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ABSTRACT

Geometric Properties of Anisotropic Gaussian Random Fields

By

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This dissertation is mainly focused on the geometric properties of two kinds of anisotropic Gaussian random fields: fractional Brownian sheets and the random string processes.

Fractional Brownian sheets arise naturally in many areas, including in stochastic partial differential equations and in studies of the most visited sites of symmetric Markov processes. We prove that fractional Brownian sheets have the property of sectorial local non-determinism. By introducing a notion of dimension, called *Hausdorff dimension contour*, we determine the Hausdorff dimensions of the images of fractional Brownian sheets for arbitrary Borel sets. Then we provide sufficient conditions for the images to be Salem sets or to have interior points.

The random string processes are specified by a stochastic partial differential equation with different initial conditions [Funaki (1983), Mueller and Tribe (2002)]. We determine the Hausdorff and packing dimensions of the level sets and the sets of double times of the random string processes. We also consider the Hausdorff and packing dimensions of the ranges and graphs of the strings.

We conclude this dissertation by describing some of our ongoing projects.

To my family.

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Introduction

The study of the geometric properties of random sets, such as the image, graph, level set and so on, associated with stochastic processes and random fields is an important subject in probability as well as in other mathematical fields. On one hand, the random sets contain very rich information about the fine structure of the sample paths. On the other hand, they provide examples of sets with special properties very much needed in other branches of mathematics such as harmonic analysis, which are very hard to construct otherwise.

The random sets are usually "thin" in a sense that their Lebesgue measure is 0 and they are not smooth or surface-like. The commonly used tools in studying the geometric properties of these sets are Hausdorff dimension and Hausdorff measure.

The first result of this kind was due to S. J. Taylor (1953), where he studied the Hausdorff dimension of the image set $B([0,1])$ of a Brownian motion $B(t)$ $(t \in \mathbb{R}_+)$ in $\mathbb{R}^d$ $(d \geq 2)$ and proved that $\dim_{\mathrm{H}} B([0,1]) = 2$ a.s. by using the potential theory developed by O. Frostman (1935) and M. Riesz (1938). Since then, many results have been obtained for Lévy processes, that is, processes with independent and stationary increments. The main ingredient is the strong Markov property, see Xiao (2004) for a survey. For non-Markovian processes, there was even difficulty in finding the Hausdorff dimension of the random sets associated to these processes since many classical techniques fail to apply [cf. Adler (1981) and Kahane (1985a)]. To overcome these difficulties in studying of the sample path properties of Gaussian processes. Berman (1973) introduced a concept, called *local nondeterminism*, and he proved that if a Gaussian process has the property of local nondeterminism, then many deep results for Brownian motion can be extended to the Gaussian process. People have developed different types of local nondeterminism for various Gaussian processes and random fields for different purposes, we refer to Xiao (2005b) for a survey. We mention here that most of the Gaussian random fields studied in the past few decades were isotropic. In general, an anisotropic Gaussian random field, that is a Gaussian random field which has different probabilistic and analytic behaviors along different directions, is not locally nondeterministic. Therefore, people had a need to develop new techniques in studying the geometric properties of the anisotropic random fields.

This dissertation focuses on the geometric properties of two kinds of anisotropic Gaussian random fields: fractional Brownian sheets and the random string processes. It is organized as following. In Chapter 1, we collect definitions and properties of fractal measures and fractal dimensions, local times, and the Brownian sheet, which will be used in the subsequent chapters. Chapter 2 studies the geometric properties of fractional Brownian sheets. The objective of this chapter is to characterize the anisotropic nature of an (N, d)-fractional Brownian sheet B^H in terms of its Hurst index H. We prove the following results:

(1) B^H is sectorially locally nondeterministic.

(2) By introducing a notion of "dimension" for Borel measures and sets, which is suitable to describe the anisotropic nature of B^H, we determine $\dim_{\mathrm{H}} B^H(E)$ for an arbitrary Borel set $E \subset (0, \infty)^N$.

(3) When $B^{\langle\alpha\rangle}$ is an (N, d)-fractional Brownian sheet with index $\langle\alpha\rangle =$

$(\alpha, \ldots, \alpha)$ $(0 < \alpha < 1)$, we prove the following uniform Hausdorff dimension result for its image sets: If $N \le \alpha d$, then with probability one,

$$\dim_{\mathrm{H}} B^{\langle\alpha\rangle}(E) = \frac{1}{\alpha} \dim_{\mathrm{H}} E \quad \text{for all Borel sets } E \subset (0, \infty)^N.$$

If $N > \alpha d$ then the uniform dimension result fails to hold. In this case we establish uniform dimensional properties for the (N, d) fractional Brownian sheet $B^{\langle\alpha\rangle}$ with $\alpha d < 1$, which extends the results of Kaufman (1989) and Khoshnevisan, Wu and Xiao (2005) for the 1-dimensional Brownian motion and the Brownian sheet, respectively.

(4) We provide sufficient conditions for the image $B^H(E)$ to be a Salem set or to have interior points.

The results in **(2)** to **(4)** describe the geometric and Fourier analytic properties of B^H. They extend and improve the previous theorems of Mountford (1989), Khoshnevisan and Xiao (2004) and Khoshnevisan, Wu and Xiao (2005) for the Brownian sheet.

In Chapter 3, we investigate the fractal properties of the random string processes. We continue the research of Mueller and Tribe (2002) by determining the Hausdorff and packing dimensions of the level sets and the sets of double times of the random string processes. We also consider the Hausdorff and packing dimensions of the range and graph of the strings.

Finally, we conclude this dissertation by describing some of our ongoing projects in Chapter 4.

CHAPTER 1

Definitions and Preliminaries

This chapter contains basic definitions and facts of Hausdorff dimension, packing dimension, Fourier dimension, local times and the Brownian sheet, which will be used in subsequent chapters.

Throughout this dissertation, the underlying parameter space is $\mathbb{R}^\ell$ or $\mathbb{R}_+^\ell = [0, \infty)^\ell$. A typical parameter, $t \in \mathbb{R}^\ell$ is written as $t = (t_1, \ldots, t_\ell)$, or as $\langle c \rangle$, if $t_1 = \cdots = t_\ell = c$. For any $s, t \in \mathbb{R}^\ell$ such that $s_j < t_j$ ($j = 1, \ldots, \ell$), we define the closed interval (or rectangle) as $[s, t] = \prod_{j=1}^{\ell} [s_j, t_j]$. We always write m_ℓ for Lebesgue's measure on $\mathbb{R}^\ell$, no matter the value of the integer ℓ. We use $\langle \cdot, \cdot \rangle$ and $|\cdot|$ to denote the ordinary scalar product and the Euclidean norm in $\mathbb{R}^\ell$ respectively, no matter the value of the integer ℓ.

We will use K to denote an unspecified positive and finite constant which may not be the same in each occurrence. More specific constants in Chapter i Section j are numbered as $K_{i,j,1}, K_{i,j,2}, \ldots$, and so on.

1.1 Hausdorff measure and Hausdorff dimension

Let Φ be the class of functions $\phi : (0, \delta) \to (0, 1)$, which are right continuous, monotone increasing with $\phi(0+) = 0$ and satisfy the following "doubling" property:

There exists a finite constant $K_{1,1,1} > 0$ for which

$$\frac{\phi(2s)}{\phi(s)} \leq K_{1,1,1}, \qquad \text{for } 0 < s < \frac{\delta}{2}. \tag{1.1}$$

For $\phi \in \Phi$, the ϕ-Hausdorff measure of $E \subseteq \mathbb{R}^N$ is defined by

$$\phi - m(E) = \liminf_{\varepsilon \to 0} \left\{ \sum_j \phi(2r_j) : E \subseteq \bigcup_{j=1}^{\infty} O(x_j, r_j), \ r_j < \varepsilon \right\}, \tag{1.2}$$

where $O(x,r)$ denotes the open ball of radius r centered at x. $\phi - m$ is a metric outer measure and every Borel set in $\mathbb{R}^N$ is $\phi - m$ measurable [cf. Rogers (1970)]. The Hausdorff dimension of E is defined by

$$\begin{aligned} \dim_{\mathrm{H}} E &= \inf\{\alpha > 0 : s^{\alpha} - m(E) = 0\} \\ &= \sup\{\alpha > 0 : s^{\alpha} - m(E) = \infty\}. \end{aligned}$$

If $0 < s^{\alpha} - m(E) < \infty$, then E is called an α-set. If there exists $\phi \in \Phi$ with $0 < \phi - m(E) < \infty$, then ϕ is called an exact Hausdorff measure function for E.

Here are some basic properties of Hausdorff dimension:

(1). Monotonicity: if $E \subseteq F$, then $\dim_{\mathrm{H}} E \leq \dim_{\mathrm{H}} F$.

(2). Hausdorff dimension is σ-stable

$$\dim_{\mathrm{H}} \left(\bigcup_{n=1}^{\infty} E_n \right) = \sup_{n \geq 1} \dim_{\mathrm{H}} E_n.$$

We refer to Falconer (1990) or Mattila (1995) for more details on Hausdorff measure and Hausdorff dimension.

The Hausdorff dimension of a Borel measure μ on $\mathbb{R}^N$ (or lower Hausdorff dimension as it is sometimes called) is defined by

$$\dim_{\mathrm{H}}\mu = \inf\left\{\dim_{\mathrm{H}} F : \ \mu(F) > 0 \text{ and } F \subseteq \mathbb{R}^N \text{ is a Borel set}\right\}. \quad (1.3)$$

Hu and Taylor (1994) proved the following characterization of $\dim_{\mathrm{H}}\mu$: if μ is a finite Borel measure on $\mathbb{R}^N$ then

$$\dim_{\mathrm{H}}\mu = \sup\left\{\gamma \geq 0 : \ \limsup_{r\to 0^+} r^{-\gamma}\mu\big(\overline{O}(t,r)\big) = 0 \quad \text{for } \mu\text{-a.e. } \ t \in \mathbb{R}^N\right\}, \quad (1.4)$$

where $\overline{O}(t,r) = \{s \in \mathbb{R}^N : \ |s-t| \leq r\}$. It is easy to see that for every Borel set $E \subset \mathbb{R}^N$, we have

$$\dim_{\mathrm{H}} E = \sup\left\{\dim_{\mathrm{H}}\mu : \ \mu \in \mathcal{M}_C^+(E)\right\}, \quad (1.5)$$

where $\mathcal{M}_C^+(E)$ denotes the family of finite Borel measures on E with compact support in E.

1.2 Packing measure and packing dimension

Packing measure was introduced by Taylor and Tricot (1985) as a dual concept to Hausdorff measure. Like Hausdorff measure and Hausdorff dimension, it is a very useful tool in analyzing fractal sets and in studying the sample path properties of stochastic processes and random fields.

For $\phi \in \Phi$, define the set function $\phi - P(E)$ on $\mathbb{R}^N$ by

$$\phi - P(E) = \lim_{\varepsilon \to 0} \sup \left\{ \sum_j \phi(2r_j) : \overline{O}(x_j, r_j) \text{ are disjoint}, \right. \\ \left. x_j \in E,\ r_j < \varepsilon \right\}. \tag{1.6}$$

$\phi - P$ is not an outer measure because it fails to be countably subadditive. However, it is a premeasure, and therefore one can get a metric outer measure $\phi - p$ on $\mathbb{R}^N$ by defining

$$\phi - p(E) = \inf \left\{ \sum_n \phi - P(E_n) :\ E \subseteq \bigcup_n E_n \right\}. \tag{1.7}$$

$\phi - p(E)$ is called the ϕ-packing measure of E. Every Borel set in $\mathbb{R}^N$ is $\phi - p$ measurable. The packing dimension of E is defined by

$$\begin{aligned} \dim_{\mathrm{P}} E &= \inf \{ \alpha > 0 :\ s^\alpha - p(E) = 0 \} \\ &= \sup \{ \alpha > 0 :\ s^\alpha - p(E) = \infty \}. \end{aligned}$$

The packing dimension of μ, denoted by $\dim_{\mathrm{P}} \mu$, can be defined by replacing $\dim_{\mathrm{H}} F$ in (1.3) by $\dim_{\mathrm{P}} F$. There is a similar identity for $\dim_{\mathrm{P}} \mu$ as (1.4) for $\dim_{\mathrm{H}} \mu$; see Hu and Taylor (1994, Corollary 4.2) and Falconer and Howroyd (1997) for further information.

For any $\varepsilon > 0$ and any bounded set $F \subseteq \mathbb{R}^d$, let

$$N(F, \varepsilon) = \text{smallest number of balls of radius } \varepsilon \text{ needed to cover } F.$$

Then the *upper box-counting dimension* of F is defined as

$$\overline{\dim}_{\mathrm{B}} F = \limsup_{\varepsilon \to 0} \frac{\log N(F, \varepsilon)}{-\log \varepsilon}. \tag{1.8}$$

The packing dimension of F can also be defined by

$$\dim_{\mathrm{P}} F = \inf \left\{ \sup_n \overline{\dim}_{\mathrm{B}} F_n : \quad F \subseteq \bigcup_{n=1}^{\infty} F_n \right\}. \tag{1.9}$$

Further information on packing dimension can be found in Tricot (1982), Falconer (1990) and Mattila (1995).

1.3 Fourier dimension and Salem set

Let us recall from Kahane (1985a, b) the definitions of Fourier dimension and Salem set. Given a constant $\beta \geq 0$, a Borel set $F \subset \mathbb{R}^d$ is said to be an M_β-set if there exists a probability measure ν on F such that

$$\widehat{\nu}(\xi) = o(|\xi|^{-\beta}) \quad \text{as} \quad \xi \to \infty, \tag{1.10}$$

where $\widehat{\nu}$ is the Fourier transform of ν.

Note that if $\beta > d/2$, then (1.10) implies that $\widehat{\nu} \in L^2(\mathbb{R}^d)$ and, consequently, F has positive d-dimensional Lebesgue measure. For any Borel set $F \subset \mathbb{R}^d$, its Fourier dimension $\dim_{\mathrm{F}} F$ is defined by

$$\dim_{\mathrm{F}} F = \sup \left\{ \gamma \geq 0 : \; F \text{ is an } M_{\gamma/2}\text{-set} \right\}. \tag{1.11}$$

It follows from Frostman's theorem [cf. Kahane (1985a, Chapter 10)] and the following formula for the γ-energy of ν:

$$I_\gamma(\nu) = \int_{\mathbb{R}^d} \left|\widehat{\nu}(\xi)\right|^2 |\xi|^{\gamma-d}\, d\xi \tag{1.12}$$

[see Kahane (1985a, Ch. 10)] that $\dim_{\mathrm{F}} F \le \dim_{\mathrm{H}} F$ for all Borel sets $F \subset \mathbb{R}^d$.

We say that a Borel set $F \subset \mathbb{R}^d$ is a *Salem set* if $\dim_{\mathrm{F}} F = \dim_{\mathrm{H}} F$. Such sets are of importance in studying the problem of uniqueness and multiplicity for trigonometric series; see Zygmund (1959, Chapter 9) and Kahane and Salem (1994) for further information.

1.4 Local times

Let $X(t)$ be a Borel vector field on $\mathbb{R}^N$ taking values in $\mathbb{R}^d$. For any Borel probability measure μ on E, the image measure of μ_X under X is defined as the following

$$\mu_X(\bullet) = \mu\left\{t \in E : X(t) \in \bullet\right\}.$$

If μ_X is absolutely continuous with respect to the Lebsgue measure m_d in $\mathbb{R}^d$, then X is said to have a local time on E. The local time $l_\mu(x)$ is defined to be the Radon–Nikodým derivative $d\mu_X/dm_d(x)$ and it satisfies the following occupation density formula: for all Borel measurable functions $f : \mathbb{R}^d \to \mathbb{R}_+$,

$$\int_E f\big(X(s)\big)\, \mu(ds) = \int_{\mathbb{R}^d} f(x) l_\mu(x)\, dx. \tag{1.13}$$

If we choose μ as the Lebesgue measure m_N in $\mathbb{R}^N$, then the corresponding image measure is called the occupation measure of X on E, denoted by μ_E. Similarly, if $\mu_E \ll m_d$, then X is said to have local times on E, defined by $d\mu_E/dm_d(x)$ and denoted by $l(x, E)$. We call x the *space variable*, and E the *time variable*. Sometimes, we write $l(x, t)$ in place of $l(x, [0, t])$. It is clear that if X has local times on E, then for every Borel set $I \subseteq E$, $l(x, I)$ also exists.

Suppose we fix a rectangle $E = \prod_{j=1}^{N}[a_j, a_j + h_j]$. Then, whenever we can choose a version of the local time, still denoted by $l(x, \prod_{j=1}^{N}[a_j, a_j + t_j])$, such that it is a continuous function of $(x, t_1, \ldots, t_N) \in \mathbb{R}^d \times \prod_{j=1}^{N}[0, h_j]$, X is said to have *jointly continuous local times* on E. When a local time is jointly continuous, $l(x, \bullet)$ can be extended to a finite Borel measure supported on the level set

$$X_E^{-1}(x) = \{t \in E : X(t) = x\},$$

see Adler (1981) for details. In other words, local times often act as a natural measure on the level set of X. Therefore, they are useful in studying the fractal properties of level sets and inverse images of the vector field X. We refer to Berman (1972), Adler (1977), Ehm (1981), Rosen (1984), Monrad and Pitt (1987) and Xiao (1997) in this regard.

For more information on local times of random, as well as non-random functions, we refer to Geman and Horowitz (1980), Geman et al (1984), Xiao (1997) and Dozzi (2003).

1.5 The Brownian sheet

Let $(S, \mathcal{S}, \mu)$ be a σ-finite measure space and let

$$\mathcal{C} = \{A \in \mathcal{S} : \mu(A) < \infty\}.$$

A random field $W : \mathcal{C} \to \mathbb{R}$ is called a *Gaussian noise with control measure* μ if

(i). $\forall A \in \mathcal{C}$, $W(A) \sim N(0, \mu(A))$.

(ii). $\forall A, B \in \mathcal{C}$, $A \cap B = \emptyset$, we have $W(A \cup B) = W(A) + W(B)$.

(iii). $\forall A, B \in \mathcal{C}$, $A \cap B = \emptyset$, $W(A)$, $W(B)$ are independent.

For a given σ-finite measure space $(S, \mathcal{S}, \mu)$, there exists a Gaussian noise on $\mathcal{C}$ with μ as its control measure. Let

$$L^2(\mu) = \left\{ f : S \to \mathbb{R} \quad \int_S |f(t)|^2 \mu(dt) < \infty \right\}.$$

We can define the stochastic integral for all $f \in L^2(\mu)$ with respect to W by

$$I(f) = \int f(t) W(dt).$$

Furthermore, it's easy to prove the following isotropic property of the stochastic integral:

$$\mathbb{E}\left[I(f)I(g)\right] = \int f(t)g(t)\mu(dt), \quad f, g \in L^2(\mu). \tag{1.14}$$

When $(S, \mathcal{S}, \mu) = (\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N), m_N)$, we call W a *Gaussian white noise*. We define a real valued Gaussian random field $B_0 = \left\{B_0(t),\ t \in \mathbb{R}_+^N\right\}$ by

$$B_0(t) = W([0,t]), \quad t \in \mathbb{R}_+^N.$$

Then B_0 is called the $(N, 1)$-Brownian sheet. It can be derived that

$$\mathbb{E}\left[B_0(t)B_0(s)\right] = \prod_{j=1}^{N} \min(t_j, s_j), \quad t,\ s \in \mathbb{R}_+^N.$$

Let $B_1, \ldots, B_d$ be d independent copies of B_0. Then the (N, d) Brownian sheet $B = \{B(t), t \in \mathbb{R}_+^N\}$, defined by

$$B^H(t) = (B_1^H(t), \ldots, B_d^H(t)), \quad t \in \mathbb{R}^N, \tag{1.15}$$

is a multiparameter extension of Brownian motion and is one of the most important Gaussian random fields.

We refer to Khoshnevisan (2002) for details on the Brownian sheet and other multiparameter processes.

CHAPTER 2

Geometric Properties of Fractional Brownian Sheets

2.1 Introduction

For a given vector $H = (H_1, \ldots, H_N)$ $(0 < H_j < 1$ for $j = 1, \ldots, N)$, an $(N, 1)$-fractional Brownian sheet $B_0^H = \{B_0^H(t), t \in \mathbb{R}^N\}$ with Hurst index H is a real-valued, centered Gaussian random field with covariance function given by

$$\begin{aligned}&\mathbb{E}\big[B_0^H(s)B_0^H(t)\big]\\&=\prod_{j=1}^{N}\frac{1}{2}\Big(|s_j|^{2H_j}+|t_j|^{2H_j}-|s_j-t_j|^{2H_j}\Big),\ s,t\in\mathbb{R}^N.\end{aligned} \tag{2.1}$$

It follows from (2.1) that $B_0^H(t) = 0$ a.s. for every $t \in \mathbb{R}^N$ with at least one zero coordinate. When $H_1 = \cdots = H_N = \alpha \in (0, 1)$, we will write $H = \langle \alpha \rangle$.

It is known that B_0^H has the following two important stochastic integral representations:

• **Moving average representation**

$$B_0^H(t) = \kappa_H^{-1} \int_{-\infty}^{t_1} \cdots \int_{-\infty}^{t_N} g(t, s) W(ds), \tag{2.2}$$

where $W = \{W(s), s \in \mathbb{R}^N\}$ is a standard real-valued Brownian sheet and

$$g(t, s) = \prod_{j=1}^{N} \left[\left((t_j - s_j)_+\right)^{H_j - 1/2} - \left((-s_j)_+\right)^{H_j - 1/2} \right]$$

with $s_+ = \max\{s, 0\}$, and where κ_H is the normalizing constant given by

$$\kappa_H^2 = \int_{-\infty}^{1} \cdots \int_{-\infty}^{1} \left[\prod_{j=1}^{N} g^2(\langle 1 \rangle, s) \right]^2 ds$$

so that $\mathbb{E}\big[(B_0^H(t))^2\big] = \prod_{j=1}^N |t_j|^{2H_j}$ for all $t \in \mathbb{R}^N$. Here $\langle 1 \rangle = (1, 1, \dots, 1) \in \mathbb{R}^N$.

• **Harmonizable representation**

$$B_0^H(t) = K_H^{-1} \int_{\mathbb{R}^N} \psi_t(\lambda)\, \widehat{W}(d\lambda), \tag{2.3}$$

where $\widehat{W}$ is the Fourier transform of the white noise in $\mathbb{R}^N$ and

$$\psi_t(\lambda) = \prod_{j=1}^{N} \frac{\exp(it_j\lambda_j) - 1}{|\lambda_j|^{H_j + \frac{1}{2}}},$$

and where K_H is the normalizing constant so that $\mathbb{E}\big[(B_0^H(t))^2\big] = \prod_{j=1}^N |t_j|^{2H_j}$ for all $t \in \mathbb{R}^N$.

Let $B_1^H, \dots, B_d^H$ be d independent copies of B_0^H. Then the (N, d)-fractional

Brownian sheet with Hurst index $H = (H_1, \ldots, H_N)$ is the Gaussian random field $B^H = \{B^H(t) : t \in \mathbb{R}^N\}$ with values in $\mathbb{R}^d$ defined by

$$B^H(t) = (B_1^H(t), \ldots, B_d^H(t)), \quad t \in \mathbb{R}^N. \tag{2.4}$$

Note that if $N = 1$, then B^H is a fractional Brownian motion in $\mathbb{R}^d$ with Hurst index $H_1 \in (0, 1)$; if $N > 1$ and $H = \langle 1/2 \rangle$, then B^H is the (N, d)-Brownian sheet, denoted by B. Hence B^H can be regarded as a natural generalization of one parameter fractional Brownian motion in $\mathbb{R}^d$ to (N, d) Gaussian random fields. as well as a generalization of the Brownian sheet. Another well known generalization is the multiparameter fractional Brownian motion $X = \{X(t),\ t \in \mathbb{R}^N\}$, which is a centered (N, d)-Gaussian random field with covariance function

$$\mathbb{E}[X_i(s)X_j(t)] = \frac{1}{2}\delta_{ij}\Big(|s|^{2H_1} + |t|^{2H_1} - |s-t|^{2H_1}\Big), \quad \forall s, t \in \mathbb{R}^N, \tag{2.5}$$

where $0 < H_1 < 1$ is a constant and $\delta_{ij} = 1$ if $i = j$ and 0 if $i \neq j$. The main difference between B^H and the multiparameter fractional Brownian motion is that the latter is isotropic and self-similar with stationary increments while B^H is anisotropic and does not have stationary increments. It follows from (2.1) that B^H is operator-self-similar in the sense that for all constants $K > 0$,

$$\{B^H(K^A t),\ t \in \mathbb{R}^N\} \stackrel{d}{=} \Big\{K^N B^H(t),\ t \in \mathbb{R}^N\Big\}, \tag{2.6}$$

where $A = (a_{ij})$ is the $N \times N$ diagonal matrix with $a_{ii} = 1/H_i$ for all $1 \leq i \leq N$

and $a_{ij} = 0$ if $i \neq j$, and $Y \stackrel{d}{=} Z$ means that the two processes have the same finite dimensional distributions. The anisotropic property and the operator-self-similarity (2.6) of B^H make it a possible model for bone structure [Bonami and Estrade (2003)] and aquifer structure in hydrology [Benson et al. (2004)].

Many authors have studied various properties of fractional Brownian sheets. For example, Dunker (2000), Mason and Shi (2001), Belinski and Linde (2002), Kühn and Linde (2002) studied the small ball probabilities of an $(N, 1)$-fractional Brownian sheet B_0^H. Mason and Shi (2001) also computed the Hausdorff dimension of some exceptional sets related to the oscillation of the sample paths of B_0^H. Ayache and Taqqu (2003) derived an optimal wavelet series expansion for the fractional Brownian sheet B_0^H; see also Kühn and Linde (2002), Dzhaparidze and van Zanten (2005) for other optimal series expansions for B_0^H. Xiao and Zhang (2002) studied the existence of local times of an (N, d)-fractional Brownian sheet B^H and proved a sufficient condition for the joint continuity of the local times. Kamont (1996) and Ayache (2002) studied the box dimension and the Hausdorff dimension of the graph set of an $(N, 1)$-fractional Brownian sheet B_0^H.

Recently, Ayache and Xiao (2005) have investigated the uniform and local asymptotic properties of B^H by using wavelet methods, and determined the Hausdorff and packing dimensions of the range $B^H([0,1]^N)$, the graph $\mathrm{Gr}B^H([0,1]^N)$ and the level set $L_x = \{t \in (0, \infty)^N : B^H(t) = x\}$. Their results suggest that, due to the anisotropy of B^H in t, the sample paths of B^H are more irregular than those of the Brownian sheet or (N, d)-fractional Brownian motion. Hence it is of interest to further describe the anisotropic properties of B^H in terms of the Hurst index $H = (H_1, \ldots, H_N)$.

The main objective of this chapter is to investigate the geometric and Fourier analytic properties of the image $B^H(E)$ of a Borel set $E \subset (0, \infty)^N$. In Section 2.2, we prove that fractional Brownian sheets satisfy a type of "sectorial local nondeterminism" [see Theorem 2.1], extending a result of Khoshnevisan and Xiao (2004) for the Brownian sheet. This property plays an important role in this chapter and it will be useful in studying other problems such as local times of fractional Brownian sheets.

In Section 2.3, we determine the Hausdorff dimension of the image $B^H(E)$. Unlike the well-known cases of fractional Brownian motion or the Brownian sheet, $\dim_{\mathrm{H}} B^H(E)$ can not be determined by $\dim_{\mathrm{H}} E$ alone due to the anisotropy of B^H [see Example 2.1]. To solve the problem, we introduce a new concept of "dimension" [we call it *Hausdorff dimension contour*] for finite Borel measures and Borel sets; and we prove that $\dim_{\mathrm{H}} B^H(E)$ can be represented in terms of the Hausdorff dimensional contour of E and the Hurst index H.

We believe that the concept of Hausdorff dimension contour is of independent interest because it carries more information about the geometric properties of Borel measures and sets than Hausdorff dimension does; and it is the appropriate notion for studying the image $B^H(E)$ and the local times of the fractional Brownian sheet B^H on E, as shown by the results in Sections 2.3, 2.5 and 2.6. Therefore it would be interesting to further study Hausdorff dimension contours and to develop ways to determine them for large classes of fractal sets.

In Section 2.4, we study the uniform dimension results for the images of the (N,d)-fractional Brownian sheet $B^{\langle\alpha\rangle}$ with index $H = \langle\alpha\rangle$, we prove the following uniform Hausdorff dimension result for its images: if $N \le \alpha d$, then with probability one,

$$\dim_{\mathrm{H}} B^{\langle\alpha\rangle}(E) = \frac{1}{\alpha}\dim_{\mathrm{H}} E \quad \text{for all Borel sets } \ E \subset (0,\infty)^N. \tag{2.7}$$

This extends the results of Mountford (1989) and Khoshnevisan, Wu and Xiao (2005) for the Brownian sheet. Our proof is based on the sectorial local nondeterminism of $B^{\langle\alpha\rangle}$ and is similar to that of Khoshnevisan, Wu and Xiao (2005). When $N > \alpha d$, the uniform dimensional results do not hold anymore. We derive weaker uniform dimensional properties for the (N, d)-fractional Brownian sheet $B^{\langle\alpha\rangle}$ with $\alpha d < 1$. Our results are extensions of the results of Kaufman (1989) and Khoshnevisan, Wu and Xiao (2005) for the 1-dimensional Brownian motion and Brownian sheet, respectively.

Let μ be a probability measure carried by E and let $\nu = \mu_{B^H}$ be the image measure of μ under the mapping $t \mapsto B^H(t)$. In Section 2.5, we study the asymptotic properties of the Fourier transform $\widehat{\nu}(\xi)$ of ν as $\xi \longrightarrow \infty$. In particular, we show that the image $B^H(E)$ is a Salem set whenever $s(H, E) \leq d$, see Section 2.3 for the definition of $s(H, E)$. These results extend those of Kahane (1985a, b) and Khoshnevisan, Wu and Xiao (2005) for fractional Brownian motion and the Brownian sheet, respectively.

In Section 2.6, we prove a sufficient condition for $B^H(E)$ to have interior points. This problem is closely related to the existence of a continuous local time of B^H on E [cf. Pitt (1978), Geman and Horowitz (1980), Kahane (1985a, b)]. Our Theorem 2.8 extends and improves the previous result of Khoshnevisan and Xiao (2004) for the Brownian sheet.

2.2 Sectorial local nondeterminism

Recently Khoshnevisan and Xiao (2004) have proved that the Brownian sheet satisfies a type of "sectorial" local nondeterminism and applied this property to study geometric properties of the Brownian sheet; see also Khoshnevisan, Wu and Xiao (2005). In the following we prove that the $(N, 1)$-fractional Brownian sheet $B_0^H = \{B_0^H(t), t \in \mathbb{R}^N\}$ satisfies a similar type of sectorial local nondeterminism. This property will play an important rôle in this chapter, as well as in studying the local times, self-intersections and other sample path properties of fractional Brownian sheets.

Theorem 2.1 (Sectorial LND) *For any fixed positive number* $\varepsilon \in (0,1)$, *there exists a positive constant* $K_{2,2,1}$, *depending on* ε, H *and* N *only, such that for all positive integers* $n \geq 1$, *and all* $u, t^1, \ldots, t^n \in [\varepsilon, \infty)^N$, *we have*

$$\mathrm{Var}\left(B_0^H(u) \mid B_0^H(t^1), \ldots, B_0^H(t^n)\right) \geq K_{2,2,1} \sum_{j=1}^{N} \min_{0 \leq k \leq n} \left|u_j - t_j^k\right|^{2H_j}, \tag{2.8}$$

where $t^0 = 0$.

Remark 2.1 The method we use for proving Theorem 2.1 is different from that in Khoshnevisan and Xiao (2004) because fractional Brownian sheets have no independent increments but the Brownian sheet does. Our proof is mainly based on the harmonizable representation (2.3) of B_0^H and is reminiscent to Kahane (1985a, Chapter 18) or Xiao (2005a).

Proof: Let $\ell \in \{1, \ldots, N\}$ be fixed and denote $r_\ell \equiv \min_{0 \leq k \leq n} \left|u_\ell - t_\ell^k\right|$. Firstly, we prove that there exists a positive constant K_ℓ such that the following inequality holds:

$$\mathrm{Var}\left(B_0^H(u) \mid B_0^H(t^1), \ldots, B_0^H(t^n)\right) \geq K_\ell\, r_\ell^{2H_\ell}. \tag{2.9}$$

Summing over ℓ from 1 to N in (2.9), we get (2.8).

In order to prove (2.9), we work in the Hilbert space setting and write the conditional variance in (2.9) as the square of the $L^2(\mathbb{P})$-distance of $B_0^H(u)$ from the subspace generated by $\{B_0^H(t^1), \ldots, B_0^H(t^n)\}$. Hence it suffices to show that for all $a_k \in \mathbb{R}$,

$$\mathbb{E}\left(B_0^H(u) - \sum_{k=1}^{n} a_k B_0^H(t^k)\right)^2 \geq K_\ell r_\ell^{2H_\ell}. \tag{2.10}$$

It follows from the harmonizable representation (2.3) of B_0^H that

$$\begin{aligned}
&\mathbb{E}\left(B_0^H(u) - \sum_{k=1}^{n} a_k B_0^H(t^k)\right)^2 \\
&= K_H^{-2} \int_{\mathbb{R}^N} \left|\psi_u(\lambda) - \sum_{k=1}^{n} a_k \psi_{t^k}(\lambda)\right|^2 d\lambda \\
&= K_H^{-2} \int_{\mathbb{R}^N} \Bigg| \prod_{j=1}^{N} (\exp(iu_j\lambda_j) - 1) \\
&\qquad - \sum_{k=1}^{n} a_k \prod_{j=1}^{N} (\exp(it_j^k\lambda_j) - 1)\Bigg|^2 f_H(\lambda) d\lambda,
\end{aligned} \tag{2.11}$$

where

$$f_H(\lambda) = \prod_{j=1}^{N} |\lambda_j|^{-2H_j - 1}.$$

Now for every $j = 1, \ldots, N$, we choose a bump function $\delta_j(\cdot) \in C^\infty(\mathbb{R})$ with values in $[0, 1]$ such that $\delta_j(0) = 1$ and strictly decreasing in $|\cdot|$ near 0 [e.g., on $(-\varepsilon, \varepsilon)$] and vanishes outside the open interval $(-1, 1)$. Let $\widehat{\delta}_j$ be the Fourier transform of δ_j. It is known that $\widehat{\delta}_j(\lambda_j)$ is also in $C^\infty(\mathbb{R})$ and decays rapidly as $\lambda_j \to \infty$. Also, the Fourier inversion formula gives

$$\delta_j(s_j) = (2\pi)^{-1} \int_{\mathbb{R}} \exp(-is_j\lambda_j)\widehat{\delta}_j(\lambda_j) d\lambda_j. \tag{2.12}$$

Let $\delta_\ell(s_\ell; r_\ell) = r_\ell^{-1}\delta_\ell(s_\ell/r_\ell)$, then by (2.12) and a change of variables, we have

$$\delta_\ell(s_\ell; r_\ell) = (2\pi)^{-1}\int_{\mathbb{R}} \exp(-is_\ell\lambda_\ell)\widehat{\delta}_\ell(r_\ell\lambda_\ell)d\lambda_\ell. \tag{2.13}$$

By the definition of r_ℓ, we have $\delta_\ell(u_\ell; r_\ell) = 0$ and $\delta_\ell(u_\ell - t_\ell^k; r_\ell) = 0$ for all $k = 1, \ldots, n$. Hence it follows from (2.12) and (2.13) that

$$\begin{aligned}
I \equiv \int_{\mathbb{R}^N} &\Big(\prod_{j=1}^N (\exp(iu_j\lambda_j) - 1) - \sum_{k=1}^n a_k \prod_{j=1}^N (\exp(it_j^k\lambda_j) - 1)\Big) \\
&\times \prod_{j=1}^N \exp(-iu_j\lambda_j)\Big(\prod_{j\neq\ell}^N \widehat{\delta}_j(\lambda_j)\Big)\, \widehat{\delta}_\ell(r_\ell\lambda_\ell)\, d\lambda \\
&= (2\pi)^N\Big[\Big(\prod_{j\neq\ell}\big(\delta_j(0) - \delta_j(u_j)\big)\Big)\Big(\delta_\ell(0; r_\ell) - \delta_\ell(u_\ell; r_\ell)\Big)\Big] \\
&- (2\pi)^N\Big[\sum_{k=1}^n a_k\Big(\prod_{j\neq\ell}\big(\delta_j(u_j - t_j^k) - \delta_j(u_j)\big)\Big) \\
&\qquad\qquad \times \Big(\delta_\ell(u_\ell - t_\ell^k; r_\ell) - \delta_\ell(u_\ell; r_\ell)\Big)\Big] \\
&\geq (2\pi)^N \prod_{j\neq\ell}\big(1 - \delta_j(\varepsilon)\big)\, r_\ell^{-1}.
\end{aligned} \tag{2.14}$$

On the other hand, by the Cauchy-Schwarz inequality, (2.11) and (2.14), we have

$$
\begin{aligned}
I^2 \le & \int_{\mathbb{R}^N} \left| \prod_{j=1}^{N} \left(\exp(i u_j \lambda_j) - 1 \right) - \sum_{k=1}^{n} a_k \prod_{j=1}^{N} \left(\exp(i t_j^k \lambda_j) - 1 \right) \right|^2 f_H(\lambda) d\lambda \\
& \times \int_{\mathbb{R}^N} \frac{1}{f_H(\lambda)} \left| \left(\prod_{j \neq \ell}^{N} \widehat{\delta}_j(\lambda_j) \right) \widehat{\delta}_\ell(r_\ell \lambda_\ell) \right|^2 d\lambda \\
= & K_H^2 \, \mathbb{E}\left(B_0^H(u) - \sum_{k=1}^{n} a_k B_0^H(t^k) \right)^2 \\
& \times r_\ell^{-2H_\ell - 2} \int_{\mathbb{R}^N} \frac{1}{f_H(\lambda)} \prod_{j=1}^{N} \left| \widehat{\delta}_j(\lambda_j) \right|^2 d\lambda \\
= & K_{2.2.2} \, r_\ell^{-2H_\ell - 2} \, \mathbb{E}\left(B_0^H(u) - \sum_{k=1}^{n} a_k B_0^H(t^k) \right)^2 .
\end{aligned}
\tag{2.15}
$$

Combining (2.14) and (2.15) yields (2.9). This finishes the proof of the theorem.

Given Gaussian random variables $Z_1, \ldots, Z_n$, we denote by $\mathrm{Cov}(Z_1, \ldots, Z_n)$ their covariance matrix. The following formula relates the determinant of $\mathrm{Cov}(Z_1, \ldots, Z_n)$ with conditional variances:

$$
\det \mathrm{Cov}(Z_1, \ldots, Z_n) = \mathrm{Var}(Z_1) \prod_{k=2}^{n} \mathrm{Var}\left(Z_k \middle| Z_1, \ldots, Z_{k-1}\right). \tag{2.16}
$$

Hence the inequality (2.8) can be used to estimate the joint distribution of the Gaussian random variables $B_0^H(t^1), \ldots, B_0^H(t^n)$. It is for this reason why the sectorial local nondeterminism is essential in this chapter and in studying local times of fractional Brownian sheets.

The following simple result will be needed in Section 2.6.

Lemma 2.1 *Let $n \geq 1$ be a fixed integer. Then for all $t^1, \ldots, t^n \in [\varepsilon, \infty)^N$ such that $t_j^1, \ldots, t_j^n$ are all distinct for some $j \in \{1, \ldots, N\}$, the Gaussian random variables $B_0^H(t^1), \ldots, B_0^H(t^n)$ are linearly independent.*

Proof: Let $t^1, \ldots, t^n \in [\varepsilon, \infty)^N$ be given as above. Then it follows from Theorem 2.1 and (2.16) that $\det\mathrm{Cov}\big(B_0^H(t^1), \ldots, B_0^H(t^n)\big) > 0$. This proves the lemma.

2.3 Dimension results for the images

In this section, we study the Hausdorff dimension of the image set $B^H(E)$ of an arbitrary Borel set $E \subset (0, \infty)^N$. When $E = [0, 1]^N$ or any Borel set with positive Lebesgue measure, this problem has been solved by Ayache and Xiao (2005). However, when $E \subset (0, \infty)^N$ is a fractal set, the Hausdorff dimension of $B^H(E)$ can not be determined by $\dim_{\mathrm{H}} E$ and H alone, as shown by Example 2.1 below. This is in contrast with the cases of fractional Brownian motion or the Brownian sheet.

To solve this problem, we will introduce a new notion of dimension, namely, *Hausdorff dimension contour*, for finite Borel measures and Borel sets. It turns out the Hausdorff dimension contour of E is the natural object in determining the Hausdorff dimension and other geometric properties of $B^H(E)$ for all Borel sets E.

We start with the following Proposition 2.2 which determines $\dim_{\mathrm{H}} B^H(E)$ when E belongs to a special class of Borel sets in $\mathbb{R}_+^N$.

Proposition 2.2 *Let $B^H = \{B^H(t), t \in \mathbb{R}^N\}$ be an (N, d)-fractional Brownian sheet with index $H = (H_1, \ldots, H_N)$. Assume that E_j $(j = 1, \ldots, N)$ are Borel sets in $(0, \infty)$ satisfying the following property: $\exists \{j_1, \ldots, j_{N-1}\} \subset \{1, \ldots, N\}$*

such that $\dim_{\rm H} E_{jk} = \dim_{\rm P} E_{jk}$ *for* $k = 1, \ldots, N-1$. *Let* $E = E_1 \times \cdots \times E_N \subset (0, \infty)^N$, *then we have*

$$\dim_{\rm H} B^H(E) = \min\left\{d; \quad \sum_{j=1}^{N} \frac{\dim_{\rm H} E_j}{H_j}\right\}, \quad \text{a.s.} \tag{2.17}$$

For proving Proposition 2.2, we need the next two lemmas that are due to Ayache and Xiao (2005).

Lemma 2.2 *Let* $B^H = \{B^H(t), t \in \mathbb{R}^N\}$ *be an* (N, d)*-fractional Brownian sheet with index* $H = (H_1, \ldots, H_N)$. *For all* $T > 0$, *there exist a random variable* $A_1 = A_1(\omega) > 0$ *of finite moments of any order and an event* Ω_1^* *of probability* 1 *such that for every* $\omega \in \Omega_1^*$,

$$\sup_{s,t \in [0,T]^N} \frac{\left|B^H(s,\omega) - B^H(t,\omega)\right|}{\sum_{j=1}^{N} |s_j - t_j|^{H_j} \sqrt{\log\left(3 + |s_j - t_j|^{-1}\right)}} \leq A_1(\omega). \tag{2.18}$$

Lemma 2.3 *Let* $B_0^H = \{B_0^H(t), t \in \mathbb{R}^N\}$ *be an* $(N, 1)$*-fractional Brownian sheet with index* $H = (H_1, \ldots, H_N)$, *then for any* $0 < \varepsilon < T$, *there exist positive and finite constants* $K_{2,3,1}$ *and* $K_{2,3,2}$ *such that for all* $s, t \in [\varepsilon, T]^N$,

$$K_{2,3,1} \sum_{j=1}^{N} |s_j - t_j|^{2H_j} \leq \mathbb{E}\left[\left(B_0^H(s) - B_0^H(t)\right)^2\right] \leq K_{2,3,2} \sum_{j=1}^{N} |s_j - t_j|^{2H_j}. \tag{2.19}$$

Proof of Proposition 2.2: As usual, the proof of (2.17) is divided into proving the upper and lower bounds separately. We will show that the upper bound in (2.17) follows from the modulus of continuity of the fractional Brownian sheet and a covering argument, and the lower bound follows from Forstman's Theorem [see e.g., Kahane (1985a, Chapter 10)] and Lemma 2.3.

For simplicity of notation, we will only consider the case $N = 2$ and $\dim_{\rm H} E_1 = \dim_{\rm P} E_1$. The proof for the general case is similar.

Upper bound: By the σ-stability of $\dim_{\rm H}$ and (1.9). it is sufficient to prove that for every Borel set $E = E_1 \times E_2$,

$$\dim_{\rm H} B^H(E) \le \min\left\{d; \quad \frac{\overline{\dim}_{\rm B} E_1}{H_1} + \frac{\dim_{\rm H} E_2}{H_2}\right\}, \quad \text{a.s.} \tag{2.20}$$

For any $\forall \gamma_1 > \overline{\dim}_{\rm B} E_1$, $\gamma_2 > \dim_{\rm H} E_2$, we choose and fix $\gamma_2' \in (\dim_{\rm H} E_2, \gamma_2)$. Then there exists an $r_0 > 0$ such that, for all $r \le r_0$, E_1 can be covered by $N(E_1, r) \le r^{-\gamma_1}$ many small intervals of length r; and there exists a covering $\{U_n, n \ge 1\}$ of E_2such that $r_n := |U_n| \le r_0$ and

$$\sum_{n=1}^{\infty} r_n^{\gamma_2'} \le 1. \tag{2.21}$$

For every $n \ge 1$ and any constant $\delta \in (0, 1)$ small enough, let $\{V_{n,m} : 1 \le m \le N_n\}$ be $N_n := N(E_1, r_n^{(H_2-\delta)/(H_1-\delta)})$ intervals of length $r_n^{(H_2-\delta)/(H_1-\delta)}$ which cover E_1. Then the rectangles $\{V_{n,m} \times U_n : n \ge 1, 1 \le m \le N_n\}$ form a covering of $E_1 \times E_2$, that is,

$$E \subset \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{N_n} V_{n,m} \times U_n,$$

and thus

$$B^H(E) \subset \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{N_n} B^H(V_{n,m} \times U_n). \tag{2.22}$$

It follows from Lemma 2.2 that, almost surely, $B^H(V_{n,m} \times U_n)$ can be covered by a ball of radius $K\, r_n^{H_2-\delta}$. By this and (2.22), we have covered $B^H(E)$ a.s. by balls of radius $K\, r_n^{H_2-\delta}$ $(n = 1, 2, \ldots)$. Moreover, recalling that $N_n \leq r_n^{-\gamma_1(H_2-\delta)/(H_1-\delta)}$, we have

$$\begin{aligned} &\sum_{n=1}^{\infty} \sum_{m=1}^{N_n} \left(r_n^{H_2-\delta}\right)^{\gamma_1/H_1+\gamma_2/H_2} \\ &\leq \sum_{n=1}^{\infty} r_n^{-\gamma_1(H_2-\delta)/(H_1-\delta)} \cdot r_n^{(H_2-\delta)\left(\gamma_1/H_1+\gamma_2/H_2\right)} \\ &= \sum_{n=1}^{\infty} r_n^{\gamma_2-\gamma_1(H_2-\delta)\left(1/(H_1-\delta)-1/H_1\right)-\delta\gamma_2/H_2}. \end{aligned} \tag{2.23}$$

Now we choose $\delta > 0$ small enough so that

$$\gamma_2 - \gamma_1(H_2 - \delta)\left(\frac{1}{H_1 - \delta} - \frac{1}{H_1}\right) - \frac{\delta\gamma_2}{H_2} > \gamma_2'.$$

Then (2.21) and (2.23) imply that

$$\sum_{n=1}^{\infty} \sum_{m=1}^{N_n} \left(r_n^{H_2-\delta}\right)^{\gamma_1/H_1+\gamma_2/H_2} \leq 1. \tag{2.24}$$

It should be clear that (2.20) follows from (2.24).

Lower bound: Choose γ_1, γ_2 such that $0 < \gamma_1 < \dim_{\mathrm{H}} E_1$, $0 < \gamma_2 <$

$\dim_{\mathrm{H}} E_2$ and $\frac{\gamma_1}{H_1} + \frac{\gamma_2}{H_2} < d$, then there exist probability measures σ_1 on E_1 and σ_2 on E_2 such that

$$\int_{E_1}\int_{E_1} \frac{\sigma_1(ds_1)\sigma_1(dt_1)}{|s_1 - t_1|^{\gamma_1}} < \infty, \quad \int_{E_2}\int_{E_2} \frac{\sigma_2(ds_2)\sigma_2(dt_2)}{|s_2 - t_2|^{\gamma_2}} < \infty. \tag{2.25}$$

Let $\sigma = \sigma_1 \times \sigma_2$. Then σ is a probability measure on E. By Lemma 2.3 and the fact that $\gamma_1/H_1 + \gamma_2/H_2 < d$, we have

$$\begin{aligned}
&\mathbb{E}\int_E\int_E \frac{\sigma(ds)\sigma(dt)}{|B^H(s) - B^H(t)|^{\gamma_1/H_1+\gamma_2/H_2}} \\
&\le K \int_E\int_E \frac{\sigma_1(ds_1)\sigma_1(dt_1)\sigma_2(ds_2)\sigma_2(dt_2)}{\left(|s_1 - t_1|^{2H_1} + |s_2 - t_2|^{2H_2}\right)^{(\gamma_1/H_1+\gamma_2/H_2)/2}} \\
&\le K \int_{E_1}\int_{E_1} \sigma_1(ds_1)\sigma_1(dt_1) \\
&\qquad \int_{E_2}\int_{E_2} \frac{\sigma_2(ds_2)\sigma_2(dt_2)}{\left(|s_1 - t_1|^{H_1} + |s_2 - t_2|^{H_2}\right)^{\gamma_1/H_1+\gamma_2/H_2}}.
\end{aligned} \tag{2.26}$$

By an inequality for the weighted arithmetic mean and geometric mean with $\beta_1 = H_2\gamma_1/(H_2\gamma_1 + H_1\gamma_2)$ and $\beta_2 = 1 - \beta_1 = H_1\gamma_2/(H_2\gamma_1 + H_1\gamma_2)$, we have

$$\begin{aligned}
&|s_1 - t_1|^{H_1} + |s_2 - t_2|^{H_2} \\
&\ge \beta_1 |s_1 - t_1|^{H_1} + \beta_2 |s_2 - t_2|^{H_2} \\
&\ge |s_1 - t_1|^{H_1\beta_1} |s_2 - t_2|^{H_2\beta_2}.
\end{aligned} \tag{2.27}$$

Therefore, the last denominator in (2.26) can be bounded from below by

$|s_1 - t_1|^{\gamma_1} |s_2 - t_2|^{\gamma_2}$. It follows from this and (2.25), (2.26) that

$$\mathbb{E} \int_E \int_E \frac{\sigma(ds)\sigma(dt)}{|B^H(s) - B^H(t)|^{\gamma_1/H_1 + \gamma_2/H_2}} < \infty.$$

This yields the lower bound in (2.17), and the proof of Proposition 2.2 is completed.

The following simple example illustrates that, in general, $\dim_{\mathrm{H}} E$ alone is not enough to determine the Hausdorff dimension of $B^H(E)$.

Example 2.1 *Let* $B^H = \{B^H(t), t \in \mathbb{R}^2\}$ *be a* $(2, d)$*-fractional Brownian sheet with index* $H = (H_1, H_2)$ *and* $H_1 < H_2$. *Let* $E = E_1 \times E_2$, $F = E_2 \times E_1$, *where* $E_1 \subset (0, \infty)$ *satisfies* $\dim_{\mathrm{H}} E_1 = \dim_{\mathrm{P}} E_1$ *and* $E_2 \subset (0, \infty)$ *is arbitrary. It is well known that under the condition* $\dim_{\mathrm{H}} E_1 = \dim_{\mathrm{P}} E_1$,

$$\dim_{\mathrm{H}} E = \dim_{\mathrm{H}} E_1 + \dim_{\mathrm{H}} E_2 = \dim_{\mathrm{H}} F,$$

[cf. Falconer (1990, p.94)]. However, by Proposition 2.2 we have

$$\dim_{\mathrm{H}} B^H(E) = \min\left\{d; \quad \frac{\dim_{\mathrm{H}} E_1}{H_1} + \frac{\dim_{\mathrm{H}} E_2}{H_2}\right\}.$$

$$\dim_{\mathrm{H}} B^H(F) = \min\left\{d; \quad \frac{\dim_{\mathrm{H}} E_2}{H_1} + \frac{\dim_{\mathrm{H}} E_1}{H_2}\right\}.$$

We see that $\dim_{\mathrm{H}} B^H(E) \neq \dim_{\mathrm{H}} B^H(F)$ *unless* $\dim_{\mathrm{H}} E_1 = \dim_{\mathrm{H}} E_2$.

Example 2.1 shows that in order to determine $\dim_{\mathrm{H}} B^H(E)$, we need to have more information about the geometry of E than its Hausdorff dimension. We have found it is more convenient to work with Borel measures carried by E.

Recall that $\dim_{\mathrm{H}}\mu$ only describes the local behavior of μ in an isotropic way [cf. (1.4)] and is not quite informative if μ is highly anisotropic as what we are dealing with in this chapter. To overcome this difficulty, we introduce the following new notion of "dimension" for $E \subset (0, \infty)^N$ that is natural for studying $B^H(E)$.

Definition 2.1 *Given a Borel probability measure μ on $\mathbb{R}^N$, we define the set $\Lambda_\mu \subseteq \mathbb{R}_+^N$ by*

$$\Lambda_\mu = \left\{ \lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{R}_+^N : \limsup_{r \to 0^+} \frac{\mu(R(t,r))}{r^{\langle \lambda, H^{-1} \rangle}} = 0, \quad \text{for } \mu\text{-a.e. } t \in \mathbb{R}^N \right\}, \tag{2.28}$$

where $R(t, r) = \prod_{j=1}^N [t_j - r^{1/H_j}, t_j + r^{1/H_j}]$ and $H^{-1} = (\frac{1}{H_1}, \ldots, \frac{1}{H_N})$.

Some basic properties of Λ_μ are summarized in the following lemma. For simplicity of notation, we assume $H_1 = \min\{H_j : 1 \le j \le N\}$.

Lemma 2.4 *Λ_μ has the following properties:*

(i) The set Λ_μ is bounded:

$$\Lambda_\mu \subseteq \left\{ \lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{R}_+^N : \langle \lambda, H^{-1} \rangle \le \frac{N}{H_1} \right\}. \tag{2.29}$$

(ii) $\forall \beta \in (0, 1]^N$, $\lambda \in \Lambda_\mu$, $\beta \circ \lambda \in \Lambda_\mu$, where $\beta \circ \lambda = (\beta_1 \lambda_1, \ldots, \beta_N \lambda_N)$ is the Hadamard product of β and λ.

(iii) Λ_μ is convex, i.e. $\forall \lambda, \eta \in \Lambda_\mu$ and $0 < b < 1$, $b\lambda + (1 - b)\eta \in \Lambda_\mu$.

(iv) For every $a \in (0, \infty)^N$, $\sup_{\lambda \in \Lambda_\mu} \langle a, \lambda \rangle$ is achieved on the boundary of Λ_μ.

Because of (iv) and its importance in this chapter, we call the boundary of Λ_μ, denoted by $\partial\Lambda_\mu$, the *Hausdorff dimension contour* of μ.

Proof: Suppose $\lambda = (\lambda_1, \ldots, \lambda_N) \in \Lambda_\mu$. Then

$$\limsup_{r\to 0^+} \frac{\mu\left(R(t,r)\right)}{r^{\langle \lambda, H^{-1}\rangle}} = 0 \qquad \text{for } \mu\text{-a.e. } t \in \mathbb{R}^N. \tag{2.30}$$

Fix a $t \in \mathbb{R}^N$ such that (2.30) holds. Since for any $a > 0$, the ball $U(t, a)$ centered at t with radius a can be covered by $R(t,\ a^{H_1})$. It follows from (2.30) that

$$\limsup_{r\to 0^+} \frac{\mu\left(U(t,a)\right)}{a^{H_1\langle \lambda, H^{-1}\rangle}} = 0 \qquad \text{for } \mu\text{-a.e. } t \in \mathbb{R}^N. \tag{2.31}$$

It follows from (1.4) and (2.31) that $\dim_{\rm H}\mu \geq H_1\langle \lambda, H^{-1}\rangle$. Hence we have $\langle \lambda, H^{-1}\rangle \leq N/H_1$. This proves (i).

Statements (ii) and (iii) follow directly from the definition of Λ_μ. To prove (iv), we note that for every $a \in (0, \infty)^N$, Property (i) implies $\sup_{\lambda\in\Lambda_\mu} \langle a, \lambda\rangle < \infty$. On the other hand, the function $\lambda \mapsto \langle a, \lambda\rangle$ is increasing in each λ_j. Hence $\sup_{\lambda\in\Lambda_\mu} \langle a, \lambda\rangle$ must be achieved on the boundary of Λ_μ.

As examples, we mention that if m is the Lebesgue measure on $\mathbb{R}^N_+$, then

$$\Delta_m = \left\{(\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^N_+ :\ \sum_{j=1}^N \frac{\lambda_j}{H_j} < \sum_{j=1}^N \frac{1}{H_j}\right\} \tag{2.32}$$

and $\sup_{\lambda\in\Lambda_m} \langle H^{-1}, \lambda\rangle = \sum_{j=1}^N \frac{1}{H_j}$. More generally we can verify that, if $\mu = \sigma_1 \times \cdots \times \sigma_N$, where σ_j $(j = 1, \ldots, N)$ are Borel probability measures on $\mathbb{R}$

such that $\dim_{\mathrm{H}}\sigma_{j_k} = \dim_{\mathrm{P}}\sigma_{j_k}$ for some $\{j_1, \ldots, j_{N-1}\} \subset \{1, \ldots, N\}$, then

$$\Delta_\mu = \left\{(\lambda_1, \ldots, \lambda_N) \in \mathbb{R}_+^N : \sum_{j=1}^N \frac{\lambda_j}{H_j} < \sum_{j=1}^N \frac{\beta_j}{H_j}\right\},$$

where $\beta_j = \dim_{\mathrm{H}}\sigma_j$ for $j = 1, \ldots, N$. In the special case of $H_1 = \cdots = H_N = \alpha \in (0, 1)$, we derive from (1.4) that for every Borel measure μ,

$$\Delta_\mu = \left\{(\lambda_1, \ldots, \lambda_N) \in \mathbb{R}_+^N : \sum_{j=1}^N \lambda_j < \dim_{\mathrm{H}}\mu\right\}. \tag{2.33}$$

For any Borel measure μ on $\mathbb{R}_+^N$, its image measure under the mapping $t \mapsto B^H(t)$ is defined by

$$\mu_{B^H}(F) = \mu\left\{t \in \mathbb{R}_+^N : B^H(t) \in F\right\} \quad \text{for all Borel sets } F \subset \mathbb{R}^d.$$

We will make use of the following result.

Proposition 2.3 *Let $B^H = \{B^H(t), t \in \mathbb{R}^N\}$ be an (N, d)-fractional Brownian sheet with index $H \in (0,1)^N$. Then for every Borel probability measure μ on $\mathbb{R}_+^N$,*

$$\dim_{\mathrm{H}}\mu_{B^H} \le s_\mu(H) \wedge d \qquad a.s. \tag{2.34}$$

where $s_\mu(H) = \sup_{\lambda \in \Lambda_\mu} \langle H^{-1}, \lambda\rangle$.

Remark 2.4 Applying a moment argument [see, e.g., Xiao (1996)] and the sectorial local nondeterminism of B^H, we can actually prove that the equality in (2.34) holds.

Since this result is not needed in this chapter and its proof is rather long, we omit it.

Proof: To prove the upper bound in (2.34), note that, without loss of generality, we may and will assume the support of μ is contained in $[\varepsilon, T]^N$ for some $0 < \varepsilon < T < \infty$. Furthermore, if $s_\mu(H) \geq d$, then $\dim_{\mathrm{H}} \mu_{B^H} \leq s_\mu(H) \wedge d$ holds trivially. Therefore, we will also assume that $s_\mu(H) < d$.

Now, for any $0 < \gamma < \dim_{\mathrm{H}} \mu_{B^H}$, by (1.4) we have

$$\limsup_{\rho \to 0} \frac{\mu_{B^H}\big(U(u, \rho)\big)}{\rho^\gamma} = 0 \quad \text{for } \mu_{B^H}\text{-a.e. } u \in \mathbb{R}^d. \tag{2.35}$$

This is equivalent to

$$\limsup_{\rho \to 0} \frac{1}{\rho^\gamma} \int_{[\varepsilon, T]^N} 1\!\!1_{\left\{|B^H(s) - B^H(t)| \leq \rho\right\}} \, \mu(ds) = 0 \quad \text{for } \mu\text{-a.e. } t \in \mathbb{R}_+^N. \tag{2.36}$$

Note that $\forall\, \delta_j \in (0, H_j)$ $(j = 1, \ldots, N)$, Lemma 2.2 implies that $\forall s, t \in [\varepsilon, T]^N$

$$\left|B^H(s) - B^H(t)\right| \leq K \sum_{j=1}^N |s_j - t_j|^{H_j - \delta_j} \qquad \text{a.s.} \tag{2.37}$$

It follows from (2.36) and (2.37) that almost surely

$$\limsup_{\rho \to 0} \frac{1}{\rho^\gamma} \mu\Big(R\big(t, \rho^{1/(H_j - \delta_j)}\big)\Big) = 0 \quad \text{for } \mu\text{-a.e. } t \in \mathbb{R}_+^N. \tag{2.38}$$

Now we choose $\delta_1, \ldots, \delta_N$ in the following way: $\forall \delta \in (0,\, H_1)$,

$$\delta_1 = \delta \quad \text{and} \quad \delta_j = \frac{H_j}{H_1}\delta, \quad \text{for } j = 2, \ldots, N. \tag{2.39}$$

Then, (2.38) implies

$$\limsup_{\rho\to 0}\frac{1}{\rho^{\gamma}}\mu\Big(R\big(t,\rho^{H_1/(H_1-\delta)}\big)\Big)=0\quad\text{for }\mu\text{-a.e. } t\in\mathbb{R}^N_+. \tag{2.40}$$

We claim that $\gamma \le s_\mu(H)$. In fact, if $\gamma > s_\mu(H)$, then there exists $\beta \notin \Lambda_\mu$ such that $\langle\beta,\ H^{-1}\rangle < \gamma$. Since $\beta \notin \Lambda_\mu$, there is a set $A \subset [\varepsilon, T]^N$ with positive μ-measure such that

$$\limsup_{r\to 0}\frac{\mu\big(R(t,r)\big)}{r^{\langle\beta,H^{-1}\rangle}}>0\quad\text{for every } t\in A. \tag{2.41}$$

Now we choose $\delta > 0$ small enough such that

$$\gamma-\frac{H_1}{H_1-\delta}\langle\beta,\ H^{-1}\rangle>0. \tag{2.42}$$

Then (2.41) and (2.42) imply that for every $t \in A$,

$$\begin{aligned}&\limsup_{\rho\to 0}\frac{1}{\rho^{\gamma}}\mu\Big(R\big(t,\rho^{H_1/(H_1-\delta)}\big)\Big)\\&=\limsup_{\rho\to 0}\frac{1}{\rho^{\gamma-H_1\langle\beta,H^{-1}\rangle/(H_1-\delta)}}\times\frac{\mu\Big(R\big(t,\rho^{H_1/(H_1-\delta)}\big)\Big)}{\rho^{H_1\langle\beta,H^{-1}\rangle/(H_1-\delta)}}=\infty.\end{aligned} \tag{2.43}$$

This contradicts (2.40). Therefore, we have proved that $\gamma \le s_\mu(H)$. Since $\gamma < \dim_{\mathrm{H}}\mu_{B^H}$ is arbitrary, we have

$$\dim_{\mathrm{H}}\mu_{B^H}\le s_\mu(H)\wedge d,\quad\text{a.s.} \tag{2.44}$$

This finishes the proof of Proposition 2.3.

The following corollary follows directly from Proposition 2.3 and (2.32). It is related to Theorem 3.1 in Ayache and Xiao (2005).

Corollary 2.1 *Let m be the Lebesgue measure on $\mathbb{R}_+^N$, then*

$$\dim_{\mathrm{H}} m_{B^H} \leq \sum_{j=1}^{N} \frac{1}{H_j} \wedge d, \quad a.s. \tag{2.45}$$

For any Borel set $E \subset (0, \infty)^N$, we define

$$\Lambda(E) = \bigcup_{\mu \in \mathcal{M}_C^+(E)} \Lambda_\mu. \tag{2.46}$$

Recall that $\mathcal{M}_C^+(E)$ is the family of finite Borel measures with compact support in E. Similar to the case of Borel measures, we call the set $\cup_{\mu \in \mathcal{M}_C^+(E)} \partial\Lambda_\mu$ the Hausdorff dimension contour of E. It follows from Lemma 2.4 that, for every $a \in (0, \infty)^N$, the supremum $\sup_{\lambda \in \Lambda(E)} \langle \lambda, a \rangle$ is determined by the Hausdorff dimension contour of E.

The following is the main result of this section.

Theorem 2.2 *Let B^H be an (N, d)-fractional Brownian sheet with index $H \in (0,1)^N$. Then, for any Borel set $E \subset (0, \infty)^N$,*

$$\dim_{\mathrm{H}} B^H(E) = s(H, E) \wedge d \qquad a.s., \tag{2.47}$$

where $s(H, E) = \sup_{\lambda \in \Lambda(E)} \langle \lambda, H^{-1} \rangle$.

We need the following lemmas to prove Theorem 2.2.

Lemma 2.5 *Let $E \subset \mathbb{R}^N$ be an analytic set and let $f : \mathbb{R}^N \longrightarrow \mathbb{R}^d$ be a continuous function. If $0 \leq \tau < \dim_{\mathrm{H}} f(E)$, then there exists a compact set $E_0 \subseteq E$ such that $\tau < \dim_{\mathrm{H}} f(E_0)$.*

Proof: The proof is the same as that of Lemma 4.1 in Xiao (1997), with packing dimension replaced by Hausdorff dimension.

Lemma 2.6 *Let $E \subset (0, \infty)^N$ be an analytic set. Then for any continuous function $f : \mathbb{R}^N \longrightarrow \mathbb{R}^d$,*

$$\dim_{\mathrm{H}} f(E) = \sup \left\{ \dim_{\mathrm{H}} \mu_f : \mu \in \mathcal{M}_C^+(E) \right\}. \tag{2.48}$$

Proof: For any $\mu \in \mathcal{M}_C^+(E)$, we have $\mu_f \in \mathcal{M}_C^+(f(E))$. By (1.5), we have

$$\dim_{\mathrm{H}} f(E) = \sup \left\{ \dim_{\mathrm{H}} \nu : \nu \in \mathcal{M}_C^+(f(E)) \right\}, \tag{2.49}$$

which implies that

$$\dim_{\mathrm{H}} f(E) \geq \sup \left\{ \dim_{\mathrm{H}} \mu_f : \mu \in \mathcal{M}_C^+(E) \right\}. \tag{2.50}$$

To prove the reverse inequality, let $\tau < \dim_{\mathrm{H}} f(E)$. By Lemma 2.5, there exists a compact set $E_0 \subset E$ such that $\dim_{\mathrm{H}} f(E) > \tau$. Hence, by (2.49), there exists a finite Borel measure $\nu \in \mathcal{M}_C^+(f(E_0))$ such that $\dim_{\mathrm{H}} \nu > \tau$. It follows from Theorem 1.20 in Mattila (1995) that there exists $\mu \in \mathcal{M}_C^+(E_0)$ such that $\nu = \mu_f$, which implies $\sup\{\dim_{\mathrm{H}} \mu_f : \mu \in \mathcal{M}_C^+(E)\} > \tau$. Since $\tau < \dim_{\mathrm{H}} f(E)$ is arbitrary, we have

$$\dim_{\mathrm{H}} f(E) \le \sup\{\dim_{\mathrm{H}} \mu_f : \mu \in \mathcal{M}_C^+(E)\}. \tag{2.51}$$

Equation (2.48) now follows from (2.50) and (2.51).

Proof of Theorem 2.2: First we prove the lower bound:

$$\dim_{\mathrm{H}} B^H(E) \ge s(H, E) \wedge d \qquad \text{a.s.} \tag{2.52}$$

For any $0 < \gamma < s(H, E) \wedge d$, there exists a Borel measure μ with compact support in E such that $\gamma < s_\mu(H) \wedge d$. Hence we can find $\lambda' = (\lambda'_1, \ldots, \lambda'_N) \in \Lambda_\mu$ such that $\gamma < \sum_{j=1}^N \frac{\lambda'_j}{H_j} \wedge d$ and

$$\limsup_{r \to 0} \frac{\mu(R(t, r))}{r^{\langle \lambda', H^{-1} \rangle}} = 0 \qquad \text{for } \mu\text{-a.e. } t \in \mathbb{R}_+^N. \tag{2.53}$$

For $\varepsilon > 0$ we define

$$E_\varepsilon = \Big\{ t \in E : \mu\big(R(t, r)\big) \le r^{\langle \lambda', H^{-1} \rangle} \quad \text{for all } \; 0 < r \le \varepsilon \Big\}. \tag{2.54}$$

Then (2.53) implies that $\mu(E_\varepsilon) > 0$ if ε is small enough. In order to prove (2.52), it suffices to show $\dim_{\mathrm{H}} B^H(E_\varepsilon) \ge \gamma$ a.s.

The proof of the latter is standard: we only need to show

$$\mathbb{E}\int_{E_\varepsilon}\int_{E_\varepsilon}\frac{\mu(ds)\mu(dt)}{|B^H(t)-B^H(s)|^\gamma}<\infty. \tag{2.55}$$

By Lemma 2.3 and the fact that $\gamma < d$, we have

$$\mathbb{E}\int_{E_\varepsilon}\int_{E_\varepsilon}\frac{\mu(ds)\mu(dt)}{|B^H(t)-B^H(s)|^\gamma}\le K\int_{E_\varepsilon}\int_{E_\varepsilon}\frac{\mu(ds)\mu(dt)}{\left(\sum_{j=1}^N|s_j-t_j|^{2H_j}\right)^{\gamma/2}}. \tag{2.56}$$

Let $t\in E_\varepsilon$ be fixed and let n_0 be the smallest integer n such that $2^{-n}\le\varepsilon$. For every $n\ge n_0$, denote

$$D_n=\left\{s\in\mathbb{R}_+^N:\ 2^{-(n+1)/H_j}<|s_j-t_j|\le 2^{-n/H_j}\ \text{ for all } 1\le j\le N\right\}.$$

Then by (2.54) we have

$$\begin{aligned}\int_{E_\varepsilon}\frac{\mu(ds)}{\left(\sum_{j=1}^N|s_j-t_j|^{2H_j}\right)^{\gamma/2}}&\le K+\sum_{n=n_0}^{\infty}\int_{D_n}\frac{\mu(ds)}{\left(\sum_{j=1}^N|s_j-t_j|^{2H_j}\right)^{\gamma/2}}\\&\le K+K\sum_{n=n_0}^{\infty}2^{-n(\sum_{j=1}^N\lambda_j'/H_j-\gamma)}\\&\le K_{2,3,3}.\end{aligned} \tag{2.57}$$

This and (2.56) yield (2.55). So we have proven (2.52).

Now we prove the upper bound in (2.47). It follows from Proposition 2.3 that for any $\mu \in \mathcal{M}_C^+(E)$ we have

$$\dim_{\mathrm{H}} \mu_{B^H} \leq s_\mu(H) \wedge d \quad \text{a.s.} \tag{2.58}$$

Hence by (2.46) and (2.48) we derive

$$\dim_{\mathrm{H}} B^H(E) \leq s(H, E) \wedge d \quad \text{a.s.} \tag{2.59}$$

Combining (2.52) and (2.59) finishes the proof.

Corollary 2.2 *Let $B^{\langle\alpha\rangle} = \{B^{\langle\alpha\rangle}(t), t \in \mathbb{R}^N\}$ be an (N, d)-fractional Brownian sheet with Hurst index $H = \langle\alpha\rangle$ and $E \subset (0, \infty)^N$ be a Borel set. Then with probability 1,*

$$\dim_{\mathrm{H}} B^{\langle\alpha\rangle}(E) = \min\left\{d, \frac{1}{\alpha}\dim_{\mathrm{H}} E\right\}. \tag{2.60}$$

Proof: This is a direct consequence of (2.33) and Theorem 2.2.

2.4 Uniform dimension results for the images

The following theorem gives us a uniform Hausdorff dimension result for the image sets of $B^{\langle\alpha\rangle}$. It extends the results of Mountford (1989) and Khoshnevisan, Wu and Xiao (2005) for the Brownian sheet.

Theorem 2.3 *If $N \leq \alpha d$, then with probability 1*

$$\dim_{\mathrm{H}} B^{\langle\alpha\rangle}(E) = \frac{1}{\alpha}\dim_{\mathrm{H}} E \textit{ for all Borel sets } E \subset (0, \infty)^N. \tag{2.61}$$

Our proof of Theorem 2.3 is reminiscent to that of Khoshnevisan, Wu and Xiao (2005) for the Brownian sheet. The key step is provided by the following lemma, which will be proven by using the sectorial local nondeterminism of $B^{\langle\alpha\rangle}$.

Lemma 2.7 *Assume* $N \leq \alpha d$ *and let* $\delta > 0$ *and* $0 < 2\alpha - \delta < \beta < 2\alpha$ *be given constants. Then with probability 1, for all integers* n *large enough, there do not exist more than* $2^{n\delta d}$ *distinct points of the form* $t^j = 4^{-n} k^j$, *where* $k^j \in \{1, 2, \ldots, 4^n\}^N$, *such that*

$$\left|B^{\langle\alpha\rangle}(t^i) - B^{\langle\alpha\rangle}(t^j)\right| < 3 \cdot 2^{-n\beta} \qquad \textit{for } i \neq j. \tag{2.62}$$

Proof: Let A_n be the event that there do exist more than $2^{n\delta d}$ distinct points of the form $4^{-n}k^j$ such that (2.62) holds. Let N_n be the number of n-tuples of distinct $t^1, \ldots, t^n$ such that (2.62) holds. Then

$$A_n \subseteq \left\{ N_n \geq \binom{[2^{n\delta d} + 1]}{n} \right\}.$$

So,

$$\mathbb{P}(A_n) \leq \frac{\mathbb{E}(N_n)}{\binom{[2^{n\delta d} + 1]}{n}}. \tag{2.63}$$

In order to estimate $\mathbb{E}(N_n)$, we write it as

$$\mathbb{E}(N_n) = \mathbb{E}\Big[\underbrace{\sum_{t^1}\sum_{t^2}\cdots\sum_{t^n}}_{\text{distinct}} \mathbf{1}_{\{(2.62)\text{ holds}\}}\Big]$$

$$= \underbrace{\sum_{t^1}\sum_{t^2}\cdots\sum_{t^n}}_{\text{distinct}} \mathbb{P}\Big\{\Big|B^{\langle\alpha\rangle}(t^i) - B^{\langle\alpha\rangle}(t^j)\Big| < 3\cdot 2^{-n\beta}, \quad \forall i \neq j \leq n\Big\}. \tag{2.64}$$

Now we fix $n-1$ distinct points $t^1, \ldots, t^{n-1}$ and estimate the following sum first:

$$\sum_{t^n} \mathbb{P}\Big\{\Big|B^{\langle\alpha\rangle}(t^i) - B^{\langle\alpha\rangle}(t^j)\Big| < 3\cdot 2^{-n\beta}, \quad \forall i \neq j \leq n\Big\}. \tag{2.65}$$

Note that for fixed $t^1 = k^1 4^{-n}, \ldots, t^{n-1} = k^{n-1}4^{-n}$, there are at most $(n-1)^N$ points $\tau^u = (\tau_1^u, \ldots, \tau_N^u)$ defined by

$$\tau_\ell^u = t_\ell^j \qquad \text{for some } j = 1, \ldots, n-1.$$

We denote the collection of τ^u's by $\Gamma_n = \{\tau^u\}$. Clearly, $t^1, \ldots, t^{n-1}$ are all included in Γ_n.

It follows from Theorem 2.1 that, for every $t^n \notin \Gamma_n$, there exists $\tau^{u_n} \in \Gamma_n$ such that

$$\text{Var}\left(B_0^{\langle\alpha\rangle}(t^n)|B_0^{\langle\alpha\rangle}(t^1), \ldots, B_0^{\langle\alpha\rangle}(t^{n-1})\right) \geq K_{2.4.1}\,|t^n - \tau^{u_n}|^{2\alpha}. \tag{2.66}$$

In this case, since $B_1^{\langle\alpha\rangle}, \ldots, B_d^{\langle\alpha\rangle}$ are the independent copies of $B_0^{\langle\alpha\rangle}$, a standard conditioning argument and (2.66) yield

$$\begin{aligned} &\mathbb{P}\Big\{ \Big|B^{\langle\alpha\rangle}(t^i) - B^{\langle\alpha\rangle}(t^j)\Big| < 3\cdot 2^{-n\beta}, \quad \forall i \neq j \leq n\Big\} \\ &\leq \mathbb{P}\Big\{ \Big|B^{\langle\alpha\rangle}(t^i) - B^{\langle\alpha\rangle}(t^j)\Big| < 3\cdot 2^{-n\beta}, \quad \forall i \neq j \leq n-1\Big\} \\ &\qquad \times \left(\frac{3\cdot 2^{-n\beta}}{K_{2.4.1}^{1/2}\, |t^n - \tau^{u_n}|^{\alpha}}\right)^d. \end{aligned} \tag{2.67}$$

If $t^n \in \Gamma_n$, then we use the trivial bound

$$\begin{aligned} &\mathbb{P}\Big\{ \Big|B^{\langle\alpha\rangle}(t^i) - B^{\langle\alpha\rangle}(t^j)\Big| < 3\cdot 2^{-n\beta}, \quad \forall i \neq j \leq n\Big\} \\ &\leq \mathbb{P}\Big\{ \Big|B^{\langle\alpha\rangle}(t^i) - B^{\langle\alpha\rangle}(t^j)\Big| < 3\cdot 2^{-n\beta}, \quad \forall i \neq j \leq n-1\Big\}. \end{aligned} \tag{2.68}$$

Hence, by combining (2.67) and (2.68), we obtain

$$\begin{aligned} &\sum_{t^n} \mathbb{P}\Big\{ \Big|B^{\langle\alpha\rangle}(t^i) - B^{\langle\alpha\rangle}(t^j)\Big| < 3\cdot 2^{-n\beta}, \quad \forall i \neq j \leq n\Big\} \\ &\leq \mathbb{P}\Big\{ \Big|B^{\langle\alpha\rangle}(t^i) - B^{\langle\alpha\rangle}(t^j)\Big| < 3\cdot 2^{-n\beta}, \quad \forall i \neq j \leq n-1\Big\} \\ &\qquad \times \left[\sum_{t^n \notin \Gamma_n} K_{2.4.2} \left(\frac{3\cdot 2^{-n\beta}}{|t^n - \tau^{u_n}|^{\alpha}}\right)^d + (n-1)^N\right]. \end{aligned} \tag{2.69}$$

Note that

$$\sum_{t^n \notin \Gamma_n} \left(\frac{3 \cdot 2^{-n\beta}}{|t^n - \tau^{u_n}|^{\alpha}}\right)^d \le \sum_{\tau^u \in \Gamma_n} \sum_{t^n \neq \tau^u} \left(\frac{3 \cdot 2^{-n\beta}}{|t^n - \tau^u|^{\alpha}}\right)^d$$

$$\le \sum_{\tau^u \in \Gamma_n} 3^d \cdot 2^{-n\beta d} \sum_{t^n \neq \tau^u} \frac{1}{|t^n - \tau^u|^{\alpha d}} \tag{2.70}$$

$$\le K_{2.4.3}\,(n-1)^{N+1}\, 2^{n(2\alpha-\beta)d},$$

where, in deriving the last inequality, we have used the fact that if $N \le \alpha d$ then for all fixed τ^u,

$$\sum_{t^n \neq \tau^u} \frac{1}{|t^n - \tau^u|^{\alpha d}} \le K \cdot 2^{2\alpha n d}\, n.$$

Plug (2.70) into (2.69), we get

$$\sum_{t^n} \mathbb{P}\left\{ \left|B^{\langle\alpha\rangle}(t^i) - B^{\langle\alpha\rangle}(t^j)\right| < 3 \cdot 2^{-n\beta}, \quad \forall i \neq j \le n \right\}$$

$$\le \mathbb{P}\left\{ \left|B^{\langle\alpha\rangle}(t^i) - B^{\langle\alpha\rangle}(t^j)\right| < 3 \cdot 2^{-n\beta}, \quad \forall i \neq j \le n-1 \right\} \tag{2.71}$$

$$\times K_{2,4,4}\,(n-1)^{N+1}\, 2^{n(2\alpha-\beta)d}.$$

Therefore, by iteration, we obtain

$$\underbrace{\sum_{t^1} \sum_{t^2} \cdots \sum_{t^n}}_{\text{distinct}} \mathbb{P}\left\{ \left|B^{\langle\alpha\rangle}(t^i) - B^{\langle\alpha\rangle}(t^j)\right| < 3 \cdot 2^{-n\beta}, \quad \forall i \neq j \le n \right\}$$

$$\le K_{2,4,5}^n \left[(n-1)!\right]^{N+1} 2^{n^2(2\alpha-\beta)d}, \tag{2.72}$$

which implies

$$\mathbb{E}(N_n) \le K_{2.4.5}^n (n-1)^{n(N+1)} 2^{n^2(2\alpha-\beta)d}. \tag{2.73}$$

By (2.63), (2.72) and the elementary inequality

$$\binom{[2^{n\delta d}+1]}{n} \ge \left(\frac{2^{n\delta d}+1}{n}\right)^n \ge \frac{2^{n^2\delta d}}{n^n},$$

we obtain

$$\mathbb{P}(A_n) \le K_{2.4.6}^n (n-1)^{n(N+2)} 2^{n^2(2\alpha-\beta-\delta)d}. \tag{2.74}$$

Since $0 < 2\alpha - \beta < \delta$, by (2.74), we get $\sum_n \mathbb{P}(A_n) < \infty$. Hence the Borel-Cantelli Lemma implies that $\mathbb{P}\left(\overline{\lim}_n A_n\right) = 0$. This completes the proof of our lemma.

For $n = 1, 2, \ldots$ and $k = (k_1, \ldots, k_N)$, where each $k_i \in \{1, 2, \ldots, 4^n\}$, define

$$I_k^n = \left\{t \in [0,1]^N : (k_i - 1)4^{-n} \le t_i \le k_i 4^{-n} \quad \text{for all } i = 1, \ldots, N\right\}. \tag{2.75}$$

The following lemma is a consequence of Lemma 2.7 and the modulus of continuity of the fractional Brownian sheet [cf. Lemma 2.2].

Lemma 2.8 *Let $\delta > 0$ and $0 < 2\alpha - \delta < \beta < 2\alpha$. Then with probability 1, for all large enough n, there exists no ball $O \subset \mathbb{R}^d$ of radius $2^{-n\beta}$ for which $B^{-1}(O)$ intersects more than $2^{n\delta d}$ cubes I_k^n.*

Now we are ready to prove Theorem 2.3.

Proof: For simplicity, we shall only prove (2.61) for all Borel sets $E \subseteq [0,1]^N$. By Lemma 2.2, we know that almost surely $B^{\langle\alpha\rangle}(t)$ satisfies a uniform Hölder condition on $[0,1]^N$ of any order smaller than α. This and Theorem 6 in Kahane (1985a) [or Proposition 2.3 in Falconer (1990)] implies that $\mathbb{P}\{\dim_{\rm H} B(E) \le \frac{1}{\alpha}\dim_{\rm H} E \quad \text{for every Borel set} \quad E \subset [0,1]^N\} = 1$.

To prove the lower bound we need only to show that almost surely for every compact set $F \subseteq \mathbb{R}^d$,

$$\dim_{\rm H}\{t \in [0,1]^N : B^{\langle\alpha\rangle}(t) \in F\} \le \alpha \dim_{\rm H} F. \tag{2.76}$$

This follows from Lemma 2.8 and a simple covering argument as in Khoshnevisan, Wu and Xiao (2005). Therefore, the proof of Theorem 2.3 is finished.

When $N > \alpha d$, (2.61) no longer holds. In fact, when $N > \alpha d$, the level sets of $B^{\langle\alpha\rangle}$ have dimension $N - \alpha d > 0$ [Ayache and Xiao (2005), Theorem 5, p. 434]. Therefore, (2.61) is obviously false for $F := \left(B^{\langle\alpha\rangle}\right)^{-1}(0)$.

In the following, we prove two weaker forms of uniform result for the images of the (N,d)-Brownian sheet $B^{\langle\alpha\rangle}$ with $\alpha d < 1$. (Of course, $\alpha d < N$ in this case.) They extend the results of Kaufman (1989) and Khoshnevisan, Wu and Xiao (2005).

Theorem 2.4 *Let $\alpha d < 1$, then with probability 1 for every Borel set $F \subseteq (0,1]^N$,*

$$\dim_{\rm H} B^{\langle\alpha\rangle}(F+t) = \min\left\{d, \frac{1}{\alpha}\dim_{\rm H} F\right\} \quad \textit{for almost all} \quad t \in [0,1]^N. \tag{2.77}$$

Define

$$H_R(x) := R^d \mathbf{1}_{[-1,\,1]^d}(Rx) \quad {}^\forall x \in \mathbb{R}^d,\ R > 0. \tag{2.78}$$

Also define

$$I_R(x,y) := \int_{[0,1]^N} H_R\Big(B^{\langle\alpha\rangle}(x+t) - B^{\langle\alpha\rangle}(y+t)\Big)\, dt \qquad {}^\forall R > 0,\ x, y \in [\varepsilon\,, 1]^N. \tag{2.79}$$

The following is the key to our proof of Theorem 2.4. Sectorial LND plays an important role in the proof of the next lemma.

Lemma 2.9 *For all $x, y \in [\varepsilon, 1]^N$, $R > 1$ and integers $p = 1, 2, \ldots,$*

$$\mathbb{E}\left[(I_R(x,y))^p\right] \le K_{2.4.7}^p (p!)^N |y-x|^{-\alpha d p}. \tag{2.80}$$

Proof: The pth moment of $I_R(x,y)$ is equal to

$$R^{pd} \int \cdots \int_{[0,1]^{Np}} \left[\mathbb{P}\left\{ \max_{1\le i\le p} \left| B_0^{\langle\alpha\rangle}(x+t^i) - B_0^{\langle\alpha\rangle}(y+t^i)\right| < R^{-1}\right\}\right]^d dt^1 \cdots dt^p. \tag{2.81}$$

We will estimate the above integral by integrating in the order $dt^p, dt^{p-1}, \ldots, dt^1$. First let $t^1, \ldots, t^{p-1} \in [0,1]^N$ be fixed and assume, without loss of generality, that all coordinates of $t^1, \ldots, t^{p-1}$ are distinct. Define

$$\Omega_i := B_0^{\langle\alpha\rangle}(x+t^i) - B_0^{\langle\alpha\rangle}(y+t^i) \qquad {}^\forall i = 1, \ldots, p. \tag{2.82}$$

We begin by estimating the conditional probabilities

$$\mathcal{P}(t^p) := \mathbb{P}\left\{ |\Omega_p| < R^{-1} \;\middle|\; \max_{1\le i\le p-1} |\Omega_i| < R^{-1} \right\}. \tag{2.83}$$

Because $B_0^{\langle\alpha\rangle}$ is sectorially LND, we have

$$\begin{aligned} &\mathrm{Var}\,(\Omega_p \,|\, \Omega_i,\ 1 \le i \le p-1) \\ &\ge \mathrm{Var}\left(\Omega_p \;\middle|\; B_0^{\langle\alpha\rangle}(x+t^i), B_0^{\langle\alpha\rangle}(y+t^i), 1 \le i \le p-1\right) \\ &\ge K_{2.4.8} \sum_{k=1}^{N} \min\left\{v_k + \bar{v}_k \ , \ |x_k - y_k|^{2\alpha}\right\}, \end{aligned} \tag{2.84}$$

where $K_{2.4.8} > 0$ is a constant which depends on ε [we have used the fact that $x_k + t_k^p \ge \varepsilon$ for every $1 \le k \le N$] and

$$\begin{aligned} v_k &:= \min_{1\le i\le p-1} \left(|t_k^p - t_k^i|^{2\alpha},\ |x_k + t_k^p - y_k - t_k^i|^{2\alpha}\right), \\ \bar{v}_k &:= \min_{1\le i\le p-1} \left(|t_k^p - t_k^i|^{2\alpha},\ |y_k + t_k^p - x_k - t_k^i|^{2\alpha}\right). \end{aligned} \tag{2.85}$$

Observe that for every $1 \le k \le N$, we have

$$v_k + \bar{v}_k \ge \min_{\substack{1\le i\le p-1 \\ \ell=1,2,3}} \left|t_k^p - z_k^{i,\ell}\right|^{2\alpha}. \tag{2.86}$$

where $z_k^{i,1} = t_k^i$, $z_k^{i,2} = t_k^i + y_k - x_k$ and $z_k^{i,3} = t_k^i + x_k - y_k$ for $k = 1, \dots, N$.

It follows from (2.84) and (2.86) that

$$\operatorname{Var}\left(\Omega_p \mid \Omega_i,\ 1 \le i \le p-1\right) \ge K_{2,4,9} \sum_{k=1}^{N} \min\left\{ \min_{\substack{1\le i\le p-1\\ \ell=1,2,3}} \left|t_k^p - z_k^{i,\ell}\right|^{2\alpha} ;\ |x_k - y_k|^{2\alpha} \right\}. \tag{2.87}$$

Therefore, by Anderson's inequality [cf. Anderson (1955)], we have that $\mathcal{P}(t^p)$ is bounded from above by

$$K_{2,4,10} R^{-1} \left[\sum_{k=1}^{N} \min\left\{ \min_{\substack{1\le i\le p-1\\ \ell=1,2,3}} \left|t_k^p - z_k^{i,\ell}\right|^{2\alpha} ,\ |x_k - y_k|^{2\alpha} \right\} \right]^{-1/2}. \tag{2.88}$$

We note that the points $t^1, \ldots, t^{p-1}$ introduce a natural partition of $[0,1]^N$. More precisely, let $\pi_1, \ldots, \pi_N$ be N permutations of $\{1, \ldots, p-1\}$ such that for every $k = 1, \ldots, N$,

$$t_k^{\pi k(1)} < t_k^{\pi k(2)} < \ldots < t_k^{\pi k(p-1)}. \tag{2.89}$$

For convenience, we define also $t_k^{\pi k(0)} := 0$ and $t_k^{\pi k(p)} := 1$ for all $1 \le k \le N$.

For every $\mathbf{j} = (j_1, \ldots, j_N) \in \{1, \ldots, p-1\}^N$, let $\tau^{\mathbf{j}} = (t_1^{\pi 1(j_1)}, \ldots, t_N^{\pi N(j_N)})$ be the "center" of the rectangle

$$I_{\mathbf{j}} := \prod_{k=1}^{N} \left[t_k^{\pi k(j_k)} - \frac{t_k^{\pi k(j_k)} - t_k^{\pi k(j_k-1)}}{2} ,\ t_k^{\pi k(j_k)} + \frac{t_k^{\pi k(j_k+1)} - t_k^{\pi k(j_k)}}{2} \right), \tag{2.90}$$

with the convention being that whenever $j_k = 1$, the left-end point of the interval is 0; and whenever $j_k = p - 1$, the interval is closed and its right-end is 1. Thus the rectangles $\{I_{\mathbf{j}}\}_{\mathbf{j}\in\{1,\dots,p-1\}^N}$ form a partition of $[0,1]^N$.

For every $t^p \in [0,1]^N$, there is a unique $\mathbf{j} \in \{1,\dots,p-1\}^N$ such that $t^p \in I_{\mathbf{j}}$. Moreover, there exists a point $s^{\mathbf{j}}$ (depending on t^p) such that for every $k = 1,\dots,N$, the k-th coordinate of $s^{\mathbf{j}}$ satisfies

$$s_k^{\mathbf{j}} \in \left\{ t_k^{\pi_k(j_k)},\ t_k^{\pi_k(j_k-1)} + |x_k - y_k|,\ t_k^{\pi_k(j_k+1)} - |x_k - y_k| \right\}, \tag{2.91}$$

[If $j_1 = 1$, then we should also include t_k^p in the right hand side of (2.91). Since this does not affect the rest of the proof, we omit it for convenience] and

$$\min_{\substack{0\le i\le p-1\\ \ell=1,2,3}} \left| t_k^p - z_k^{i,\ell} \right| = \left| t_k^p - s_k^{\mathbf{j}} \right| \tag{2.92}$$

for every $k = 1,\dots,N$.

Hence, for every $t^p \in I_{\mathbf{j}}$, (2.88) can be rewritten as

$$\mathcal{P}(t^p) \le K_{2,4,10}\, R^{-1} \left[\sum_{k=1}^{N} \min\left\{ |t_k^p - s_k^{\mathbf{j}}|^{2\alpha}\ ,\ |x_k - y_k|^{2\alpha} \right\} \right]^{-1/2}. \tag{2.93}$$

Note that, as t^p varies in $I_{\mathbf{j}}$, there are at most 3^N corresponding points $s^{\mathbf{j}}$. Define $I_{\mathbf{j}}^{\mathcal{G}} := \{t^p \in I_{\mathbf{j}} :\ |x_k - y_k| \le |t_k^p - s_k^{\mathbf{j}}| \text{for all } k = 1,\dots,N\}$ as the set of "$\mathcal{G}$ood" points, and $I_{\mathbf{j}}^{\mathcal{B}} := I_{\mathbf{j}} \setminus I_{\mathbf{j}}^{\mathcal{G}}$ be the collection of "$\mathcal{B}$ad points." For every

$t^p \in I_{\mathbf{j}}^{\mathcal{G}}$, (2.93) yields

$$\mathcal{P}(t^p) \leq K_{2.4.10}\, R^{-1} \left[\sum_{k=1}^{N} |x_k - y_k|^{2\alpha} \right]^{-1/2} \leq K_{2.4.10}\, R^{-1}\, |y - x|^{-\alpha}. \tag{2.94}$$

If $t^p \in I_{\mathbf{j}}^{\mathcal{B}}$, then $|t_k^p - s_k^{\mathbf{j}}| < |x_k - y_k|$ for some $k = 1, \ldots, N$. We denote the collection of those indices by $\mathcal{U}$. Then, for every $k \notin \mathcal{U}$, $|x_k - y_k| \leq |t_k^p - s_k^{\mathbf{j}}|$, and we have

$$\mathcal{P}(t^p) \leq K_{2.4.10}\, R^{-1} \left[\sum_{k \in \mathcal{U}} \left| t_k^p - s_k^{\mathbf{j}} \right|^{2\alpha} + \sum_{k \notin \mathcal{U}} |x_k - y_k|^{2\alpha} \right]^{-1/2}. \tag{2.95}$$

It follows from (2.94) and (2.95) that $\int_{I_{\mathbf{j}}} [\mathcal{P}(t^p)]^d \, dt^p$ is at most

$$\begin{aligned} &\int_{I_{\mathbf{j}}^{\mathcal{G}}} K_{2.4.11}\, R^{-d}\, |y - x|^{-\alpha d}\, dt^p \\ &\quad + \int_{I_{\mathbf{j}}^{\mathcal{B}}} K_{2.4.11}\, R^{-d} \left[\sum_{k \in \mathcal{U}} \left| t_k^p - s_k^{\mathbf{j}} \right|^{2\alpha} + \sum_{k \notin \mathcal{U}} |x_k - y_k|^{2\alpha} \right]^{-d/2} dt^p \\ &\leq K_{2,4,12}\, R^{-d}\, |y - x|^{-\alpha d}. \end{aligned} \tag{2.96}$$

Note that we are able to neglect the integral over $I_{\mathbf{j}}^{\mathcal{B}}$ since $\alpha d < 1$. Hence, we have

$$\int_{[0,1]^N} [\mathcal{P}(t^p)]^d \, dt^p = \sum_{\mathbf{j}} \int_{I_{\mathbf{j}}} [\mathcal{P}(t^p)]^d \, dt^p \leq K_{2,4,12}\, p^N\, R^{-d}\, |y - x|^{-\alpha d}. \tag{2.97}$$

Continue integrating $dt^{p-1}, \ldots, dt^1$ in (2.81) in the same way, we finally obtain (2.80) as desired.

Remark 2.5 For later use in the proof of Theorem 2.5, we remark that the method of the proof of Lemma 2.9 can be used also to prove that

$$\int_{[0,1]^{2Np}} \cdots \int \left[\mathbb{P}\left\{ \max_{1\le j\le 2p} |B_0^{\langle\alpha\rangle}(x+t^j) - B_0^{\langle\alpha\rangle}(y+t^j)| \le 2^{-(1-\varepsilon)n} \right\}\right]^{N/\alpha} d\mathbf{t}$$

$$\le K_{2.4.13}^p \left[(2p)!\right]^N \left(2^{-\frac{2(1-\varepsilon)nNp}{\alpha}} n^{2p} + 2^{-\frac{2(1-\varepsilon)nNp}{\alpha}} |x-y|^{-2Np} \right). \tag{2.98}$$

In fact, by taking $R := 2^{(1-\varepsilon)n}$ in (2.88), we obtain that $\mathcal{P}(t^{2p})$ is bounded from above by

$$2^{-(1-\varepsilon)n} K_{2,4,14} \left[\sum_{k=1}^{N} \min\left\{ \min_{1\le i\le 2p-1} \left| t_k^{2p} - z_k^{i,\ell} \right|^{2\alpha} \ , \ |x_k - y_k|^{2\alpha} \right\} \right]^{-1/2}. \tag{2.99}$$

Based on (2.99) and the argument in the proof of Lemma 2.9, we follow through (2.94), (2.95), and (2.96). This leads us to (2.98).

With the help of Lemma 2.9, we can modify the proof of Theorem 1 in Kaufman (1989) to prove our Theorem 2.4.

Proof of Theorem 2.4: Almost surely, $\dim_{\mathrm{H}} B^{\langle\alpha\rangle}(F+t) \le \min\left\{d, \frac{1}{\alpha}\dim_{\mathrm{H}} F\right\}$ for all Borel sets F and all $t \in [0,1]^N$. Thus, we need to prove only the lower bound.

We first demonstrate that there exists a constant $K_{2,4,15}$ and an a.s.-finite random variable $n_0 = n_0(\omega)$ such that almost surely for all $n > n_0(\omega)$,

$$I_{2^n}(x,y) \le K_{2.4.15} n^N |y-x|^{-\alpha d} \quad \forall x, y \in [\varepsilon, 1]^N. \tag{2.100}$$

Let θ be an integer such that $\theta > 2^{1/\alpha}$ and consider the set $Q_n \subseteq [0,1]^N$ defined by

$$Q_n := \left\{\theta^{-n}\mathbf{k} : \; k_j = 0, 1, \ldots, \theta^n, \; ^\forall j = 1, \ldots, N\right\}. \tag{2.101}$$

The number of pairs $x, y \in Q_n$ is at most $K\theta^{2Nn}$. Hence for $u > 1$, Lemma 2.9 implies that

$$\begin{aligned} &\mathbb{P}\left\{I_{2^n}(x,y) > un^N|y-x|^{-\alpha d} \text{ for some } \; x, y \in Q_n \cap [\varepsilon\,, 1]^N\right\} \\ &\quad \leq \theta^{2Nn} K^p_{2,4,10}(p!)^N(un^N)^{-p}. \end{aligned} \tag{2.102}$$

By choosing $p := n$, $u := K_{2,4,10}\,\theta^N$, and owing to Stirling's formula, we know that the probabilities in (2.102) are summable. Therefore, by the Borel-Cantelli lemma, a.s. for all n large enough,

$$I_{2^n}(x,y) \leq K_{2,4,16}n^N|y-x|^{-\alpha d} \qquad ^\forall x, y \in Q_n \cap [\varepsilon, \; 1]^N. \tag{2.103}$$

Now we are ready to prove (2.100). This is a trivial task unless $n^N\,2^{-nd} < |y-x|^{\alpha d}$, which we assume is the case. For $x, \; y \in [\varepsilon\,, 1]^N$, we can find $\bar{x}$ and $\bar{y} \in Q_{n-1} \cap [\varepsilon, \; 1]^N$ so that $|x-\bar{x}| \leq \sqrt{N}\theta^{-n}$ and $|y-\bar{y}| \leq \sqrt{N}\theta^{-n}$, respectively. By the modulus of continuity of $B_0^{(\alpha)}$, we see that $I_{2^n}(x,y) \leq I_{2^{n-1}}(\bar{x},\bar{y})$ for all n large enough. On the other hand, by (2.103) and the assumption $n^N\,2^{-nd} <$

$|y - x|^{\alpha d}$, we have

$$I_{2^{n-1}}(\bar{x}, \bar{y}) \le (n-1)^N |\bar{x} - \bar{y}|^{-\alpha d} \le K_{2.4.15} n^N |y - x|^{-\alpha d}. \qquad (2.104)$$

Equation (2.100) follows.

For any Borel set $F \subset (0, 1]^N$ and all $\gamma \in (0, \dim_{\rm H} F)$, we choose $\eta \in (0, d \wedge \frac{\gamma}{\alpha})$. Then F carries a probability measure μ such that

$$\mu(S) \le K_{2.4.17} (\operatorname{diam} S)^\gamma \quad \text{for all measurable sets } S \subset (0, 1]^N. \qquad (2.105)$$

By Theorem 4.10 in Falconer (1990), we may and will assume μ is supported on a compact subset of F. Hence (2.100) is applicable.

Let ν_t be the image measure of μ under the mapping $x \mapsto B_0^{\langle\alpha\rangle}(x + t)$ $(x, t \in (0, 1]^N)$. By Frostman's Theorem, in order to prove $\dim_{\rm H} B_0^{\langle\alpha\rangle}(F + t) \ge \eta$, it suffices to prove that

$$\iint_{\mathbb{R}^{2d}} \frac{\nu_t(du)\, \nu_t(dv)}{|u - v|^\eta} < \infty. \qquad (2.106)$$

Now we follow Kaufman (1989), and note that the left-hand side is equal to

$$\begin{aligned}
&\iint \frac{\mu(dx)\, \mu(dy)}{\left|B_0^{\langle\alpha\rangle}(x + t) - B_0^{\langle\alpha\rangle}(y + t)\right|^\eta} \\
&= \eta \int_0^\infty \iint H_R\left(B_0(x + t) - B_0(y + t)\right) R^{\eta - d - 1}\, \mu(dx)\, \mu(dy)\, dR \\
&\le 1 + \int_1^\infty \iint H_R\left(B_0^{\langle\alpha\rangle}(x + t) - B_0^{\langle\alpha\rangle}(y + t)\right) R^{\eta - d - 1}\, \mu(dx)\, \mu(dy)\, dR.
\end{aligned} \qquad (2.107)$$

To prove that the last integral is finite for almost all $t \in [0,1]^N$, we integrate it over $[0,1]^N$ and prove that

$$\iint \int_1^\infty I_R(x,y)\, R^{\eta-d-1}\, dR\, \mu(dx)\, \mu(dy) < \infty. \tag{2.108}$$

We split the above integral over $D = \{(x,y) : |x-y| \le R^{-\frac{1}{\alpha}}\}$ and its complement, and denote them by J_1 and J_2, respectively. Since $(\mu \times \mu)(D) \le K_{2,4,17}\, R^{-\frac{\gamma}{\alpha}}$ and $\eta \in (0, \frac{\gamma}{\alpha})$, we have

$$J_1 \le K_{2,4,17} \int_1^\infty R^{-\frac{\gamma}{\alpha}+\eta-1}\, dR < \infty. \tag{2.109}$$

On the other hand, $|x-y|^{-\alpha} < R$ for all $(x,y) \in D^c$. Moreover, by (2.100), $I_R(x,y) < c(\omega)(\log R)^N |x-y|^{-\alpha d}$. It follows that

$$\begin{aligned} J_2 &\le K_{2,4,18}(\omega) \iint \frac{\mu(dx)\,\mu(dy)}{|x-y|^{\alpha d}} \int_{|x-y|^{-\alpha}}^\infty R^{\eta-d-1} (\log R)^N\, dR \\ &< K_{2,4,19}(\omega) \iint \frac{\log^N(1/|y-x|)}{|x-y|^{\alpha\eta}}\, \mu(dx)\,\mu(dy) < \infty, \end{aligned} \tag{2.110}$$

where the last inequality follows from (2.105). Combining (2.109) and (2.110) gives (2.108). This completes the proof of Theorem 2.4.

Theorem 2.5 *Let $\alpha d < 1$, then with probability 1, $m_d(B^{\langle\alpha\rangle}(F+t)) > 0$ for almost all $t \in [0,1]^N$, for every Borel set $F \subset (0,1]^N$ with $\dim_{\mathrm{H}} F > \alpha d$.*

Proof: Since $\dim_{\mathrm{H}} F > \alpha d$, there exists a Borel probability measure μ on F such that

$$\iint \frac{\mu(ds)\,\mu(dt)}{|s-t|^{\alpha d}} < \infty. \tag{2.111}$$

Again, we will assume that μ is supported by a compact subset of F.

Let ν_t denote the image-measure of μ, as it did in the proof of Theorem 2.4. It suffices to prove that

$$\int_{[0,1]^N} \int_{\mathbb{R}^d} |\widehat{\nu_t}(u)|^2 \, du \, dt < \infty, \qquad \text{a.s.,} \tag{2.112}$$

where exception null set does not depend on μ. Here, $\widehat{\nu_t}$ denotes the Fourier transform of ν_t; i.e.,

$$\widehat{\nu_t}(u) := \int_{\mathbb{R}_+^N} \exp(i\langle u, B^{(\alpha)}(x+t)\rangle)\mu(dx). \tag{2.113}$$

We choose and fix a smooth function $\psi \geq 0$ on $\mathbb{R}^d$ such that $\psi(u) = 1$ when $1 \leq |u| \leq 2$ and $\psi(u) = 0$ outside $1/2 < |u| < 5/2$, and satisfying $\psi(b_1 u_1, \ldots, b_d u_d) = \psi(u)$, where $b_\ell = \pm 1$, $\forall \ell = 1, \ldots, d$ and $u = (u_1, \ldots, u_d)$. Then $\int_{|u|>1} |\widehat{\nu_t}(u)|^2 \, du$ is bounded above by

$$\begin{aligned} &\sum_{n=0}^{\infty} \int_{\mathbb{R}^d} \psi(2^{-n}u) \, |\widehat{\nu_t}(u)|^2 \, du \\ &= \sum_{n=0}^{\infty} 2^n \int_{\mathbb{R}_+^{2N}} \widehat{\psi} \left(2^n B^{(\alpha)}(x+t) - 2^n B^{(\alpha)}(y+t)\right) \, \mu(dx) \, \mu(dy). \end{aligned} \tag{2.114}$$

Consequently, it suffices to show that

$$\sum_{n=0}^{\infty} 2^n \int_{[0,1]^N} \int_{\mathbb{R}_+^{2N}} \widehat{\psi}\left(2^n B^{\langle\alpha\rangle}(x+t) - 2^n B^{\langle\alpha\rangle}(y+t)\right) \mu(dx)\,\mu(dy)\,dt \tag{2.115}$$

is finite. To this end, we define

$$J(x,y,n) := \int_{[0,1]^N} \widehat{\psi}\left(2^n B^{\langle\alpha\rangle}(x+t) - 2^n B^{\langle\alpha\rangle}(y+t)\right) dt. \tag{2.116}$$

Lemma 2.10 *There exist positive and finite constant $K_{2,4,20}$ and β such that, with probability 1, for all $n \geq n(\omega)$ and $\sqrt{N} \geq |x-y| \geq K_{2,4,20} 2^{-n/\alpha} n^{1/\alpha}$,*

$$|J(x,y,n)| \leq (2+\beta)^{-n} |x-y|^{-\alpha d}. \tag{2.117}$$

Proof: It suffices to prove that there are positive constants $K_{2,4,21}$, $K_{2,4,22}$ and β such that for all integer $n \geq 1$ and $\sqrt{N} \geq |x-y| \geq K_{2,4,20} 2^{-n/\alpha} n^{1/\alpha}$,

$$\mathbb{E}\left[J(x,y,n)^{2n}\right] \leq K_{2,4,21}^{2n} n^{K_{2,4,22} n} (2+\beta)^{2n^2} |x-y|^{-2n\alpha d}. \tag{2.118}$$

Then (2.117) will follow from a Borel-Cantelli argument as in the proof of Theorem 2.4.

Note that the moment in (2.118) can be written as

$$\begin{aligned}
&\mathbb{E}\left[\int_{[0,1]^{2Nn}} \prod_{j=1}^{2n} \widehat{\psi}\big(2^n B^{\langle\alpha\rangle}(x+t^j) - 2^n B^{\langle\alpha\rangle}(y+t^j)\big)\, d\mathbf{t}\right] \\
&= \mathbb{E}\left[\int_{S_n} \prod_{j=1}^{2n} \widehat{\psi}\big(2^n B^{\langle\alpha\rangle}(x+t^j) - 2^n B^{\langle\alpha\rangle}(y+t^j)\big)\, d\mathbf{t}\right] \\
&\quad + \mathbb{E}\left[\int_{[0,1]^{2Nn}\setminus S_n} \prod_{j=1}^{2n} \widehat{\psi}\big(2^n B^{\langle\alpha\rangle}(x+t^j) - 2^n B^{\langle\alpha\rangle}(y+t^j)\big)\, d\mathbf{t}\right],
\end{aligned} \tag{2.119}$$

where $\mathbf{t} := (t^1, \ldots, t^{2n})$ and

$$\begin{aligned}
S_n := \bigcup_{k=1}^{2n} \bigcup_{\ell=1}^{N} \Bigg\{ \mathbf{t} \in [0,1]^{2Nn} :\ &\left|t_\ell^k - t_\ell^j\right| > r_n \ \text{ and} \\
&\left|x_\ell + t_\ell^k - t_\ell^j - y_\ell\right| > r_n \ \forall j \neq k \Bigg\},
\end{aligned} \tag{2.120}$$

and $r_n := K_{2,4,20} 2^{-n/\alpha}(n+1)^{1/\alpha}$, where $K_{2.4.20} > 0$ is a constant whose value will be determined later.

We consider the integral over S_n first; it can be rewritten as

$$\begin{aligned}
&\mathbb{E}\left[\int_{S_n}\int_{\mathbb{R}^{2nd}} \prod_{j=1}^{2n} \exp(i\langle \xi^j,\, 2^n B^{\langle\alpha\rangle}(x+t^j) - 2^n B^{\langle\alpha\rangle}(y+t^j)\rangle)\psi(\xi^j)\, d\xi\, d\mathbf{t}\right] \\
&= \int_{S_n}\int_{\mathbb{R}^{2nd}} \exp\left[-\frac{1}{2}\sum_{\ell=1}^{d} \operatorname{Var}\left(\sum_{j=1}^{2n} \xi_\ell^j 2^n [B_0^{\langle\alpha\rangle}(x+t^j) - B_0^{\langle\alpha\rangle}(y+t^j)]\right)\right] \\
&\qquad \times \prod_{j=1}^{2n} \psi(\xi^j)\, d\xi\, d\mathbf{t},
\end{aligned} \tag{2.121}$$

where $\xi := (\xi^1, \dots, \xi^{2n})$.

Note that for every $\mathbf{t} \in S_n$, there is a $k \in \{1, \dots, 2n\}$ such that $|t^k - t^j| > r_n$ for all $j \neq k$ and $|x + t^k - t^j - y| > r_n$ for all $0 \leq j \leq 2n$. Since $|\xi^k| \in [\frac{1}{2}, \frac{5}{2}]$, there exists $\ell_0 \in \{1, \dots, d\}$ such that $|\xi^k_{\ell_0}| \geq (2\sqrt{d})^{-1}$. We derive that

$$\begin{aligned}
&\operatorname{Var}\Bigg(\sum_{j=1}^{2n} \xi^j_{\ell_0} \big[2^n B_0^{\langle\alpha\rangle}(x + t^j) - 2^n B_0^{\langle\alpha\rangle}(y + t^j)\big]\Bigg) \\
&\geq \operatorname{Var}\Big(\xi^k_{\ell_0} \big[2^n B_0^{\langle\alpha\rangle}(x + t^k) - 2^n B_0^{\langle\alpha\rangle}(y + t^k)\big] \\
&\qquad\qquad\qquad \big| B_0^{\langle\alpha\rangle}(x + t^j), B_0^{\langle\alpha\rangle}(y + t^j),\, j \neq k\Big) \\
&\geq \frac{1}{4d}\, 2^{2n} \operatorname{Var}\Big(B_0^{\langle\alpha\rangle}(x + t^k) \big| B_0^{\langle\alpha\rangle}(x + t^j),\ \forall j \neq k;\ B_0^{\langle\alpha\rangle}(y + t^j),\ \forall j\Big) \\
&\geq K_{2,4,23}\, 2^{2n} \sum_{\ell=1}^{N} \min_{1 \leq j \neq k \leq 2n} \Big\{ |t^k_\ell - t^j_\ell|^{2\alpha}\ ,\ \ |x_\ell + t^k_\ell - y_\ell - t^j_\ell|^{2\alpha} \Big\} \\
&\geq K_{2,4,23} 2^{2n} r_n^{2\alpha} = K_{2,4,23} K_{3,20}^{2\alpha} (n+1)^2.
\end{aligned} \tag{2.122}$$

In the above $K_{2,4,23} > 0$ is a constant depending on ε and again we have used the fact that $x_\ell + t^k_\ell \geq \varepsilon$ for every $1 \leq \ell \leq N$.

By combining (2.121) and (2.122), we obtain

$$\begin{aligned}
&\mathbb{E}\left[\int_{S_n} \int_{\mathbb{R}^{2nd}} \prod_{j=1}^{2n} \exp(i\langle \xi^j,\, 2^n B^{\langle\alpha\rangle}(x + t^j) - 2^n B^{\langle\alpha\rangle}(y + t^j)\rangle)\, \psi(\xi^j)\, d\xi\, d\mathbf{t}\right] \\
&\leq \exp(-K_{2,4,24} n^2).
\end{aligned} \tag{2.123}$$

Now, we consider the integral in (2.119) over $T_n := [0,1]^{2Nn} \backslash S_n$, which can be written as

$$
\begin{aligned}
T_n = \Big\{ \mathbf{t} \in [0,1]^{2Nn} : {}^{\forall}k \in \{1,\dots,2n\},\ {}^{\forall}\ell \in \{1,\dots,N\}, \\
{}^{\exists}j_{\ell,1} \neq k \ \text{ s.t. } \left|t_\ell^k - t_\ell^{j_{\ell,1}}\right| \le r_n \\
\text{or } {}^{\exists}j_{\ell,2} \neq k \text{ s.t. } \left|x_\ell + t_\ell^k - y_\ell - t_\ell^{j_{\ell,2}}\right| \le r_n \Big\} \\
= \bigcap_{k=1}^{2n} \bigcap_{\ell=1}^{N} \Bigg(\Big\{ \mathbf{t} \in [0,1]^{2Nn} : \min_{j_{\ell,1} \neq k} \left|t_\ell^k - t_\ell^{j_{\ell,1}}\right| \le r_n \Big\} \\
\cup \Big\{ \mathbf{t} \in [0,1]^{2Nn} : \min_{j_{\ell,2} \neq k} \left|x_\ell + t_\ell^k - y_\ell - t_\ell^{j_{\ell,2}}\right| \le r_n \Big\} \Bigg).
\end{aligned}
\tag{2.124}
$$

From (2.124), we can see that T_n is a union of at most $(4n)^{2Nn}$ sets of the form:

$$
A_{\mathbf{j}} = \left\{ \mathbf{t} \in [0,1]^{2Nn} : \max_{\substack{1 \le k \le 2n \\ 1 \le \ell \le N}} \left| z_\ell + t_\ell^k - t_\ell^{j_{\ell,k}} \right| \le r_n \right\}, \tag{2.125}
$$

where $\mathbf{j} := (j_{\ell,k} : 1 \le k \le 2n,\ 1 \le \ell \le N)$ has the property that $j_{\ell,k} \neq k$ and where $z_\ell = 0$ or $x_\ell - y_\ell$.

The following lemma from Khoshnevisan, Wu and Xiao (2005) estimates the Lebesgue measure of T_n.

Lemma 2.11 *For any positive even number m, all $z_1, \dots, z_L \in \mathbb{R}$, every sequence $\{\ell_1, \dots, \ell_L\} \subseteq \{1, \dots, L\}$ satisfying $\ell_j \neq j$, and for each $r \in (0,1)$, we have*

$$
m_L \left\{ s \in [0,1]^L : \max_{k \in \{1,\dots,L\}} |z_k + s_k - s_{\ell_k}| \le r \right\} \le (2r)^{L/2}. \tag{2.126}
$$

Now, it follows from (2.124), (2.125) and Lemma 2.11 that

$$m_{2Nn}(T_n) \le (4n)^{2Nn} (2r_n)^{Nn}. \tag{2.127}$$

We proceed to estimate the integral in (2.119) over T_n. It is bounded above by

$$\begin{aligned}
&\int_{T_n} \mathbb{E}\left[\prod_{j=1}^{2n} \left|\widehat{\psi}(2^n B^{\langle\alpha\rangle}(x+t^j) - 2^n B^{\langle\alpha\rangle}(y+t^j))\right|\right] d\mathbf{t} \\
&= \int_{T_n} \mathbb{E}[\cdots ;\ D_n]\, d\mathbf{t} + \int_{T_n} \mathbb{E}[\cdots ;\ D_n^c]\, d\mathbf{t} =: I_1 + I_2,
\end{aligned} \tag{2.128}$$

where

$$D_n := \left\{ \max_{1\le j\le 2n} \left| B^{\langle\alpha\rangle}(x+t^j) - B^{\langle\alpha\rangle}(y+t^j)\right| > 2^{-(1-\varepsilon)n} \right\}. \tag{2.129}$$

Since $\widehat{\psi}$is a rapidly decreasing function, we derive from (2.127) that

$$\begin{aligned}
I_1 &\le m_{2Nn}(T_n)\, \mathbb{P}(D_n) \exp\{-K_{2.4.25} n\} \\
&\le (4n)^{2Nn} (2r_n)^{Nn} \exp\{-K_{2,4,25} n\} \\
&= K_{2.4.26}^n\, (n)^{Nn(2+\frac{1}{\alpha})}\, 2^{-\frac{Nn^2}{\alpha}} \exp\{-K_{2,4,25} n\}.
\end{aligned} \tag{2.130}$$

Note that $K_{2.4.25} > 0$ can be chosen arbitrarily large, I_1 is very small. On the other hand, by the Cauchy-Schwarz inequality, I_2 is at most

$$\begin{aligned}
&\int_{T_n} \mathbb{P}\Big\{|B^{\langle\alpha\rangle}(x+t^j) - B^{\langle\alpha\rangle}(y+t^j)| \le 2^{-(1-\varepsilon)n}, \forall j = 1, \ldots, 2n\Big\}\, d\mathbf{t} \\
&\le \int_{[0,1]^{2Nn}} \mathbf{1}_{T_n}(\mathbf{t}) \\
&\times \Big(\mathbb{P}\Big\{|B_0^{\langle\alpha\rangle}(x+t^j) - B_0^{\langle\alpha\rangle}(y+t^j)| \le 2^{-(1-\varepsilon)n}, \forall j = 1, \ldots, 2n\Big\}\Big)^d\, d\mathbf{t} \\
&\le \Big(m_{2Nn}(T_n)\Big)^{(N-\alpha d)/N} \Big\{\int_{[0.1]^{2Nn}} \\
&\qquad \Big(\mathbb{P}\Big\{|B_0^{\langle\alpha\rangle}(x+t^j) - B_0^{\langle\alpha\rangle}(y+t^j)| \le 2^{-(1-\varepsilon)n}, \forall j\Big\}\Big)^{N/\alpha} d\mathbf{t}\Big\}^{\alpha d/N} \\
&\le K_{2,4,27}^n\, n^{K_{2,4,28}n}\, 2^{-n^2\left(\frac{N}{\alpha}+(1-2\varepsilon)d\right)}\, |x-y|^{-2\alpha n d},
\end{aligned} \tag{2.131}$$

where the last inequality follows from (2.127) and (2.98) in Remark 2.5 with $p = n$ and where $K_{2,4,28} > 0$ is a constant depending on α, d and N only.

Combining (2.119), (2.123) with $K_{2,4,25}$ large, (2.128), (2.130) and (2.131), we obtain

$$\mathbb{E}\big[J(x,y,n)^{2n}\big] \le K_{2,4,21}^n\, n^{K_{2,4,22}n}\, 2^{-n^2\left(\frac{N}{\alpha}+(1-2\varepsilon)d\right)}\, |x-y|^{-2\alpha n d}. \tag{2.132}$$

We choose and fix $0 < \varepsilon < \frac{N+\alpha d-2\alpha}{2\alpha d}$. This guarantees that $2^{-\frac{n}{2}\left(\frac{N}{\alpha}+(1-2\varepsilon)d\right)}$ for some constant $\beta > 0$. By using (2.132), the Borel-Cantelli lemma and the modulus of continuity of B, we can derive (2.117) in the same way as in the proof of Theorem 2.4.

Now we conclude the proof of Theorem 2.5. Thanks to Lemma 2.10, we have

$$2^n \iint |J(x,y,n)|\, \mu(dx)\, \mu(dy) \le 2^n(2+\beta)^{-n} \iint \frac{\mu(dx)\, \mu(dy)}{|y-x|^{\alpha d}} \le K_{2.4.28} \left(\frac{2}{2+\beta}\right)^n. \tag{2.133}$$

This implies (2.115), and finishes our proof of Theorem 2.5.

Remark 2.6 When $N > \alpha d \ge 1$, our proof of Lemma 2.9 breaks down. See (2.96), where the integral on $I_{\mathbf{j}}^{\beta}$ can not be neglected anymore. In this general case, we do not know whether Theorems 2.4 and 2.5 remain valid.

The following question was raised by Kaufman (1989) for Brownian motion in $\mathbb{R}$. It is still open, and we reformulate it for the fractional Brownian sheet with Hurst index $\langle\alpha\rangle$.

Question 2.7 Suppose $N > \alpha d$. Is it true that, with probability 1, $B^{\langle\alpha\rangle}(F+t)$ has interior points for some $t \in [0,1]^N$ for every Borel set $F \subset (0,\infty)^N$ with $\dim_{\mathrm{H}} F > \alpha d$?

2.5 Salem set

Let $X = \{X(t),\, t \in \mathbb{R}^N\}$ be a centered Gaussian random field with values in $\mathbb{R}^d$. When X is an (N,d)-fractional Brownian motion of index $\gamma \in (0,1)$, Kahane (1985a, b) studied the asymptotic properties of the Fourier transforms of the image measures of X and proved that, for every Borel set $E \subset \mathbb{R}^N$ with $\dim_{\mathrm{H}} E \le \gamma d$, $X(E)$ is a Salem set almost surely. Kahane (1993) further raised the question of studying the Fourier dimensions of other random sets. Recently, Shieh and Xiao

(2005) extended Kahane's results to a large class of Gaussian random fields with stationary increments and Khoshnevisan, Wu and Xiao (2005) proved similar results for the Brownian sheet.

In this section, we study the asymptotic properties of the Fourier transforms of the image measures of the (N, d)-fractional Brownian sheet B^H. The main result of this section is Theorem 2.6 below, whose proof depends crucially on the ideas of sectorial local nondeterminism and Hausdorff dimension contour. Moreover, by combining Theorems 2.2 and 2.6 we show that, for every Borel set $E \subset (0, \infty)^N$, $B^H(E)$ is almost surely a Salem set whenever $s(H, E) \leq d$. Recall that $s(H, E)$ is defined in Theorem 2.2.

Let B_0^H be an $(N, 1)$-fractional Brownian sheet with index $H = (H_1, \ldots, H_N)$. Let $0 < \varepsilon < T$ be fixed. For all $n \geq 2$, $t^1, \ldots, t^n, s^1, \ldots, s^n \in E \subset [\varepsilon, T]^N$, denote $\mathbf{s} = (s^1, \ldots, s^n)$, $\mathbf{t} = (t^1, \ldots, t^n)$ and

$$\Psi(\mathbf{s}, \mathbf{t}) = \mathbb{E}\left[\sum_{k=1}^{n} \left(B_0^H(t^k) - B_0^H(s^k)\right)\right]^2. \tag{2.134}$$

For $\mathbf{s} \in E^n$ and $r > 0$, we define

$$F(\mathbf{s}, r) = \bigcup_{i_1=1}^{n} \cdots \bigcup_{i_N=1}^{n} \bigcap_{j=1}^{N} \left\{u \in E : \left|u_j - s_j^{i_j}\right| \leq r^{1/H_j}\right\}.$$

This is a union of at most n^N rectangles of side-lengths $2r^{1/H_1}, \ldots, 2r^{1/H_N}$, centered at $(s_1^{i_1}, \ldots, s_N^{i_k})$. Let

$$G(\mathbf{s}, r) = \left\{\mathbf{t} = (t^1, \ldots, t^n) : t^k \in F(\mathbf{s}, r) \text{ for } 1 \leq k \leq n\right\}. \tag{2.135}$$

The following lemma is essential for the proof of Theorem 2.6.

Lemma 2.12 *There exists a positive constant* $K_{2,5,1}$, *depending on* ε, T, H, N *only, such that for all* $r \in (0, \varepsilon]$ *and all* $\mathbf{s}, \mathbf{t} \in E^n$ *with* $\mathbf{t} \notin G(\mathbf{s}, r)$, *we have* $\Psi(\mathbf{s}, \mathbf{t}) \geq K_{2,5,1}\, r^2$.

Proof: Since $\mathbf{t} \notin G(\mathbf{s}, r)$, there exist $k_0 \in \{1, \ldots, n\}$ and $j_0 \in \{1, \ldots, N\}$ such that $\left|t_{j0}^{k0} - s_{j0}^{k}\right| > r^{1/H_{j0}}$ for all $k = 1, \ldots, n$. It follows from (2.3) that

$$\Psi(\mathbf{s}, \mathbf{t}) = K_H^{-2} \int_{\mathbb{R}^N} \left| \sum_{k=1}^{n} \left(\prod_{j=1}^{N} \left(\exp(it_j^k \lambda_j) - 1 \right) - \prod_{j=1}^{N} \left(\exp(is_j^k \lambda_j) - 1 \right) \right) \right|^2 f_H(\lambda)\, d\lambda, \tag{2.136}$$

where

$$f_H(\lambda) = \prod_{j=1}^{N} |\lambda_j|^{-2H_j - 1}.$$

Let $\delta_j(\cdot) \in C^\infty(\mathbb{R})$ $(1 \leq j \leq N)$ be the bump functions in the proof of Theorem 2.1. We define

$$\delta_{j0}^{r}(u_{j0}) = r^{-1/H_{j0}} \delta_{j0}(r^{-1/H_{j0}} u_{j0})$$

and

$$\delta_j^{\varepsilon}(u_j) = \varepsilon^{-1} \delta_j(\varepsilon^{-1} u_j), \quad \text{if } j \neq j_0.$$

Then, by using the Fourier inversion formula again, we have

$$\delta^r_{j0}(u_{j0}) = (2\pi)^{-1} \int_{\mathbb{R}} \exp(-iu_{j0}\lambda_{j0})\widehat{\delta}_{j0}(r^{1/H_{j0}}\lambda_{j0})d\lambda_{j0}$$

and similar identities holds for $\delta^\varepsilon_j(u_j)$ with $j \neq j_0$.

Since $r \in (0, \varepsilon)$, we have $\delta^r_{j0}(t^{k0}_{j0}) = 0$ and $\delta^\varepsilon_j(t^{k0}_j) = 0$ for all $j \neq j_0$. Similarly, $\delta^r_{j0}(t^{k0}_{j0} - s^k_{j0}) = 0$ for all $k = 1, \ldots, n$. Hence,

$$\begin{aligned} J \equiv & \int_{\mathbb{R}^N} \left[\sum_{k=1}^{n} \left(\prod_{j=1}^{N} (\exp(it^k_j\lambda_j) - 1) - \prod_{j=1}^{N} (\exp(is^k_j\lambda_j) - 1) \right) \right] \\ & \times \prod_{j=1}^{N} \exp(-it^{k0}_j\lambda_j) \left(\prod_{j \neq j0}^{N} \widehat{\delta}_j(\varepsilon\lambda_j) \right) \widehat{\delta}_{j0}(r^{1/H_{j0}}\lambda_{j0})\, d\lambda \\ = & (2\pi)^N \sum_{k=1}^{n} \left(\prod_{j \neq j0}^{N} (\delta^\varepsilon_j(t^{k0}_j - t^k_j) - \delta^\varepsilon_j(t^{k0}_j)) \right) \left(\delta^r_{j0}(t^{k0}_{j0} - t^k_{j0}) - \delta^r_{j0}(t^{k0}_{j0}) \right) \\ - & (2\pi)^N \sum_{k=1}^{n} \left(\prod_{j \neq j0}^{N} (\delta^\varepsilon_j(t^{k0}_j - s^k_j) - \delta^\varepsilon_j(t^{k0}_j)) \right) \left(\delta^r_{j0}(t^{k0}_{j0} - s^k_{j0}) - \delta^r_{j0}(t^{k0}_{j0}) \right) \\ \geq & (2\pi)^N \varepsilon^{-(N-1)} r^{-1/H_{j0}}. \end{aligned} \tag{2.137}$$

In the above, all the terms in the first sum are non-negative and the second sum equals 0.

On the other hand, by the Cauchy-Schwarz inequality, (2.136) and (2.137), we get

$$
\begin{aligned}
J^2 &\le K_H^2 \Psi(\mathbf{t}, \mathbf{s}) \int_{\mathbb{R}^N} \frac{1}{f_H(\lambda)} \prod_{j \ne j0}^{N} \left|\widehat{\delta}_j(\varepsilon \lambda_j)\right|^2 \left|\widehat{\delta}_{j0}\left(r^{1/H_{j0}} \lambda_{j0}\right)\right|^2 d\lambda \\
&= K_H^2 \Psi(\mathbf{t}, \mathbf{s})\, \varepsilon^{-2(N-1)-2\sum_{j \ne j0} H_j}\, r^{-2-2/H_{j0}} \\
&\qquad \times \prod_{j=1}^{N} \int_{\mathbb{R}} |\lambda_j|^{2H_j+1} \left|\widehat{\delta}_j(\lambda_j)\right|^2 d\lambda_j \\
&= K_{2.5.2} r^{-2-2/H_{j0}} \Psi(\mathbf{t}, \mathbf{s}).
\end{aligned}
\tag{2.138}
$$

Square the both sides of (2.137) and combine it with (2.138), the lemma follows.

For any Borel probability measure μ on $\mathbb{R}_+^N$, let $\nu = \mu_{B^H}$ be the image measure of μ under B^H. The Fourier transform of ν can be written as

$$
\widehat{\nu}(\xi) = \int_{\mathbb{R}_+^N} \exp(i\langle \xi, B^H(t)\rangle)\, \mu(dt). \tag{2.139}
$$

The following theorem describes the asymptotic behavior of $\widehat{\nu}(\xi)$ as $\xi \longrightarrow \infty$. Contrast to the results for the fractional Brownian motion and the Brownian sheet mentioned above, the behavior of $\widehat{\nu}(\xi)$ is anisotropic.

Theorem 2.6 *Let $B^H = \{B^H(t), t \in \mathbb{R}^N\}$ be an (N, d)-fractional Brownian sheet. Assume that, for every $j = 1, \ldots, N$, the function $\tau_j : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is non-decreasing such that $\tau_j(0) = 0$ and $\tau_j(2r) \le K_{2.5.3} \tau_j(r)$ for all $r \ge 0$ [i.e., τ_j satisfies the doubling property]. If μ is a Borel probability measure on $[\varepsilon, T]^N$ such* *that*

$$\mu\big(R(t,r)\big) \le K_{2.5.4} \prod_{j=1}^{N} \tau_j(r^{1/H_j}), \quad \forall\, t \in \mathbb{R}_+^N, \tag{2.140}$$

where $R(t,r) = \prod_{j=1}^{N}[t_j - r^{1/H_j}, t_j + r^{1/H_j}]$. *Then there exists a positive and finite constant* ϱ *such that almost surely,*

$$\limsup_{|\xi|\to\infty} \frac{|\widehat{\nu}(\xi)|}{\sqrt{\Big(\prod_{j=1}^{N} \tau_j\big(|\xi|^{-1/H_j}\big)\Big) \log^{\varrho}|\xi|}} < \infty. \tag{2.141}$$

Proof: The argument is similar to that of Kahane (1985a). First note that by considering the restriction of μ on subsets of its support and the linearity of the Fourier transform, we see that, without loss of generality, we may and will assume μ is supported on a Borel set $E \subset [\varepsilon, T]^N$ with diam$E < \varepsilon^{1/H_1}$ [we have assumed that $H_1 = \min\{H_j : 1 \le j \le N\}$]. The reason for this reduction will become clear below.

For any positive integer $n \ge 1$, (2.139) yields

$$\begin{aligned}
&\mathbb{E}\big(|\widehat{\nu}(\xi)|^{2n}\big) \\
&= \mathbb{E}\int_{\mathbb{R}_+^{nN}}\int_{\mathbb{R}_+^{nN}} \exp(i\langle \xi, \sum_{k=1}^{n}(B^H(t^k) - B^H(s^k))\rangle)\, \mu^n(d\mathbf{s})\mu^n(d\mathbf{t}) \\
&= \int_{\mathbb{R}_+^{nN}}\int_{\mathbb{R}_+^{nN}} \exp\Big(-\frac{1}{2}|\xi|^2\Psi(\mathbf{s},\mathbf{t})\Big)\, \mu^n(d\mathbf{s})\mu^n(d\mathbf{t}),
\end{aligned} \tag{2.142}$$

where $\mu^n(d\mathbf{s}) = \mu(ds^1)\cdots\mu(ds^n)$.

Let $\mathbf{s} \in [\varepsilon, T]^{nN}$ be fixed and we write

$$\begin{aligned}
&\int_{\mathbb{R}_+^{nN}} \exp\Big(-\frac{1}{2}|\xi|^2\Psi(\mathbf{s},\mathbf{t})\Big)\mu^n(d\mathbf{t}) \\
&= \int_{G(\mathbf{s},r)} \exp\Big(-\frac{1}{2}|\xi|^2\Psi(\mathbf{s},\mathbf{t})\Big)\mu^n(d\mathbf{t}) \qquad (2.143)\\
&\quad + \sum_{m=1}^{\infty} \int_{G(\mathbf{s},r2^m)\setminus G(\mathbf{s},r2^{m-1})} \exp\Big(-\frac{1}{2}|\xi|^2\Psi(\mathbf{s},\mathbf{t})\Big)\mu^n(d\mathbf{t}).
\end{aligned}$$

Since μ is supported on E with $\mathrm{diam}E < \varepsilon^{1/H_1}$, the above summation is taken over the integers m such that $r2^m \le \varepsilon$. Hence we can apply Lemma 2.12 to estimate the integrands.

By (2.140), we always have

$$\int_{G(\mathbf{s},r)} \exp\Big(-\frac{1}{2}|\xi|^2\Psi(\mathbf{s},\mathbf{t})\Big)\mu^n(d\mathbf{t}) \le \Big(K_{2,5,4}\, n^N \prod_{j=1}^{N} \tau_j(r^{1/H_j})\Big)^n. \qquad (2.144)$$

Given $\xi \in \mathbb{R}^d\setminus\{0\}$, we take $r = |\xi|^{-1}$. It follows from Lemma 2.12, the doubling property of functions τ_j, and (2.140) that

$$\begin{aligned}
&\int_{G(\mathbf{s},r2^m)\setminus G(\mathbf{s},r2^{m-1})} \exp\Big(-\frac{1}{2}|\xi|^2\Psi(\mathbf{s},\mathbf{t})\Big)\mu^n(d\mathbf{t}) \\
&\le \exp\Big(-\frac{1}{2}K_{2,5,1}\,|\xi|^2(r2^{m-1})^2\Big)\cdot\Big(K_{2,5,4}\, n^N \prod_{j=1}^{N} \tau_j(2^m r^{1/H_j})\Big)^n \\
&\le \Big(K_{2,5,4}\, n^N \prod_{j=1}^{N} \tau_j(r^{1/H_j})\Big)^n \exp\Big(-K_{2,5,5}\,2^{2m}\Big)\cdot K_{2,5,3}^{Nmn}.
\end{aligned} \qquad (2.145)$$

Note that

$$1+\sum_{m=1}^{\infty}\exp\Big(-K_{2,5,5}\,2^{2m}\Big)\cdot K_{2,5,3}^{Nmn}\le K_{2,5,6}^{n}n^{\rho n}, \tag{2.146}$$

where $\rho = N/(2\log K_{2,5,3})$.

Combining (2.144), (2.145) and (2.146), we derive an upper bound for the integral in (2.143):

$$\int_{\mathbb{R}_+^{nN}}\exp\Big(-\frac{1}{2}|\xi|^2\Psi(\mathbf{s},\mathbf{t})\Big)\mu^n(d\mathbf{t})\le K_{2,5,7}^{n}\cdot n^{(N+\rho)n}\cdot\Big(\prod_{j=1}^{N}\tau_j\big(|\xi|^{-1/H_j}\big)\Big)^n. \tag{2.147}$$

Integrate the both sides of (2.147) in $\mu^n(d\mathbf{s})$, we get

$$\mathbb{E}\big(\big|\widehat{\nu}(\xi)\big|^{2n}\big)\le K_{2,5,7}^{n}\cdot n^{\varrho n}\cdot\Big(\prod_{j=1}^{N}\tau_j\big(|\xi|^{-1/H_j}\big)\Big)^n, \tag{2.148}$$

where $\varrho = N+\rho$.

The same argument as in Kahane (1985a, pp. 254–255) using (2.148) and the Borel-Cantelli lemma implies that almost surely

$$\limsup_{z\in\mathbb{Z}^d,|z|\to\infty}\frac{\big|\widehat{\nu}(z)\big|}{\sqrt{\Big(\prod_{j=1}^{N}\tau_j\big(|z|^{-1/H_j}\big)\Big)\log^{\varrho}|z|}}<\infty. \tag{2.149}$$

Therefore (2.141) follows from (2.149) and Lemma 1 of Kahane (1985a, p.252). This finishes the proof of Theorem 2.6.

Theorem 2.7 *Let $B^H = \{B^H(t), t\in\mathbb{R}^N\}$ be an (N,d)-fractional Brownian* *sheet with Hurst index* $H \in (0,1)^N$. *Then for every Borel set* $E \subset (0,\infty)^N$ *with* $s(H,E) \le d$, $B^H(E)$ *is almost surely a Salem set with Fourier dimension* $s(H,E)$.

Proof: It follows from (2.47) and the fact that $\dim_{\mathrm{F}} F \le \dim_{\mathrm{H}} F$ for all Borel sets $F \subset \mathbb{R}^d$ that for every Borel set $E \subset (0,\infty)^N$ satisfying $s(H,E) \le d$, we have $\dim_{\mathrm{F}} B^H(E) \le \dim_{\mathrm{H}} B^H(E) = s(H,E)$ a.s.

To prove the reverse inequality, it suffices to show that if $s(H,E) \le d$ then for all $\gamma \in (0, s(H,E))$ we have $\dim_{\mathrm{F}} B^H \ge \gamma$ a.s.

Note that for any $0 < \gamma < s(H,E)$, there exists a Borel probability measure μ with compact support in E such that $\gamma < s_\mu(H)$. Hence we can find $\lambda' = (\lambda'_1, \ldots, \lambda'_N) \in \Lambda_\mu$ such that $\gamma < \sum_{j=1}^N \frac{\lambda'_j}{H_j}$ and (2.53) holds. Let μ_ε be the restriction of μ to the set E_ε defined by (2.54). Then μ_ε satisfies the condition (2.140) with $\tau_j(r) = r^{\lambda'_j}$ $(j = 1, \ldots, N)$.

Let ν be the image measure of μ_ε under B^H. Then by Theorem 2.6 we have almost surely,

$$\left|\widehat{\nu}(\xi)\right| = O\left(\sqrt{|\xi|^{-\sum_{j=1}^N \frac{\lambda'_j}{H_j}} \log^\varrho |\xi|}\right), \qquad \text{as } \xi \to \infty. \tag{2.150}$$

This and (1.11) imply that $\dim_{\mathrm{F}} B^H(E) \ge \sum_{j=1}^N \frac{\lambda'_j}{H_j}$ a.s., which yields $\dim_{\mathrm{F}} B^H(E) \ge \gamma$ a.s. Therefore $\dim_{\mathrm{F}} B^H(E) = \dim_{\mathrm{H}} B^H(E)$, i.e., $B^H(E)$ is a Salem set.

Applying Theorems 2.6 and 2.7 to $B^{\langle\alpha\rangle}$, we have the following result.

Corollary 2.3 *Let* $B^{\langle\alpha\rangle} = \{B^{\langle\alpha\rangle}(t), t \in \mathbb{R}^N\}$ *be an* (N,d)*-fractional Brownian sheet, and let* $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ *be a non-decreasing function satisfying* $\tau(0) = 0$ *and* *the doubling property. If μ is a probability measure on $[\varepsilon, T]^N$ such that*

$$\mu(B(x,r)) \le K_{2,5,8}\, \tau(2r), \qquad \forall x \in \mathbb{R}_+^N,\ r \ge 0. \tag{2.151}$$

Then there exists a positive and finite constant ϱ such that

$$\limsup_{|\xi|\to\infty} \frac{|\widehat{\nu}(\xi)|}{\sqrt{\tau\big(|\xi|^{-1/\alpha}\big) \log^{\varrho} |\xi|}} < \infty. \tag{2.152}$$

Moreover, for every Borel set $E \subset (0,\infty)^N$ with $\dim_{\rm H} E \le \alpha d$, $B^{\langle\alpha\rangle}(E)$ is almost surely a Salem set with Fourier dimension $\dim_{\rm H} E/\alpha$.

2.6 Interior points

By using the Fourier analytic argument of Kahane (1985a), it is easy to show the following: If a Borel set $E \subset (0,\infty)^N$ carries a probability measure μ such that

$$\int_E \int_E \frac{1}{\big(\sum_{j=1}^N |s_j - t_j|^{2H_j}\big)^{d/2}}\, \mu(ds)\mu(dt) < \infty, \tag{2.153}$$

then almost surely, $B^H(E)$ has positive d-dimensional Lebesgue measure. In particular, it follows from the proof of Theorem 2.2 that, if $E \subset (0,\infty)^N$satisfies $s(H,E) > d$, then $B^H(E)$ has positive d-dimensional Lebesgue measure. It is a natural question to further ask when $B^H(E)$ has interior points. This question for Brownian motion was first considered by Kaufman (1975), and then extended by Pitt (1978) and Kahane (1985a, b) to fractional Brownian motion and by Khoshnevisan and Xiao (2004) to the Brownian sheet. Recently, Shieh and Xiao (2005) proved similar results under more general conditions for a large class of Gaussian random fields.

In the following, we prove that a condition similar to that in Shieh and Xiao (2005) is sufficient for $B^H(E)$ to have interior points almost surely. This theorem extends and improves the result of Khoshnevisan and Xiao (2004) mentioned above.

Theorem 2.8 *Let $B^H = \{B^H(t), t \in \mathbb{R}^N\}$ be an (N,d)-fractional Brownian sheet with index $H \in (0,1)^N$. If a Borel set $E \subset (0,\infty)^N$ carries a probability measure μ such that*

$$\sup_{t\in\mathbb{R}_+^N} \int_{\mathbb{R}_+^N} \frac{1}{\left(\sum_{j=1}^N |s_j - t_j|^{2H_j}\right)^{d/2}} \times \log_+^{(N+1)\gamma}\left(\frac{1}{\sum_{j=1}^N |s_j - t_j|^{2H_j}}\right) \mu(ds) \le K_{2.6.1} \tag{2.154}$$

for some finite constants $K_{2.6.1} > 0$ and $\gamma > N$, where $\log_+ x = \max\{1, \log x\}$, then $B^H(E)$ has interior points almost surely.

From Theorem 2.8 we derive the following corollaries.

Corollary 2.4 *If $E \subset (0,\infty)^N$ is a Borel set with $s(H,E) > d$, then $B^H(E)$ a.s. has interior points.*

Proof: It follows from the proof of Theorem 2.2 that, if $s(H,E) > d$, then there is a Borel probability measure μ on E satisfying (2.154). Hence the conclusion follows from Theorem 2.8.

Corollary 2.5 *Let $B^{\langle\alpha\rangle} = \{B^{\langle\alpha\rangle}(t), t \in \mathbb{R}^N\}$ be an (N,d)-fractional Brownian sheet with Hurst index $H = \langle\alpha\rangle$. If a Borel set $E \subset (0,\infty)^N$ carries a probability* *measure μ such that*

$$\sup_{t\in\mathbb{R}_+^N}\int_{\mathbb{R}_+^N}\frac{1}{|s-t|^{\alpha d}}\log_+^{(N+1)\gamma}\Big(\frac{1}{|s-t|}\Big)\,\mu(ds)\le K_{2.6.2} \tag{2.155}$$

for some finite constants $K_{2.6.2}>0$ and $\gamma>N$. Then $B^{\langle\alpha\rangle}(E)$ has interior points almost surely.

The existence of interior points in $B^H(E)$ is related to the regularity of the local times of B^H on E. In order to prove Theorem 2.8, we will need to use the following continuity lemma of Garsia (1972).

Lemma 2.13 *Assume that $p(u)$ and $\Psi(u)$ are two positive increasing functions on $[0,\infty)$, $p(u)\downarrow 0$ as $u\downarrow 0$, $\Psi(u)$ is convex and $\Psi(u)\uparrow\infty$ as $u\uparrow\infty$. Let D denote an open hypercube in $\mathbb{R}^d$. If the function $f(x):D\mapsto\mathbb{R}$ is measurable and*

$$A:=A(D,f)=\int_D\int_D\Psi\Big(\frac{|f(x)-f(y)|}{p(|x-y|/\sqrt{d})}\Big)dxdy<\infty, \tag{2.156}$$

then after modifying $f(x)$ on a set of Lebesgue measure 0, we have

$$|f(x)-f(y)|\le 8\int_0^{|x-y|}\Psi^{-1}\Big(\frac{A}{u^{2d}}\Big)\,dp(u)\qquad \text{for all } x,y\in D. \tag{2.157}$$

We take the function $p(u)$ in Garsia's lemma as follows: Let γ be the constant in (2.154) and define

$$p(u) = \begin{cases} 0, & \text{if } u = 0, \\ \log^{-\gamma}(e/u), & \text{if } 0 < u \le 1, \\ \gamma u - \gamma + 1, & \text{if } u > 1. \end{cases} \tag{2.158}$$

Clearly, the function $p(u)$ is strictly increasing on $[0, \infty)$ and $p(u) \downarrow 0$ as $u \downarrow 0$.

Proof of Theorem 2.8: First note that, since μ is a Borel probability measure on E, without loss of generality, we can assume that E is compact. Hence there are constants $0 < \varepsilon < T < \infty$ such that $E \subseteq [\varepsilon, T]^N$. Since $B^H(E)$ is a compact subset of $\mathbb{R}^d$, (1.13) implies that $\{x : l_\mu(x) > 0\}$ is a subset of $B^H(E)$. Hence, in order to prove our theorem, it is sufficient to prove that the local time $l_\mu(x)$ has a version which is continuous in x; see Pitt (1978, p.324) or Geman and Horowitz (1980, p.12). This will be proved by deriving moment estimates for the local time l_μ and by applying Garsia's continuity lemma.

Secondly, as in Khoshnevisan and Xiao (2004), we may and will assume that the Borel probability measure μ in (2.154) has the following property: For any constant $c > 0$ and $\ell = 1, \ldots, N$,

$$\mu\{t = (t_1, \ldots, t_N) \in E : \; t_\ell = c\} = 0. \tag{2.159}$$

Otherwise, we can replace B^H by an $(N-1, d)$-fractional Brownian sheet $\widetilde{B}^H$ and prove the desired conclusion for $\widetilde{B}^H(E_\ell)$, where E_ℓ is the set in (2.159) with positive μ-measure.

Consider the set $\widetilde{E}_n$ defined by

$$\left\{\bar{\mathbf{t}} = (t^1, \ldots, t^n) \in E^n : \; t_\ell^j = t_\ell^i \text{ for some } i \neq j \;\text{ and }\; 1 \leq \ell \leq N\right\}. \tag{2.160}$$

It follows from (2.159) and the Fubini–Tonelli theorem that $\mu^n(\widetilde{E}_n) = 0$.

The following lemma provides estimates on high moments of the local time, which is the key for finishing the proof of Theorem 2.8.

Lemma 2.14 *Let μ be a Borel probability measure on $E \subset [\varepsilon, T]^N$ satisfying (2.154) and (2.159) and let $p(u)$ be defined by (2.158). Then for every hypercube $D \subset \mathbb{R}^d$ there exists a finite constant $K_{2.6.3} > 0$, depending on N, d, γ, μ and D only, such that for all even integers $n \geq 2$,*

$$\mathbb{E}\int_D\int_D \left(\frac{l_\mu(x) - l_\mu(y)}{p(|x-y|/\sqrt{d})}\right)^n dxdy \leq K_{2.6.3}^n\,(n!)^N\,\log^{n(N+1)\gamma} n. \tag{2.161}$$

We now continue with the proof of Theorem 2.8 and defer the proof of Lemma 2.14 to the end of this section.

Let $\Psi(u) = u\exp(u^\theta)$, where $\theta \in \left(\frac{1}{\gamma}, \frac{1}{N}\right)$ is a constant. Then Ψ is increasing and convex on $(0, \infty)$. It follows from Jensen's inequality, the Fubini–Tonelli theorem and Lemma 2.14 that for all closed hypercubes $D \subset \mathbb{R}^d$ and all integers n with

$\theta + 1/n < 1$,

$$\begin{aligned}
&\mathbb{E}\int_D\int_D \left(\frac{|l_\mu(x) - l_\mu(y)|}{p(|x-y|/\sqrt{d})}\right)^{n\theta+1} dxdy \\
&\leq \left(m_d(D)\right)^{2-\theta-1/n} \left\{\mathbb{E}\int_D\int_D \left(\frac{|l_\mu(x) - l_\mu(y)|}{p(|x-y|/\sqrt{d})}\right)^{n} dxdy\right\}^{\theta+1/n} \\
&\leq K^n \,(n!)^{N(\theta+1/n)} \,(\log n)^{n(N+1)\gamma(\theta+1/n)} \\
&\leq K_{2,6,4}^n \,(n!)^{N\theta} \,\log^{n(N+1)\gamma\theta} n,
\end{aligned} \tag{2.162}$$

where $K_{2,6,4}$ is a finite constant depending on N, d, θ, D and $K_{2,6,3}$ only.

Expanding $\Psi(u)$ into a power series and applying the inequality (2.162), we derive

$$\begin{aligned}
&\mathbb{E}\int_D\int_D \Psi\left(\frac{|l_\mu(x) - l_\mu(y)|}{p(|x-y|/\sqrt{d})}\right) dxdy \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}\,\mathbb{E}\int_D\int_D \left(\frac{|l_\mu(x) - l_\mu(y)|}{p(|x-y|/\sqrt{d})}\right)^{n\theta+1} dxdy \\
&\leq K_{2,6,5} < \infty,
\end{aligned} \tag{2.163}$$

the last inequality follows from the fact that $N\theta < 1$. Hence Garsia's lemma implies that there are positive and finite random variables A_1 and A_2 such that for almost all x, $y \in D$ with $|x-y| \leq e^{-1}$,

$$\begin{aligned}
|l_\mu(x) - l_\mu(y)| &\leq \int_0^{|x-y|} \Psi^{-1}\left(\frac{A_1}{u^{2d}}\right) dp(u) \\
&\leq A_2\Big[\log\left(1/|x-y|\right)\Big]^{-(\gamma-1/\theta)}.
\end{aligned}$$

Note that, by our choice of θ, we have $\theta > 1/\gamma$ and hence B^H has almost surely a local time $l_\mu(x)$ on E that is continuous for all $x \in D$. Finally, by taking a sequence of closed hypercubes $\{D_n,\ n \geq 1\}$ such that $\mathbb{R}^d = \cup_{n=1}^{\infty} D_n$, we have proved that almost surely $l_\mu(x)$ is continuous for all $x \in \mathbb{R}^d$. This completes the proof of Theorem 2.8.

It remains to prove Lemma 2.14. Our proof relies on the sectorial local nondeterminism of B^H and on an argument which improves those in Khoshnevisan and Xiao (2004) and Shieh and Xiao (2005).

We will need several lemmas. Lemma 2.15 is essentially due to Cuzick and DuPreez (1982), where the extra condition on g is dropped in Khoshnevisan and Xiao (2004). Lemma 2.16 is a slight modification of Lemma 4 in Cuzick and DuPreez (1982).

Lemma 2.15 *Let $Z_k\ (k = 1, \ldots, n)$ be linearly independent centered Gaussian variables. If $g : \mathbb{R} \longrightarrow \mathbb{R}_+$ is a Borel measurable function, then*

$$\begin{aligned}\int_{\mathbb{R}^n} g(v_1) \exp\left(-\frac{1}{2}\mathrm{Var}\left(\langle v, Z\rangle\right)\right) dv \\ = \frac{(2\pi)^{(n-1)/2}}{Q^{1/2}} \int_{-\infty}^{\infty} g(z/\sigma_1) \exp\left(-\frac{1}{2}z^2\right) dz,\end{aligned} \tag{2.164}$$

where $\sigma_1^2 = \mathrm{Var}(Z_1 \,|\, Z_2, \ldots, Z_n)$ and $Q = \det \mathrm{Cov}(Z_1, \ldots, Z_n)$ is the determinant of the covariance matrix of $Z_1, \ldots, Z_n$.

Lemma 2.16 *If $\alpha \geq e^2/2$, then*

$$\int_1^{\infty} \log^{\alpha} x \exp\left(-\frac{x^2}{2}\right) dx \leq \sqrt{\pi} \log^{\alpha} \alpha. \tag{2.165}$$

Consider the non-decreasing function $\Lambda(u) = 2\min\{1, u\}$ on $[0, \infty)$. Later we will make use of the elementary inequality

$$|e^{iu} - 1| \le \Lambda(|u|), \qquad \forall u \in \mathbb{R}. \tag{2.166}$$

Lemma 2.17 *Assume $h(y)$ is any positive and non-decreasing function on $[0, \infty)$ such that $h(0) = 0$, $y^n/h^n(y)$ is non-decreasing on $[0, 1]$, and $\int_1^\infty h^{-2}(y)dy < \infty$. Then there exists a constant $K_{2.6.6}$ such that for all integers $n \ge 1$ and $v \in (0, \infty)$,*

$$\int_0^\infty \frac{\Lambda^n(vy)}{h^n(y)}\, dy \le K_{2.6.6}^n\, h_+^{-n}\Big(\frac{1}{v}\Big), \tag{2.167}$$

where $h_+(y) = \min\{1, h(y)\}$ so that $h_+^{-n}(y) = \max\{1, h^{-n}(y)\}$.

Proof: The proof is the same as that of Lemma 3 in Cuzick and DuPreez (1982).

The following result is about the function $p(u)$ defined by (2.158).

Lemma 2.18 *Let $p(u)$ be defined as in (2.158). Then for all $\sigma > 0$ and integers $n \ge 1$,*

$$\int_0^\infty p_+^{-n}\Big(\frac{\sigma}{v}\Big) \exp\Big(-\frac{v^2}{2}\Big) dv \le K_{2.6.7}^n \left[\log^{n\gamma} n + \log_+^{n\gamma}\Big(\frac{e}{\sigma}\Big)\right]. \tag{2.168}$$

Proof: Since

$$p_+^{-n}(x) = \begin{cases} \log^{n\gamma}\left(\frac{e}{x}\right), & \text{if } 0 < x < 1, \\ 1 & \text{if } x \ge 1 \end{cases}$$

and $\log_+^\alpha(xy) \le 2^\alpha(\log_+^\alpha x + \log_+^\alpha y)$ for all $\alpha \ge 0$, we deduce that the integral in (2.168) is at most

$$\begin{aligned}\int_{\sigma/v\ge 1} \exp\Big(-\frac{v^2}{2}\Big)\,dv + 2^{n\gamma}\int_{\sigma/v<1} \log_+^{n\gamma}(v)\exp\Big(-\frac{v^2}{2}\Big)\,dv \\ + 2^{n\gamma}\int_{\sigma/v<1} \log_+^{n\gamma}\Big(\frac{e}{\sigma}\Big)\exp\Big(-\frac{v^2}{2}\Big)\,dv.\end{aligned} \tag{2.169}$$

It follows from (2.169) and Lemma 2.16 that

$$\begin{aligned}\int_0^\infty p_+^{-n}\Big(\frac{\sigma}{v}\Big)\exp\Big(-\frac{v^2}{2}\Big)dv &\le c^n\Big[\log^{n\gamma}(n\gamma) + \log_+^{n\gamma}\Big(\frac{e}{\sigma}\Big)\Big] \\ &\le K_{2.6.7}^n\Big[\log^{n\gamma} n + \log_+^{n\gamma}\Big(\frac{e}{\sigma}\Big)\Big].\end{aligned} \tag{2.170}$$

This completes the proof of Lemma 2.18.

Now we proceed to prove Lemma 2.14.

Proof of Lemma 2.14: By (25.7) in Geman and Horowitz (1980), we have that for every $x, y \in \mathbb{R}^d$, and all even integers $n \ge 2$,

$$\begin{aligned}&\mathbb{E}\left[(l_\mu(x) - l_\mu(y))^n\right] \\ &= (2\pi)^{-nd}\int_{E^n}\int_{\mathbb{R}^{nd}}\prod_{k=1}^n\left[\exp(-i\langle u^k, x\rangle) - \exp(-i\langle u^k, y\rangle)\right] \\ &\qquad\times\exp\Big[-\frac{1}{2}\mathrm{Var}\Big(\sum_{k=1}^n\langle u^k, B^H(t^k)\rangle\Big)\Big]\,d\mathbf{u}\,\mu^n(d\mathbf{t}).\end{aligned} \tag{2.171}$$

In the above, $\mathbf{u} = (u^1, \ldots, u^n)$, $u^k \in \mathbb{R}^d$ for each $k = 1, \ldots, n$ and we will write it coordinate-wise as $u^k = (u_1^k, \ldots, u_d^k)$.

Note that for $u^1, \ldots, u^n,\ y \in \mathbb{R}^d$, the triangle inequality implies

$$\begin{aligned}
&\prod_{k=1}^{n} \Big| \exp(-i\langle u^k, y\rangle) - 1 \Big| \\
&\le \prod_{k=1}^{n} \Big| \sum_{j=1}^{d} \Big[\exp\Big(-i \sum_{\ell=0}^{j} u_\ell^k\, y_\ell \Big) - \exp\Big(-i \sum_{\ell=0}^{j-1} u_\ell^k\, y_\ell \Big) \Big] \Big| \\
&\le \prod_{k=1}^{n} \Big[\sum_{j=1}^{d} \big| \exp(-iu_j^k\, y_j) - 1 \big| \Big] \\
&= \sum{}' \prod_{k=1}^{n} \big| \exp(-iu_{j_k}^k\, y_{j_k}) - 1 \big|,
\end{aligned} \tag{2.172}$$

where $y_0 = u_0^j = 0$ in the first inequality and the last summation $\sum'$ is taken over all sequences $(j_1, \cdots, j_n) \in \{1, \ldots, d\}^n$.

It follows from (2.171), (2.172), (2.166) and the Fubini–Tonelli theorem that for any fixed hypercube $D \subset \mathbb{R}^d$ and any even integer $n \ge 2$, we have

$$
\begin{aligned}
&\mathbb{E}\int_D\int_D\left(\frac{l_\mu(x)-l_\mu(y)}{p(|x-y|/\sqrt{d})}\right)^n dxdy\\
&\le \sum{}' \int_D\int_D\int_{E^n}\int_{\mathbb{R}^{nd}}\prod_{k=1}^n \frac{|\exp(iu_{j_k}^k(y_{j_k}-x_{j_k}))-1|}{p(|y-x|/\sqrt{d})}\\
&\qquad\times \exp\Big[-\frac{1}{2}\mathrm{Var}\Big(\sum_{k=1}^n\langle u^k,\ B^H(t^k)\rangle\Big)\Big]\,d\mathbf{u}\ \mu^n(d\mathbf{t})\,dx\,dy\\
&\le m_d(D)\sum{}' \int_{D\ominus D}\int_{E^n}\int_{\mathbb{R}^{nd}}\prod_{k=1}^n \frac{\Lambda(|u_{j_k}^k y_{j_k}|)}{p(|y|/\sqrt{d})}\\
&\qquad\times \exp\Big[-\frac{1}{2}\mathrm{Var}\Big(\sum_{k=1}^n\langle u^k,\ B^H(t^k)\rangle\Big)\Big]\,d\mathbf{u}\ \mu^n(d\mathbf{t})\,dy.
\end{aligned}
\tag{2.173}
$$

In the above, we have made a change of variables and $D\ominus D=\{x-y : x,y\in D\}$. By our assumptions on μ, we see that the integral in (2.173) with respect to μ^n can be taken over the set $E^n\backslash\widetilde{E}_n$, where $\widetilde{E}_n$ is defined by (2.160).

Now we fix $\mathbf{t}\in E^n\backslash\widetilde{E}_n$, a sequence $\mathbf{j}=(j_1,\ldots,j_n)\in\{1,\ldots,d\}^n$ and define $\mathcal{M}_n(\mathbf{t})\equiv\mathcal{M}_n(\mathbf{j},\mathbf{t})$ by

$$
\begin{aligned}
\mathcal{M}_n(\mathbf{t}) = &\int_{D\ominus D}\int_{\mathbb{R}^{nd}}\prod_{k=1}^n \frac{\Lambda(|u_{j_k}^k y_{j_k}|)}{p(|y|/\sqrt{d})}\\
&\times \exp\Big[-\frac{1}{2}\mathrm{Var}\Big(\sum_{k=1}^n\langle u^k,\ B^H(t^k)\rangle\Big)\Big]\,d\mathbf{u}\,dy.
\end{aligned}
\tag{2.174}
$$

Then the last integral in (2.173) corresponding to the sequence $\mathbf{j}=$

$(j_1, \ldots, j_n)$ can be written as

$$\mathcal{N}_{\mathbf{j}} \equiv \int_{E^n \setminus \widetilde{E}_n} \mathcal{M}_n(\mathbf{t})\, \mu^n(d\mathbf{t}). \tag{2.175}$$

We will estimate the above integral by integrating in the order $\mu(dt^n)$, $\mu(dt^{n-1}), \ldots, \mu(dt^1)$. For this purpose, we need to derive an upper bound for $\mathcal{M}_n(\mathbf{t})$. Observe that for any positive numbers $\beta_1, \ldots, \beta_n$ satisfying $\sum_{k=1}^n \beta_k = n$, we can write

$$\begin{aligned} \mathcal{M}_n(\mathbf{t}) = \int_{D \cdots D} \int_{\mathbb{R}^{nd}} \prod_{k=1}^{n} \frac{\Lambda\big(|u^k_{j_k} y_{j_k}|\big)}{p^{\beta_k}(|y|/\sqrt{d})} \\ \times \exp\left[-\frac{1}{2}\mathrm{Var}\Big(\sum_{k=1}^{n} \langle u^k,\ B^H(t^k)\rangle\Big)\right] d\mathbf{u}\, dy. \end{aligned} \tag{2.176}$$

Later it will be clear that the flexibility in choosing β_k in (2.176) is essential to our proof. More precisely, by choosing the constants $\beta_k\,(1 \le k \le n)$ appropriately, we minimize the effect of "bad points" [see (2.188) below].

For any n points $t^1, \ldots, t^n \in E^n \setminus \widetilde{E}_n$, Lemma 2.1 implies that the Gaussian random variables $B_j^H(t^k)$ $(j = 1, \ldots, d,\ k = 1, \ldots, n)$ are linearly independent. By applying the generalized Hölder's inequality, Lemma 2.15 and a change of variables, we see that $\mathcal{M}_n(\mathbf{t})$ is bounded by

$$\prod_{k=1}^{n}\left\{\int_{D\ominus D}\int_{\mathbb{R}^{nd}} \frac{\Lambda^n(|u_{j_k}^k y_{j_k}|)}{p^{n\beta_k}(|y|/\sqrt{d})} \times \exp\left[-\frac{1}{2}\mathrm{Var}\left(\sum_{k=1}^{n}\sum_{\ell=1}^{d} u_\ell^k B_\ell^H(t^k)\right)\right] d\mathbf{u}\, dy\right\}^{1/n}$$

$$= \frac{K_{2.6.8}^{nd}}{[\det\mathrm{Cov}(B_0^H(t^1), \ldots, B_0^H(t^n))]^{d/2}} \times \prod_{k=1}^{n}\left\{\int_{D\ominus D}\int_{\mathbb{R}} \frac{\Lambda^n(|u_{j_k}^k y_{j_k}|/\sigma_k)}{p^{n\beta_k}(|y|/\sqrt{d})} \exp\left(-\frac{(u_{j_k}^k)^2}{2}\right) du_{j_k}^k dy\right\}^{1/n}, \tag{2.177}$$

where, for every $1 \le k \le n$, $\sigma_k^2 \equiv \sigma_k^2(\mathbf{t})$ is the conditional variance of $B_{j_k}^H(t^k)$ given $B_\ell^H(t^m)$ ($\ell \ne j_k$ and $1 \le m \le n$, or $\ell = j_k$ and $m \ne k$).

Denote the n integrals in the last product of (2.177) by $\mathfrak{J}_1, \ldots, \mathfrak{J}_n$, respectively. In order to estimate them, we will apply the sectorial local nondeterminism of B_0^H. Since $B_1^H, \ldots, B_d^H$ are independent copies of B_0^H, we have

$$\sigma_k^2(\mathbf{t}) = \mathrm{Var}\left(B_0^H(t^k)\,\Big|\,\{B_0^H(t^m)\}_{m\ne k}\right). \tag{2.178}$$

It follows from Theorem 2.1 that for every $1 \le k \le n$,

$$\sigma_k^2(\mathbf{t}) \ge K_{2.2.1}\sum_{\ell=1}^{N}\min_{m\ne k}\left|t_\ell^m - t_\ell^k\right|^{2H_\ell}. \tag{2.179}$$

In order to estimate the sum in (2.179) as a function of t^n, we introduce N permutations $\Gamma_1, \ldots, \Gamma_N$ of $\{1, \ldots, n-1\}$ such that for every $\ell = 1, \ldots, N$,

$$t_\ell^{\Gamma_\ell(1)} < t_\ell^{\Gamma_\ell(2)} < \ldots < t_\ell^{\Gamma_\ell(n-1)}. \tag{2.180}$$

This is possible since t_ℓ^k $(1 \leq k \leq n-1,\ 1 \leq \ell \leq N)$ are distinct. For convenience, we denote $t_\ell^{\Gamma_\ell(0)} = \varepsilon$ and $t_\ell^{\Gamma_\ell(n)} = T$ for all $1 \leq \ell \leq N$.

For every sequence $(i_1, \ldots, i_N) \in \{1, \ldots, n-1\}^N$, let $\tau_{i_1, \cdots, i_N} = (t_1^{\Gamma_1(i_1)}, \ldots, t_N^{\Gamma_N(i_N)})$ be the "center" of the rectangle

$$I_{i_1, \cdots, i_N} = \prod_{\ell=1}^N \left[t_\ell^{\Gamma_\ell(i_\ell)} - \frac{1}{2}\big(t_\ell^{\Gamma_\ell(i_\ell)} - t_\ell^{\Gamma_\ell(i_\ell - 1)}\big),\ t_\ell^{\Gamma_\ell(i_\ell)} + \frac{1}{2}\big(t_\ell^{\Gamma_\ell(i_\ell+1)} - t_\ell^{\Gamma_\ell(i_\ell)}\big)\right) \tag{2.181}$$

with the convention that the left-end point of the interval is ε whenever $i_\ell = 1$; and the interval is closed and its right-end is T whenever $i_\ell = n-1$. Thus the rectangles $\{I_{i_1, \cdots, i_N}\}$ form a partition of $[\varepsilon, T]^N$.

For every $t^n \in E$, let $I_{i_1, \cdots, i_N}$ be the unique rectangle containing t^n. Then (2.179) yields the following lower bound for $\sigma_n^2(\mathbf{t})$:

$$\sigma_n^2(\mathbf{t}) \geq K_{2,2,1} \sum_{\ell=1}^N \left| t_\ell^n - t_\ell^{\Gamma_\ell(i_\ell)} \right|^{2H_\ell}. \tag{2.182}$$

For every $k = 1, \ldots, n-1$, we say that $I_{i_1, \cdots, i_N}$ *cannot see* t^k *from* *direction* ℓ if

$$t_\ell^k \notin \left[t_\ell^{\Gamma_\ell(i_\ell)} - \frac{1}{2}\left(t_\ell^{\Gamma_\ell(i_\ell)} - t_\ell^{\Gamma_\ell(i_\ell-1)}\right),\ t_\ell^{\Gamma_\ell(i_\ell)} + \frac{1}{2}\left(t_\ell^{\Gamma_\ell(i_\ell+1)} - t_\ell^{\Gamma_\ell(i_\ell)}\right)\right]. \tag{2.183}$$

We emphasize that if $I_{i_1,\cdots,i_N}$ cannot see t^k from all N directions, then

$$\left|t_\ell^k - t_\ell^n\right| \geq \frac{1}{2} \min_{m \neq k,n} \left|t_\ell^k - t_\ell^m\right| \qquad \text{for all } 1 \leq \ell \leq N. \tag{2.184}$$

Thus t^n does not contribute to the sum in (2.179). More precisely, the latter means that

$$\sigma_k^2(\mathbf{t}) \geq K_{2,6,9} \sum_{\ell=1}^{N} \min_{m \neq k,n} \left|t_\ell^k - t_\ell^m\right|^{2H_\ell}. \tag{2.185}$$

The right hand side of (2.185) only depends on $t^1, \ldots, t^{n-1}$, which will be denoted by $\widetilde{\sigma}_k^2(\mathbf{t})$. Because of this, t^k is called a "good" point for $I_{i_1,\cdots,i_N}$ when (2.183) holds for every $\ell = 1, \ldots, N$.

Let $1 \leq k \leq n-1$. If $I_{i_1,\cdots,i_N}$ sees the point t^k from a direction and $t^k \neq \tau_{i_1,\cdots,i_N}$, then it is impossible to control $\sigma_k^2(\mathbf{t})$ from below as in (2.182) or (2.185). We say that t^k is a "bad" point for $I_{i_1,\cdots,i_N}$ [Note that by definition $n \notin \Theta^n{}_{i_1,\cdots,i_N}$]. It is important to note that, because of (2.180), the rectangle $I_{i_1,\cdots,i_N}$ can only have at most N bad points t^k ($1 \leq k \leq n-1$), i.e., at most one in each direction. We denote the set of bad points for $I_{i_1,\cdots,i_N}$ by

$$\Theta^n{}_{i_1,\cdots,i_N} = \{1 \le k \le n-1 : \ t^k \ \text{ is a bad point for } \ I_{i_1,\cdots,i_N}\}$$

and denote its cardinality by $\#(\Theta^n{}_{i_1,\cdots,i_N})$. Then $\#(\Theta^n{}_{i_1,\cdots,i_N}) \le N$.

Now we choose the constants $\beta_1, \ldots, \beta_n$ [they depend on the sequence $(i_1, \ldots, i_N)$] as follows: $\beta_k = 0$ if t^k is a bad point for $I_{i_1,\cdots,i_N}$; $\beta_k = 1$ if t^k is a good point for $I_{i_1,\cdots,i_N}$ and

$$\beta_n = 1 + \#(\Theta^n{}_{i_1,\cdots,i_N}).$$

Clearly, $\beta_n \le N+1$.

By Lemma 2.17 and Lemma 2.18, we have

$$\begin{aligned}
\mathfrak{J}_n &= \int_{D\ominus D}\int_{\mathbb{R}} \frac{\Lambda^n\big(|u^n_{j_n} y_{j_n}|/\sigma_n(\mathbf{t})\big)}{p^{n\beta_n}(|y|/\sqrt{d})} \exp\Big(-\frac{(u^n_{j_n})^2}{2}\Big)\, du^n_{j_n} dy \\
&\le K \int_{\mathbb{R}} \exp\Big(-\frac{v^2}{2}\Big)\, dv \int_0^\infty \frac{\Lambda^n(v y_{j_n}/\sigma_n(\mathbf{t}))}{p^{n\beta_n}(y_{j_n}/\sqrt{d})}\, dy_{j_n} \\
&\le K^n_{2,6,10} \int_{\mathbb{R}} p_+^{-n\beta_n}\Big(\frac{\sigma_n(\mathbf{t})}{v}\Big) \exp\Big(-\frac{v^2}{2}\Big)\, dv \\
&\le K^n_{2,6,11}\Big[\log^{n(N+1)\gamma} n + \log_+^{n(N+1)\gamma}\Big(\frac{e}{\sigma_n(\mathbf{t})}\Big)\Big].
\end{aligned} \tag{2.186}$$

In the above, we have also used the fact that $p(|y|/\sqrt{d}) \ge p(|y_j|/\sqrt{d})$ for all $j = 1, \ldots, d$.

If t^k is a good point for $I_{i_1,\cdots,i_N}$, then by the monotonicity of the function Λ we have

$$\begin{aligned}\mathfrak{J}_k &= \int_{D\ni D}\int_{\mathbb{R}} \frac{\Lambda^n\big(|u_{jk}^k y_{jk}|/\sigma_k(\mathbf{t})\big)}{p^{n\beta_k}(|y|/\sqrt{d})} \exp\Big(-\frac{(u_{jk}^k)^2}{2}\Big)\, du_{jk}^k\, dy \\ &\le \int_{D\ni D}\int_{\mathbb{R}} \frac{\Lambda^n(u_{jk}^k y_{jk}/\widetilde{\sigma}_k(\mathbf{t}))}{p^n(|y|/\sqrt{d})} \exp\Big(-\frac{(u_{jk}^k)^2}{2}\Big)\, du_{jk}^k\, dy.\end{aligned} \tag{2.187}$$

When t^k is a bad points for $I_{i_1,\cdots,i_N}$, we use the inequality $\Lambda(u) \le 2$ to obtain

$$\mathfrak{J}_k \le 2^n \int_{D\ni D}\int_{\mathbb{R}} \exp\Big(-\frac{(u_{jk}^k)^2}{2}\Big)\, du_{jk}^k\, dy \le K_{2,6,12}^n. \tag{2.188}$$

Since there are at most N bad points for $I_{i_1,\cdots,i_N}$, their total contribution to $\mathcal{M}_n(\mathbf{t})$ is bounded by a constant $K_{2,6,13}$, which depends on D, d and N only.

Combining (2.177), (2.186), (2.187) and (2.188), we derive that

$$
\begin{aligned}
&\mathcal{M}_n(\mathbf{t}) \\
&\leq \frac{K_{2.6.14}^n}{[\det\mathrm{Cov}(B_0^H(t^1), \ldots, B_0^H(t^n))]^{d/2}} \left[\log^{(N+1)\gamma} n + \log_+^{(N+1)\gamma}\left(\frac{e}{\sigma_n(\mathbf{t})}\right)\right] \\
&\times \prod_{k \notin \Theta_{i_1,\ldots,i_N}^n} \left\{ \int_D \cdots \int_D \int_{\mathbb{R}} \frac{\Lambda^n(u_{jk}^k y_{jk}/\widetilde{\sigma}_k(\mathbf{t}))}{p^n(|y|/\sqrt{d})} \exp\left(-\frac{(u_{jk}^k)^2}{2}\right) du_{jk}^k \, dy \right\}^{1/n} \\
&= \frac{K_{2.6.15}^n}{[\det\mathrm{Cov}(B_0^H(t^1), \ldots, B_0^H(t^{n-1}))]^{d/2}} \\
&\times \prod_{k \notin \Theta_{i_1,\ldots,i_N}^n} \left\{ \int_D \cdots \int_D \int_{\mathbb{R}} \frac{\Lambda^n(u_{jk}^k y_{jk}/\widetilde{\sigma}_k(\mathbf{t}))}{p^n(|y|/\sqrt{d})} \exp\left(-\frac{(u_{jk}^k)^2}{2}\right) du_{jk}^k \, dy \right\}^{1/n} \\
&\qquad \times \frac{1}{\sigma_n(\mathbf{t})^d} \left[\log^{(N+1)\gamma} n + \log_+^{(N+1)\gamma}\left(\frac{e}{\sigma_n(\mathbf{t})}\right)\right].
\end{aligned}
\tag{2.189}
$$

Note that Condition (2.154) implies

$$
\begin{aligned}
&\int_{I_{i_1,\ldots,i_N}} \frac{1}{\sigma_n(\mathbf{t})^d} \left[\log^{(N+1)\gamma} n + \log_+^{(N+1)\gamma}\left(\frac{e}{\sigma_n(\mathbf{t})}\right)\right] \mu(dt^n) \\
&\quad \leq K_{2.6.16} \int_{I_{i_1,\ldots,i_N}} \frac{1}{\left(\sum_{\ell=1}^N \left|t_\ell^n - t_\ell^{\Gamma_\ell(i_\ell)}\right|^{2H_\ell}\right)^{d/2}} \\
&\qquad \times \left[\log^{(N+1)\gamma} n + \log^{(N+1)\gamma}\left(\frac{1}{\sum_{\ell=1}^N \left|t_\ell^n - t_\ell^{\Gamma_\ell(i_\ell)}\right|^{2H_\ell}}\right)\right] \mu(dt^n) \\
&\quad \leq K_{2.6.17} \log^{(N+1)\gamma} n.
\end{aligned}
\tag{2.190}
$$

Integrating $\mathcal{M}_n(\mathbf{t})$ as a function of t_n with respect to μ on $I_{i_1,\ldots,i_N}$ and using (2.189) and (2.190), we obtain

$$\int_{I_{i_1,\dots,i_N}} \mathcal{M}_n(\mathbf{t})\,\mu(dt^n) \le \frac{K_{2.6.18}^n \log^{(N+1)\gamma} n}{[\det\mathrm{Cov}(B_0^H(t^1),\dots,B_0^H(t^{n-1}))]^{d/2}}$$
$$\times \prod_{k\notin\Theta^n_{i_1,\dots,i_N}} \left\{ \int_{D\ominus D}\int_{\mathbb{R}} \frac{\Lambda^n(u^k_{jk}y_{jk}/\widetilde{\sigma}_k(\mathbf{t}))}{p^n(|y|/\sqrt{d})} \exp\left(-\frac{(u^k_{jk})^2}{2}\right) du^k_{jk}\,dy \right\}^{1/n}. \tag{2.191}$$

It is important that the right hand side of (2.191) depends on $t^1,\dots,t^{n-1}$ only and is similar to (2.177).

Summing (2.191) over all the sequences $(i_1,\dots,i_N)\in\{1,\dots,n-1\}^N$, we derive that the integral $\mathcal{N}_{\mathbf{j}}$ in (2.175) is bounded by

$$K_{2.6.18}^n \log^{(N+1)\gamma} n \sum_{i_1,\dots,i_N} \int_{E^{n-1}} \frac{\mu(dt^{n-1})\dots\mu(dt^1)}{[\det\mathrm{Cov}(B_0^H(t^1),\dots,B_0^H(t^{n-1}))]^{d/2}}$$
$$\times \prod_{k\notin\Theta^n_{i_1,\dots,i_N}} \left\{ \int_{D\ominus D}\int_{\mathbb{R}} \frac{\Lambda^n(u^k_{jk}y_{jk}/\widetilde{\sigma}_k(\mathbf{t}))}{p^n(|y|/\sqrt{d})} \exp\left(-\frac{(u^k_{jk})^2}{2}\right) du^k_{jk}\,dy \right\}^{1/n}. \tag{2.192}$$

Note that, for different sequences $(i_1,\dots,i_N)$, the index sets $\Theta^n_{i_1,\dots,i_N}$ may be the same. We say that a set $\Theta^n\subseteq\{1,\dots,n-1\}$ is *admissible* if it is the set of bad points for some $I_{i_1,\dots,i_N}$. It can be seen that every admissible set Θ^n has the following properties:

(i) $\#(\Theta^n)\le N$ [recall that there are at most N bad points for each $I_{i_1,\dots,i_N}$];

(ii) Denote by $\chi(\Theta^n)$ the number of sequences $(i_1, \ldots, i_N)$ such that $\Theta^n_{i_1,\ldots,i_N} = \Theta^n$. If $\#(\Theta^n) = p$, then $\chi(\Theta^n) \le c n^{N-p}$.

It follows from (i), (ii) and an elementary combinatorics argument that

$$\sum_{\Theta^n} \chi(\Theta^n) = \sum_{p=1}^{N} \sum_{\#(\Theta^n)=p} \chi(\Theta^n) \le K_{2.6.19} n^N, \tag{2.193}$$

where the first summation is taken over all admissible sets $\Theta^n \subseteq \{1, \ldots, n-1\}$.

By regrouping $\Theta^n_{i_1,\ldots,i_N}$ in (2.192), we can rewrite

$$\begin{aligned}\mathcal{N}_{\mathrm{j}} \le{}& K^n_{2.6.20} \log^{(N+1)\gamma} n \\ &\times \sum_{\Theta^n} \chi(\Theta^n) \int_{E^{n-1}} \frac{\mu(dt^{n-1}) \ldots \mu(dt^1)}{[\det\mathrm{Cov}(B^H_0(t^1), \ldots, B^H_0(t^{n-1}))]^{d/2}} \\ &\times \prod_{k \notin \Theta^n} \left\{ \int_{D \ominus D} \int_{\mathbb{R}} \frac{\Lambda^n(u^k_{jk} y_{jk}/\widetilde{\sigma}_k(\mathbf{t}))}{p^n(|y|/\sqrt{d})} \exp\left(- \frac{(u^k_{jk})^2}{2} \right) du^k_{jk}\, dy \right\}^{1/n},\end{aligned} \tag{2.194}$$

where, as in (2.193), the summation is taken over all admissible sets $\Theta^n \subseteq \{1, \ldots, n-1\}$.

We now carry out the procedure iteratively. In order to simplify the computation, we will make some further reductions:

(iii) Increasing the number of elements in Θ^n changes only the integrals in (2.194) by a constant factor [recall (2.188) and the fact that we have used $\#(\Theta^n) \le N$ in deriving the first inequality in (2.189)].

Hence we may just consider the admissible sets Θ^n with $\#(\Theta^n) = N$ and

(iv) $\det \operatorname{Cov}\big(B_1(t^1), \ldots, B_1(t^{n-1})\big)$ is symmetric in $t^1, \ldots, t^{n-1}$. Hence we can further assume $\Theta^n = \{1, \ldots, N\}$.

Based on the above observations we can repeat the preceding argument and integrate $\mu(dt^{n-1})$ [we define $n - N - 1$ constants β_k, $k \in \{N+1, \ldots, n-1\}$ accordingly] and then, in the same way, continue to integrate $\mu(dt^{n-2}), \ldots, \mu(dt^1)$ respectively. We obtain

$$\begin{aligned} \mathcal{N}_{\mathbf{j}} &\leq K_{2.6.21}^n \, \log^{n(N+1)\gamma} n \sum_{\Theta^n} \cdots \sum_{\Theta^1} \chi(\Theta^n) \cdots \chi(\Theta^1) \\ &\leq K_{2.6.22}^n \, (n!)^N \, \log^{n(N+1)\gamma} n, \end{aligned} \tag{2.195}$$

where last inequality follows from (2.193) and, moreover, the positive constant $K_{2.6.22}$ is independent of $\mathbf{j}$.

By combining (2.173), (2.174), (2.175) and (2.195) we derive that

$$\mathbb{E} \int_D \int_D \left(\frac{l_\mu(x) - l_\mu(y)}{p(|x-y|/\sqrt{d})} \right)^n dx dy \leq K_{2.6.23}^n \, (n!)^N \, \log^{n(N+1)\gamma} n. \tag{2.196}$$

This finishes the proof of Lemma 2.14.

CHAPTER 3

Fractal Properties of the Random String Processes

3.1 Introduction

Consider the following model of a random string introduced by Funaki (1983):

$$\frac{\partial u_t(x)}{\partial t} = \frac{\partial^2 u_t(x)}{\partial x^2} + \dot{W}, \tag{3.1}$$

where $\dot{W}(x,t)$ is a space-time white noise in $\mathbb{R}^d$, which is assumed to be adapted with respect to a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, where $\mathcal{F}$ is complete and the filtration $\{\mathcal{F}_t, t \geq 0\}$ is right continuous. The components $\dot{W}_1(x,t), \ldots, \dot{W}_d(x,t)$ of $\dot{W}(x,t)$ are independent space-time white noises, which are generalized Gaussian processes with covariance given by

$$\mathbb{E}\big[\dot{W}_j(x,t)\dot{W}_j(y,s)\big] = \delta(x-y)\delta(t-s), \quad (j = 1, \ldots, d).$$

That is, for every $1 \leq j \leq d$, $W_j(f)$ is a random field indexed by functions $f \in L^2([0,\infty) \times \mathbb{R})$ and, for all $f, g \in L^2([0,\infty) \times \mathbb{R})$, we have

$$\mathbb{E}\big[W_j(f)W_j(g)\big] = \int_0^\infty \int_{\mathbb{R}} f(t,x)g(t,x)\,dxdt.$$

Hence $W_j(f)$ can be represented as

$$W_j(f) = \int_0^\infty \int_{\mathbb{R}} f(t,x)\,W_j(dx\,dt).$$

Note that $W(f)$ is $\mathcal{F}_t$-measurable whenever f is supported on $[0,t] \times \mathbb{R}$.

Recall from Mueller and Tribe (2002) that a solution of (3.1) is defined as an $\mathcal{F}_t$-adapted, continuous random field $\{u_t(x) : t \geq 0,\ x \in \mathbb{R}\}$ with values in $\mathbb{R}^d$ satisfying the following properties:

(i) $u_0(\cdot) \in \mathcal{E}_{\exp}$ almost surely and is adapted to $\mathcal{F}_0$, where $\mathcal{E}_{\exp} = \cup_{\lambda>0}\mathcal{E}_\lambda$ and

$$\mathcal{E}_\lambda = \Big\{ f \in C(\mathbb{R}, \mathbb{R}^d) : \ |f(x)|\,e^{-\lambda|x|} \to 0 \ \text{ as } |x| \to \infty \Big\};$$

(ii) For every $t > 0$, there exists $\lambda > 0$ such that $u_s(\cdot) \in \mathcal{E}_\lambda$ for all $s \leq t$, almost surely;

(iii) For every $t > 0$ and $x \in \mathbb{R}$, the following Green's function representation holds

$$u_t(x) = \int_{\mathbb{R}} G_t(x-y)u_0(y)dy + \int_0^t G_{t-r}(x-y)\, W(dy\, dr), \quad (3.2)$$

where $G_t(x) = \frac{1}{\sqrt{4\pi t}} \exp(-\frac{x^2}{4t})$ is the fundamental solution of the heat equation.

We call each solution $\{u_t(x) : t \geq 0, x \in \mathbb{R}\}$ of (3.1) a random string process with values in $\mathbb{R}^d$, or simply a random string as in Mueller and Tribe (2002). Note that, whenever the initial conditions u_0 are deterministic, or are Gaussian fields independent of $\mathcal{F}_0$, the random string processes are Gaussian. We refer to Mueller and Tribe (2002) and Funaki (1983) for information on stochastic partial differential equations (SPDEs) related to the random string processes.

Funaki (1983) investigated various properties of the solutions of semi-linear type SPDEs which are more general than (3.1). In particular, his results [cf. Lemma 3.3 in Funaki (1983)] imply that every solution $\{u_t(x) : t \geq 0, x \in \mathbb{R}\}$ of (3.1) is Hölder continuous of any order less than $\frac{1}{2}$ in space and $\frac{1}{4}$ in time. This anisotropic property of the process $\{u_t(x) : t \geq 0, x \in \mathbb{R}\}$ makes it a very interesting object to study. Recently Mueller and Tribe (2002) have found necessary and sufficient conditions [in terms of the dimension d] for a random string in $\mathbb{R}^d$ to hit points or to have double points of various types.

They have also studied the question of recurrence and transience for $\{u_t(x) : t \geq 0,\ x \in \mathbb{R}\}$. Note that, in general, a random string may not be Gaussian, a powerful step in the proofs of Mueller and Tribe (2002) is to reduce the problems about a general random string process to those of the stationary pinned string $U = \{U_t(x), t \geq 0,\ x \in \mathbb{R}\}$, obtained by taking the initial functions $U_0(\cdot)$ in (3.2) to be defined by

$$U_0(x) = \int_0^\infty \int (G_r(x-z) - G_r(z))\, \widetilde{W}(dzdr), \tag{3.3}$$

where $\widetilde{W}$ is a space-time white noise independent of the white noise $\dot{W}$. One can verify that $U_0 = \{U_0(x) : x \in \mathbb{R}\}$ is a two-sided $\mathbb{R}^d$ valued Brownian motion satisfying $U_0(0) = 0$ and $\mathbb{E}[(U_0(x) - U_0(y))^2] = |x - y|$. We assume, by extending the probability space if needed, that U_0 is $\mathcal{F}_0$-measurable. As pointed out by Mueller and Tribe (2002), the solution to (3.1) driven by the noise $W(x, s)$ is then given by

$$\begin{aligned} U_t(x) &= \int G_t(x-z)U_0(z)dz + \int_0^t \int G_r(x-z)W(dzdr) \\ &= \int_0^\infty (G_{t+r}(x-z) - G_r(z))\, \widetilde{W}(dzdr) + \int_0^t \int G_r(x-z)W(dzdr). \end{aligned} \tag{3.4}$$

A continuous version of the above solution is called *a stationary pinned string.*

The components $\{U_t^j(x) : t \geq 0,\ x \in \mathbb{R}\}$ for $j = 1, \ldots, d$ are independent and identically distributed Gaussian processes. In the following we list some basic properties of the processes $\{U_t^j(x) : t \geq 0, x \in \mathbb{R}\}$, which will be needed for proving the results in this chapter. Lemma 3.1 below is Proposition 1 of Mueller and Tribe (2002).

Lemma 3.1 *The components* $\{U_t^j(x) : t \geq 0, x \in \mathbb{R}\}$ $(j = 1, \ldots, d)$ *of the stationary pinned string are mean-zero Gaussian random fields with stationary increments. They have the following covariance structure: for* $x, y \in \mathbb{R}$, $t \geq 0$,

$$\mathbb{E}\Big[\big(U_t^j(x) - U_t^j(y)\big)^2\Big] = |x - y|, \tag{3.5}$$

and for all $x, y \in \mathbb{R}$ *and* $0 \leq s < t$,

$$\mathbb{E}\Big[\big(U_t^j(x) - U_s^j(y)\big)^2\Big] = (t-s)^{1/2} F\big(|x-y|(t-s)^{-1/2}\big), \tag{3.6}$$

where

$$F(a) = (2\pi)^{-1/2} + \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} G_1(a-z)G_1(a-z')\big(|z| + |z'| - |z-z'|\big)\, dz dz'.$$

$F(x)$ *is a smooth function, bounded below by* $(2\pi)^{-1/2}$, *and* $F(x)/|x| \to 1$ *as* $|x| \to \infty$. *Furthermore there exists a positive constant* $K_{3,1,1}$ *such* *that for all* $s, t \in [0, \infty)$ *and all* $x, y \in \mathbb{R}$,

$$K_{3,1,1}\left(|x-y|+|t-s|^{1/2}\right) \le \mathbb{E}\left[\left(U_t^j(x)-U_s^j(y)\right)^2\right] \le 2\left(|x-y|+|t-s|^{1/2}\right). \tag{3.7}$$

It follows from (3.6) that the stationary pinned string has the following scaling property [or operator-self-similarity]: For any constant $c > 0$,

$$\{c^{-1}U_{c^4 t}(c^2 x) : t \ge 0, x \in \mathbb{R}\} \stackrel{d}{=} \{U_t(x) : t \ge 0,\ x \in \mathbb{R}\}, \tag{3.8}$$

where $\stackrel{d}{=}$ means equality in finite dimensional distributions; see Corollary 1 in Mueller and Tribe (2002).

We will also need more precise information about the asymptotic property of the function $F(x)$. By a change of variables we can write it as

$$F(x) = -(2\pi)^{-1/2} + \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} G_1(z)G_1(z')\left(|z-x|+|z'-x|\right) dzdz'. \tag{3.9}$$

Denote the above double integral by $H(x)$. Then it can be written as

$$H(x) = \int_{\mathbb{R}} G_1(z)\,|z-x|\,dz. \tag{3.10}$$

The following lemma shows that the behavior of $H(x)$ is similar to that of $F(x)$, and the second part describes how fast $H(x)/|x| \to 1$ as

$x \to \infty$.

Lemma 3.2 *There exist positive constants* $K_{3,1,2}$ *and* $K_{3,1,3}$ *such that*

$$\begin{aligned} K_{3,1,2}\left(|x-y| + |t-s|^{1/2}\right) &\le |t-s|^{1/2} H\left(|x-y||t-s|^{-1/2}\right) \\ &\le K_{3,1,3}\left(|x-y| + |t-s|^{1/2}\right). \end{aligned} \tag{3.11}$$

Moreover, we have the limit:

$$\lim_{x\to\infty} |H(x) - x| = 0. \tag{3.12}$$

Proof: The inequality (3.11) follows from the proof of (3.7) in Mueller and Tribe (2002, p.9). Hence we only need to prove (3.12).

By (3.10), we see that for $x > 0$,

$$\begin{aligned} H(x) - x &= \int_{\mathbb{R}} G_1(z)\left(|z-x| - x\right) dz \\ &= \int_x^{\infty} (z - 2x)\, G_1(z)\, dz - \int_{-\infty}^{x} z\, G_1(z)\, dz. \end{aligned} \tag{3.13}$$

Since the last two terms tend to 0 as $x \to \infty$, (3.12) follows.

The following lemmas indicate that, for every $j \in \{1, 2, \dots, d\}$, the Gaussian process $\{U_t^j(x), t \ge 0, x \in \mathbb{R}\}$ satisfies some preliminary forms of sectorial local nondeterminism; see Section 2.2 for more information on the latter. Lemma 3.3 is implied by the proof of Lemma 3 in Mueller and Tribe (2002, p.15), and Lemma 3.4 follows from the proof of Lemma

4 in Mueller and Tribe (2002, p.21).

Lemma 3.3 *For any given $\varepsilon \in (0,1)$, there exists a positive constant $K_{3,1,4}$, which depend on ε only, such that*

$$\operatorname{Var}\left(U_t^j(x)\middle|U_s^j(y)\right) \geq K_{3,1,4}\left(|x-y| + |t-s|^{1/2}\right) \tag{3.14}$$

for all $(t,x), (s,y) \in [\varepsilon, \varepsilon^{-1}] \times [-\varepsilon^{-1}, \varepsilon^{-1}]$.

Lemma 3.4 *For any given constants $\varepsilon \in (0,1)$ and $L > 0$, there exist a constant $K_{3,1,5} > 0$ such that*

$$\begin{aligned}\operatorname{Var}&\left(U_{t_2}^j(x_2) - U_{t_1}^j(x_1)\middle|U_{s_2}^j(y_2) - U_{s_1}^j(y_1)\right)\\ &\geq K_{3,1,5}\left(|x_1-y_1| + |x_2-y_2| + |t_1-s_1|^{1/2} + |t_2-s_2|^{1/2}\right)\end{aligned} \tag{3.15}$$

for all $(t_k, x_k), (s_k, y_k) \in [\varepsilon, \varepsilon^{-1}] \times [-\varepsilon^{-1}, \varepsilon^{-1}]$, where $k \in \{1,2\}$, such that $|t_2 - t_1| \geq L$ and $|s_2 - s_1| \geq L$.

Note that in Lemma 3.4, the pairs t_1 and t_2, s_1 and s_2, are well separated. The following lemma is concerned with the case when $t_1 = t_2$ and $s_1 = s_2$.

Lemma 3.5 *Let $\varepsilon \in (0,1)$ and $L > 0$ be given constants. Then there* *exist positive constants* $h_0 \in (0, \frac{L}{2})$ *and* $K_{3.1.6}$ *such that*

$$\begin{aligned}
&\mathrm{Var}\Big(U_t^j(x_2) - U_t^j(x_1)\Big| U_s^j(y_2) - U_s^j(y_1)\Big) \\
&\qquad \geq K_{3.1.6}\left(|s-t|^{1/2} + |x_1 - y_1| + |x_2 - y_2|\right)
\end{aligned} \tag{3.16}$$

for all $s, t \in [\varepsilon, \varepsilon^{-1}]$ *with* $|s-t| \leq h_0$ *and all* $x_k, y_k \in [-\varepsilon^{-1}, \varepsilon^{-1}]$, *where* $k \in \{1, 2\}$, *such that* $|x_2 - x_1| \geq L$, $|y_2 - y_1| \geq L$ *and* $|x_k - y_k| \leq \frac{L}{2}$ *for* $k = 1, 2$.

Remark 3.1 Note that, in the above, it is essential to only consider those $s, t \in [\varepsilon, \varepsilon^{-1}]$ such that $|s-t|$ is small. Otherwise (3.16) does not hold as indicated by (3.5). In this sense, Lemma 3.5 is more restrictive than Lemma 3.4. But it is sufficient for the proof of Theorem 3.7.

Proof: Using the notation similar to that in Mueller and Tribe (2002), we let $(X, Y) = \big(U_t^j(x_2) - U_t^j(x_1), U_s^j(y_2) - U_s^j(y_1)\big)$ and write $\sigma_X^2 = \mathbb{E}(X^2)$, $\sigma_Y^2 = \mathbb{E}(Y^2)$ and $\rho_{X,Y}^2 = \mathbb{E}\big[(X-Y)^2\big]$. Recall that, for the Gaussian vector (X, Y), we have

$$\mathrm{Var}(X|Y) = \frac{\big(\rho_{X,Y}^2 - (\sigma_X - \sigma_Y)^2\big)\big((\sigma_X + \sigma_Y)^2 - \rho_{X,Y}^2\big)}{4\sigma_Y^2}. \tag{3.17}$$

Lemma 3.1 and the separation condition on x_k and y_k imply that both σ_X^2 and σ_Y^2 are bounded from above and below by positive constants. Similar to the proofs of Lemmas 3 and 4 in Mueller and Tribe (2002), we only need to derive a suitable lower bound for $\rho_{X,Y}^2$. By using the identity

$$(a-b+c-d)^2 = (a-b)^2+(c-d)^2+(a-d)^2+(b-c)^2-(a-c)^2-(b-d)^2$$

and (3.5) we have

$$\begin{aligned}
\rho_{X,Y}^2 &= |t-s|^{1/2} F\big(|x_2-y_2||t-s|^{-1/2}\big) \\
&+ |t-s|^{1/2} F\big(|y_1-x_1||t-s|^{-1/2}\big) \\
&+ |x_2-x_1| - |t-s|^{1/2} F\big(|x_2-y_1||t-s|^{-1/2}\big) \\
&+ |y_1-y_2| - |t-s|^{1/2} F\big(|x_1-y_2||t-s|^{-1/2}\big).
\end{aligned} \tag{3.18}$$

By (3.9), we can rewrite the above equation as

$$\begin{aligned}
\rho_{X,Y}^2 &= |t-s|^{1/2} H\big(|x_2-y_2||t-s|^{-1/2}\big) \\
&+ |t-s|^{1/2} H\big(|y_1-x_1||t-s|^{-1/2}\big) \\
&+ |x_2-x_1| - |t-s|^{1/2} H\big(|x_2-y_1||t-s|^{-1/2}\big) \\
&+ |y_1-y_2| - |t-s|^{1/2} H\big(|x_1-y_2||t-s|^{-1/2}\big).
\end{aligned} \tag{3.19}$$

Denote the algebraic sum of the last four terms in (3.18) by S and we need to derive a lower bound for it. Note that, under the conditions of our lemma, $|x_2-y_1| \geq \frac{L}{2}$ and $|x_1-y_2| \geq \frac{L}{2}$. Hence Lemma 3.2 implies that, for any $0 < \delta < K_{3,1,2}/2$, there exists a constant $h_0 \in (0, \frac{L}{2})$ such that

$$|t-s|^{1/2}H\left(|x_2-y_1||t-s|^{-1/2}\right) \leq |x_2-y_1| + \frac{\delta}{2}|t-s|^{1/2} \tag{3.20}$$

whenever $|t-s| \leq h_0$; and the same inequality holds when $|x_2-y_1|$ is replaced by $|x_1-y_2|$. It follows that

$$\begin{aligned} S &\geq \left(|x_2-x_1| - |x_2-y_1| + |y_1-y_2| - |x_1-y_2|\right) - \delta\,|t-s|^{1/2} \\ &= -\delta\,|t-s|^{1/2}, \end{aligned} \tag{3.21}$$

because the sum of the four terms in the parentheses equals 0 under the separation condition. Combining (3.18), (3.19) and (3.11) yields

$$\rho^2_{X,Y} \geq \frac{K_{3.1.2}}{2}\left(|t-s|^{1/2} + |x_1-y_1| + |x_2-y_2|\right) \tag{3.22}$$

whenever x_k, y_k $(k=1,2)$ satisfy the above conditions.

By (3.5), we have $(\sigma_X-\sigma_Y)^2 \leq c\left(|y_1-x_1|+|x_2-y_2|\right)^2$. It follows from (3.17) and (3.22) that (3.16) holds whenever $|y_1-x_1|+|x_2-y_2|$ is sufficiently small. Finally, a continuity argument as in Mueller and Tribe (2002, p.15) removes this last restriction. This finishes the proof of Lemma 3.5.

The present chapter is a continuation of the paper of Mueller and Tribe (2002). Our objective is to study the fractal properties of various random sets generated by the random string processes. In Section 3.2, we determine the Hausdorff and packing dimensions of the range $u\left([0,1]^2\right)$ and the graph Gr$u\left([0,1]^2\right)$. We also consider the Hausdorff dimension of the range $u(E)$, where $E \subseteq [0,\infty) \times \mathbb{R}$ is an arbitrary Borel set. In Section 3.3, we consider the existence of the local times of the random string process and determine the Hausdorff and packing dimensions of the level set $L_{\mathbf{u}} = \{(t,x) \in (0,\infty) \times \mathbb{R} : u_t(x) = \mathbf{u}\}$, where $\mathbf{u} \in \mathbb{R}^d$. Finally, we conclude this chapter by determining the Hausdorff and packing dimensions of the sets of two kinds of double times of the random string in Section 3.4.

3.2 Dimension results of the range and graph

In this section, we study the Hausdoff and packing dimensions of the range $u\big([0,1]^2\big) = \big\{u_t(x) : (t,x) \in [0,1]^2\big\} \subset \mathbb{R}^d$ and the graph Gr$u\big([0,1]^2\big) = \big\{ ((t,x), u_t(x)) : (t,x) \in [0,1]^2\big\} \subset \mathbb{R}^{2+d}$.

Theorem 3.1 *Let $\{u_t(x) : t \geq 0,\ x \in \mathbb{R}\}$ be a random string process taking values in $\mathbb{R}^d$. Then with probability 1,*

$$\dim_{\mathrm{H}} u\big([0,1]^2\big) = \min\big\{d;\, 6\big\} \tag{3.23}$$

and

$$\dim_{\rm H} {\rm Gr} u\big([0,1]^2\big) = \begin{cases} 2 + \frac{3}{4}d & \textit{if } \ 1 \le d < 4, \\ 3 + \frac{1}{2}d & \textit{if } \ 4 \le d < 6, \\ 6 & \textit{if } \ 6 \le d. \end{cases} \tag{3.24}$$

Proof: Corollary 2 of Mueller and Tribe (2002) states that the distributions of $\{u_t(x) : \ t \ge 0, \ x \in \mathbb{R}\}$ and the stationary pinned string $U = \{U_t(x) : \ t \ge 0, \ x \in \mathbb{R}\}$ are mutually absolutely continuous. Hence it is enough for us to prove (3.23) and (3.24) for the stationary pinned string $U = \{U_t(x) : \ t \ge 0, \ x \in \mathbb{R}\}$. This is similar to the proof of Theorem 4 of Ayache and Xiao (2005). We include a self-contained proof for reader's convenience.

As usual, the proof is divided into proving the upper and lower bounds separately. For the upper bound in (3.23), we note that clearly $\dim_{\rm H} U\big([0,1]^2\big) \le d$ a.s., so we only need to prove the following inequality:

$$\dim_{\rm H} U\big([0,1]^2\big) \le 6 \qquad \text{a.s.} \tag{3.25}$$

Because of Lemma 3.1, one can use the standard entropy method for estimating the tail probabilities of the supremum of a Gaussian process to establish the modulus of continuity of $U = \{U_t(x) : \ t \ge 0, \ x \in \mathbb{R}\}$. See, for example, Kôno (1975). It follows that, for any constants $0 < \gamma_1 < \gamma_1' < 1/4$ and $0 < \gamma_2 < \gamma_2' < 1/2$, there exist a random variable

$A > 0$ of finite moments of all orders and an event Ω_1 of probability 1 such that for all $\omega \in \Omega_1$,

$$\sup_{(s,y),(t,x)\in[0,1]^2} \frac{|U_s(y,\omega) - U_t(x,\omega)|}{|s-t|^{\gamma_1'} + |x-y|^{\gamma_2'}} \le A(\omega). \tag{3.26}$$

Let $\omega \in \Omega_1$ be fixed and then suppressed. For any integer $n \ge 2$, we divide $[0,1]^2$ into n^6 sub-rectangles $\{R_{n,i}\}$ with sides parallel to the axes and side-lengths n^{-4} and n^{-2}, respectively. Then $U\big([0,1]^2\big)$ can be covered by the sets $U(R_{n,i})$ $(1 \le i \le n^6)$. By (3.26), we see that the diameter of the image $U(R_{n,i})$ satisfies

$$\mathrm{diam} U(R_{n,i}) \le K_{3.2.1}\, n^{-1+\delta}, \tag{3.27}$$

where $\delta = \max\{1 - 4\gamma_1', 1 - 2\gamma_2'\}$. We choose $\gamma_1' \in (\gamma_1, 1/4)$ and $\gamma_2' \in (\gamma_2, 1/2)$ such that

$$(1-\delta)\left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2}\right) > 6.$$

Hence, for $\gamma = \frac{1}{\gamma_1} + \frac{1}{\gamma_2}$, it follows from (3.27) that

$$\sum_{i=1}^{n^6} \Big[\mathrm{diam} U(R_{n,i})\Big]^{\gamma} \le K_{3,2,2}\, n^6\, n^{-(1-\delta)\gamma} \to 0 \tag{3.28}$$

as $n \to \infty$. This implies that $\dim_{\rm H} U([0,1]^2) \le \gamma$ a.s. By letting $\gamma_1 \uparrow 1/4$ and $\gamma_2 \uparrow 1/2$ along rational numbers, respectively, we derive (3.25).

Now we turn to the proof of the upper bound in (3.24) for the stationary pinned string U. We will show that there are three different ways to cover ${\rm Gr}U([0,1]^2)$, each of which leads to an upper bound for $\dim_{\rm H}{\rm Gr}U([0,1]^2)$.

- For each fixed integer $n \ge 2$, we have

$$
{\rm Gr}U([0,1]^2) \subseteq \bigcup_{i=1}^{n^6} R_{n,i} \times U(R_{n,i}). \tag{3.29}
$$

  It follows from (3.27) and (3.29) that ${\rm Gr}U([0,1]^2)$ can be covered by n^6 cubes in $\mathbb{R}^{2+d}$ with side-lengths $K_{3.2.3}\, n^{-1+\delta}$ and the same argument as the above yields

$$
\dim_{\rm H}{\rm Gr}U([0,1]^2) \le 6 \quad \text{a.s.} \tag{3.30}
$$

- Observe that each $R_{n,i} \times U(R_{n,i})$ can be covered by $\ell_{n,1}$ cubes in $\mathbb{R}^{2+d}$ of sides n^{-4}, where by (3.26)

$$
\ell_{n,1} \le K_{3.2.4}\, n^2 \times \left(\frac{n^{-1+\delta}}{n^{-4}}\right)^d .
$$

Hence $\mathrm{Gr}U\big([0,1]^2\big)$ can be covered by $n^6 \times \ell_{n,1}$ cubes in $\mathbb{R}^{2+d}$ with sides n^{-4}. Denote

$$\eta_1 = 2 + (1-\gamma_1)d.$$

Recall from the above that we can choose the constants γ_1, γ'_1 and γ'_2 such that $1-\delta > 4\gamma_1$. Therefore

$$n^6 \times \ell_{n,1} \times \big(n^{-4}\big)^{\eta_1} \;\le\; K_{3,2,5}\, n^{-(1-\delta-4\gamma_1)d} \to 0$$

as $n \to \infty$. This implies that $\dim_{\mathrm{H}} \mathrm{Gr}U\big([0,1]^2\big) \le \eta_1$ almost surely. Hence,

$$\dim_{\mathrm{H}} \mathrm{Gr}U\big([0,1]^2\big) \le 2 + \frac{3}{4}d, \qquad \text{a.s.} \tag{3.31}$$

- We can also cover each $R_{n,i} \times U(R_{n,i})$ by $\ell_{n,2}$ cubes in $\mathbb{R}^{2+d}$ of sides n^{-2}, where by (3.26)

$$\ell_{n,2} \le K_{3,2,6} \left(\frac{n^{-1+\delta}}{n^{-2}}\right)^d.$$

Hence $\mathrm{Gr}U\big([0,1]^2\big)$ can be covered by $n^6 \times \ell_{n,2}$ cubes in $\mathbb{R}^{2+d}$ with sides n^{-2}. Denote $\eta_2 = 3 + (1-\gamma_2)d$. Recall from the above that we can choose the constants γ_2, γ'_1 and γ'_2 such that $1-\delta > 2\gamma_2$.

Therefore

$$n^6 \times \ell_{n,2} \times \left(n^{-2}\right)^{\eta_2} \;\le\; K_{3,2,7}\, n^{-(1-\delta-2\gamma_2)d} \longrightarrow 0$$

as $n \to \infty$. This implies that $\dim_{\mathrm{H}} \mathrm{Gr} U\big([0,1]^2\big) \le \eta_2$ almost surely. Hence,

$$\dim_{\mathrm{H}} \mathrm{Gr} U\big([0,1]^2\big) \le 3 + \frac{1}{2}d, \qquad \text{a.s.} \tag{3.32}$$

Combining (3.30), (3.31) and (3.32) yields

$$\dim_{\mathrm{H}} \mathrm{Gr} U\big([0,1]^2\big) \le \min\left\{6,\ 2 + \frac{3}{4}d,\ 3 + \frac{1}{2}d\right\}, \quad \text{a.s.} \tag{3.33}$$

and the upper bounds in (3.24) follow from (3.33).

To prove the lower bound in (3.23), by Frostman's theorem it is sufficient to show that for any $0 < \gamma < \min\{d, 6\}$,

$$\mathcal{E}_\gamma = \int_{[0,1]^2} \int_{[0,1]^2} \mathbb{E}\left(\frac{1}{|U_s(y) - U_t(x)|^\gamma}\right) ds dy dt dx < \infty. \tag{3.34}$$

See, e.g., Kahane (1985a, Chapter 10). Since $0 < \gamma < d$, we have $0 < \mathbb{E}(|\Xi|^{-\gamma}) < \infty$, where Ξ is a standard d-dimensional normal vector. Combining this fact with Lemma 3.1, we have

$$\mathcal{E}_\gamma \le K_{3,2,8} \int_0^1 ds \int_0^1 dt \int_0^1 dy \int_0^1 \frac{1}{\big(|s-t|^{1/2} + |x-y|\big)^{\gamma/2}}\, dx. \tag{3.35}$$

Recall the weighted arithmetic-mean and geometric-mean inequality: for all integer $n \geq 2$ and $x_i \geq 0$, $\beta_i > 0$ ($i = 1, \ldots, n$) such that $\sum_{i=1}^{n} \beta_i = 1$, we have

$$\prod_{i=1}^{n} x_i^{\beta_i} \leq \sum_{i=1}^{n} \beta_i x_i. \tag{3.36}$$

Applying (3.36) with $n = 2$, $\beta_1 = 2/3$ and $\beta_2 = 1/3$, we obtain

$$|s-t|^{1/2} + |x-y| \geq \frac{2}{3}|s-t|^{1/2} + \frac{1}{3}|x-y| \geq |s-t|^{1/3}|x-y|^{1/3}. \tag{3.37}$$

Therefore, the denominator in (3.35) can be bounded from below by $|s-t|^{\gamma/6}|x-y|^{\gamma/6}$. Since $\gamma < 6$, by (3.35), we have $\mathcal{E}_\gamma < \infty$, which proves (3.34).

For proving the lower bound in (3.24), we need the following lemma from Ayache and Xiao (2005).

Lemma 3.6 *Let α, β and η be positive constants. For $a > 0$ and $b > 0$, let*

$$J := J(a, b) = \int_0^1 \frac{dt}{(a + t^\alpha)^\beta (b + t)^\eta}. \tag{3.38}$$

Then there exist finite constants $K_{3,2,9}$ and $K_{3,2,10}$, depending on α, β, η only, such that the following hold for all reals $a, b > 0$ satisfying $a^{1/\alpha} \leq K_{3,2,9}\, b$:

(i) if $\alpha\beta > 1$, then

$$J \leq K_{3,2,10} \frac{1}{a^{\beta - \alpha^{-1}} b^\eta}; \tag{3.39}$$

(ii) if $\alpha\beta = 1$, *then*

$$J \leq K_{3.2.10}\,\frac{1}{b^{\eta}}\log\left(1 + ba^{-1/\alpha}\right); \tag{3.40}$$

(iii) if $0 < \alpha\beta < 1$ *and* $\alpha\beta + \eta \neq 1$, *then*

$$J \leq K_{3.2.10}\left(\frac{1}{b^{\alpha\beta+\eta-1}} + 1\right). \tag{3.41}$$

Now we prove the lower bound in (3.24). Since $\dim_{\mathrm{H}}\mathrm{Gr}U\big([0,1]^2\big) \geq \dim_{\mathrm{H}}U\big([0,1]^2\big)$ always holds, we only need to consider the cases $1 \leq d < 4$ and $4 \leq d < 6$, respectively.

Since the proof of the two cases are almost identical, we only prove the case when $1 \leq d < 4$ here. Let $0 < \gamma < 2 + \frac{3}{4}d$ be a fixed, but arbitrary, constant. Since $1 \leq d < 4$, we may and will assume $\gamma > 1 + d$. In order to prove $\dim_{\mathrm{H}}\mathrm{Gr}U([0,1]^2) \geq \gamma$ a.s., again by Frostman's theorem, it is sufficient to show

$$\mathcal{G}_{\gamma} = \int_{[0,1]^2}\int_{[0,1]^2}\mathbb{E}\left[\frac{ds\,dy\,dt\,dx}{\big(|s-t|^2 + |x-y|^2 + |U_s(y) - U_t(x)|^2\big)^{\gamma/2}}\right] < \infty. \tag{3.42}$$

Since $\gamma > d$, we note that for a standard normal vector Ξ in $\mathbb{R}^d$ and any number $a \in \mathbb{R}$,

$$\mathbb{E}\left[\frac{1}{\big(a^2 + |\Xi|^2\big)^{\gamma/2}}\right] \leq K_{3,2,11}\, a^{-(\gamma-d)},$$

see e.g. Kahane (1985a, p.279). Consequently, by Lemma 3.1, we derive that

$$\mathcal{G}_\gamma \le K_{3,2,12} \int_{[0,1]^2}\int_{[0,1]^2} \frac{ds dy dt dx}{\left(|s-t|^{1/2}+|x-y|\right)^{d/2}\left(|s-t|+|x-y|\right)^{\gamma-d}}. \tag{3.43}$$

By Lemma 3.6 and a change of variable and noting that $d < 4$, we can apply (3.41) to derive

$$\begin{aligned}\mathcal{G}_\gamma &\le K_{3,2,13}\int_0^1 dx \int_0^1 \frac{1}{(t^{1/2}+x)^{d/2}(t+x)^{\gamma-d}} dt \\ &\le K_{3,2,14}\int_0^1 \left(\frac{1}{x^{d/4+\gamma-d-1}}+1\right) dx < \infty,\end{aligned} \tag{3.44}$$

where the last inequality follows from $\gamma - \frac{3}{4}d - 1 < 1$. This completes the proof of Theorem 3.1.

By using the relationships among the Hausdorff dimension, packing dimension and the box dimension [see Falconer (1990)], Theorem 3.1 and the proof of the upper bounds, we derive the following analogous result on the packing dimensions of $u([0,1]^2)$ and $\mathrm{Gr}u([0,1]^2)$.

Theorem 3.2 *Let $\{u_t(x) : t \ge 0, x \in \mathbb{R}\}$ be a random string process taking values in $\mathbb{R}^d$. Then with probability 1,*

$$\dim_{\mathrm{P}} u([0,1]^2) = \min\{d; 6\} \tag{3.45}$$

and

$$\dim_{\mathrm{P}} \mathrm{Gr}u\big([0,1]^2\big) = \begin{cases} 2 + \frac{3}{4}d & \textit{if } 1 \le d < 4, \\ 3 + \frac{1}{2}d & \textit{if } 4 \le d < 6, \\ 6 & \textit{if } 6 \le d. \end{cases} \tag{3.46}$$

Theorems 3.1 and 3.2 show that the random fractals $u\big([0,1]^2\big)$ and $\mathrm{Gr}u\big([0,1]^2\big)$ are rather regular because they have the same Hausdorff and packing dimensions.

Now we will turn our attention to find the Hausdorff dimension of the range $u(E)$ for an arbitrary Borel set $E \subseteq [0,\infty) \times \mathbb{R}$.

For this purpose, we mention the related results for an (N,d)-fractional Brownian sheet $B^H = \{B^H(\mathbf{t}) : \mathbf{t} \in \mathbb{R}_+^N\}$ with Hurst index $H = (H_1, \ldots, H_N) \in (0,1)^N$ [cf. Section 2.3]. What the random string process $\{u_t(x) : t \ge 0,\, x \in \mathbb{R}\}$ and a $(2,d)$-fractional Broanian sheet B^H with $H = (\frac{1}{4}, \frac{1}{2})$ have in common is that they are both anisotropic.

As Section 2.3 pointed out, the Hausdorff dimension of the image $B^H(F)$ cannot be determined by $\dim_{\mathrm{H}} F$ and H alone for an arbitrary fractal set F, and more information about the geometry of F is needed. To capture the anisotropic nature of B^H, we have introduced a new notion of dimension, namely, the *Hausdorff dimension contour*, for finite Borel measures and Borel sets and showed that $\dim_{\mathrm{H}} B^H(F)$ is determined by the Hausdorff dimension contour of F. It turns out that we can use the same technique to study the images of the random string.

We start with the following Proposition 3.2 which determines $\dim_{\mathrm{H}} u(E)$ when E belongs to a special class of Borel sets in $[0, \infty) \times \mathbb{R}$. Its proof is the same as that of Proposition 2.2.

Proposition 3.2 *Let $\{u_t(x) : t \geq 0,\ x \in \mathbb{R}\}$ be a random string in $\mathbb{R}^d$. Assume that E_1 and E_2 are Borel sets in $[0, \infty)$ and $\mathbb{R}$, respectively, which satisfy $\dim_{\mathrm{H}} E_1 = \dim_{\mathrm{P}} E_1$ or $\dim_{\mathrm{H}} E_2 = \dim_{\mathrm{P}} E_2$. Let $E = E_1 \times E_2 \subset [0, \infty) \times \mathbb{R}$, then we have*

$$\dim_{\mathrm{H}} u(E) = \min\{d;\ 4\dim_{\mathrm{H}} E_1 + 2\dim_{\mathrm{H}} E_2\}, \quad a.s. \tag{3.47}$$

In order to determine $\dim_{\mathrm{H}} u(E)$ for an arbitrary Borel set $E \subset [0, \infty) \times \mathbb{R}$, we recall from Section 2.3 the following definition. Denote by $\mathcal{M}_C^+(E)$ the family of finite Borel measures with compact support in E.

Definition 3.1 *Given $\mu \in \mathcal{M}_C^+(E)$, we define the set $\Lambda_\mu \subseteq \mathbb{R}_+^2$ by*

$$\Lambda_\mu = \left\{ \lambda = (\lambda_1, \lambda_2) \in \mathbb{R}_+^2 : \overline{\lim_{r \to 0^+}} \frac{\mu\left(R((t,x), r)\right)}{r^{4\lambda_1 + 2\lambda_2}} = 0, \right.$$
$$\left. \text{for } \mu\text{-a.e. } (t,x) \in [0, \infty) \times \mathbb{R} \right\}, \tag{3.48}$$

where $R((t,x), r) = [t - r^4, t + r^4] \times [x - r^2, x + r^2]$.

The properties of set Λ_μ can be found in Lemma 2.4. The boundary of Λ_μ, denoted by $\partial\Lambda_\mu$, is called the Hausdorff dimension contour of μ.

Define

$$\Lambda(E) = \bigcup_{\mu \in \mathcal{M}_C^+(E)} \Lambda_\mu.$$

and define the Hausdorff dimension contour of E by $\bigcup_{\mu \in \mathcal{M}_C^+(E)} \partial\Lambda_\mu$. It can be verified that, for every $\beta \in (0,\infty)^2$, the supremum $\sup_{\lambda \in \Lambda(E)} \langle \lambda, \beta \rangle$ is achieved on the Hausdorff dimension contour of E (Lemma 2.4).

Theorem 3.3 *Let $u = \{u_t(x) : t \geq 0, x \in \mathbb{R}\}$ be a random string process with values in $\mathbb{R}^d$. Then, for any Borel set $E \subset [0,\infty) \times \mathbb{R}$,*

$$\dim_{\rm H} u(E) = \min\big\{d;\ s(E)\big\}, \qquad a.s. \tag{3.49}$$

where $s(E) = \sup_{\lambda \in \Lambda(E)}(4\lambda_1 + 2\lambda_2)$.

Proof: By Corollary 2 of Mueller and Tribe (2002), one only needs to prove (3.49) for the stationary pinned string $U = \{U_t(x) : t \geq 0, x \in \mathbb{R}\}$. The latter follows from the proof of Theorem 2.2.

3.3 Existence of the local times and dimension results for level sets

In this section, we will first give a sufficient condition for the existence of the local times of a random string process on any rectangle $I \in \mathcal{A}$, where $\mathcal{A}$ is the collection of all the rectangles in $[0, \infty) \times \mathbb{R}$ with sides parallel to the axes. Then, we will determine the Hausdorff and packing dimensions for the level set $L_{\mathbf{u}} = \{(t, x) \in [0, \infty) \times \mathbb{R} : u_t(x) = \mathbf{u}\}$, where $\mathbf{u} \in \mathbb{R}^d$ is fixed.

The following theorem is concerned with the existence of local times of the random string.

Theorem 3.4 *Let $\{u_t(x) : \ t \geq 0, \ x \in \mathbb{R}\}$ be a random string process in $\mathbb{R}^d$. If $d < 6$, then for every $I \in \mathcal{A}$, the string has local times $\{l(\mathbf{u}, I), \ \mathbf{u} \in \mathbb{R}^d\}$ on I, and $l(\mathbf{u}, I)$ admits the following L^2 representation:*

$$l(\mathbf{u}, I) = (2\pi)^{-d} \int_{\mathbb{R}^d} \exp(-i\langle \mathbf{v}, \, \mathbf{u}\rangle) \int_I \exp(i\langle \mathbf{v}, \, u_t(x)\rangle) dt dx d\mathbf{v}, \forall \, \mathbf{u} \in \mathbb{R}^d. \tag{3.50}$$

Proof: Because of Corollary 2 of Mueller and Tribe (2002), we only need to prove the existence for the stationary pinned string $U = \{U_t(x) : \ t \geq 0, \ x \in \mathbb{R}\}$.

Let $I \in \mathcal{A}$ be fixed. Without loss of generality, we may assume

$I = [\varepsilon, 1]^2$. By (21.3) in Geman and Horowitz (1980) and using the characteristic functions of Gaussian random variables, it suffices to prove the integral $\mathcal{J}(I)$ defined by

$$\int_I dtdx \int_I dsdy \int_{\mathbb{R}^d} d\mathbf{u} \int_{\mathbb{R}^d} \left|\mathbb{E}\exp\left(i\langle \mathbf{u}, U_t(x)\rangle + i\langle \mathbf{v}, U_s(y)\rangle\right)\right| d\mathbf{v} \tag{3.51}$$

is finite. Since the components of U are i.i.d., it is easy to see that

$$\mathcal{J}(I) = (2\pi)^d \int_I dtdx \int_I \left[\det\mathrm{Cov}\left(U_t^1(x), U_s^1(y)\right)\right]^{-d/2} dsdy. \tag{3.52}$$

By Lemma 3.3 and noting that $I = [\varepsilon, 1]^2$, we can see that

$$\begin{aligned} \det\mathrm{Cov}\big(U_t^j(x), U_s^j(y)\big) &= \mathrm{Var}\big(U_s^j(y)\big)\mathrm{Var}\big(U_t^j(x)\big|U_s^j(y)\big) \\ &\geq K_{3,3,1}\left(|x-y| + |t-s|^{1/2}\right). \end{aligned} \tag{3.53}$$

The above inequality, (3.37) and the fact that $d < 6$ lead to

$$\mathcal{J}(I) \leq K_{3,3,2} \int_\varepsilon^1 \int_\varepsilon^1 |s-t|^{-d/6} dtds \int_\varepsilon^1 \int_\varepsilon^1 |x-y|^{-d/6} dxdy < \infty, \tag{3.54}$$

which proves (3.51), and therefore Theorem 3.4.

Remark 3.3 It would be interesting to study the regularity properties of the local times $l(\mathbf{u}, \mathbf{t})$, $(\mathbf{u} \in \mathbb{R}^d, \mathbf{t} \in [0, \infty) \times \mathbb{R})$ such as joint continuity and moduli of continuity. One way to tackle these problems is to establish sectorial local nondeterminism [cf. Section 2.2] for the stationary pinned string $U = \{U_t(x) : t \geq 0, x \in \mathbb{R}\}$. This will have to be pursued elsewhere. Some results of this nature for certain isotropic Gaussian random fields can be found in Xiao (1997).

Mueller and Tribe (2002) proved that for every $\mathbf{u} \in \mathbb{R}^d$,

$$\mathbb{P}\{u_t(x) = \mathbf{u} \text{ for some } (t,x) \in [0,\infty) \times \mathbb{R}\} > 0$$

if and only if $d < 6$. Now we study the Hausdorff and packing dimensions of the level set $L_{\mathbf{u}} = \{(t,x) \in [0,\infty) \times \mathbb{R} : u_t(x) = \mathbf{u}\}$.

Theorem 3.5 *Let $\{u_t(x) : t \geq 0, x \in \mathbb{R}\}$ be a random string process in $\mathbb{R}^d$ with $d < 6$. Then for every $\mathbf{u} \in \mathbb{R}^d$, with positive probability,*

$$\dim_{\mathrm{H}}\big(L_{\mathbf{u}} \cap [0,1]^2\big) = \dim_{\mathrm{P}}\big(L_{\mathbf{u}} \cap [0,1]^2\big) = \begin{cases} 2 - \frac{1}{4}d & \text{if } 1 \leq d < 4, \\ 3 - \frac{1}{2}d & \text{if } 4 \leq d < 6. \end{cases} \tag{3.55}$$

Proof: As usual, it is sufficient to prove (3.55) for the stationary pinned string $U = \{U_t(x) : t \geq 0, x \in \mathbb{R}\}$. We first prove the almost sure upper bound

$$\dim_{\mathrm{P}}\big(L_{\mathbf{u}} \cap [0,1]^2\big) \leq \begin{cases} 2 - \frac{1}{4}d & \text{if } 1 \leq d < 4, \\ 3 - \frac{1}{2}d & \text{if } 4 \leq d < 6. \end{cases} \tag{3.56}$$

By the σ-stability of $\dim_P$, it is sufficient to show (3.56) holds for $L_{\mathbf{u}} \cap [\varepsilon, 1]^2$ for every $\varepsilon \in (0, 1)$. For this purpose, we construct coverings of $L_{\mathbf{u}} \cap [0, 1]^2$ by cubes of the same side length.

For any integer $n \geq 2$, we divide the square $[\varepsilon, 1]^2$ into n^6 sub-rectangles $R_{n,\ell}$ of side lengths n^{-4} and n^{-2}, respectively. Let $0 < \delta < 1$ be fixed and let $\tau_{n,\ell}$ be the lower-left vertex of $R_{n,\ell}$. Then

$$\begin{aligned}
&\mathbb{P}\{\mathbf{u} \in U(R_{n,\ell})\} \\
&\leq \mathbb{P}\left\{ \max_{(s,y),(t,x) \in R_{n,\ell}} |U_s(y) - U_t(x)| \leq n^{-(1-\delta)};\ \mathbf{u} \in U(R_{n,\ell}) \right\} \\
&\qquad + \mathbb{P}\left\{ \max_{(s,y),(t,x) \in R_{n,\ell}} |U_s(y) - U_t(x)| > n^{-(1-\delta)} \right\} \\
&\leq \mathbb{P}\left\{ |U(\tau_{n,\ell}) - \mathbf{u}| \leq n^{-(1-\delta)} \right\} + \exp(-c\, n^{2\delta}) \\
&\leq K_{3,3,3}\, n^{-(1-\delta)d}.
\end{aligned} \tag{3.57}$$

In the above we have applied Lemma 3.1 and the Gaussian isoperimetric inequality [cf. Lemma 2.1 in Talagrand (1995)] to derive the second inequality.

Since we can deal with the cases $1 \leq d < 4$ and $4 \leq d < 6$ almost identically, we will only consider the case $1 \leq d < 4$ here and leave the case $4 \leq d < 6$ to the interested readers.

Define a covering $\{R'_{n,\ell}\}$ of $L_{\mathbf{u}} \cap [\varepsilon, 1]^2$ by $R'_{n,\ell} = R_{n,\ell}$ if $\mathbf{u} \in U(R_{n,\ell})$ and $R'_{n,\ell} = \emptyset$ otherwise. Note that each $R'_{n,\ell}$ can be covered by n^2 squares of side length n^{-4}. Thus, for every $n \geq 2$, we have obtained a covering of the level set $L_{\mathbf{u}} \cap [\varepsilon, 1]^2$ by squares of side length n^{-4}. Consider the sequence of integers $n = 2^k$ $(k \geq 1)$, and let N_k denote the minimum number of squares of side-length 2^{-4k} that are needed to cover $L_{\mathbf{u}} \cap [\varepsilon, 1]^2$. It follows from (3.57) that

$$\mathbb{E}(N_k) \leq K_{3.3.3}\, 2^{6k} \cdot 2^{2k} \cdot 2^{-k(1-\delta)d} = K_{3.3.3}\, 2^{k(8-(1-\delta)d)}. \tag{3.58}$$

By (3.58), Markov's inequality and the Bore-Cantelli lemma we derive that for any $\delta' \in (0, \delta)$, almost surely for all k large enough,

$$N_k \leq K_{3.3.3}\ 2^{k(8-(1-\delta')d)}. \tag{3.59}$$

By the definition of box dimension and its relation to $\dim_{\mathrm{P}}$ [cf. Falconer (1990)], (3.59) implies that $\dim_{\mathrm{P}}\big(L_{\mathbf{u}} \cap [\varepsilon, 1]^2\big) \leq 2-(1-\delta')d/4$ a.s. Since $\varepsilon > 0$ is arbitrary, we obtain the desired upper bound for $\dim_{\mathrm{P}}\big(L_{\mathbf{u}} \cap [\varepsilon, 1]^2\big)$ in the case $1 \leq d < 4$.

Since $\dim_{\mathrm{H}} E \leq \dim_{\mathrm{P}} E$ for all Borel sets $E \subset \mathbb{R}^2$, it remains to prove the following lower bound: for any $\varepsilon \in (0, 1)$, with positive probability

$$\dim_{\mathrm{P}}\big(L_{\mathbf{u}} \cap [\varepsilon, 1]^2\big) \geq \begin{cases} 2 - \frac{1}{4}d & \text{if } 1 \leq d < 4, \\ 3 - \frac{1}{2}d & \text{if } 4 \leq d < 6. \end{cases} \tag{3.60}$$

We only prove (3.60) for $1 \leq d < 4$. The other case is similar and is omitted. Let $\delta > 0$ such that

$$\gamma := 2 - \frac{1}{4}(1+\delta)d > 1. \tag{3.61}$$

Note that if we can prove that there is a constant $K_{3,3,4} > 0$ such that

$$\mathbb{P}\big\{\dim_{\rm H}\big(L_{\mathbf{u}} \cap [\varepsilon, 1]^2\big) \geq \gamma\big\} \geq K_{3,3,4}, \tag{3.62}$$

then the lower bound in (3.60) will follow by letting $\delta \downarrow 0$.

Our proof of (3.62) is based on the capacity argument due to Kahane [see, e.g., Kahane (1985a)]. Similar methods have been used by Adler (1981), Testard (1986), Xiao (1995), Ayache and Xiao (2005) to various types of stochastic processes.

Let $\mathcal{M}_\gamma^+$ be the space of all non-negative measures on $[0,1]^2$ with finite γ-energy. It is known [cf. Adler (1981)] that $\mathcal{M}_\gamma^+$ is a complete metric space under the metric

$$\|\mu\|_\gamma = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\mu(dt, dx)\mu(ds, dy)}{(|t-s|^2 + |x-y|^2)^{\gamma/2}}. \tag{3.63}$$

We define a sequence of random positive measures μ_n on the Borel sets of $[\varepsilon, 1]^2$ by

$$\mu_n(C) = \int_C (2\pi n)^{d/2} \exp\Big(- \frac{n\,|U_t(x) - \mathbf{u}|^2}{2}\Big) dt dx$$
$$= \int_C \int_{\mathbb{R}^d} \exp\Big(- \frac{|\xi|^2}{2n} + i\langle \xi, U_t(x) - \mathbf{u}\rangle\Big) d\xi\, dt dx, \qquad (3.64)$$
$$\forall\, C \in \mathcal{B}([\varepsilon, 1]^2).$$

It follows from Kahane (1985a) or Testard (1986) that if there are positive constants $K_{3,3,5}$ and $K_{3,3,6}$, which may depend on $\mathbf{u}$, such that

$$\mathbb{E}\big(\|\mu_n\|\big) \ge K_{3,3,5}, \qquad \mathbb{E}\big(\|\mu_n\|^2\big) \le K_{3,3,6}, \qquad (3.65)$$

$$\mathbb{E}\big(\|\mu_n\|_\gamma\big) < +\infty, \qquad (3.66)$$

where $\|\mu_n\| = \mu_n([\varepsilon, 1]^2)$, then there is a subsequence of $\{\mu_n\}$, say $\{\mu_{n_k}\}$, such that $\mu_{n_k} \to \mu$ in $\mathcal{M}_\gamma^+$ and μ is strictly positive with probability $\ge K_{3,3,5}^2/(2K_{3,3,6})$. It follows from (3.64) and the continuity of U that μ has its support in $L_{\mathbf{u}} \cap [\varepsilon, 1]^2$ almost surely. Hence Frostman's theorem yields (3.62).

It remains to verify (3.65) and (3.66). By Fubini's theorem we have

$$
\begin{aligned}
&\mathbb{E}(\|\mu_n\|) \\
&= \int_{[\varepsilon,1]^2}\int_{\mathbb{R}^d} \exp(-i\langle\xi,\mathbf{u}\rangle)\exp\Big(-\frac{|\xi|^2}{2n}\Big)\,\mathbb{E}\exp\Big(i\langle\xi,U_t(x)\rangle\Big)d\xi\,dtdx \\
&= \int_{[\varepsilon,1]^2}\int_{\mathbb{R}^d} \exp(-i\langle\xi,\mathbf{u}\rangle)\exp\Big(-\frac{1}{2}(n^{-1}+\sigma^2(t,x))|\xi|^2\Big)\,d\xi\,dtdx \\
&= \int_{[\varepsilon,1]^2}\Big(\frac{2\pi}{n^{-1}+\sigma^2(t,x)}\Big)^{d/2}\exp\Big(-\frac{|\mathbf{u}|^2}{2(n^{-1}+\sigma^2(t,x))}\Big)\,dtdx \\
&\geq \int_{[\varepsilon,1]^2}\Big(\frac{2\pi}{1+\sigma^2(t,x)}\Big)^{d/2}\exp\Big(-\frac{|\mathbf{u}|^2}{2\sigma^2(t,x)}\Big)\,dt := K_{3,3,5},
\end{aligned}
\tag{3.67}
$$

where $\sigma^2(t,x) = \mathbb{E}\left[\left(U_t^1(x)\right)^2\right]$.

Denote by I_{2d} the identity matrix of order $2d$ and by $\mathrm{Cov}(U_s(y),U_t(x))$ the covariance matrix of the Gaussian vector $(U_s(y),U_t(x))$. Let $\Gamma = n^{-1}I_{2d} + \mathrm{Cov}(U_s(y),U_t(x))$ and let $(\xi,\eta)'$ be the transpose of the row vector (ξ,η). As in the proof of (3.51), we apply (3.14) in Lemma 3.3 and the inequality (3.36) to derive

$$
\begin{aligned}
&\mathbb{E}(\|\mu_n\|^2)\\
&= \int_{[\varepsilon,1]^2}\int_{[\varepsilon,1]^2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \exp(-i\langle \xi+\eta, \mathbf{u}\rangle)\\
&\qquad \times \exp\Big(-\frac{1}{2}(\xi,\eta)\,\Gamma\,(\xi,\eta)'\Big)\, d\xi d\eta\, dsdydtdx\\
&= \int_{[\varepsilon,1]^2}\int_{[\varepsilon,1]^2} \frac{(2\pi)^d}{\sqrt{\det\Gamma}} \exp\Big(-\frac{1}{2}(\mathbf{u},\mathbf{u})\,\Gamma^{-1}\,(\mathbf{u},\mathbf{u})'\Big)\, dsdy\, dtdx\\
&\le \int_{[\varepsilon,1]^2}\int_{[\varepsilon,1]^2} \frac{(2\pi)^d}{\big[\det\mathrm{Cov}(U_s^1(y), U_t^1(x))\big]^{d/2}}\, dsdy\, dtdx\\
&\le K_{3.3.7}\int_\varepsilon^1\int_\varepsilon^1 |s-t|^{-d/6} dtds \int_\varepsilon^1\int_\varepsilon^1 |x-y|^{-d/6} dxdy := K_{3.3.6} < \infty.
\end{aligned}
\tag{3.68}
$$

Similar to (3.68) and by the same method as in proving (3.43), we have

$$
\begin{aligned}
\mathbb{E}(\|\mu_n\|_\gamma) &= \int_{[\varepsilon,1]^2}\int_{[\varepsilon,1]^2} \frac{dsdy\, dtdx}{(|s-t|^2+|x-y|^2)^{\gamma/2}}\\
&\quad\times \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-i\langle \xi+\eta, \mathbf{u}\rangle} \exp\Big(-\frac{1}{2}(\xi,\eta)\,\Gamma\,(\xi,\eta)'\Big)\, d\xi d\eta\\
&\le K_{3.3.8}\int_{[\varepsilon,1]^2}\int_{[\varepsilon,1]^2} \frac{dsdy\, dtdx}{\big(|s-t|^{1/2}+|x-y|\big)^{d/2}\big(|s-t|+|x-y|\big)^{\gamma}}\\
&< \infty,
\end{aligned}
\tag{3.69}
$$

where the last inequality follows from Lemma 3.6 and the facts that $d < 4$ and $d/4+\gamma-1 < 1$. This proves (3.66) and thus the proof of

Theorem 3.5 is finished.

3.4 Hausdorff and packing dimensions of the sets of double times

Mueller and Tribe (2002) found necessary and sufficient conditions for an $\mathbb{R}^d$-valued string process to have double points. In this section, we determine the Hausdorff and packing dimensions of the sets of double times of the random string.

As in Mueller and Tribe (2002), we consider the following two kinds of double times for the string process $\{u_t(x) : t \geq 0, x \in \mathbb{R}\}$.

- Type I double times:

$$L_{\mathrm{I},2} = \Big\{ \big((t_1, x_1), (t_2, x_2)\big) \in \big((0, \infty) \times \mathbb{R}\big)^2_{\neq} : \ u_{t_1}(x_1) = u_{t_2}(x_2) \Big\}, \tag{3.70}$$

where $\big((0,\infty) \times \mathbb{R}\big)^2_{\neq} = \big\{\big((t_1, x_1), (t_2, x_2)\big) \in \big((0,\infty) \times \mathbb{R}\big)^2 : (t_1, x_1) \neq (t_2, x_2)\big\}$.

In order to determine the Hausdorff and packing dimensions of $L_{\mathrm{I},2}$, we introduce a $(4, d)$-random field $\Delta u = \{\Delta u(t_1, x_1; t_2, x_2)\}$

defined by

$$\Delta u(t_1, x_1; t_2, x_2) = u_{t_2}(x_2) - u_{t_1}(x_1), \\ \forall\ (t_1, x_1, t_2, x_2) \in \big((0,\infty) \times \mathbb{R}\big)^2. \tag{3.71}$$

Then $L_{\mathrm{I},2}$ can be viewed as the zero set, denoted by $(\Delta u)^{-1}(0)$, of $\Delta u(t_1, x_1; t_2, x_2)$ and its Hausdorff and packing dimensions can be studied by using the method in Section 3.3.

- Type II double times:

$$L_{\mathrm{II},2} = \Big\{(t, x_1, x_2) \in (0,\infty) \times \mathbb{R}^2_{\neq} :\ u_t(x_1) = u_t(x_2)\Big\},$$

where $\mathbb{R}^2_{\neq} = \{(x_1, x_2) \in \mathbb{R}^2 :\ x_1 \neq x_2\}$.

In order to determine the Hausdorff and packing dimensions of $L_{\mathrm{II},2}$, we will consider the $(3, d)$-random field $\widetilde{\Delta} u = \{\widetilde{\Delta} u(t; x_1, x_2)\}$ defined by

$$\widetilde{\Delta} u(t; x_1, x_2) = u_t(x_2) - u_t(x_1),\ \forall\ (t, x_1, x_2) \in (0,\infty) \times \mathbb{R}^2. \tag{3.72}$$

Then we can see that $L_{\mathrm{II},2}$ is nothing but the zero se of $\widetilde{\Delta} u$:

$$L_{\mathrm{II},2} = \Big\{(t, x_1, x_2) \in (0,\infty) \times \mathbb{R}^2_{\neq} :\ \widetilde{\Delta} u(t; x_1, x_2) = 0\Big\}.$$

For any constants $0 < a_1 < a_2$ and $b_1 < b_2$, consider the squares

$J_\ell = [a_\ell, a_\ell + h] \times [b_\ell, b_\ell + h]$ ($\ell = 1, 2$). Let $J = \prod_{\ell=1}^{2} J_\ell \subset ((0, \infty) \times \mathbb{R})^2$ denote the corresponding hypercube. We choose $h > 0$ small enough, say,

$$h < \min\left\{\frac{a_2 - a_1}{3}, \frac{b_2 - b_1}{3}\right\} \equiv L.$$

Thus $|t_2 - t_1| > L$ for all $t_2 \in [a_2, a_2 + h]$ and $t_1 \in [a_1, a_1 + h]$. We will use this assumption together with Lemma 3.4 to prove Theorem 3.6 below. We denote the collection of the hypercubes having the above properties by $\mathcal{J}$.

The following theorem gives the Hausdorff and packing dimensions of the Type I double times of a random string.

Theorem 3.6 *Let $u = \{u_t(x) : t \geq 0, x \in \mathbb{R}\}$ be a random string process in $\mathbb{R}^d$. If $d \geq 12$, then $L_{\mathrm{I},2} = \emptyset$ a.s. If $d < 12$, then, for every $J \in \mathcal{J}$, with positive probability,*

$$\dim_{\mathrm{H}}(L_{\mathrm{I},2} \cap J) = \dim_{\mathrm{P}}(L_{\mathrm{I},2} \cap J) = \begin{cases} 4 - \frac{1}{4}d & \text{if } 1 \leq d < 8, \\ 6 - \frac{1}{2}d & \text{if } 8 \leq d < 12. \end{cases} \tag{3.73}$$

Proof: The first statement is due to Mueller and Tribe (2002). Hence, we only need to prove the dimension result (3.73).

Thanks to Corollary 2 of Mueller and Tribe (2002), it is sufficient to prove (3.73) for the stationary pinned string U. This will be done by working with the zero set of the $(4, d)$-Gaussian field

$\Delta U = \{\Delta U(t_1, x_1; t_2, x_2)\}$ define by (3.71). That is, we will prove (3.73) with $L_{1,2}$ replaced by the zero set $(\Delta U)^{-1}(0)$. The proof is a modification of that of Theorem 3.5. Hence, we only give a sketch of it.

For an integer $n \geq 2$, we divide the hypercube J into n^{12} sub-domains $T_{n,p} = R^1_{n,p} \times R^2_{n,p}$, where $R^1_{n,p}$, $R^2_{n,p} \subset (0, \infty) \times \mathbb{R}$ are rectangles of side lengths $n^{-4}h$ and $n^{-2}h$, respectively. Let $0 < \delta < 1$ be fixed and let $\tau^k_{n,p}$ be the lower-left vertex of $R^k_{n,p}$ $(k = 1, 2)$. Then the probability $\mathbb{P}\big\{0 \in \Delta U(T_{n,p})\big\}$ is at most

$$\begin{aligned}
&\mathbb{P}\Big\{ \max_D \big|\Delta U(t_1, x_1; t_2, x_2) - \Delta U(s_1, y_1; s_2, y_2)\big| \leq n^{-(1-\delta)}; 0 \in \Delta U(T_{n,p})\Big\} \\
&+ \mathbb{P}\Big\{ \max_D \big|\Delta U(t_1, x_1; t_2, x - \Delta U(s_1, y_1; s_2, y_2)\big| > n^{-(1-\delta)}\Big\} \\
&\leq \mathbb{P}\Big\{ \big|\Delta U(\tau^1_{n,p}; \tau^2_{n,p})\big| \leq n^{-(1-\delta)}\Big\} \\
&\quad + \mathbb{P}\Big\{ \max_D \big|\Delta U(t_1, x_1; t_2, x_2) - \Delta U(s_1, y_1; s_2, y_2)\big| > n^{-(1-\delta)}\Big\},
\end{aligned} \tag{3.74}$$

where $D = \{(t_1, x_1; t_2, x_2), (s_1, y_1; s_2, y_2) \in T_{n,p}\}$.

By the definition of J, we see that $\Delta U(\tau^1_{n,p}, \tau^2_{n,p})$ is a Gaussian random variable with mean 0 and variance at least $K\, L^{1/2}$. Hence the first term in (3.74) is at most $K_{3,4,1}\, n^{-(1-\delta)d}$.

On the other hand, note that

$$\left|\Delta U(s_1, y_1; s_2, y_2) - \Delta U(t_1, x_1; t_2, x_2)\right| \le K \sum_{k=1}^{2} \left|U_{s_k}(y_k) - U_{t_k}(x_k)\right|,$$

we have

$$\begin{aligned}
&\mathbb{P}\left\{ \max_{D} \left|\Delta U(t_1, x_1; t_2, x_2) - \Delta U(s_1, y_1; s_2, y_2)\right| > n^{-(1-\delta)} \right\} \\
&\le \sum_{k=1}^{2} \mathbb{P}\left\{ \max_{(s_k, y_k), (t_k, x_k) \in R^k_{n,p}} \left|U_{s_k}(y_k) - U_{t_k}(x_k)\right| > \frac{n^{-(1-\delta)}}{2c} \right\} \qquad (3.75) \\
&\le \exp(-K_{3.4.2}\, n^{2\delta}),
\end{aligned}$$

where the last inequality follows from Lemma 3.1 and the Gaussian isoperimetric inequality [cf. Lemma 2.1 in Talagrand (1995)].

Combine (3.74) and (3.75), we have

$$\mathbb{P}\left\{ 0 \in \Delta U(T_{n,p}) \right\} \le K_{3.4.1}\, n^{-(1-\delta)d} + \exp(-K_{3.4.2}\, n^{2\delta}). \qquad (3.76)$$

Hence the same covering argument as in the proof of Theorem 3.5 yields the desired upper bound for $\dim_{\rm P}\left((\Delta U)^{-1}(0) \cap J\right)$. This proves the upper bounds in (3.73).

Now we prove the lower bound for the Hausdorff dimension of $(\Delta U)^{-1}(0) \cap J$. We will only consider the case $1 \le d < 8$ here and leave the case $8 \le d < 12$ to the interested readers.

Let $\delta > 0$ such that

$$\gamma := 4 - \frac{1}{4}(1+\delta)d > 2. \tag{3.77}$$

As in the proof of Theorem 3.5, it is sufficient to prove that there is a constant $K_{3,4,3} > 0$ such that

$$\mathbb{P}\{\dim_{\mathrm{H}}(L_{\mathrm{I},2} \cap J) \geq \gamma\} \geq K_{3,4,3}. \tag{3.78}$$

Let $\mathcal{N}_\gamma^+$ be the space of all non-negative measures on $[0,1]^4$ with finite γ-energy. Then $\mathcal{N}_\gamma^+$ is a complete metric space under the metric

$$\|\nu\|_\gamma = \int_{\mathbb{R}^4}\int_{\mathbb{R}^4} \frac{\nu(dt_1dx_1dt_2dx_2)\nu(ds_1dy_1ds_2dy_2)}{(|t_1-s_1|^2+|x_1-y_1|^2+|t_2-s_2|^2+|x_2-y_2|^2)^{\gamma/2}}; \tag{3.79}$$

see Adler (1981). We define a sequence of random positive measures ν_n on the Borel set J by

$$\begin{aligned}\nu_n(C) &= \int_C (2\pi n)^{d/2} \exp\left(-\frac{n\,|\Delta U(t_1,x_1;t_2,x_2)|^2}{2}\right) dt_1dx_1dt_2dx_2 \\ &= \int_C\int_{\mathbb{R}^d} \exp\left(-\frac{|\xi|^2}{2n} + i\langle \xi, \Delta U(t_1,x_1;t_2,x_2)\rangle\right) d\xi\, dt_1dx_1dt_2dx_2.\end{aligned} \tag{3.80}$$

It follows from Kahane (1985a) or Testard (1986) that (3.78) will follow if there are positive constants $K_{3.4.4}$ and $K_{3.4.5} > 0$ such that

$$\mathbb{E}(\|\nu_n\|) \geq K_{3.4.4}, \qquad \mathbb{E}(\|\nu_n\|^2) \leq K_{3.4.5}, \tag{3.81}$$

$$\mathbb{E}(\|\nu_n\|_\gamma) < +\infty, \tag{3.82}$$

where $\|\nu_n\| = \nu_n(J)$.

The verifications of (3.81) and (3.82) are similar to those in the proof of Theorem 3.5. By Fubini's theorem we have

$$\begin{aligned}
&\mathbb{E}(\|\nu_n\|)\\
&= \int_J \int_{\mathbb{R}^d} \exp\Big(-\frac{|\xi|^2}{2n}\Big)\, \mathbb{E}\exp\Big(i\langle \xi, \Delta U(t_1, x_1; t_2, x_2)\rangle\Big)\, d\xi\, dt_1 dx_1 dt_2 dx_2\\
&= \int_J \int_{\mathbb{R}^d} \exp\Big(-\frac{1}{2}\xi\big(n^{-1}\mathrm{I}_d\\
&\qquad\qquad + \mathrm{Cov}(\Delta U(t_1, x_1; t_2, x_2))\big)\xi'\Big)\, d\xi\, dt_1 dx_1 dt_2 dx_2\\
&= \int_J \frac{(2\pi)^{d/2}}{\sqrt{\det\big(n^{-1}\mathrm{I}_d + \mathrm{Cov}(\Delta U(t_1, x_1; t_2, x_2))\big)}}\, dt_1 dx_1 dt_2 dx_2\\
&\geq \int_J \frac{(2\pi)^{d/2}}{\sqrt{\det\big(\mathrm{I}_d + \mathrm{Cov}(\Delta U(t_1, x_1; t_2, x_2))\big)}}\, dt_1 dx_1 dt_2 dx_2 := K_{3,4,4}.
\end{aligned} \tag{3.83}$$

Denote by $\mathrm{Cov}\big(\Delta U(s_1, y_1; s_2, y_2), \Delta U(t_1, x_1; t_2, x_2)\big)$ the covariance matrix of the Gaussian vector $\big(\Delta U(s_1, y_1; s_2, y_2), \Delta U(t_1, x_1; t_2, x_2)\big)$ and let

$$\Gamma = n^{-1}\mathrm{I}_{2d} + \mathrm{Cov}\big(\Delta U(s_1, y_1; s_2, y_2), \Delta U(t_1, x_1; t_2, x_2)\big).$$

Then by the definition of J and (3.15) in Lemma 3.4, we have

$$\begin{aligned}
&\mathbb{E}\big(\|\nu_n\|^2\big) = \\
&\int_J\int_J\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \exp\Big(-\frac{1}{2}(\xi,\eta)\,\Gamma\,(\xi,\eta)'\Big)\, d\xi d\eta\, ds_1 dy_1 ds_2 dy_2 dt_1 dx_1 dt_2 dx_2 \\
&= \int_J\int_J \frac{(2\pi)^d}{\sqrt{\det\Gamma}}\, ds_1 dy_1 ds_2 dy_2 dt_1 dx_1 dt_2 dx_2 \\
&\le \int_J\int_J \frac{(2\pi)^d ds_1 dy_1 ds_2 dy_2 dt_1 dx_1 dt_2 dx_2}{\big[\det\mathrm{Cov}(\Delta U^1(s_1,y_1;s_2,y_2),\Delta U^1(t_1,x_1;t_2,x_2))\big]^{d/2}} \\
&\le K_{3.4.6}\int_J\int_J \frac{ds_1 dy_1 ds_2 dy_2 dt_1 dx_1 dt_2 dx_2}{\big[|x_1-y_1|+|x_2-y_2|+|t_1-s_1|^{1/2}+|t_2-s_2|^{1/2}\big]^{d/2}} \\
&\le K_{3.4.7}\int_J\int_J \frac{dx_1 dy_1 dx_2 dy_2 dt_1 ds_1 dt_2 ds_2}{\big[|x_1-y_1||x_2-y_2||t_1-s_1||t_2-s_2|\big]^{d/12}} := K_{3.4.5} < \infty,
\end{aligned} \tag{3.84}$$

where the last inequality follows from $d < 12$. In the above, we have also applied the inequality (3.36) with $\beta_1 = \beta_2 = 1/6$ and $\beta_3 = \beta_4 = 1/3$.

Similar to (3.84) and by the same method as in proving (3.43), we have that $\mathbb{E}(\|\nu_n\|_\gamma)$ is, up to a constant factor, bounded by

$$\begin{aligned}
&\int_J\int_J \frac{ds_1dy_1ds_2dy_2dt_1dx_1dt_2dx_2}{(|x_1-y_1|+|x_2-y_2|+|t_1-s_1|+|t_2-s_2|)^\gamma} \\
&\qquad \times \int_{\mathbb{R}^{2d}} \exp\Big(-\frac{1}{2}(\xi,\eta)\,\Gamma\,(\xi,\eta)'\Big)\,d\xi d\eta \\
&\le K_{3,4,8}\int_J\int_J \frac{1}{(|x_1-y_1|+|x_2-y_2|+|t_1-s_1|+|t_2-s_2|)^\gamma} \\
&\qquad \times \frac{dx_1dy_1dx_2dy_2dt_1ds_1dt_2ds_2}{\big(|x_1-y_1|+|x_2-y_2|+|t_1-s_1|^{1/2}+|t_2-s_2|^{1/2}\big)^{d/2}} \\
&\le K_{3,4,9}\int_0^1\int_0^1\int_0^1\int_0^1 \frac{dx_2dx_1dt_2dt_1}{(t_1^{1/2}+t_2^{1/2}+x_1+x_2)^{d/2}(t_1+t_2+x_1+x_2)^\gamma} \\
&< \infty,
\end{aligned} \tag{3.85}$$

where the last inequality follows from Lemma 3.6, $d < 8$ and the definition of γ [We need to consider three cases: $d < 4$, $d = 4$ and $4 < d < 8$, respectively]. This proves (3.82) and hence Theorem 3.6.

For $a > 0$ and $b_1 < b_2$, let $K = [a, a+h] \times [b_1, b_1+h] \times [b_2, b_2+h] \subset (0, \infty) \times \mathbb{R}^2$. We choose $h > 0$ small enough, say,

$$h < \frac{b_2 - b_1}{3} \equiv \kappa.$$

Then $|x_2 - x_1| > \kappa$ for all $x_2 \in [b_2,\, b_2 + h]$ and $x_1 \in [b_1,\, b_1 + h]$. We denote the collection of all the cubes K having the above properties by $\mathcal{K}$.

By using Lemma 3.5 and a similar argument as in the proof of Theorem 3.6, we can prove the following dimension result on $L_{\text{II},2}$. We leave the proof to the interested readers.

Theorem 3.7 *Let* $u = \{u_t(x) : \ t \geq 0, \ x \in \mathbb{R}\}$ *be a random string process in* $\mathbb{R}^d$. *If* $d \geq 8$, *then* $L_{\text{II},2} = \emptyset$ *a.s. If* $d < 8$, *then for every* $K \in \mathcal{K}$, *with positive probability,*

$$\dim_{\text{H}}\big(L_{\text{II},2} \cap K\big) = \dim_{\text{P}}\big(L_{\text{II},2} \cap K\big) = \begin{cases} 3 - \frac{1}{4}d & \text{if } 1 \leq d < 4, \\ 4 - \frac{1}{2}d & \text{if } 4 \leq d < 8. \end{cases} \tag{3.86}$$

Remark 3.4 Rosen (1984) studied k-multiple points of the Brownian sheet and multiparameter fractional Brownian motion by using their self-intersection local times. It would be interesting to establish similar results for the random string processes.

CHAPTER 4

Concluding Remarks

In this dissertation, we have studied the geometric properties of two kinds of anisotropic Gaussian random fields. We have determined the Hausdorff dimensions of their image sets and provided sufficient conditions for the image sets to be Salem sets and to have interior points for fractional Brownian sheets, and studied the fractal properties for the random string processes. This work is of importance in studying the statistical properties of these random fields.

We have been working on the following problems:

(1) Study the regularity property of the local times and self-intersection local times of fBs B^H, find the Hausdorff dimensions of the sets of multiple points and multiple times, and establish sharp local and uniform Hölder conditions for the local times and self-intersection local times.

(2) For a large class of Gaussian random fields with stationary increments, investigate the regularity property of the self-intersection local times, find the Hausdorff dimensions of the sets of multiple points and multiple times, and establish sharp local and uniform Hölder conditions for the self-intersection local times.

(3) Investigate the sectorial local non-determinism of random string processes [cf. Chapters 2 and 3] and study their sample path properties.

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