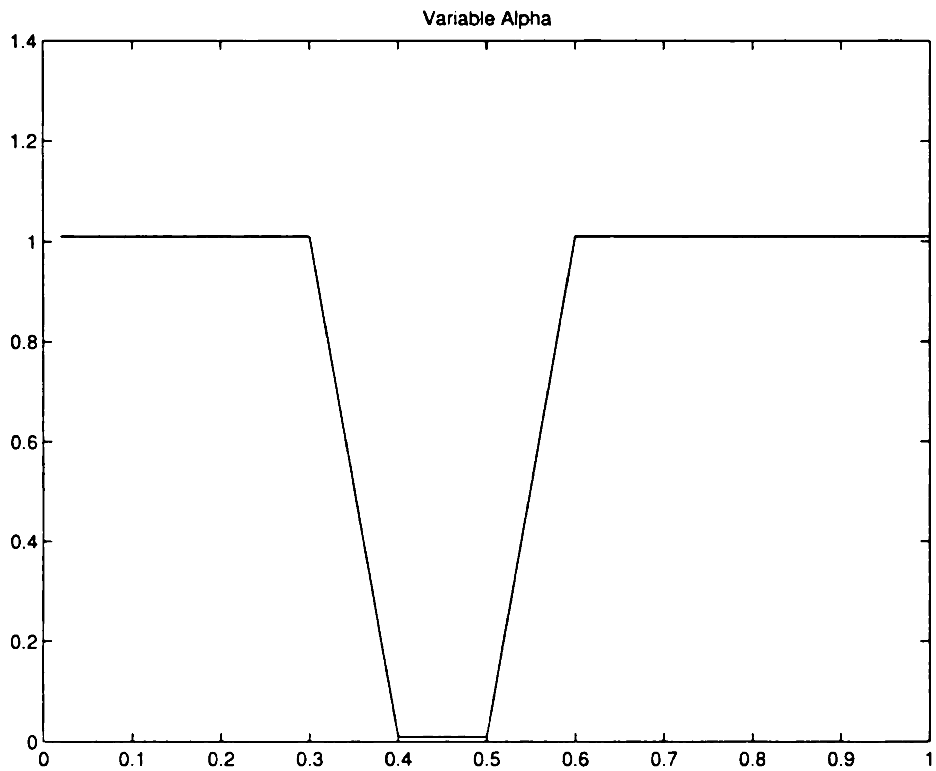
Local regularrization generates the following results:


Locally regularized solution
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
N = 50 , r= 0.02
alpha = 0.96 , iteration 1

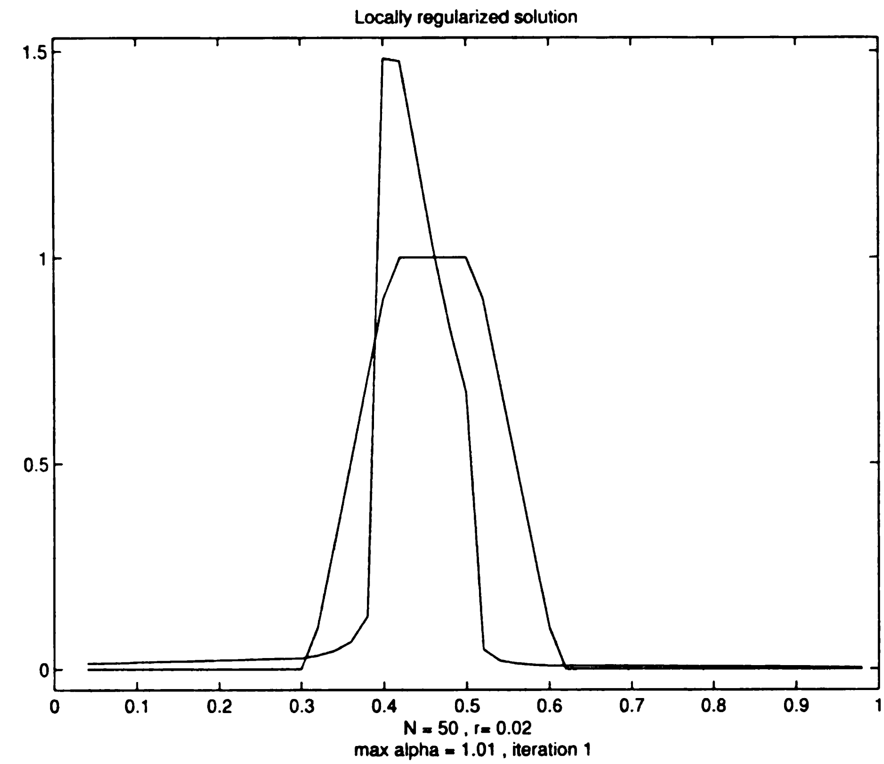

Locally regularized solution
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
N = 50 , r= 0.02
alpha = 0.96 , iteration 2

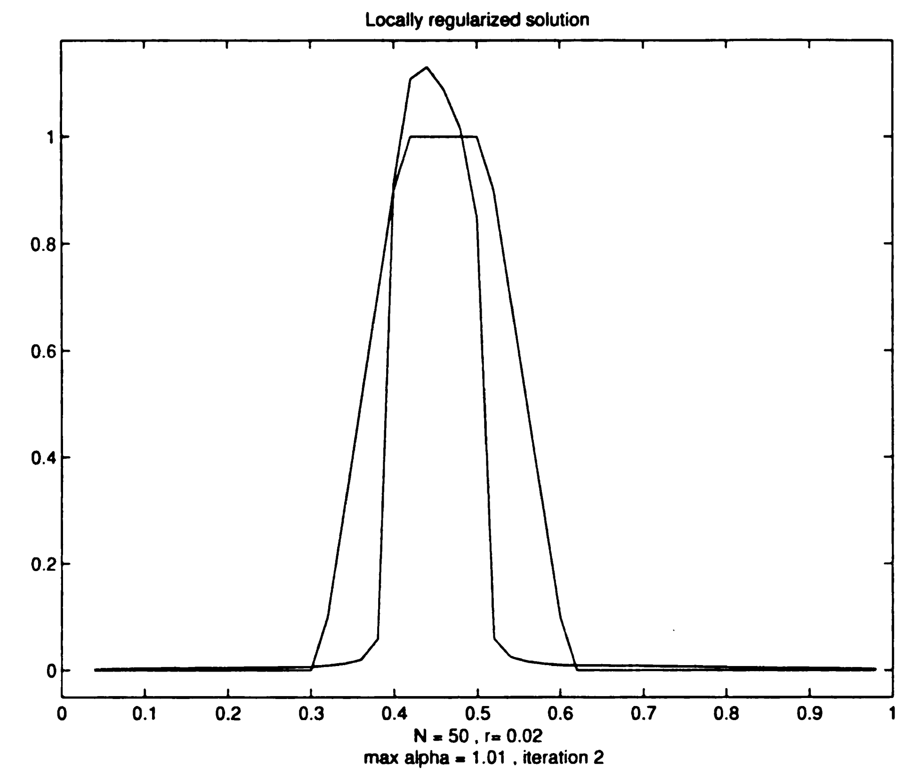


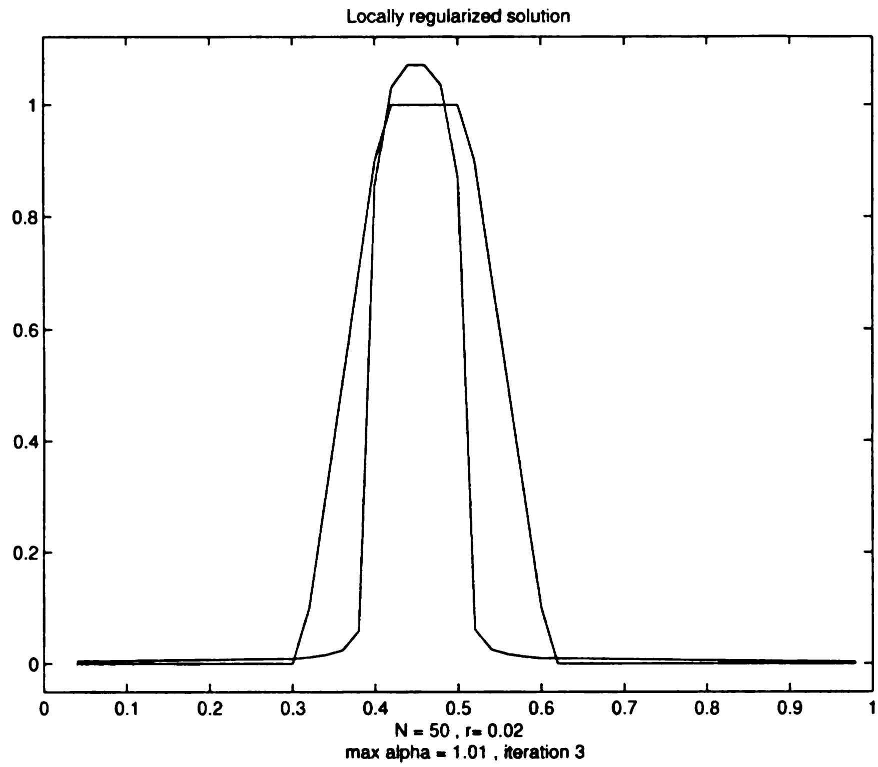
In our second try, the regularization parameter $\alpha(\mathbf{x})$ is depicted in the following graph:



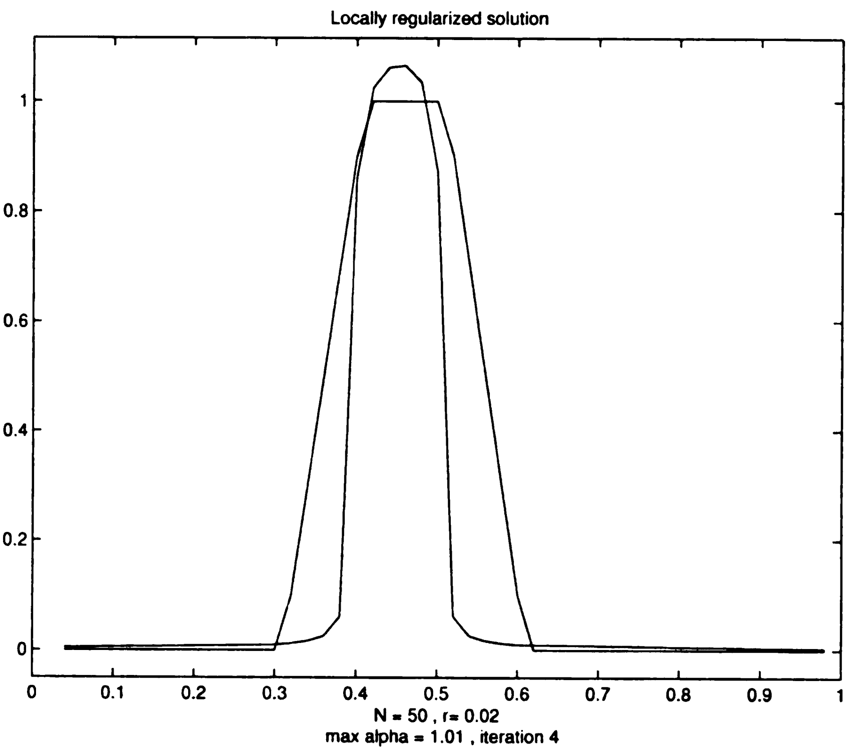
Results generated from our local regularization algorithm:


Locally regularized solution
1.5
1
0.5
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
N = 50 , r= 0.02
max alpha = 1.01 , iteration 1

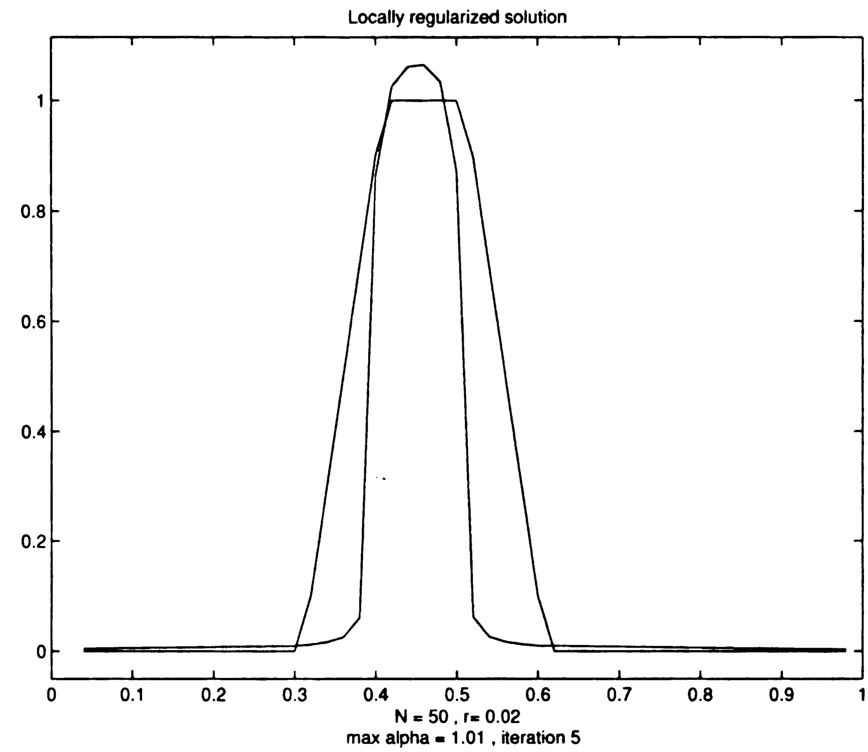

Locally regularized solution
1
0.8
0.6
0.4
0.2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
N = 50 , r= 0.02
max alpha = 1.01 , iteration 2



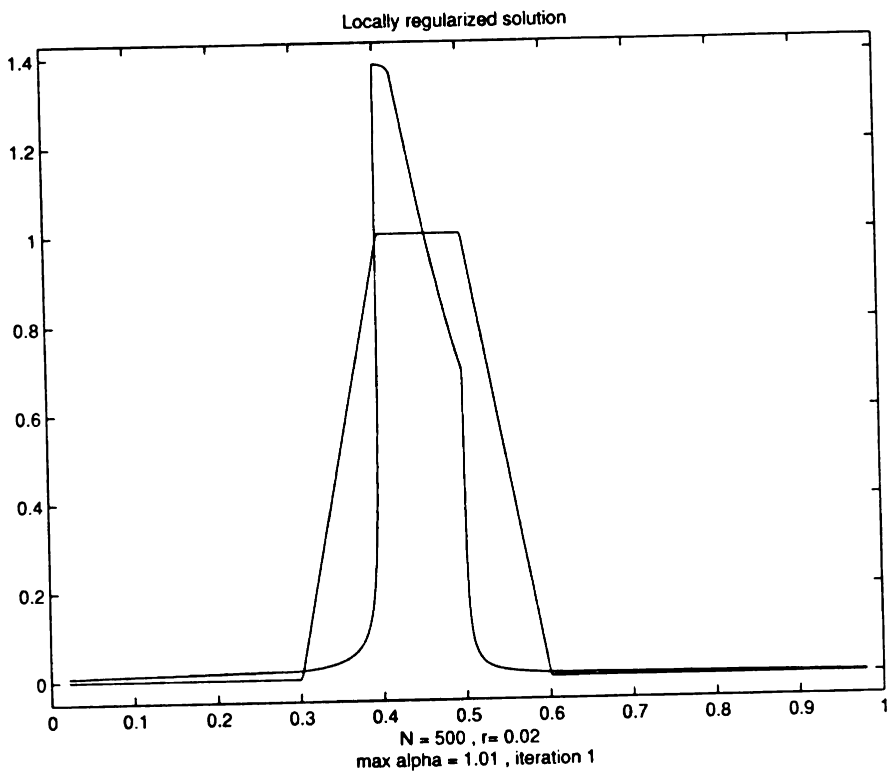
Locally regularized solution
1
0.8
0.6
0.4
0.2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
N = 50 , r= 0.02
max alpha = 1.01 , iteration 3

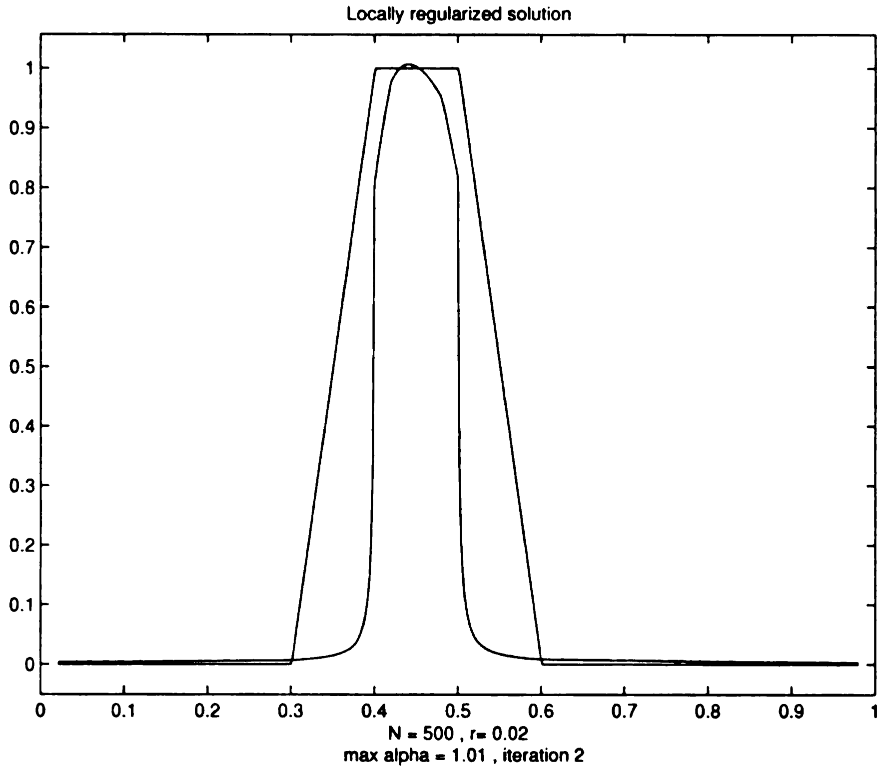


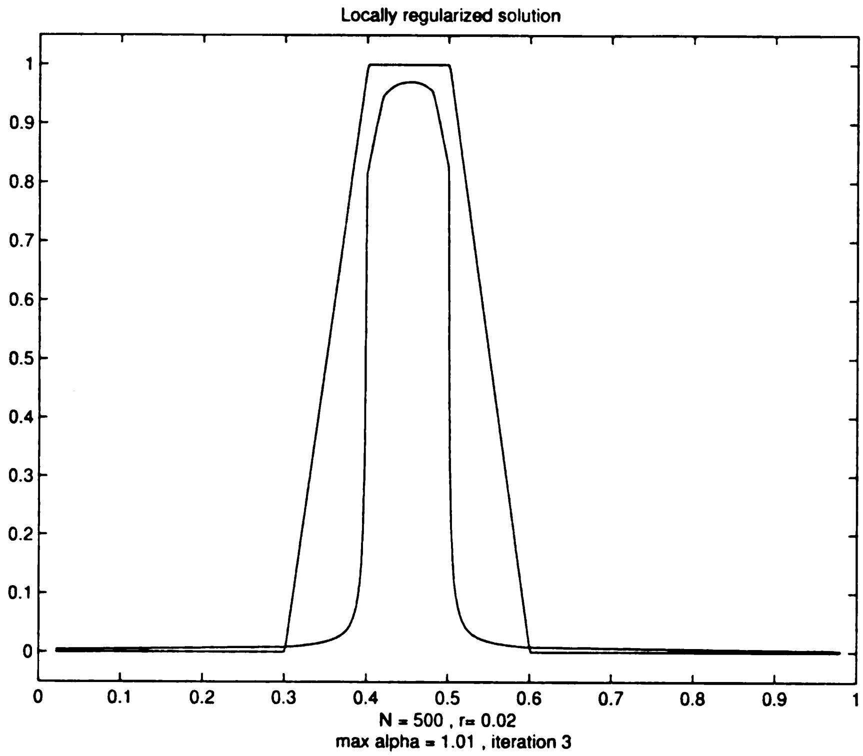
Locally regularized solution
1
0.8
0.6
0.4
0.2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
N = 50 , r= 0.02
max alpha = 1.01 , iteration 4

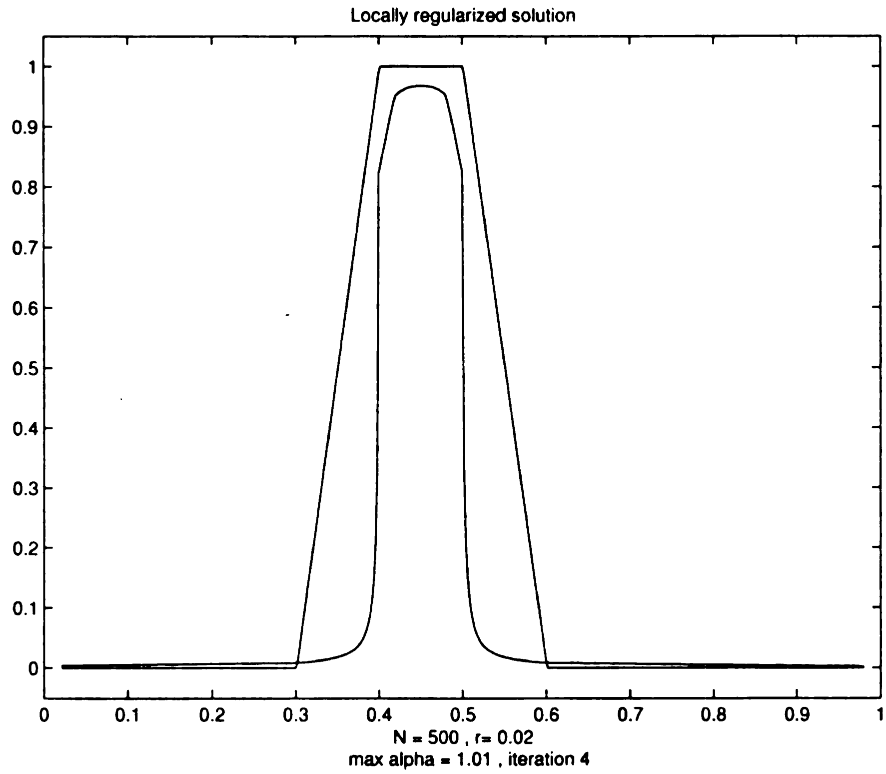


Using the same α, with $N = 500$, the results are shown below:


Locally regularized solution
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
N = 500 , r= 0.02
max alpha = 1.01 , iteration 1


Locally regularized solution
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
N = 500 , r= 0.02
max alpha = 1.01 , iteration 2


Locally regularized solution
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
N = 500 , r= 0.02
max alpha = 1.01 , iteration 3


Locally regularized solution
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
N = 500 , r= 0.02
max alpha = 1.01 , iteration 4

BIBLIOGRAPHY

BIBLIOGRAPHY

[1] R. Acar and C. R. Vogel. Analysis of bounded variation penalty methods for ill-posed problems. *Inverse Problems*, 10:1217–1229, 1994.

[2] A. Chambolle and P.-L. Lions. Image recovery via total variation minimization and related problems. *Numer Math.*, 72:167–188, 1997.

[3] D. C. Dobson and C. R. Vogel. Convergence of an iterative method for total variation denoising. *SIAM J. Numerical Analysis*, 34:1779–1971, 1997.

[4] David C. Dobson and Fadil Santosa. Recovery of blocky images from noisy and blurred data. SIAM J. Applied Math., 56:1181–1198, 1996.

[5] H. W. Engl, M. Hanke, and A. Neubauer. *Regularization of Inverse Problems.* Kluwer Academic Publishers, Dordrecht, The Netherlands, 1996.

[6] C. W. Groetsch. *The Theory of Tikhonov regularization for Fredholm equations of the first kind.* Pitman, Boston, 1984.

[7] P. K. Lamm. Full convergence of sequential local regularization methods for Volterra inverse problems.

[8] P. K. Lamm. Future-sequential regularization methods for ill-posed Volterra equations: Applications to the inverse heat conduction problem. *J. Math. Anal. Appl.*, 195:469–494, 1995.

[9] P. K. Lamm. On the local regularization of inverse problems of Volterra type. In *Proc. 1995 ASME Design Engineering Technical Conferences – Parameter Identification Symposium*, September 1995, Boston, MA., 1995.

[10] P. K. Lamm. Solution of ill-posed Volterra equations via variable-smoothing Tikhonov regularization. In H. W. Engl, A. K. Louis, and W. Rundell, editors, *Inverse Problems in Geophysical Applications*, pages 92 – 108. SIAM, 1997.

[11] P. K. Lamm. Variable-smoothing local regularization methods for first-kind integral equations. *Inverse Problems*, 19:195–216, 2003.

[12] P. K. Lamm and Lars Eldén. Numerical solution of first-kind Volterra equations by sequential Tikhonov regularization. *SIAM J. Numerical Analysis*, 34:1432–1450, 1997.

[13] P. K. Lamm and T. L. Scofield. Sequential predictor-corrector methods for the variable regularization of Volterra inverse problems. *Inverse Problems*, 16:373–399, 2000.

[14] P. K. Lamm and T. L. Scofield. Local regularization methods for the stabilization of ill-posed Volterra problems. *Numer. Funct. Analy. Opt.*, 22:913–940, 2001.

[15] L. I. Rudin, S. Osher, and E. Fatemi. Nonlinear total variation based noise removal algorithms. *Physica D*, 60:259–268, 1992.