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WALKING A STRAIGHT LINE:

INTRODUCTORY DISCOURSE ON LINEARITY

IN CLASSROOMS AND CURRICULUM

By

Christopher Danielson

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

2005

ABSTRACT

WALKING A STRAIGHT LINE:

INTRODUCTORY DISCOURSE ON LINEARITY

IN CLASSROOMS AND CURRICULUM

BY

Christopher Danielson

The current curricular reform in US mathematics education has changed many aspects of classroom teaching. Commonly, discussions about this curricular reform presume an unproblematic relationship between textbooks and classroom instruction. This study contributes to the understanding of the relationship between one published reform curriculum, *Connected Mathematics* (CMP) (Lappan, Fey, Fitzgerald, Friel & Phillips, 2001) and classroom instruction. The study characterizes teaching and learning in terms of communication patterns—*discourse*—and analyzes the discourse of CMP, of a traditional US curriculum, *Mathematics, Structure and Method* (Dolciani, Sorgenfrey & Graham, 1992), and of two teachers in urban classrooms—focusing on the introductory lessons on linear relationships in each case. Results include full descriptions of the introductory discourse on linearity in the textbooks and changes that the CMP textbook discourse undergoes as the curriculum is implemented in these two classrooms.

For the students of Humboldt Junior High School—past, present and future.

My questions come from you.

ACKNOWLEDGMENTS

Thanks go first and foremost to Rachel and to Griffin for your love and your patience. Of course, thanks are also due to the rest of my family: my parents and my in-laws have all been a great source of support over the long haul.

My committee members—Glenda Lappan, Jack Smith, Jacob Plotkin, Joan Ferrini-Mundy and Anna Sfard—have all been essential in this work. Thank you for your questions, your critiques and your suggestions at every phase of my work.

I need to thank Anne Bartel, without whom none of this work would have begun, nor would it have been finished.

Judith Martus Miller deserves more gratitude than this page can contain. She provided support, short-term housing, friendship and countless favors.

Finally, thanks go to fellow graduate students who have been so supportive along the way: Sarah Sword, Harrison Hwang, Amy Kuiper, Kelly Rivette, Marie Turini and Deb Johanning. It has been a long road that you each helped to smooth for me in some way. Thank you.

Ms. M and Ms. H... Je vous remercie mille fois.

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Chapter I: Introduction

Consider the following two algebra problems, each from an American junior-high school mathematics curriculum:

Problem 1

Use a graph to solve the system of equations;

$$y - x = 4$$
$$3y + x = 8$$

(Dolciani, Sorgenfrey and Graham, 1992, p. 284)

Problem 2

In Mr. Goldberg's gym class, Emile finds out that his walking rate is 2.5 meters per second. When he gets home from school, he times his little brother Henri, as Henri walks 100 meters. He figures out that Henri's walking rate is 1 meter per second.

Henri challenges Emile to a walking race. Because Emile's walking rate is faster, Emile gives Henri a 45-meter head start.

Emile knows his brother would enjoy winning the race, but he does not want to make the race so short that it is obvious his brother will win.

What would be a good distance to make the race so that Henri will win in a close race? Describe your strategy, and give evidence to support your answer. (Lappan, Fey, Fitzgerald, Friel & Phillips, 2001c, p. 21)

Each of these problems asks students to solve a linear system. In this sense, they have the same mathematical goals. Yet the two problems have profound differences that

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exemplify aspects of the ongoing mathematics curriculum reform in U.S. classrooms spurred by the publication of the National Council of Teachers of Mathematics' *Curriculum and Evaluation Standards* (NCTM, 1989). The surface difference is an important one. Problem 1 uses mathematical vocabulary: *graph*, *solve*, and *equation*. Problem 2 uses everyday vocabulary: *walking rate*, *head start* and *finish line*.

Deeper differences can be found, too. Problem 1 explicitly states the system to be solved, in the form $ax + by = c$ (called *standard form*). Problem 2 requires students to write the system, and the story is more naturally represented in the form $y = mx + b$ (called *slope-intercept form*).

The model solution to Problem 1 is demonstrated in the Student Edition, and it involves finding three solutions to each equation, listing the solutions in a table and using these solutions to graph two lines. The point of intersection of the lines is estimated, then checked in the equation.

In contrast, there are three model solutions to Problem 2, and they are in the Teacher Edition: 1. guessing and checking, 2. generating a table that continues until the solution is found, 3. making a graph to find the point of intersection.

These two problems suggest some of the changes the curriculum reform has brought to classrooms. In reform materials, mathematical ideas are commonly introduced by having students reason about stories (called *contexts* in this study). Algebra tends to be taught as a way of understanding changing variables, rather than a set of techniques for solving equations. Students are expected to develop and choose algorithms, rather than practice given algorithms.

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In institutionalizing the introduction of contexts before technique, and intuition prior to formalization, United States middle school curriculum has undergone significant change. There is empirical evidence that this change has had little effect on the traditional focus of instruction: algorithmic skill, but that it has had substantial effect on students' ability to solve problems (e.g. Ben Chaim, Fey, Fitzgerald, Benedetto & Miller, 1998; Riordan & Noyce, 2001). The quantitative study of student achievement is important, but in order to truly understand instruction in this new curricular context, with the hope of improving both published curriculum and classroom instruction, we need to study it qualitatively as well. This type of understanding of instruction is the focus of the present study.

In particular, this study is designed to further our understanding of two issues:

1. How, exactly, reform curriculum differs from traditional curriculum, and
2. How classroom instruction of a reform curriculum differs from what is printed on the page.

The opening examples point to the kinds of differences this study uncovers in curriculum; the roles of mathematical and everyday vocabulary, the use of mathematical representations (in algebra–the context for this study—these are representations of functions—tables, graphs and equations), and the mathematical behaviors encouraged in texts written for students and for teachers. Additionally, this study finds differences in the *content* of introductory algebra instruction.

Together, the analysis of vocabulary, of the use of representations, of content and of the mathematical behaviors encouraged in curriculum and in classrooms make up a form of discourse analysis. The details of this analysis are spelled out in Chapters III and

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IV. The important point here is that the communicative activity in classrooms is what is meant by discourse, and that analysis of this activity is the focus of this study.

Finally, some unexpected findings are reported about the differences between published and implemented curriculum. While the major themes of algebra in CMP are represented in each classroom, the treatment of equations in each classroom differs from the curricular treatment.

Chapter II of this paper reviews research related to the present study. Chapter III presents a theoretical framework for the study. Teaching and learning are framed as problems of communication and the tools of discourse analysis are outlined as they pertain to the problems in this study. Chapter IV presents the research questions and outlines the methods that will be used to answer them. Chapter V analyzes one traditional and one reform curriculum with the goal of reaching a precise characterization of the reform curriculum's approach to introductory algebra instruction. Chapter VI compares this characterization to the instruction in two classrooms, seeking how the discourse of the printed page changes as teachers bring it to life in classrooms. Finally, Chapter VII revisits the major results of the study, considers the contributions of the theoretical framework and highlights limitations of the study.

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Chapter II: Literature Review

Chapter I introduced this study as concerning the nature of a particular reform mathematics curriculum—*Connected Mathematics* (CMP) (Lappan, Fey, Fitzgerald, Friel & Phillips, 2001) and the relationship between this curriculum and classroom instruction. The chapter also hinted that the study of communication patterns—discourse—would form the heart of the analysis. As such, this study sits at the intersection of several large fields of inquiry in mathematics education, including language/linguistics, discourse/discourse analysis, the teaching and learning of linear relationships (an algebra topic), and curriculum and implementation. In this chapter, relationships between the present study and each of the fields listed above are briefly addressed in order to situate subsequent chapters.

Language

The language used in teaching and learning mathematics has been much studied—both empirically and theoretically. One of the fundamental concepts in the analysis of language in mathematics classrooms is the notion of a *register*. According to Halliday (1978),

> A register is a set of meanings that is appropriate to a particular function of language, together with the words and structures which express these meanings. (p. 195)

Halliday defines *register* in order to discuss how mathematical language differs from everyday language. While he includes brief mention that a register is not just about words, but is also about "modes of argument," he is more concerned with the meanings

and creation of words. For instance, *set* is a word with a technical mathematical meaning that is related to, but different from, its meaning in the everyday English register.

Halliday's discussion is theoretical and analytical. Pimm (1987) picks up Halliday's ideas and gives them practical application. Pimm too is concerned with words and their meaning in his discussion of register. He offers careful analysis of the use of such words as *any*, *if-then*, and *diagonal* and suggests that *register confusion* is the source of what are often considered mathematical misconceptions. Register confusion involves participants in conversation having different expectations about the meanings of words due to operating in different registers. For example, when a student asserts that a rectangle oriented with one vertex at the bottom of the page has "four diagonals," meaning that none of the four sides is horizontal or vertical, the mathematical meaning of *diagonal* conflicts with the everyday meaning of the term—this is register confusion. Pimm is correct that register confusion is one source of misconceptions, but surely there are others not accounted for by this idea.

The notion of a mathematics register is important, and it applies to parts of the present study—those parts that seek to understand how mathematical language (a mathematical register) develops from the everyday language (a non-mathematical register) that students bring to the classroom, and that CMP exploits through the contextual problems it presents.

The relationship between everyday language and mathematical language has been studied by several others, including Herbel-Eisenmann (2000 & 2002) and Mamokgethi, Adler, Reed and Bapoo (2002). Herbel-Eisenmann describes the practice of a CMP teacher who encourages and adopts the use of students' everyday language for

mathematical ideas before introducing more formal mathematical language and suggests that this practice makes mathematics accessible to a wide variety of students. For example, this teacher uses the phrase *swoopy down curve* to describe graphs of exponential decay relationships. This phrase was introduced by her students.

Mamokgethi, et al., describe a more complex linguistic environment in which the everyday language of students is different from the language of instruction. Correspondingly, these researchers' findings are complicated. They find teachers making efforts to navigate the "journey" from informal talk in main (home) languages to formal talk and writing in English (the language of instruction). They find teachers working to incorporate into their instruction opportunities for students to learn from talk, but they find that journey to be treacherous, "stranding" many students and teachers. The researchers introduce the concept of *code switching* in which the teacher moves between languages (e.g. formal mathematical English to informal Venda) in order to make concepts more understandable to students. The linguistic environments in my study are not as complex as described by Mamokgethi, et al., in that all speakers described are fluent in English and English pervades the surrounding culture in ways that it does not in the South African schools they describe. Nonetheless, the metaphor of a journey from the familiar to the formal is a useful one for the process I wish to investigate in my study—the process of deriving mathematical meaning from contextual problem solving experiences.

Discourse

The language studies just cited focus primarily on words and the way that words are used. The study of discourse concerns itself with communication patterns more

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broadly. Certainly many of the concerns of these language studies are also concerns in the analysis of discourse, but discourse analysis tends to take a larger view and to ask questions about communication systems as a whole. In educational research, discourse has been used to characterize teaching and learning (e.g. Greeno, 2003 and Sfard, 2001), in contrast to more behaviorist notions of transfer (e.g. Thorndike & Woodworth, 1901) or to cognitive characterizations that focus on acquiring cognitive structures (described nicely by Sfard, 1998).

Resnick (1988) is among those who argue for such a characterization in mathematics (as is Lampert, 1990). Resnick argues that mathematics traditionally has been taught as a "well-structured discipline" in which truths are pre-determined and organized for the learner. She argues instead that mathematics should be taught as an "ill-structured discipline," in which participation in discourse is the learning goal, rather than the mastery of some set of rote skills. This participation would involve seeking and organizing mathematical truth. My study does not make assertions about how mathematics should be taught, but Resnick's argument about the nature of mathematical activity resonates with my characterization in Chapter V of CMP's mathematical discourse as *heuristic*—meaning that the discourse is focused on strategies for making and testing conjectures.

Sfard (2001) characterizes teaching and learning in terms of discourse by rephrasing the question *What is to be learned?* as *In what ways should students' communication patterns change?* This shift clarifies the question and makes it operative. In addition, Sfard provides a framework (the *communicational framework*) with which to

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distinguish mathematical discourse from other discourse genres. Together, these tools provide the framework for my study, which is detailed in Chapter III.

One consequence of the communicational framework is a characterization of an important (although perhaps not necessary) condition for learning: *communicational conflict*. When knowledge is conceived of cognitively, we may refer to *cognitive conflict* in which a learner's conceptions do not match an objective reality or other people's ideas. This disagreement, when noticed, can lead to a reformulation of these conceptions. But there is a difficulty with the concept of cognitive conflict from a research perspective; we cannot directly observe a learner's conceptions. In the communicational framework, we may similarly talk about communicational conflict. This is when a learner's communicational patterns disagree with the patterns of others. Without using Sfard's terminology, Moschkovich (1996) describes communicational conflict and its resolution in her study of students trying to decide whether the line for $y = x + 5$ is steeper than the line for $y = x$. One student focuses on the y-intercept, while the other focuses on the angle the line makes with the x-axis. This narrative conflict arises only when the pair is asked to show what a steeper line would look like. The first student, using a pen to represent the new line, moves the pen higher. The second rotates the pen counter-clockwise.

Sfard and Avgil (2001) detail communicational conflict in the learning of negative numbers. For instance, they cite a conversation in which students and teacher disagree about where to find the rules for multiplying integers with opposite signs. Students argue for extending the rules from addition of integers in which the sign of the number with the greater absolute value is the sign of the sum, while the teacher tries to

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help students to apply more formal rules for deciding the sign of the product. In this case, both the procedure (*how do I multiply integers?*) and the modes of justification (*how do I know my procedure is correct?*) are in conflict. My study uses a special case of communicational conflict—narrative conflict. *Narrative conflict* occurs when two mathematical assertions are in opposition, such as when the two students in Moschkovich's study disagree about whether steeper lines have larger y-intercepts or greater angles with the x-axis. Teachers, textbook authors and (often) students seek to resolve narrative conflicts once they are identified and these instances become rich opportunities for learning.

Linear Relationships

There have been several studies involving the teaching and learning of linear relationships in reform mathematics classrooms. Because these studies examine students' ways of speaking and thinking about linear relationships in realistic problem situations, the analytic focus is on how students make sense of these situations and the tools they are given to analyze them (tables, graphs and equations). Herbel-Eisenmann's classroom study (2000), cited above, documents the use of contextual and transitional language for slope, for instance. The phrase *goes up by*, describing the increment in a linear table, is an example of transitional language. A different classroom study in a high school reform curricular context (Lobato, Ellis & Muñoz, 2003) confirms that this language is widespread, rather than idiosyncratic and argues that, depending on how teachers focus student attention, the phrase *goes up by* can lead to a variety of correct and incorrect generalizations. A study of CMP student understanding of linear relationships (Smith, Herbel-Eisenmann, Star & Jansen, 2000) argues that this language is pervasive in student

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talk and thinking. Both the Smith, et al., and the Lobato, et al., studies find that students develop an understanding of the $y = mx + b$ form of linear equations that involves a *starting point* (b) and an amount that x *goes up by* (m). However, both studies suggest that when students talk about what a table *goes up by*, they may see slope as an isolated increment (a difference) rather than as a ratio of two related values. In the graphical representation of linear relationships, the importance of *steepness* as a qualitative (i.e. descriptive, but non-numeric) measure of slope has been observed in several studies (e.g. Smith, et al., 2000 and Moschkovich, 1996). Each of these issues—understanding the ratio, not just the difference nature of a table's increment; starting points; interpretations of the equation $y = mx + b$; and the isolation and identification of steepness—plays a role in the present study.

Generally, these studies show that students use the contextual problems in reform curriculum to generalize mathematical ideas, but that this process can be problematic. The studies further demonstrate that there are various conceptions of the important aspects of linear relationships (especially slope and y-intercept) that students develop, but that vary by representation.

Curriculum and Implementation

This is a study of curriculum in classrooms. As is acknowledged by the National Research Council (2004), the meaning of the term *curriculum* can vary widely in research. In some cases, curriculum is a list of district topics; in others it is a set of realized learning outcomes. This study follows the NRC's definition.

> A curriculum consists of a set of materials for use at each grade level, a set of teacher guides, and accompanying classroom assessments. It may include a listing

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of prescribed or preferred classroom manipulatives or technologies, materials for parents, homework booklets, and so forth. (p. 38)

The NRC study is a survey of existing studies of the effectiveness of K–12 mathematics curricula—studies that have in large part been spurred by the curriculum reform that followed the 1989 NCTM *Standards*. The NRC study categorizes curriculum effectiveness studies in four categories: Content Analysis, Comparative Studies, Case Studies and Syntheses. The study concludes that there is insufficient evidence to support the effectiveness of particular programs. Instead of arguing for this effectiveness, the study's report gives guidelines for judging the quality of current and future studies of curricular effectiveness. My study is not a study of the effectiveness of curriculum. Methodologically, my study combines elements of the NRC categories of Content Analysis and Case Studies. Chapter V of my study compares two curricula in order to do a careful analysis of the content and instructional support in CMP, but the scope of my analysis is too limited to meet the NRC guidelines for a Content Analysis study. These guidelines call for an analysis of the topics for at least a full grade level. My study instead focuses on depth—looking at the introduction to a single topic. Chapter VI of my study analyzes classroom instruction in comparison to published curriculum, but here too, the study is too limited in scope to be considered a Case Study for effectiveness under the NRC guidelines. In any case, the present study is not concerned with effectiveness per se, although this is always an important concern. Instead, this study focuses on careful description of curriculum (as defined by NRC) and of teaching in order to better understand classroom instruction in a reform mathematics curriculum environment.

A study of instruction in US mathematics and science classrooms titled *Looking inside the Classroom* (Weiss, Pasley, Smith, Banilower & Heck, 2003) evaluates lesson quality in these classrooms. Most of the components the Weiss, et al., study evaluates in each lesson are relevant to my study, including lesson design, lesson implementation and the mathematics content of the lesson. Weiss, et al., however, are concerned with evaluating lesson effectiveness on a national scale rather than with rich description of a small number of classrooms, as in the case of my study. Additionally, Weiss et al. report that the influence of curriculum on the topics in mathematics and science lessons is reported by only about half of the teachers in their study, whereas the observations in my study are of classrooms where the curriculum very much dictates the topics of instruction.

The Senk and Thompson text, *Standards-Based School Mathematics Curricula* (2003) includes studies of student achievement in reform curricula at each level: elementary, middle school and high school. As with the two studies just cited, the emphasis in the Senk and Thompson text is on evaluation rather than description of curriculum and classroom practice. The evaluation of CMP in this text (Ridgway, Zawojewski, Hoover & Lambdin, 2003) is a synthesis of studies and finds that CMP students outperform students in curriculum that adheres less closely to NCTM's *Standards* and that CMP students make progress in technical skills, but noted a lack of information about the implementation of the curriculum in the schools in the studies.

Two studies conducted at Michigan State University (also, that is, in addition to my study) come closer to the focus of my study. Theule-Lubienski (1996) studied a year's worth of CMP instruction (her own) in order to understand how students of different socio-economic backgrounds participated in and learned from the discussions of

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mathematical ideas that are part of the CMP Launch-Explore-Summarize instructional model. She carefully analyzed classroom discussions and raised questions about a variety of issues. In particular, her questions about how students generalize their solutions in particular contexts to mathematical algorithms applicable on a wide variety of problems informed the questions in my study. 7Yet there are significant differences between our two studies. Theule-Lubienski's study is concerned with class differences in participation and learning. These differences are not factors that I analyze. Her data set spans a year's worth of instruction while mine spans a few weeks. Theule-Lubienski is concerned with the social aspect of discourse, and especially how certain groups are privileged in or excluded from the discourse, while I am concerned with characterizing the mathematical nature of classroom discourse. Herbel-Eisenmann's study (2000), cited above, examines discourse patterns in two CMP classrooms, and in this respect is similar to my own study. The main results of her analysis relate to the source of mathematical authority in the two classrooms. This parallels one aspect of discourse analyzed in my study—the routines for establishing mathematical truth. When the teacher is the source of mathematical authority, we might expect the routine of telling described in my study to be more prominent than when the source of authority is individual or class investigation.

In this brief review of related literature, I have shown how the present study stands in relation to existing theory and empirical research. This study is descriptive, not evaluative. In looking at communication patterns in general, rather than staying at the sentence level, this study is in line with other research on discourse in mathematics classrooms. The content of the lessons studied—linear relationships—means that this

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study's findings add to an existing body of research on the teaching and learning of algebra in reform mathematics classrooms.

Chapter III: Theoretical Framework

In Chapter I, I introduced the problem underlying this study: understanding teaching and learning in U.S. middle school classrooms in the wake of a particular and significant curriculum reform. In particular, this study is designed to examine the teaching part of this problem. In this chapter, I argue in favor of the analysis of discourse as a construct for making progress on this problem.

Background

Teaching and learning are intertwined in obvious ways and attempts to separate them are questionable. The questions in this study arise from published curricula, which by nature make choices about teaching while making assumptions about learning. Published curriculum is written with the intention of helping students to learn something new—with the intention of teaching. Authors of curriculum make choices about which topics to include, the order of these topics, the models to be used in explaining and understanding, etc. These are all choices about teaching that rest on assumptions about the students who are learning, e.g. assumptions such as the usefulness of concrete models, or the importance of real-life contexts to learners. Because the authors write at some distance in time from the learners (a distance of years, typically), the authors can never truly know their students. Instead, authors of curriculum write to idealized imagined students. In many cases these idealizations are informed by the experiences of actual students, but they are inarguably idealizations.

In this way, teaching is emphasized in curriculum over learning. The teaching strategies in curriculum can be planned in substantial detail, but the learning can never be known in advance. This study examines teaching in the carefully controlled curricular

sense and in the messier world of real-time classroom interactions. The study takes a curricular perspective on classroom interactions—looking at instances of teaching without overt focus on the learning that results, for this is a major question of curriculum design: what is the relationship between the written and the delivered curriculum? (Lemke, 1989) Questions of learning are also important (perhaps more so) but understanding the relationship between curriculum and classroom teaching is an important goal of its own.

So this study seeks to understand what is being taught in mathematics curriculum and classrooms. Sfard (2001) has observed that a difficulty in educational research is unacknowledged disagreement about the meaning of terms such as teaching and learning. In particular, she notes that these terms are frequently used in non-operative ways—that definitions are not made in ways that other researchers would be able to pick up and use. If Sfard's call for operative terminology in educational research is taken seriously, the statement, *this study seeks to understand what is being taught in mathematics curriculum and classrooms* needs to be made operative. This chapter summarizes a vision of teaching and learning in which participation in a discourse is the goal. The vision makes *teaching in mathematics curriculum and classrooms* operative.

This chapter introduces important background material before defining the term *discourse*. The analysis in this study rests on a characterization of mathematical discourse, which is described in detail.

Introductory Notions

Cobb (1994) shares Sfard's view that difficulties in communication between constructivists and those adhering to a sociocultural view of mathematics learning are

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exacerbated by different uses of the same words, including the word *meaning*. He prefers to see the two perspectives pragmatically—each is useful for understanding certain kinds of problems and achieving different ends, but neither tells the whole story.

The constructivist view of learning asserts that learners construct understanding and that they construct meaning in their minds. This view rests on a metaphor: that ideas are like buildings and learners like construction workers. While this metaphor is useful in talking about education and is an improvement on that of the *tabula rasa*[1], it does not easily lead to operative research questions. We cannot directly access people's ideas: we can only make inferences about them. These inferences are based on behaviors—especially communicational behaviors, such as answers people write on test forms, or responses they give to interview questions. If, instead of attempting to draw inferences about people's ideas through their communicational behaviors, we instead examine the behaviors themselves closely, we may characterize learning in an operative way. This is the spirit in which the present study uses Sfard's communicational framework (2001). In an attempt to describe teaching and learning in an operative way, mathematics knowledge is characterized as participation in mathematical discourse. Then learning is defined as change in this participation. A teacher is someone who models a discourse that learners are attempting to participate in. Teaching is defined as engaging learners in the model discourse.

In this communicational framework, the question *What is being taught?* is tantamount to asking, *What is the model discourse?* In classrooms, there are typically two discourses being modeled for students: that of a textbook, which is static and

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unresponsive to students, and that of the teacher, which is dynamic and responsive to students' participation.

This study uses tools analyzes discourse (in a process referred to as *discourse analysis*) in order to characterize the relationship between curriculum and teaching in reform mathematics classrooms. To summarize, the use of discourse analysis for this study is rooted in an attempt to operationalize the question *what is being taught?*

In particular, the goal of this study is to better understand the workings of classrooms in which a curriculum, *Connected Mathematics* (CMP, for *Connected Mathematics Project*), is implemented. CMP introduces mathematical ideas through the study of contextual problems, putting mathematical language into a non-standard role in the curriculum. Mathematical language in CMP tends to be introduced after students have used the ideas that the language names to solve contextual problems. *Slope*, for instance, is introduced only after students have solved many problems with rates and have talked about the steepness of lines on graphs. As we shall see in Chapter V, many aspects of CMP that distinguish it from traditional US curricula can be framed as linguistic differences.

This study seeks to describe part of CMP's textbook discourse—the introductory discourse on linearity. A variety of theoretical and pragmatic considerations dictated this choice; these are detailed in Chapter IV. Linear relationships are the simplest predictable algebraic relationships: for each unit change in one variable, the other variable changes by a constant amount. Linear equations can be solved with the four arithmetic operations: addition, subtraction, multiplication and division. These are among the reasons that linearity is an introductory algebra topic in CMP.

In order to keep the study manageable, the focus is not on the entire discourse on linearity, but only on the introductory discourse on linearity. In a study of this kind, there is a choice between focusing on one or two features of the discourse over a long stretch of time and focusing on a larger number of features of the discourse for a short time. In allocating resources to this study, depth had to be weighed against breadth. The choice was made to focus on a larger set of features of the discourse for a short time, rather than fewer features over a longer time span.

The introductory lessons on linearity in two curricula were chosen for study. There are several reasons that focusing on the introduction to a topic is valuable. The introductory lessons set the tone for later study. Focusing on the introductory lessons should give insights on the discourse as a whole. The introduction to a topic tells students what is valued in the discourse.

Before proceeding to research methods in the next chapter, it is important to make certain terms operational. In particular, the term *discourse* needs definition.

Discourse

Discourse is defined as *communicative activity* and may take many forms: oral speech, written text and non-linguistic communication among them. This is largely consistent with other definitions of discourse, including Grimshaw's (2003):

> This...perspective sees discourse as spoken or written text in a language, intended for use in the accomplishment of social ends of users (speakers, hearers, writers, readers). (p. 27)

Surely Grimshaw would acknowledge pauses, shrugs, nods and other non-linguistic communication as part of discourse. Such communicative activity has some standardized

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forms, called *genres* (ibid), that may or may not be explicitly understood as such by the users. One problem in the study of discourse is to identify the characteristics of discourse that distinguish one genre from another in a reliable way. Of particular interest to this study is the genre of mathematical discourse.

A related perspective to this is that there exist specialized patterns of communicational activity; each such pattern is *a discourse*. Thus, within the genre of mathematical discourse, there may be many patterns of communicational activity—many discourses. In this sense, mathematical discourse does not refer to a single discourse, but to a collection of discourses with some set of commonalities. One problem in discourse analysis is to identify these commonalities.

Earlier, I defined teaching as engagement with learners in the model discourse. By model discourse, I mean the disciplinary (here mathematical) communication patterns of the teacher. Thus the everyday discourse in which the teacher engages students in the hallway is not the model discourse in this study.

This study concerns itself with two types of mathematical discourse that are closely related: textbook and classroom mathematical discourses. In Chapter IV, I argue that comparison of these two discourses is warranted on the grounds that textbook writers have an idealized classroom discourse in mind as they write; their goal is to influence the classroom discourse. While they are comparable along important dimensions, the two are indeed different types of discourse. Textbooks are static and unresponsive to students. A teacher's classroom discourse is dynamic and co-constructed with students, at least in part. In the analysis that forms the heart of this study, these differences are clear, as I

demonstrate that several aspects of the teacher's discourse are influenced by students' questions and ideas.

Sfard (2005a) argues that mathematical discourse has four distinguishing features: *word use*, *visual mediators*, *(endorsed) narratives* and *routines*. This study uses these features of mathematical discourse to characterize the discourse on linearity in two texts and in two classrooms. This presumes a meaningful relationship between disciplinary discourse and classroom discourse. Lampert (1990) argues that "content" per se is not the only goal of schooling, but that enculturation into the discourse community of school is another. If we take seriously the idea of disciplines as communities of discourse, then it makes sense to investigate the ways in which school discourse resembles mathematical discourse.

Discourse analysis can attend to many dimensions of discourse. Cazden (1988) is concerned with the structures of classroom discourse—identifying its patterns in order to understand schooling as a social and educational phenomenon. In this case, power relations, turn taking and participation patterns (among other possible dimensions) are central to the analysis. Herbel-Eisenmann (2000) uses discourse analysis in her study of two *Connected Mathematics* classrooms to understand the center of mathematical authority—whether it is the teacher, the class as a whole or the textbook that decides mathematical truth. The present study is interested in describing the relationships between the mathematical discourses of textbooks and classrooms.

Sfard's description is of *formal* mathematical discourse, while this study is concerned with classroom mathematical discourse. The relationship between these is complex, as is the relationship between any discipline and the school subject bearing the

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same name. Schwab (1978) argues that the structures of a discipline, which may be roughly described as the relationships between the objects of study and the procedures for dealing with and verifying these relationships, ought to be preserved when the disciplines are taught as school subjects, but that too often it is only the *fruits* of the discipline—the facts the discipline produces—that are taught. One way of characterizing Schwab's structures is as the narratives and routines of discourse in Sfard's framework. But even in a classroom that is sympathetic to Schwab's viewpoint, it is inappropriate to assume the classroom discourse to be identical to that of the discipline.

Lampert (1990) describes her attempts to teach mathematics with the explicit goal of developing students as participants in a mathematical discourse with particular practices, which she contrasts with traditional classroom practices. Without citing Schwab, her teaching was in this spirit: teaching students to think like mathematicians. The title of her article, "When the problem is not the question and the answer is not the solution," speaks to this emphasis on mathematical discourse practices (posing problems and making conjectures about solutions) over finding answers to routine questions.

An important difference between formal mathematical discourse and classroom mathematical discourse is power. The idealized formal mathematical discourse disregards power relations among participants. This difference is explored in more depth below, in the section titled *Narratives*. For now, the obvious difference that a teacher directs the discourse of a classroom and may share or retain the power to do so in varying degrees suffices to distinguish classroom discourse from the larger mathematical discourse. Another difference is the idiosyncratic nature of classrooms. Students and teachers may use language to refer to prior shared experiences that will be incomprehensible to

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outsiders (e.g. references to "Sean numbers" in Ball's work (Ball, 1992)). Thus, using the characteristics of mathematical discourse to study classroom discourse will surely turn up some mismatches: middle school students and teachers use words less formally from the way mathematicians would use them, for instance. At the same time, the use of this analytical tool does indeed reveal that the textbook and classroom discourses under study are justifiably categorized as mathematical. In particular, the ways of finding or determining mathematical truth are recognizable from formal mathematical discourse, in the case of one textbook, and from heuristic (exploratory) mathematical discourse in the case of the other textbook in this study, and in the cases of the two classrooms in this study.

The intention of this technique is not to demonstrate that classroom mathematical discourse differs from formal mathematical discourse, for this is a truism. Instead, the intention is to treat classroom mathematical discourse as a kind of mathematical discourse. That is, this study presumes that the discourse in textbooks and classrooms is mathematical and considers it accordingly. The result is perhaps a different understanding of classroom discourse than if it is considered as a social phenomenon as in Cazden or Herbel-Eisenmann.

Word Use

In mathematics discourse more generally, words are used differently from their use in everyday discourse. This is true in two senses:

1. A particular word may have a different meaning in mathematics,
2. The rules governing word use in mathematics are different, and

There are words—such as *similar*—that are used in everyday discourse as well as in mathematics, but which are used differently in the two domains—that is to say; they have different meanings. In everyday discourse, we might say that rectangles are similar to squares. By this we mean that the two shapes have some unspecified properties in common. In mathematical discourse, the statement is false. The only rectangle *mathematically* similar to a square is another square. Similar is an example of word that is used differently in mathematics than it is used elsewhere, and there are many such examples.

The second sense in which the claim above (*words are used differently in mathematical discourse*) is true is that the rules governing word use are different in mathematics. It is characteristic of mathematical discourse that words are carefully defined, and then used in ways consistent with the original definition. It is common for classroom mathematical discourse to be less rigorous about word use than is formal mathematical discourse (as the present study will demonstrate). This second sense of *word use* is less of a focus in this study. Instead, the focus will be on the use (meaning) of three words, *equation*, *rate* and *linear*. The choices of these terms are detailed in Chapter IV.

The way in which each word is used is the major aspect of word use in this study. A third aspect to word use, relevant to this study, but not unique to mathematical discourse, is identifying which words are used to describe a mathematical idea. This issue is of lesser importance in this study, but there are categories of words that will be useful for characterizing aspects of discourse. Herbel-Eisenmann (2000, 2002) describes four categories of language in *Connected Mathematics* classrooms: Official Mathematical

Language (she uses the acronym, OML), Transitional Mathematical Language (TML), Classroom Generated Language (CGL) and Contextual Language (CL).

OML consists of mathematical terms that would be recognized by participants in an English-speaking mathematical discourse outside K-12 classrooms. Included here would be *slope*, *equation* and *graph*. TML is informal language that might be expected to be understood by a wider audience. The use of *steepness* for slope is an example of TML. CGL is idiosyncratic. Herbel-Eisenmann cites a classroom where "swoopy down curve" is used to describe graphs of exponential decay relationships; this is an example of CGL. CL is vocabulary that arises from the contexts in which mathematics is studied. *Walking rate* and *head start* in Problem 2 in Chapter I are examples of CL. Each may refer to a more general concept, or it may not, but the vocabulary originates in a contextual problem about a walking race.

For the purposes of this study, the distinction between TML and CGL, namely whether it is idiosyncratic or likely to be in widespread use, is not particularly relevant. As such, the two are combined into a single category: transitional language. This leaves three categories: *mathematical language*, *transitional language* and *contextual language*. These terms—especially the first and the last—are used in this study to describe word use in the final sense discussed above; the equivalent use of multiple words. Mamokgethi, Adler, Reed & Bapoo (2002) make a similar distinction between OML and CL. They refer to these two types of language as *formal mathematical language* and *informal language*.

Visual Mediators

In everyday discourse, the objects of the discourse are frequently concrete and can be pointed to in the real world. If we are discussing a movie, we can *see* the movie. If we are having a mathematical conversation about functions, though, we are discussing something that can only be said to exist in a more abstract sense. We cannot point to and look at *functions* as we talk about them[2]. Instead, we use representations of these functions: visual mediators such as tables, graphs and equations. In mathematical discourse, then, the visual mediators are often *representations* that capture some aspects of the object of the discourse. In many other discourses, the objects of the discourse are concrete; they need not be represented because they can be looked at directly.

In classroom instruction, questions are raised and ideas explored. Visual mediators are useful for answering mathematical questions and for testing mathematical ideas. Closely related groups of questions and ideas are referred to in this study as *themes*. Themes are different from narratives (discussed below) in that themes raise the questions that narratives answer. Narratives are more general than themes and are declarative, while themes tend to be interrogative, hypothetical or tentative. A sample theme is the idea that *rates are represented in tables*. There is a corresponding narrative to describe *how* the rates are represented.

A simple listing of themes in the discourse gives little information about the importance of each theme, and thus makes it difficult to characterize the discourse as a whole. In an attempt to make such principled distinctions, this study uses the language of *recurrent*, *passing* and *absent* themes. Absent themes are those we might *expect* to find in

[2] Mathematics is certainly not the only discourse of which this is a feature; theological and philosophical discourses have this in common with mathematical discourse.

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a discourse—in comparison with another discourse, for instance—but that are not present. Passing themes are only briefly or rarely mentioned. Recurrent themes are found repeatedly in the discourse and are emphasized. Specific measures of these terms vary slightly between the textual analysis and the classroom analysis; each is discussed in the appropriate place below.

Narratives

Narratives are the properties of and the relationships between the objects of the discourse, such as *The slope of a line is the change in* y*-coordinates divided by the change in* x*-coordinates* or *In a right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the two legs.* In formal mathematical discourse, narratives take particular forms, primarily axioms, definitions and theorems. These forms, and the degree of absolute truth the discourse attributes to them, distinguish formal mathematical discourse. In particular, formal mathematical discourse does not admit conflicting narratives (as in the Law of the Excluded Middle).

Sfard's communicational framework (2005a), around which the present study is built, uses the term, *endorsed narratives* to identify those relationships and procedures that are accepted as true in the mathematical community. In Sfard's work, *endorsed* is important in that she is interested in comparing classroom discourses to formal mathematical discourses. By using the term endorsed, she points to those narratives in the classroom discourse that are accepted in the larger mathematical community, "*Endorsed narratives* are sets of propositions that are accepted and labeled as *true* by the mathematical community." (Sfard, 2005a, p. 6) Thus, mathematicians endorse narratives in Sfard's description. In the present study, the correspondence with formal mathematical

discourse is less important. Indeed, there will be times where narratives under consideration in classrooms are *not* endorsed by the mathematical community. For these reasons, this study refers to narratives rather than to endorsed narratives.

As alluded to earlier, an important distinction between formal mathematical discourse and classroom mathematical discourse is power. The idealized mathematical discourse disregards power relations in endorsing mathematical narratives. There are routines for determining the truth of a narrative and only the true narratives are endorsed. The status of the person who produces the narrative is supposed to be irrelevant. In classroom discourses, this is not so. The teacher retains the authority to endorse narratives. The refrain regarding division of fractions, "Yours is not to question why; yours is to invert and multiply," is a play on this power dynamic. Indeed, it is not just the teacher who may endorse classroom narratives by dint of a position of power, as Sfard (2005a) demonstrates in a classroom scenario in which a charismatic and intelligent student establishes an incorrect rule for the multiplication of integers, which the class accepts and which the teacher is unable to dislodge with mathematical questioning.

Because classroom narratives are sometimes not endorsed mathematical narratives, this study refers to *establishing* narratives and to *reinforcing* them. A teacher establishes a narrative by indicating for the first time that she believes it to be correct. A teacher need not be conscious of a narrative in order to establish it. After establishing a narrative, a teacher may reinforce it[3]. A routine, in the sense described in the next section,

[3] Establishing and reinforcing can be seen as two types of endorsing of narratives—assuming a more local definition of an endorsed narrative. In this alternate definition, endorsement is not just done in the larger community; a classroom, a teacher or a student can endorse a narrative. In this case, an endorsed narrative is distinguished from a narrative in that the former is incorporated into the set of true statements for use in the

might be employed to establish or to reinforce the narrative, but it is important to note that this might be explicit– as in stating a definition– and it might be implicit—as in a change of topics after a student correctly answers the teacher's question. A more detailed version of these possibilities is in Greeno (2003), but the fact that established narratives might not be explicitly stated is what is important here.

Routines

Most often, the routines for establishing narratives in a discourse are implicit. In order for the participants to be conscious of these routines, they would need to consider the discourse, not just to participate in it. It is in this sense that Sfard refers to the rules that govern the use of routines as *meta-discursive* (2005a). As an example, CMP asks questions to lead students to narratives, but never tells students that answering these questions will help them to understand something. The use of questions to establish narratives is implicit. One place where these routines are closest to being explicit is when there are conflicting narratives needing resolution, as in the following example, where M is the teacher, R and TA are students, and ellipses indicate that material is omitted from the transcript.

M: Who's the faster walker. Yeah, R?

R: Yves.

M: How come? Why do you think so?

R: Cause he has whole numbers, not halves.

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discourse. Indeed, Sfard (2005b) indicates that this is the more productive definition. The use of *narrative* together with *establishing* and *reinforcing* will serve to make the necessary distinctions in the present study.

TA: The numbers are higher (Ms. M Activity 1, 10/27/04)

In this example, two conflicting narratives are introduced by students: *Fractional numbers in a table indicate smaller rates* and *Larger numbers in a table indicate greater rates*. Because there exist fractions greater than one, the two clearly conflict with each other, in addition to conflicting with the curricular narrative, *Greater increments in the dependent variable (given a constant increment in the independent variable) in a table represent greater rates*. In each case, Ms. M asks questions of the student stating the narrative and of the class. She uses the routine of Asking Questions.

While Lampert (1990) does not use the term *routine* to describe the practices by which she and her students verified mathematical claims in her classroom, the distinctions she draws are paralleled in the present study.

> Knowing mathematics in school... comes to mean having a set of unexamined beliefs, whereas Lakatos and Polya suggest that the knower of mathematics needs to be able to stand back from his or her own knowledge... (p. 32)

The generalization in this passage is strong; that school mathematics is about accepting ideas without question, which stands in stark contrast to Lakatos's (1976) and Polya's (1973) characterizations of knowing mathematics, in which reflection and the questioning of ideas are paramount. Nonetheless, these two contrasting sets of routines have relevance in the findings of the present study. One textbook discourse asks students to accept narratives as given, the other asks them to derive narratives through questions and reasoning. Two classroom discourses are seen to emphasize the latter, but also to give narratives in certain circumstances.

This chapter has posed the central problem of the study in operative terms with the use of a communicational framework in which teaching and learning are seen as statements about participation in discourse. The next chapter states the study's research questions in the language of this framework. It also lays out the rules for the analysis of the textbook and classroom data.

Chapter IV: Research Methods

This study arose from questions about classroom instruction in light of a substantial reform in K-12 mathematics instruction. In particular, these questions were about the classroom interactions in classrooms using *Connected Mathematics* (CMP) (Lappan Fey, Fitzgerald, Friel & Phillips, 2001a). This curriculum is the focus of this study. The preceding chapter laid out the theoretical framework in which this study takes place; teaching is conceived of as engagement with learners in a model discourse. In this framework, the question *What is being taught in CMP classrooms?* is equivalent to the question, *What is the model discourse in CMP classrooms?* The study breaks into three analytical pieces: analysis of textbook discourse and analysis of teachers' classroom discourse and the comparison of the two discourse types. These pieces correspond to three research questions:

1. What characterizes the introductory discourse on linearity in *Connected Mathematics?*
2. What characterizes each teacher's introductory classroom discourse on linearity?
3. What changes does the textbook discourse undergo in becoming classroom discourse?

In the first piece (curriculum analysis), a second curriculum *Mathematics, Structure and Method* (Dolciani, Sorgenfrey & Graham, 1992), referred to here as *Dolciani* is used as a contrast in order to better understand CMP. This curriculum was chosen under an assumption that the analytical tools being applied in this study would reveal sharp differences with CMP. It was also chosen for its reputation as a well-

constructed curriculum that has a consistent vision of mathematics and of mathematics learning. These two curricula are described next.

Texts

CMP is one of several National Science Foundation funded mathematics curriculum projects developed in response to the National Council of Teachers of Mathematics *Curriculum and Evaluation Standards* (NCTM, 1989). CMP was first published in 1996. It is a grades six through eight curriculum consisting of 8 units at each grade level and covering 4 strands of mathematics: number, geometry and measurement, algebra, and probability and statistics. A typical unit is about four to six weeks of classroom instruction. Each unit focuses on one strand. For instance, the sixth-grade unit *Bits and Pieces I* is a number unit, while another sixth-grade unit, *Shapes and Designs* is a geometry unit. A unit consists of four to seven investigations, which in turn consist of problems, homework questions and reflection questions. In general, a problem corresponds to one classroom lesson.

CMP is based on an explicit teaching model, called Launch-Explore-Summarize. This three-phase lesson structure takes place around "Problems," which consist of a series of tasks and questions, often in context. By context, I mean everyday situations with which most students are familiar outside of their school experience, such as bumper cars, walking races and fund-raisers. Janvier (1981) uses the term *situation* to describe contexts, while Boaler (1993) refers to contexts. Both researchers are concerned with calling attention to the limitations of contexts for mathematics instruction, and these concerns are relevant to instruction in CMP.

In the Launch phase of a lesson, teachers engage students with the problem—both its context and its mathematical challenge. In the Explore phase, students work on the problem, typically in pairs or small groups, but sometimes individually or as a whole class. In the Summarize phase, the teacher directs a discussion of students' solutions and helps the class to reach the mathematical goals of the problem.

This study focuses on one seventh-grade unit, *Moving Straight Ahead*, whose subtitle is *Linear Relationships*. *Moving Straight Ahead* includes six investigations. Investigation 1 consists of two experiments that students are to perform. The data collected from these experiments is analyzed with tables, graphs and equations and students are asked questions that require interpolation and extrapolation based on an assumption that the relationship in each experiment is linear. Investigation 2 is the focus of this study and introduces the term *linear relationship*. Students analyze a series of contextual problems to explore the properties of linear relationships, especially slope and y-intercept. Investigation 3 uses graphing calculators to analyze linear relationships that are not easily modeled by contextual problems, such as with negative coefficients and y-intercepts. The investigation also introduces solutions to linear equations. Investigation 4 focuses on solving equations. Investigation 5 formally defines *slope*. Finally, Investigation 6 consists of a series of contextual problems in which students practice the ideas of the preceding five investigations.

Dolciani is an older text that has been revised several times over the years. This study focuses on the 1992 edition. A note at the beginning of the Teacher Edition of the text explains how the revised text meets the 1989 NCTM *Standards*. The topics, language and presentation are similar to a preceding 1979 edition. As the title, *Mathematics,*

Structure and Method implies, Dolciani's presentation focuses on mathematical structure—definitions, theorems and rules—over contexts and real-world applications. There are two courses to the series, intended for 7th and 8th grades. This study focuses on Course 2, which is for 8th grade.

The teaching model in Dolciani is less explicit than in CMP, since they do not directly tell teachers how to structure lessons. But the notes to the teacher suggest that the American model of teacher exposition followed by student practice (documented in Stigler (1999)) is assumed. These notes are in a section at the beginning of each chapter in the Teacher Edition called *Teaching Suggestions*. There is a set of Teaching Suggestions for each lesson. The suggestions for each lesson consist of about a page of text. Objectives are listed for each lesson, while some lessons have additional commentary or ideas for teachers under such headings as, *Exploration*, *Computer Activity* and *Thinking Skills*.

This study focuses on Chapter 8 of Dolciani, titled, "The Coordinate Plane". This is the third of four algebra chapters in the text. Other earlier chapters have focused on number (four chapters) and geometry (one chapter). A chapter in Dolciani contains between 5 and 10 lessons and is roughly two weeks of instruction.

Dolciani is included in this study as a contrast. The study is not interested in the Dolciani text on its own terms. Instead, the characterization of CMP requires a contrasting curriculum in order to make the claims about the CMP discourse stand out as important. In this sense, this study uses Dolciani to better understand the discourse in CMP. Once Dolciani has been used to characterize the CMP discourse, the study leaves

Dolciani behind. Attention turns back to CMP, and in particular to the implementation of CMP by two teachers: Ms. M and Ms. H, who are described in the next section.

Teachers

Ms. M and Ms. H are junior high school teachers in two urban Midwestern public school districts. My teaching experience is in urban junior high schools, so this was the context for research in which I was interested. Teachers using CMP were chosen for this study based on these additional criteria:

1. Experience
2. Time on task
3. Attention to mathematical generality
4. Scheduling

Having identified the geographic area in which the study would take place, curriculum leaders familiar with one or both districts were consulted and approximately 10 teachers identified as likely candidates for participation in this study. Each of these classrooms was visited by the researcher at least once in the spring prior to the study. During these visits, the criteria for selection were considered. The two selected teachers best met the given criteria and agreed to participate.

The intention of this study is to describe a curriculum, CMP, in practice in a standard urban mathematics classroom. While there are many important questions about teachers using this curriculum to learn to teach, or to change their practice, this study has a different focus.

Participating teachers needed at least three years of experience in junior high school classrooms and with CMP. The year of the study is Ms. M's 15th year of teaching

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mathematics, her 8th year teaching junior high and also of teaching CMP. It is Ms. H's 7th year of teaching junior high math, all of them using CMP.

High time on task was a requirement for this study. A classroom in which a substantial amount of time were lost to discipline and off-task behaviors would be unlikely to yield useful information about the workings of the curriculum. Disruptions were few in each of Ms. M's and Ms. H's classrooms during the pre-observations and each teacher efficiently used class time for mathematics instruction.

Participating teachers needed to show some evidence of attending to the mathematical goals of the lessons. Ms. M and Ms. H were observed to use mathematical language and to attempt to elicit prior mathematical vocabulary from their students in the pre-observations. Finally, Ms. M and Ms. H each had plans to teach the unit *Moving Straight Ahead* in the window of time available for data collection for this study: late September through late November.

In a study of two classrooms, a representative sample of CMP classrooms is impossible. This minimum set of conditions was important to ensure that the curriculum was a major influencing force in these classrooms, but was not intended to create a set of idealized conditions for maximally effective implementation. Such conditions might have included an examination of the assignment of students to courses, the professional development experiences of the teachers, the support for the curriculum in their buildings and their districts, and other factors (NRC, 2004). Typical classrooms with good conditions were sought rather than exceptional classrooms and the result is a description of what might very well be happening in many classrooms around the country rather than an "existence proof" (Lampert, 1990) of what is possible.

School Contexts

Ms. M's school has approximately 450 students in seventh and eighth grade and follows a seven-period day in which classes last 48 minutes. Ms. M is one of four math teachers in the school. Students change schedules each semester and they study *Moving Straight Ahead* in eighth grade. As a result, some of the students in Ms. M's class in this study were her students the previous year, as seventh graders, while some were not. In addition, the student turnover rate at her school is high: 49% of the student population transferred in or out of the school in the first semester of the 2003-04 school year (this is the mid-year mobility rate)[4]. Students who were at her school as seventh graders studied CMP, whether with Ms. M or another teacher. Students who were not at her school had a variety of curricular backgrounds. The school's enrollment is 86% minority.

Ms. M taught *Moving Straight Ahead* beginning in the eighth week of school. The first seven weeks, she taught from a curriculum that her school was piloting, *College Preparatory Mathematics* (CPM, 2002) During this time a variety of topics was studied, including integer concepts, operations on integers, and functions. During the instruction on functions, the word *linear* was used in association with graphs. Students wrote equations and referred to them as *rules*. These rules were generated from tables of *Input-Output* values. A typical input-output table from this period of instruction is in Figure 1. Ms. M's students therefore came to the study of *Moving Straight Ahead* with different instructional experiences than a typical CMP student brings to the unit. Because this CPM instruction was not observed, these differences cannot be characterized beyond the citation of topics provided above.

[4] This is the most recent available data.

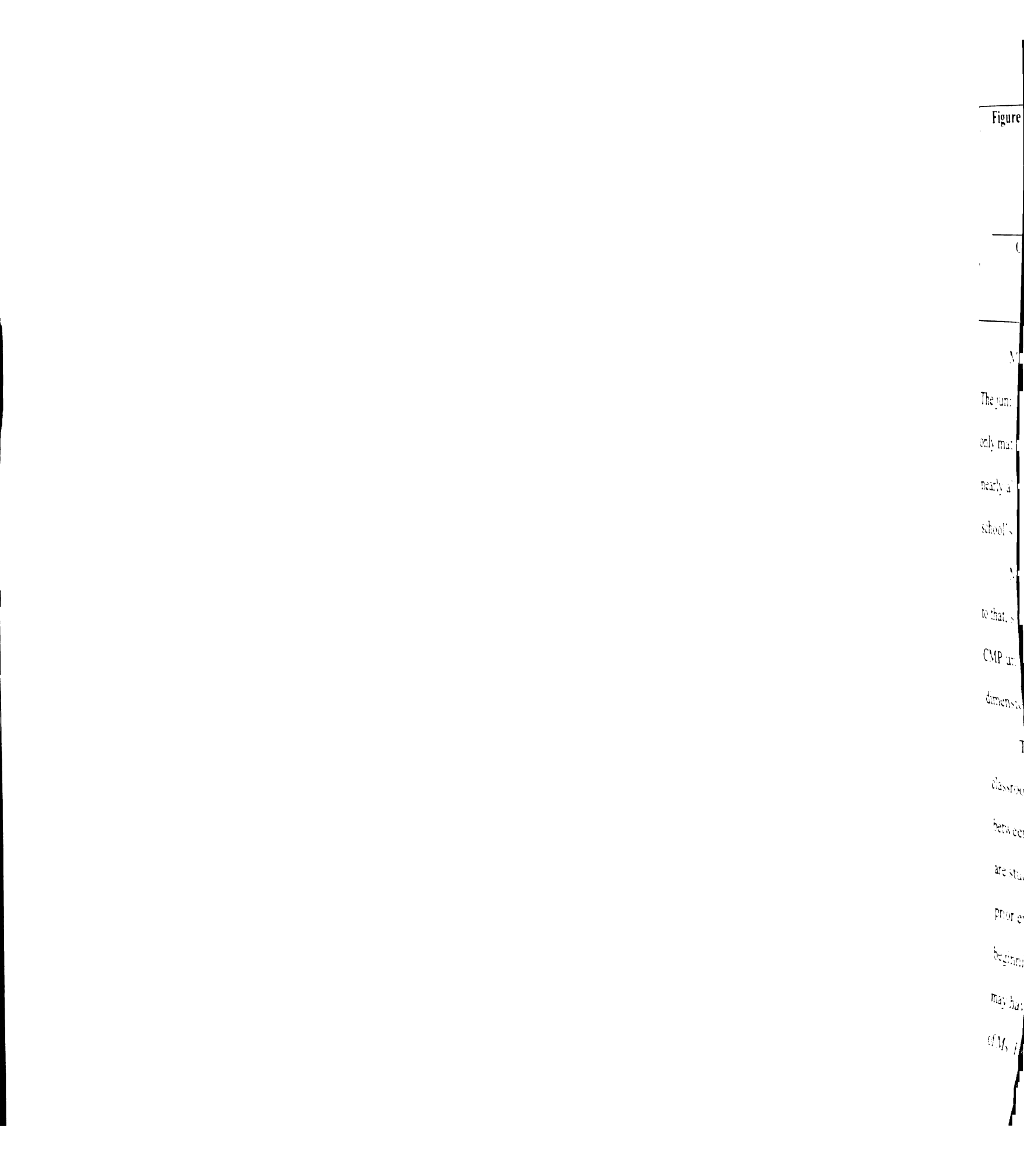

Figure 1: An Input-Output table from Ms. M's instruction with her pilot curriculum, *College Preparatory Mathematics* (CPM, 2002).

Input	5	3	9	1	0	7	100	4	10	x
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Ms. H's school is K-8, with approximately 480 students in all grades combined. The junior high students follow a six-period day with 53-minute classes. Ms. H is the only math teacher in her building. Student turnover is low (6% midyear mobility rate), so nearly all of the 8th graders in Ms. H's classes studied CMP with her in 7th grade. The school's enrollment is 44% minority.

Ms. H taught *Moving Straight Ahead* beginning in the tenth week of school. Prior to that, she did some introductory and review work with students and taught the 7th grade CMP unit *Filling and Wrapping* which focuses on volume and surface area of 3-dimensional figures.

These two classroom contexts—Ms. M's room and Ms. H's room—are urban classrooms with experienced CMP teachers, but I have described several differences between them. The student populations are diverse, but differently so. Both classrooms are studying *Moving Straight Ahead* in the fall of eighth grade, but their immediately prior experiences are different—Ms. M's students studied another curriculum at the beginning of the year while Ms. H's student studied CMP. A typical student of Ms. M's may have had as many as three junior high mathematics teachers, while a typical student of Ms. H's has had only Ms. H for junior high math. These two cases represent common

urban school structures. Together they provide a rich context for examining *Connected Mathematics* instruction in urban junior high schools.

Research Questions

This study was designed to examine the introduction of mathematical ideas through work on contextual problems, using discourse to frame the research questions. The theoretical framework outlined in Chapter III makes operative the research questions about this process. This section details the research questions and the methods for answering them.

This study was designed to be a detailed look at one instance of the introduction of mathematical concepts through work on contextual problems in the CMP curriculum. The introduction to linearity in *Moving Straight Ahead* is one place where this process is cleanest: Investigation 2 contains five contextual problems focused on a small set of mathematical ideas, but there is little mathematical language introduced until later in the unit. While there are several other such places in the curriculum, the existing body of research of research on this investigation and associated content in CMP (e.g. Smith, et al., 2000; Herbel-Eisenmann, 2000) made this instance particularly appealing for research.

Table 1 below states the research questions that guide this study, as well as the sources of data that are consulted in answering each one. Each question will be then be taken in turn and elaborated.

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Table 1: Research Questions and Data Sources	
Overarching question: How does the introductory discourse on linearity change from curriculum to classroom implementation?	
Research question	**Data source**
1. What characterizes the introductory textbook discourse on linearity in *Connected Mathematics*?	Teacher and Student Edition of *Connected Mathematics* Teacher and Student Edition of *Mathematics: Structure and Method*
2. What characterizes each teacher's introductory classroom discourse on linearity?	Videotaped observation Field notes Teacher Interviews
3. What changes does the textbook discourse undergo in becoming classroom discourse?	Analyses of research questions 1 and 2

Question 1: What characterizes the introductory textbook discourse on linearity in Connected Mathematics*?*

This study seeks to understand the relationship between the textbook discourse on linearity in *Connected Mathematics* and the classroom discourse that results in its implementation. The first step in understanding this relationship is to understand the textbook discourse on its own terms. This is the subject of the first research question.

Brief Overview of Algebra in Each Curriculum

In particular, this study examines the unit *Moving Straight Ahead*, a unit in grade 7 of the curriculum. This unit is the second in the algebra strand[5] and the only unit that focuses on linear relationships[6]. More specifically yet, this study focuses most carefully on the introduction to linearity. The first Investigation of *Moving Straight Ahead* consists of two experiments that are intended to produce real-world data that is roughly (although

[5] The first is also at grade 7: *Variables and Patterns*.
[6] At grade 8, there are also algebra units on non-linear relationships in general, on quadratic relationships, on exponential relationships and on symbolic reasoning.

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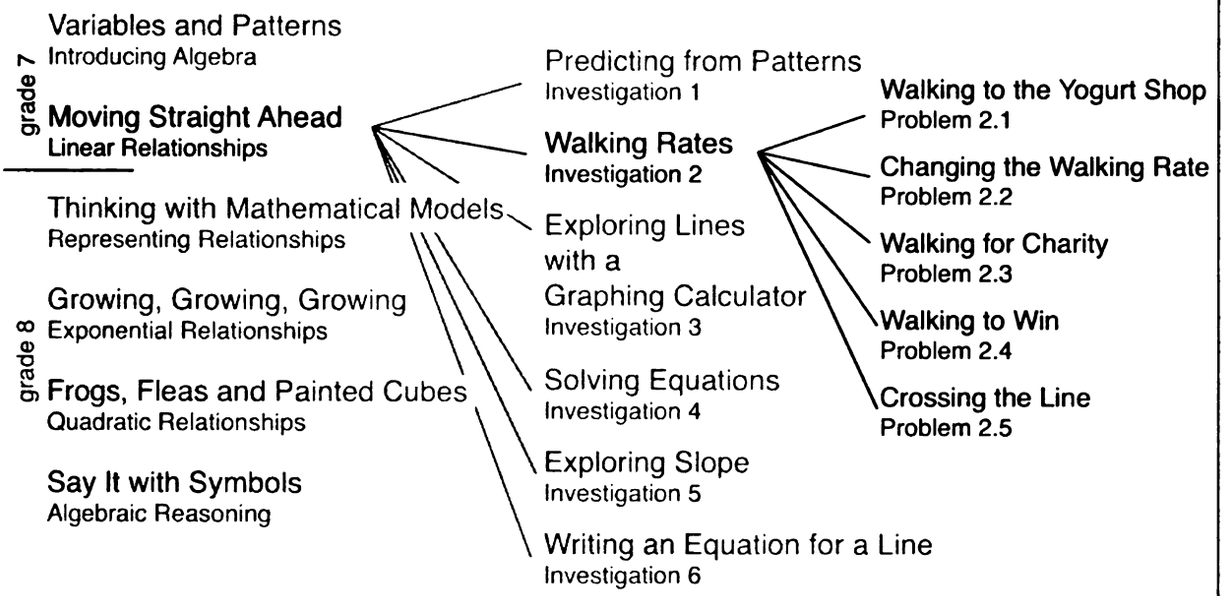
imperfectly) linear. The focus is on interpolation and extrapolation of the collected data, rather than on defining or specifically discussing linearity. For this reason, this study focuses instead on Investigation 2, in which *linear* is introduced. Investigation 2 consists of a series of five contextual problems that introduce students to the major themes of the unit. The focus on Investigation 2 of *Moving Straight Ahead,* then, is a choice based on the topic of the unit and on the introductory nature of the investigation. The location of this investigation in CMP's algebra strand is shown in Figure 2.

Figure 2: Algebra units in CMP

The algebra units in CMP are listed on the left, together with their subtitles. Next are the Investigations and their titles in *Moving Straight Ahead*, then the problems and their titles in Investigation 2.



In early attempts at characterizing the discourse in CMP, it was difficult to do much more than recite boilerplate phrases from the NCTM *Standards*. The characterization felt obvious rather than enlightening. In order to improve the characterization—to make it new rather than trite—a contrast is needed. In this spirit, I

apply the same analysis to another beginning algebra text, *Mathematics: Structure and Method* (Dolciani, Sorgenfrey & Graham, 1992), referred to henceforth as *Dolciani*. This curriculum was chosen for being widely used in U.S. classrooms and for its contrasting patterns of discourse on linear relationships. Dolciani is a representative of what is often called *traditional* curriculum. Also important is the fact that Dolciani shares with CMP an uncluttered layout, few colors or illustrations and an unwavering focus on the curricular principles to which it is committed. These two curricula stand in contrast to many of the widely available published curricula that are lumped, together with Dolciani, in the *traditional* category[7]. The traditional curriculum has been characterized as having a focus on procedural skills and as taking an incremental approach to the development of these skills[8].

Comparisons between CMP classrooms and Dolciani classrooms also would be informative, but this study focuses on classroom implementation of CMP and uses the curriculum analysis of Dolciani only as a tool for understanding the CMP textbook discourse. This contrast is in the spirit of Lemke's (1998) argument that,

> The basis of discourse analysis is comparison. If you are interested in covariation between text features and context features, you should not collect data only for the cases of interest, but also for cases you believe will stand in contrast with them.

Instead of investigations, Dolciani is divided into chapters. Correspondingly, instead of *Problems*, Dolciani has *Lessons*. These are roughly equivalent units of

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[7] Indeed, detractors of reform curriculum may deride current versions of traditional curriculum. Referring to one such textbook, Klein writes, "It is a book that I would wish on no one...The book is full of pointless color photographs, much like a web page. The effect is to distract the reader and discourage focused attention." (n.d.)

[8] A side benefit of this study is an improved understanding of the traditional curriculum resulting from the focus on its language.

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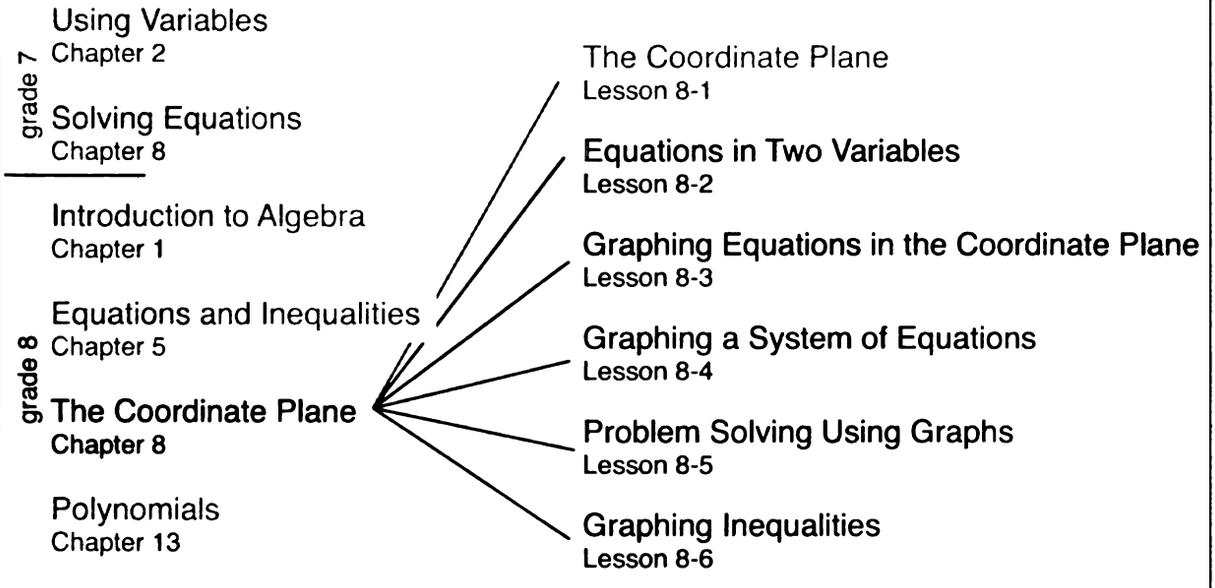
instruction: in CMP one problem generally takes one class period and in Dolciani one lesson generally takes one class period. There is no chapter with a singular focus on linear relationships. However, all mentions of linearity are in three lessons in a single chapter: Chapter 8. I focus on five lessons from this chapter: the three that use the term linear (Lessons 8-3, 8-4 and 8-6), together with the lesson immediately preceding these (Lesson 8-2) and the one lesson in Dolciani that includes the term *slope* (Lesson 8-5). The location of these lessons in the Dolciani 7-8 curriculum is shown in Figure 3.

Figure 3: Algebra chapters in Dolciani

The algebra chapters are shown, together with their titles on the left, the lessons in this study, together with their titles on the right.



The focus of the analysis of each curriculum is restricted to the body of the lesson, by which I mean the part of the lesson intended for the classroom, around which the teacher and students interact. In CMP, this means the Problem, together with its introductory text and follow-ups. Not included are the ACE questions or the Mathematical Reflections. In Dolciani, the body of the lesson means the explanatory text,

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the examples and the Class Exercises. Not included are the Written Exercises (the homework questions). Assessments and any supplementary materials are omitted from the analysis. The focus on the body of the lesson rests on an assumption that the classroom lesson is the relevant unit of instruction. Research question number 3 will compare the discourse on linearity in the curriculum with that in classroom instruction. Accordingly, I analyze the parts of the curriculum that are intended for use in the classroom lesson.

Data for the analysis of the discourse in the two curricula comes from the Student and Teacher Editions. The two together provide evidence for the complete picture of the discourse in each curriculum. There are important surface differences between the two Student Editions and between the two Teacher Editions. These are outlined next in order to better understand the context of the curricular study. The following consists of a glimpse at the surface features of each curriculum, not a set of claims about the discourse of either one. Discourse findings are reserved for the next chapter.

Structure of Each Curriculum

As discussed at the beginning of this chapter the CMP Student Edition consists of a series of problems, each intended to be worked over the course of a single lesson. Consider a typical example, Problem 2.3. This problem was chosen because it includes all of the important features of a typical CMP problem. The problem number and title come first (Problem 2.3 Walking for Charity), followed by some introductory text about walkathons. Then there is a set of questions that makes up the problem. These questions are set off in a shaded box. Under the shaded box is a set of follow-up questions.

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The introductory text is sometimes about the context (as in the example), sometimes about mathematics. Occasionally, there is an introduction of mathematical terminology. The problem generally consists of a set of questions, categorized as parts (A, B, C, etc.) and subparts (A1, etc.). Commonly there are at least three and as many as a dozen questions. The Follow-Up questions are similar in nature to those in the problem, but are typically fewer in number. The important point is that there is no standard place in the CMP Student Edition for explanations or examples. Instead, the mathematical point of the lesson is communicated through introductory text that focuses student attention, but does not explain, and through a series of questions in the shaded box.

Dolciani is a contrast to this structure. A typical Dolciani Lesson is Lesson 8-2. The lesson consists of expository text with examples and is followed by a set of practice problems. In contrast to the CMP model, Dolciani gives explicit instructions for procedures in the expository text (e.g. "To find a solution of a given equation in x and y, we can choose any value for x, substitute it in the equation, and solve for the corresponding value for y.") and provides examples for students to follow. The number of questions included in a Lesson is quite small compared to a CMP Problem. More to the point, in Lesson 8-2, even including the Written Exercises, there are no questions. Instead, Dolciani gives instructions and tells students to perform actions (e.g. "Tell", "Solve", etc.)

The two Teacher Editions are also quite different. The *wraparound*, in which a reduced-size copy of each student page is surrounded by answers and commentary for the teacher, is standard in the U.S. textbook industry, and both curricula include it. Dolciani overlays answers onto the reproduced student pages, offsetting them in color. CMP does

not overlay answers, but includes some in the wraparound margins and some on later pages. CMP and Dolciani each include some notes to the teacher in the wraparound section. CMP includes at least a page—sometimes as many as three or four pages—of commentary specifically for the teacher on each Problem. I will refer to this as the *LES section* for Launch, Explore, Summarize—the three phases of the CMP teaching model, and the three main parts of each of these commentaries. The equivalent in Dolciani is roughly one page of *Teaching Suggestions*. As with the Student Editions, a notable difference between these two Teacher Editions is the number of questions. The LES section for the problems in Investigation 2 averages more than 8 questions the teacher might ask the students per problem[9]. The Dolciani Teaching Suggestions for the lessons in this study include two questions: one in Lesson 8-2 and one in Lesson 8-5. In both cases, the answers in the Teacher Edition are left out of the analysis.

Next, I turn to methods for the analysis itself. Chapter III laid out four distinguishing characteristics of mathematical discourse: word use, visual mediators, narratives and routines. Each of these is taken in turn, focusing on the methods for analyzing each characteristic.

Word Use. A word use analysis needs to focus on a small set of words with theoretical importance. The set needs to be small in order to keep the analysis to a reasonable size, but it needs to be large enough to describe several aspects of the discourse. This study focuses on the use of three words, *equation*, *linear* and *rate*. *Equation* was chosen because any study of linear relationships is an algebraic study and algebra suggests equations (certainly school algebra does, anyway). The use and

[9] Forty-four questions over the five problems.

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treatment of equations is also a topic of the visual mediators section below. In this section, only the word *equation*[10] is considered. Under visual mediators, the use of equations is considered, whether *equation* is spoken or not. *Rate* was chosen because it is a central idea in the CMP approach to linear relationships. Relationships between variables constitute the focus of CMP algebra instruction. The word *rate* suggests the dynamic nature of the functional relationships in CMP. The hypothesis was that the use of this word would be substantially different in Dolciani, which focuses on solving equations, rather than on expressing relationships between changing variables. Studying the use of the word *rate* brings characteristics of the CMP discourse into sharper focus. Initially, the third unit for the word use analysis was to be the term *linear relationship*—an obvious choice because it is the subtitle of the CMP unit in this study—but this term does not appear in Dolciani. Instead, the word *linear* on its own is analyzed, for this term appears in both curricula. It was hypothesized that studying the use of *linear relationships*, or simply of *linear* in the two curricula would yield useful comparisons between them, and this study is of the discourse on linearity. The analysis includes derivatives of *linear*, such as *nonlinear* and *linearity* but not the word's root: line, which applies mainly to graphs rather than to relationships more broadly.

There is a perspective that the meaning of a word is equivalent to its use in communication (Wittgenstein, 1953; Sfard, 2005a). Acceptance of such a position necessitates the study of the use of the word *linear* in this study; for the *meaning* of linearity is what is sought in two textbooks and in two classrooms. If meaning is

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[10] In order to distinguish *equation*, the word, from equation, the thing (e.g. $y=2x$), *equation* will be italicized when it refers to the word and in plain text when it refers to the thing.

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The data for the analysis of word use consists of the sentences in which the words are used. For each curriculum, these sentences were compiled in three tables; one for each word. In considering the sentences in isolation, there were two questions: 1. *What patterns arise with respect to other words in these sentences?* and 2. *What is being said in this sentence about the word?* For example, in the CMP sentence, "How does the walking rate affect the equations?" (Student Edition, p. 18), rate and equation are in the same sentence, which did not occur at all in Dolciani, suggesting a pattern in question 1. With respect to question 2, the sentence suggests that rates do affect equations, and this implies that rates are represented in equations.

In some cases, a simple count of the number of times a word is used in a particular way provides a quick, rough contrast between discourses. When this is the case, word counts are used as an introduction to the finer-grained analysis of word use.

Attention is paid to words that are equivalent to each other in the sense of being used in the same ways, and to those that are equivalent only in certain situations (partially equivalent). For instance, *function* and *relationship* are equivalent in the introductory text in *Moving Straight Ahead* because the relationships studied there form the basis of the informal definition of function.

From word use, we may also gain clues about the philosophy of the nature of mathematics that underlies curriculum and instruction. For instance, if a teacher consistently talks with her students about *writing* equations, and if she talks about the

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equation as the generalization of a set of computational procedures, we may infer that she is modeling an operational conception (Sfard, 1991) of equations in algebra. If, on the other hand, she speaks of *finding* equations, and if she talks about equations as objects with parts (e.g. *y*-intercepts and slopes) that may be separated and analyzed, we may infer that she is modeling a structural conception of equations (i.e. equations are objects). Each of these is a well-accepted (while greatly debated) conception of the nature of mathematical object, so neither will define one teacher's discourse as more or less mathematical than another, but such observations will help to characterize each discourse.

Visual mediators. In mathematical discourse, visual mediators are used to discuss the abstract objects (functions, in this case) that are not physically present. In this study, three of these mediators are tables, graphs and equations, the standard algebraic representations. These mediators are not chosen for the analysis, so much as they are a product of it. When the non-textual elements of the curricula are considered, these three representations predominate. Any instances of these representations are analyzed, as are any diagrams in the text. Also included are any representations or diagrams that the text explicitly instructs students to make.

The analysis of the use of visual mediators in each curriculum began with a simple list of everything that is not a word or a number, but that is intended to be part of the mathematical discourse. In each curriculum, this included tables, graphs and equations, but *ordered pairs* also showed up, as well as one diagram in Dolciani. Next, places were noted where students are instructed to make one of these visual mediators. Then a list was made of what each visual mediator appeared to be used for each time it

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appeared. For example, a graph might be included to illustrate how to find slope, to define linearity, or for one of several other purposes. Finally, patterns were identified within each curriculum, with special focus on repetitions of the purposes.

Summary and discussion of these patterns are in the visual mediators section of the analysis of each curriculum. Each of the purposes of each of the visual mediators is referred to as a theme and the themes are identified as being *recurrent*, *passing* and *absent* in each textbook's discourse. Themes describe patterns in the use of visual mediators in the textbook, and therefore the meaning the authors intend to convey to students.

Recurrent themes are mentioned through expository text, questioning or examples in more than one sentence in more than one lesson. The Teacher and Student Editions are considered collectively for this purpose, so one mention might be in the Teacher Edition, another in the Student Edition. However, the mentions need to be distinct from each other. If the Teacher Edition mentions that the Student Edition asks about *y*-intercepts, for example, this is not counted as a second mention, as in this example from Problem 2.5, "Question 5 informally introduces the concept of the *y*-intercept by asking questions about the point where the line crosses the y-axis." (Teacher Edition, p. 34l)

Passing themes might be mentioned more than once in a single lesson, or once in each of multiple lessons, or a combination of these. Absent themes are not present in the discourse of the textbook. These terms have corresponding meanings in the teacher discourses, and these are discussed in that section.

In most instances, *mentions* are straightforward, even when the precise mathematical term is not used. In the following example, *y*-intercept is mentioned when

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the teacher focuses student attention on a specific part of a graph, "At what points do Emile's and Henri's graphs cross the y-axis?" (Student Edition, p. 22). Here the teacher mentions *y*-intercept without using the term by asking a question about the point that is the *y*-intercept. However, there are more ambiguous situations. For instance, a follow-up question to Problem 2.5 asks, "Which of the following equations gives the relationship between Yvette's distance from the starting line, *d*, and the time, *t*?" (Student Edition, p. 23) There are four choices, below (ibid).

i. $d = 20 + 2t$
ii. $d = 2 + 20$
iii. $d = 20t + 2$
iv. $d = 20 + t$

Because this question does not specifically direct students' attention to the *y*-intercept's representation in the equation, it is not counted as a mention. What makes this ambiguous is the fact that choices *i* and *iii* reverse the roles of slope and *y*-intercept, making each idea likely to be raised in student thinking or classroom discussion. In the end, though, the curriculum does not direct student attention to the concept and so this situation is not counted as a mention.

To summarize, visual mediators, primarily the algebraic representations of table, graph and equation, are central objects in the discourse on linearity. The ways in which these objects are used by the text, and the procedures the text offers to students for using these mediators are analyzed as themes in the visual mediators section for each curriculum. The relationships between these visual mediators are narratives and are discussed in the next section.

Narratives. Narratives are statements about the objects under study, and the relationships of these objects to each other. In mathematical discourse, narratives are

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statements about the status and relationships of mathematical objects to one another. I use the term *endorsed narrative*, to point to those relationships generally accepted as true by the larger (here, mathematical) community, but the term *endorsed* may also be applied to a narrative accepted as true by a subset of a community, including a single person. In most cases, a narrative in a classroom is endorsed by someone—the teacher, the class or a single student. Rarely do participants in mathematical discourse knowingly assert falsities. For this reason, this study limits the use of the term *endorsed* to describe narratives accepted as true in the larger mathematical community. *Definitions* and *theorems* are examples of endorsed narratives in mathematical discourse.

In the case of linearity, the primary objects of discourse are functions, as represented in tables, graphs and equations—the visual mediators, but the objects also include rates, slopes, *y*-intercepts, etc.—the features of linear functions. In the visual mediators sections, each of these representations is considered on its own. The relationships between these objects of discourse, their properties and the methods for working with them are the narratives in this study. A theme in the preceding section addresses one aspect of the use of one representation. A narrative in this section addresses commonalities and differences among the representations and their uses.

There are countless possible narratives in any text. Some, such those that appear in the form of a definition, are explicit. Some are implicit. Some are important and some are trivial. This study set out to identify the major narratives directly related to the study of linear relationships in each of the two curricula. The themes in the use of visual mediators are measured for their importance to the discourse. The narratives are not, for their complexity precludes this. Instead, narratives are identified as present or absent,

without measure of their frequency. If such a measure were devised, we would expect to find similar patterns in the narratives to the ones found in the use of visual mediators.

CMP is the primary curriculum under study, so these narratives were identified first. The Dolciani narratives, then, were identified with an eye to making reasonable comparisons with CMP. For instance, all three algebraic representations[11] are studied in detail in CMP, but they are not in Dolciani. If Dolciani were being studied on its own merits, there would likely not be a narrative about tables—the topic is not that important to the Dolciani treatment. Yet, for the purposes of this study, it is enlightening to ferret out what Dolciani *does* have to say about this relatively minor topic—because it is important in CMP.

In general, declarative statements about mathematical objects refer to narratives (e.g. "[T]here are *infinitely* many ordered pairs that satisfy this equation." (Dolciani, p. 276)). Questions, though, can also point to narratives, as in, "How does the change in walking rate affect the graph?" This question contributes to the narrative, *Rates are represented in tables, graphs and equations.*

Of particular importance in the case of CMP is that narratives should not be stated in the language of the contexts. That is, *faster walkers have steeper lines* is not a mathematical narrative, but *rate is represented by the slope of a graph* is. Practically, this is important in order to make meaningful comparisons with other curricula. Even another curriculum heavily reliant on contexts might not use the context of walking rates, but would be expected to address rates and slopes. As a theoretical matter, it is important to state narratives in mathematical language because this study originated in questions about

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the relationship between contextual problem solving and the formalization of mathematical ideas in classrooms. It is important, then, to identify in this study the formal mathematical ideas on which the curriculum focuses. As a result, identifying the narratives in CMP is a different task from identifying them in Dolciani. To identify them in CMP, the problems were considered together with their follow-ups. For each part of each problem, the mathematical goal of the task was identified. The example below is Part A from Problem 2.2,

> In Problem 2.1, each student walked at a different rate. Use the walking rates given in that problem to make a table showing the distance walked by each student after different numbers of seconds. How does the walking rate affect the data in the table? (Student Edition, p. 18)

There are two tasks in Problem 2.2: (1) Make a table and (2) Answer a question about the table. Students are shown the beginnings of a table with a constant increment of 1 in the time column (reproduced in Figure 4 below.) The point of the first task is to represent the relationship between two variables. The point of the second is that rates are represented in tables (as the increment in the dependent variable when the independent variable has a constant increment of 1), which is a narrative in the CMP textbook discourse.

Figure 4: The model table in CMP Problem 2.2

Time (seconds)	Distance (meters)		
	Terry	Jade	Jerome
0	0	0	0
1	1	2	2.5
2			
3			
.			
.			
.			

The introductory text to the Investigation and to each Problem was considered, checking for additional narratives. Finally, this list of narratives was compared to the Teacher Edition Goals. The final list of narratives in Investigation 2 of CMP is given in Table 2.

To identify the narratives in Dolciani, the process was quite different. Dolciani asks few questions of students, and the tasks to be performed during the Lesson are also few. Instead, the expository text communicates the narratives, in a more direct manner than CMP does through its questions. Each paragraph of the expository text in each of the five Dolciani lessons was considered. The main points of each paragraph were identified. When a point was a mathematics concept, it was deemed a narrative. When a point was procedural, it was noted as a procedural narrative (discussed at the end of this section).

When a narrative was not directly related to linear relationships, it was ignored[12]. Careful attention was paid to definitions.

As an example of using the main point(s) of a paragraph, this paragraph from Dolciani establishes the narrative, *A table lists solutions to an equation.*

> The equation in two variables $x + 2y = 6$ has infinitely many solutions. The **table of values**...lists some of the solutions. When we graph the solutions on a coordinate plane, we find that they all lie on a straight line. (Student Edition, p. 280)

The paragraph reinforced the narratives, *A solution to a linear equation is an ordered pair* and *A graph is the set of all points that are graphs of solutions to an equation.*

The following paragraph was deemed procedural and did not establish any narratives about the relationships between tables, graphs and equations.

> In order to graph a linear equation, we need to graph only two points whose coordinates satisfy the equation and then join them by means of a line. It is wise, however, to graph a third point as a check. (Student Edition, p. 281)

However, the paragraph does reinforce the narrative, *A linear equation has a straight-line graph.*

Next. this list of narratives was compared to the purposes of each example, to the purposes of the Class Exercises and to the goals for each Lesson in the Teacher Edition. In the end, this produced 9 narratives related to linearity. These are listed in Table 3.

[12] An example of a narrative in Dolciani, but not included in this study is *The point of intersection of two graphs is the solution to a system of equations.* This is not directly relevant to the introductory discourse on linear relationships.

The narratives described so far are about relationships between the objects of the discourse on linearity (e.g. between rates and tables). It has been argued (e.g. Sfard, 2005a) that another category of narrative comprises the procedural statements about mathematics. The preceding quotation from Dolciani contains a narrative about graphs: *plotting two points is sufficient for determining the graph of a linear equation.* This is a *procedural* narrative because it describes how to do something, rather than a relationship between objects. In particular, the sentence gives the procedure for graphing a linear equation: plot two points (and join them by means of a line). Procedural narratives are not analyzed in this study, although the procedures for identifying them would be identical to those discussed so far.

Routines. In the Student Edition of each curriculum, the pieces of the text that establish each of the narratives, whether implicitly or explicitly, were considered. In each case, the basis on which students are to be convinced of the existence and the truth of the narrative was noted. For instance the narrative, *Rates are represented in tables, graphs and equations,* is raised multiple times in CMP through questions of the form, *What effect does the walking rate have on the table/graph/equation?* This question form implicitly suggests that the rate *is* represented somehow, but that it is for students to figure out how. Additionally, the process of figuring out *how* rates appear in each representation involves inductive reasoning. Students consider three walking rates in Problem 2.2 and three pledge rates in Problem 2.3. Through six examples[13], students are to discern that rates affect the steepness of a graph.

[13] Together with a non-example in Problem 2.2,

When narratives conflict, resolution is needed. In classrooms, students often introduce narratives that conflict with those of the teacher's model discourse (e.g. Sfard's negative numbers example (2005a)). Examining how a teacher resolves this conflict can reveal routines for establishing and reinforcing narratives. In a textbook, the student's voice is absent. Teacher Editions often give examples of ideas that students find difficult, or of student misconceptions. Such mentions of narrative conflict were sought in each curriculum. An example is this from the Dolciani Teacher Edition, "...students should be warned to substitute correctly when checking a given number pair in an equation." (p. 271b) In each case, I considered the basis for the suggestion (e.g. *Why do students need to be warned about substitution?*) and the method suggested for resolving the difficulty or conflict (e.g. *What is the justification for substituting correctly?* and *How do students know whether they have substituted correctly?*).

Mathematical discourse is characterized by its word use, its use of visual mediators, its narratives and its routines for establishing and reinforcing these narratives. Textbook discourse is treated as mathematical discourse and these features are used to characterize a particular textbook mathematical discourse: that of CMP. A contrasting curriculum—Dolciani—is used to refine this characterization. While each feature contributes to the understanding of the CMP discourse, it is only through a full examination of the features together that the full picture emerges. Care is taken in the analysis to avoid getting lost in the minutiae of word counts and prioritization of themes. Instead, these details are used to contribute to the understanding of the discourse as a whole.

Question 2: What characterizes each teacher's discourse on linearity?

Just as the first research question deals with understanding the textbook discourse on its own terms, this second question deals with understanding discourse in CMP classrooms, as modeled by teachers. The same structure is used for analysis, but it must be acknowledged that textbook discourse and classroom discourse are different from each other—for instance in the fact that textbook discourse cannot be responsive to students, while classroom discourse changes in response to student ideas and questions. The elaboration of the third research question deals with reconciling these differences and argues for the comparability of the two discourse types.

Prior to the classroom observations, each teacher was interviewed about her goals for students in the investigation and her experiences in past years with the curriculum. Each classroom was observed for the duration of the instruction in Investigation 2 of *Moving Straight Ahead.* In Ms. M's classroom, this consisted of 7 lessons. In Ms. H's classroom, this was 5 lessons. Each lesson was videotaped with a stationary tripod-mounted camera that captured approximately 2/3 of the students as well as the teacher and her visual aids (e.g. overhead projector screen, chalkboard, etc.) Microphones were placed at two locations on student desks to capture student discussion as well as the teacher's language.

Detailed field notes were taken by the researcher during each lesson. These notes divided each lesson into activities. Every time the nature of student activity changed (e.g. from listening to writing), a new activity was recorded. Lessons consisted of between 3 and 7 activities. These activities were put into five categories: Launch, Explore, Summary, warm-up and other. The Launch, Explore and Summary categories correspond

to the CMP instructional model in which the Launch engages students with the contextual and mathematical tasks in a problem, the Explore activities have students answering the questions in the problem and the teacher leads the whole class in discussion of solutions and procedures in the Summary. Each teacher used warm-up activities at the beginning of some periods. These warm-up activities tended to focus on topics from the investigation, but were of each teacher's own design. Activities categorized as "other" tended to be administrative (e.g. collecting homework or listening to school announcements). This study focused on the warm-up, Launch and Summary activities because these are the activities richest in teacher language.

Word Use. As with the analysis of word use in the two curricula, the analysis of word use in each teacher's discourse began with compiling the sentences using the words in question. Once these sentences were compiled, patterns of word use from the CMP discourse were sought. For instance, the association of *relationship* with the use of the word equation was a pattern in CMP's discourse, so the frequency of this association was analyzed in each teacher's discourse. After identifying where there were similarities and differences with CMP, the sentences were examined again to identify additional patterns, not present in the CMP discourse.

Visual Mediators. The use of visual mediators is more complicated in the classroom than it is in a text. In a text, one can look at a page and find the non-linguistic elements at a glance. In a classroom, a visual representation might be present, but not discussed and it might be discussed but not present. Further, the transcripts from which a researcher typically works do not include the representations themselves. As a result, the analysis of visual mediators in classroom discourse had two components. The first

involved working solely with the transcripts; all passages in which each representation was under discussion were compiled. As with word use, these compiled passages were examined for the CMP themes and then again for additional themes. The second component involved watching the videos and capturing what visual mediators the teacher was focusing students' attention on. In this component of the analysis, the words a teacher spoke were less important than what she displayed and pointed to. Again, the analysis considered first the patterns already identified and then looked for new patterns.

The themes associated with each of the visual mediators were sorted into three categories as in the curriculum analysis: recurrent, passing and absent. *Recurrent* themes are present in more than one teacher turn in each of more than one whole-class activity. *Absent* themes cannot be identified in the teacher's whole-class instruction. *Passing* themes may be present in single teacher turns in multiple whole-class activities, in multiple turns in a single activity, or in a combination of these.

Because the discourse on linear relationships in CMP relies on contexts, the mathematical topic need not be named to be present in the discourse. For instance, the theme of identifying a linear relationship from a graph is present when Ms. H discusses the connection between *straight lines* and *walking at a steady pace.* A steady walking pace is a constant rate, which is the hallmark of linear relationships. As with the narratives in CMP, mathematical ideas are often stated in contextual language. The analysis in this paper reflects that.

Narratives. The narratives in each teacher's classroom discourse were identified in comparison with the narratives in CMP. That is, the analysis began with the list of CMP narratives and the lessons were examined for evidence of the presence of each. In

CMP's textbook discourse, the identification of narratives involved identifying the mathematical goal of each part of each problem, and identifying the point of each paragraph of introductory text. In the teachers' classroom discourses, the identification of narratives involved identifying the mathematical point of each teacher turn, whether interrogative or expository. In general, these two teachers more frequently state narratives outright than does the CMP curriculum, making the task of identifying narratives more direct. A combination of the techniques applied to the two textbooks was employed. As with CMP, the tasks and questions were analyzed for their mathematical goals. As with Dolciani, the exposition in each lesson (i.e. teacher assertions) was analyzed for their main points. As with the textbook discourse, procedural narratives were considered separately from those that deal with relationships between objects.

Routines. In the transcripts of each lesson, teacher turns were again considered as the unit of analysis. As in the case of the texts, the basis on which students are expected to be convinced of the existence and truth of any narratives embedded in the teacher turn was inferred. In addition, routines can be identified when students introduce conflicting narratives. When two narratives are in direct conflict, the teacher can be expected to take steps to resolve the conflict. These steps highlight the routines by which narratives are established and reinforced. In some instances, the teacher might ask questions to make the student see the conflict. In other instances, the teacher might ask another to reiterate a previously established narrative (e.g. "What did we say yesterday?" (Ms. H Activity 2, 11/09/04)).

Just as textbook discourse is treated as mathematical, teachers' classroom discourse is treated as a mathematical discourse as well. In theory this is justified by

assumptions about the narratives to be found in mathematics classrooms: whether they are about procedures or complex relationships between mathematical concepts, we should expect the discourse of mathematics classrooms to be populated with mathematical narratives. Also, we might expect mathematical words to be used and for visual mediators to be used to discuss mathematical ideas. While we also expect differences from formal mathematical discourse, it is reasonable to expect prior to data collection that classroom discourse will conform to norms of mathematical discourse in these ways. This study bears out these expectations, as is argued in Chapter VI.

Question 3: What changes does the discourse of the curriculum undergo in its implementation by each teacher?

A teacher's classroom discourse contains narratives just as a curriculum's discourse does. In each case, the model discourse establishes relationships among the objects of the discourse. The narratives may be different in the classroom discourse from those in the text, and their importance may vary, but the basic nature of a narrative is unchanged. In this sense, the discourses are comparable.

Similarly, the routines in each type of discourse are of the same nature, although the presence or absence of specific routines varies across the discourses. The routines manifest themselves differently in the two types of discourse as well. For instance a textbook does not respond to student turns. Therefore extended exploration of competing narratives, in which a student's narrative is questioned and compared to the model narrative, is rare in a textbook while it may be common in classrooms.

In this sense, a teacher's classroom discourse is incommensurate with textbook discourse. Classroom discourse is co-constructed by teachers and students (e.g. the

Classroom-Generated Language that Herbel-Eisenmann (2000) found was often coined by students) while textbook discourse is not. Yet, if each is taken as a kind of mathematical discourse, it is reasonable to make comparisons. While each discourse is generated in a different way, and by participants with different roles, each can be expected to share some features of mathematical discourse.

The problem for the research question at hand, then, is not so much justifying the comparison of the two kinds of discourse, but ensuring that the descriptions of them are justifiably commensurate so that meaningful comparisons can be made. This is the task of this section.

Word Use. In the analysis of the textbook discourses, the terms *linear*, *equation* and *rate* were analyzed in three ways: an examination of patterns of use with respect to other words, counts of the appearances of these patterns, and consideration of what is being said to students about each word when it is used. It is reasonable to expect that patterns in association in word use would appear in both teacher speech and written textbooks. In particular, it would be significant if patterns prevalent in a textbook did not show up in the teacher discourse and patterns peculiar to a particular teacher's discourse are likewise informative.

It is not reasonable, however, to assume that the word counts will correspond between textbooks and classrooms. The number of words spoken in a classroom lesson is in general far greater than the number of words on the page. Further, because the teacher's discourse is constructed in response to students, the allocation of word use would be expected to vary greatly. Word counts are a crude tool when the discourse types

are alike. When the discourse types vary, they can be meaningful only when the differences they reveal are great.

Finally, because the question, *What is being said about this word in this sentence?* is a way of asking about a word's meaning, we should expect similarities between textbook and classroom discourse.

Visual Mediators. As with the use of words, it is reasonable to expect that visual mediators are put to the same uses in textbook discourse and in classroom discourse. In each analysis, this study attempts to measure the importance of each use of visual mediators in each discourse by identifying themes and labeling each theme as recurrent, passing or absent in each discourse. That these three labels are commensurate in each type of discourse needs justification.

The purpose of the measure is the same in each discourse. I wish to evaluate the importance of a theme to a discourse. There is an assumption that the importance of a theme is related to its frequency. A theme that is mentioned several times in a lesson is more important than one that is mentioned only once. Similarly, a theme that appears in multiple lessons is assumed to be more important than one that appears in only a single lesson. Thus, recurrent themes are assumed to be more important in a discourse than passing themes, which are obviously more important than absent themes. This is not a fine-grained measure and it is not used to make subtle distinctions across types of discourse. Instead, the measure reveals similarities and differences in broad patterns across discourse types.

The measure in each discourse type is intuitively equivalent. In textbook discourse, a recurrent theme is mentioned in more than one sentence in more than one

lesson. In a teacher's classroom discourse, a recurrent theme is mentioned in more than one teacher turn in more than one lesson. In each case, the goal is to isolate repetition within a lesson and then across lessons. Because teacher turns are not edited, they often use multiple sentences and sentence fragments to express an idea that would be stated in a single sentence in a textbook. Sentences in the two discourses are incommensurate, so instead a textbook sentence is equated with a teacher turn.

Narratives. Because narratives are the relationships among the objects of the discourse, these relationships should be stable across discourse types. Indeed it is a fundamental assumption of mathematics curriculum writing that the relationships among the objects outlined in textbooks are communicated in classrooms. This is the goal. We should expect comparison of narratives across the discourse types to be unproblematic and any differences revealed to be informative.

Routines. Routines are likewise commensurate. Each of the textbooks in this study gives explicit instructions to teachers for establishing narratives, and pays attention to the issue of resolving conflicting narratives. The main problem for research, which has been addressed above, is that these explicit instructions in curriculum do not cover all of the routines in a textbook and teachers' discourse is rarely explicit about the routines used for establishing and reinforcing narratives. Instead, inferences need to be made in each discourse type. No specific comparative claims will be made about frequency or importance of these routines in either discourse type, so measurement is not an issue. Instead, the task is to identify and compare the routines that are present.

This third research question involves a comparison across two discourse types: textbook and classroom. While there are differences inherent in their nature (especially in

the degree of responsiveness to student learning), I have argued that the analytical tools applied to the two types of discourse can yield comparable data. Where the data are incommensurate (e.g. fine distinctions in word counts), comparison is not meaningful. Instead, this study has been designed to use comparable data to make meaningful comparisons across the discourse types, and to answer the question *What is being taught?* in each one.

The Researcher

My background is relevant to this study, so I briefly summarize it here. I am an experienced CMP teacher. I taught CMP in the Saint Paul Public Schools for six years prior to starting graduate school. I taught seventh and eighth grade students, but used nearly all of the sixth, seventh and eighth grade units with these students at various times. While in the classroom, I conducted professional development workshops for teachers new to CMP and have continued this work while a graduate student. My graduate assistantship has been funded by the National Science Foundation grant for the revision of *Connected Mathematics* that began in the year 2000. My work for that revision has included editing, production of pilot units, a small amount of writing for students and some substantial writing for the revised Teacher Editions. I have conducted and published one curriculum analysis in a spirit similar to the comparison of CMP and Dolciani in this study (Danielson, 2005). This analysis focused on differing curricular approaches to perimeter.

I originally came to Dolciani with fresh eyes. As I worked and analyzed this curriculum more closely, I became more and more sure that an earlier edition of this series was the text from which I studied algebra in ninth grade in the mid-1980's.

Summary

In Chapter III, I modified the question, *What is being taught in CMP classrooms?* to make it operative by phrasing it in the language of discourse analysis, *What is the model discourse in CMP classrooms?* In particular, this study concerns itself with understanding the relationship between two model discourses in CMP classrooms—that of the published curriculum and that of the classroom teacher. This chapter has posed three related research questions and detailed the methods used to answer them. While the third research question, in which the two discourse types are compared, is the original driving force behind the dissertation, each of the other two questions yields useful and interesting results on its own merits. This is especially true of the analysis in Chapter V, which is a careful comparison of two curricular discourses in order to characterize the textbook discourse on linearity in *Connected Mathematics*. This analysis is next.

Chapter V: Textbook Discourse

The 1990's saw the introduction of a host of new mathematics curricula written in response the National Council of Teachers of Mathematics (NCTM) 1989 *Curriculum and Evaluation Standards*, most of them with National Science Foundation development money. These curricula, together often called *reform* curricula, tend to have more than surface differences with the standard curricula in place in the 1980's, called *traditional* curricula. These reform curricula, in responding to the *Standards* changed the emphases on topics, as well as the approaches to topics across the grade levels. In order to improve instruction in this new curricular environment—either through revisions to existing curricula, or through professional development for new and experienced teachers—it is important to understand the classroom implementation of these curricula. One piece of this is developing precise language and frameworks for the analysis of curriculum.

Standards documents (e.g. NCTM, 1989 and 2000) make policy recommendations for curriculum in the form of specifying answers to the questions, "What should be taught?" and "What should be learned?" Indeed, these questions are the starting point for any coherent curriculum writing. This chapter and the following one turn the first question around and ask, "What is being taught?" by looking carefully first at published curriculum, and then at classroom teaching. A communicational framework (Sfard, 2001) is used to formulate meanings for terms such as *teaching* and *learning* in the language of discourse. Discourse is defined as communicative activity. Knowledge is characterized in terms of participation in discourse and learning as changes in that participation. Teaching is defined as engagement with learners in the model discourse and so the question, *What is being taught?* is asked a different way: *What is the model discourse?*

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In particular, the present chapter seeks to describe the communication patterns in one curriculum, *Connected Mathematics* (CMP). These communication patterns are referred to as the *textbook discourse*. In earlier chapters, I have characterized CMP as relying on contexts for the construction of mathematical meaning. In order to study this process, several places in the curriculum were identified where the process can be most clearly seen. The introduction to linear relationships in the seventh-grade unit *Moving Straight Ahead* was chosen from among these places due to an existing research base on student knowledge and understanding related to this unit (see Smith, et al., 2000 and Herbel-Eisenmann, 2000). Thus, the research question underlying this chapter is *What characterizes the introductory textbook discourse on linearity in CMP?*

In order to highlight the characterization of the discourse on linearity in CMP, the analysis continues with a contrasting curriculum, *Mathematics, Structure and Methods* (Dolciani, 1992). This curriculum was chosen with the hypothesis that sharp differences would be seen between it and CMP. In early attempts at characterizing the CMP textbook discourse, the claims resembled boilerplate reform rhetoric rather than research findings. In order to bring meaning to the findings, a contrast was necessary; we needed to see how things could be otherwise. In this sense, Dolciani brings richer meaning to the claims about the CMP discourse by making salient the fact that the claims about CMP are not idle generalizations, but real patterns that can be delineated in a systematic and meaningful way. This chapter does not directly speculate on these consequences, nor bring empirical evidence to bear on them. Instead, the focus here is on the textbooks themselves. The next chapter looks at how the CMP textbook discourse on linearity changes as it becomes teachers' classroom discourse in implementation with students.

A discourse on linearity is a subgenre of discourse on functions. There are two broad categories of conceptions (and hence, discourse) on functions (see e.g. Kieran, 1993): 1. a *relational-dependency* conception and 2. a *set-theoretic* conception. In the relational-dependency conception, functions are viewed as relationships between variables. Rates and change are central to this conception. Graphs are considered for their slopes and tables for their patterns of change. The set-theoretic conception emphasizes correspondence between x-values and y-values (we are speaking here of functions of one variable, but there are analogous ideas for multi-variable functions). In each case, functions may or may not be described by equations. When a function is described by an equation, the relational conception emphasizes how patterns of change in the two variables can be identified, while the set-theoretic conception emphasizes finding solutions (i.e. completing the correspondence).

A central claim of this chapter is that CMP promotes a relational-dependency conception of functions. This can be seen in every aspect of the CMP discourse on linearity: in the way words are used (and *which* words are used) in the discourse, in the use of visual mediators, in the narratives and in the routines used to establish these narratives. Dolciani, for its part, emphasizes a set-theoretic conception of functions. The various aspects of mathematical discourse help to highlight these differences.

CMP

Investigation 2 of *Moving Straight Ahead* includes five problems. Titles and descriptions of each of these problems are in Table 4 below. As can be seen in Table 4, each of the five problems in Investigation 2 is set in a context. Two claims will be substantiated in this section: 1. that CMP has a relational-dependency approach to

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algebra, and 2. that this approach is founded on the use of contexts to develop mathematical meaning. In order to facilitate this second claim, an extended description of the contexts is important.

Problems 2.1 and 2.2 share a common context. Three students—Terry, Jade and Jerome—are walking to a nearby yogurt shop. Each student has a different walking rate: Terry walks 1 meter per second, Jade walk 2 meters per second and Jerome walks 2.5 meters per second. The three students begin from the same place and at the same time. In Problem 2.1 (titled *Walking to the Yogurt Shop*), students are asked a number of questions about various times and distances for each walker. These questions can all be answered with arithmetic computation; there is no explicit call for students to use algebraic representations or techniques. There are two follow-up questions. The first asks students to explain their strategies for answering the questions in the problem. The second follow-up asks whether Problem 2.1 involves linear relationships.

Table 2: The Problems in *Moving Straight Ahead* Investigation 2

Problem Number	Problem Title	Description
2.1	Walking to the Yogurt Shop	Three characters, each with a different walking rate, leave school for the local yogurt shop 750 meters away. Students answer questions about distances covered in given amounts of time and about the time taken to cover given distances.
2.2	Changing the Walking Rate	Students are directed to analyze the situation from Problem 2.1 using tables, graphs and equations.
2.3	Walking for Charity	Three new characters are participating in a walkathon. Each has a different pledge plan that relates miles walked to money raised. Students are directed to make tables, graphs and equations to analyze the situation.
2.4	Walking to Win	Two new characters have a walking race. One character is given a head start because he walks more slowly. Students are asked to decide how long the race should be in order for the finish to be close.
2.5	Crossing the Line	Students make tables, graphs and equations to answer a series of questions about the walking race from Problem 2.4.

In Problem 2.2 (*Changing the Walking Rate*), the context is the same, but students are told to make a table, a graph and an equation for each walker. In the case of each representation, students are asked a question of the form, "How does changing the

walking rate affect the [insert representation here]?" Additionally, there are two follow-up questions based on a different, but related context. In the follow-up, a character named Abby is examining some time and distance data and wonders whether the relationship is linear (it is not). The first question asks students to state a rule for knowing whether data in a table will produce a straight-line graph. The second question asks students to interpret the data in terms of the story of the race that produced it.

Problem 2.3 (*Walking for Charity*) introduces a new context—a walkathon. Again, there are three main characters: Leanne, Gilberto and Alana. Each character develops a pledge plan for their participation in the walkathon. Leanne decides to ask each sponsor for 1 dollar per mile that she walks. Gilberto asks for 2 dollars per mile that he walks and Leanne asks for a five-dollar donation and an additional 50 cents for each mile that she walks. The questions from Problem 2.2 are repeated; students are asked to make a table, a graph and to write an equation for each walkathon participant showing the relationship between the miles he/she walks and the money raised from each sponsor. Students are asked to identify how increasing the pledge rate affects the table, the graph and the equation. A series of questions asks students to go back and forth between the independent and dependent variables, first finding the money raised by walking 8 miles and then finding the distance required to raise 10 dollars. The effect of the donation on the table, graph and equation is examined. The follow up questions ask about the meaning of points on the graph and about the relationship between each equation and the steepness of its corresponding graph.

Problem 2.4 (*Walking to Win*) introduces the third new context. In this problem two brothers, Emile (the older brother) and Henri, are having a walking race. Emile's

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walking rate is 2.5 meters per second, while Henri's is 1 meter per second. Emile gives his younger brother a head start of 45 meters. Students are challenged to decide how long the race should be so that the younger brother, Henri, wins but only by a little bit, so that it is not obvious that Emile has let him win. As with Problem 2.1, students are not required to use any particular method for solving the problem and it need not be done using algebraic tools.

Problem 2.5 (*Crossing the Line*) revisits the same context, but requires students to make tables, graphs and equation to solve the problem. Students are asked questions about the intersection point of the graphs of the two boys' equations. The follow-up questions ask students to find other time and distance values and to explain how the representations can be used to do so; they point student attention to the y-intercept and its representation in each of the three representations; and they introduce a third walker with an intermediate walking rate and head start. In this last follow-up question, students make a table and choose an equation from among several possibilities to represent the relationship between distance and time for this third walker.

This section analyzes the introductory discourse on linearity in *Connected Mathematics* using four distinguishing features of mathematical discourse: word use, visual mediators, narratives and routines. These features are isolated and addressed in turn. At the end of this section, the features are considered together in order to characterize the discourse as a whole. The following section uses the same structure to characterize Dolciani's discourse.

Word Use

The analysis of CMP's discourse begins, as with each discourse in this study, with word use. Three words are analyzed, as described in the Methods chapter, *equation*, *rate* and *linear*. *Equation* is analyzed because of the expectation that equations would be central objects of any algebraic discourse. *Rate* is chosen for being important to the CMP approach to linearity, while less important in traditional American approaches, including Dolciani. *Linear* is chosen for the quick impression its use should give about the discourse on linearity. In each case, patterns in the use of the word are identified and supported with evidence. In some cases, counts of the word's frequency, or its appearance in particular situations or in combination with other words will be useful for demonstrating that the identified pattern is significant. In other cases, quotations from the discourse are used to substantiate this claim.

Equation

There are two important aspects that describe the use of *equation* in Investigation 2. The first is the association of *equation* with the term *relationship*. The second is the status of *equation* relative to that of *graph* and *table*. Each of these supports the assertion that CMP's approach to functions is a relational-dependency one.

Relationship. The predominant use of equation is as a representation of a relationship. In the introductory text in the student materials, nearly every use of equation (5 out of 6) is in the same sentence as the term relationship (or *relating*), as in the example below.

> How does increasing the average speed at which the van travels affect the table, the graph and the equation of the relationship between distance and time?

(Lappan, et al., 2001b, p. 16; referred to as *Student Edition* henceforth.)

From this and other uses of the term *relationship*, it can be inferred that its meaning is close to the mathematical definition of *function*[14]. In the phrase, *the relationship between distance and time*, the implication is that for each time value, there is exactly one distance value. Yet, *relationship* is used only to describe continuous, predictable functions, and in this sense, the terms are not synonymous. So an equation represents what might be described as a "well-behaved" function, and indeed the term *function* appears in the introduction to the investigation.

> For example, the total cost to rent bikes depends on the number of people on the tour. We say that the rental cost is a *function of* the number of people on the tour. (Student Edition, p. 15, emphasis in original)

Here, *function* informally describes a dependency relationship, rather than being mathematically defined in a formal way. Nonetheless, the introduction of the term here is important. It sets the tone for the relational-dependency approach to linearity.

This association of *equation* with *relationship* continues with Problem 2.2, the first time in the unit that students write equations. "For each student, write an equation that gives the relationship between the time and the distance walked." (p. 18)

In the introductory text and in Problem 2.2, the relationship is the central object of the discourse. In Problem 2.3, a different aspect of equations is introduced. Here, equation and relationship do not appear together at all. Students are instructed to, "Check your equation by graphing it..." (p. 20) The suggestion to students is that the equation itself is worthy of study and discussion. There is a subtle shift here as relationships yield

[14] I.e. a set of ordered pairs such that no two pairs have the same first element, but distinct second elements.

center stage to equations. Leading up to this problem, an equation has been a representation of a relationship. In this problem, an equation can itself be represented (i.e. graphed) and studied.

In Problem 2.5, *relationship* returns in the Student Edition with sentences such as, "Write an equation for each brother showing the relationship between the time and the distance from the starting line." (p. 22) At the same time, the problem maintains the idea that graphs and equations are objects of study without reference to a relationship, "How can you predict where a graph will cross the *y*-axis...from an equation?" (p. 22) Overall, 43% of the sentences in which *equation* appears in Investigation 2 of the Student Edition also include the word *relationship*. Through the use of the words *equation* and *relationship*, the Student Edition has established a relational-dependency approach to functions.

The Teacher Edition does not use *relationship* in any sentence with *equation*. Nonetheless, the shift from emphasis on relationships as the objects of study early in the investigation to equations as the objects of study in Problem 2.3 can still be observed there, albeit more subtly. The relationship discussed repeatedly in the Student Edition is between variables in contextual situations. For this reason, we may consider sentences using *equation* that also include the word *situation* or the names of the contextual variables (*time*, *distance*, *walking rate*, etc). Sixty-three percent of the sentences using *equation* also use one of these terms in Problem 2.2 (7 out of 11 uses), while 30% use them in Problem 2.3 (3 out of 10 uses). The frequency of allusion to relationships when discussing equations is roughly half what it was in Problem 2.2.

Typical of the sentences in Problem 2.2 is, "What effect will increasing your walking rate have on the table, the graph and the equation?" (p. 34b), while typical of those in Problem 2.3 is, "How do the tables, graphs, and equations from Problem 2.2 compare to the tables, graphs, and equations for this problem?" (p.34g). The shift from comparing relationships to comparing equations (as well as tables and graphs) is notable. In the example from Problem 2.2, the language of the context is included (*walking rate*), implying that the equation represents the relationship between the variables there. In the example from Problem 2.3, the context is stripped away and the equation is considered in comparison to another equation. That is, the equations are objects of study in their own right in this example, not just as representations of some contextual relationship.

The greater theme, though, is of equations as representations of relationships, which is consistent with a relational-dependency approach to functions. A careful reader might notice that the examples in the preceding paragraph were also statements about tables and graphs, as well as equations. This is the next topic.

Table and Graph. Related to the preceding discussion is the fact that equation is but one of three basic algebraic representations of functions, together with table and graph. In a relational-dependency approach to functions, tables, graphs and equations are of equal importance—none is said to be closer to the real function. There are two ways that CMP suggests that the three representations are of equal importance. The first is through sentences such as, "The students used tables, graphs and equations to look for patterns relating variables such as cost, income and profit." (Student Edition, p. 15) In these sentences, table, graph and equation are listed together. The second way that CMP communicates the equivalent value of the three representations is through series of

repeated questions, such as, "How does the walking rate affect the data in the table?", "How does the walking rate affect the graphs?" and "How does the walking rate affect the equations?" (p. 18). This repetition suggests that tables, graphs and equations are equivalent; they receive the same treatment.

These two patterns appear throughout the investigation. Two-thirds of the mentions of *equation* in Investigation 2 are of one of these two forms—in a sentence listing *table*, *graph* and *equation*[15]; or in a sentence that is one of a series mentioning each of *table*, *graph* and *equation.*

To summarize, the use of the word equation in Investigation 2 of *Moving Straight Ahead* has two important components:

1. *Equation* is associated with *relationship*, and
2. *Equation* has a status equivalent to that of *table* and *graph.*

Each of these components supports the relational-dependency view of functions in CMP's introduction to linearity.

Rate

There are three major patterns in the use of the word *rate* in Investigation 2 of *Moving Straight Ahead*: association with *walking*, the assertion that rates *affect* tables, graphs and equations, and the phrase *rate of change.* The term *rate* suggests a relationship between variables; miles per hour, dollars per mile and dollars per gallon are all rates. In each case, we tend to think of one variable (e.g. miles) changing in response

[15] Listing is different from simply being in the same sentence. I do not include "Write an equation for a pledge plan whose graph is less steep..." (p. 20) in the analysis because graph and equation are not equivalent items in a list.

to the other (hours). That is, the frequent use of *rate* supports the relationship pattern in the use of the word *equation*.

Each use of *rate* in the Student Edition of Investigation 2 is associated with walking. Most of these instances use the term *walking rate*; others are of a slightly different form as in, "each student walked at a different rate." (p. 18) Rates are also strongly associated with walking in the Teacher Edition; 63% of the mentions of *rate* are associated with walking. The rate in Problem 2.3 is not a walking rate, but a fund-raising rate: dollars per mile. CMP does not use the word *rate* in Problem 2.3. Instead, the text refers to the *pledge amount* and the *amount pledged per mile.*

The Student and the Teacher Editions suggest that rates *affect* the three algebraic representations. Forty-two percent of the mentions of rate in the Student Edition are of this form, while 34% of those in the Teacher Edition are. The idea is that greater rates imply greater increments, steeper lines and greater coefficients. All three of these assertions are true for non-negative rates and indeed these are the only rates students see in Investigation 2.[16]

The Teacher Edition is alone in using the construction *rate of change,* the generalized form of *walking rate.* The phrase appears four times, in three of the 32 sentences using *rate.* The use of the word *rate* is consistent with the use of *equation* in that each supports CMP's relational-dependency approach to functions.

Linear

Despite being the central idea of the unit[17], the word linear only appears in two places in the Student Edition's Investigation 2: the introductory text, and Problem 2.2. It

[16] Steepness of lines in fact depends on the *absolute value* of the rate.
[17] Recall that the unit's subtitle is *Linear Relationships.*

appears three times in each of these places. In nearly all cases, linear accompanies *relationship*, although in one instance it accompanies *situation* instead. In general, use of the word *linear* is in the context of testing whether a relationship is linear, as in, "How can you determine whether a situation is linear by examining a table of data or an equation?" (p. 16) and, "She wonders whether the data represent a linear relationship." (p. 19).

The use of *linear* is much more frequent in the Teacher Edition, where it is mentioned 21 times. The contrasting idea of non-linearity is directly introduced here as well. Teachers are instructed that, "To fully understand linear relationships, students must also see examples of linear and nonlinear situations." (p. 34d) In the Teacher Edition linear modifies *situation* more frequently than in the Student Edition. Additionally, it modifies *set of data* in the Teacher Edition. All three of these terms are used synonymously: relationship, situation and set of data (that is, the terms are equivalents) as in the following examples.

How can we use the table to prove that this is not a linear relationship? (p. 34e)

The follow-up raises the question of whether *every* situation is linear. (p. 34c)

How can we use a graph to determine whether a set of data is linear? (p. 34e)

This section has looked at the word use in CMP. Even at this micro level, the communication patterns are becoming clear: the prevalence of the terms *relationship* and *rate* point to the relational-dependency conception of functions in the curriculum. The use of these two terms highlights the fact that CMP is concerned with how one variable changes in response to another (*relationship*) and with the measurement and

characterization of that change (*rate*). Further, the second contention about CMP—that it uses contexts to develop mathematical meaning—can be seen in the discussion of rates. Rate is a central idea in the CMP discourse on linearity, and the word *rate* is nearly always associated with contextual language (especially *walking*). Thus, the mathematical conception of function as a dependency relationship between two variables is founded on students' classroom experiences with contexts. Similar discourse patterns will be featured in the next section, which looks at the discourse on a slightly larger scale. The next section analyzes the use of visual mediators—non-linguistic means of communication.

Visual Mediators

In the student materials, the three algebraic representations (tables, graphs and equations) are the primary visual mediators. Over the course of the investigation, students are shown three tables, two graphs and eight equations, while they are asked to make four tables, six graphs[18] and to write ten equations. One important point here is that students are more frequently asked to create the visual mediators than they are to consider pre-made visual mediators.

There are two aspects common to the use of all three of the algebraic representations: they represent contextual information and they reveal linearity (or non-linearity). The first aspect is broken into three: individual values, y-intercepts and rates. Together with revealing linearity, these are referred to as *themes* for CMP's use of visual mediators in this study. A theme describes how a visual mediator is used in discourse. It describes the purpose of the visual mediator and the kinds of information it can convey.

[18] Three of these tables and three of these graphs represent multiple relationships, so students make four tables to represent nine relationships and six graphs to represent 11 relationships.

A theme does not state *how* the mediator conveys this information, nor do themes address relationships among the visual mediators. These more complex messages are communicated by narratives, which are analyzed in the next section.

In this section, the importance of each theme to the discourse is measured using one of three descriptors: a *recurrent* theme is mentioned in more than one sentence in more than one lesson; a *passing* theme is mentioned only once in each of multiple lessons, or multiple times in a single lesson (or a combination of these); an *absent* theme is not in the discourse. Absent themes are identified by theory (e.g. if a theme appears in the use of tables and of graphs, the theory would suggest analyzing the corresponding theme for equations).

Each of the three main contexts in Investigation 2 is based on a linear relationship. The contextual information represented in tables, graphs and equations is of 3 kinds: individual values (such as the distance for a given time), relationships between values (such as the walking rate) and initial value (such as a head start). While the initial value is of course simply a particular individual value, it has a special status owing to the slope-intercept form of the equation, and so is analyzed on its own. The emphasis on the relationship between values is evidence of the relational-dependency approach to functions, as is the fact (to be substantiated in this section) that the three visual mediators receive nearly equivalent treatment in CMP.

The most important of these kinds of contextual information in Investigation 2 is the relationship between values. This is the most frequently recurrent theme and it is recurrent for all three representations. Students are repeatedly asked how the rate affects each representation, as described in the Word Use: Rate section earlier. Problem 2.2, for

instance, asks "How does the walking rate affect the data in the table?", "How does the walking rate affect the graphs?" and "How does the walking rate affect the equations?" (p. 18) This is reinforced by the teacher materials, which suggest discussion of the table for Problem 2.2 along these lines, "Ask the class to indicate where this [the increase of one second of time causing an increase of 2 meters in a walker's distance] is shown in the table." (p. 34c) The Teacher Edition also refers to graphs,

> It should be fairly easy for the class to recognize the effect that different walking rates have on a graph: the steepness of the lines are different. (p. 34d)

and to equations, "The rate, r, is the coefficient of t; it is the number by which the variable is multiplied." (ibid)

This pattern is repeated in Problem 2.3. The Student Edition asks the question, "What effect does increasing the amount pledged per mile have on the table? On the graph? On the equation?" (p. 20) The Teacher Edition then reinforces the importance of the idea by directing teachers to address it in the summary.

> The effects of the cost per mile on the representations are similar to the effect of the walking rate in Problem 2.2. Help students to see this similarity. (p. 34g)

Y-intercepts are a passing theme with respect to all three representations. In Problem 2.3, students are asked how the y-intercept[19] affects each representation.

> Alana suggested that each sponsor make a \$5 donation and then pledge 50¢ per mile. How is this fixed \$5 donation represented in the table? In the graph? In the equation? (Student Edition, p. 20).

[19] Note that the language is contextual, so the questions involve *donation* and *head starts*. The term y-intercept does not appear until the end of Investigation 3.

The problem asks one question of each representation, without reinforcement in the Teacher Edition. Similarly, the *y*-intercept arises again in Problem 2.5. In this problem, each representation is briefly mentioned. Graphs are mentioned in two sentences, "At what points do Emile's and Henri's graphs cross the y-axis? What do these points mean in terms of the race?" (Student Edition p. 22) Tables and equations are each mentioned once, "How can you predict where a graph will cross the *y*-axis from a table? From an equation?" (ibid). Because these themes appear, but are not mentioned more than once, in more than one lesson, they are passing themes.

For tables and graphs, the topic of finding individual values is a recurrent theme. In each of Problems 2.3 and 2.5, questions are asked about each representation. The following examples establish multiple mentions in Problem 2.3.

> On the graph of a pledge plan, the point (2,6) means that a student who walks 2 miles earns $6 from each sponsor. On which of the graphs is the point (2,6)? (Student Edition, p. 20)
>
> If a student walks 8 miles in the walkathon, how much would a sponsor owe under each pledge plan? Explain how you got your answer. (Student Edition, p. 20)
>
> For part C, discuss, or have students demonstrate, how to use each representation– the table, the graph, and the appropriate equation– to answer the question. (Teacher Edition, p. 34g)

The last example above refers to the preceding one. That is, the combination of the Student Edition question and the Teacher Edition instruction to teachers counts as a

mention of finding individual values in each of the three representations. Problem 2.5 asks twice about this topic with respect to tables and to graphs.

> How far from the starting line will Emile overtake Henri? Explain how you can use the table and the graph to answer this question. (Student Edition p. 22)
>
> After how many seconds will Emile overtake Henri? Explain how you can use the table and the graph to answer this question. (ibid)

As in these examples, these questions generally name a representation and ask students explicitly how to find the information in that representation. Through this kind of questioning, CMP indicates that these representations are useful for answering quantitative questions arising from the context. Finding individual values is a passing theme for equations.

Testing for linearity is a recurrent theme for all three representations. The introductory text cites having a straight-line graph as a criterion for linearity.

> The graphs of $C = 300 + 2n$ and $d = 60t$ are straight lines. From the graphs, it is easy to see that the relationships between the number of people and the rental cost and between the miles traveled and the time are *linear relationships*. (Student Edition, p. 16, emphasis in original.)

The same introductory text mentions tables and equations in the next sentence.

> In this investigation, you will consider this question: *How can you determine whether a situation is linear by examining a table of data or an equation?* (ibid)

Finally, the idea that each representation can be used to determine linearity is reiterated, establishing multiple mentions for these three themes in the introductory text.

> Once you have determined–from the table, the graph, or the equation– that a

relationship is linear, you can explore this question, *How does changing one of the quantities in a situation affect the table, the graph, or the equation?* (ibid)

In Problem 2.2, students are told, "Abby knows that if the relationship is linear, the data will lie on a straight line when graphed." They are then asked whether a table will generate a linear relationship.

> Use the table to determine how the distance changes as the time increases. How can you use this information to predict whether or not the data will lie on a straight line when graphed? (Student Edition, p. 19)

The Teacher Edition suggests that this question is important to ask about equations as well as about tables and graphs, but that it will be more difficult for students to answer in this investigation than the same question about the other two representations. After suggesting that teachers ask students, "How can we use an equation [to determine whether a relationship is linear]?" (p. 34e), the Teacher Edition states,

> Most students will not be able to answer this last question for now...ask that they continue to think about it as they work through the next few problems. (ibid)

Nonetheless, the message is that each of the representations can be used to determine whether a relationship is linear. Together the Teacher Edition and Student Edition mention these themes more than once in Problem 2.2, establishing that they are recurrent.

There are also relationships between pairs of the representations in CMP. An equation can be graphed, for instance. This particular example is discussed above, in the section titled *Word Use*, where it is noted that in some instances in CMP, equation is the primary object of study, while in most instances relationship is. A graph has a direct relationship with a table as well, ""How can you predict where a graph will cross the *y*-

axis from a table?" In this example, graph is the central object and a table can inform us about the graph. There is no direct relationship suggested between tables and equations in the student materials.

There are few other visual mediators in the CMP Student Edition. Ordered pairs are shown twice (but this vocabulary is not used) and there are several illustrations. The ordered pairs are referred to as *points* and students are asked about the meaning of these points in the walkathon context. The illustrations do not carry mathematical meaning[20], and so are outside the discourse on linearity, except for reinforcing the role of context.

The examination of the use of visual mediators in CMP has taken place at a slightly higher level than the examination of word use (i.e. it is a somewhat more macro-view of the discourse). That is to say, the unit of analysis is larger. In considering word use, we look at the sentence level, while when considering the of visual mediators, we look at the paragraph level. In addition, the word use analysis only attends to the word (for example) *equation*, while the analysis of the use of visual mediators attends to equations, even when the word is not in use. The same patterns emerge at this level. There is an emphasis on rate and on identifying the *relationship* between the variables in each representation. The *Word Use* section claimed that the treatment of the three representations in the introductory discourse is equivalent. This is compatible with the relational-dependency conception of functions, and it is borne out by the evidence in this section as well. Using the measures of *recurrent, passing* and *absent* we see that three of the four major themes (rates, *y*-intercepts and linearity) receive equivalent treatment in

[20] There is a picture of a bicycle when the mathematical relationships of a bike trip are discussed, cones of soft serve frozen yogurt in the problem about walking to the yogurt shop, etc.

each of the three representations. The only difference is a decreased emphasis on finding individual values in equations, which is a passing theme in contrast to being a recurrent theme for tables and for graphs. The analysis in this section also supports the claim that CMP has students use contexts to develop mathematical meaning, as all of the visual mediators in the text and all of the mediators that students are asked to create represent contexts while the themes have mathematical meaning (i.e. the representation of rates in equations is a mathematical idea, not a contextual one). Yet, this meaning is created through students' interactions with the contextual problems.

Narratives

The word use in the opening section and the themes in the use of visual mediators in the previous section support the contentions that 1. CMP has a relational-dependency approach to functions, and 2. CMP builds mathematical meaning on students' experiences with problem solving in contexts. The narratives identified in this section bolster these arguments. Recall that a theme in the use of a visual mediator states the uses of the visual mediator, but does not outline *how* to go about finding or representing information. Similarly, themes do not identify relationships among the visual mediators. These are important roles of narratives. A narrative states a relationship between objects of a discourse (e.g. between rates and equations), a property of an object or a procedure for operating on an object. In the CMP student text, narratives are rarely stated explicitly. Instead the curriculum focuses on asking questions that lead students to establish narratives themselves. This and other techniques for establishing the narratives in this section will form the focus of the next section, *Routines*.

Investigation 2 includes at least six important narratives related to linearity (listed in Table 3 below). This section elaborates on the mathematical significance of each narrative and supports the contention that each narrative is part of CMP's discourse on linearity using examples from the Student and the Teacher Editions. The analysis of narratives takes place at a higher level (i.e. gives a more macro view) than either the word use or the visual mediator analyses. At this level, the same patterns can be seen: CMP has a relational-dependency approach to functions in the introductory discourse on linearity and CMP uses contexts to build mathematical meaning.

Table 3: Narratives in Investigation 2 of the CMP unit *Moving Straight Ahead.*	
1	Mathematical variables represent contextual variables.
2	Tables, graphs and equations represent relationships between variables.
3	Tables with constant increments in the dependent variable (given a constant increment in the independent variable) represent linear relationships.
4	Straight-line graphs represent linear relationships.
5	Equations of form $y = mx + b$ or $y = b + mx$ represent linear relationships.
6	Rates are represented in tables by the constant increment of the dependent variable (given a constant unit increment in the independent variable).
7	Rates are represented by the steepness (slope) of a line on a graph.
8	Rates are represented by the coefficient of x in an equation representing a linear relationship.
8	*Y*-intercepts are represented in tables by the value of the dependent variable when the independent variable is zero.
9	*Y*-intercepts are represented in graphs by the point where the graph crosses the *y*-axis.
10	*Y*-intercepts are represented in equations by the constant term.
11	Analysis of tables, graphs and equations can answer contextual questions.

Mathematical Variables Represent Contextual Variables

This first narrative is a theme throughout the investigation, beginning with the introductory text excerpted below.

If we let C be the total cost to rent the bikes and n the number of people who go on the tour, we can write this equation to show the relationship between the

number of people on the tour and the total rental cost: $C = 300 + 2n$. (Student Edition, p. 15)

By *mathematical variable* (which is my own term), I mean the independent and dependent variables that are encoded in tables, graphs and equations. Sometimes, these variables retain their contextual names in the representation (as in a table with headings *Time* and *Distance*), sometimes they are abbreviations of the contextual names (as in the example cited just above) and sometimes they are abstractions such as x and y. The distinguishing feature of a mathematical variable is that it is used in an algebraic representation. A *contextual variable* (again, my own terminology) is a changing quantity around which a story is told. For instance, Emile's distance in the walking race of Problem 2.4 is a contextual variable that may be represented variously by the mathematical variables *Emile's distance*, E, d_{Emile}, y, etc.

It should already be clear that contextual variables abound in Investigation 2 of CMP. Mathematical variables also appear frequently. They are in the tables, graphs and equations that are shown to students, but they are also referred to in the text as in, "Let d represent the distance in meters and t the time in seconds." (Student Edition, p. 18) There are 24 direct references to mathematical variables in the student materials[21]. Twenty-one (88%) of these refer directly to contextual variables. CMP establishes through repetition that the variables used in tables, graph and equations represent changing contextual quantities.

[21] When two variables are used, as in $d = 60t$, only one reference is counted.

Tables, Graphs and Equations Represent Relationships between Variables

This second narrative is supported by combining two aspects of the use of the word *equation* discussed above. If equations have the same status (are used for the same purposes) as tables and graphs, and equations represent relationships, then tables, graphs and equations all represent relationships.

The next three narratives form the major focus of Investigation 2 of *Moving Straight Ahead*; they specify how linearity is represented in tables, in graphs and in equations. The introductory text to the investigation sets up these narratives by telling students how to identify linear relationships in graphs, then asking the question, "How can you determine whether a situation is linear by examining a table of data or an equation?" (Student Edition, p. 16). The three representations are taken in turn.

Tables with Constant Increments in the Dependent Variable (Given a Constant Increment in the Independent Variable) Represent Linear Relationships.

This narrative is explicitly stated in one of the Teacher Edition goals, "To recognize linear relationships from tables: for each unit change in one variable, there is a constant rate of change in the other variable." (Teacher Edition, p. 14f) The narrative is implied in the Student Edition. The introductory text to the investigation asks, "How can you determine whether a situation is linear by examining a table...?" (p. 16) The answer is implied through the introductory text to Problem 2.2, "Your findings will give you some important clues about how to identify linear relationships from tables, graph, and equations."(p. 18). Here, CMP suggests to students that the table they make will reveal characteristics of linear relationships. The curriculum then sets up the table for students, with the independent variable (time in seconds) increasing by a unit increment. This table

is reproduced in Figure 5. This assures that the dependent variable will have a constant increment. By examining three sets of data, students use inductive reasoning to see the narrative (see *Routines*, next section).

Figure 5: The model table in CMP Problem 2.2

Time (seconds)	Distance (meters)		
	Terry	Jade	Jerome
0	0	0	0
1	1	2	2.5
2			
3			
.			
.			
.			

Straight-Line Graphs Represent Linear Relationships

As with the preceding narrative, this is a Teacher Edition goal, "To determine whether a set of data is linear by examining its graph." (p. 14f) Note that this is not as explicit as the corresponding goal for tables, which mentions the criterion: constant increment. Nonetheless, the intention is clear. The criterion is a straight line. In the introductory text, the Student Edition also does not quite state the narrative.

> The graphs of $C = 300 + 20n$ and $d = 60t$ are straight lines. [Graphs are shown] From the graphs, it is easy to see that the relationship between the number of people and the rental cost and between the miles traveled and the time are *linear relationships*. (p. 16, emphasis in original)

At this point students are left to piece together that the fact of the graphs being straight lines implies the relationship being represented is linear. In Problem 2.2, however, the principle is stated directly, "Abby knows that if the relationship is linear, the data will lie on a straight line when graphed." (Student Edition, p. 19)

Equations of Form $y = mx + b$ or $y = b + mx$ Represent Linear Relationships

This is explicitly *not* a goal of this investigation. After suggesting that teachers ask students, "How can we use an equation [to determine whether a relationship is linear]?"(p. 34e) teachers are told, "Most students will not be able answer this last question now." Instead, this narrative is established implicitly through example. All eight of the equations shown to students in the investigation are of these forms (in two cases, $b = 0$). Additionally, this is the general form used in the Teacher Edition to refer to linear equations, as in the Teacher Edition goal, "To interpret the meaning of the coefficient of x and the y-intercept of a graph of $y = mx + b$." (p. 14f)

Note that $y = mx + b$ is intended here to represent a form where y represents the dependent variable, m the slope, x the independent variable and b the y-intercept. Thus, $d = 2t$ is of this form since d is the dependent variable, t the independent variable, 2 is the slope and 0 is the y-intercept. The equation $d - 2t = 0$, although equivalent, is not of this form. The use of $y = mx + b$ to refer to the slope-intercept form is not intended to suggest that only the variables y and x are used.

The next three narratives, like the preceding three, have a parallel structure. The narratives are alluded to in the Teacher Edition goal, "To recognize how the rate of change between two variables is associated with its representations," (p. 14f) and more directly by another Teacher Edition goal, "To recognize that a change in rate will change

the steepness of a line and the coefficient of x." (ibid) While tables are not mentioned in this Teacher Edition goal, the Student Edition indicates that they also represent rates. Thus, there are three parallel narratives here as well—one for each algebraic representation:

Rates Are Represented in Tables by the Constant Increment of the Dependent Variable (Given a Constant Unit Increment in the Independent Variable).

Rates Are Represented by the Steepness (Slope) of a Line on a Graph.

Rates Are Represented by the Coefficient of x *in an Equation Representing a Linear Relationship.*

These \narratives are established implicitly in the Student Edition through questioning. Students are repeatedly asked questions of this form, "How does the walking rate affect the table/graph/equation?" (e.g. in each of Problems 2.2 and 2.3). These questions also appear repeatedly in the Teacher Edition.

The point of these questions is to help students to notice that the contextual rate is represented in each of the representations. It is conceivable that a student making a table, for instance, simply sees it as a list of values. CMP intends for students to notice that the relationships between these values (e.g. between two consecutive distance entries in the table, or the vertical distance between points on the graph) are dependent on the rate in the relationship.

Next in the analysis is another set of parallel narratives. This set deals with the representation of y-intercepts in tables, graphs and equations.

Y-*Intercepts Are Represented in Tables by the Value of the Dependent Variable When the Independent Variable Is Zero.*

Y-*Intercepts Are Represented in Graphs by the Point Where the Graph Crosses the* Y-*Axis.*

Y-*Intercepts Are Represented in Equations by the Constant Term.*

These narratives are not stated directly in the Teacher Edition goals. The closest goal is, "To interpret the meaning of the coefficient of x and the y-intercept of a graph of $y = mx + b$." (p. 14f) The goal does imply that y-intercepts are represented in graphs and equations, but it neglects tables and the focus is on the meaning of the y-intercept, not on how it is represented. A great deal less time is spent on y-intercepts, in the investigation as a whole, in comparison to rates. Nonetheless, questions such as, "How is this fixed $5 donation represented in the table? In the graph? In the equation?" (Student Edition, p. 20) parallel the earlier cited questions about rates, and serve to point student attention to the concept of y-intercept and to its representation in tables, graphs and equations.

Analysis of Tables, Graphs and Equations Can Answer Contextual Questions

There is no reference to this in the Teacher Edition goals. However, there is a form of question in the Student Edition that points to this narrative. For example this question is asked after students have made tables and graphs and written equations in Problem 2.3, "For a sponsor to owe a student $10, how many miles would the student have to walk under each pledge plan?" (Student Edition, p. 20) In case students do not use one of the three representations to answer the question, there is this question,

On the graph of a pledge plan, the point (2,6) means that a student who walks two

miles earns $6 from each sponsor. On which of the graphs is the point (2,6)? (ibid)

Notice that "a student who walks 2 miles earns $6" is the form of answer to the earlier question about how many miles a student would have to walk to earn $10. With the question about the graph, CMP is endorsing the graph as a way to find answers to contextual questions. A question in Problem 2.5 similarly endorses the idea that any of the three representations can be used to answer a contextual question.

> Explain how you can use the table, the graph, and the equations to determine how far from the starting line each brother will be after 5 minutes. (Student Edition, p. 22)

It should be noted at this point that in the CMP discourse, there is a very close correspondence between the themes in the use of visual mediators and the narratives. This is due to the nature of the content. Functions are complex and abstract objects for which we require visual mediators in order to communicate effectively. Further, the introductory nature of the discourse necessitates narratives about the representations, rather than about the abstract objects. At some later point in the curriculum, functions might be discussed as abstract objects, but at the introductory level, the representations—being visible and directly accessible—are the focus.

Most of the narratives in CMP have one or more visual mediators as their subject. The classroom research, however, illuminates how the themes in the *Visual Mediators* section differ from the narratives. A theme in the use of tables, for instance, is that tables can be used to identify linear relationships. A narrative says how to do this. In a textbook, there is generally only a single narrative. In a classroom, there may be competing

narratives. In this study, a competing narrative is introduced in one classroom when a student asks, "Wouldn't she not be linear?" The student is referring to the table representing Alana's pledge plan, which has a non-zero *y*-intercept. This implicitly introduces the narrative *Linear relationships have zero y-intercept*. This competes with CMP's (and the teacher's) narrative, *Tables with constant increment represent linear relationships* in which the *y*-intercept is irrelevant. Competing narratives will be examined in more detail in the classroom research in Chapter VI. Textbooks do not generally introduce competing narratives, but the Teacher Editions of the two curricula in this study give teachers information about likely competing narratives that their students will establish.

The analysis of narratives in CMP is consistent with the findings at the more microscopic levels of word use and the use of visual mediators. There is a near equivalence in the narratives among tables, graphs and equations, as all six of the narratives deal with the three representations. The only exception is the warning to teachers that formalizing the form of a linear equation (in order to distinguish it from a non-linear equation) is not a goal of the investigation. This (near) equivalence of the three representations is also consistent with a relational-dependency approach to functions, as is the first narrative, *Mathematical variables represent contextual variables*. The variables take center stage in this narrative, which allows the curriculum to focus student attention on the relationship between the variables. This narrative also emphasizes the role of contexts in developing mathematical meaning. This process is examined in more detail in the next section.

Routines

Routines are the ways in which narratives are established and reinforced. Routines are rarely explicit; they need to be inferred from patterns in texts and from patterns in behavior when examining classrooms. In formal mathematical discourse, definition and proof are the major routines for establishing narratives. In less formal mathematical discourse, there are various forms of heuristics. In middle school classrooms and curriculum, definition sometimes plays a role, proof rarely. There are three main routines for establishing and reinforcing mathematical narratives in the CMP Student Edition: asking questions, inductive reasoning and finding mathematical meaning in a context.

Student Edition: Asking Questions

Many of the narratives in CMP are established through questions. Consider *Rates are represented in tables, graphs and equations* as an example. Nowhere is this directly stated in the Student Edition. Yet, this is clearly a theme throughout the investigation, and it arises from questions. Indeed, the only questions in the shaded box in Problem 2.2 are of this form, "How does the walking rate affect the table/graph/equation?" Through repeated questioning about relationships between the context and the representations, CMP indicates to students that rates are represented in tables, graphs and equations and that the only issue is figuring out how.

Other narratives established through questioning include *Y-intercepts are represented in tables, graphs and equations* ("How is this fixed $5 donation represented in the table?" (p. 20)) and *Analysis of tables, graphs and equations can answer contextual questions* ("At what point do Emile's and Henri's graphs cross the *y*-axis? What do these points mean in terms of the race? (p. 22)).

Student Edition: Inductive Reasoning

Some narratives are established through repeated examples; inductive reasoning. In order to establish that *Tables with constant increments represent linear relationships*, students are given the linear relationship between distance and time when walking at a constant pace in Problem 2.2. Students are asked to complete a table, begun in the text with a constant unit increment in the time column. In the problem's introductory text, CMP alerts students that, "Your findings will give you some important clues about how to identify linear relationships from tables, graphs, and equations." (p. 18) Students know before they make the table that doing so will result in noticing how linear relationships appear in tables. The essential element is that in a single table, students keep track of the distances for three walkers (see Figure 5, p. 97). While the distances are different for each walker, and thus the increments are as well, each walker has a constant increment from second to second and this increment is the same as the walker's walking rate. One example would not be enough to discern a pattern, so three examples are provided. Students are to use inductive reasoning to conclude that constant increment is a feature of the table of a linear relationship.

The role of non-example is important to the inductive reasoning. In the follow-up questions to Problem 2.2, students are shown a table with irregular increments and asked how to determine whether it represents a linear relationship.

The process is repeated in Problem 2.3, when students keep track of the pledge plans for three walkers in a walkathon. In this case, an anomaly is introduced: a walker who asks for an initial donation. All of the preceding relationships have initial values of 0; the walkers in Problem 2.2 start together at 0 meters and the other two walkers in the

walkathon in Problem 2.3 each raise a set amount per mile. The initial donation (corresponding to $5 raised for 0 miles walked) sets up the study of *y*-intercepts. Without this anomaly students might draw the conclusion that linear relationships must have graphs that go through the origin (i.e. have *y*-intercepts of 0)[22].

Student Edition: Deriving Mathematical Meaning from a Context

This last example points to the third routine in the Student Edition: finding mathematical meaning in a context. I have written the narratives in CMP in mathematical language. One need not know the curriculum to understand the terms of these narratives. Further, I have argued that these narratives *as written*[23] are goals of Investigation 2 of *Moving Straight Ahead.* Yet, much of the writing in Investigation 2 uses contextual language such as *miles*, *walking rates*, *pledge plans* and the names of the various walkers. It must be that students are expected to use contextual language to develop understanding of the mathematical narratives. *Head start* and *initial donation* are intended to give meaning to *y*-intercepts, for instance and walking rates stand in for the more general concept of rate. This study makes no claims about the effectiveness of this routine. That is, it is beyond the scope of this study to claim that the expectation described above is met. Instead, this is an analysis of the model (i.e. intended) discourse. In this model discourse, students and teachers use contextual language to develop mathematical ideas.

Teacher Edition: Asking Questions

[22] Indeed the classroom example cited earlier illustrates this. A student asked, "Wouldn't she not be linear, Alana?" and the teacher responded, "It doesn't matter what it starts at. It just matters that it goes up by the same amount every time. So hers is linear."

[23] With the one exception of the vocabulary item y-*intercept*. Students are not expected to learn this term until Investigation 3.

The Teacher Edition supports these three routines. Teachers are often told to ask students questions in order to help them understand something, as in the following passage.

> Ask questions that help students notice that for each unit change in time, there is a constant change in the distance walked by Terry, Jade, and Jerome. (Teacher Edition, p. 34c)

In Investigation 2, teachers are told seven times to ask questions of students, either to help point out an important relationship or to help clarify something with which students may struggle. Sample questions are provided for these purposes. The LES section for Investigation 2 includes 47 questions for teachers to ask students.

It is worth parsing the phrase, "Ask questions to help students notice," for this phrase differentiates the routine of asking questions. The CMP Student Edition rarely states narratives explicitly. Instead, the text is full of questions for students to consider. The Teacher Edition likewise contains far more directions for teachers to ask questions than to tell students things. The Initiation-Response-Evaluation pattern of teacher questioning in classrooms is well documented (e.g. Cazden, 1988) and can be characterized as a form of teacher telling. In this case, teacher questions depend on the ensuing evaluation of student responses to communicate meaning, as in:

> Teacher: What is 2+2?
>
> Student: 4
>
> Teacher: Very good.

In this simplistic scenario, the question does not establish the narrative, *2+2=4*, the teacher response does. When the Teacher Edition implores teachers to "Ask questions

that help students notice," the indication is that the questions themselves (together, presumably with the ensuing student thinking) will produce the desired narrative (here that a constant increment in a table is the hallmark of a linear relationship).

Teacher Edition: Inductive Reasoning

In contrast to the Student Edition, the Teacher Edition explicitly advocates the consideration of examples and non-examples, in the ways described in the above discussion of induction.

> To fully understand linear relationships, students must also see examples of linear and nonlinear situations. The follow-up questions pose a situation that is not linear. It is important to discuss this example in class and compare it to linear situations. (p. 34d)

Teacher Edition: Deriving Mathematical Meaning from a Context

Finally, the Teacher Edition includes an introductory paragraph for the Launch-Explore-Summary text of each problem. The following, from Problem 2.2 is typical.

> In this reprise of Problem 2.1, students examine the effect of different rates on graphs, tables, and equations, and consider the question of linearity more directly. (p. 34b)

As in the example, these paragraphs state the mathematical subject of the problem without using contextual language. Similarly, there is a set of Mathematical and Problem-Solving Goals for the investigation that states mathematical goals in non-contextual language. In these ways, the Teacher Edition communicates to teachers that students are to construct mathematical meaning from the contexts in the investigation.

The text is rarely explicit about precisely how students are to derive mathematical meaning from the contexts. Three (of perhaps many) possibilities are:

1. Mathematical vocabulary is defined in contextual terms.
2. A mathematical question is asked (by the teacher or by the student) and the teacher models or encourages seeking the answer by thinking about the context.
3. The mathematical language is developed as a generalization of the specific contextual ideas.[24]

An example of the first might be, "The *y*-intercept in a linear relationship is the walker's head start." In an example of the second possibility, a student might ask, "How do I find the *y*-intercept in an equation?" and the teacher might respond, "Think of this as an equation for a walker; where would you find his head start?" In the third possibility, a teacher might say, "We have seen two examples of *y*-intercepts; head starts and donations. In each case, we had to add something to the *x* term in the equation. In general, this added term is called the *y*-intercept."

The student text focuses almost exclusively on the contexts, and so does not give indications about deriving mathematical generalities. The Teacher Edition has no examples of the first two possibilities and only hints of the third in Investigation 2. In Problem 2.2, there is a follow-up question in the Student Edition that presents a table with nonlinear data. The Teacher Edition suggests the following,

> If none of the students uses the data directly from the table to argue that they do not represent a straight line, ask: *How can we use the* table *to prove that this is not*

[24] Of course, these are conjectural.

a linear relationship?(As the time increases 1 unit, there is not constant increase in the distance. From 0 seconds to1 second, the distance increases 2 meters; from 2 to 3 seconds, the distance increases 4 meters.)

Help the class to generalize what they have discovered.

How can we use a graph to determine whether a set of data is linear?... How can we use a table?.(There will be a constant rate of change between the variables.)

How can we use an equation? (p. 34e)

Teachers are instructed that students are to generalize tests for linearity. The first time testing for linearity is raised, teachers are told to ask about "*this* relationship [emphasis added]". The second time, they are told to ask about "*a* set of data [emphasis added]". The answer the first time is in contextual language: *time* and *distance*. The answer the second time is in mathematical language: *rate of change* between variables.

Another example of moving from the specific (contextual language) to the general (mathematical language) occurs in the summary for Problem 2.5.

It is fundamentally important that students understand that this [the ordered pair (20,50)] is a pair of values from the table for Emile that shows how many meters Emile walks in 20 seconds. From this idea, students can build the understanding that *every pair of values in the table is a point on the graph* and that *every pair of values in the table and on the graph is a pair of values that fits the equation.*

(Teacher Edition, p. 34k, emphasis in original)

It is reasonable to interpret, "the table", "the graph" and "the equation" to mean tables, graphs and equations in general, not just those describing Emile's distance. Yet, the text is indirect about this, in contrast with the preceding example. Later investigations

are more frequently explicit about deriving mathematical meaning from contexts, but at the introductory level that is the focus of this study, the process is implicit in the Student Edition and infrequently discussed in the Teacher Edition.

Three routines for establishing and reinforcing narratives have been identified in CMP: *asking questions*, *inductive reasoning* and *deriving mathematical meaning from a context*. These routines align poorly with formal mathematical discourse in which exposition, proof and definition are central. The CMP routines, however, do align with a heuristic mathematical discourse—the kind of mathematical discourse that Lampert (1990) and Lakatos (1976) argue is essential to the work of mathematicians.

To summarize, in CMP, the study of linearity is a study of relationships. The more abstract idea of algebraic relationships is based in situations with contextual meaning. There are multiple ways to represent these relationships whether they are being considered in context or abstractly. This focus on relationship is at the heart of the relational-dependency approach to functions.

The discourse on linearity is grounded in contexts. The use of the words *equation* and *rate* is heavily associated with the storylines in the problems. *Equation* is associated with relationships between contextual variables. *Rate* is almost exclusively associated with *walking*. All three visual mediators are used to represent and to find contextual information of various kinds. Most of the mathematical narratives have contextual equivalents within the investigation. The narrative, *Straight-line graphs represent linear relationships* could be rephrased in contextual language as *Straight-line graphs represent walkers with a steady pace*, which is the focus of several problems and a follow-up question. This grounding of classroom mathematical discourse in contexts is not the only

form for current textbooks, and the relational-dependency approach to functions is not the only approach. To highlight these aspects of the CMP discourse, the analysis turns to a curriculum with a different approach to classroom mathematical discourse, the Dolciani text.

Dolciani

The introductory discourse on linearity in *Connected Mathematics* is only one of several possibilities for such a discourse. To illustrate this point, and to highlight aspects of the CMP discourse, I now turn to *Mathematics, Structure and Method*, (Dolciani, 1992). All four components of Dolciani's mathematical discourse prove to be substantially different from those of CMP. Each of the words in the word use analysis is used differently. The patterns in visual mediator use vary greatly among the three representations of functions. The narratives put tables and graphs in service of equations, rather than treating each of these three in parallel. The routines are more closely aligned with those of formal mathematical discourse than the heuristic discourse described in the section on CMP.

In Dolciani, the introductory lessons on linearity are in Chapter 8, titled *The Coordinate Plane.* Chapter 8 of Dolciani consists of six lessons, numbered 8-1 through 8-6. The chapter's focus is on graphing the equations and inequalities that are the subject of study earlier in the text. Lesson 8-1 is omitted from this study because it deals with content that is not particular to the study of linear relationships: plotting points on the coordinate plane. This is assumed knowledge in the CMP lessons under study and the analysis of Dolciani picks up in Lesson 8-2, where point plotting is also an assumed skill.

Lesson 8-2 (titled, *Equations in Two Variables*) introduces equations in two variables. Students are not told this until the next lesson, but all of the equations in this lesson are linear equations in standard form ($ax + by = c$). Students are told that solutions to these equations are ordered pairs and that there are infinitely many solutions to an equation in two variables.[25] There are three examples. The first is about determining whether a given ordered pair is a solution to a given equation. This is done by substitution. The second example is about finding a solution (an ordered pair) to a given equation. Again, this is done by substitution. The third example is about finding y-values for given x-values for a given equation. The class exercises are of three forms: determining whether a given ordered pair is a solution to a given equation, solving a given equation for y and determining y-values for a given equation for given x-values.

Lesson 8-3 (titled, *Graphing Equations in the Coordinate Plane*) introduces linear equations in two variables. All of the equations are given in standard form. Tables of values are used to determine a set of solutions for an equation. These solutions are graphed and the text notes that the solutions to linear equations form straight lines in the plane. As a result, two solutions are needed to draw the graph of an equation, but students are told to include a third point to check their work. There is one example in this lesson, concerning graphing a given linear equation in standard form. The class exercises have students find three solutions for each given equation, including the x-intercept and the y-intercept for each.

Lesson 8-4 (*Graphing a System of Equations*) introduces the term *system of equations* and has students find the point of intersection of the two graphs of the

[25] In fact, the statement is made about a particular equation, but it is true for any of the class of equations for which it is an example: linear equations.

equations of the system. There are three examples. In the first example, the solution to a system is first approximated from the graph, then checked in the equations. In the second example, the lines are parallel. This justifies the observation that there is no solution to this system. In the third example, the two equations in the system are equivalent and so the system has infinitely many solutions. These three examples are summarized at the end of the lesson:

> If the graphs intersect, the system has one solution. The coordinates of the point of intersection form the solution.
>
> If the graphs are parallel, the system has no solution.
>
> If the graphs coincide, the system has infinitely many solutions. (Student Edition, p. 286)

The class exercises have students determine whether a given ordered pair solves a given equation and restate several of the salient points of the lesson.

Lesson 8-5 (*Problem Solving: Using Graphs*) introduces the topic of slope. Slope is defined as,

> [T]he ratio of the change in the y-coordinate to the change in the x-coordinate when moving from one point on the line to another. (Student Edition, p. 288)

There are two examples, both contextual. In the first example, temperatures are changing at a constant rate from a given temperature at a given time. The example is about extrapolating and interpolating temperatures at other times. In the second example Maryanne is burning calories while she stretches and then at a given rate while she runs. The example is about using this information to determine how long she must run in order to burn a given number of calories. An equation is written, a line plotted and the solution

is read from the graph. One set of class exercises has students find the slopes of lines that are already plotted, The other set asks students to plot points, connect the points with a line, find other points on the line and to determine the slope of the line.

Lesson 8-6 (*Graphing Inequalities*) considers the graph of a linear equation as the boundary line between two regions, each defined by an inequality. Inequalities are defined as relations, and linear equations as belonging to a special class of relations: functions. There are two examples; each is about graphing an inequality by determining the boundary line and then one point that satisfies the inequality in order to know which side of the line to shade (the shading indicates the solutions to the inequality). There are two sets of class exercises. The first has students rewrite a given inequality in $y > mx + b$ or $y < mx + b$ form and to state the equation of the boundary line in slope-intercept form. The second set has students state whether given points satisfy given inequalities.

In the first half of this chapter, the CMP approach to functions is characterized as a relational-dependency approach in which variables, relationships between variables and equivalence of algebraic representations are in the foreground. CMP indicates that a function is a dependency relationship and does not offer a formal definition. The Dolciani approach is different. Dolciani's approach to functions is *set-theoretic*. In this approach, functions are treated according to a more formal definition, such as, "a set of ordered pairs such that no two ordered pairs with the same first coordinate have different second coordinates." Here, the relationship between the variables is de-emphasized in favor of careful attention to the correspondence between coordinates (i.e. members of the ordered pairs). In both approaches, linear functions are described by equations. The focus on equations in the relational-dependency approach is on how rates (i.e. change) are shown

in the equation. The focus on equations in a set-theoretic approach is on finding one coordinate of the pair when the other is given, that is, on solving equations.

This section examines the Dolciani textbook discourse on linearity. One focus of the analysis will be supporting the contention that Dolciani has a set-theoretic approach to functions. The same structure as the analysis of CMP's discourse is followed here; word use, visual mediators, narratives and routines.

Word Use

Equation

In CMP, the use of equation has two important aspects: *equation* is used regularly with *relationship* and many of its uses are identical to the uses of the words *table* and *graph*. These aspects are both absent from Dolciani. Instead, the mentions of *equation* in Dolciani almost always coincide with the term *solution*.

Relationship. The Dolciani student materials do not use *equation* in the same sentence with *relationship*. The closest the materials come to doing so is quoted below.

> Write an equation that relates the total number of Calories (y) that she uses when she stretches and goes for a run to the number of minutes (x) that she spends running. (p.288)

There is a subtle distinction here. While CMP frequently uses *relationship*, a noun, Dolciani uses *relate*, a verb. An equation, in CMP, represents a relationship between two variables. In Dolciani, an equation relates two variables. In CMP, there is a sense that the relationship is primary and that the equation serves the relationship. In Dolciani, this is reversed. The equation is primary and the relationship secondary. The CMP sense of *relationship* appears once in the Dolciani teacher materials, "When graphs and equations

of lines are used to represent the relationship between real quantities, the numerical constants have actual meanings." (p. 271e) The implication in the word *when* is that this representation is not the primary use of equations. This implication is substantiated by the fact that this is the only use of *relationship* that parallels CMP's use.

Once, Dolciani uses the mathematical term *relation* together with *equation*, "The equation $y = x + 1$ is a special kind of relation because it defines a function." (p. 295) But here, *relation* has a technical meaning unrelated to the meaning of *relationship* intended in CMP[26].

Table and Graph. A common pattern in CMP is to list *equation* as one of three representations, with each of the representations having the same usage, or alternatively to ask the same question three times; once about each of the representations. This never happens in the lessons on linearity in Dolciani.

There are three sentences in these lessons (all in the Student Edition) that use all three representations. Each is of this form, "Make a table of values for each equation and graph both equations on one set of axes." (p. 285). In each case, the equation is given and the table is created to generate ordered pairs for graphing. In these lessons, students do not begin with one of the other representations. Tables are not analyzed. Thus, in contrast with CMP, equations have a different status from tables and graphs throughout the introductory lessons on linearity in Dolciani. Equations, unlike the other two representations, are the starting point for mathematical analysis. This is consistent with the set-theoretic approach to functions of Dolciani.

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[26] Recall the inference that the CMP meaning of relationship is close to *function*, but implies a predictability that is not inherent in the mathematical definition of *function*. A relation in Dolciani follows the formal mathematical definition: any set of ordered pairs whatsoever.

Solutions

Where CMP uses the word *equation* as one of three representations of relationships between variables, Dolciani uses *equation* in association with *solution*. At first glance, 51% of the sentences in the student materials that use *equation* also use either *solve* or *solution*[27]. Dolciani says that solutions *satisfy* equations. If we include this term (which is equivalent to *is a solution of*), two-thirds (34 out of 51) of the sentences using *equation* are about solutions.

A solution to an equation is an ordered pair that, when substituted into the equation, makes the equation true. *Solution* is defined in the first chapter—well before the lessons in this study—in this way,

> When a value of the variable makes an open sentence a true statement, we say that the value is a **solution** of, or **satisfies**, the sentence. (p. 9, emphasis in the original).

In an earlier lesson, an equation is defined to be a kind of sentence; one which uses an equal sign to relate two expressions.

Together, these three features of the use of *equation* in the introductory linearity discourses of CMP and of Dolciani highlight major differences. In Dolciani, equations are generally given and their fundamental purpose is to be solved. This is consistent with a set-theoretic approach to functions. In CMP's relational-dependency approach, equations are to be derived from contextual relationships and their fundamental use is to represent these relationships. In Dolciani, the emphasis is not on representation, nor on

[27] The proportion is smaller in the teacher materials: 43%, but this is still the major theme.

rates, but on correspondence. When we solve an equation, we find the x-value that corresponds to the given y-value (or vice versa).

Rate

Rate appears twice in the Dolciani Student Edition; three times in the Teacher Edition. This stands in contrast to the twelve mentions in the CMP Student Edition and the 32 mentions in the CMP Teacher Edition. In CMP, there were three patterns in the use of the word *rate*: *walking rates*, the effect rates have on the three algebraic representations, and the use of the phrase *rate of change.* As with the use of the word *equation*, the use of *rate* differs markedly in Dolciani; none of the CMP patterns is present.

The three Student Edition mentions of *rate* are in Lesson 8-5, titled, "Using Graphs to Solve Problems." This is the one lesson among those in this study to focus on contextual problems. The word *rate* appears in the explanatory text (cited below) and then twice in an example.

> A straight-line graph sometimes expresses the relationship between two physical quantities. For example, such a graph can represent the conditions of temperature falling at a constant rate or of a hiker walking at a steady pace. (p. 288)

All three Student Edition citations are about the falling or rising of temperatures. In a sense, the use of *rate* here is quite similar to that in CMP. While the context is different, every use of *rate* in the Dolciani student materials is based in a context, just as it was in CMP. In Dolciani, the context is changing temperature; in CMP the context is a walking race. Yet, the first sentence of this excerpted text states an important difference between the discourses of the two curricula. The discourse on linearity in CMP is

founded on the contexts. Dolciani, however, builds a formal[28] discourse on linearity and then tells students that one aspect of this discourse (a visual mediator—a straight line graph) "*sometimes* expresses the relationship between two physical quantities." (p. 288, emphasis added) As with the Teacher Edition's use of *when* to suggest that equations are not fundamentally about describing contexts, the use of *sometimes* here reinforces the notion that contextual relationships are not the primary motivation for the study of linear equations.

The CMP Teacher Edition uses the phrase *rate of change* several times. I have argued that this is a generalization of the contextual rates in the Student Edition, and that this generalization is not present in the student materials. Dolciani operates in a similar fashion, i.e. contextual uses of *rate* in the Student Edition and generalized uses in the Teacher Edition. The Dolciani Teacher Edition uses the word *rate* three times in the Lesson Commentaries for the introductory lessons on linearity, all three in the same paragraph, part of which is reproduced below.

> On the other hand, the 10 is a rate, specifically the rate of energy consumption per unit of time. The quantity $10x$, the energy used in running only, is a variable amount, depending on the number of minutes. Note that the rate is also the slope of the line. In general, slopes are rates. (p. 271e)

The first mention of *rate* here is contextual: rate of energy consumption. But the other two make a more abstract connection: slopes and rates are equivalent. In the introductory lessons on linear relationships, CMP does not explicitly state this. Indeed the

[28] In the sense of *formalist*—the view that mathematics is a game of symbols and rules, driven by logic and that any correspondence with the world of observation is a side benefit.

word *slope* does not appear in the student materials until Investigation 5, and it does not appear with *rate* in the teacher materials in Investigation 2. In the end, *rate* is rarely mentioned in the Dolciani discourse on linearity, consistent with the set-theoretic approach to functions.

Linear

The dominant feature of the use of the word *linear* in CMP is its use to modify *relationship*. The major tasks for students are: to determine whether a given relationship is linear and to isolate the features of a relationship that help to determine this. As with the other two words in this analysis, Dolciani's use of *linear* is quite different.

Every use of *linear* in Chapter 8 of Dolciani modifies *equation*[29] as in the following passage.

> In order to graph a linear equation, we need to graph only two points whose coordinates satisfy the equation and then join them by means of a line. (Student Edition, p. 281)

In the sections on the use of the word *equation*, I argued that one distinction between the introductory discourses on linearity in these two curricula is the central object of study; in CMP this is *relationship*, while in Dolciani it is *equation*. The use of the word *linear* supports this distinction. In CMP, relationships are linear. In Dolciani, equations are linear.

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[29] With one exception: in the teacher materials, there is the suggestion that *linear programming* is a challenging topic for "abler students" (p. 271f). Curiously, an example of a linear programming problem is worked out, but a definition of linear programming is not stated.

Where CMP has students spend a good deal of time establishing rules for determining whether a relationship is linear, Dolciani gives a definition based on the form of the equation, as in this passage,

> In general, any equation that can be written in the form $ax + by = c$ where x and y are variables and a, b, and c are numbers (with a and b not both zero), is called a **linear equation in two variables** because its graph is always a straight line in the plane. (pp. 280-1)

Note that the definition of *linear equation* rests on the form of the equation, while the shape of the graph justifies its name. CMP does not state a formal definition in the Student Edition. Instead, having a straight-line graph is informally stated as a criterion for being a linear relationship. Again, the contrast between the two approaches to algebra is clear.

Following this definition (above), the other uses of linear in the Dolciani student materials involve graphing linear equations. The teacher materials add the idea of non-linearity to the discourse, "To strengthen students' understanding of linear equations, you might point out that not all equations are linear." (p. 271d).

As with CMP, the microscopic view of the Dolciani discourse afforded by the close examination of word use reveals a distinctive approach to functions. *Equation* is associated with solving—that is, with completing the correspondence between x and y at a particular point, rather than on the dynamic relationship between the two variables. *Rate* is nearly absent in the teacher and student materials. This too points to the de-emphasis on the relationship between the variables. Finally, *linear* modifies *equation* in Dolciani, rather than modifying *relationship*, pointing to the increased emphasis on

equations over the other representations, which is to be expected in a set-theoretic approach to functions. This differential emphasis on the representations will be seen in more detail in the next section.

Visual Mediators

As in CMP, tables, graphs and equations are important visual mediators in Dolciani. Additionally, Dolciani includes ordered pairs as a frequently used visual mediator. In contrast with CMP, though, students are rarely asked to generate the representations. Students are shown 10 tables, 19 graphs and 39 equations[30] in the classroom portion of the Dolciani Lessons in this study[31]. Students are asked to make five graphs, but are not asked to make tables, nor to write equations[32]. That is, students are shown all three of the algebraic representations, but in class are asked only to generate one: graphs. This observation supports the contention in the word use section that equations are generally given, rather than derived from contexts, in Dolciani. Of course, the much greater frequency of equations in the student materials supports the contention that equations are the central objects of study in Dolciani's discourse on linearity.

This section first examines the themes in the use of visual mediators previously identified in CMP and then highlights themes unique to Dolciani. The analysis begins with the representation of rate in tables, graphs and equations.

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[30] The figure for equations does not include equivalent equations generated in the process of solving equations.

[31] Recall that this compares to three tables, two graphs and eight equations in CMP.

[32] This is partly, but not entirely, an artifact of this study's analysis method. A Dolciani lesson has a different structure from a CMP lesson, so it is plausible that these numbers would look quite different if we include Dolciani's written exercises, which are intended as homework (as are the ACE questions in CMP). Indeed, a student who did all of the written exercises would make 117 graphs and write 14 equations. Such a student would not be asked to make a table.

In contrast to their use in CMP, the tables in Dolciani are used to keep orderly lists of solutions to equations. These solutions are often used to create a graph, as in Lesson 8-3.

> The **table of values** ... lists some of the solutions. When we graph the solutions on a coordinate plane, we find that they all lie on a straight line. (Student Edition, p. 280)

Tables themselves are not analyzed for patterns of change as they are in CMP, and they are not discussed as a representation. Instead, they are intermediaries between equations and graphs. Because tables are used to keep track of solutions, rather than to represent change, there is no reference to increments in tables, and in fact the solutions are not in any consistent order within the tables. Not surprisingly, then, the theme of the representation of rates in tables is absent in Dolciani. The corresponding themes for graphs and equations are passing, being mentioned only in Lesson 8-5, the one lesson that focuses on the real-world applications of linear relationships.

Consistent with the set-theoretic approach of Dolciani, the identification of individual values is a recurrent theme for each of tables, graphs and equations. Tables are organized lists of individual values and appear multiple times in each of Lessons 8-3 and 8-4. Each of these tables corresponds to a graph, on which are labeled the values from the table. In addition, graphs are used to solve systems of linear equations. In this case, the individual value is the point of intersection of the two lines on the graph. An estimate is made from the graph and then the value is checked in the corresponding equations. Lessons 8-2 through 8-4 each have examples where multiple equations are solved (generating individual values).

As in CMP, the themes of the representation of *y*-intercepts in tables, graphs and equations are less emphasized than most of the others. The theme is absent for tables and passing for graphs and equations. The *y*-intercept is defined and discussed in terms of the graph in Lesson 8-3, "The *y*-coordinate of a point where a graph crosses the *y*-axis is called the ***y*-intercept** of the graph." (Student Edition, p. 280, emphasis in original) Equations are used in this lesson to find *y*-intercepts in order to make a graph. The one other place that *y*-intercepts appear, although not by name, is in Lesson 8-5, "We can see that the line crosses the vertical axis at –1." (p. 289)

The final theme identified in CMP's use of visual mediators is the identification of linearity in each of the three representations. Consistent with the use of tables discussed in relation to rates, linearity is an absent theme in Dolciani's use of tables. Linearity, in Dolciani, pertains mainly to graphs. There are multiple mentions of the *straight-line graphs* and *connecting points by means of a line* in Lessons 8-3 and 8-5, so the theme is recurrent. With respect to equations, the only mention of linearity is in the definition of a *linear equation in two variables* in Lesson 8-3. This theme is passing.

The treatment of these themes in the two curricula is summarized in Tables 6–8. From these tables, it can be seen that the two curricula have similar emphases on the themes with respect to graphs, but vary greatly on their treatment of tables and of equations.

Table 4: Themes in each discourse on tables.

	Rate	Individual Value	*Y*-intercept	Linearity
CMP	recurrent	recurrent	passing	recurrent
Dolciani	absent	recurrent	absent	absent

Table 5: Themes in each discourse on graphs.

	Rate	Individual Value	*Y*-intercept	Linearity
CMP	recurrent	recurrent	passing	recurrent
Dolciani	passing	recurrent	passing	recurrent

Table 6: Themes in each discourse on equations

	Rate	Individual Value	*Y*-intercept	Linearity
CMP	recurrent	passing	passing	recurrent
Dolciani	passing	recurrent	passing	passing

I have made the argument several times that equations are the central objects of study in Dolciani's introductory discourse on linear relationships. A further piece of evidence is that four of the five lessons in this study begin with an equation, as in Lesson 8-2, "The equation $x + y = 5$ has two variables, x and y." (p. 276)

A fourth major visual mediator in Dolciani is ordered pairs. Over the course of the five lessons, students are shown 89 ordered pairs[33] of the form (x,y) where x and y are usually integers, but may be rational numbers or variables. In general, these ordered pairs are solutions to given equations, reinforcing again that equations are the central objects of discourse in Dolciani. The use of ordered pairs to display solutions to equations is a recurrent theme in Dolciani.

There is one last visual mediator in these lessons. The diagram reproduced in Figure 6 shows a hill with varying steepness. The diagram is included to give students an example for thinking about steepness, and then slope.

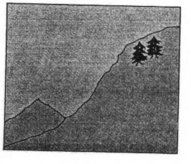


Figure 6: A visual mediator in Dolciani

To summarize, the use of visual mediators in Dolciani reinforces the set-theoretic approach to functions in several ways. First, the three algebraic representations of table,

[33] Recall that this compares to two ordered pairs in CMP. The figure for Dolciani does not include the ordered pairs that label points on graphs.

graph and equation receive differential treatment, with much more emphasis on equations than the other two. Second, the theme of identifying and representing rates in the representations is at most passing, being mentioned in one lesson. Third, the abundance of ordered pairs is reminiscent of the formal definition of function in which a function is defined as being a set of ordered pairs with particular properties. Similar patterns emerge at a higher level in the next section, when the narratives in Dolciani are considered.

Narratives

In contrast with CMP, Dolciani's narratives are more explicitly stated in the student materials (see Table 9). CMP tends to point to true mathematical statements through repeated questioning, while Dolciani tends to do so through definition. As an example of this second characteristic, consider the case of the first of these narratives, *A linear equation is of the form* $ax + by = c$.

Table 7: Narratives in the study of linear relationships in Lessons 8-2 through 8-6 of Dolciani. Italics indicate narratives that are verbatim from the text.	
1	A linear equation is an equation of the form $ax + by = c$.
2	*A solution to a linear equation is an ordered pair.*
3.	Solutions are found by substituting values into an equation.
4	A table lists solutions to an equation.
5	*A graph is the set of all points that are graphs of solutions to an equation.*
6	A straight-line graph can represent a relationship between contextual variables.
7	A straight-line graph represents a constant rate.
8	A linear equation has a straight-line graph.
9	Slope is the ratio of the change in x-coordinates to the change in y-coordinates on a graph.

A Linear Equation is of the Form $\boldsymbol{ax + by = c}$

Recall that in CMP, we have to infer the form of a linear equation in the introductory lessons on linearity. While this form is made explicit later in the unit, the introductory lessons suggest that linear equations are of the form $y = mx + b$ or $y = b + mx$ through examples and the emphasis on how rate and y-intercept are represented in the equation. Dolciani is more direct.

> In general, any equation that can be written in the form $ax + by = c$ where x and y are variables and a, b, and c are numbers (with a and b not both zero), is called a **linear equation in two variables** because its graph is always a straight line in the plane. (pp. 280–1, emphasis in original)

This is the first mention of the term *linear*, and the definition is explicit.

A Solution to a Linear Equation Is an Ordered Pair

This narrative is stated explicitly in the Dolciani Student Edition, in the paragraph cited below.

> The equation $x + y = 5$ has two variables. A solution to this equation consists of two numbers, one for each variable. The solution can be expressed as an ordered pair of numbers (x,y). (p. 276)

Once this fact is established, Dolciani relies on it throughout the remaining lessons on linearity, asking students to determine whether given ordered pairs are solutions to given equations (or systems of equations) in Lessons 8-2, 8-3 and 8-4. Ordered pairs are discussed as solutions in Lessons 8-5 and 8-6 as well.

Solutions Are Found by Substituting Values into an Equation

Lesson 8-2 is titled *Equations in Two Variables.* The Objective stated in the Teacher Edition is, "To find solutions for an equation in two variables." (p. 271b) The introductory paragraph in the Student Edition establishes that a solution of a linear equation is an ordered pair. These facts establish that this lesson is about solving equations. There are three examples worked out in the text. The central aspect of each of these examples is a substitution. There is one paragraph between Example 2 and Example 3, cited below.

> To find the value of y corresponding to any given value of x, we could substitute the value of x into a given equation and solve for y, as in Example 2. An easier method is to solve for y in terms of x first, and then substitute, as in Example 3. (Student Edition, p. 277)

The essential message here is that substitution is the technique necessary to find solutions. The implication in this paragraph is that the only issue worth debating is at what stage in the problem-solving process to make a substitution.

A Table Lists Solutions to an Equation

As discussed above, under the heading Visual Mediators, tables are used differently in Dolciani than they are in CMP. There are four narratives in CMP involving tables; that they represent relationships, that rates and y-intercepts can be discerned from them and that they can be analyzed to answer contextual problems. None of these is suggested in Dolciani. Instead the Student Edition states,

> The **table of values** ... lists some of the solutions. When we graph the solutions on a coordinate plane, we find that they all lie on a straight line. (Student Edition, p. 280)

In addition, each of the 10 tables in the Student Edition is an unorganized list of solutions, usually generated in order to graph an equation. The only consistency in the choices of the x-values in these tables is that they are single digit integers. Consider these three lists of x-values—each is in the order that the values appear in a table—{-9, -3, 0, 3, 6}, {0, 6, 3} and {-1, 5, 2}. In each case, the corresponding y-values are listed as well, but these are unorganized lists of solutions, not representations of a relationship. Students are not expected to extract meaning from these tables; they are expected to use the tables to make graphs.

A Graph is the Set of All Points that Are Graphs of Solutions to an Equation

This narrative is pretty much the definition of *graph of an equation* in Lesson 8-3. The contrast with CMP is strong. In CMP, a graph is a representation with its own status.

In Dolciani, a graph depends on an equation. More fundamentally, a graph is composed of solutions in Dolciani, while solutions are not introduced in CMP until students have a great deal of experience with graphs.

A Straight-Line Graph Can Represent a Relationship between Contextual Variables

Lesson 8-5 is titled *Problem Solving: Using Graphs*. This lesson is the only one among the five lessons in this study whose examples include contextual variables. The relationship between mathematical variables and contextual variables is essential to the CMP approach, but is much less essential in Dolciani. Yet, the relationship is not neglected. Students are told, "A straight-line graph sometimes expresses the relationship between two physical quantities." (p. 288) The word *sometimes* is an acknowledgment of the difference between this lesson, which emphasizes contextual variables, and the rest of the chapter, which does not.

A Straight-Line Graph Represents a Constant Rate

This is most directly suggested in Chapter 8 of the Student Edition in the following sentence.

> For example, such a graph can represent the conditions of temperature falling at a constant rate or of a hiker walking at a steady pace. (p. 288)

This is less direct than most Dolciani narratives, which tend to be definitional. But this one suggests that *temperature falling at a constant rate* and *hiker walking at a steady pace* are among the things that can be represented by straight-line graphs, rather than defining straight-line graphs as necessarily being about constant rates and steady paces. The narrative appears implicitly again in this passage from the same lesson.

> Since we know that the temperature climbed at a constant rate, we can first graph

the two points and then draw a straight line through them. (p. 289)

A Linear Equation Has a Straight-Line Graph

This narrative is directly stated in Dolciani's Lesson 8-3 when the term *linear* is defined,

> In general, any equation that can be written in the form $ax + by = c$ where x and y are variables and a, b, and c are numbers (with a and b not both zero), is called a **linear equation in two variables** because its graph is always a straight line in the plane. (pp. 280-1)

Slope is the Ratio of the Change in Y*-coordinates to the Change in* X*-coordinates on a Graph*

In these lessons, slope is a matter with relevance only to graphs. This is in contrast to CMP, where rates (contextual rates, although these are later tied to slope) are studied with respect to tables and to equations, as well as to graphs. There is one lesson in Dolciani that concerns itself with slope, Lesson 8-5.

> The **slope of a line** is the ratio of the change in the y-coordinate to the change in the x-coordinate when moving from one point on the line to another. (p. 288)

In CMP, all narratives but one have visual mediators as subjects. In Dolciani, all of the narratives have visual mediators as their subject. CMP's narratives have a parallelism that was closely related to a pattern noted in the use of the word *equation*: tables, graphs and equations have roughly equivalent status. This shows up in the CMP narratives in the repeated construction, "tables, graphs and equations," and in the narrative *Linearity is represented in tables, graphs and equations* 4, 5 and 6, which state how to identify linearity in each representation. This is consistent with the relational-

dependency approach to functions. No such parallelism is present in the Dolciani narratives. Equations and graphs are the major subjects of narratives, with tables appearing only once. There is no narrative that describes all three representations. In the CMP *Narratives* section, the argument is made that the narratives have the visual mediators as subjects because of the abstract nature of functions. This claim is also trued for the Dolciani narratives. Dolciani gives a much different treatment of linear relationships and yet the narratives are still about the representations. This is due to the nature of the content, rather than indicating other similarities in the two discourses.

Routines

As with the other aspects of the Dolciani discourse on linearity, the routines for establishing narratives differ markedly from those of CMP. The three main discourse routines of CMP– asking questions, inductive reasoning and finding mathematical meaning in a context– are mostly absent from Dolciani. Instead, Dolciani has four main routines: defining, telling, calculating and examples. Three of these– defining, telling and examples– are prominent in the Student Edition. Two are prominent in the Teacher Edition, defining and calculating.

Student Edition: Defining

Defining is the most common routine for establishing the narratives in this study. Dolciani definitions follow the mathematical form quite closely. In this form, the defined term is highlighted (here in bold type), followed by the word *is*, and then the meaning of the term, as in:

> The **slope of a line** is the ratio of the change in the y-coordinate to the change in the x-coordinate when moving from one point on the line to another. (Student

Edition, p. 288)

In this example, the Dolciani form is distinguished from the formal mathematical form in that "when moving from one point to another" is an operation that is supposed to have an intuitive meaning rather than a strictly mathematically meaning. A definition that follows the mathematical form more closely is this one,

> In general, any equation that can be written in the form $ax + by = c$ where x and y are variables and a, b, and c are numbers (with a and b not both zero), is called a **linear equation in two variables** because its graph is always a straight line in the plane. (pp. 280-1)

This is equivalent to "A linear equation is any equation..." This example establishes two narratives: *A linear equation is of the form* $ax + by = c$ and *A linear equation has a straight-line graph.* The first of these narratives is established by definition. The second of these narratives is slightly different. The narrative *A linear equation has a straight line graph* tells students about the relationship between two defined objects. This is the second routine in Dolciani.

Student Edition: Telling

Definitions in Dolciani establish the meaning of a term the first time it is used. Telling involves terms that have already been defined[34] and are already in use. Three narratives are established by telling students a relationship between mathematical objects: *A solution to a linear equation is an ordered pair, A straight-line graph can represent a relationship between variables* and *A linear equation has a straight-line graph.* In each case, two previously defined objects (solution and ordered pair, graph and relationship,

[34] Although possibly very recently.

and linear equation and graph) have a relationship that is told to students. This is akin to a theorem in formal mathematical discourse. Dolciani does not prove the told narratives, as must be done with theorems in a more formal discourse, but the emphasis on the relationship between two defined objects is analogous to the structure of a theorem. One example of telling is in the preceding section, immediately after *linear equation* is defined. Dolciani justifies the term *linear* by stating that linear equations always have straight-line graphs. Note the difference between telling and defining in that passage. *Linear equation* is defined in terms of its form as an equation; students are told that such equations have straight-line graphs.

Student Edition: Examples

In contrast to CMP, every Dolciani Lesson in this study has at least one example. The five Lessons together have 11 numbered examples[35]. A Dolciani example has a particular form, and the use of the term *example* is different from its use in the analysis of CMP and inductive reasoning. In Dolciani, an example is a well-defined task with the answer given together with a set of steps leading to the answer. In general, many of the Written Exercises (which are not analyzed in this study) are tasks similar to the examples. The examples can be used by students as a reference when working on the Written Exercises. In the analysis of CMP's use of inductive reasoning, examples are instances of a relationship that has not been explicitly stated. The CMP examples are not labeled as such. They are often the product of a task, and their form changes depending on the relationship to be demonstrated.

[35] As with questions in the CMP problems, these Examples often have subparts, so there are really more than 11 problems worked out in the 11 Examples.

In Dolciani, examples are used to establish two narratives. The first, *Solutions are found by substituting values into an equation*, is established in Lesson 8-2. Here, the task in the example is to, "Tell whether each ordered pair is a solution of the equation $2x + y = 7$." (p. 276) Under the heading "Solution," the procedure is to, "Substitute the given values of x and y in the equation." (ibid) There are 6 Examples that explicitly involve solutions. Of these, three use the word *substitute*, while the remaining three demonstrate substitution without using the term.

The second of the narratives established by example is *A straight-line graph represents a constant rate.* In the *Narratives* section, I argue that this narrative is established indirectly. This happens in the midst of an Example with this task,

> The temperature at 8 a.m. was 3°C. At 10 a.m., it was 7°C. If the temperature climbed at a constant rate from 6 a.m. to 12 noon, what was it at 6 a.m.? What was it at 12 noon? (p. 288)

This task involves extrapolation. Two data points are given, along with the fact that the rate of change is constant. It has not previously been established that constant rates and straight-line graphs have a close relationship, but the text indicates that this is so.

> Since we know that the temperature climbed at a constant rate, we can first graph the two points and then draw a straight line through them. (p. 289)

This is an indirect way of saying "Straight-line graphs represent constant rates."

Teacher Edition: Defining

When establishing or reinforcing the narratives in this study, the Dolciani Teacher Edition sometimes refers to previous definitions, as in this passage, which gives teachers

a strategy for reinforcing an important aspect of the narrative, *A solution to a linear equation is an ordered pair*.

> By this time students find it quite natural to substitute for variables. They need to be reminded, however, that we have agreed to the convention that ordered pairs are (x,y) pairs. (p. 271b)

The text continues to demonstrate that $(4,-1)$ is a solution to $2y + x = 7$, but that $(-1,4)$ is not. In order to fully establish the narrative about ordered pairs, the definition (convention) of an ordered pair is reinforced.

Teacher Edition: Calculating

The *Word Use* section demonstrates that in Dolciani, equations are *number sentences*. Thus, the various statements about equations in Dolciani are statements about equality of expressions that represent numbers. Therefore, finding the numerical value of these expressions is one way to verify statements about equations. Calculation is a common strategy for establishing narratives in the Teacher Edition. As an example, consider the follow-up to the example above, under Definition. That example revisited the definition of an ordered pair in order to reinforce the narrative, *A solution to a linear equation is an ordered pair*. The routine of calculation immediately follows,

> Conversely, students should be warned to substitute correctly when checking a given number pair in an equation. In Example 1, $(4,-1)$ is a solution, but $(-1,4)$ is not because $2(-1)+4 \neq 7$. (p. 271b)

In this example the assertion is that the order of the pair matters, and this is established through calculation.

These routines point to Dolciani's set-theoretic approach to functions, but also to the mathematically formal nature of the discourse in Dolciani. A formal definition of *function*, which has been stated here, involves sets of ordered pairs. It is sensible, then, that Dolciani's set-theoretic approach to functions would involve more formal mathematical routines for establishing narratives; especially the routine of defining. It is a notable contrast that Dolciani offers seven definitions in the five lessons in this study, while CMP offers none.

Comparison

To summarize, the contrast between CMP's discourse and Dolciani's discourse is strong. The use of all three words in the *Word Use* analysis of Dolciani differs markedly with the use of these words in CMP. *Equation* is associated in Dolciani with *solution*, rather than with *relationship*. *Rate* is rarely mentioned in Dolciani, while it is a central object in CMP. *Linear* is restricted to a single lesson in Dolciani. Each visual mediator in Dolciani has a unique purpose and use, in contrast to CMP, where the representations are nearly equivalent. Dolciani's narratives focus on equations, while CMP's narratives encompass all three representations equally. The routines for establishing narratives are more formal and direct in Dolciani than in CMP.

Another major difference that has consequences in each of these categories is the relationship between contexts and mathematics in the two curricula. In Dolciani, contexts are introduced in a single lesson: Lesson 8-5 and students are told that linear equations sometimes represent relationships between real-world variables. In CMP, the analysis of change in contexts is the focus of the discourse. The study of linear relationships in CMP begins with the study of contexts. In Dolciani, the study of linear relationships mentions

contexts. This can be seen in the routine *Deriving mathematical meaning from a context*, which is absent in Dolciani and it can be seen in the differing uses of equations; describing relationships versus solving.

At the beginning of this chapter is a claim that equations are important objects in the study of algebra, and this has certainly been confirmed in the analysis of these two curricula. I conclude the chapter using an equation, together with contrasting material from Dolciani to characterize the nature of the mathematical discourse in CMP.

The equation $y = mx + b$ is the general form of linear equations in CMP. In Dolciani, the form is $ax + by = c$. In the CMP form (called *slope-intercept*), it is easy to find the slope (or rate of change) and the y-intercept. The slope is m, the y-intercept is b.

There is a direct relationship with the contexts in which the study of linearity takes place. There are basically two contexts in CMP around which the introductory discourse on linearity in CMP takes place: walking races and raising money in a walkathon. One important feature of each context is highlighted by b and one by m. The y-intercept in a walking race is the starting point. If everyone has the same starting point, then all of the equations will have the same b. In a walkathon, it is the initial donation—the amount of money someone raises without walking a step. In a walking race, m is the walking rate; in a walkathon m is the pledge rate. Important contextual information can be read directly from the slope-intercept form of the equation.

Recall that CMP has a relational-dependency approach to functions. In practice, this means that instruction should be focused on the relationship between variables, and that one should think of one variable depending on the other. The chosen contexts support this way of thinking, and together the contexts and this way of thinking support

the slope-intercept form of linear equations. I am not arguing that a curriculum with a relational-dependency approach to equations could not use the standard form for linear equations. Instead I claim that there is a natural connection between the approach CMP has taken to functions, the contexts in which students learn about linear functions and the form of linear equations that CMP endorses. The curriculum is consistent in this approach.

It is also notable that CMP endorses the slope-intercept form of equations, in this investigation, through example and through the contexts, rather than through definition. Formal mathematical discourse—as captured in textbooks, journals and lectures—has a character different from the everyday work of mathematicians. Lampert (1990) makes this argument in the study of her classroom, and the mathematical activity described by Lakatos (1976) is quite different from the linear *definition-theorem-proof* pattern of formal mathematical discourse. Behind the clean formality is difficult messy conjectural activity. CMP encourages students to engage in this messy activity by having them write equations before explicitly stating a general form. Students are guided to a particular form by the nature of the problems in the curriculum, but as in a mathematician's work, the definition comes later. I have argued that Dolciani holds a contrasting view of the nature of mathematical activity. Dolciani lessons tend to begin with a definition and with theorem-like statements in the manner of mathematics journal articles or lectures.

A final way to characterize these differences is to examine assumptions about learning in the two discourses. Learning a discourse involves imitation to some degree (enculturation). The teacher is modeling a discourse and the student's job is to participate in this discourse, which she can only do by imitating the teacher's discourse patterns. In

particular, students imitate (and to a certain degree negotiate) the routines for establishing narratives. In the Dolciani form, students imitate procedures for using and verifying pre-established truths—calculating and the working of problems similar to examples. In CMP, students imitate routines for finding mathematical truths that may be unknown—asking questions, reasoning inductively and finding mathematical meaning in real-world problems (a form of metaphor) are all heuristic techniques that advanced students of mathematics use when attempting to solve new problems. This is the justification for considering these two textbook discourses as mathematical discourses, as discussed in Chapter IV. These two textbook discourses encourage different types of mathematical discourse: Dolciani the formal discourse of journals and lectures and CMP the informal discourse of mathematical discovery.

This chapter has described the introductory discourse on linearity in two curricula, using the discourse in Dolciani to highlight important features of the CMP discourse. The analysis in this chapter is instructive about the differences in reform curriculum. The theoretical framework involving communication has turned up substantial differences in the content and teaching procedures of CMP as compared with Dolciani. Also, the analysis has been instructive for future study of reform curriculum. It has revealed that assumptions about the content of two curricula—even when the topic is the same—may be invalid. Finally, this chapter has set the stage for answering the larger question about the relationship between published curriculum and classroom instruction by characterizing the published curriculum CMP. The next chapter shifts to an analysis of the discourse in two classrooms using Investigation 2 of *Moving Straight Ahead.* The

ways in which the textbook discourse of CMP changes in classroom implementation will be the focus of the next chapter.

Chapter VI: Classroom Discourse

The curriculum *Connected Mathematics* (Lappan, et al., 2001a) is part of a larger curriculum reform effort in US K-12 mathematics instruction. Lively debate has taken place surrounding this reform and its goals. Much of this debate has been based on presumed correspondence between the printed curriculum and classroom instruction. This chapter examines this correspondence in detail, using one set of lessons (those analyzed in Chapter V) as realized in two classrooms. There are two research questions addressed in this chapter: *What characterizes each teacher's discourse on linearity* and *What changes does the discourse of the curriculum undergo in its implementation by each teacher?* Together, these questions are designed to develop our understanding of instruction in the wake of a significant curriculum reform, and to provide information that will be useful in improving mathematics instruction in this context.

The preceding chapters have characterized teaching and learning in a communicational framework using the language of discourse analysis. *Knowledge* has been characterized as participation in a discourse; *learning* as change in that participation and *teaching* as engagement with learners in the model discourse. These definitions are offered in order to make the question, *What is being taught in reform mathematics classrooms?* a researchable question. Rephrased as *What is the model discourse?* the question is operative in the sense that the terminology can be turned into straightforward research tools. These tools are detailed in Chapter IV.

Chapter V examines the *Connected Mathematics* curriculum (CMP; Lappan, et al., 2001a), specifically the curriculum's introduction to linearity. Another curriculum, *Mathematics, Structure and Method* (Dolciani, 1992) is used as a contrast in order to

highlight important aspects of the CMP discourse. The introductory textbook discourse on linearity of CMP is characterized by its relational-dependency approach to functions and adherence to a heuristic—rather than formal—mathematical discourse.

CMP's approach to functions differs from Dolciani's approach, which is set-theoretic. The relational-dependency approach emphasizes the variation underlying the term *variable*, the relationships between variables, and multiple representations of these relationships. This is contrasted with a set-theoretic approach that emphasizes correspondence between variables, variables as unknowns, solving equations and places a priority on equations and ordered pairs over graphs and tables.

A heuristic mathematical discourse involves the techniques people use when they are trying to solve novel problems in mathematics. For instance, reasoning inductively from examples, asking questions and employing various forms of metaphor are ways that mathematicians and advanced students of mathematics attempt to understand new domains. This is contrasted with formal mathematical discourse, where the heuristics, once used, are hidden and in their place, terms are given careful definitions and theorems are stated and proven. Formal mathematical discourse is the discourse of lectures and journals. Heuristic mathematical discourse is what takes place around the chalkboards in the offices of mathematics professors and graduate students.

The question that drives the present study is, *What is being taught in reform mathematics classrooms?* Having phrased this question in terms of discourse, it is important to note that there are two model discourses in most mathematics classrooms: that of the textbook and that of the teacher. The present chapter has two goals that correspond to the research questions:

(1) Characterize the introductory classroom discourses on linearity in two classrooms, and

(2) Examine the changes that the textbook discourse undergoes as it is translated into classroom discourse by each teacher.

Chapter IV addresses some of the difficulties inherent in comparing these two discourse types (e.g. that students have input into the classroom discourse, but not the textbook discourse), and these difficulties will be kept in mind and addressed again in the comparison section of this chapter.

The two classrooms in this study were chosen based on the following characteristics: they are located in city schools with diverse student populations, the teachers have experience with the student population and with the curriculum, there is high time on task in the classrooms, the teachers pay attention to the mathematical generality behind the contextual problems in the curriculum, and the time of year the unit *Moving Straight Ahead* (CMP, 2001b) would be taught. These conditions were intended to frame a study of typical classrooms with good conditions, rather than ideal classrooms. This study does not seek to provide an "existence proof" of what is possible (Lampert, 1990), but to describe situations that are likely to be replicated in many classrooms in this country.

Sfard (2001) argues that four aspects distinguish mathematical discourse from other discourse genres: word use, the use of visual mediators, endorsed narratives and the routines for endorsing these narratives. These aspects are analyzed in the textbook mathematical discourses of CMP and of Dolciani in Chapter V. This chapter uses the same set of analytical tools (modified for the different discourse form of classrooms

instead of textbooks) to characterize the classroom mathematical discourse of the teachers in this study. This chapter has three sections. Each teacher is considered separately (two sections), and then the changes in the textbook discourse as it is implemented in the two classrooms are detailed. In each of the first two sections, the characteristics of mathematical discourse (word use, visual mediators, narratives and routines) are considered in turn, giving a close, medium and long range view of Ms. M's discourse, and then Ms. H's discourse. The chapter ends with a comparison of the textbook discourse on linearity (in CMP) with the classroom discourses in this study.

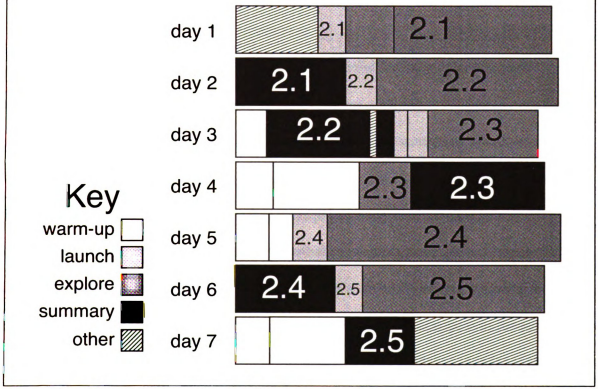
Ms. M's Discourse

Ms. M teaches in a school with approximately 450 students in 7th and 8th grades. The school's enrollment is 86% minority. The school follows a seven-period day in which classes last 48 minutes. Students change schedules each semester, staying with some of their previous teachers and getting some new ones in an unsystematic way. *Moving Straight Ahead*, the CMP unit that introduces linearity, is studied in 8th grade. Some of Ms. M's students were in her classes the previous year, as 7th graders, while some were not. The student mobility rate is high: 49% of the student population transferred in or out of the school in the first semester of the 2003-04 school year, the most recent year for which figures are available. Ms. M's students who were at her school as 7th graders studied CMP, either with Ms. M or another teacher. Students who were not at her school had a variety of curricular backgrounds.

Ms. M. spends seven class periods working through the problems of Investigation 2. Her instruction includes the three phases of a CMP lesson (Launch, Explore, Summarize), but generally not all three in the same day. The sequencing of these phases

is shown in Figure 7. The typical sequence is to Launch and Explore a problem one day and to have the Summary near the beginning of class the next day, followed by the Launch for the next problem.

Figure 7: Major activities[36] in Ms. M's Investigation 2 lessons.



In addition, she spends a significant amount of time on warm-up activities (referred to as *warm-ups*, these are indicated by the white areas in Figure 7). These warm-ups are related mathematically and contextually to the problems in the investigation, but the tasks are of her own design. In general, Ms. M uses these warm-ups to discuss individual narratives from the list of narratives in CMP. The major contribution that warm-ups make to the classroom discourse is to establish narratives that are absent

[36] *Activity* is defined in Chapter III. A new activity is recorded whenever the answer to the question, *What are students supposed to be doing now?* changes.

from the Launch-Explore-Summary (LES) part of the lessons. In the following analysis, claims are first made about each aspect of Ms. M's discourse using evidence from the LES sequence for the problems from the text. When appropriate, separate claims follow about how the warm-ups contribute to each aspect of the discourse. That is, the warm-ups are addressed here only when they contribute to the understanding of Ms. M's discourse.

Word Use

Equation

In this study, *word use* refers to the patterns in the appearance of words in discourse. Occasionally, the frequency of these patterns is telling, so frequency counts appear in parts of this section. Frequency counts, though, are a crude tool—particularly when comparing across discourse types. The main foci of the analysis of word use are identifying the patterns, considering what these patterns say about the meaning of the words, and considering how the patterns contribute to the characterization of each classroom mathematical discourse. The first patterns to be analyzed are the patterns that appear in the CMP textbook discourse, then any new patterns that appear in the teacher's classroom discourse, but not the textbook discourse, are analyzed.

Ms. M's use of the word *equation* shares with CMP's use the association with tables and graphs, but she talks about these differently. She puts less emphasis on the term *relationship* than does CMP. In addition, there is a pattern unique to Ms. M's use of *equation*: that of *possession*.

Relationship. In CMP's discourse, the predominant use of *equation* is as a representation of a relationship. In contrast, Ms. M is less explicit about this. Where 43% of the Student Edition sentences using the word *equation* also used the term *relationship*,

only 16% of Ms. M's sentences using *equation* use either *relationship* or the term *situation.*

The association between *equation* and *relationship* is less prominent in Ms. M's discourse at least in part because she discusses the writing of equations at length. Even if the number of sentences associating *equation* and *relationship* were identical in the two discourses, the extended discussions of writing equations in Ms. M's discourse drives down the percentage (more on this in the next section, *Visual Mediators*).

Table and Graph. Ms. M. opens two of her summaries by discussing tables, graphs and equations, as in this passage,

> Yesterday what you were asked to do was to make a table, graph and come up with an equation to represent this walking race situation where Henri walked a meter per second, Emile walked two and a half meters per second. (Ms. M. Activity 3, 11/02/04)

Additionally, she follows the lead of the text in discussing tables, graphs and equations in turn, but her questions tend to be different in nature for each of the three representations. Ms. M. asks questions that suggest students will be able to notice and articulate how rates affect tables, as in[37]:

> You also looked at the table between the three of them. Like we saw in the warm-up today, how can you tell in a table which person has the faster rate? (Ms. M. Activity 2 10/27/04)

[37] When transcript excerpts involve multiple turns, *M* indicate Ms. M's turns, while students' initials (or *Student* when the student is not identified) indicate students' turns. Ellipses within turns indicate pauses. Ellipses between turns indicate that content has been omitted from the transcript.

This is a rephrasing of the text's question, *How does the walking rate affect the graph?* in which Ms. M asks for a general principle ("how can you tell in a table...?") that might be applied to any table of walking rates. The question she asks about graphs is different, "Looking at those three graphs, where is Jerome's graph? What color is it?" (Ms. M. Activity 2, 10/27/04) In this case, the students had remembered aloud that Jerome was the fastest walker and Ms. M asks about locating this particular walker, rather than about a general principle. She asks students to say which is Jerome's graph, rather than to say how to tell who has the fastest rate when looking at a graph.

Finally, Ms. M tends to ask questions that lead to the writing of equations, rather than to the comparison of equations. This is in contrast to her questions about tables, in which students are asked to state principles for rate comparisons, and to her questions about graphs, which implicitly ask students to identify graphs with particular properties (i.e. with the fastest rate).

> Now the final part, part C, you were asked to find the equations that displayed the relationship between time and distance for these three walkers, and that's always kind of the challenge part, but at this point, you guys are really good at finding equations.
>
> ...
>
> What can you tell me about time and distance? (Ms. M Activity 2 10/27/04)

The line of questioning that begins with, "What can you tell me about time and distance," continues as students state that time and distance are the same for one walker (hence the equation $d = t$), that distance is twice the time for another walker ($d = 2t$), etc.

Once the equations are written in this summary, she finishes the conversation by saying, "So we have tables, graphs and equations, all that can represent this linear situation." (ibid). This final statement reiterates the text's message that tables, graphs and equations are equivalent representations. Yet it comes at the end of lengthy discussion in which the representations have been used in non-equivalent ways.

In CMP, the same questions are asked of students about tables, graphs and equations, requiring parallel answers: larger rates yield larger increments in tables, steeper lines in graphs and larger coefficients in equations. In Ms. M's discourse, the comparison of the coefficients is left out. Equations get written, but they do not get compared to each other in the way that tables and graphs do.

Possession. The most common pattern in Ms. M's use of *equation* is the use of possessive adjectives. In Problem 2.2, she refers to "Jade's equation." In Problem 2.3, she asks students, "...what's a good equation for Leanne?" and then, "So what's her equation?" This use pattern shows up in one problem in the text, Problem 2.5, "Write an equation for each brother showing the relationship between the time and the distance from the starting line." (CMP Student Edition, p. 22) By contrast, Ms. M uses a possessive adjective or talks about an equation *for* someone in each of Problems 2.2 through 2.5, and the CMP construction *showing the relationship* is absent, as can be seen in the preceding examples.

The tension between relational-dependency and set-theoretic approaches to functions has been summarized by Kieren (1993) as a *process-object* tension. The *process-object* distinction captures some, but not all of the differences between the two approaches. When a function is seen as a process, the computational procedures that it

encompasses are highlighted; these are the rules for moving from values of the independent variable to values of the dependent variable. This is Sfard's (1991) *operational description* of functions. When a function is seen as an object, the correspondence and ordered pairs are highlighted. This is Sfard's *structural description* of functions. The way the word *function* is used, too, can give clues about whether functions are seen as processes or objects. It is in this light that Ms. M's *possession* pattern is important. When she speaks of "Alana's equation," equation is being used linguistically in the manner that an object would, such as Alana's *shoe* or Alana's *home*. At the microscopic level of the analysis of word use, this is evidence of an object view of functions, which is consistent with a set-theoretic approach. Thus, we can see the *process-object* tension at play in Ms. M's classroom at the level of word use.

The term *tension* applies in Ms. M's discourse because the two approaches coexist, but are not reconciled. That is, equations have somewhat different treatment from tables and graphs, but Ms. M does not discuss this differential treatment. The two approaches are unresolved; they are in tension. In CMP, the relational-dependency approach is the norm. Problem 2.3 briefly places equations at the center as objects of the discourse, suggesting set-theoretic notions. Like Ms. M's discourse, this transition is not directly discussed. However, unlike Ms. M's discourse, this occurs once and briefly; this second approach does not linger. Instead, the discourse highlights set-theoretic ideas in the follow-up questions and then returns to the relational-dependency approach in the next problem. There is no ongoing tension between the two approaches.

One question about the classroom implementation of CMP is how a traditionally trained teacher teaches a reform curriculum. One might conjecture that a teacher with

experience in a traditional curriculum (and thus with a set-theoretic approach to functions) would struggle to use tables and graphs with students—that equations might continue to take the prominent place in the teacher's discourse. This is not the case with Ms. M's discourse. Another conjecture might be that the use of *equation* would correlate with *solving* and *correspondence* (set-theoretic notions) over *variation* and *relationship* (relational-dependency notions). This second conjecture is a bit closer to the mark in Ms. M's discourse. While she does not heavily emphasize the set-theoretic ideas, she does speak of functions as objects in at least one way and she notably de-emphasizes the relational-dependency ideas. The conjecture that this is related to her teaching experience prior to teaching CMP is one for future research.

Ms. M's use of *equation* reveals two co-existing patterns: she asks questions about tables and graphs together with her questions about equations, and so emphasizes some form of equivalence among the three representations, but she does not emphasize the relationships that CMP does, and she speaks of functions as objects in at least one way. This points to a tension in her discourse between the set-theoretic and the relational-dependency approaches to functions.

Rate

There are two patterns in Ms. M's use of the term *rate*: association with *walking*, and the identification of rates in the three representations. Recall that in CMP, the use of *rate* is associated with *walking,* with *affecting* the three algebraic representations, and with the term *rate of change.*

Sixty-nine percent of the sentences (25 out of 36) that include rate include the word *walking* or a variation. This usage is comparable to the discourse in the CMP text,

where all of the mentions of rate in the Student Edition are associated with walking and 63% of those in the Teacher Edition are.

The use of *rate* in Ms. M's instruction parallels the use of *equation*. When discussing equations, Ms. M alters the CMP discourse. Where CMP focuses on comparing equations, Ms. M emphasizes writing them. Similarly, the CMP discussion of how rates *affect* tables, graphs and equations is downplayed in favor of *identification* of rates in tables, graphs and equations. The comparison task is changed to an identification task in each case. In the Student Edition, 42% of the mentions of *rate* asked how the rate affects one or more representations, while the figure is 34% in the Teacher Edition. By contrast, Ms. M's figure is 17% (6 out of 36). Of these six mentions, she is quoting directly from the Student Edition five times—only once does Ms. M suggest on her own that rates affect tables, graphs and equations. Instead, she more frequently identifies rates, or has students identify rates in the representations.

A lesser pattern in the CMP discourse is the use of the term *rate of change* as a generalized form of walking rates. Ms. M does not use this term. Ms. M's use of *rate* is consistent with CMP's use in its association with walking—*rate* is a contextual term in Ms. M's discourse, just as it is in the CMP Student Edition. Yet, Ms. M deviates from CMP's use of *rate* in a manner similar to her deviation in the use of *equation*; comparison tasks are changed to identification tasks.

Linear

The term *linear* arises in four passages (9 sentences altogether) in Ms. M's lessons, in contrast to the Teacher Edition of CMP, which uses the term 21 times. Ms. M links linearity strongly with graphs, as in, "They make linear graphs. Are all three of

them linear? Are all three of these graphs linear?" (Ms. M Activity 5, 10/28/05) While Ms. M takes time to identify rates in tables, graphs and equations, the only representation associated with her use of *linear* is graphs. She mentions *linear relationships* and *linear situations* one time each. Ms. M does not use the term *nonlinear*.

In the microscopic view afforded by this word-use analysis, differences between CMP's textbook discourse on linearity and Ms. M's classroom discourse begin to emerge. It is relevant and important that the differences noted are not inherent in the two types of discourse; instead, they point to different approaches to functions and to linearity. These differences speak to differences in discourse patterns. In particular, Ms. M's use of the word *equation*, with its de-emphasis on relationships and emphasis on possession, points subtly in the direction of a set-theoretic approach to functions characteristic of traditional American algebra curriculum, where CMP consistently takes a relational-dependency approach. Yet, Ms. M is also consistent with the curriculum in a number of ways, most notably in her use of *rate* and in the association of *equation* in sentences with *tables* and *graphs*. The comparison with CMP, of course, highlights where there are deviations in Ms. M's word use from those we would expect in a strict relational-dependency approach to functions. Overall, it is more accurate to characterize Ms. M's discourse in terms of deviating from a relational-dependency approach in the particular ways outlined here, than to characterize it as an example of a set-theoretic approach.

Visual Mediators

In mathematical discourse, we use visual mediators to communicate about abstract objects that cannot be physically present. In the case of functions, three algebraic

representations—tables, graphs and equations—are the primary visual mediators. In CMP, these three representations are used in nearly equivalent ways. In Chapter V, this claim is substantiated by considering *themes* in the use of the representations and characterizing them as *recurrent*, *passing* or *absent*. In textbook discourse, a recurrent theme is mentioned in more than one sentence in more than one lesson. Textbooks are carefully edited while teacher talk is not, so this measure is too fine for the analysis of classroom discourse. Instead, when analyzing teachers' classroom discourse, a recurrent theme appears in more than one teacher turn in more than one lesson. A passing theme appears in a single teacher turn in multiple lessons, or in multiple turns in a single lesson, or a combination of these. Absent themes are not present[38].

Ms. M uses the same visual mediators as CMP: tables, graphs and equations. She shares the curriculum's expectation that students will create these mediators more often than they will be shown them. In the Launch-Explore-Summary sequence, the only time she shows students tables, graphs or equations is during the summary—after students have created their own visual mediators. Further, she frequently uses the student-created mediators for her summaries. Ms. M also shares the two uses of the representations with the curriculum: representing contextual information and revealing linearity.

Tables

There are three important kinds of contextual information that can be found in each representation: rates, individual values and the starting value (or y-intercept). The use of tables to identify rates and find individual values are recurrent themes in Ms. M's discourse, but the role of y-intercepts in tables and the use of tables to identify linear

[38] This is a non-trivial matter, as the themes identified in the curriculum are considered in the classroom discourse. Thus, the set of absent themes is finite.

relationships are both passing themes. The first time she discusses the representation of rates in tables is in her summary of Problem 2.2.

M: ...how can you tell in a table which person has the faster rate?

Student: bigger numbers

M: Bigger numbers. OK. Bigger numbers, and where else can you look besides just the individual number itself? Because here, yeah, we end with the bigger numbers here. Think about what E was doing when she found the rate in the table.

Remember you can also look in between. What's the change here?

Student: 2.5

M: 2.5. What's the change here?

Students: 2

M: 2. What's the change here?

Students: 1

M: 1. OK. So a faster rate in a table is going to show up in a larger change between values. Let's say it like that. (Ms. M Activity 2, 10/27/04)

In this case, Ms. M's first question suggests that tables can be used to compare rates. When the student answers are not quite mathematically correct, she focuses the students' attention on the important feature of the table ("What's the change here?") and then states the relationship, "a faster rate is going to show up in a larger change between values."

Ms. M revisits the topic of rates in tables in her summary for Problem 2.4, where she asks questions of a group whose table is displayed on poster paper at the front of the

classroom. In this case, she adopts students' language for increment ("adding each time") and uses this to elicit from students an explicit statement about how rates appear in tables. This establishes the theme as recurrent.

In Problems 2.3 and 2.4, Ms. M uses the tables to identify individual contextual values. In Problem 2.3, Ms. M has students identify points that would be on each walker's graph. The discussion begins on the graph, but moves to the table when the points are difficult to read on the graph on the overhead projector.

In Problem 2.4, Ms. M has groups of students tell how long the race between Emile and Henri should be. Of particular importance for the present analysis is that each of these groups has used a table to solve the problem and Ms. M focuses the conversation on the part of the table that shows the finish line determined by each group. These two examples show that finding individual values in tables is a recurrent theme.

The role of y-intercepts in tables, however, is a passing theme for Ms. M discusses it in a single problem. She has students use tables to find the initial value in Problem 2.3. She asks, referring to Alana's pledge plan, "Where's that five dollar donation represented in the table?" (Ms. M, Activity 5, 10/28/04). There are several turns making up this part of the discussion, but the topic is not raised again in the investigation.

The representation of rates in tables is a recurrent theme. Ms. M has students compare the rates across multiple tables in the investigation. However, in the Launch-Explore-Summary sequence of the lesson, she does not directly discuss the idea of a constant rate, nor of linearity with respect to tables. This theme would be absent if only this part of the lesson were analyzed. Instead, it appears in the warm-up on 11/2/04, so the theme is passing. This episode is examined in more detail in the *Narratives* section.

In Problem 2.4, there is a subtle difference between Ms M's use of tables and CMP's use. In the problem, students are looking for the point in the race where Emile catches Henri. Ms. M has student groups put their solution strategies on poster paper and Ms. M conducts the entire summary using these posters. In all five cases, the students used a table to solve the problem. It should be noted that these students had prior experience from a pilot unit in another curriculum[39] in making tables and writing equations from the patterns in tables. This other curriculum emphasized the relationship between x and y in each pair, rather than the increment in the table. Many students made tables that did not involve a constant increment for the independent variable. Three student tables are reproduced in Figure 8.

Figure 8: Three student produced tables used in Ms. M's summary of Problem 2.4

Time	Emile	Henri
0	0	45
10	25	55
20	50	65
25	63	70
29	72	74

Guess Number	Henri + 45 meters	Emile x2.5 meters
10	55	25
20	65	50
30	75	75
29	74	72.5

Sec.	Henri	Emile
1	45	0
2	46	2.5
3	47	5.0
...[40]		
29	[blank]	[blank]
30	[blank]	[blank]

[39] College Preparatory Mathematics, (CPM, 2002). See Chapter II for a more detailed description of these students' curricular path.

[40] These students continued the unit increment all the way to 30 seconds. They continued keeping track of Henri's distance through 20 seconds and Emile's distance through 9 seconds.

This type of table (with non-constant increment) is only implicitly sanctioned by the Student Edition, which for this problem tells students to, "[m]ake a table showing the distance each brother is from the starting line at several different times during the first 40 seconds." (p. 22). This indicates that students are free to choose any values they wish for time, and that there need not be a pattern to their choices. The answer in the Teacher Edition, however, shows a table with constant increment of 10 (i.e. the time values are 0, 10, 20, 30 and 40). Further, a follow-up question introduces a third walker, with a different rate and different head start and asks students to complete a table started in the text. As with the other tables shown in the Student Edition, this one has a constant unit increment for the independent variable. In this instance, Ms. M's discourse seems to vary from that of CMP; Ms. M explicitly endorses tables with non-constant increments[41], while the CMP Student and Teacher Editions model only tables with constant increments.

Graphs

The role of rates in graphs, the finding of individual values in graphs and the representation of the *y*-intercept in a graph are all recurrent themes in Ms. M's discourse. The use of graphs to identify linearity is also a recurrent theme. Ms. M adds one aspect to CMP's treatment of graphs: a manipulation that helps to highlight the meaning of *steepness*.

Ms. M addresses the topic of rates in graphs in three Problems: 2.2, 2.3 and 2.5. In Problem 2.2, Ms. M not only indicates *that* rates are represented in graphs, she tells students how.

It's steeper, OK? And that's a word that I want you to keep thinking about. That

[41] By using such tables as the focus of her summary, for instance.

the person who had the fastest walking rate... what kind of line did he have? He had a much steeper line than the other two. (Ms. M, Activity 2, 10/27/04)

She returns to this theme in Problem 2.3. *Faster* does not apply in that problem, but the idea of an increased rate (i.e. more money *per mile*) showing up as a steeper line is the focus in this problem.

Finally, in Problem 2.5, Ms. M discusses the follow-up question, which addresses a third person who walks faster then Henri, but slower than Emile and who gets a smaller head start than Henri does. She directly addresses how the rate will affect the steepness of the graph.

Ms. M raises the subject of representing individual values in the graph in two Problems: 2.3 and 2.5. In Problem 2.3, a student indicates that he found dollar amounts for given distances using his graph. Ms. M pursues this conversation for several turns to elicit an explicit statement of the student's procedure.

In her summary for Problem 2.5, Ms. M uses the graph to locate the point where Emile overtakes Henri. Students have not raised the topic, so Ms. M uses several turns to focus student attention on this point. This establishes the theme of using tables to identify individual values as recurrent.

The representation of y-intercepts in graphs is raised by Ms. M in two Problems: 2.3 and 2.5. In Problem 2.3, Ms. M asks, "Where's that five dollars represented on the graph?" (Ms. M Activity 5, 10/28/04) From prior instruction, her students know the term y-*intercept*, so she follows this question with another, "What's that point called?" (ibid)

Ms. M discusses a follow-up question to Problem 2.5 involving a third walker with intermediate rate and head start. The brief discussion of the graph begins by determining where the line should start (i.e. the location of the y-intercept).

M: How would that equation fit in up here? Where would it be starting on the x-axis [sic]?

Student: At 20

M: At 20 meters, good. So we start it right here. (Ms. M Activity 3, 11/02/04)

Ms. M introduces the topic of linearity in graphs in Problem 2.1. In the Launch for that problem, she points her students' attention to a set of graphs on chart paper at the back of the room. These are graphs that students made in their instruction with the CPM curriculum (see footnote, page 156). In that experience, students made graphs and learned names for the patterns in the graphs, *linear* among them. Ms. M asks which of these graphs is linear. Students answer correctly. She asks, "How do you know a linear relationship?" and students answer, "Straight line." (Ms. M Activity 2, 10/25/04) In Problem 2.3, Ms. M revisits the topic of linearity in graphs. She asks a question about the relationship between the rate and the graph.

M: Ma, can you keep going with *how does it affect the graph?*

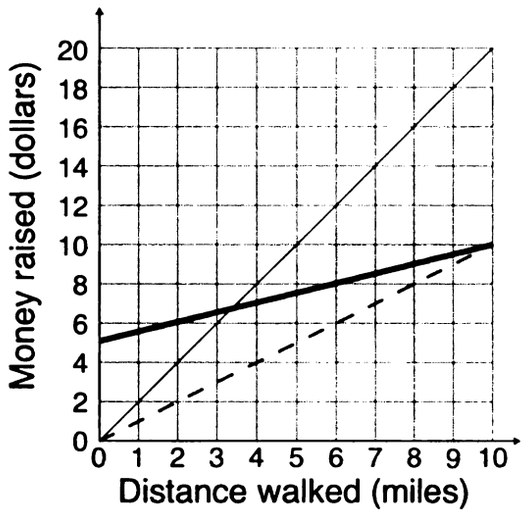
Ma: It just makes a linear... (Ms. M Activity 5, 10/28/04)

Ms. M follows up with several turns, asking questions to clarify whether all three graphs are linear (they are).

Ms. M adds to the CMP discussion of graphs a manipulation of this visual mediator. In the following passage, she is struggling to help her students verbalize a distinction between *higher points* on a graph and a *steeper line*. Figure 9 reproduces the graph under discussion. In the figure the bold line represents Leanne's pledge plan, the

narrow line represents Gilberto's pledge plan and the dotted line represents Alana's pledge plan.

Figure 9: Graph of the three pledge plans in Problem 2.3



1 M: But what we're trying to figure out is how come these lines look different?

2 A: Because one's higher and the other one's lower.

3 M: Why is this green one higher than the red one? Why is it steeper than the red one?

4 A: Because I think that's more money.

5 M: More money! You're there, A. What was the pledge for this green graph? How much did they get per mile?

6 A: Two dollars.

7 M: Two dollars. How much did Leanne get per mile?

8 A: One dollar.

9 M: One dollar. And imagine everybody, if I take this black line,

and let's forget that they got that five-dollar jump. If I would just move it down to here,

10 A: It'd be less.

11 M: It'd be even less steep than this one. The only reason it seems higher for a little bit is it got that five dollar jump in there.

12 A: But if it wouldn't have had the five dollar jump, it would be even less.

13 M: Say that again.

14 A: But if it wouldn't have had that five-dollar jump, it would be less.

15 M: Right. Because this was only 50 cents per mile, right there. So M, back to your question. How does that dollar per mile change the graph? What happens to it as that dollar per mile increases?

16 Ma: The black line decreases...

17 M: The black line is flatter. What happens to one dollar per mile?

18 A: (whispers to Ma) It gets higher.

19 M: What's that word we used yesterday everybody?

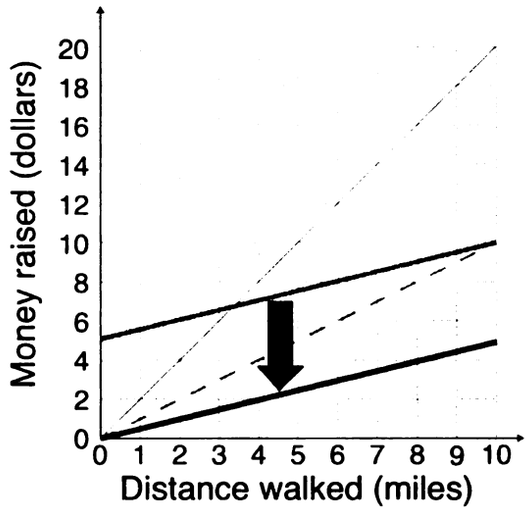
20 Student: Steeper

21 M: Steeper. One dollar per mile it gets steeper. (Ms. M Activity 5, 10/28/04)

When Ms. M says, "If I would just move it down to here," in turn 9, she indicates the transformation in Figure 10. Her students' language (e.g. in turn 2) suggests they are

seeing only the height of the line, rather than its slope. This is an improvisation in response to her students' repeated use of *higher*. After she performs the transformation, she seems to be convinced that the problem is no longer students not seeing the phenomenon she wants them to see, but the words they are using to describe it. She asks, "What's that word we used yesterday..." to clarify that she is seeking a particular word. Transforming graphs is a passing theme, appearing only this one time.

Figure 10: Ms. M transforms the graph showing Alana's pledge plan.



There are two main differences between CMP's treatment of graphs and Ms. M's treatment, but neither points to fundamental differences in the two discourses. Ms. M puts more emphasis on the representation of *y*-intercepts in graphs than does CMP and she introduces the theme of transformations, which is absent in CMP. This is not the case with equations. Ms. M uses equations differently from CMP. She indicates to students that equations represent contextual information, but she downplays the discussion of how equations can be used to decide whether a relationship is linear.

Equations

In Ms. M's discourse involving equations, rates and *y*-intercepts are recurrent themes. The finding of individual values is a passing theme. Discussion of identifying linear relationships from equations is absent from Ms. M's discourse in the problems of the investigation, but it does appear in the warm-ups.

Ms. M addresses the topic of rates in equations in three Problems: 2.2, 2.3 and 2.5. In Problem 2.2, she uses the table students have worked on to help them to notice the relationship between the two variables; distance and time.

Ms. M revisits the representation of rates in the equation in Problem 2.3. In this case, the equations have already been written. She asks students to articulate how the rate appears in each. This consumes several turns.

The follow-up question to Problem 2.5 has been analyzed in the two preceding subsections, *tables* and *graphs*. In this follow-up, Ms. M also raises the topic of rates in equations for a final time in the investigation.

M: Ok, if Paulette is at 20 meters head start and walks 2 meters per second, what would be her equation? Yeah, T?

T: [inaudible]

M: Talk a little louder. d equals 20 plus 2 t [inaudible; *presumably the student is saying each part of this equation, but it cannot be heard*] Excellent.

M: 20 is the amount added on. Two meters is our rate.

The topic of using equations to find or represent individual values is a passing theme. It surfaces in one problem in Ms. M's instruction: 2.3. After she quotes the text's question, *For a sponsor to owe a student 10 dollars, how many miles would a student*

have to walk under each pledge plan? (CMP, Student Edition, p. 20), a student tells how she computed her answers and Ms. M summarizes,

> ...she was using these equations here. Because she was saying, *well my distance is 10, one times 10 is gonna give me my money.* For this one, my distance is 10, two times 10 is gonna give me my money. Then for Alana, the trickier one, she went *50 cents times 10 is five dollars* and added on the 5. (Ms. M Activity 5, 10/28/04)

The *y*-intercept's relationship with equations is a recurrent theme in Ms. M's discourse. She raises the question of how the *y*-intercept is represented in the equation in two Problems: 2.3 and 2.5. In Problem 2.3, as the class is discussing the writing of Alana's equation, Ms. M presses a student to state the constant term in the equation in a voice loud enough for the whole class to hear, then draws attention to this term by discussing its form.

> M: Keep talking, R. What'd you say?
>
> R: *m* equals point fifty *d*...
>
> M: point fifty *d*...
>
> K: plus five
>
> M: Plus five. Yeah, K, yeah, Ma. Now when I wrote it here, I didn't write point fifty, I just wrote point five. Does everybody agree that five tenths is the same as fifty hundredths?
>
> Students: yeah
>
> M: Point five is the same as point five zero? OK. *d* plus that five dollars. All right. (Ms. M Activity 4, 10/28/04)

Collapsing Ms. M's last turn in the excerpt to read, "point five *d* plus that five dollars," shows that one purpose of the sentence is to summarize the place of the *y*-intercept (in context, the initial donation in the pledge plan) in the equation. The *y*-intercept in an equation is raised again in Problem 2.5.

In the Launch-Explore-Summary sequence of the five problems of Investigation 2, Ms. M does not discuss identifying linear relationships from equations. Equations are discussed in four of the five summaries, but the word *linear* does not appear in these summaries.

As indicated in the section on the use of the word *equation*, the main focus of Ms. M's instruction involving equations is on writing them, with a lesser emphasis on comparing them. This emphasis on writing equations is revealed in an analysis of her discussion of equations as visual mediators (as opposed to the analysis of her sentences using the word *equation*). In these discussions, the focus is on how the rate appears in the equation, with questions and statements such as the following:

> What's the relationship between those two columns? (Ms. M Activity 2, 10/26/04)
>
> So the rate this time shows up in the equation as well. (ibid)
>
> Where is the amount per mile in this equation? (Ms. M Activity 4, 10/28/04)
>
> Keep that in mind, that rate is always multiplied to the variable, OK? (ibid)
>
> *d* equals 20 plus 2*t*. Excellent. Twenty is the amount added on. Two meters is our rate. (Ms. M Activity 3, 11/02/04)

These examples come from three different problems: Problem 2.2, 2.3 and 2.5. These are the three problems in which equations were a focus of the summary. This means that every time Ms. M discusses equations, she raises the issue of how the rate is represented in equations. In the first example, "What's the relationship between those two columns?" Ms. M asks the question to help students notice that the number of seconds Jerome (in Problem 2.2) walks is multiplied by 2.5 to find the number of meters he walks in that time. This leads to writing the equation $d = 2.5t$.

An extended example helps to clarify how Ms. M discussed equations differently from tables and graphs. Consider the following transcript, from Ms. M's summary of Problem 2.3 (Ms. M Activity 5 10/28/04).

M: We had this group over here, M, E and A, your question is, *what effect does increasing the amount pledged per mile; what does the effect have on the table, on the graph and on the equation?*[42]

...

E: The table, it, uh... it increases, so if... the numbers it increases...

...

E: OK, so... the amount pledged on the graph, per mile, it goes up and it increases.

M: OK

E: And that's what the effect has, it increases for the pledge.

M: So you're saying if the amount pledged per mile increases, the amount on the table is...

[42] The italics here indicate that she is quoting from the text.

E: increases

M: Increases as well. OK. Does Alana's throw anything off at all?

E: It increases.

M: It increases, but she's only getting fifty cents a mile, where these guys are getting two dollars a mile.

A: It's decreasing?

E: Well, I think it's the same cause she has the five dollars.

M: Cause she had that five dollar jump. OK. It's still going up, but not quite as fast, maybe.

To this point, Ms. M is pushing the student (E) to distinguish between the number in the table and the increment. When she asks, "Does Alana throw anything off?" she indicates that the student's seeming assertion that greater rates yield larger numbers in the table is incorrect. When this doesn't work, she supplies the answer, "It's still going up, but not quite as fast." Throughout this segment on tables, Ms. M keeps the focus on comparison of the tables for walkers with different rates. The conversation then turns to graphs.

M: Ma, can you keep going with how does it affect the graph?

Ma: It just makes a linear...

M: They make linear graphs. Are all three of them linear? Are all three of these graphs linear?

A: No.

... [the class revisits the definition of linear]

M: There are three straight lines, so M they're all linear. How does the

changing...

[interruption]

M: But what we're trying to figure out is how come these lines look different?

A: Because one's higher and the other one's lower.

M: Why is this green one higher than the red one? Why is it steeper than the red one?

Here, Ms. M introduces the term that she wants students to use, *steeper*, and pushes students to explain the term in context (i.e. to state what the steepness of a line says about the rate). As with the discussion of tables, the conversation stays focused on comparing the three graphs. The conversation continues a few moments later,

M: So Ma, back to your question. How does that dollar per mile change the graph? What happens to it as that dollar per mile increases?

Ma: The black line decreases...

M: The black line is flatter. What happens to one dollar per mile?

A: (whispers to Ma) It gets higher.

M: What's that word we used yesterday everybody?

Student: Steeper

M: Steeper. One dollar per mile it gets steeper. Two dollars per mile,

A: Gets higher

M: It gets even more steep. It gets even steeper than before. As you increase that amount per mile, the graph seems to be getting steeper.

Even though the nature of the questions is different here than it was for tables (e.g. she is asking for word recall), the focus remains on comparison. The comparison is lost when the conversation moves to equations.

M: All right, and then… the equations. Where do you see that amount per mile in the equation everybody? Or, sorry, you guys over here, that's still your question.

Where is the amount per mile in this equation?

A: In the graph?

M: The 1.

M: Where's the amount per mile in this equation?

A: The 2?

M: Where's the amount per mile in this equation?

A: The two-fifty.

M: The fifty. Right there, the point 5. That number that is always multiplied to that d. That distance. Keep that in mind, that rate is always multiplied to the variable, OK?

This extended example shows a profound change in discourse patterns from the patterns in CMP. Recall that CMP asks the same questions about each of the three representations, either in a list as in, "How can you tell whether a set of data is linear by examining a graph? A table? An equation?" (Teacher Edition, p. 34g), or by asking the same question three times; once for each representation. This example has demonstrated that Ms. M's discourse has a different pattern: tables and graphs are compared, equations are written, but not compared.

In the four warm-ups Ms. M conducts over the course of this Investigation, these patterns of use of visual mediators are largely consistent. One consistent difference is that Ms. M shows students visual mediators more frequently in warm-ups than she does in the course of the lessons. In her warm-up on 10/27/04, she asks students to use two given tables and two given graphs to determine which of two walkers is faster. On 10/28, she gives students a table and asks them to make a graph. On 10/29, she gives students a graph and has them make a table and write an equation. Finally, on 11/2, she shows students 2 tables, 3 graphs and 3 equations, and asks students to identify which representations represent linear relationships. In the course of these four warm-ups, students are shown 5 tables, 6 graphs and 3 equations while they make one of each. This contrast to the numbers in the lessons speaks to the different use of warm-ups in Ms. M's room. She uses the warm-ups to discuss one feature of linear relationships at a time. On 10/27, she focuses on comparing rates in tables, on 10/28, she focuses on using a table to identify a linear relationship, on 10/29, the warm-up is focused on making a table and writing an equation for a relationship when the graph is given and on 11/2, it focuses on identifying linearity in all three representations. She keeps these warm-ups brief by showing the representations rather than having students make them.

The warm-up on 11/2 is significant in that it adds to the classroom discourse a theme that would otherwise be absent. There is an extended discussion in this warm-up of how to identify linear relationships from their equations. As a result, this is a passing, rather than an absent theme.

Ms. M's use of visual mediators, in comparison with the themes in CMP's use, is summarized in Tables 10-12. The significant difference is in the use of equations. There

are two themes that receive different treatment from Ms. M than from CMP: y-intercepts and linearity. Ms. M places more emphasis on y-intercepts than does CMP, but less emphasis on identifying linearity in equations. It is not clear that these differences point to fundamental differences in the two discourses. However, in the *Word Use* section, I observed that Ms. M's differential use of the word *equation* from the words *table* and *graph* could signify a tension between the set-theoretic approach to functions of the traditional American curriculum and the relational-dependency approach in CMP.

Table 8: Themes in each discourse on tables.				
	Rate	Individual Value	*Y*-intercept	Linearity
CMP	**recurrent**	**recurrent**	**passing**	**recurrent**
Ms. M	**recurrent**	**recurrent**	**passing**	**passing**

Table 9: Themes in each discourse on graphs.				
	Rate	Individual Value	*Y*-intercept	Linearity
CMP	**recurrent**	**recurrent**	**passing**	**recurrent**
Ms. M	**recurrent**	**recurrent**	**recurrent**	**recurrent**

Table 10: Themes in each discourse on equations				
	Rate	Individual Value	*Y*-intercept	Linearity
CMP	**recurrent**	**passing**	**passing**	**recurrent**
Ms. M	**recurrent**	**passing**	**recurrent**	**passing**

This observation is bolstered by the use patterns of the three representations. Graphs, which receive scant attention in the set-theoretic approach to functions (e.g. in Dolciani, 1992, see Chapter V), receive equal attention through each of the four themes: rates, individual values, y-intercepts and linearity. The themes vary in their emphasis with respect to equations. In a pure set-theoretic approach, we would expect the theme of finding and representing individual values in equations (i.e. solving them) to be recurrent. This is not the case in Ms. M's discourse. Instead, slopes and y-intercepts are recurrent while individual values and linearity are passing. This does not support a claim that Ms. M's approach to functions is set-theoretic, but this assertion is not being made.

Instead, the claim is that one dimension along which Ms. M's discourse varies from that of CMP is the relative importance of set-theoretic and relational-dependency approaches to functions. On the one hand, Ms. M's approach seems to lack some qualities that would be important to the relational-dependency approach, such as equivalent treatment of tables, graphs and equations. On the other hand, her approach has some elements—lacking in CMP—that contribute to the relational-dependency approach, such as asking questions about the contextual meaning of the transformation of a graph, and consistency in treatment of the y-intercept across the three algebraic representations.

One more aspect of Ms. M's use of visual mediators deserves comment. When Ms. M endorses the use of tables with non-constant increments in her summary of Problem 2.4, the emphasis shifts from using tables to identify rates (a relational-dependency notion) to using tables to identify the correspondence between x and y (a set-theoretic notion).

Narratives

Narratives express the properties of and the relationships between the objects in a discourse. Narratives are different from the themes of the previous section in that themes in the use of visual mediators state *that* visual mediators have particular uses, but do not outline how to use them, nor do themes express relationships. In formal mathematical discourse, narratives are generally definitions and theorems. In classroom mathematical discourse, the forms for expressing relationship between objects of the discourse are less standardized.

The narratives in teachers' classroom discourse are first compared to the CMP narratives. That is, evidence is sought for the presence of the CMP narratives in each teacher's discourse. Second, transcripts are analyzed for the presence of additional narratives related to linearity. In this analysis, tasks, questions and exposition are studied and the main mathematical points compiled. The overall result is that each of the CMP narratives is also a narrative in Ms. M's discourse, but additionally there is one narrative in Ms. M's discourse that does not appear in CMP. Each of these narratives is listed next, together with at least one example from Ms. M's instruction.

Mathematical Variables Represent Contextual Variables

Recall the distinction made in the CMP Discourse section between mathematical variables and contextual variables. The former are used in mathematical representations such as tables, although they may retain contextual names (as in the heading "distance"), while the latter is a changing quantity in the context (as in a walker's distance). Recall too that Dolciani's mathematical variables tend to be unrelated to contexts. That is, x and y are largely studied on their own terms in Dolciani, while they tend to represent

contextual variables in CMP. In this, Ms. M's discourse closely mirrors that of CMP. as in, "$m = 2d$. The money equals two times the distance." (Ms. M Activity 4, 10/28/04). She writes the mathematical variables on the overhead projector, m and d, and names the contextual variables that they represent. There are no examples in Ms. M's summaries of mathematical variables that are not tied to contextual variables.

Tables, Graphs and Equations Represent Relationships between Variables

As in the case of CMP, combining two patterns in the use of the word equation in Ms. M's discourse yields this narrative. The fact that equations represent relationships together with the fact that equation is used similarly to graphs and tables implies that all three representations represent relationships between variables. This syllogism is true in CMP and it is true in Ms. M's discourse. The difference in Ms. M's use of equation—the emphasis on writing over comparing—does not pertain to this narrative, for she still speaks of equations representing relationships.

Tables with Constant Increments Represent Linear Relationships

As discussed in the *Visual Mediators* section, this narrative is absent from Ms. M's discourse in the Launch-Explore-Summary sequence. Instead, Ms. M establishes it in a warm-up. On 11/2, the final day of the investigation, Ms. M opens class with a warm-up that shows students 2 tables, 3 graphs and 3 equations. For each representation, students are asked to decide whether the relationship represented is linear. The case of the tables (reproduced in Figure 11) gives an opportunity to see how Ms. M's warm-ups contribute to the classroom discourse on linearity. Without this warm-up, this narrative would not be a part of the discourse. Ms. M has a discussion with students in which she

asks students to determine which of the tables or equations represents a linear relationship.

Figure 11: Ms. M's tables from the warm-up on 11/2/05.

x	2	4	6	8
y	-4	-8	-12	-16

d	M
0	10
1	12
2	15
3	18
4	17

A student, E, initially suggests the narrative by saying, "it [the (x,y) table in Figure 11] has an even pattern." Ms. M later refines this by equating the term *increment* with *continuous pattern*, which is equivalent to *even pattern*. In turning to the nonlinear table, Ms. M establishes first that there is a constant increment for the independent variable by asking, "What's happening in the x-column?" She walks students through the increments in the dependent variable, observing that they are non-constant, "Do we always add two?...We add three, then we add..?" With her use of the student's language of *pattern* and the nonlinear example she has chosen here, she leaves the constant increment condition implicit in this discussion. The narrative, *Tables with predictable patterns represent linear relationships* is an incomplete narrative. There are non-linear relationships in which we can identify predictable patterns in the dependent variables; quadratic relationships, for instance. This idea is revisited a few minutes later, when the class uses tables to check for linearity of the equation, $y = x^2$.

M: Do we have a linear pattern here?

Ma: No

M: Why do you say *no*, Ma?

Ma: Because they don't go, like in a constant rate. Because... I don't know. Yeah.

M: You're saying they don't go in a constant rate? What's the increase here?

Ma: Because the change is like 3 right there, and then it goes like 5, and then...

...

M: OK, it's not linear. So this one is not.

In this sequence, the language of *constant rate* is introduced by a student, Ma. Ms. M pushes her to be explicit about what this means and so establishes that linear relationships must have a pattern in the table, *and* that this pattern is a constant increment in the dependent variable (assuming a constant increment in the independent variable).

Straight-Line Graphs Represent Linear Relationships

In her launch for Problem 2.1, Ms. M introduces linear relationships by referring to a set of *x-y* graphs displayed at the back of the room[43]. She asks students which of the graphs is linear and then asks, "How do you know a linear relationship?" (Ms. M Activity 2, 10/25/04), accepting the answer *straight line*. In Problem 2.3, Ms. M revisits the question of linearity in graphs when a student struggles to decide whether all three graphs in the problem are linear. When she asks, "Who can explain what a linear graph looks like" (Ms. M Activity 5, 10/28/04), Ms. M re-introduces the criterion for linearity: the graph is a straight line.

Equations of Form $y = mx + b$ *or* $y = b + mx$ *Represent Linear Relationships*

[43] Due to technical difficulties, this conversation was not audio-recorded. It has been reconstructed from field notes.

As with CMP, Ms. M does not use the form of an equation to distinguish linear relationships from nonlinear ones in this investigation. However, given that a relationship is linear she does explicitly work on this preferred form. In addition to writing equations in slope-intercept form, $y = mx + b$[44], Ms. M pushes students to do so also, as in Problem 2.2,

> So Jade's equation, distance equals *t* times 2. What's the other way we could write that equation?
>
> ...
>
> [U]sually in our rules, we like to put the variables after the number, so you could rewrite them as $d = 1t$, $d = 2t$, $d = 2.5t$. (Ms. M Activity 2, 10/27/04)

In this passage, Ms. M encourages the writing of equations in the form $y = mx$ instead of $y = xm$. Nowhere does she write or discuss equations in standard form.

Consistent with this discussion of the slope-intercept form, the *y*-intercept is added on, as *b*, in Ms. M's discourse. In Problem 2.3, she discusses what equation would have a graph that is less steep than the ones in the problem. A student suggests $m = 0.5d$ where *m* is the money raised by walking *d* miles. Ms. M responds, "Oh, that's kind of like Alana's plan. But without the added five?" (Ms. M Activity 5, 10/28/04) In Problem 2.5, she discusses a follow-up question with her students, who have suggested writing the equation $d = 20 + 2t$. She summarizes, "20 is the amount added on. Two meters is our rate." This discussion is based in the slope-intercept form of a linear equation. While Ms. M uses neither the name nor the general form, she models it and reinforces its structure.

Rates Are Represented in Tables, Graphs and Equations

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[44] Recall from the discussion of this narrative in the case of CMP that $y = mx + b$ does not imply that the variables need to be *x* and *y*.

Ms. M's discourse parallels CMP's discourse with respect to this narrative, except that she is more explicit than CMP in endorsing it. Where CMP repeatedly asks, "How does the rate affect the table/graph/equation?" Ms. M asks this question and follows it up with specific statements and questions about each representation. For example, in Problem 2.2, she says, "Think about what E was doing when she found the rate in the table. Remember you can also look in between. What's the change here?" (Ms. M Activity 2, 10/26/04) In Problem 2.3, she asks:

> How does that dollar per mile change the graph [in comparison to a rate of 0.5 dollars per mile]?

and:

> Where is the amount per mile in this equation? (Ms. M Activity 5, 10/28/04)

In each of these examples, Ms. M is not discussing generalized comparisons, but the representation of a specific rate. The implication is that the rate *can* be found in the table, the graph and the equation because it is represented there. There is a slight difference between graphs on one hand and tables and equations on the other in Ms. M's discourse. With graphs Ms. M asks how an increase in a specific rate affects the graph. She is looking for a qualitative description (e.g. *steeper*), not a quantitative measure. She does not ask for students to identify the rate in the graph, only to make a comparison. That is, she does not expect students to be able to find slope, as the ratio $\frac{\Delta y}{\Delta x}$, from a graph at this point in the unit, while she does expect them to find the rate in a table (as the *change* between successive values) and in an equation (as the *multiplier* of the variable).

Y-intercepts Are Represented in Tables, Graphs and Equations

As in the CMP discourse, the representation of *y*-intercepts in tables, graphs and equations is less central than the representation of rates, but Ms. M does address the question. In Problem 2.3, Ms. M asks these questions,

> Where's that five-dollar donation represented in the table?
>
> Where's that five dollars represented on the graph?
>
> ...*y*-intercept. OK. Where is it represented in her equation? (Ms. M Activity 5, 10/28/04)

Although CMP does not use the vocabulary of *y*-intercept, but rather refers to the *y*-intercept in contextual language, it is a term that Ms. M had used with students in the curriculum she was piloting. She revisits it here. The use of *y*-intercept was initiated by Ms. M, not by her students.

Analysis of Tables, Graphs and Equations Can Answer Contextual Questions

Ms. M paraphrases the text's questions that establish this narrative. In Problem 2.3, she emphasizes when a student has used the equation to find a contextual value, Ms. M quotes this question from the text, and "...for a sponsor to owe a student 10 dollars, how many miles would a student have to walk under each pledge plan?" (CMP Student Edition, p. 20) In the ensuing conversation, Ms. M addresses all three representations.

There is little difference between the narratives in Ms. M's introductory discourse on linearity and those of CMP. All of the CMP narratives appear in Ms. M's discourse, although one of them does not appear in the Launch-Explore-Summary sequence of her lessons.

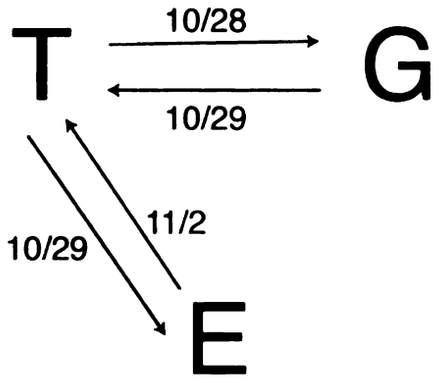
The observation was made in Chapter III that any mathematical text, whether written or spoken, contains many narratives. One difficult job for research is determining

which of these is significant and worthy of inclusion in analysis. Correspondingly there are many narratives in Ms. M's discourse that have not yet been listed, but most will not be analyzed here, just as there were many narratives that were left out of the analysis of the CMP and Dolciani discourses. All of Ms. M's tasks, questions and assertions related to linearity are represented by one of the CMP narratives, together with one more. There is one narrative in Ms. M's discourse that is closely related to the other narratives, but which is absent from CMP. Because this narrative, *Any of the three representations of a linear relationship can be generated from any of the others* is related topically to the others and is of similar intellectual power, it is included here and shows one dimension along which Ms. M's discourse differs from that of CMP.

Any of the Three Representations of a Linear Relationship Can Be Generated from Any of the Others

Three of Ms. M's warm-ups contribute to this narrative. On 10/28, Ms. M shows students a table and has them make a graph. On 10/29, she shows students a graph and has them generate the table and the equation. On 11/2, she leads students through the generation of a table from given equations in order to test for linearity. Figure 12 shows a diagram of these interrelationships. As a result, we can begin with an equation, make the table and then use the table to make the graph (as in Dolciani). We can begin with a graph, make a table and use this table to write an equation. Finally, we can move directly from a table to either an equation or a graph. Note from the diagram that the direct moves between equation and graph are not made in Ms. M's discourse.

Figure 12: Any representation can be generated from any of the others.



The narratives in Ms. M's discourse closely parallel those of CMP, with the exception of an added narrative that affords transitions between representations. This is not inconsistent with the findings in the *Visual Mediators* section. In that section, all of the themes in the use of visual mediators from CMP are identified in Ms. M's discourse, as are the narratives in this section. The themes are measured for their importance to the discourse, which the narratives are not. Narratives are identified as present or absent, without measure of their frequency. If such a measure were devised, we would expect to find similar patterns in the narratives to the ones found in the use of visual mediators.

The final section analyzing Ms. M's discourse pays attention to the ways of establishing and reinforcing these narratives—routines.

Routines

The routines for establishing and reinforcing narratives in discourse are often implicit. The problem for research is identifying routines through inference. The major question in the analysis of teachers' classroom discourse is, *On what basis are students to notice a narrative or to accept it as true?*

The routines for establishing narratives in CMP's discourse are also present in Ms. M's discourse. Ms. M asks questions, she provides multiple examples to lead students to conclusions through inductive reasoning and she expects them to find mathematical meaning in the contexts in the investigation. These are all consistent with the heuristic mathematical discourse described in the analysis of CMP. In addition to these heuristic routines, Ms. M explicitly states (tells) some narratives.

Asking Questions

One important difference between textbook discourse and classroom discourse is the presence of students in the latter. When CMP asks questions, students do not respond in the text; student responses occur in the classroom. The Initiation-Response-Evaluation (IRE, see Cazden, 1988) sequence of classroom discourse is absent from textbook discourse. As a result, the routines *asking questions* and *telling* (telling is described below) may be more similar to each other than the corresponding textbook routines. In considering the routine of asking questions in classroom discourse, I include instances of the IRE sequence, even when it veers close to telling. The distinguishing feature is whether the teacher is attempting to have students establish a narrative, or whether the teacher is establishing the narrative on her own. When Ms. M asks, "Where is that five dollar donation represented in the table?" (Ms. M Activity 5, 10/28/04), she is attempting to get students to establish the narrative, Y-*intercepts in tables are represented by the value of* x *when* y *is zero*, and she is using the routine of asking questions. When students respond correctly, the evaluation can be seen as a form of telling. When students respond incorrectly, Ms. M sometimes employs explicit telling in which she states the narrative herself, rather than evaluating a student's narrative. In any case, the use of questions to

push students to state narratives is a routine that is different from the routine of telling, even though the routines may be closely associated in many classroom situations.

Ms. M asks all of the same questions as in the shaded boxes for the problems in the Student Edition, either quoting directly or paraphrasing. As a result, the implicit endorsement of narratives through questioning that is outlined in the section on CMP's routines applies to Ms. M's discourse as well. She also asks many questions that are not in the curriculum. Of particular note, though, is that Ms. M tells narratives when her questions do not lead students to them.

Telling

The following transcript excerpt was used as an example for the narrative *Analysis of tables, graphs and equations can answer contextual questions* in Ms. M's discourse.

M: And we could check all these points on our table, too. Here's the point (3,3). Here's the point (2,4) for Gilberto. What was your other one? (4,6)? Four miles...

Student: Six dollars.

M: Ooh...four miles would be how much? (Ms. M Activity 5, 10/28/04)

In this excerpt, Ms. M establishes the narrative through telling. While she asks the question, "four miles would be how much?" she has already told students that the answer can be found by analyzing the table. Note that she asks for the contextual interpretation of the answer (six *dollars*), but she tells students that the table can be used to answer contextual questions, "[W]e could check all these points on our table, too."

A common pattern in Ms. M's instruction is to ask a series of questions and then to tell when students do not provide a correct and complete answer. Two examples follow. The first is from Problem 2.2.

M: Looking at those three graphs, where is Jerome's graph? What color is it?

Students: black

M: Black. Where is it in relation to the other two?

Student: Above?

M: OK, it's above the other two graphs. Imagine everybody starting to ride your bike. You're down at this point. You have three hills to go up. This hill, this hill and this hill. How does Jerome's hill compare to Jade's and Terry's?

G: Steep

M: It's steeper, OK? And that's a word that I want you to keep thinking about. That the person who had the fastest walking rate... what kind of line did he have? He had a much steeper line than the other two. (Ms. M Activity 2, 10/27/04)

Students are paying attention to the height of the points on the graph, rather than to the steepness. Ms. M introduces the word *steep*, since the students did not provide it, and then tells them, "...the person who had the fastest walking rate...had a much steeper line than the other two."

Inductive Reasoning

Ms. M follows the same structure as CMP in this regard. She has students make the same tables showing multiple walkers in order to highlight the importance of constant increment. She uses the same non-examples to ensure that students do not over generalize

the patterns they detect. She reinforces with telling the inductive reasoning that she sets students up to do (see preceding discussion of the routine, telling).

Deriving Mathematical Meaning from a Context

Just as CMP does, Ms. M has mathematical goals for students that are not based in contexts, yet much of her instruction takes place using contextual language, such as walking rate and head start. Recall three possibilities from Chapter V for how teachers might encourage students to derive mathematical meaning from a context: defining terms in contextual language, answering math questions by referring to the context and generalizing. Because the bulk of the language in Ms. M's lessons, like that in the student text, is contextual, only the third possibility is plausible. As with CMP, there are a few places in Ms. M's discourse where generalizing is explicitly encouraged. One example is the following.

T: Where's the five-dollar donation?

DE: Under her table.

T: Under the table, where under the table? There's lots of points here under hers.

DE: Across from the zero.

T: Across from the zero. Because remember she gets that donation of 5 bucks even if she doesn't walk. Where's that 5 dollars represented on the graph?

DE: Um... on the y-axis.

T: On the y-axis. Which point?

DE: 5

T: At 5. What's that point called? ER told us a little bit ago.

Student: y-intercept.

In this case, Ms. M asks students to give the general name, *y*-intercept, to the contextual value, donation. In this sense, she is asking them to generalize. Note, though, that y-*intercept* is previously established mathematical vocabulary so the process is functioning differently in Ms. M's classroom than in the CMP text. Ms. M asks students to see *donation* as an instance of a known pattern: *y*-intercepts. CMP asks students to use instances (e.g. donations and head starts) to construct the general pattern.

In the preceding examples of routines, Ms. M has been establishing new narratives or reinforcing established ones. It is also illuminating to observe the routines used when two or more conflicting narratives are under consideration in a classroom. In the first example of conflicting narratives, Ms. M has students working on a warm-up that asks which of two walkers whose time and distance data are displayed in a table is faster. The two tables are reproduced in Figure 13.

Figure 13: Ms. M's tables from the warm-up on 10/27/05.

	Distance (meters)	
Time (sec)	Xavier	Yves
0	0	0
1	1.5	2
2	3	4
3	4.5	6
4	6	8
5	7.5	10

In the following excerpt, four conflicting narratives arise:

1. Tables with whole numbers represent greater rates than do those with fractions.
2. Tables with greater numbers represent greater rates.
3. The value of the dependent variable when the independent variable is 1 is the rate.
4. Tables with greater increments for the dependent variable (given a common increment for the independent variable) have a greater rate.

The first three are raised by students, while the fourth is by Ms. M. Note that this fourth narrative is part of the narrative, *Rates are represented in tables, graphs and equations.* In the transcript that follows, the turns are numbered to facilitate the discussion.

1	M:	Who's the faster walker. Yeah, R?
2	R:	Yves.
3	M:	How come? Why do you think so?
4	R:	Cause he has whole numbers, not halves.
5	M:	He has whole numbers not halves. 2, 4, 6, 8, 10, yup there's some halves in here. Anybody agree with R? You shook your head, why not, TA?
6	TA:	I went like this!
7	M:	OK, so you agree. Why do you agree? Same reason? TA, same reason?
8	TA:	(unintelligible)
9	M:	Xavier? Yves?
10	TA:	The numbers are higher

11 M: These person's numbers are higher. OK. These numbers are higher. Are they always higher than what Xavier's walking at?

12 Student: Yeah.

13 M: Yeah. OK. Anything to add. E?

14 E: It's because Yves walks 2 meters per second and Xavier only walks... walks 1.5 per second.

15 M: How do you know that Yves walks two meters per second?

16 E: Because if you look at one second, it equals two.

17 M: And after one more second...

18 E: 4

19 M: And after one more second..

20 E: 6

21 M: What's happening each second?

22 E: It goes up by two.

23 M: It goes up by two.

24 M: What's happening over here?

25 E: It goes up by 1.5

26 M: It goes up by 1.5 each time. OK, so we saw bigger numbers, we saw the rates, E actually picked out, of 1.5 meters per second vs. 2 meters per second. We saw that change here of going by 2 each time, here we're only going by 1.5 each time.

27 So, R, I just want to get back to you. Halves vs. whole numbers. I don't, I don't quite see what you were at, what you

were saying.

28 R: (unintelligible)... is going up by halves...

29 M: So this first guy, the X guy, goes up by 1.5? OK, so not really a half but by one and a half. And this guy's going by..

30 R: two

31 M: Two, so a whole number which happens to be a bigger number than this one. Not so much that they're going up by half, half numbers, but that there's a bigger increase.

32 All right, so Yves seems to be the faster walker. (Ms. M Activity 1, 10/27/04)

In turn 4, R says, "Cause he has whole numbers, not halves." Rewritten in language equivalent to Ms. M's narratives, this narrative says, *Tables with whole numbers represent greater rates than do those with fractions*. Clearly, this competes with the narrative Ms. M seeks to establish, *Tables with greater increments for the dependent variable (given a common increment for the independent variable) have greater rates*, as whole numbers are not necessarily larger than mixed numbers and the conflicting narrative refers to the values, rather than to the increment. After examination of other conflicting narratives, Ms. M returns to R, who has now adopted the language of the intervening conversation, "...is *going up by* halves...(emphasis added)."

Ms. M tells the student (turns 29 and 31) that Xavier's increment is $1\frac{1}{2}$, not $\frac{1}{2}$, and that the fractional nature of the increment is irrelevant in any case. It is not clear whether this resolves the narrative conflict for the student, as he does not speak again, but it is a clear attempt by Ms. M to resolve the conflict by telling.

The second conflicting narrative, *Tables with greater numbers represent greater rates*, is introduced by a different student, TA, in turn 10. Again the narrative conflicts with the one Ms. M is trying to establish (in the general case, although the two are consistent when the y-intercept is zero). This can be seen in the table for Problem 2.3, for instance, where many of Alana's pledge totals are greater than Leanne's totals for the same distances, although Alana's pledge rate is less. Only when two linear relationships have a common y-intercept does a greater rate correspond directly with greater values for the dependent variable. Ms. M's attempt to resolve this conflict is subtler than in the previous example. She does not say that the student is wrong. Instead she pursues another narrative introduced by another student, E, in turn 14.

Ms. M signals that this narrative, *The value of the dependent variable when the independent variable is 1 is the rate* is more valued (presumably because it is closer to the desired narrative) by the time she spends on follow-up questions. In fact, it rests on the same assumption as TA's narrative: a zero y-intercept, but E uses the language of walking rates. Ms. M uses turn 17, "And after one more second..." to shift the focus from the individual value corresponding to 1 second in the table to the increment between consecutive values in the table. This is an example of the routine *asking questions*, although in this case the question, "What would the value be after one more second?" is implied. Ms. M uses her questions to guide the student to the desired narrative. When she asks, "What's happening over here?" in turn 24, E's reply, "It goes up by 1.5" indicates that Ms. M has successfully shifted this student's narrative.

In sum, like CMP's discourse on linearity, Ms. M's discourse is grounded in contexts and visual mediators. Ms. M's discussions of equations are focused on writing

them for the contextual problems in the investigation. She tends to attribute possession of the equations to the characters in the contexts, as in a discussion of "Jerome's equation." Rates are primarily associated with walking rates.

Ms. M's use of visual mediators differs more markedly from CMP's use in that she treats tables, graphs and equations differently. She uses tables to identify rates and individual values, with a lesser emphasis on *y*-intercepts. She steps outside the problems of CMP in order to discuss linearity in tables. She consistently uses graphs to identify rates, individual values, *y*-intercepts and linearity. She emphasizes rates and *y*-intercepts, consistent with her focus on writing equations, and deemphasizes using equations to find individual values and to identify linearity. While the emphasis on each representation is different, she draws connections across the representations in a way that CMP does not, as evidenced by the narrative unique to her discourse, *Any of the three representations of linear relationship can be generated from any of the others.*

CMP's textbook discourse on linearity is characterized in Chapter V in two ways; by its approach to functions, and by the nature of its mathematical discourse. CMP takes a relational-dependency approach to functions, and evidence of this could be seen at the microscopic level of word use as well as with a more macroscopic view of the narratives. Classroom discourse, of course, is more complicated.

In the microscopic view afforded by word use, Ms. M's introductory classroom discourse on linearity diverges from CMP's textbook discourse. A major pattern in CMP, fundamental to the relational-dependency approach to functions, is the association of *equation* with *relationship*. This pattern is not absent from Ms. M's discourse, but it is less prominent than in CMP. The prominence of *relationship* is lessened in part because

of an increased emphasis on the writing of equations in Ms. M's discourse. Ms. M has lengthy discussions with students about strategies for writing equations. Some of these conversations focus on computing one value (e.g. distance) when the other (e.g. time) is known, and others focus on using a given rate (e.g. 2 m/s) to write an equation. These two examples point to the dual nature of functions; in Ms. M's discourse, functions can be seen both as processes and as objects. At a more macroscopic view, we may consider equations as visual mediators. Of the three representations, equations in Ms. M's discourse differ the most from their role in CMP's discourse. Here, too, the discourse cannot be neatly characterized as relational-dependency or as set-theoretic. At the top-level analysis of narratives, there is more consistency with the CMP discourse. Ms. M and CMP have similar goals for students, and these reflect a relational-dependency approach to functions; all three representations come into play in Ms. M's discourse.

Chapter V gives us a second curriculum to compare against, and doing so makes clear that Ms. M is closer to a pure relational-dependency approach to functions than to a set-theoretic approach. While correspondence appears in Ms. M's classroom discourse on linearity, it is not a major idea. While Ms. M does not treat the three representations exactly equivalently, all three are used for a variety of purposes and there is no special priority afforded to functions, in contrast with the Dolciani discourse.

Finally, returning to the idea of learning a discourse as changing one's participation patterns in the discourse, we should consider the characteristics of participation in the model discourse. Chapter V asserted that the CMP model discourse is akin to a heuristic mathematical discourse, in which questions are asked, examples tested, generalizations made, and forms of metaphor employed. These techniques are present in

Ms. M's model discourse. She asks questions to get students to state narratives before she states them herself. Ms. M employs the examples from CMP, and creates her own in the warm-ups in order to lead students to use inductive reasoning. She encourages generalization from the contexts in the unit. Ms. M does employ the routine of telling when students do not state narratives in response to her questioning. Similarities and differences with the CMP discourse are examined in more detail in the third section of this chapter. First, the analysis turns to Ms. H's discourse.

Ms. H's Discourse

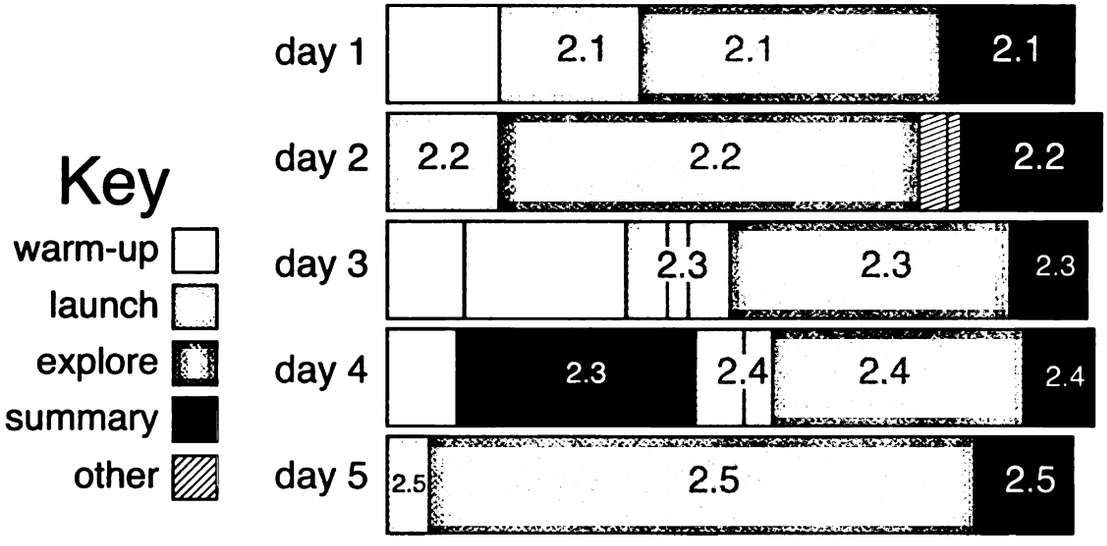
Ms. H teaches at a K-8 school that has approximately 480 students. The junior high (i.e. grades 7 and 8) students follow a six-period day in which each class last 53 minutes. Ms. H is the only math teacher in her building. This, together with a low turnover rate– the midyear mobility rate was 6% in 2003-2004– means nearly all of the 8^{th} graders in Ms. H's classes studied CMP with her in 7^{th} grade. The school's enrollment is 44% minority.

Ms. H spends 5 class periods working through Investigation 2 (see Figure 14). Prior to the lessons in the study, Ms. H opened the unit by having her students measure their own walking rates in yards per second, and then convert these to meters per second to match the walking rates in the investigation.

Unlike Ms. M's instruction, Ms. H typically fits the Launch-Explore-Summary sequence into a single class period. The one exception is with Problem 2.3, where she begins the summary at the end of the period and finishes it at the beginning of the next day's period. In addition, Ms. M has a warm-up three out of the five days. These warm-ups are related mathematically, and sometimes contextually to the problems in the

investigation. As with the analysis of Ms. M's discourse, the main analysis for Ms. H relies primarily on the Launch-Explore-Summary sequence and incorporates evidence from the warm-ups as necessary to fully describe the discourse.

Figure 14: Major activities in Ms. H's Investigation 2 lessons.



Word Use: Equation

Recall that *word use* refers to the patterns in the appearance of words in discourse. Frequency counts appear in parts of this section, when the crude comparisons they afford are useful and informative. Because the main comparison in this chapter is across discourse types (i.e. the comparison is between carefully edited textbooks and real-time teacher talk), frequency counts are used sparingly. The main thrust of the analysis is identifying the patterns. Once identified, the patterns are considered for what they say about the meaning of the words, and for how they contribute to teacher's classroom mathematical discourse. The first patterns to be analyzed are the patterns that appear in the CMP textbook discourse, then any new patterns that appear in Ms. H's classroom discourse, but not the textbook discourse, are analyzed.

Ms. H's use of the word *equation* shares with CMP frequent listing with *table* and *graph*. Ms. H does not use the words *relationship* or *situation* in any sentences together with equation. She shares with Ms. M the tendency to attribute possession of equations to the characters in the curriculum's contextual narratives.

Relationship

In the lessons of Investigation 2, Ms. H does not use the word *relationship*, nor the word *situation* when talking about equations. Together with the next pattern, *possession*, the absence of *relationship* when discussing equations points to the same tension in Ms. H's discourse that was present in Ms. M's discourse; the tension between a relational-dependency approach to functions and a set-theoretic approach.

Table and Graph

Ms. H frequently lists table, graph and equation in a manner similar to CMP. Indeed, the frequency is high enough for her students to be able to recite the trio, as in this example from her Launch of Problem 2.5.

H: Today we're all going to make a...

Student: Graph

H: And a...

Student: Table

H: And a...

Student: Equation.

H: All three of those. (Ms. H Activity 1, 11/10/04)

In the repeated use of the words *table*, *graph* and *equation*, Ms. H's word use is consistent with CMP's word use.

Possession

Ms. H shares with Ms. M the tendency to attribute possession of equations to the contextual characters. In passages cited above, Ms. H refers to "Jerome's equation" and "Emile's equation," among others.

In the analysis of Ms. M's discourse, I pointed out that speaking about equations as being possessions is a small indication that equations are objects, rather than processes. Equations as objects are consistent with a structural description of functions, and the set-theoretic approach. This pattern is present in Ms. H's discourse as well. By the same token, the absence of the term *relationship* in association with *equation* in Ms. H's discourse downplays the relational-dependency approach of CMP. At the microscopic view afforded by the word use analysis, Ms. H does associate *table* and *graph* with *equation* and so does not privilege *equation* over the other two terms—a pattern that is consistent with CMP's approach.

Word Use: Rate

Ms. H frequently associates *walking* with *rate*, as do CMP and Ms. M. She addresses how walking rates *relate* to equations, but does not use the CMP language of *affecting*. Finally, Ms. H does not use the term *rate of change*.

Eighty-five percent of Ms. H's sentences including the word *rate* include a reference to walking (recall that for CMP, the figure is 100% in the Student Edition and for Ms. M the figure is 69%), as in the following two examples.

> If Jerome is wondering how long it's gonna take people to get to the yogurt shop if they walk at different rates, then that means that they really have to walk at those different rates. (Ms. H Activity 2, 11/03/04)

We started off class talking about walking rate and speed. (Ms. H Activity 5, 11/04/04)

In her Summary of Problem 2.2, Ms. H asks questions about how the walking rates affect equations as in CMP, but she uses the word *relate* instead.

What...how...anything connecting the walking rate to the equation?

...

Can we relate the walking rate to the equation?

...

So just for Terry. Just for Terry, how does his walking rate relate to his equation? (Ms. H Activity 5, 11/04/04)

Note that *relate* leads to considering one equation at a time, while *affect* leads to consideration of a class of equations. That is, Ms. H asks how one ("his") rate affects one ("his") equation. The text asks how changing a walking rate affects the (general) equation, without reference to a particular rate or equation.

Finally, in contrast to both CMP and Ms. M, Ms. H does not use the word *rate* after Problem 2.2[45]. For example, when discussing the two boys' graphs in Problem 2.5, she says that one boy's line is, "Steeper. He's going... he walks... faster." (Ms. H Activity 3, 11/10/04) In this case, rates are referred to, but the word *rate* is not used.

Word Use: Linear

Ms. H uses the word *linear* in two problems in Investigation 2, Problems 2.3 and 2.5. In the summaries for these problems, she uses the term seven times. The term arises from student use in Problem 2.3, when a student asks during discussion of Alana's table,

[45] Although the word is vocalized when a student reads Problem 2.4 aloud.

"Wouldn't she not be linear?" Ms. H corrects her, referring to the previously established[46] fact that, "It doesn't matter what it starts at, it just matters that it goes up by the same amount each time." (Ms. H Activity 2, 11/09/04)

In each of Ms. H's uses of the word *linear* in Problem 2.5, the term modifies *relationship*, as in the example below.

> So you could tell from the graph that Henri had, that his distance was a linear relationship with his time. (Ms. H Activity 3, 11/10/04)

As exemplified in this passage, her use of linear is associated with identifying linearity in tables and graphs.

To summarize, Ms. H's use of the word *equation* differs from CMP's use in that it is not associated with *relationship* and she regularly discusses functions as possessions (i.e. objects). However consistent with CMP, she uses *equation* together with *table* and *graph*. The next section, *Visual Mediators*, affords a more macroscopic view which allows us to see greater differences in the treatment of equations than is revealed by examining only the word *equation*. Ms. H's use of *rate* is closer to CMP's use, especially the use in the Student Edition, in that it is largely contextual and related to walking. Ms. H's use of *linear* mirrors that of CMP, being associated with *relationship* and with a lesser emphasis on the term *nonlinear*.

The word use analysis, then, reveals some attention to both approaches to functions that were seen in curriculum: a relational-dependency approach and a set-theoretic approach. As is the case with Ms. M however, Ms. H's word use is best

[46] This was established not in a problem, but in a warm-up. More on this below.

characterized as deviating somewhat from a pure relational-dependency approach to functions than as a set-theoretic approach.

Visual Mediators

Taken as a group, Ms. H uses visual mediators similarly to CMP; the only deviation from CMP's use of visual mediators is in the use of equations. The themes in Ms. H's use of visual mediators are summarized in Tables 13–15. She uses the visual mediators to identify and represent three kinds of contextual information: rates, individual values and *y*-intercepts. She also uses visual mediators to identify linear relationships. Taken individually, the mediators have substantially different roles.

In the problems for the investigation, Ms. H is consistent with Ms. M and with CMP in showing only representations that she expects students already to have made. She does not show students tables, graphs or equations in the Launch, but reserves them for the Summary. Less frequently than Ms. M does, Ms. H sometimes displays a representation made by a student (e.g. as a poster or overhead transparency). Ms. H has one warm-up task that is not from the investigation, and as Ms. M's warm-ups do, it includes the display and analysis of the representations, but made by the teacher, not by the students.

Table 11: Themes in each discourse on tables.				
	Rate	Individual Value	*Y*-intercept	Linearity
CMP	**recurrent**	**recurrent**	**passing**	**recurrent**
Ms. H	**recurrent**	**recurrent**	**passing**	**recurrent**

Table 12: Themes in each discourse on graphs.				
	Rate	Individual Value	*Y*-intercept	Linearity
CMP	**recurrent**	**recurrent**	**passing**	**recurrent**
Ms. H	**recurrent**	**recurrent**	**passing**	**recurrent**

Table 13: Themes in each discourse on equations				
	Rate	Individual Value	*Y*-intercept	Linearity
CMP	**recurrent**	**passing**	**passing**	**recurrent**
Ms. H	**recurrent**	**absent**	**recurrent**	**absent**

Tables

Identifying rates in tables is a recurrent theme in Ms. H's discourse. The identification of individual values and *y*-intercepts in tables are passing themes. The use of tables to identify linear relationships is a recurrent theme.

In Problems 2.3 and 2.5, Ms. M has extended conversations with students about how rates appear in tables. In Problem 2.3, the discussion is about comparing the rates in tables, as in this passage.

> H: All right. I want to see if from the table I can tell who's getting the most per mile. What's this going up by? Well, you tell me. How can I, How can I look at the table and know who's getting the most money per mile?
>
> Student: Whoever goes up the most.
>
> H: What do you mean *goes up the most*? This one, I mean they're all going up. How can I tell which one goes up the most? Just explain a little more.
>
> ...
>
> H: They all go up by some number, but Gilberto's number... the number that he goes up by is the highest, which means he has the highest pledge amount, and [Alana's] is the smallest. (Ms. H Activity 5, 11/04/04)

This passage introduces the phrase *goes up by* which becomes the standard way for Ms. H and her students to discuss the increment in a table. The relationship between what a variable *goes up by* and the rate is clarified in Problem 2.5, where Ms. H uses a constant increment of 4 for the independent variable. Ms. H says, "I'm gonna go by 4s" as she makes a table on the overhead projector, referring to the increment in the time column (independent variable). This means that the rate will not be the increment in the

distance (dependent variable) column. Accordingly, she challenges students to convert the rate to the correct increment in turn 2, "How should I do this here? I went by different numbers than you did."

This substantiates the claim that identifying rates in tables is a recurrent theme in Ms. H's discourse. Ms. H spends less time discussing the appearance of individual values of the variables in tables than she does discussing the rates; it is a passing theme. She discusses individual values in a single turn in Problem 2.2, telling students, "[W]e can try to read exact values on the graph; sometimes it's difficult, so then we can always rely on our table." (Ms. H Activity 5, 11/04/04) The topic is the center of the summary of Problem 2.4.

Nearly the entire summary of Problem 2.4 focuses on finding individual values in tables. Yet, the fact that it does not reappear in multiple teacher turns in other places keeps it from being recurrent. Indeed, Ms. H is explicit about her preference for finding individual values in graphs in Problem 2.3.

> OK. On to letter C. And I like using the graph for these. I noticed a lot of people were using their table when they were doing their warm up. The question was, letter C, *how far does each person walk in... how much money do they get if they walk 8 miles?* (Ms. H Activity 2, 11/09/04)

The role of the y-intercept in a table is also a passing theme in the Launch-Explore-Summary sequence of Ms. H's instruction in Investigation 2. At the end of an extended discussion of equations in Problem 2.3, including how rates and y-intercepts appear in equations, Ms. H asks about tables.

> H: Ooh look back at your table that someone got a five-dollar donation. Si.

Si: Um, all the zeroes. Leanne, Gilberto, they got zero, but for Alana, they got five... she got five dollars when the distance was zero. (Ms. H Activity 2, 11/09/04)

All of the tables that Ms. H discusses or shows students have constant increments for the independent variables, with constant unit increments being the general rule. An exception to the latter pattern is the discussion from Problem 2.4 cited above, in which Ms. H uses a constant increment of 4 for the independent variable. In the overall pattern, this use of tables with constant increments is consistent with CMP, where the Student Edition shows only constant unit increments and the Teacher Edition shows one table with a constant non-unit increment. Recall that this contrasts with Ms. M, who sanctions tables made with non-constant increments. Tables with constant increments focus student attention on the rates—the relationships between successive values of the dependent variable. This is representative of a relational-dependency approach to functions.

The issue of identifying linearity in tables is recurrent, arising in the summaries for two problems in the investigation. Both times the theme is addressed, however, the exchange is brief. In the summary of Problem 2.3, one student has in mind that linear relationships have *y*-intercept equal to zero.

B: Wouldn't she not be linear, Alana?

H: Wouldn't she not be linear? Well if it goes... what did we say yesterday?

Student: If it goes up by the same amount, that's linear.

H: It doesn't matter what it starts at. It just matters that it goes up by the same amount every time. So hers is linear. And it looked linear, right? (Ms. H Activity 2, 11/09/04)

Recall that Alana's pledge plan includes a donation, which results in a non-zero y-intercept. Ms. H refers to the criterion for linearity in a table: a constant increment in the dependent variable (assuming a constant increment in the independent variable). She also briefly refers to the criterion for linearity in a graph when she says, "it looked linear, right?" Here, she refers to the fact that the graph of Alana's pledge plan is a straight line. The topic arises again in the summary for Problem 2.5, at the end of a discussion of rates.

H: So, everybody, what was Emile's distance going up by here?

Student: 10

H: Every time?

Students: Yup.

H: Yup. So does Emile have a linear relationship?

Students: yes.

H: Ok. Is it still linear? Here, they were both going up by four. Here this is going up by four and this is going up by 10. That's OK?... Yes. As long as they both go up by something and stick with that, then the relationship is linear. (Ms. M Activity 3, 11/02/04)

Three aspects to the use of tables are important to the relational-dependency approach to functions: that they are used to represent functions (as in CMP)—and not only to keep an organized list of solutions (as in Dolciani)—that rates can be identified in them, and that they can be used to determine the type of function being represented. All three are emphasized in Ms. H's discourse. The lesser aspects of Ms. H's discourse—identifying and representing y-intercepts and other individual values—are also less essential to CMP's relational-dependency approach to functions.

Graphs

The representation of rates and individual values in graphs are recurrent themes in Ms. H's discourse, while *y*-intercepts are a passing theme. Identifying linear relationships in graphs is a recurrent theme in Ms. H's discourse. In addition, she performs a manipulation with graphs that is quite similar to the one Ms. M performs.

Ms. H discusses the representation of rates in graphs in three problems; Problems 2.2, 2.3 and 2.5. In Problem 2.2 the discussion is brief. One of Ms. H's students expresses the same idea about rates in graphs as Ms. M's students do: higher points are related to faster rates.

H: I would like some observations about these graphs. S.

S: Uh...The faster they walked, the higher they went up, like that Terry, he had really, he was far away from everybody else and Jerome had the tallest graph. (Ms. H Activity 5, 11/04/04)

Following this student's observation, several turns are spent verifying the implied claim that Jerome is the fastest walker. The fact that faster walkers have *steeper,* rather than higher graphs remains implicit in this discussion, which then turns to whether the graphs represent linear relationships.

A more substantial discussion of the relationship between rates and graphs is in Problem 2.3. The term *steeper* is introduced first by a student in response to Ms. H's request for observations about the graphs. Ms. H reinforces this language and later ties the steepness of the lines to the rates.

H: [W]ho had the highest pledge amount?

J: Gilberto.

H: And what was it?

Student: Twenty, at ten

H: No his amount. His pledge amount... per mile

Students: Two dollars

H; Two dollars per mile. And how can I tell that from the graph that Gilberto has the highest pledge amount? J?

J: Cause it keeps going up.

H: It's the steepest one. Yup. And whose is the least steep?

Students: Leanne

Student: Alana (ibid)

Although Ms. H, like the CMP text, does not use the term *rate* in the discussion of this problem, this is the point of her question about "pledge amount...per mile." Notice that the term *pledge amount* leads a student to respond with an individual value. Amount per mile is a term for rate and Ms. H ties it to steepness in this passage.

Ms. H returns to the topic a third time later in the summary. In this case, she poses an alternate scenario in which one of the walkers, Leanne, decides to ask for an initial donation in addition to her $1 pledge per mile. Ms. H plots the points on the overhead projector as she discusses them with the class.

1 H: How's her graph gonna change?

2 Si: It's gonna start at three

3 Student: Won't it be steeper?

4 H: Will it be steeper?

5 Student: No

6	J:	It'll be bigger
7	Student:	It'll start at three.
8	H:	So, say she doesn't walk at all, C, how much money is she gonna get?
9	J:	Three dollars.
10	C:	Three dollars
11	H:	Three dollars for the three dollar donation, so I'm gonna go right up here, and then she walks a mile, she gets...
12	Students:	Four dollars.
13	H:	Another buck, that's four dollars. She walks two miles, she still gets her three-dollar donation, plus two, she's up at five. Let's look at it. Is it steeper? ...
14	H:	They're parallel lines, so they have the same steepness, but it starts up higher because of that donation. (ibid)

In turn 3, a student suggests that the donation will make Leanne's graph steeper. Ms. H makes a graph in order to demonstrate that, in fact, the donation has no effect on the steepness of the graph.

Ms. H discusses finding individual values in two Problems: 2.2 and 2.3. In each case, there is an extended use of the graphs to identify individual values of the contextual variables.

The relationship between the *y*-intercept and the graph is a passing theme in Ms. H's discourse. In Problem 2.2, she asks, "What do all the graphs have in common?" (ibid) One response from a student prompts this brief exchange,

K: They all start at zero.

H: They all start at zero. What else, N? (ibid)

This is a single teacher turn. The topic is discussed at more length in Problem 2.3, where Ms. M asks,

> How do we see the five dollar donation on the graph? How can I look at the graph and know that somebody got a five-dollar donation and somebody didn't? (Ms. H Activity 2, 11/09/04)

Several teacher and student turns follow, where the discussion is about where each graph "starts" and the amount each walker has "at zero [miles]".

Ms. M uses the form of the graph as a criterion for the linearity of a relationship. This topic is recurrent in Ms. H's use of visual mediators; she discusses it in two Problems: 2.2 and 2.5. While the term *linear* does not come up in Problem 2.2, linearity is the focus of a brief conversation touched off by a student observation about the graphs in the problem—that they all form straight lines. The theme is more explicit in Problem 2.5.

H: Now you already graphed Henri's line. Was it a line?

Students: yeah

H: I just called it a line, so yeah it was a line. So you could tell from the graph that Henri had, that his distance was a linear relationship with his time. (Ms. H Activity 3, 11/10/04)

To summarize the role of graphs in Ms. H's discourse, their use in finding two kinds of contextual information: rates and individual values are recurrent themes, as is their use for identifying linearity. The use of graphs in finding *y*-intercepts is a passing theme. This nearly matches the use of tables in Ms. H's discourse. Therefore, a consistent pattern in the use of visual mediators is emerging: with tables and graphs, the aspects essential to a relational-dependency approach to functions are emphasized. Equations have a different role.

Equations

In her discussion of equations, rates and *y*-intercepts are recurrent themes. The identification of individual values is absent. Likewise, the identification of a linear relationship from an equation is absent from Ms. H's discourse.

The topic of identifying rates in equations comes up in two problems in Ms. H's classroom. The first time this happens, she asks students to express how rates are related to equations, which her students struggle to do.

H; Can we relate the walking rate to the equation?

...

Does everybody see a 1 here?

Students: yes.

H: And a 1 here? and a 2 here? 2 here? 2.5? 2.5?

What's going on?

...

H: So just for Terry. Just for Terry, how does his walking rate relate to his equation?

S: Uh because.. the distance he's going is 1 second... meter

C: Going the same distance.

D: One second per meter

S: yeah.

H: They both have, it's... we'll come back to this on Monday. (Ms. H Activity 5, 11/04/04)

While the student who says, "One second per meter" is technically correct, this is not generalizable (i.e. the rate as expressed in meters per second, not seconds per meter in the equations) and thus the answer is not accepted as correct. Yet it is the closest to a correct answer that the students produce. Ms. H reinforces the importance of the theme in Problem 2.3. In the summary for that problem, a student suggests that the equation $m = 0.5 + 5d$ could express Alana's pledge plan. Ms. H replies, "So here, I'm gonna take the distance I walk... For every mile I walk, do I get five dollars?" (Ms. H Activity 7, 11/08/04) With this question, she suggests to the student that the coefficient in the equation (5 in the student's example) is the rate. She makes this explicit a moment later, "Point five and the d are next to each other because that's what I multiply. That's what the plan is. It's fifty cents a mile." (ibid)

Ms. H discusses the role of the y-intercept in two problems in the investigation, Problems 2.3 and 2.5. In Problem 2.3, Ms. H asks, "What ... if Leanne does not get out of the house that day, does not get off the couch, she walks zero. How much money is she going to get?" (Ms. M Activity 7, 11/08/04) The answer, five dollars, brings to the conversation the first non-zero y-intercept that the class has discussed. This takes place in the context of discussing the equations for the three pledge plans. Just prior, the class has

sorted out the coefficients and Ms. H has asked, referring to the equations for Gilberto and Leanne's pledge plans, "How come these guys don't have anything added on?" (ibid). She then goes on to ask, "So why did these guys not have a plus in the equation?" This discussion is targeted at the fact that the *y*-intercept shows up as the constant term in the slope-intercept form of a linear equation.

Finally, in Problem 2.5, Ms. H relates the *y*-intercept in that problem with the one in Problem 2.3, through discussion of the equation.

> H: So, you've seen having a plus five it was the other day it was a donation. Here a plus 45 is a... What is this plus 45?
>
> Students: Head start. (Ms. H Activity 3, 11/10/04)

In the cited discussions of rates in equations, Ms. H, like Ms. M, does not have students compare equations. Students compare tables ("the number that [Gilberto] goes up by is the highest, which means he has the highest pledge amount") and they compare graphs (with the language of *steeper*), but the discussion of equations is about writing them, not about comparing them.

In the analysis of Ms. M's discourse is a conjecture about how a traditionally trained teacher approaches the topics of tables, graphs and equations when teaching a reform curriculum; that such a teacher would struggle to use tables and graphs as extensively as CMP does and that equations might continue to take the prominent place in the teacher's discourse. This is not the case with Ms. M's discourse and it does not fit Ms. H's discourse either. A second conjecture, that the use of *equation* would correlate with *solving* and *correspondence* (set-theoretic notions) over *variation* and *relationship* (relational-dependency notions) more closely matches the evidence in each of these

cases—especially the lessened emphasis on relationships and the increased emphasis on equations as objects, and on the writing of equations to express a correspondence. Neither teacher strongly emphasizes the set-theoretic notions of function, but each shows a set-theoretic influence when addressing equations that she does not show when discussing the other two representations.

In summary, I reinforce this point. Ms. H consistently uses the three representations equivalently to identify rates. Beyond that however, she asks different kinds of questions about equations than she does about tables and graphs. When discussing tables in her Summaries, Ms. H asks questions that involve finding contextual information. When a graph proves difficult to read, she suggests that they turn to the table for a more precise answer. When discussing graphs in the summaries, Ms. H focuses on finding contextual information, as has just been suggested, but she also spends time having students use this information to identify the relationship that a graph represents. In Problem 2.2, she asks students to determine which of the unlabeled graphs on the overhead projector represents which walker. Neither of these questioning patterns– finding contextual information, or identifying characters– is present in Ms. H's discussion of equations. Instead, the focus is on writing equations, as in the following example.

H: Uh... equations... d equals...

Student: $1t$

H: t for Terry. Some people wrote $1t$...and, um... Jade's equation?

Student: d equals ...2

H: $2t$. Jerome's equation?

Student: 2 point 5 *t* (Ms. H Activity 5, 11/04/04)

In this passage, the contrast in the use of the representations can be seen. Tables and graphs represent aspects of the context, including rates, and time is spent discussing this fact. Equations, however, are written and then the conversation moves on.

Narratives

In the *Word Use* and *Visual Mediators* sections, I have argued that Ms. H's introductory discourse on linearity alters the CMP discourse—especially on the topic of equations. In the use of tables and graphs as visual mediators, Ms. H's discourse is similar to CMP's textbook discourse, but equations are treated differently from these other representations in terms of their emphasis. This differential emphasis also appears in the wider view of the discourse afforded by the focus on narratives.

Most of the CMP narratives are also narratives in Ms. H's discourse, with one exception, *Linearity is represented by tables, graphs and equations*. Each narrative is listed next, together with at least one example as evidence of this claim.

Mathematical Variables Represent Contextual Variables

This is a theme throughout Ms. H's instruction in this investigation. There are no examples in the summaries of mathematical variables that do not represent contextual variables. Instead, mention of variables is rooted in the contexts, as in this passage.

> The money's based on the distance. So it's always gonna be.. money equals, the money equals, the money equals. For Leanne, her money equaled her distance: a dollar per mile. Does that remind you of somebody else's equation from last week?(Ms. H Activity 7, 11/08/04)

Tables, Graphs and Equations Represent Relationships between Variables

In the cases of CMP's and Ms. M's discourses, two patterns in the use of the word *equation* establish this narrative: equations represent relationships and equations have a status roughly equivalent to tables and graphs. In Ms. H's discourse, the second of these patterns holds: equations are equivalent (in some ways) to tables and to graphs as representations, but the first does not hold: Ms. H does not use the term *relationship.* Instead, the narrative is implicit in how Ms. H discusses the representations. In the previous example, when Ms. H says, "The money's based on the distance," she is pointing to the relationship between the two variables, money and distance. This leads into a discussion of how to represent the relationship with an equation. Indeed, this is a common theme in Ms. H's discussions of equation writing; the goal is to express how one variable is based on, or found from the other. In a similar vein, Ms. H establishes through the discussion of rates in each of the representations that relationships between variables are under study. Nonetheless, the narrative is implicit in Ms. H's discourse.

Tables with Constant Increments Represent Linear Relationships

Ms. H explicitly states this narrative in response to the student who asks in Problem 2.3, "Wouldn't she not be linear?" (Ms. H Activity 2, 11/09/04). The student is referring to the walker with a non-zero y-intercept. Ms. H elicits the working definition of linear from a student, then restates it herself.

Student: If it goes up by the same amount, that's linear.

H: It doesn't matter what it starts at. It just matters that it goes up by the same amount every time. So hers is linear. And it looked linear, right? (ibid)

Straight-Line Graphs Represent Linear Relationships

As discussed in the visual mediators section, Ms. H uses graphs to identify linear relationships, and the criterion is that the graph must be a straight line, as in this example.

> I just called it a line, so yeah it was a line. So you could tell from the graph that Henri had, that his distance was a linear relationship with his time. (Ms. H Activity 3, 11/10/04)

In this teacher turn, the condition of being a line is equated with a linear relationship.

The major work on linearity in Ms. H's instruction takes place in a warm-up before Problem 2.3, where she also introduces the idea that a relationship can be "not linear". For this warm-up, Ms. H displays six tables. Each table is of the form in Figure 15, with independent variable values 1, 2, 3 and 4, with dependent variable values for the first three of these and the instructions, "Find the 4^{th} term and the equation." In the discussion of the warm-up, Ms. H addresses several aspects of linearity: how to identify linear relationships from tables and from graphs, and the idea of non-linearity.

Figure 15: The basic form of the tables in Ms. H's warm-up preceding Problem 2.3

d	*t*
1	3
2	6
3	9
4	
d=	

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The discussion of identifying linearity in a table uses the language of *goes up by*. This refers to the increment in either the independent variable or the dependent variable, as in this passage,

> It went by one point five every time. There's my equation... again, this is a linear relationship. Every time this went up by one, this went up by one point five. Did they go up by the same number? Did *t* and *d* go up by the same number? (ibid)

Similarly, Ms. H points out that non-linearity can be identified in a table (Figure 16) by what the variables go up by, "...it's not linear. It [the dependent variable] can't go up by different amounts here when this [the independent variable] is going up by the same amount every time. "(Ms. H Activity 2, 11/08/04)

Figure 16: A non-linear table from Ms. H's instruction

t	*d*
1	4
2	10
3	12
4	
d=	

Ms. H also focuses on identifying linearity in a graph. Here the requirement is that the points lie on a line. In the following example, Ms H plots the three points given in one of the tables from the warm-up as she speaks.

> This one is not linear. I graphed it. I tried doing this last hour. At 1, it's at 4. At 2

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it's at 5, 6, 7, 8, 9, 10. And then at 3, all of a sudden it's only at...12. Look at this graph. Is that a line? (ibid)

Because the graph is not a line, the relationship is deemed not linear. Later in the warm-up, Ms H reminds students of the importance of studying nonlinear relationships.

we talked about this. We said, if our whole book is about linear relationships, sometimes we're gonna need to look at some that...are **not** linear to help us recognize ... to appreciate the ones that are. (ibid)

Even in this warm-up, however, the identification and representation of linearity in equations is absent.

Equations of Form $y = mx + b$ *or* $y = b + mx$ *Represent Linear Relationships*

As noted in the *Visual Mediators* section, the representation of linearity in equations is an absent theme in Ms. H's discourse. It follows, then, that *how* linearity is represented must also be missing, and indeed it is.

However, all of the equations that Ms. M writes and discusses are of these two forms. She writes $y = mx + b$, but she explicitly endorses a student's suggestion that one could also write $y = b + mx$ in this passage,

Si: I got m=5+.5d

H: You could, you could um you could put the donation first and then the rate, that would be fine too.

Therefore, Ms. H introduces the appropriate forms for linear equations, but she never directly addresses the idea that this form is relevant to determining linearity. In doing so, she leaves open the possibility for competing narratives, such as *All equations must be of the form* $y = mx + b$ or $y = b + mx$.

Rates Are Represented in Tables, Graphs and Equations

As discussed in the visual mediators section, Ms. H's discourse includes rates as a recurrent theme with all three representations. Additionally, the means of determining rates in all three representations are addressed.

The y-intercept in a table is represented by the x-value when y is zero

The y-intercept in a graph is where the graph starts

The y-intercept in an equation is the amount added on

While these three narratives receive less emphasis than those about rates, these too are discussed in the visual mediators section, where *y*-intercepts were shown to be at least passing themes in each representation.

Analysis of Tables, Graphs and Equations Can Answer Contextual Questions

While the three representations are not equivalent in this narrative, it is the case that each representation can be used to answer some contextual questions according to Ms. H's discourse. Analysis of tables can answer questions about individual values, for instance, "[W]e can try to read exact values on the graph; sometimes it's difficult, so then we can always rely on our table." (Ms. H Activity 5, 11/04/04) This example also establishes that graphs can answer questions about individual values. This idea, however is absent from the discussion of equations. Instead, equations are useful for identifying starting values, "How can I tell in my equation that someone got a five dollar donation? And that someone else didn't?" (Ms. H Activity 2, 11/09/04)

Ms. H does not establish the final narrative that appears in Ms. M's discourse, *Any of the Three Representations of a Linear Relationship Can Be Generated from Any of the Others*. Instead, like CMP, she treats the three representations in parallel. Each has

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several purposes, many of them overlapping, but there is not an indication that one can be used to generate the others. The only exception is the warm-up on 11/08, in which the final line of each table calls for the writing of an equation (see Figures 15 and 16).

The narratives in Ms. H's parallel those of CMP, with the exception of the narrative describing how to identify linear relationships from equations. This is consistent with the findings in the *Visual Mediators* section, where equations are observed to be given different treatment from the other two representations. It is impossible to know the reasons for this difference. The speculation above about an interaction between Ms. H's mathematical education and experience prior to teaching CMP (which is also unknown, but likely involved a set-theoretic approach to functions) and the relational-dependency approach in CMP is just that: speculation. Regardless of the origins of the noted differences, they are clear. Equations have a different role from tables and graphs in Ms. H's discourse. Equations are less strongly associated with relational-dependency notions of function, while these notions are emphasized in her treatment of tables and graphs.

Routines

The routines for establishing and reinforcing narratives in CMP's discourse are also present in Ms. H's discourse. Ms. H asks questions, provides multiple examples to support inductive reasoning and she expects students to find mathematical meaning in the contexts of the problems. Additionally, Ms. H mirrors Ms. M in explicitly stating, or telling, narratives in a way that the CMP Student Edition does not.

Asking Questions

Ms. H asks the questions in the shaded boxes in the problems in the investigation, sometimes quoting directly, as in, "letter B. What effect does increasing the amount

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pledged per mile have on the table, on the graph and on the equation?" (Ms. H Activity 2, 11/09/04) and sometimes paraphrasing, as in, "Letter D, what if they each just wanna get 10 dollars? I'm just gonna walk till I get 10 dollars, then I'm quitting. OK?" (ibid) As in the case of Ms. M, it is notable that Ms. H often tells at the end of a series of questions, but it is also important to distinguish between asking questions and the telling that may follow.

Telling

Recall that the routine of telling differs from the routine of asking questions in that the latter routine gives students the opportunity to state narratives in their own terms; to contribute their ideas to the discourse. Telling states narratives in the teachers' language. When a teacher tells, the usual result is a lack of further narratives. When a teacher asks questions, multiple competing narratives often enter the discourse. Ms. H commonly asks a series of questions, and then tells at the end. Frequently, as in the following example, Ms. H's telling is a response to her students not stating it, or to their giving incorrect answers to her questions. In this example, Ms. H is trying to get students to compare equations. When her initial question goes unanswered, she asks a series of questions intended to highlight a part of the narrative, *Rates are represented in tables, graphs and equations,* namely: *The larger coefficient represents the greater rate.* She ends the exchange by stating this in contextual language.

H: How in the equation can I see who has the steepest... who has the highest pledge amount?

...

H: Does this two mean... where does this 2 come from?

Students: The pledge!

H: Two dollars per mile. And this 1 comes from?

J: From the one dollar

H: One dollar per mile, and this point five comes from?

J: Fifty cents per mile

H: Fifty cents per mile. So who has the highest pledge amount?

J: Gilberto!

H: Gilberto and you can see it in his equation because you see this 2 next to the d. And this only has a ...

Students: One

H: One next to the *d*, which is a smaller number. And this has a point five next to the *d*, which is the smallest number of all. So even if you had never seen the graph, to know that Gilberto's was the steepest line, you could look at the equation and see the biggest number that you're multiplying times the distance. So that means that he is getting the most per mile. (Ms. H Activity 2, 11/09/04)

In this example, Ms. H establishes a narrative by telling, *Larger rates are represented by larger coefficients in the equation.* She also tells when reinforcing a narrative. This is best seen when there are conflicting narratives, as in an example cited earlier (in the *Visual Mediators* section, under the subheading *Tables*).

B: Wouldn't she not be linear, Alana?

H: Wouldn't she not be linear? Well if it goes... what did we say yesterday?

Student: If it goes up by the same amount, that's linear.

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H: It doesn't matter what it starts at. It just matters that it goes up by the same amount every time. So hers is linear. And it looked linear, right?

...

H: Good point. They're all linear. They all go up by some number. (Ms. H Activity 2, 11/09/04)

The student introduces the narrative, *Linear relationships have zero y-intercept*. This conflicts with Ms. H's intended narrative, *Tables with constant increments represent linear relationships*, which does not depend on the *y*-intercept. In order to resolve the conflict, Ms. H asks a student to restate the rule, "it goes up by the same amount," which is a rephrasing of the intended narrative. Ms. H then restates it herself; she tells.

Inductive Reasoning

Ms. H follows the same structure here as CMP. Students make the same tables, graphs and equations and she uses the same non-examples of linearity that are in the student texts. The preceding example, under the routine *Telling* demonstrates this. In her attempt to lead students to the observation that the coefficient is the same as the rate, she raises the three examples in turn before she tells students the narrative.

Deriving Mathematical Meaning from a Context

Ms. H's instruction follows the student text quite closely. One result of this is that asking questions is one of the routines for establishing narratives in her discourse. Another result is that much of the discourse in her classroom is grounded in contexts, just as is the case with CMP. Yet, also like CMP, she has mathematical goals for her students. For instance, in the interview, Ms. H states that in addition to knowing how to write an equation,

> The other goals would be, I guess what I was just saying about them recognizing, you know, not having formal language for *y*-intercept, but recognizing that when the graph has a *y*-intercept other than zero, that means that there's something that's being added on independent of the variable. And again, they wouldn't have that formal language, but they might think of it as the head start in the race, or the donation part of the pledge sheets. (Ms. H Interview, 10/21/04)

Logically, then, it follows that she expects students to derive mathematical meaning from the contexts. In the problems in this investigation, this process remains implicit. Ms. H follows CMP's lead in not introducing the mathematical vocabulary of *slope* and y-*intercept*, so she cannot define these in terms of the contexts. She also follows the text's lead in asking questions about the contexts in the investigation, so there was no opportunity to refer back to a context in order to answer a non-contextual question. There is no place in the investigation that the student text refers to purely mathematical goals and there is no place where Ms. H does either.

Because Ms. H follows the CMP text closely, her mathematical discourse is founded on the contexts in the curriculum, and the three algebraic representations of functions play a prominent role as well. The detail-oriented analysis of Ms. H's word use reveals some differences that appear at the other levels of analysis—in particular the differential role played by equations in her discourse. Ms. H focuses on writing equations over comparing them, and spends no time on using them to identify linearity. The use of the word *equation* is not associated, in Ms. H's discourse, with *relationship*.

While this difference is an important one, it should not be overstated. The goal of the analysis is to characterize each teacher's introductory classroom discourse on

linearity. This necessitates characterizing the approach to functions in each discourse. Using a standard characterization of this topic, in which there are two approaches to functions—a relational-dependency approach and a set-theoretic approach—the commonalities of Ms. H's discourse with the relational-dependency approach attributed to CMP in the previous chapter are the stronger result than the differences with the CMP approach. A set-theoretic approach would privilege equations over the other two representations, which is clearly not the case in Ms. H's discourse. A set-theoretic approach would use the other two representations in service of equations, which again is not the case in Ms. H's discourse. Instead, Ms. H could be said to have a relational-dependency approach to functions when represented by tables and graphs, but a more set-theoretic approach to functions represented by equations.

Finally, learning is characterized in this study as changing one's participation patterns in a model discourse. We may consider the characteristics of participation in Ms. H's model discourse. In chapter IV the CMP model discourse is related to a heuristic mathematical discourse, in which questions are asked, examples tested, generalizations made, and forms of metaphor employed. This is in contrast to a formal mathematical discourse in which the heuristics are hidden and the focus is on straightforward, logical development. The heuristic techniques are present in Ms. H's model discourse. She asks questions to get students to state narratives before she states them herself. Like Ms. M, Ms. H uses the examples from CMP, and creates her own in the warm-ups in order to lead students to use inductive reasoning. The exception to the heuristics characterization is that Ms. H tells when students do not state narratives in response to her questioning. Telling is a classroom equivalent to stating a theorem in formal mathematical discourse.

Discussion

This final section has three parts. The first two parts address the research question, *What characterizes each teacher's introductory classroom discourse on linearity?* by recapping and summarizing the classroom discourses of each teacher. The focus is on characterizing the study of linearity, the use of contexts and the mathematical character of each discourse. The last part addresses the research question, *What changes does the discourse of the curriculum undergo in its implementation by each teacher?*

Ms. M's Discourse

The introductory lessons on linearity in Ms. M's classroom clearly indicate that the study of linearity is a study of relationships. At the level of word use, *linear* is not frequently used, but when it is, it is associated with *relationship* and with *situation*[47]. *Rate* is a central idea, mentioned 36 times by Ms. M in these lessons. As I argue in Chapter V, use of the term *rate* implies a focus on relationships, since rates involve two variables (e.g. time and distance). At the level of visual mediators, relationships appear through the identification of rates in discussions of each of the three representations. The narratives in Ms. M's discourse similarly reinforce the idea that linearity is about relationships through an emphasis on rates. Rates, in Ms. M's discourse, affect graphs by changing their steepness. They show up as increments in tables and multipliers of the dependent variable in equations. With respect to each of these three narratives, Ms. M uses language that includes both variables (e.g. money and distance), as in the example, "Where's the amount per mile in this equation? ... That number that is always multiplied to that *d*." (Ms. M Activity 5, 10/28/04) This contrasts, for instance, with a student turn in

[47] *Relationship* and *situation* are equivalent-used in the same way—as is the case in CMP.

the same lesson that mentions only one variable in trying to account for the differences in two graphs, "Because I think that's more money." (ibid)

Sfard's (2005a) four distinguishing features of mathematical discourse—word use, visual mediators, (endorsed) narratives and routines—are used in this study primarily to complement each other in characterizing each teacher's discourse and to compare discourses. The cases of *equation* as a word and equations as visual mediators provide an opportunity to make comparisons within a discourse. That is, this study considers the use of the word *equation* and it considers the use of equations as representations of functions. These give different views of the role of equations in Ms. M's discourse. Her use of the word *equation* is associated with the use of *table* and of *graph*—at the level of word use, there is an equivalence among the three representations, as evidenced in, "So we have tables, graphs and equations, all that can represent this linear situation." (Ms. M Activity 2, 10/27/04) Yet, when the use of equations as visual mediators is considered, there are differences among the three representations. Most notably, when Ms. M discusses equations, she focuses on writing them in contrast to CMP, where the focus is on describing how equations are affected by changes in walking rates. Ms. M asks long series of questions about the writing of specific equations, but does not ask these kinds of questions about making tables or graphs. Similarly, the themes in her use of equations as visual mediators differ from her themes in the uses of tables and of graphs. Without considering the uses of equations as mediators, in addition to the use of the word *equation*, these differences might go undetected.

Ms. M's introductory discourse on linearity also relies heavily on contexts—the same contexts as in CMP. In closely following the sequence of problems in CMP's

Investigation 2, Ms. M introduces slope through contextual language (e.g. *walking rate*) in each of the three algebraic representations. She also uses this contextual language to reinforce the mathematical term y-*intercept*, which had been introduced in a pilot curriculum prior to the teaching of *Moving Straight Ahead.* The contexts are not mere artifacts of the curriculum, however. When Ms. M steps out of the curricular framework of Launch-Explore-Summarize and gives students tasks of her own design (in the phase of the lesson she calls *warm-ups*), she uses the walking rate context to address mathematical topics. That is, this context is a central aspect of Ms. M's discourse on linearity that is retained outside the specific structure of the curriculum's tasks. There is one warm-up in which Ms. M uses the mathematical variables x and y instead of d and t. But in this case, there is no reference to the walking rate context. In each of the other warm-ups, walking rates are either an explicit part of the original task or Ms. M asks students to compare the tasks to the walking rate tasks from the curriculum.

Lampert (1990) describes her attempts to bring the practices, rather than only the products, of the discipline of mathematics to her students through conjecturing, testing and asking questions. Ms. M's discourse has elements of this kind of classroom mathematical discourse; she asks many questions and sets students up to use inductive reasoning (i.e. to generalize from examples). In Ms. M's discourse, there is also an indication that mathematics consists of some previously established facts; she asserts mathematical relationships (narratives) through telling, as in, "[T]he person who had the fastest walking rate...had a much steeper line than the other two." (Ms. M Activity 2, 10/27/04). The routine of telling is distinguished from the evaluation of a student's answer to a question by the person who states the narrative; when Ms. M asks a question

and then confirms a student's answer, the student has stated the narrative. When Ms. M states a narrative, she is telling. This routine is a pervasive pattern in Ms. M's discourse and it is generally associated with the routine of asking questions—Ms. M tends to tell only when her questions do not result in students stating narratives she wishes to endorse. As a result of the mixing of the CMP routines with the routine of telling, Ms. M's discourse has characteristics of a heuristic mathematical discourse, with some elements of a formal mathematical discourse. The more consistent emphasis is on the former.

Ms. H's Discourse

Ms. H's introductory discourse on linearity is also about relationships between variables. In her word use, this shows up in her frequent use of the term *rate* (20 mentions in the five lessons), and in the association, in Problem 2.5, between *linear* and *relationship*. Ms. H's use of visual mediators demonstrates the importance of relationships as well, with the identification and/or representation of rates being a recurrent theme for each of the three algebraic representations.

As with Ms. M's discourse above, we may take the opportunity to compare the use of the term *equation* with the use of equations as visual mediators in Ms. H's discourse. At the level of word use, Ms. H treats *table*, *graph* and *equation* equivalently. She lists these three terms together often enough for her students to chorally recite them in this order. When we pay attention to the use of equations, the objects, not just to the use of the word *equation*, we see differences that might not be detected at the word use level. Ms. H has students compare tables to each other ("The number that Gilberto goes up by is the highest"), and compare graphs to each other ("Let's look at it [the graph], is it steeper?"). But Ms. H does not have students compare equations to each other. Instead,

she focuses on writing equations. There are two contrasts here. The comparison task is missing from Ms. H's discourse, but also the writing of equations is emphasized in a way that the making of tables and graphs is not. Further differences between the use of *equation* and the use of equations emerge by considering the themes in the use of the visual mediators. The identification of individual values is a recurrent theme for tables and for graphs in Ms. H's discourse, but the corresponding theme is absent from her uses of equations. The same pattern holds for her the theme of linearity—the theme is recurrent for tables and for graphs, but absent for equations.

Ms. H's introductory discourse on linearity relies heavily on contexts because she closely follows the sequence of problems in the curriculum, resulting in the use of extensive contextual language: *walking rate*, *donation*, *head start*, etc. The ideas of slope, *y*-intercept and solutions are introduced by Ms. H in this series of lessons, but they are discussed by students and teacher alike in contextual language. Ms. H uses three warm-ups over the five lessons; two of them rely on the walking rate context, while one of them does not, except for the use of the variables *d* and *t*. These suggest *distance* and *time,* but the discussion of the task does not rely on this interpretation.

Ms. H's routines for establishing and reinforcing narratives are similar to Ms. M's. The heuristic nature of mathematical discourse in CMP's textbook discourse, and advocated by Lampert (1990) is present in the questions Ms. H asks, and in the examples used to encourage inductive reasoning. These examples (and corresponding non-examples) appear in the Launch-Explore-Summary sequence from the CMP text, but they also appear in the warm-ups that Ms. H creates to establish and reinforce narratives outside of this sequence. Ms. H indicates to students that some mathematical

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relationships are previously established by asserting (telling) narratives, as in this example, "It doesn't matter what it [the dependent variable] starts at [in a table]. It just matters that it goes up by the same amount every time." (Ms. H Activity 2, 11/09/04) Ms. H tends to tell when asking questions does not result in students stating narratives she endorses. As with Ms. M, the emphasis in Ms. H's discourse is on the heuristic nature of the CMP mathematical discourse, but the use of the routine of telling is an element of a more formal mathematical influence.

Changes in the CMP Discourse in Implementation

The introductory discourse on linearity in *Connected Mathematics* is founded on contexts. Students are expected to derive mathematical meaning from these contexts. The linear relationships embedded in these contexts are expressed through tables, graphs and equations. Each of these three visual mediators can be used interchangeably to convey contextual information of various kinds, including rates and individual values. The two teachers in this study, Ms. M and Ms. H similarly base their discourse on contexts—the ones in CMP. Each teacher follows the student text with few departures, but there are several notable changes that the curricular discourse undergoes as it is implemented in these classrooms.

There are differences in the format of the lessons—differences from the CMP format that are strikingly similar in the two classrooms. Not every narrative in these two classrooms is established or reinforced through the problems in Investigation 2. Each teacher uses warm-ups to establish narratives that are not established in the Launch-Explore-Summary (LES) sequence, or to reinforce narratives. It is not clear whether the decision to establish narratives outside of the LES sequence is intentional for either

teacher, but it is again remarkable that the teachers independently use the warm-ups, which are a deviation from the *form* of the curriculum, to keep their discourses on linearity consistent with the discourse of the text. Without these warm-ups, for instance, Ms. M's discourse would lack the narrative, *Linearity is represented in tables by a constant increment in the dependent variable*. By contrast, this narrative is prominent in Investigation 2 of *Moving Straight Ahead.*

Ms. M and Ms. H both use what they call *warm-up* activities. This name is a generic term for mathematical tasks given to students prior to the main body of the lesson. But for both, the warm-ups are used to introduce important content, not just to prime students for the ensuing instruction. Both teachers use the curriculum's contexts most of the time for their warm-ups (Ms. M: 4 out of 5 warm-ups, Ms. H: 2 out of 3), but each teacher has one warm-up that does not rely on a context.

In the analysis of curriculum and of classrooms, this study attempts to measure the importance of each use of visual mediators in each discourse by identifying themes and labeling each theme as *recurrent*, *passing* or *absent* in each discourse. Recurrent themes are more important than passing themes, which are obviously more important than absent ones. The most striking change in the textbook discourse occurs in each classroom: Themes in the use of equations have an importance different from these same themes in the use of tables and of graphs.

In CMP, the uses of the three representations have nearly identical importance—with the one exception being that finding individual values in equations is de-emphasized relative to finding individual values in the other two representations. In both Ms. M's and Ms. H's discourses, tables and graphs are used for the same purposes, and these uses have

similar importance as in CMP. Table 14 shows that the treatment of equations in each teacher's classroom differs markedly from the treatment in CMP. When we consider the two approaches to functions outlined in this study, we see that tables and graphs play a much more prominent role in the relational-dependency approach than in the set-theoretic approach, while equations are prominent in both. Due to the lack of emphasis on the other two representations, equations take center stage in a set-theoretic approach to functions. The set-theoretic approach, which is equation-centric, is standard in the traditional curriculum, such as Dolciani, while the relational-dependency approach (which is variable-centric) is more common in newer curriculum like CMP. Equations, while not center-stage, have different uses in a relational-dependency approach from their uses in a set-theoretic approach.

Table 14: Themes in each discourse on equations

	Rate	Individual Value	*Y*-intercept	Linearity
CMP	**recurrent**	**passing**	**passing**	**recurrent**
Ms. M	**recurrent**	**passing**	**recurrent**	**passing**
Ms. H	**recurrent**	**absent**	**recurrent**	**absent**

In these two teachers' discourses, we see the expansion of the roles of tables and graphs from the traditional approach, but not the full expansion of the role of equations. Recall the conjecture about approaches to functions by teachers who did not learn algebra in a reform curriculum. The initial conjecture was that such teachers would de-emphasize

tables and graphs, with which they were conjectured to be less familiar, while emphasizing the more familiar representation: equations. This conjecture is not supported by the data from these two classrooms. The uses of equations are, in fact, more limited in one of the classrooms. Ms. H does not address the finding of individual values in equations (which corresponds to solving—the main purpose of a traditional approach like Dolciani's), nor does she address the identification of linear relationships from their equations. Consider that this (identifying linear relationships in equations) depends on the form of the equation, and that in the traditional approach exemplified in this study by Dolciani, this form is the defining feature of linearity. In this sense, Ms. H's treatment of equations is in opposition to the set-theoretic approach, but also leaves out an important aspect of CMP's relational-dependency approach—identifying linear relationships from the form of equations.

In addition to the themes in the uses of equations as visual mediators, the results in this chapter show that each teacher emphasizes the writing of equations, while de-emphasizing the comparison tasks in the curriculum. Some form of comparison is highlighted for tables and for graphs in each teacher's classroom; for instance the graphs for three walkers are displayed on the same set of axes in both classrooms, and each teacher uses the word *steeper*. The discussions of equations in both classrooms focus on writing equations rather than on comparing them. It is unclear why this is so. Perhaps the teachers have made a choice to develop the comparison of equations in later investigations, perhaps they each perceive that their students struggle with equation writing and so focus their efforts on this topic.

Ms. M and Ms. H each make a change to the graph in Problem 2.3, either putting in or removing a donation to a walker's pledge plan and examining the effect of doing so on the graphs. In each case, the teacher does this in order to isolate the steepness of a line from its height. Each teacher demonstrates the change and then asks a form of the question, "How does this affect the situation?" In so doing, each teacher models two routines from the CMP discourse: asking questions and the non-example part of inductive reasoning. The student-generated narrative, *Larger rates yield higher points on the graph* can be seen as an overgeneralization from examples with zero y-intercept. By providing an example that has the same steepness, but different height, each teacher introduces a counterexample to the implicit inductive reasoning. It is remarkable that these two teachers make the same improvisational instructional move—CMP does not include instructions to do so.

Two aspects of the teachers' instruction—the use of warm-ups and the improvisation with graphs in Problem 2.3 —demonstrate that the teachers see the CMP discourse as the model for their own discourse. They modify the form of instruction to keep this model discourse intact.

Recall the research question, *What changes does the discourse of the curriculum undergo in its implementation by each teacher?* I have argued that answering this question is important because the writing and public discussion of curriculum often presumes an unproblematic relationship between published curriculum and classroom teaching. This study turns up ways in which this relationship is indeed unproblematic. The content of CMP, as expressed through the narratives in discourse, is largely unchanged in these two classrooms, for instance. Ms. M adds a narrative about transitions

among the three algebraic representations and Ms. H leaves out the form of linear equations, but otherwise the major narratives in CMP's textbook discourse on linearity are the same as the narratives in the classroom discourses.

Each teacher models the routines by which mathematical narratives are established and reinforced in CMP's textbook discourse. There is one important change to these routines that appears in each classroom, however. Each teacher uses the routine of telling—primarily at the end of a series of questions in which students do not state a narrative that the teacher endorses. When the routine of asking questions does not produce a student-stated narrative, or when it produces only narratives that conflict with the teacher's narratives, Ms. M and Ms. H tell. Under these conditions, each teacher states the desired narrative herself.

A significant goal of the introductory instruction on linearity in CMP, however, is changed in each teacher's classroom; the treatment of equations. Given the traditionally central role of equations in US curriculum (as exemplified in Dolciani), this may not be a surprising finding. But it is a significant finding. Some major changes in algebra instruction in the current curricular reform have taken hold in the two classrooms studied: an increased use of tables and graphs and the study of rates and relationships between variables among them. One major change has not: the equivalent status and use of tables, graphs and equations as representations of linear relationships. Full implementation of the curriculum reform would require this. The study of the consequences of this difference, and the problem of addressing it in the classrooms of practicing teachers, is a topic addressed in the concluding chapter.

Chapter VII: Conclusion

This study began with a question about classroom instruction in the wake of significant reforms in U.S. mathematics curriculum. Chapter III frames this question in the language of discourse analysis, using a communicational framework in which knowledge is characterized as participation in a discourse. The question, *What is being taught?* becomes *What is the model discourse?* Chapter IV turns this question into two research questions, *What characterizes the textbook discourse?* and *What characterizes the classroom discourse?*, and it adds a third research question, *What changes does the discourse of the curriculum undergo in its implementation by each teacher?* That same chapter makes an argument that these two discourse types—textbook and classroom—have some different properties, but that they can be reconciled and compared in a principled way. Chapter V uses a traditional American curriculum, *Mathematics, Structure and Method* (Dolciani, 1992) as a way of understanding the claims made about the reform curriculum under study, *Connected Mathematics* (CMP; Lappan, et al., 2001a). Finally, Chapter VI characterizes the teacher's model discourse in each of two classrooms and compares these to the textbook discourse of CMP.

Findings

This section recaps the important findings of this study. It points out the audiences who should care about these findings and suggests ways in which some of these findings might be put to use in the service of improving teaching in reform mathematics classrooms.

Textbook Discourse

The introductory textbook discourse in CMP is characterized as having a relational-dependency approach to functions in which variables and the relationships between them are the primary foci of instruction. This is contrasted with the set-theoretic approach to functions found in Dolciani, in which equations and their solutions are the primary foci. Most important among the differences between these two curricula (and therefore most important in the characterization of CMP) are those in the narratives and in the routines for establishing and reinforcing narratives.

Narratives

The narratives in CMP's introductory discourse on linearity have a parallel structure: tables, graphs and equations are treated as equivalent representations. Each can be used to represent rates and y-intercepts and to determine linearity. The equivalence of these representations pervades Investigation 2 of *Moving Straight Ahead.* Absent from the CMP narratives are relationships among these three representations. Dolciani, for its part, emphasizes particular relationships between the three representations, but these relationships establish priorities—equations are the most important representations, graphs have a lesser importance and tables are used only to create graphs.

Routines

Narratives depend on the content of the curriculum in that a unit on quadratic relationships would necessarily have different narratives (although their structure might be similar to those described here). Routines, on the other hand, can be expected to apply more generally. We should expect to see the routines identified in CMP in this study to

appear in other CMP units when students study other content, and likewise for the Dolciani routines. The differences between these two curricula likely exist throughout.

The routines in CMP model the actions that advanced knowers of mathematics take when working on a new and non-routine problem, and for this reason I have characterized CMP as having a heuristic mathematical discourse. CMP students are asked questions, encouraged to make conjectures from sets of examples and given contexts in which to find mathematical meaning—a form of metaphor. The routines in Dolciani, by contrast, model the formal presentation of mathematics in lectures, texts and journals. Students are given definitions and worked-out examples.

These differences in narratives and routines should be of interest to anyone selecting curriculum for classroom implementation. Principled selection pays attention to the mathematical content of a curriculum. The characterization of this content through narratives and routines provides a richer look at curriculum than a list of correspondence to state standards. An analysis such as this one could make a useful contribution to a curriculum selection process.

In the current curriculum reform, teachers—like the two in this study—are generally being asked to teach mathematics in a way different from how they learned it. Teachers are aware of this, but often discuss these differences in pedagogical language. They identify the use of calculators and the increased use of small groups in reform mathematics instruction as important differences, for instance. The analysis of curriculum in this study makes precise some of the differences between the *mathematics* teachers learned as students and the mathematics they are being asked to teach when they adopt new curriculum. Such insight should allow teachers to develop their own mathematical

knowledge. It should help them in their instruction by allowing them to make decisions about when to bring their own prior knowledge to bear and when to work within the curricular expectations. Without a precise understanding of the differences between what they learned as students and what the curriculum is trying to teach, teachers are unable to make such decisions consciously.

Classroom Discourse

Classroom discourse is not so neatly characterized as textbook discourse is, but each teacher in this study is seen to encourage a relational-dependency approach to functions, with modifications. Each teacher treats tables and graphs in a manner consistent with a purely relational-dependency approach to functions. Each teacher also treats equations differently from tables and graphs. This would not be predicted in a classroom with a relational-dependency approach to functions. In both classrooms, the goal in discussions of equations is to get to the equation—to write it. This contrasts with the teachers' goals in discussions of tables and graphs, where getting to the tables and graphs is assumed to be unproblematic and instead, there is comparison. When there are three walkers in a race, all three tables are shown simultaneously and the class discusses what each walker's distance *goes up by*. All three graphs are shown simultaneously and the class discusses which line is *steeper*. By contrast, equations are considered one at a time and there is no corresponding language for comparing them. Ms. H uses equations for fewer purposes than she uses tables and graphs; she omits discussion of identifying linearity and of finding individual values in equations. She uses tables and graphs for both of these purposes.

These results suggest some important considerations for several audiences. While the full analysis undertaken in this study is too detailed to be practical for practicing teachers to use in considering their own teaching, the results from these two classrooms might serve as case studies that point teachers to areas for reflection and possibly small action-research projects. For example, knowing some dimensions along which Ms. M and Ms. H treat equations differently from tables and graphs should lead teachers to consider how they might help students make comparisons among equations.

Curriculum developers can learn from the changes that curriculum undergoes as it is implemented in classrooms. In the introduction, I suggested that curriculum writing addresses idealized learners. In a similar way, it addresses idealized teachers as well. This study's findings add to the body of knowledge about teachers available to curriculum developers and so refine their image of the model teacher. In particular, the findings suggest that teachers of reform curriculum need some form of support in addressing the role of equations in the introduction to linear relationships. This support might come in the form of restructured student tasks, or in additional or restructured writing in the teacher materials.

Finally, many teachers new to CMP receive some form of professional development. The findings of this study are informative for those who provide this professional development. In identifying where teachers deviate from the curricular vision, this study should help to focus instructional time with teachers. For example, a leader of professional development might give an explicit model for comparing equations, since this is something that was lacking in each teacher's discourse.

Additional Findings

This section considers findings from this study that do not neatly fit as answers to the original research questions. Instead, these are observations, supported by evidence, on issues tangentially related to the research questions, and which could be developed further with the data from the study, or with some additional data collection.

Role of Warm-Ups

It is observed in Chapter VI that Ms. M and Ms. H each use warm-ups in their classrooms to begin a typical day of instruction. Some form of warm-up– in which students work on a small set of practice problems before the main lesson– is common in U.S. math classrooms. Warm-ups can be used in many ways, including practice for standardized tests, review of previously learned material and preparation for the present day's lesson.

CMP does not suggest the use of warm-ups in either the Student Edition or the Teacher Edition. Thus, it is remarkable that these two teachers use warm-ups in very similar ways. Ms. M uses four warm-ups over seven lessons, while Ms. H uses three warm-ups over five lessons. Each of these teachers uses her warm-ups to reinforce ideas in the curriculum and to introduce ideas that did not surface in the main body of previous lessons. For instance, Ms. M uses a warm-up to address how to identify and represent linearity in a table. While the other themes involving the use of tables as visual mediators are discussed in the lessons, she does not discuss this one until the last day of the investigation. On that last day, Ms. M shows students tables, graphs and equations on the overhead projector and has students determine which represent linear relationships and which do not. This leads to establishing the criterion *Linear relationships are represented by constant change in a table.*

This use of warm-ups to establish themes and narratives that would otherwise be absent from a teacher's discourse suggests insight and reflection on the part of these teachers. These seem not to be teachers blindly following a curricular path, but teachers who understand the path well enough to adjust their lessons in ways that keep them on the path. Ms. M does not use the warm-up described above only for practice and reinforcement; she uses this warm-up– at this time– to establish an important narrative. That both teachers use warm-ups in this way is remarkable. This similarity raises a question, as well. One may wonder how widespread this use of warm-ups is, and at a higher level, one may wonder how common the reflection attributed to Ms. M and to Ms. H are among American mathematics teachers.

The Relational-Dependency Approach to Functions in the Classroom

CMP's textbook discourse on linearity is representative of a relational-dependency approach to functions in which variables and rates are center stage and the three algebraic representations are of equal importance. In the two classroom discourses in this study, the relational-dependency approach to functions is evident, but there are modifications. In particular, the role of equations changes as the textbook discourse becomes classroom discourse.

Equations do not take the prominent role that they would in a set-theoretic approach to functions, but they are also not treated equivalently to the other representations. Equations, in both Ms. M's and Ms. H's classrooms, are less strongly associated with the representations of relationships than are tables and graphs, less strongly associated with linearity than the other two representations and less likely to be analyzed in comparison tasks than the other two. In Chapter VI, I speculate about the

causes for this—that it might stem from an interaction between these teachers' traditional training and the expectations of reform curriculum, for instance—but it remains speculation. At its root, this speculation asks about the relationship between teacher knowledge and classroom instruction—a deep and lively current research domain in mathematics education (see, e.g. Ball, 2002).

The modifications of the relational-dependency approach to functions in the transition from published curriculum to classroom are remarkably similar in the two cases examined in this study. As with the preceding topic, further research could investigate questions about whether this is coincidental or indicative of some more widespread phenomenon by looking at a larger set of classrooms.

Contributions of the Theoretical Framework

I have characterized the decision to adapt Sfard's communicational framework for this study as a pragmatic one. Characterizing teaching and learning in terms of discourse makes these terms operative and means that the data for the study is directly observable. Communicational behavior in the classroom can be observed and captured (in large part) on audio and videotape in a way that structures in the mind cannot. Yet, the framework brings other advantages.

Systematic Look at Curriculum

From my classroom experience, I know the CMP unit *Moving Straight Ahead* very well. The analysis of this unit was guided by substantial prior knowledge about its structure, style and content. Using the four characteristics of mathematical discourse—word use, visual mediators, narratives and routines—provided a way to write about this

unit that could be repeated by someone without my prior knowledge. The framework provided a sense of validity to the CMP analysis.

Just as important, the framework gave me a principled way to examine a curriculum I did not know well: Dolciani. The four characteristics of mathematical discourse are independent of my deep knowledge of CMP so this knowledge did not taint my examination of Dolciani. Instead, the communicational framework gave me the same tool to apply in the two cases, producing comparable results.

Textbooks and Classrooms

On a related note, the framework provided a way to compare textbook discourse to classroom discourse by considering them as two instances of the same phenomenon: discourse. Because textbook writing was considered as communicational behavior in this study, it could be examined in the same way as the communicational behavior of the two teachers, using the same characteristics to describe it. While the precise methods differed to account for some inherent differences between the two discourse types (e.g. that students help to shape the classroom discourse, but not textbook discourse), the four characteristics of mathematical discourse could be applied equally well to describing each discourse type.

Multiple Perspectives on Teaching

I have argued that three of the four characteristics of mathematical discourse in this study give views at different levels—word use is a close-up view of discourse, the use of visual mediators is at a medium range, and narratives gives a long-range view of discourse. These three views provided some useful contrasts. For instance, I have analyzed the use of the word *equation* in each teacher's discourse and the use of

equations as representations in each teacher's discourse. In this case, the word use is not consistent with the visual mediator use. That is, the word *equation* is used in ways similar to *graph* and to *table*, but equations themselves are put to different uses than are tables and graphs, as discussed above. The communicational framework gave me a way to find and describe these differences.

Limitations

This study investigates the relationship between published curriculum and classroom instruction. There are three major limitations of this study: student learning is not examined, teaching is not examined in a Dolciani classroom, and the research questions are not examined longitudinally.

Student Learning

In the analysis of teachers' classroom discourse, I have noted several times that this discourse differs from textbook discourse in that textbook discourse is unaffected by students and their contributions to classroom discourse. While the teacher models the mathematical discourse, there is some degree of negotiation between the students and the teacher that can be seen in this study when teachers adopt student language such as *goes up by* or *constant pattern* for the increment in a table. More fundamentally, though, this study has left out a close examination of the changes in student discourse participation over the course of the investigation—an examination of student learning.

In Chapter III, I argue for the importance of studying teacher discourse in isolation from student discourse on the grounds that a major question of curriculum design is how the written curriculum impacts teachers' instruction. The results of this study are informative for curriculum design, but it would also be useful to know what

these students have learned. Data was collected in the course of this study that could be brought to bear on this question. The classroom audiotapes contain student talk during the Explore phase of each of the lessons in this study, and there is a small amount of pencil and paper assessment data that could be analyzed and used to understand students' discourse patterns in these two classrooms.

This study is important because it helps us to understand how to improve teaching in a reform mathematics curriculum environment and we can expect this to lead to increased student learning. A limitation of this study is the lack of analysis of student learning in these classrooms and with these textbooks.

Dolciani classrooms

Chapter V provides a meaningful set of contrasts in curricular approaches. In particular, CMP's relational-dependency approach to functions contrasts with Dolciani's set-theoretic approach and differences between these appear in each of the four aspects of mathematical discourse. This study does not examine classroom practice using the Dolciani curriculum. There are two further questions that could be answered with data from Dolciani classrooms. *How does the discourse on linearity in Dolciani change as it is implemented?* is the analog of research question 3 about the changes from the CMP text to the classroom. *How does classroom discourse differ in a Dolciani classroom from that in a CMP classroom?* is the classroom analog of the textbook comparison. Each of these questions is pertinent to the important issues of mathematics curriculum change that have been raised elsewhere in this dissertation.

A Longitudinal Look

The routine of *deriving mathematical meaning from a context* is what sparked the original ideas for this study. When I propose, in Chapters V and VI, that there are at least three possibilities for how students are to derive mathematical meaning from a context, these possibilities are based on theory, observation and reflection on my own teaching practice. Recall that the three suggested possibilities are,

1. Mathematical vocabulary is defined in contextual terms.
2. A mathematical question is asked (by the teacher or by the student) and the teacher models or encourages seeking the answer by thinking about the context.
3. The mathematical language is developed as a generalization of the specific contextual ideas.

In Investigation 2 of *Moving Straight Ahead,* little new mathematical language is generally developed. In particular, the vocabulary items *slope* and y-*intercept* are not developed until later in the unit. This means that inferences needed to be made for this study about the direction the text and the teacher *might* be headed as the contextual language is developed. A study of a classroom that lasted through the entire unit would be informative about this process. If the same research techniques were applied to 4–6 weeks worth of instruction, the researcher would expect to find moments where mathematical vocabulary is introduced, or where contextual language seems to be pointing to a concept that goes beyond the context in question. In a week's worth of instruction in which all of the problems are tied to contexts, these processes were not directly observed. Thus, it is a limitation of this study that it does not look at a larger

section of the curriculum. In Chapter II I describe the National Research Council (NRC) report on studies of curriculum effectiveness (2004). This report calls for studies that examine at least an entire grade. This study is much more limited.

Another potential benefit of studying a larger section of the curriculum would be investigation of the hypothesis that the discourse patterns in *Moving Straight Ahead* are representative of those in the curriculum as a whole. My informal knowledge of the curriculum suggests that this is the case, but I have not applied the techniques of this study to other portions of the curriculum.

Conclusion

This chapter has summarized the important findings of this study as a whole, pertaining to two types of classroom discourse—textbook and teacher. A small number of additional findings have been addressed. The chapter has suggested how the communicational framework contributed to progress on the study's central problem of describing the use of contexts to develop mathematical meaning in a reform mathematics curriculum. Finally, limitations of the present study have been considered, including the singular focus on the teacher in the classroom data and the small segment of the curriculum that is studied closely. These limitations are described in a way that points to important analyses of existing data, as well as to future data collection and analysis. Such future research should expand our understanding of the mathematical goals of curriculum materials and how these materials and goals interact with classroom instruction.

References

Ball, D. L. (1992). Magical hopes: Manipulatives and the reform of mathematics education. *American Educator, 16*(2), 14-18, 46-47.

Ball, D. L. (2002). Knowing mathematics for teaching: Relations between research and practice. *Mathematics and education reform newsletter, 14*(3), 1–5.

Ben-Chaim, D., Fey, J.T., Fitzgeraled, W.M., Benedetto, C., & Miller, J. (1998). Proportional reasoning among 7th grade students with different curricular experiences. *Educational Studies in Mathematics, 36,* 247–273.

Boaler, J. (1993). The role of contexts in the mathematics classroom: Do they make mathematics more "real"?. *For the Learning of Mathematics, 13*(2), 12–17.

Cazden, C. (1988). *Classroom discourse: The language of teaching and learning.* Portsmouth, NH: Heinemann.

Cobb, P. (1994). Where is the mind? Constructivist and sociological perspectives on mathematical development. *Educational Researcher, 23*(7). 13–20.

College Preparatory Mathematics (CPM). (2002). *Foundations for algebra, year 2.* Sacramento, CA: CPM Educational Program.

Danielson, C. (2005). Perimeter in the curriculum. *For the learning of mathematics. 25*(1), 30-33.

Dolciani, M. P., Sorgenfrey, R. H. & Graham, J.A. (1992). *Mathematics: Structure and method, course 2.* Boston: Houghton Mifflin.

Greeno, J. (2003). Situative research relevant to standards for school mathematics. In J. Kilpatrick, W. G. Martin & D. Schifter (Eds.) *A research companion to principles and standards for school mathematics.* Reston, VA: National Council of Teachers of Mathematics.

Grimshaw, A. (2003). Genres, registers, and contexts of discourse. *Handbook of discourse processes.* A. Graesser, M. Gernsbacher & S. Goldman (Eds.). Malwah, NJ: Lawrence Erlbaum Associates.

Halliday, M.A.K. (1978). *Language as a social semiotic.* Baltimore: University Park Press.

Herbel-Eisenmann, B. (2000). *How discourse structures norms: A tale of two middle school mathematics classrooms.* Unpublished doctoral dissertation. Michigan State University: East Lansing.

Herbel-Eisenmann, B. (2002). Using student contributions and multiple representations to develop mathematical language. *Mathematics Teaching in the Middle School, 8*(2), 100–104.

Janvier, C. (1981). Use of situations in mathematics education. *Educational Studies in Mathematics, 12*(1), 113–122.

Kieran, C. (1993). Functions, graphing and technology: Integrating research on learning and instruction. In T. Romberg, E. Fennema & T. Carpenter (Eds.) *Integrating research on the graphical representation of functions.* Hillsdale, NJ: Lawrence Erlbaum Associates.

Klein, D. (n.d.) Retrieved from www.csun.edu/~vcmth00m/noyce.htm July 21, 2005.

Lakatos, I. (1976). *Proofs and refutations: The logic of mathematical discovery.* New York: Cambridge University Press.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal, 27*(1), 29–63.

Lappan, G., Fey, J.T., Fitzgerald, W., Friel, S. & Phillips, E. (2001a). *Connected Mathematics.* Upper Saddle River, NJ: Prentice Hall.

Lappan, G., Fey, J.T., Fitzgerald, W., Friel, S. & Phillips, E. (2001b). *Connected Mathematics: Moving Straight Ahead, Student Edition.* Upper Saddle River, NJ: Prentice Hall.

Lappan, G., Fey, J.T., Fitzgerald, W., Friel, S. & Phillips, E. (2001c). *Connected Mathematics: Moving Straight Ahead, Teacher Edition.* Upper Saddle River, NJ: Prentice Hall.

Lemke, J.L. (1998) Analysing verbal data: principles, methods, and problems, in : K. Tobin & B. Fraser, eds. *International handbook of science education* . Dordrecht: Kluwer.

Lobato, J., Ellis, A.B. & Muñoz, R. (2003). How "focusing phenomena" in the instructional environment support individual students' generalizations. *Mathematical Thinking and Learning, 5*(1), 1–36.

Mamokgethi, S., Adler, J., Reed, Y. & Bapoo, A. (2002). Incomplete journeys: Code-switching and other language practices in mathematics, science and English-language classrooms in South Africa. *Learning and education, 16*, 128–149.

Moschkovich, J. (1996). Moving up and getting steeper: Negotiating shared descriptions of linear graphs. *Journal of the Learning Sciences, 5*(3), 239–277.

National Council of Teachers of Mathematics (NCTM). (1989). *Curriculum and Evaluation Standards for School Mathematics.* Reston, VA: NCTM.

National Council of Teachers of Mathematics (NCTM). (2000). *Principles and Standards for School Mathematics.* Reston, VA: NCTM.

National Research Council (NRC). (2004). *On evaluating curricular effectiveness: Judging the quality of K-12 mathematics evaluations.* Washington, D.C.: National Academies Press.

Pimm, D. (1987). *Speaking mathematically.* New York: Routledge.

Polya, G. (1973). *How to solve it: A new aspect of mathematical method.* Princeton, NJ: Princeton University Press.

Resnick, L. B. (1988). Treating mathematics as an ill-structured discipline. In R. I. Charles & E. A. Silver (Eds.), *Research agenda for mathematics education: Vol. 3. The teaching and assessing of mathematical problem solving* (pp. 32-60). Hillsdale, NJ: Erlbaum.

Ridgway, J.E., Zawojewski, J.S., Hoover, M.N. & Lambdin, D.V. (2003). Student attainment in the *Connected Mathematics* curriculum. In S.L. Senk & D. R. Thompson (Eds.) *Standards-based school mathematics curricula: What are they? What do students learn?* (pp. 181–224). Mahway, NJ: Lawrence Erlbaum Associates.

Riordan, J. E., & Noyce, P. E. (2001). The impact of two standards-based mathematics curricula on student achievement in Massachusetts. *Journal for Research in Mathematics Education*, 32(4), 368-398.

Schwab, J. (1978). Education and the structure of disciplines. *Science, curriculum and liberal education.* Chicago: University of Chicago Press.

Senk, S.L. & Thompson, D.R. (Eds.). (2003). *Standards-based mathematics curricula: What are they? What do students learn?* Mahwah, NJ: Lawrence Erlbaum Associates.

Sfard, A. (1998). On two metaphors for learning and the dangers of choosing just one. *Educational Researcher, 27*(2), 4–13.

Sfard, A. (2001). Learning mathematics as developing a discourse. In R. Speiser, C. Maher, C. Walter (Eds), Proceedings of 21st Conference of PME-NA (pp. 23-44).

Columbus, Ohio: Clearing House for Science, Mathematics, and Environmental Education

Sfard, A. (2005a). When the rules of discourse change, but nobody tells you—The case of a class learning about negative numbers. Submitted for publication.

Sfard, A. (2005b). Personal communication.

Smith, J., Herbel-Eisenmann, B., Star, J. & Jansen, A. (2000). Quantitative pathways to understanding and using algebra: Possibilities, transitions and disconnects. Paper presented at 2000 Annual Meeting of the National Council of Teachers of Mathematics.

Stigler, J. W. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom.* New York: Free Press.

Theule-Lubienski, S.A. (1996). Mathematics for all? : Examining issues of class in mathematics teaching and learning. Unpublished doctoral dissertation. Michigan State University: East Lansing.

Thorndike, E. L. & Woodworth, R.S. (1901). The influence of improvement in one mental function upon the efficiency of other functions (I). *Psychological Review, 8*, 247–261.

Weiss, I. R., Pasley, J.D., Smith, P.S., Banilower, E.R., & Heck, D.J. (2003). *Looking inside the classroom: A study of K-12 mathematics and science education in the United States.* Chapel Hill, N.C.: Horizon Research, Inc.

Wittgenstein, L. (1953). *Philosophical investigations.* Oxford: Blackwell.

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