THE CLOSURE PROBLEM FOR THE BACKWARD SHIFT OPERATOR IN THE HARDY p-CLASSES $1 \leq P < \infty$

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This is to certify that the

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ABSTRACT

THE CLOSURE PROBLEM FOR THE BACKWARD SHIFT OPERATOR IN THE HARDY p-CLASSES $1 \leqq p < \infty$.

By

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The Hardy p-class of the disc is the set of all functions $f(z)$ holomorphic on the open unit disc for which

$$\sup_{0 \le r < 1} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

For p fixed, $1 \le p < \infty$, the class H^p becomes a separable Banach space under the norm

$$\|f\|_{H^p} = \sup_{0 \le r < 1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right\}^{1/p}.$$

The left shift operator U^* on H^p is defined by

$$(U^* f)(z) = \frac{f(z) - f(0)}{z}, \quad \text{for } |z| < 1.$$

A function $f(z) \in H^p$ is said to be cyclic for U^* if and only if the linear manifold spanned by $\{U^{*^n} f\}_{n=0}^{\infty}$ is dense in H^p. A function $f(z)$ is said to be non-cyclic for U^* iff f is not cyclic for U^*. The closure problem[1] for U^* is to characterize the cyclic or non-cyclic vectors for U^*.

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A function $f(z)$ holomorphic on the open unit disc is said to have a pseudocontinuation across $\{z: |z| = 1\}$ if and only if $f(z)$ has non-tangential limit $f(e^{i\theta})$ for almost all $\theta \in [0,2\pi]$, there exists a function $\tilde{f}(z)$ meromorphic on the complement of the closed unit disc with boundary values $\tilde{f}(e^{i\theta})$ for almost all $\theta \in [0,2\pi]$, and with $f(e^{i\theta}) = \tilde{f}(e^{i\theta})$ almost everywhere.

Douglas, Shapiro and Shields[2] were able to prove the following theorem:

$f(z) \in H^2$ is non-cyclic for U^* if and only if

(1) $f(z)$ has a pseudocontinuation $\tilde{f}(z)$

and

(2) $\tilde{f}(z)$ is of bounded Nevanlinna characteristic.

We have extended this theorem to the classes H^p, $1 \leq p < \infty$.

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[1] Beurling, A. On Two Problems Concerning Linear Transformations in Hilbert Space. Acta Math., 81(1949), 239-255.

[2] Douglas, R. G., Shapiro, H. S. and Shields, A. L. Cyclic Vectors and Invariant Subspaces for the Backward Shift Operator. Ann. Inst. Fourier, Grenoble, 20, 1(1970), 37-76.

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To my wife

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CHAPTER I

Introduction

We define the Hardy class $H^2(D)$ to be the class of all functions $f(z)$ holomorphic on $D = \{z: |z| < 1\}$ such that

$$\sup_{0 \le r < 1} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty.$$

It is well known [19] that f belongs to $H^2(D)$ if and only if the sequence of Taylor coefficients of f belong to ℓ^2, the class of all square summable sequences of complex numbers.

In his 1949 paper, "On Two Problems Concerning Linear Transformations in Hilbert Space" Beurling [1] was concerned with the shift operators on the Hilbert space $H^2(D)$. The shift operators are u the forward (or right) shift and u^* the backward (or left) shift operator. The action of the shift operators u and u^* on functions in $H^2(D)$ is given by

(1) $$u(f)(z) = zf(z), \text{ for } f(z) \in H^2(D),$$

and

(2) $$u^*(f)(z) = \frac{f(z)-f(0)}{z}, \text{ for } f(z) \in H^2(D).$$

The names forward and backward (or right and left) shift come from the action of u and u^* respectively on functions in $H^2(D)$ represented by their Taylor series about the origin, that is,

$$u(a_0+a_1z+a_2z^2+\cdots) = 0+a_0z+a_1z^2+a_2z^3+\cdots,$$

and

$$u^*(a_0+a_1z+a_2z^2+\cdots) = a_1+a_2z+a_3z^2+\cdots,$$

or equivalently by the action of u and u^* on the sequences of Taylor coefficients considered as elements of ℓ^2.

The problem with which we are concerned was called the closure problem by Beurling [1]. To describe the closure problem, we first define a function $f(z) \in H^2(D)$ to be cyclic for a continuous linear operator A on $H^2(D)$ if and only if the linear manifold spanned by $\{A^n f\}_{n=0}^{\infty}$ is dense in $H^2(D)$. A function $f(z) \in H^2(D)$ is said to be non-cyclic for an operator A on $H^2(D)$ if and only if $f(z)$ is not cyclic for A. The closure problem for an operator A on $H^2(D)$ is to characterize the cyclic (or non-cyclic) functions for A.

Beurling [1] was able to obtain an elegant characterization of the cyclic functions for the right shift u as well as a description of all of the closed invariant subspaces for u, namely,

(3) a closed subspace S of $H^2(D)$ is invariant under u if and only if there exists an inner function $\varphi(z)$ with $S = \varphi H^2(D)$,

and

(4) a function f in $H^2(D)$ is cyclic for u if and only if f is an outer function,

where an inner function φ is a function holomorphic on the unit disc with boundary values $|\varphi(e^{i\theta})| = 1$ almost everywhere on the circle $\{z: |z| = 1\}$, while an outer function is a function $F(z)$ holomorphic on the unit disc, of the form

$$F(z) = \lambda \exp\left[\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\, k(t)\,dt\right],$$

where $k(t)$ is real valued and integrable on $[0, 2\pi]$, and $|\lambda| = 1$.

Since on the Hilbert space $H^2(D)$, u^* is the adjoint operator of u, Beurling's characterization of the invariant subspaces and cyclic functions for u also gives a characterization of the invariant subspaces for u^* [3], namely, the invariant subspaces for u^* are the orthogonal complements of the subspaces $\varphi H^2(D)$, with φ an inner function, while f is non-cyclic for u^* if and only if f lies in a subspace of the form $\{\varphi H^2(D)\}^{\perp}$ for some non-constant inner function φ.

Douglas, Shapiro and Shields [3] were able to give a different characterization of the non-cyclic functions which permits one to more readily identify classes of cyclic and non-cyclic functions for the left shift on $H^2(D)$.

One can speak of the shift operators in spaces other than $H^2(D)$. Indeed, the literature on these operators is extensive, see for example, Wells and Kellog [25], Helson and Lowdenslager [11], Gamelin [8] and [9], deLeeuw and Rudin [2], Hoffman [13], Hasumi and Srinivasan [10], and Srinivasan and Wong [20].

The spaces with which we are concerned are the Hardy p-classes of the disc. (We will formally state all definitions and theorems in Chapters II and III.) On the Hardy p-classes we can define the shift operators u and u^* as in (1) and (2) by merely replacing the condition $f(z) \in H^2(D)$ with $f(z) \in H^p$. The operators so defined are continuous linear operators on $H^p(D)$ (p is considered to be fixed).

De Leeuw and Rudin [2] have shown that Beurling's characterization of the cyclic vectors for u is the same in $H^1(D)$ as in $H^2(D)$, namely, a function $f(z) \in H^1(D)$ is cyclic for the right shift u (either the norm topology or the weak* topology on $H^1(D)$) if and only if $f(z)$ is an outer function.

We will extend the theorem of Douglas, Shapiro and Shields [3] characterizing the non-cyclic functions for the left shift u^* to the Hardy p-classes $H^p(D)$, $1 \le p < \infty$.

In Chapter II we will present some of the well known results on the structure of the Hardy p-classes. We will state these theorems in forms which we can readily use in the proof of the characterization theorem.

In Chapter III we prove the main theorem characterizing the non-cyclic functions for the left shift and give a brief discussion of applications of this theorem to certain classes of functions.

CHAPTER II

Classical H^p Spaces

1. Basic Definitions and Theorems

The purpose of this chapter is to present some background material on the Hardy p-classes and related topics for use in the subsequent chapters. We begin with some notations and definitions.

Definition 2.1:

The following notation will be used throughout this thesis:

C = set of all complex numbers

$\not{\mathcal{C}} = C \cup \{\infty\}$

$D = \{z \in C: |z| < 1\}$

$T = \{z \in C: |z| = 1\}$

$D_e = \{z \in C: |z| > 1\} \cup \{\infty\}$.

Let $u(z)$ be a real valued continuous function on D, possessing continuous second order partial derivatives with respect to x and y where $z = x + iy$, x and y real numbers. $u(z)$ is said to be harmonic on D if and only if $u(z)$ satisfies Laplace's equation on D,

$$u_{xx}(z) + u_{yy}(z) = 0$$

in Cartesian coordinates, or

$$u_{rr}(re^{i\theta})+\frac{1}{r}u_r(re^{i\theta}) + \frac{1}{r^2}\, u_{\theta\theta}(re^{i\theta}) = 0$$

in polar coordinates.

A complex valued function $u(z)$ continuous on D is said to be harmonic on D if and only if both the real and imaginary parts of $u(z)$ are harmonic on D.

For $0 < p \leqq \infty$ and for $u(re^{i\theta})$ harmonic on D, we define the L^p means of u on $|z| = r$ by

$$m_p(r,u) = \begin{cases} \left\{\frac{1}{2\pi}\int_0^{2\pi} |u(re^{i\theta})|^p d\theta\right\}^{1/p} & \text{if } 0 < p < \infty \\ \sup\limits_{0\leq\theta\leq 2\pi} |u(re^{i\theta})| & \text{if } p = \infty \end{cases}$$

If $f(z)$ is holomorphic, and hence harmonic, on D we adopt the notation

$$M_p(r,f) = m_p(r,f).$$

For $0 < p \leq \infty$, we define the Hardy p-classes:

$h^p(D) = \{u(z)\colon u(z)$ is harmonic on D and $\sup\limits_{0\leq r<1} m_p(r,u) < \infty\}$

$H^p(D) = \{f(z)\colon f(z)$ is holomorphic on D and $\sup\limits_{0\leq r<1} M_p(r,f) < \infty\}$

$H^p(D_e) = \{f(z)\colon f(z)$ is holomorphic on D_e and if $g(z) = f(\frac{1}{z})$ for $|z| < 1$, then $g(z) \in H^p(D)\}$

where we have adopted the convention that $\frac{1}{0} = \infty$.

We define the Nevanlinna classes:

$N(D) = \{f(z)\colon f(z)$ is meromorphic on D and $\sup\limits_{0\leq r<1} \int_0^{2\pi} \ln^+ |f(re^{i\theta})|\, d\theta < \infty\}$

$N(D_e) = \{f(z)\colon f(z)$ is meromorphic on D_e and if $g(z) = f(\frac{1}{z})$, then $g(z) \in N(D)\}$,

where $\ln^+|t| = \max\{0, \ln|t|\}$.

We will also adopt the notations $h^p = h^p(D)$, $H^p = H^p(D)$ and $N = N(D)$.

We point out that it follows from the definitions that if $p < q$, then $h^q \subset h^p$ and $H^q \subset H^p$. We also note that $H^1 \subset h^1$.

For $0 < p < \infty$ the inequality $\ln|t| < |t|^p$ immediately shows that $H^p \subset N$ while $f(z) \in H^\infty$ implies that $|f(z)| \leqq \sup_{0 \le r < 1} M_\infty(r,f) < \infty$ for each $z \in D$ and hence $\frac{1}{2\pi}\int_0^{2\pi} \ln^+|f(re^{i\theta})|d\theta \leqq \sup_{0 \le r < 1} M_\infty(r,f) < \infty$. Thus $H^p \subset N$ for $0 < p \leqq \infty$.

We will be working with functions defined on the boundary of the unit disc and thus find it convenient to denote such functions in the form $f(e^{i\theta})$ for $0 \le \theta \leqq 2\pi$ rather than a notation of the form $F(\theta)$ for $0 \le \theta \leqq 2\pi$. We shall always assume that functions of one variable θ, $0 \le \theta \le 2\pi$ are periodic of period 2π and when we use the terms measure or almost everywhere we shall mean one dimensional Lebesgue measure.

For $f(e^{i\theta})$ a measurable function of θ, $0 \le \theta \leqq 2\pi$ we define the L^p "norm" of $f(e^{i\theta})$ by

$$\|f(e^{i\theta})\|_{L^p} = \begin{cases} \underset{\theta\in[0,2\pi]}{\text{ess sup}} \ |f(e^{i\theta})| & \text{if } p = \infty \\ \left[\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^p d\theta\right]^{1/p} & \text{if } 0 < p < \infty. \end{cases}$$

We say that $f(e^{i\theta}) \in L^p[0,2\pi]$ if and only if $\|f(e^{i\theta})\|_{L^p} < \infty$. For $f(re^{i\theta}) \in H^p$, we define the H^p "norm" of f by

$$\|f\|_{H^p} = \sup_{0\le r<1} M_p(r,f).$$

For $u(re^{i\theta}) \in h^p$, we define the h^p "norm" of u by

$$\|u\|_{h^p} = \sup_{0\le r<1} m_p(r,u).$$

We note that for $1 \le p \le \infty$, the H^p, L^p and h^p "norms" are norms in the usual sense [6] and that the spaces H^p, L^p and h^p are Banach spaces [6]. For $0 < p < 1$, the H^p, L^p and h^p "norms" are not norms since the triangle inequality fails. We do know however, that for $0 < p < 1$, these spaces form translation invariant, complete metric spaces [6] or Frechet spaces [4] under the metrics $\rho(f,g) = \|f-g\|_{H^p}^p$, $\rho(f,g) = \|f-g\|_{L^p}^p$ and $\rho(f,g) = \|f-g\|_{h^p}^p$ respectively.

We observe that if $f(re^{i\theta}) \in H^p$ with r fixed, $0<r<1$, then $f_r(e^{i\theta}) \equiv f(re^{i\theta})$ is a function of θ, $f_r(e^{i\theta}) \in L^p[0,2\pi]$ and $\|f_r(e^{i\theta})\|_{L^p} = M_p(r,f)$.

We have defined h^p as a class of complex valued functions, while some authors restrict the class h^p to be only real valued functions.

Definition 2.2:

Let $f(z)$ be defined and single valued on D. We say that $f(z)$ has non-tangential boundary value $f(e^{i\theta})$ at $e^{i\theta} \in T$ if and only if for any fixed α, $0 < \alpha < \pi$,

$$\lim_{n\to\infty} f(z_n) = f(e^{i\theta})$$

for any sequence $\{z_n\}_{n=0}^{\infty} \subset S(\alpha, e^{i\theta})$ with $\lim_{n\to\infty} z_n = e^{i\theta}$ where $S(\alpha, e^{i\theta})$ is the domain common to the unit disc and the sector with vertex at $e^{i\theta}$, of angle α, symmetric with respect to the radius from the origin to $e^{i\theta}$.

For $f(z)$ defined and single valued on D_e, we say that $f(z)$ has non-tangential boundary value $f(e^{i\theta})$ at $e^{i\theta} \in T$ if and only if $g(z) = f(1/z)$ has non-tangential boundary value $f(e^{i\theta})$ at $e^{-i\theta}$.

We first state the Theorem of Lusin and Priwalow.

Theorem 2.1:

[14, p. 212] Let $f_1(z)$ and $f_2(z)$ be meromorphic on D (or D_e) and both possess equal non-tangential boundary values on the same set $E \subset T$ with $m(E) > 0$. Then $f_1(z) = f_2(z)$ for each $z \in D$ (or D_e).

Motivated by this theorem we identify any two boundary functions which are equal for almost all $e^{i\theta} \in T$.

It is well known that every function in H^p, $p > 0$ and every function in h^q, $q \geq 1$ possess unique non-tangential boundary values for almost all $e^{i\theta} \in T$ [14], and that given the boundary function we can reconstruct the original function. For completeness we shall develop these facts here. For the most part we shall follow the development of Duren [5], Hoffman [13] and Priwalow [14].

The most natural place to begin this task seems to be with harmonic functions in classes h^q, $q \geq 1$ and then to extend these results to the H^p classes, even for $0 < p < 1$.

We begin with a very important harmonic function, the Poisson kernel, $P(r,\theta)$.

Definition 2.3:

$$P(r,\theta) = \frac{1}{2\pi}\left(\frac{1-r^2}{1-2r\cos\theta + r^2}\right), \quad 0 \leq r < 1, \quad 0 \leq \theta \leq 2\pi.$$

We list some of the more important and well known properties of the Poisson kernel. See, for example, Hille [12] or Hoffman [13].

(i) $P(r,\theta) = \sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta}$, for $0 \leq r < 1$ and $0 \leq \theta \leq 2\pi$ with convergence of the series absolute on D, and uniform on compact subsets of D;

(ii) $P(r,\theta) > 0$, for $0 \leq r < 1$ and any $\theta \in [0,2\pi]$;

(iii) $\int_0^{2\pi} P(r,\theta)d\theta = 1$, for any $r \in [0,1]$;

(iv) For any real number ϵ, $0 < \epsilon < \pi$, $\lim\limits_{r\to 1^-} \sup\limits_{|\theta|>\epsilon} |P(r,\theta)| = 0$.

Perhaps the most important property of the Poisson kernel is furnished by the following theorem which says, among other things, that the Poisson kernel can be used to produce a harmonic function on the disc, given an integrable function defined on the unit circle.

<u>Definition 2.4</u>:

If $f(e^{i\theta})$ is a Lebesgue integrable function of θ on $[0,2\pi]$ we define the Poisson integral of f to be

$$\int_0^{2\pi} f(e^{it})\ P(r,\theta-t)dt.$$

If $\mu(t)$ is a complex Baire measure on $[0,2\pi]$, the Poisson integral of μ is

$$\int_0^{2\pi} P(r,\theta-t)d\mu(t).$$

<u>Theorem 2.2</u>:

[13, pp. 33-34] Let f be a complex-valued harmonic function on the open unit disc, and write

$$f_r(e^{i\theta}) = f(re^{i\theta}),$$

then

(i) If $1 < p \leq \infty$, then f is the Poisson integral of an L^p function on T if and only if $f(re^{i\theta}) \in h^p$.

(ii) f is the Poisson integral of an integrable function φ on T if and only if the functions f_r converge to φ, as $r \to 1^-$, in the L^1 norm.

(iii) f is the Poisson integral of a continuous function on the unit circle if and only if the functions f_r converge uniformly.

(iv) f is the Poisson integral of a finite complex Baire measure on the unit circle if and only if $f \in h^1$.

(v) f is the Poisson integral of a finite positive Baire measure if and only if f is non-negative.

Theorem 2.3:

(Fatou's Theorem) [13, p. 34] Let μ be a finite complex Baire measure on the unit circle and let f be the harmonic function on D defined by

$$f(re^{i\theta}) = \int_0^{2\pi} P(r,\theta-t)\,d\mu(t).$$

Let θ_0 be any point where μ is differentiable with respect to Lebesgue measure. Then

$$\lim_{r\to 1^-} f(re^{i\theta_0}) = \mu'(\theta_0),$$

and in fact $\lim f(re^{i\theta}) = \mu'(\theta_0)$ as the point $z = re^{i\theta}$ approaches $e^{i\theta_0}$ along any path in the open unit disc which is not tangent to the unit circle.

There is much contained in these theorems. First of all we wish to point out that Theorem 2.2 (iv) says that any function $u(z) \in h^1$ can be represented as the Poisson integral of a finite Baire measure μ on the unit circle T.

Theorem 2.3 says that $u(z) = u(re^{i\theta}) \to \mu'(\theta_0)$ as $z = re^{i\theta} \to e^{i\theta_0}$ non-tangentially for almost all $\theta_0 \in [0,2\pi]$. When no confusion can result, we will denote the boundary values of $u(re^{i\theta}) \in h^1$ by the function $u(e^{i\theta})$ where it is to be understood that $u(e^{i\theta})$ is unique up to sets of Lebesgue measure zero on $[0,2\pi]$.

We also note that since $h^p \subset h^1$ for $p \geq 1$, every function $u(z) \in h^p$, $p \geq 1$ has non-tangential boundary values almost everywhere on T. The same is true for H^1 as $H^1 \subset h^1$ and thus for H^p, $p \geq 1$ since $H^p \subset H^1$ whenever $p \geq 1$. Also it is a corollary to the proof of Theorems 2.2 and 2.3 that for $1 < p \leq \infty$ if $f(z) \in h^p$ with boundary function $f(e^{i\theta})$, then $f(z)$ is the Poisson integral of $f(e^{i\theta})$ [13]. This is not true in h^1, but it is still true for H^1. [26].

We shall need some results for the classes H^p, $0 < p < 1$ which are not true for the corresponding h^p classes, but one of the classical methods of deriving these results is to obtain them from the results true for H^p or h^p, $1 < p \leq \infty$. To this end we state the Nevanlinna Theorem.

<u>Theorem 2.4</u>:

[5], [15] A function $f(z)$ holomorphic on D is in class N if and only if $f(z)$ is the quotient of two functions from H^∞. (We do not allow the function in the denominator of the quotient to be the zero function.)

Since every function in H^{∞} has non-tangential boundary values almost everywhere on T, Theorem 2.4 suggests the same may be true for all of class N. However, the possibility that the numerator and denominator functions from Theorem 2.4 could both have boundary value zero creates a problem. Fortunately the class N is a class of "well behaved" functions as the next theorem shows. First we state a Lemma.

<u>Lemma 2.5</u>:

(Jensen's Formula) [12, II, p. 189] If $\varphi(z)$ is holomorphic on D, then $\int_0^{2\pi} \ln|\varphi(re^{i\theta})|d\theta$ increases monotonically with r, $0 \le r < 1$ and

$$\frac{1}{2\pi}\int_0^{2\pi} \ln\,|\varphi(re^{i\theta})|d\theta = \ln\,|\varphi(0)| + Z(\varphi, r),$$

where $Z(\varphi,r) = \sum\limits_{r_n \le r} \ln \frac{r}{r_n}$, with $\{r_n\}_{n=0}^{\infty}$ the sequence of modulae of the zeros of φ repeated according to multiplicity.

Now we can proceed.

<u>Theorem 2.6</u>:

[5], [14] If $f(z) \in N$, then $f(z)$ has non-tangential limit $f(e^{i\theta})$ almost everywhere. Furthermore, $\ln|f(e^{i\theta})|$ is integrable unless $f(z) \equiv 0$ and $f(z) \in H^p$ for any $p > 0$ implies that $f(e^{i\theta}) \in L^p$.

Proof:

Because of the importance of this theorem we include the proof as found in Duren [5].

Let $f(z) \in N$ be given. Assume $f(z) \not\equiv 0$. By Theorem 2.4 there exist two functions $\alpha(z), \beta(z) \in H^\infty$ such that $f(z) = \frac{\alpha(z)}{\beta(z)}$, for each $z \in D$.

Without loss of generality we assume that $\|\alpha\|_{H^\infty} \leq 1$ and that $\|\beta\|_{H^\infty} \leq 1$ since if $\|\alpha\|_{H^\infty} > 1$ or $\|\beta\|_{H^\infty} > 1$ we could replace $\alpha(z)$ and $\beta(z)$ respectively by

$$\alpha'(z) = \frac{\alpha(z)}{\max\{\|\alpha\|_{H^\infty}, \|\beta\|_{H^\infty}\}}$$

and

$$\beta'(z) = \frac{\beta(z)}{\max\{\|\alpha\|_{H^\infty}, \|\beta\|_{H^\infty}\}}$$

which are well defined since $\|\beta\|_{H^\infty} \neq 0$ and which satisfy

(i) $\|\alpha'\|_{H^\infty} \leq 1$ and $\|\beta'\|_{H^\infty} \leq 1$,

and

(ii) $f(z) = \frac{\alpha'(z)}{\beta'(z)}$.

Now by Theorem 2.3 $\alpha(z)$ and $\beta(z)$ have non-tangential limits $\alpha(e^{i\theta})$ and $\beta(e^{i\theta})$ respectively almost everywhere.

Noting that $-\ln|\alpha(re^{i\theta})| \geq 0$ and that

$$\lim_{r\to 1^-}[-\ln|\alpha(re^{i\theta})|] = -\ln|\alpha(e^{i\theta})| \text{ a.e.}$$

we apply Fatou's lemma [15] obtaining

$$\int_0^{2\pi} -\ln|\alpha(e^{i\theta})|d\theta \leq \liminf_{r\to 1^-} \left\{- \int_0^{2\pi} \ln|\alpha(re^{i\theta})|d\theta\right\}.$$

By Lemma 2.5 (Jensen's Theorem $\int_0^{2\pi} \ln|\alpha(re^{i\theta})|d\theta$ increases with r and hence $-\int_0^{2\pi} \ln|\alpha(re^{i\theta})|d\theta$ is a non-negative decreasing function of r. Thus

$$\liminf_{r\to 1^-} \int_0^{2\pi} - \ln|\alpha(re^{i\theta})|d\theta = \lim_{r\to 1^-} \int_0^{2\pi} - \ln|\alpha(re^{i\theta})|d\theta$$

is finite. Hence

$$\int_0^{2\pi} \Big|\ln|\alpha(e^{i\theta})|\Big|d\theta < \infty,$$

that is,

$$\ln|\alpha(e^{i\theta})| \in L^1[0,2\pi].$$

Similarly $\ln|\beta(e^{i\theta})| \in L^1[0,2\pi]$. In particular, neither $\alpha(e^{i\theta})$ nor $\beta(e^{i\theta})$ can vanish on a set of positive measure since $\ln|\alpha(e^{i\theta})| \in L^1$ and $\ln|\beta(e^{i\theta})| \in L^1$. Thus the non-tangential limit $f(e^{i\theta})$ exists almost everywhere, and since $\ln|f(e^{i\theta})| = \ln|\alpha(e^{i\theta})| - \ln|\beta(e^{i\theta})|$ a.e. we have $\ln|f(e^{i\theta})| \in L^1$.

Thus, in particular, if $0 < p \leq \infty$ and if $f \in H^p$, $\lim_{r\to 1^-} f(re^{i\theta}) = f(e^{i\theta})$ exists a.e. and Fatou's lemma [15] gives

$$\int_0^{2\pi} |f(e^{i\theta})|^p d\theta \leq \liminf_{r\to 1^-} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty$$

since $f \in H^p$.

Now there are times when zeros of an H^p function cause difficulties in proofs, so we state some theorems which enable us to in some sense factor out zeros.

Theorem 2.7:

[5] If $f(z) \not\equiv 0$ is holomorphic on D with zeros $a_1, a_2, \cdots$ repeated according to multiplicity, then

$$\sup_{0 \leq r < 1} \int_0^{2\pi} \ln|f(re^{i\theta})|d\theta < \infty \text{ if and only if } \sum_{n=1}^{\infty} (1-|a_n|) < \infty.$$

Proof:

The proof of this theorem follows from Jensen's Theorem (Lemma 2.4).

Corollary 2.8:

Let $f(z) \in N$, $f(z) \not\equiv 0$, then if $\{a_n\}$ are the zeros of f, $\Sigma(1-|a_n|) < \infty$.

Theorem 2.9:

[13] If $\{a_n\}_{n=1}^{\infty}$ is a sequence of non-zero complex numbers with no limit point in D and if $\sum_{n=1}^{\infty} (1-|a_n|) < \infty$, then the product

$$B(z) = \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{1-\bar{a}_n z} .$$

converges uniformly in each disc $|z| \leq R < 1$. Each a_n is a zero of $B(z)$ with multiplicity equal to the number of times it occurs in the sequence $\{a_n\}$. $B(z)$ has no other zeros in D. $|B(z)| < 1$ for each $z \in D$ and $|B(e^{i\theta})| = 1$ a.e.

Definition 2.5:

A function of the form

$$e^{i\gamma} z^k \prod \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \bar{a}_n z} \quad ,$$

where γ is a fixed real number, k is a non-negative integer and $\{a_n\}$ is a sequence (finite or infinite) of complex numbers in D satisfying $\Sigma\,(1 - |a_n|) < \infty$, is called a Blaschke product.

Theorem 2.10:

[5] Any function $f(z) \in H^p$, $0 < p \leq \infty$ can be factored in the form $f(z) = B(z)g(z)$ where $B(z)$ is a Blaschke product and $g(z)$ is an H^p function which has no zeros in D.

Theorem 2.11:

[16] If $f(z) \in H^p$, $0 < p < \infty$, then

$$\lim_{r \to 1^-} \int_0^{2\pi} |f(re^{i\theta}) - f(e^{i\theta})|^p \, d\theta = 0,$$

and

$$\lim_{r \to 1^-} M_p(r, f) = \left[\frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^p d\theta\right]^{1/p},$$

or

$$\|f_r(e^{i\theta}) - f(e^{i\theta})\|_{L^p} \to 0, \quad \text{as} \quad r\to 1^-,$$

where

$$f_r(e^{i\theta}) = f(re^{i\theta}),$$

and

$$\|f(z)\|_{H^p} = \|f(e^{i\theta})\|_{L^p}.$$

We comment that Theorem 2.11 is a stronger result for H^p than Theorem 2.2 (i). For $1 < p < \infty$ the conclusions are the same. For $0 < p < 1$, Theorem 2.11 tells us that $f_r(e^{i\theta}) = f(re^{i\theta}) \in H^p$ converges in L^p norm to the boundary function $f(e^{i\theta})$, while for $0 < p < \infty$ an h^p function need not have non-tangential boundary values. We now wish to characterize the class of all boundary functions of H^p.

<u>Definition 2.6</u>:

Let $\mathfrak{H}^p(D) = \mathfrak{H}^p$ denote the set of all boundary functions $f(e^{i\theta})$ of functions $f(z) \in H^p$, $0 < p \leq \infty$.

We note that by Theorem 2.6 $\mathfrak{H}^p \subset L^p$, $0 < p \leq \infty$. Since the H^p classes are linear spaces, it is clear that $\mathfrak{H}^p$ is a linear manifold in L^p. Also $\mathfrak{H}^p$ must contain all polynomials in $e^{in\theta}$, $n \geq 0$, that is, functions of the form $\sum_{k=0}^{N} a_k e^{ik\theta}$ since each H^p class contains the polynomials $\sum_{k=0}^{N} a_k z^k$.

Lemma 2.12:

[5] If $f(z) \in H^p$, $0 < p < \infty$, then

$$|f(z)| \leq \|f\|_{H^p} \left(\frac{1}{1-|z|}\right)^{1/p}$$

Now a characterization of $\mathfrak{H}^p$ will be given.

Theorem 2.13:

[5] Let p be fixed, $0 < p < \infty$. $\mathfrak{H}^p$ is the L^p closure of the set of polynomials in $e^{i\theta}$.

Proof:

We show first that $\mathfrak{H}^p$ is closed in L^p. Let $\{f_n(e^{i\theta})\} \subset \mathfrak{H}^p$. Assume $\|f_n(e^{i\theta}) - \varphi(e^{i\theta})\|_{L^p} \to 0$ as $n \to \infty$ for some $\varphi \in L^p$. We must show $\varphi \in \mathfrak{H}^p$.

Now $\|f_n(e^{i\theta})\|_{L^p} \leq 2^p\|f_n(e^{i\theta}) - \varphi(e^{i\theta})\|_{L^p} + 2^p\|\varphi\|_{L^p}$

and hence there exists a constant M such that

$$\|f_n(e^{i\theta})\|_{L^p} \leq M \quad \text{for} \quad n = 0,1,2,\cdots$$

Now fix R, $0 < R < 1$. By lemma 2.12, if $|z| < R$,

$$|f_n(z)| \leq M\left(\frac{1}{1-R}\right)^{1/p}$$

where $f_n(e^{i\theta})$ is the boundary function of $f_n(z)$. Thus the sequence $\{f_n(z)\}$ is uniformly bounded on $\{z: |z| \leq R < 1\}$. Consequently $\{f_n(z)\}$ forms a normal family of functions [12, II, p. 242]. Hence there exists a subsequence $\{f_{n_k}(z)\}$

which converges uniformly to a holomorphic function $f(z)$ on compact subsets of D. By Fatou's lemma [15] $f(z) \in H^p$. It remains to be shown that $\varphi(e^{i\theta})$ is the boundary function of $f(z)$.

Let $\epsilon > 0$ be given. Choose N so that if $m, n, \geq N$, $\|f_n(z) - f_m(z)\| < \epsilon$. Let $r \in (0,1)$ be fixed and $m \geq N$ be fixed.

$$\int_0^{2\pi} |f(re^{i\theta}) - f_m(re^{i\theta})|^p d\theta = \lim_{k\to\infty} \int_0^{2\pi} |f_{n_k}(re^{i\theta}) - f_m(re^{i\theta})|^p d\theta$$

since $f_{n_k}(z)$ converges uniformly to $f(z)$ on $\{z:|z| \leq r\}$. Now

$$\lim_{k\to\infty} \int_0^{2\pi} |f_{n_k}(re^{i\theta}) - f_m(re^{i\theta})|^p d\theta \leq \overline{\lim_{k\to\infty}} \int_0^{2\pi} |f_k(re^{i\theta}) - f_m(re^{i\theta})|^p d\theta.$$

But since $m \geq N$,

$$\overline{\lim_{k\to\infty}} \int_0^{2\pi} |f_k(re^{i\theta}) - f_m(re^{i\theta})|^p d\theta \leq \epsilon^p \cdot 2\pi$$

Hence

$$\int_0^{2\pi} |f(re^{i\theta}) - f_m|(re^{i\theta})|^p d\theta < \epsilon^p \cdot 2\pi$$

Now by Fatou's lemma [15], letting $r \to 1^-$ we obtain

$$\int_0^{2\pi} |f(e^{i\theta}) - f_m(e^{i\theta})|^p d\theta \leq \epsilon^p \cdot 2\pi$$

or since $\epsilon > 0$ was arbitrary,

$$\lim_{m\to\infty} \|f(e^{i\theta}) - f_m(e^{i\theta})\|_{L^p} \to 0$$

Hence

$$f(e^{i\theta}) = \varphi(e^{i\theta}) \quad \text{a.e.}$$

Thus $\mathfrak{H}^p$ is closed in L^p.

It remains to be shown that the polynomials are dense in $\mathfrak{H}^p$. Let $f(z) \in H^p$ and $\epsilon > 0$ be given. Choose an R, $0 < R < 1$ so that $\|f(Re^{i\theta}) - f(e^{i\theta})\|_{L^p} < \epsilon/4$. We can pick such an R by Theorem 2.11. Let $s_n(z)$ denote the $n^{\underline{th}}$ partial sum of the Taylor series of f about the origin. On the compact set $\{z: |z| \le R < 1\}$, $s_n(z)$ converges uniformly to $f(z)$ so pick N such that $n \geqq N$ implies that

$$\sup_{|z| \le R} |f(z) - s_n(z)| < \epsilon/4 .$$

Define $p(e^{i\theta})$, a polynomial in $e^{i\theta}$ by $p(e^{i\theta}) = s_n(Re^{i\theta})$. Now

$$\begin{aligned} \|p(e^{i\theta}) - f(e^{i\theta})\|_{L^p} &\leqq \|p(e^{i\theta}) - f(Re^{i\theta}) + f(Re^{i\theta}) - f(e^{i\theta})\|_{L^p} \\ &\le 2\left[\|p(e^{i\theta}) - f(Re^{i\theta})\|_{L^p} + \|f(Re^{i\theta}) - f(e^{i\theta})\|_{L^p}\right] \\ &< 2 \cdot \epsilon/4 + 2 \cdot \epsilon/4 \\ &= \epsilon . \end{aligned}$$

Since $\epsilon > 0$ was arbitrary we have shown that the polynomials in $e^{i\theta}$ are dense in $\mathfrak{H}^p$.

We comment that Theorems 2.11 and 2.13 enable us to define a linear isometry between H^p and $\mathfrak{H}^p$, namely, the correspondence between a boundary function in $\mathfrak{H}^p$ and the holomorphic function in H^p, $0 < p < \infty$.

2. The J Operator

In Chapter III we will need to know something about the relationship between functions holomorphic inside the unit disc and functions holomorphic on the complement (with respect to the Riemann Sphere) of the closed unit disc. The method which we will employ was chosen primarily to simplify notation.

Definition 2.7:

Let $f(z)$ be holomorphic on D. We define

$$(Jf)(z) = \overline{f(1/\bar{z})} \text{ for } 1 < |z| \leq \infty,$$

where we adopt the convention that $\frac{1}{\infty} = 0$.

If $f(z)$ is holomorphic on D_e, we define

$$(J^{-1}f)(z) = \overline{f(1/\bar{z})} \text{ for } 0 \leq |z| < 1,$$

with the convention $\frac{1}{0} = \infty$.

We point out that if $f(z)$ is holomorphic in D with Taylor series about the origin $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then

$$(Jf)(z) = \sum_{n=0}^{\infty} \bar{a}_n z^{-n}.$$

If $g(z)$ is holomorphic on D_e with Laurent series $g(z) = \sum_{n=0}^{\infty} a_n z^{-n}$, then

$$(J^{-1}g)(z) = \sum_{n=0}^{\infty} \bar{a}_n z^n.$$

We also observe that if $f(z)$ is holomorphic on D and if

$f(z)$ has non-tangential boundary values $f(e^{i\theta})$ at $e^{i\theta} \in T$, then $(Jf)(z)$ has non-tangential boundary value $\overline{f(e^{i\theta})} = (Jf)(e^{i\theta})$ at $e^{i\theta} \in T$.

Definition 2.8:

Let p be fixed, $0 < p \leq \infty$. We define the classes

$$H^p(D_e) = \{Jf: f \in H^p(D)\}$$

$$N(D_e) = \{Jf: f \in N(D)\}.$$

We can now prove results for $H^p(D_\epsilon)$ corresponding to most of the theorems about $H^p(D)$.

Theorem 2.3′:

A function $f(z)$ holomorphic on D_e is in class $N(D_e)$ if and only if $f(z)$ is the quotient of two functions from $H^\infty(D_e)$.

Proof:

Let $f(z) \in N(D_e)$, then there exists a function $h(z) \in N(D)$ such that $(Jh)(z) = f(z)$ for each $z \in D_e$. By Theorem 2.3, there exists two functions $\alpha(z), \beta(z) \in H^\infty(D)$ such that

$$h(z) = \frac{\alpha(z)}{\beta(z)}.$$

Thus

$$(Jh)(z) = \overline{h(1/\bar{z})} = \left\lfloor \overline{\frac{\alpha(1/\bar{z})}{\beta(1/\bar{z})}} \right\rfloor$$

$$= \frac{\overline{\alpha(1/\bar{z})}}{\overline{\beta(1/\bar{z})}} = \frac{(J\alpha)(z)}{(J\beta)(z)} .$$

Theorem 2.6':

If $f(z) \in N(D_e)$, then $f(z)$ has non-tangential boundary values $f(e^{i\theta})$ almost everywhere. Furthermore, $\ln|f(e^{i\theta})|$ is integrable unless $f(z) = 0$ and $f(z) \in H^p(D_e)$ for any $p > 0$ implies that $f(e^{i\theta}) \in L^p$.

Proof:

Let $f(z) \in N(D_e)$. Let $h(z) \in N(D)$ be such that $(Jh)(z) = f(z)$ for any z with $|z| > 1$. Let $h(e^{i\theta})$ be the boundary function of $h(z)$. Then $f(z)$ has non-tangential boundary values $f(e^{i\theta}) = \overline{h(e^{i\theta})}$ a.e. Also by Theorem 2.5, $\ln|h(e^{i\theta})|$ is integrable and hence $\ln|\overline{h(e^{i\theta})}| = \ln|f(e^{i\theta})|$ is integrable unless $h(z) \equiv 0$, that is, $f(z) \equiv 0$. Furthermore, $f(z) \in H^p(D_e)$ implies that $h(z) \in H^p(D)$. By Theorem 2.5, $h(z) \in H^p(D)$ implies that $h(e^{i\theta}) = f(e^{i\theta}) \in L^p$.

Definition 2.9:

Let p be fixed, $0 < p \leq \infty$. We define the class $\mathfrak{H}^p(D_e)$ to be the class of all boundary functions of functions in class $H^p(D_e)$.

Theorem 2.12′:

Let p be fixed, $0 < p < \infty$. $\mathfrak{H}^p(D_e)$ is the L^p closure of polynomials in $e^{-i\theta}$. (Polynomials in $e^{-i\theta}$ means linear combinations of non-negative integer powers of $e^{-i\theta}$).

Proof:

By Theorem 2.5′ we have $\mathfrak{H}^p(D_e) \subset L^p[0,2\pi]$. Since each polynomial in powers of $e^{-i\theta}$ is in $\mathfrak{H}^p(D_e)$, it remains to be shown that $\mathfrak{H}^p(D_e)$ is closed in L^p and that the polynomials in $e^{-i\theta}$ are dense in $\mathfrak{H}^p(D_e)$. Now from the definition of the J operator and the discussion following that definition, $\mathfrak{H}^p(D_e)$ contains exactly those functions which are complex conjugates of functions in $\mathfrak{H}^p(D)$. By Theorem 2.12 $\mathfrak{H}^p(D)$ is closed in $L^p[0,2\pi]$ and hence $\mathfrak{H}^p(D_e)$ is also closed in L^p. Since complex conjugates of polynomials in $e^{i\theta}$ are polynomials in $e^{-i\theta}$, by Theorem 2.12 we can conclude that polynomials in $e^{-i\theta}$ are dense in $\mathfrak{H}^p(D_e)$.

3. Conjugate Harmonic Functions

If $u(z)$ is harmonic on D, we say that a function $v(z)$ harmonic on D is a harmonic conjugate of $u(z)$ if and only if $u(z) + iv(z)$ is holomorphic on D. Any given $u(z)$ harmonic on D has many harmonic conjugates all differing by constants.

<u>Definition 2.10</u>:

If $u(z)$ is harmonic on D we say that $v(z)$ is the normalized harmonic conjugate of $u(z)$ if and only if $v(z)$ is harmonic on D, $v(0) = 0$ and $u(z) + iv(z)$ is holomorphic on D.

The problem with which we are concerned is the following: Given $u(z) \in h^p$ can we claim that $v(z)$, the normalized harmonic conjugate of $u(z)$, is in any h^p class? The question is answered in part by the following theorems.

<u>Theorem 2.14</u>:

[26, p. 253] (Theorem of Riesz) If $u(z)$ is real valued, $u(z) \in h^p$, $1 < p < \infty$, then $v(z)$, the normalized harmonic conjugate of $u(z)$ is in h^p and there exists a constant A_p depending only on p such that $\|v(z)\|_{h^p} \leq A_p \|u(z)\|_{h^p}$.

The Theorem of M. Riesz is false for $p = 1$, a counterexample being the Poisson kernel [13]. In the case $p = 1$ we do have the following theorem.

Theorem 2.15:

[5] or [26, p. 254] (Theorem of Kolomogrov)

If $u(z)$ is real valued and $u(z) \in h^1$, then $v(z)$ the normalized harmonic conjugate of $u(z)$ is in h^p for any $p \in (0,1)$ and there exists a constant B_p depending only on p such that

$$\|v(z)\|_{h^p} \leq B_p \|u(z)\|_{h^1}.$$

Harmonic functions are very closely related to Fourier series. We observe that if $f(e^{i\theta}) \in L^1[0,2\pi]$, then for each integer n,

$$\frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-in\theta}d\theta = c_n$$

exists, and $|c_n| \to 0$ as $|n| \to \infty$; however, it is sufficient for our purposes to know the c_n's are well defined and uniformly bounded.

Definition 2.11:

If $f(e^{i\theta}) \in L^1[0,2\pi]$ we define the Fourier series of f to be the formal power series $\sum_{n=-\infty}^{\infty} c_n e^{in\theta}$, where

$$c_n \equiv \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-in\theta}d\theta,$$

and we write

$$f(e^{i\theta}) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\theta}.$$

The complex number c_n is called the Fourier coefficient of index n, and will be denoted by $c_n = \hat{f}(n)$.

Now the connections between Fourier series and harmonic functions with which we are concerned are the following:

<u>Theorems 2.16</u>:

If $f(e^{i\theta}) \in L^1[0,2\pi]$ has the Fourier series $\sum_{n=-\infty}^{\infty} c_n e^{in\theta}$, then $u(z) = u(re^{i\theta}) = \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{in\theta}$ is harmonic on D.

<u>Theorem 2.17</u>:

If $u(e^{i\theta}) \in L^1[0,2\pi]$ has the Fourier series $\sum_{n=-\infty}^{\infty} c_n e^{in\theta}$, then,

(i) $u(re^{i\theta}) = \int_0^{2\pi} P(r,\theta-t)\, u(e^{it})dt$ is harmonic on D;

(ii) $u(re^{i\theta}) \in h^1$;

(iii) $u(re^{i\theta}) = \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{in\theta}$, $0 \le r \le 1$, $\theta \in [0,2\pi]$;

(iv) $g(z) = \sum_{n=0}^{\infty} c_n z^n$ is holomorphic on D;

(v) $g(z) \in H^{1/2}$;

and

(vi) $\|g\|_{H^{1/2}} \leq C\|u(e^{i\theta})\|_{L^1}$, where C is a constant independent of $u(e^{i\theta})$.

<u>Proof</u>:

(i) and (ii) are restatements of Theorem 2.2 (ii). Since $P(r,\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}$, we have by substitution

$$u(re^{i\theta}) = \int_0^{2\pi} \sum_{n=-\infty}^{\infty} r^{|n|} e^{in(\theta-t)} u(e^{it})dt.$$

Fix r, $0 \leq r < 1$. Then, by uniform convergence,

$$\int_0^{2\pi} \sum_{n=-\infty}^{\infty} r^{|n|} e^{in(\theta-t)} u(e^{it})dt = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} \int_0^{2\pi} e^{-int} u(e^{it})dt$$

$$= \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} c_n.$$

Thus (iii) holds.

Since $|c_n| \leq \|u(e^{i\theta})\|_{L^1}$, $g(z)$ is holomorphic on D. (v) will follow from (vi)

To prove (vi) we treat the case where $u(e^{i\theta})$ is real valued first, and then extend to the complex valued case.

Case I:

Assume $u(e^{i\theta})$ is real valued. Since $u(e^{i\theta})$ is real valued $c_n = \overline{c}_{-n}$. Let v be the normalized harmonic conjugate of $u(re^{i\theta})$. Then $2g(re^{i\theta}) = u(re^{i\theta}) + iv(re^{i\theta}) + c_0$.

Now for $0 < p < 1$, Theorem 2.15 implies that

$$\sup_{0\leq r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |v(re^{i\theta})|^p d\theta\right)^{1/p} \leq B_p \sup_{0\leq r<1} \frac{1}{2\pi} \int_0^{2\pi} |u(re^{i\theta})|d\theta$$

$$= B_p\|u(re^{i\theta})\|_{h^1} = B_p\|u(e^{i\theta})\|_{L^1}.$$

Since

$$c_0 = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta})d\theta,$$

we also have $|c_0| \leq \|u(e^{i\theta})\|_{L^1}$. Thus, if we fix r, $0 \leq r < 1$,

$$\frac{1}{2\pi}\int_0^{2\pi}|g(re^{i\theta})|^p d\theta \leq \frac{1}{2\pi}\int_0^{2\pi}|u(re^{i\theta}) + iv(re^{i\theta}) + c_0|^p d\theta$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi}|u(re^{i\theta})|^p d\theta + \frac{1}{2\pi}\int_0^{2\pi}|v(re^{i\theta})|^p d\theta$$

$$+ |c_0|^p.$$

$$\leq \left[\frac{1}{2\pi}\int_0^{2\pi}|u(re^{i\theta})|d\theta\right]^p + B_p{}^p\|u(e^{i\theta})\|_{L^1}^{\ p}$$

$$+ \|u(e^{i\theta})\|_{L^1}^{\ p}$$

$$= \ c_p{}^p\|u(e^{i\theta})\|_{L^1}^{\ p}.$$

where c_p is a constant depending on p alone. Hence $\|g(z)\|_{H^p} \leq c_p\|u(e^{i\theta})\|_{L^1}$.

Case II:

If $u(e^{i\theta})$ is complex valued, then set $u_1(e^{i\theta}) = \operatorname{Re} u(e^{i\theta})$ and $u_2(e^{i\theta}) = \operatorname{Im} u(e^{i\theta})$, and thus $u(re^{i\theta}) = u_1(re^{i\theta}) + u_2(re^{i\theta})$.

Let $g(z) = g_1(z) + ig_2(z)$ where $g_1(z)$ and $g_2(z)$ are the holomorphic functions in (iv) corresponding to $u_1(e^{i\theta})$ and $u_2(e^{i\theta})$ respectively.

Now by case I, for any r, $0 \leq r < 1$ we have

$$\frac{1}{2\pi}\int_0^{2\pi}|g(re^{i\theta})|^p \leq \frac{1}{2\pi}\int_0^{2\pi}|g_1(re^{i\theta}) + ig_2(re^{i\theta})|^p d\theta$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi}|g_1(re^{i\theta})|^p d\theta + \frac{1}{2\pi}\int_0^{2\pi}|g_2(re^{i\theta})|^p d\theta$$

$$\leq c_p{}^p \|u_1(e^{i\theta})\|_{L^1}^p + c_p{}^p \|u_2(e^{i\theta})\|_{L^1}{}^p =$$

$$= c_p{}^p \left[\frac{1}{2\pi}\int_0^{2\pi} |u_1(e^{i\theta})|\,d\theta\right]^p + c_p{}^p \left[\frac{1}{2\pi}\int_0^{2\pi} |u_2(e^{i\theta})|\,d\theta\right]^p$$

$$\leq 2c_p{}^p \left[\frac{1}{2\pi}\int_0^{2\pi} |u(e^{i\theta})|\,d\theta\right]^p.$$

Since r was arbitrary, taking $1/p$ powers and a supremum we obtain

$$\|g(z)\|_{H^p} \leq 2^{1/p} c_p \|u(e^{i\theta})\|_{L^1}.$$

4. The Shift Operators

We are primarily concerned with the Hardy p-classes for $1 \leq p < \infty$. On these spaces we define the left and right shifts as follows:

Definition 2.12:

Let p be fixed, $1 \leq p < \infty$. Let $f(z) \in H^p$ be given. We define

$$(Uf)(z) = z \cdot f(z), \quad \text{for } z \in D$$

and

$$(U^*f)(z) = \frac{f(z) - f(0)}{z}, \quad \text{for } z \in D.$$

We call U the forward or right shift on H^p and U^* the left or backward shift. As we noted in the introduction the action of U on a function $f(z)$ represented by its Taylor series about the origin is to shift the Taylor coefficients forward or to the right, that is, if

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \text{for } |z| < 1$$

then

$$(Uf)(z) = \sum_{n=0}^{\infty} a_n z^{n+1}, \quad \text{for } |z| < 1.$$

The action of U^* is similar, that is, if

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \text{for } |z| < 1,$$

then

$$(U^*f)(z) = \sum_{n=1}^{\infty} a_n z^{n-1}, \quad \text{for} \quad |z| < 1.$$

Definition 2.13:

Let X and Y be two complex Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ respectively. A mapping $T: X \to Y$ is called a linear operator or operator if and only if

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$

for each $x, y \in X$ and each α, β complex numbers.

An operator T is said to be continuous if and only if T is continuous in the norm topologies on X and Y. An operator from X to Y is said to be bounded if and only if there exists a real number M such that

$$\sup_{\|x\|_X \le 1} \|Tx\| \le M.$$

It is well known [4, pp. 59-60] that a linear operator is bounded if and only if it is continuous.

Theorem 2.18:

Let p be fixed, $1 \le p < \infty$. The mappings U and U^* of definitions 2.12 are continuous linear operators from H^p to H^p.

Proof:

Linearity follows immediately from the definition of U and U*. For continuity, let $f(z) \in H^p$ be given. Then

$$\|Uf\|_{H^p} = \|z\cdot f(z)\|_{H^p} =$$

$$= \lim_{r\to 1^-} \left[\frac{1}{2\pi} \int_0^{2\pi} |re^{i\theta} f(re^{i\theta})|^p d\theta \right]$$

$$= \lim_{r\to 1^-} \left[\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right]$$

$$= \|f\|_{H^p}.$$

Thus $\sup_{\|f\|_{H^p}\le 1} \|Uf\|_{H^p} = 1$ or $\|U\| = 1$.

Similarily,

$$\|U^*f\|_{H^p} = \left\|\frac{f(z) - f(0)}{z}\right\|_{H^p}$$

$$= \lim_{r\to 1^-} \left[\frac{1}{2\pi}\int_0^{2\pi} |r^{-1}e^{-i\theta}(f(re^{i\theta}) - f(0))|^p d\theta\right]^{1/p}$$

$$= \lim_{r\to 1^-} \left[\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta}) - f(0)|^p d\theta\right]^{1/p}$$

$$= \|f(z) - f(0)\|_{H^p}$$

$$\le \|f\|_{H^p} + |f(0)|$$

$$= \|f\|_{H^p} + \left|\frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})d\theta\right|$$

$$\leq \|f\|_{H^p} + \|f\|_{H^p} = 2\|f\|_{H^p}.$$

Thus,

$$\sup_{\|f\|_{H^p} \leq 1} \|U^*f\| \leq 2, \quad \text{or } \|U^*\| \leq 2.$$

Theorem 2.19:

Let p be fixed, $1 \leq p < \infty$. Let $f(z) \in H^p$ be given. Let n be a positive integer. If $f(z) = \sum_{k=0}^{\infty} a_k z^k$, then

$$(U^{*n}f)(z) = [f(z) - p_n(z)]z^{-n}, \quad 0 < |z| < 1,$$

where $p_n(z) = \sum_{k=0}^{n-1} a_k z^k$.

5. Pseudocontinuations

From the theorem of Lusin and Priwalow, Theorem 2.1, we know that if $f(z)$ is meromorphic on D and if $f(z)$ has non-tangential boundary values $f(e^{i\theta})$ on a set $E \subset T$ with $m(E) > 0$, then $f(z)$ is uniquely determined by these boundary values. Now if there is a function $\tilde{f}(z)$ meromorphic on D_e, which also has non-tangential boundary values $\tilde{f}(e^{i\theta})$ on the same set $E \subset T$ with $f(e^{i\theta}) = \tilde{f}(e^{i\theta})$ then in some sense $f(z)$ and $\tilde{f}(z)$ uniquely determine one another. We formulate this in a more precise manner in the following definition [21].

Definition 2.14:

If $f(z)$ is meromorphic in D we say that $f(z)$ is pseudocontinuable across T onto D_e if and only if the following hold:

(i) $f(z)$ has non-tangential boundary values $f(e^{i\theta})$ for almost all $\theta \in [0,2\pi]$,

(ii) there exists a function $\tilde{f}(z)$ meromorphic on D_e,

(iii) $\tilde{f}(z)$ has non-tangential boundary values $\tilde{f}(e^{i\theta})$ for almost all $\theta \in [0,2\pi]$, and

(iv) $\tilde{f}(e^{i\theta}) = f(e^{i\theta})$ a.e.

We remark that we have defined a pseudocontinuation across T onto D_e and in this definition we require that the pseudocontinuation be meromorphic on all of D_e. The reason for requiring D_e to be the domain for a pseudocontinuation for our purposes will be made clear in Chapter III. In general one could define a pseudocontinuation across a subarc of T (with positive one-dimensional Lebesgue measure) onto a subarc of D_e having the arc as part of its boundary. See Shapiro [21] for a discussion of Pseudocontinuations.

We note that if $f(z)$ is holomorphic on D and if $f(z)$ can be continued analytically across T onto D_e with the continuation meromorphic on D_e, then the analytic continuation is a pseudocontinuation across T.

A pseudocontinuation may exist even though the original function is nowhere analytically continuable [3] or [21]. To see this we first consider inner functions.

Theorem 2.20:

[3] If $f(z)$ is an inner function, then $f(z)$ is pseudocontinuable across T.

Proof:

Let $f(z)$ be an inner function, that is, $|f(e^{i\theta})| = 1$ a.e. The function $(Jf)(z) = \overline{f(1/\bar{z})}$ has boundary values $\tilde{f}(e^{i\theta}) = \overline{f(e^{i\theta})}$ of modulus 1 for almost all $e^{i\theta} \in T$, hence $\tilde{f}(e^{i\theta}) = \dfrac{1}{f(e^{i\theta})}$ a.e. Thus the function $\tilde{f}(z) = \dfrac{1}{(Jf)(z)}$ has boundary values $\tilde{f}(e^{i\theta}) = f(e^{i\theta})$ a.e. Finally, $\tilde{f}(z)$ is holomorphic on D_e except at the zeros of $(Jf)(z)$, that is $\tilde{f}(z)$ is meromorphic on D_e. We also note that if $f(z)$ is a singular inner function (no zeros on D), then the pseudocontinuation $\tilde{f}(z)$ is holomorphic on D_e.

Theorem 2.21:

[13, p. 68] If $S_\mu(z)$ is the singular inner function determined by the positive singular measure μ, then S_μ is analytically continuable everywhere in the complex plane except at those points which are in the closed support of μ. The function S_μ (or even $|S_\mu|$) is not continuable from the interior of the disc to any point in the closed support of μ.

Now take a measure μ which is positive and singular with respect to Lebesgue measure on T and with the closed support of μ all of T. The singular inner function $S_\mu(z)$ is pseudocontinuable across T onto D_e, the pseudocontinuation is holomorphic on D_e and $S_\mu(z)$ is not analytically continuable across any subarc of T onto any subdomain of D_e.

A question which arose when Shapiro [21] defined pseudocontinuations was: Are there any functions which do not admit a pseudocontinuation? The answer given by Shapiro was: The function

$$f(z) = \sum_{n=0}^{\infty} \left(\frac{z^{2^n}}{2^n}\right), \quad |z| \le 1,$$

is not pseudocontinuable across any subarc of T.

6. Continuous Linear Functionals on H^p.

We shall have need of an integral representation of the continuous linear functionals on the H^p classes.

Definition 2.15:

For $0 < p \leq \infty$, a mapping $\Phi: H^p \to C$ such that

$$\Phi(\alpha f + g) = \alpha\Phi(f) + \Phi(g)$$

for each $\alpha \in C$ and $f,g \in H^p$ for which there exists a real number M satisfying

$$\sup_{\|f\|_{H^p} \leq 1} |\Phi(f)| \leq M$$

is called a bounded linear functional on H^p.

For $1 \leq p \leq \infty$, H^p is a Banach space,,and it is well known that a linear functional is bounded if and only if it is continuous. For $0 < p < 1$, H^p with the metric $\rho(f,g) = \|f-g\|^p_{H^p}$ is a Frechet space as we have noted. It is known [6] that a linear functional on H^p, $0 < p < 1$, is continuous in the Frechet space topology if and only if it is bounded.

For $0 < p < \infty$, we can regard H^p as a subspace of $L^p[0,2\pi]$ by identifying $f(z)$ with the corresponding boundary function $f(e^{i\theta}) \in \mathfrak{H}^p \subset L^p$. For $1 \leq p < \infty$ this approach is quite fruitful for considering the continuous linear functionals on H^p since $\mathfrak{H}^p$ is a closed subspace of L^p and for $1 \le p$ the spaces L^p have many continuous linear functionals. In the case $0 < p < 1$, however, only the zero functional is continuous on L^p [6] while for H^p we still have enough continuous linear functionals to separate the points.

Theorem 2.22:

[22] If ϕ is a continuous linear functional on H^p, $1 < p < \infty$, then there exists a function $g(z) \in H^q$, $q = \frac{p}{p-1}$, such that

$$\phi(f) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\,d\theta, \text{ for each } f \in H^p,$$

and conversely, each $g \in H^q$ so defines a continuous linear functional on H^p.

We restate this theorem in a form which we will find more useful.

Theorem 2.23:

If ϕ is a continuous linear functional on H^p, $1 < p < \infty$, then there exists a function $G(z) \in H^q(D_e)$, $q = \frac{p}{p-1}$, such that

$$\phi(f) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})G(e^{i\theta})\,d\theta, \text{ for each } f \in H^p$$

and conversely each $G \in H^q(D_e)$ so defines a continuous linear functional on H^p.

Proof:

$H^q(D_e)$ was defined in terms of the J operator on $H^q(D)$. Given $\phi \in (H^p)^*$, take the $g(z) \in H^q(D)$ guaranteed to exist by Theorem 2.22. Define $G(z) = (Jg)(z)$. Then $G(z) \in H^q(D_e)$ and $G(e^{i\theta}) = \overline{g(e^{i\theta})}$. Similarily for the converse.

Theorem 2.24:

If ϕ is a continuous linear functional on H^1, then there exist two functions $G(e^{i\theta})$, $g(z)$ such that

(i) $G(e^{i\theta}) \in L^\infty [0, 2\pi]$;

(ii) $g(z) \in H^p$, for any $p < \infty$;

(iii) $\phi(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})\overline{G(e^{i\theta})}\, d\theta$;

(iv) $\phi(f) = \lim_{r\to 1^-} \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta})\overline{g(e^{i\theta})}\, d\theta$;

and

(v) if $G(e^{i\theta})$ has the Fourier series $\sum_{n=-\infty}^{\infty} c_n e^{in\theta}$, then $g(z)$ has the Taylor series $\sum_{n=0}^{\infty} c_n z^n$ about the origin and $g(e^{i\theta})$ has the Fourier series $\sum_{n=0}^{\infty} c_n e^{in\theta}$.

We shall need to know something about the linear functionals on H^p for $0 < p < 1$. We first define several classes of functions.

Definition 2.16:

[26, p. 42] Let A denote the class of functions holomorphic on D and continuous on the closed unit disc. Let $f(e^{i\theta})$ be defined for $\theta \in [0,2\pi]$. We define the modulus of continuity of f by

$$\omega(h;f) = \sup_{\substack{|t-s| \leq h \\ t,\, s \in [0,2\pi]}} |f(e^{it}) - f(e^{is})|.$$

For $f \in A$, we say that $f \in \Lambda_\alpha$ $(0 < \alpha \leq 1)$ if and only if

$$\omega(h;f(e^{i\theta})) = (O(h^\alpha) \quad \text{as} \quad h \to 0,$$

and we say that $f \in \Lambda_*$ if and only if

$$|f(e^{i(t+h)}) - 2f(e^{it}) + F(e^{i(t-h)})| = O(h),$$

uniformly in t as $h \to 0$.

Theorem 2.25:

[6] Let $\theta \in (H^p)^*$, $0 < p < 1$. Then there is a unique function $g \in A$ such that

$$(1) \quad \theta(f) = \lim_{r \to 1^-} \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) g(e^{-i\theta})\, d\theta, \ f \in H^p.$$

If $\frac{1}{n+1} < p < \frac{1}{n}$ $(n=1,2,\cdots)$, then $g^{(n-1)} \in \Lambda_\alpha$, where $\alpha = \frac{1}{p} - n$.

Conversely, for any g with $g^{(n-1)} \in \Lambda_\alpha$, the limit (1) exists for all $f \in H^p$ and defines a functional $\theta \in (H^p)^*$. In the case $p = \frac{1}{n+1}$, $g^{(n-1)} \in \Lambda_*$; and conversely, any g with $g^{(n-1)} \in \Lambda_*$ defines through (1) a bounded linear functional on H^p.

Corollary 2.26:

If $f \in H^p$ has Taylor series $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then for any fixed n $(n = 0,1,2,\cdots)$, the mapping $P_n f = a_n$ is a bounded linear functional on H^p.

This corollary implies that if we have a sequence of functions $\{f_n\} \subset H^p$ $(0 < p < 1)$ with $f_n \to 0$ in H^p metric as $n \to \infty$, then the Taylor coefficients converge to zero. We state this in a corollary.

Corollary 2.27:

Let p be fixed, $0 < p < 1$. Let $\{f_n\}_{n=1}^{\infty} \subset H^p$. If there exists a $g \in H^p$ such that $\|f_n - g\|_{H^p} \to 0$ as $n \to \infty$, then if

$$f_n(z) = \sum_{k=0}^{\infty} a_{n,k} z^k, \quad |z| < 1,$$

and

$$g(z) = \sum_{k=0}^{\infty} b_k z^k, \quad |z| < 1,$$

then $\lim_{n\to\infty} a_{n,k} = b_k$, for $n = 1,2,3,\cdots$.

CHAPTER III

The Closure Problem for U^*

1. Characterization Theorem.

In this chapter we will present a characterization of the non-cyclic vectors for the left shift in the H^p spaces, $1 \leq p < \infty$. The main result of Douglas, Shapiro and Shields [3] is a characterization of the non-cyclic vectors in H^2 for the left shift in terms of pseudocontinuations across T.

The specific techniques used by Douglas, Shapiro and Shields involved identifying $H^2(D)$ with ℓ^2 (the space of all square summable sequences of complex numbers $\{a_n\}$ with $\|\{a_n\}\|_{\ell^2}^2 = \Sigma |a_n|^2$) and the dual space of ℓ^2 with itself. There is little difficulty in extending their result to the H^p spaces, $1 < p < \infty$. The extension to H^1 is quite difficult due to several factors. First of all the dual space of H^1 is not as neatly described as that of H^2 or H^p, $1 < p < \infty$. Secondly, in the H^1 case one has problems with sequences of L^1 functions which converge in L^μ, $0 < \mu < 1$, but perhaps not in L^1. This latter convergence causes problems as we will have a sequence of L^1 functions for which the Fourier coefficients of index $n = 0, \pm 1, \pm 2, \cdots$ converge to zero but the sequence may only converge in the $L^{1/2}$ metric. We will want to be able to conclude that the limit function is the zero function but we will need to use much of the structure of the H^p spaces as Fourier coefficients do not make much sense in the L^p spaces when $0 < p < 1$.

We shall first state and prove the characterization theorem for H^p with $1 < p < \infty$, basically repeating the methods of [3], but changing the notation.

We recall that the left shift operator U^* is defined by

$$(U^*f)(z) = \frac{f(z) - f(0)}{z}$$

and that $f(z)$ is said to be non-cyclic for U^* if and only if span $\{U^{*n}f\}_{n=0}^{\infty}$ is not dense in H^p. It is to be understood that when speaking of the left shift operator one has a fixed space (p) in mind.

In the proofs of Theorems 3.3 and 3.4 we will need some relatively straightforward results whose proofs are simply computations. In order to keep the notation somewhat reasonable we state these results in the form of two Lemmas.

Lemma 3.1:

Let $H(z) \in H^p(D_e)$, $1 \le p \le \infty$ have boundary values $H(e^{i\theta})$. Then $\hat{H}(n) = 0$ for $n = +1, +2, \cdots$, that is, the positive Fourier coefficients of $H(e^{i\theta})$ vanish.

<u>Proof</u>:

Since $H^p(D_e) \subset H^1(D_e)$ for $1 \le p$ it is sufficient to prove the Lemma for $H^1(D_e)$.

Let $H(z) \in H^1(D_e)$ be given. Denote the boundary values of $H(z)$ by $H(e^{i\theta})$. Let $\varepsilon > 0$ be given. By Theorem 2.12′ there exists a polynomial $Q(z)$ such that $\|H(e^{i\theta}) - Q(e^{-i\theta})\|_{L^1} < \varepsilon$. Let n be any positive integer. Now

$$|\hat{H}(n)| = \left|\frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} H(e^{i\theta})\,d\theta\right|$$

$$= \left|\frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}\left[H(e^{i\theta}) - Q(e^{-i\theta})\right]d\theta\right|$$

$$\le \frac{1}{2\pi}\int_0^{2\pi} |H(e^{i\theta}) - Q(e^{-i\theta})|\,d\theta$$

$$= \|H(e^{i\theta}) - Q(e^{-i\theta})\|_{L^1} < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $\hat{H}(n) = 0$.

<u>Lemma 3.2</u>:

Let $\overline{G(e^{i\theta})}$ and $\overline{g(e^{i\theta})}$ be any two non-zero functions in $L^2[0,2\pi]$ with Fourier series $\overline{G(e^{i\theta})} \sim \sum_{n=-\infty}^{\infty} \hat{G}(n)e^{in\theta}$ and $\overline{g(e^{i\theta})} \sim \sum_{n=-\infty}^{\infty} \hat{g}(n)e^{in\theta}$ respectively. Let $\hat{G}(n) = \hat{g}(n)$, for $n = -1,-2,\cdots$. Let $f(z) = f(re^{i\theta})$ be a holomorphic function on D. For each value of r, $0 \le r < 1$ define

$$h_r(e^{i\theta}) = f(re^{i\theta})\,\overline{g(e^{i\theta})}$$

and

$$k_r(e^{i\theta}) = f(re^{i\theta})\,\overline{G(e^{i\theta})}.$$

Then $\hat{h}_r(n) = \hat{k}_r(n)$, for $n = -1,-2,-3,\cdots$.

<u>Proof</u>:

For any negative integer n,

$$\hat{h}_r(n) = \sum_{k=0}^{\infty} \hat{f}(k) r^k \hat{\underline{g}}(n-k)$$

$$= \sum_{k=0}^{\infty} \hat{f}(k) r^k \hat{\underline{G}}(n-k) =$$

$$= \hat{k}_r(n).$$

<u>Theorem 3.3</u>:

Let p be fixed, $1 < p < \infty$. A necessary and sufficient condition for $f(z) \in H^p$ to be non-cyclic for U^* is that the following two conditions hold:

(i) $f(z)$ has pseudocontinuation, $\tilde{f}(z)$, across T, with $\tilde{f}(z)$ meromorphic on D_e.

(ii) $\tilde{f}(z) \in N(D_e)$, i.e., $\tilde{f}(z)$ is of bounded Nevanlinna characteristic or type on D_e.

<u>Remark</u>:

Note that this proof of sufficiency is also valid for the case where $p = 1$.

Proof:

(Sufficiency) Let $f(z) \in H^p$ be given. Assume $f(z)$ has pseudocontinuation $\tilde{f}(z)$ across T, where $\tilde{f}(z)$ is meromorphic in D_e and of bounded Nevanlinna characteristic on D_e. Let $\sum_{n=0}^{\infty} a_n z^n$ be the Taylor series about the origin for $f(z)$, and let $G(z)$, $H(z)$ be two functions in $H^{\infty}(D_e)$, with $H(z) \not\equiv 0$ for $z \in D_e$, such that $\tilde{f}(z) = \frac{G(z)}{H(z)}$.

Without loss of generality we may assume that $G(\infty) = 0$. Indeed, if $G(\infty) \neq 0$, we replace $H(z)$ and $G(z)$ with $H^*(z) = \frac{1}{z}H(z)$ and $G^*(z) = \frac{1}{z}G(z)$, respectively. This gives

$$\tilde{f}(z) = \frac{G(z)}{H(z)} = \frac{\frac{1}{z}G(z)}{\frac{1}{z}H(z)} = \frac{G^*(z)}{H^*(z)}, \quad \text{for } z \in D_e$$

and

$$G^*(z),\ H^*(z) \in H^{\infty}(D_e) \quad \text{with} \quad G^*(\infty) = 0,$$

since $\frac{1}{z}$ is bounded and holomorphic for $|z| > 1$.

Define a continuous linear functional ϕ on H^p by

$$\phi(k) = \frac{1}{2\pi}\int_0^{2\pi} k(e^{i\theta})H(e^{i\theta})\,d\theta, \quad \text{for any } k(z) \in H^p,$$

where $k(e^{i\theta})$ and $H(e^{i\theta})$ denote the boundary functions of $k(z)$ and $H(z)$ respectively.

Now by Theorem 2.6 and Theorem 2.5′ we have

$$k(e^{i\theta}) \in L^p[0,2\pi] \quad \text{since} \quad k(z) \in H^p, \quad \text{and}$$

$$H(e^{i\theta}) \in L^q[0,2\pi], \quad \text{for } \frac{1}{p} + \frac{1}{q} = 1,$$

since

$$H(z) \in H^{\infty}(D_e) \subset H^{q}(D_e).$$

Thus by Theorem 2.23 ϕ is a continuous linear functional on H^p. Observe that ϕ is not the zero functional since $H(z) \not\equiv 0$.

We claim that ϕ annihilates f and all of its left shifts. We proceed by induction.

$$\phi(f) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})H(e^{i\theta})d\theta = \frac{1}{2\pi}\int_0^{2\pi} G(e^{i\theta})d\theta,$$

since $G(e^{i\theta}) = f(e^{i\theta})H(e^{i\theta})$ a.e. $[0,2\pi]$.

Now since $G(z) \in H^1(D_e)$, $\lim\limits_{r\to 1^-}\int_0^{2\pi} G(re^{i\theta})d\theta = \int_0^{2\pi} G(e^{i\theta})d\theta$, but $\int_0^{2\pi} G(re^{i\theta})d\theta = G(\infty) = 0$. Thus $\phi(f) = 0$. Let n be a fixed non-negative integer. Assume that $\phi((U^*)^k f) = 0$, for $k = 0,1,\cdots,n$. We wish to show that $\phi((U^*)^{n+1}f) = 0$. Set $K(z) = [(U^*)^n f](z)$, $p(z) = \sum\limits_{k=0}^{n-1} a_k z^k$. Thus,

$$z^n K(z) = f(z) - p(z) \quad \text{or} \quad f(z) = z^n K(z) + p(z).$$

We must show that $\phi(U^*K) = 0$.

Now

$$\phi(U^*K) = \frac{1}{2\pi}\int_0^{2\pi} e^{-i\theta}[K(e^{i\theta})-K(0)]\,H(e^{i\theta})d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{-i\theta}K(e^{i\theta})H(e^{i\theta})d\theta - \frac{1}{2\pi}\int_0^{2\pi} e^{-i\theta}K(0)H(e^{i\theta})d\theta.$$

We claim that both of the integrals above are zero.

By lemma 3.1, $\int_0^{2\pi} e^{-i\theta}H(e^{i\theta})d\theta = 0$.

Now fix r, $0 < r < 1$ and set $z = re^{i\theta}$. From the definitions of $K(z)$ and $p(z)$ it follows that

$$f(z)H(e^{i\theta}) = [z^n K(z) + p(z)]H(e^{i\theta}) = z^n K(z)H(e^{i\theta}) + p(z)H(e^{i\theta}).$$

Since $z \neq 0$ we obtain

$$z^{-n-1}[f(z)H(e^{i\theta})] = z^{-1}K(z)H(e^{i\theta}) + (z^{-n-1}p(z))H(e^{i\theta}),$$

and hence

$$\int_0^{2\pi} r^{-n-1}e^{(-n-1)i\theta}f(re^{i\theta})H(e^{i\theta})d\theta = \int_0^{2\pi} r^{-1}e^{-i\theta}K(re^{i\theta})H(e^{i\theta})d\theta$$

$$+ \int_0^{2\pi} r^{-n-1}e^{(-n-1)i\theta}p(re^{i\theta})H(e^{i\theta})d\theta$$

Now r is fixed, $0 < r < 1$ so that $r^{-n-1}e^{-(n+1)i\theta}p(ne^{i\theta})$ is a finite linear combination of negative powers of $e^{i\theta}$ since $p(z)$ is a polynomial in z of degree at most $n-1$. Thus, $\int_0^{2\pi} r^{-n-1}e^{-(n+1)i\theta}\, H(e^{i\theta})d\theta$ is a finite linear combination of Fourier coefficients of positive index of $H(e^{i\theta})$. Since $H(z) \in H^\infty(D_e)$, Lemma 3.1 says that the Fourier coefficients of positive index of $H(e^{i\theta})$ are all zero, hence,

$$\int_0^{2\pi} r^{-n-1}e^{-(n+1)i\theta}p(re^{i\theta})H(e^{i\theta})d\theta = 0, \quad \text{for} \quad 0 < r < 1.$$

Next, a straight forward application of the Lebesgue dominated convergence theorem gives

$$\lim_{r\to 1^-}\int_0^{2\pi} r^{-n-1}e^{-(n+1)i\theta}f(re^{i\theta})H(e^{i\theta})d\theta = \int_0^{2\pi} e^{-(n+1)i\theta}f(e^{i\theta})H(e^{i\theta})d\theta$$

$$= \int_0^{2\pi} e^{-(n+1)i\theta} G(e^{i\theta})\, d\theta = 0,$$

since the last integral above is a Fourier coefficient of positive index of $G(e^{i\theta})$, and $G(z) \in H^\infty(D_e)$. Hence,

$$0 = \lim_{r \to 1^-} \int_0^{2\pi} e^{-i\theta} r^{-1} K(re^{i\theta}) H(e^{i\theta})\, d\theta = \int_0^{2\pi} e^{-i\theta} K(e^{i\theta}) H(e^{i\theta})\, d\theta = \phi(U^* K)$$

Therefore $\phi((U^*)^{n+1} f) = \phi(U^* K) = 0$ which completes the induction step. Hence under the hypothesis that f has a pseudocontinuation across T, of bounded Nevanlinna characteristic on D_e, we have shown that there exists $\phi \in (H^p)^*$, $\phi \not\equiv 0$ such that $\phi[(U^{*^n}) f] = 0$, for $n = 0, 1, 2, \cdots$. Thus span $\{U^{*n} f\}_{n=0}^\infty \subset$ Kernel (ϕ). Since ϕ is continuous, Kernel (ϕ) is closed in H^p. Since $\phi \not\equiv 0$, Kernel $(\phi) \not\equiv H^p$. Thus span $\{U^{*n} f\}_{n=0}^\infty$ is not dense in H^p or equivalently f is non-cyclic for U^*.

(Necessity):

Let p be fixed, $1 < p < \infty$ and let $f(z) \in H^p(D)$ be a given non-cyclic vector for U^*. Let $\sum_{n=0}^\infty a_n z^n$ be the Taylor series about the origin for $f(z)$. We must show that f has a pseudocontination across T of bounded Nevanlinna characteristic on D_e.

If f is the zero function, or indeed any constant function we are clearly done, so assume f is non-constant.

From the definition of non-cyclic, the closure in H^p of span $\{U^{*n} f\}_{n=0}^\infty$ is a closed proper subspace of H^p.

Hence by the Hahn-Banach theorem there exists a continuous linear functional $\phi \neq 0$ on H^p such that $\phi[\text{span}\,\{U^{*n}f\}_{n=0}^{\infty}] = 0$.

By the Riesz Theorem [19] there exists $g(z) \in H^q$, $q = \frac{1}{p-1}$ such that

$$(3.1)\quad \phi(h) = \frac{1}{2\pi}\int_0^{2\pi} h(e^{i\theta})\overline{g(e^{i\theta})}\,d\theta, \quad \text{for any } h \in H^p.$$

Now define $G(z) \equiv (Jg)(z) \equiv \overline{g(1/\bar{z})}$ for $\infty \geqq |z| > 1$. By definition of $H^q(D_e)$, we have $G(z) \in H^q(D_e)$, $G(e^{i\theta}) = \overline{g(e^{i\theta})}$ and $G(e^{i\theta}) \in L^q[0,2\pi]$, where $G(e^{i\theta})$ is the boundary function of $G(z)$. Now define

$$(3.2)\quad H(e^{i\theta}) = f(e^{i\theta})G(e^{i\theta}) \text{ a.e.}$$

By the Hölder inequality [15] $H(e^{i\theta}) \in L^1[0,2\pi]$. We claim that $H(e^{i\theta}) \in \mathfrak{H}^1(D_e)$. This will follow from the hypothesis that f is non-cyclic for U^* and a theorem of F. and M. Riesz. We first show that the non-negative Fourier coefficients of $H(e^{i\theta})$ are all zeros. Now note that

$$\int_0^{2\pi} H(e^{i\theta})\,d\theta = \int_0^{2\pi} f(e^{i\theta})G(e^{i\theta})\,d\theta = \int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\,d\theta = 2\pi\,\phi(f) = 0,$$

or $\hat{H}(0) = 0$. We now show by induction that the positive Fourier coefficients of H are all zero. Let n be a non-negative integer and assume that

$$\int_0^{2\pi} e^{-ik\theta}H(e^{i\theta})\,d\theta = 0 \quad \text{for} \quad k = 0,1,\cdots,n.$$

Define $p(z) = \sum_{k=0}^{n} a_k z^k$. Recall that $(U^{*n+1}f)(z) = \frac{f(z) - p(z)}{z^{n+1}}$, for $0 < |z| < 1$. Also note that

$$(U^{*n+1}f)(e^{i\theta}) = [f(e^{i\theta}) - p(e^{i\theta})]e^{-i(n+1)\theta} \quad \text{a.e.}$$

Now by hypothesis,

$$0 = \phi(U^{*n+1}f) = \frac{1}{2\pi}\int_0^{2\pi} [f(e^{i\theta}) - p(e^{i\theta})]e^{-i(n+1)\theta}\overline{g(e^{i\theta})}\,d\theta =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{-i(n+1)\theta} f(e^{i\theta})\overline{g(e^{i\theta})}\,d\theta - \frac{1}{2\pi}\sum_{k=0}^{n} a_k \int_0^{2\pi} e^{i(k-n-1)\theta}\overline{g(e^{i\theta})}\,d\theta$$

Now $g(e^{i\theta}) \in \mathfrak{H}^q(D)$ so that

$$\int_0^{2\pi} e^{i(k-n-1)\theta}\overline{g(e^{i\theta})}\,d\theta = \overline{\int_0^{2\pi} e^{i(n+1-k)\theta} g(e^{i\theta})\,d\theta} = 0,$$

for $k = 0, 1, 2, \cdots, n$.

Thus

$$0 = \frac{1}{2\pi}\int_0^{2\pi} e^{-i(n+1)\theta} f(e^{i\theta})\overline{g(e^{i\theta})}\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{-i(n+1)\theta} H(e^{i\theta})\,d\theta.$$

Hence by induction all of the non-negative Fourier coefficients of $H(e^{i\theta})$ are zero.

Now the function $h(e^{i\theta}) = \overline{H(e^{i\theta})}$ is in $L^1[0, 2\pi]$ and has negative Fourier coefficients all zero, and thus by the theorem of F. and M. Riesz [13], $h(e^{i\theta})$ is the boundary function of $h(z) \in H^1(D)$. But $H(e^{i\theta})$ is the boundary function of $H(z) = (Jh)(z)$, that is $H(e^{i\theta}) \in H^1(D_e)$.

Thus $H(e^{i\theta}) = f(e^{i\theta})G(e^{i\theta})$ a.e., where $\overline{G(e^{i\theta})} = g(e^{i\theta}) \in \mathfrak{H}^q(D)$, with $g(e^{i\theta})$ not the zero function and thus $g(e^{i\theta}) = 0$ at most on a set of measure zero. Thus $f(e^{i\theta}) = \frac{H(e^{i\theta})}{G(e^{i\theta})}$ a.e. Now $H(z)$ and $G(z)$ are both in $H^1(D_e)$ and thus each is the quotient of two functions from $H^\infty(D_e)$; that is, $\frac{H(z)}{G(z)} \in N(D_e)$ has the boundary function $\frac{H(e^{i\theta})}{G(e^{i\theta})} = f(e^{i\theta})$ a.e.

This completes the proof.

In an attempt to generalize the proof of the previous theorem to the case H^1, the first place that trouble begins is in line (3.1): the function $g(e^{i\theta})$ is not necessarily the boundary function of an $H^\infty(D)$ function. About the most we can conclude is that $g(e^{i\theta}) \in \mathfrak{H}^p(D)$ for each $p < \infty$, which is an unfortunate consequence of fact that there is no bounded projection from L^∞ onto H^∞. Now in line (3.2) we can define $H(e^{i\theta}) = f(e^{i\theta})G(e^{i\theta})$ a.e., however, $H(e^{i\theta})$ is the product of an L^1 function with a function not necessarily in L^∞. The resulting $H(e^{i\theta})$ is in L^μ, for any $\mu \in (0,1)$, but this does not help if one wishes to speak of the Fourier coefficients of $H(e^{i\theta})$. We can get around these problems, but the proof becomes a bit complicated.

Theorem 3.4:

A necessary and sufficient condition for $f(z) \in H^1$ to be non-cyclic for U^* is that the following two conditions hold:

(i) f has pseudocontinuation $\tilde{f}$ across T with $\tilde{f}$ meromorphic on D_e.

(ii) $\tilde{f} \in N(D_e)$.

<u>Proof</u>:

As we have noted the proof of sufficiency in Theorem 3.3 holds for the case $p = 1$.

<u>Necessity</u>:

Assume $f(z) \in H^1$ is non-cyclic for U^*. We wish to show that conditions (i) and (ii) hold. Let $\sum_{n=0}^{\infty} a_n z^n$ be the Taylor series about the origin of $f(z)$. Assume $f(z) \not\equiv 0$ since if $f(z) \equiv 0$ the theorem is trivially true. By definition of non-cyclic the closure in H^1 of span $\{U^{*n}f\}_{n=0}^{\infty}$ is a proper closed subspace of H^1. Thus, by the Hahn-Banach theorem there exists a continuous linear functional $\phi \not\equiv 0$ on H^1 such that ϕ [span $\{U^{*n}f\}_{n=0}^{\infty}] = 0$.

By the Riesz representation theorem [15] and Theorem 2.24 there exist two functions $G(e^{i\theta})$, $g(z)$ such that

$$G(e^{i\theta}) \in L^{\infty}[0,2\pi],\ g(z) \in H^p, \quad \text{for all } p < \infty,$$

$$(3.3)\quad \phi(k) = \frac{1}{2\pi}\int_0^{2\pi} k(e^{i\theta})\overline{G(e^{i\theta})}\,d\theta, \quad \text{for all } k \in H^1$$

and

$$(3.4)\quad \phi(k) = \lim_{r\to 1^-} \frac{1}{2\pi}\int_0^{2\pi} k(re^{i\theta})\overline{g(e^{i\theta})}\,d\theta, \quad \text{for each } k \in H^1$$

We remark that $G(e^{i\theta})$ can be obtained by extending ϕ from H^1 to Φ on L^1 by the Hahn Banach theorem, and taking $G(e^{i\theta})$ to be the "Riesz representation theorem function" for the integral representation of Φ in 3.3. $G(e^{i\theta})$ is by no means unique. We also note that it was shown in Theorem 2.24 that

$$\hat{G}(n) = \hat{g}(n), \quad \text{for} \quad n = 0,1,+2,+3,\cdots$$

Now by Theorem 2.23, $\overline{g(e^{i\theta})}$ is the boundary function of $(Jg)(z) = \overline{g(1/\bar{z})}$. Also $(Jg) \in H^p(D_e)$ for any $p < \infty$ since $g \in H^p(D)$. Since $\lim_{r\to 1^-} f(re^{i\theta}) = f(e^{i\theta})$ exists for almost all $\theta \in [0,2\pi]$, we can define a function $h(e^{i\theta}) = \lim_{r\to 1^-} f(re^{i\theta})\overline{g(e^{i\theta})}$ a.e.

In Theorem 3.3 we defined a function similar to $h(e^{i\theta})$. In that theorem we had $f \in H^p$ and $g \in H^q$, $1 < p < \infty$, $q = \frac{p}{p-1}$ and thus $h(e^{i\theta})$ was in L^1 by the Hölder inequality [15]. The most we can get from the Hölder inequality here is that $h(e^{i\theta}) \in L^\mu$ for any $\mu \in (0,1)$. In subsequent portions of the proof we will need to know that $h(e^{i\theta})$ is in some fixed L^μ class, so we pick $L^{1/2}$. There is nothing special about this choice of μ as any other number in $(0,1)$ will work just as well.

Now for each value of r, $0 \le r < 1$ define

$$h_r(e^{i\theta}) = f(re^{i\theta})\overline{g(e^{i\theta})}$$

and

$$k_r(e^{i\theta}) = f(re^{i\theta})\overline{G(e^{i\theta})}$$

We stress that we are regarding $h_r(e^{i\theta})$ and $k_r(e^{i\theta})$ as functions of θ and we want to know what happens to these functions of θ as $r \to 1^-$. That $\lim_{r\to 1^-} h_r(e^{i\theta}) = h(e^{i\theta})$ exists a.e. has already been noted as has the fact that $h(e^{i\theta}) \in L^{1/2}[0,2\pi]$. Similarly $\lim_{r\to 1^-} k_r(e^{i\theta}) = \lim_{r\to 1^-} f(re^{i\theta})\overline{G(e^{i\theta})} = f(e^{i\theta})\overline{G(e^{i\theta})} = k(e^{i\theta})$ exists a.e. since $f \in H^1$. Furthermore,

$$\|k_r(e^{i\theta}) - k(e^{i\theta})\|_{L^1} = \|f(re^{i\theta})\overline{G(e^{i\theta})} - f(e^{i\theta})\overline{G(e^{i\theta})}\|_{L^1}$$

$$\leq \|G(e^{i\theta})\|_{L^\infty}\|f(re^{i\theta}) - f(e^{i\theta})\|_{L^1}$$

But by Theorem 2.11, $\|f(re^{i\theta}) - f(e^{i\theta})\|_{L^1} \to 0$ as $r \to 1^-$ and thus $\|k_r(e^{i\theta}) - k(e^{i\theta})\|_{L^1} \to 0$ as $r \to 1^-$.

We now use the hypothesis that f is non-cyclic for U^* to show that the non-negative Fourier coefficients of $h_r(e^{i\theta})$ converge to zero as $r \to 1^-$.

If we denote the Fourier coefficient of index n of $h_r(e^{i\theta})$ by $\hat{h}_r(n)$, then

$$0 = \phi(f) = \lim_{r\to 1^-} \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})\overline{g(e^{i\theta})}\,d\theta$$

$$= \lim_{r\to 1^-} \frac{1}{2\pi}\int_0^{2\pi} h_r(e^{i\theta})\,d\theta$$

$$= \lim_{r\to 1^-} \hat{h}_r(0)$$

Next,

$$0 = \phi(U^*f) = \lim_{r\to 1^-} \frac{1}{2\pi}\int_0^{2\pi} (U^*f)(re^{i\theta})\overline{g(e^{i\theta})}\,d\theta$$

$$= \lim_{r\to 1^-} \frac{1}{2\pi}\int_0^{2\pi} [f(re^{i\theta}) - f(0)]r^{-1}e^{-i\theta}\overline{g(e^{i\theta})}\,d\theta$$

$$= \lim_{r\to 1^-}\Big[\frac{1}{2\pi}r^{-1}\int_0^{2\pi} e^{-i\theta}f(re^{i\theta})\overline{g(e^{i\theta})}\,d\theta -$$

$$\frac{1}{2\pi}r^{-1}f(0)\int_0^{2\pi} e^{-i\theta}g(e^{i\theta})\,d\theta\Big]$$

$$= \lim_{r\to 1^-}[r^{-1}\hat{h}_r(+1) - 0]$$

$$= \hat{h}_r(+1)$$

since by Theorem 2.24 and Lemma 3.2 the Fourier coefficient of index +1 of $\overline{g(e^{i\theta})}$ is zero.

We proceed by induction. Assume that $\lim_{r\to 1^-} \hat{h}_r(k) = 0$ for $k = 0,1,\cdots,n-1$.

Set

$$p(z) = \sum_{n=0}^{n-1} a_j z^j$$

Then, by Theorem 2.19:

$$(U^{*n}f)(z) = \frac{f(z) - p(z)}{z^n}, \quad \text{for } 0 < |z| < 1.$$

so that

$$0 = \phi(U^{*n}f) = \lim_{r\to 1^-} \frac{1}{2\pi}\int_0^{2\pi} (U^{*n}f)(re^{i\theta})\overline{g(e^{i\theta})}\,d\theta$$

$$= \lim_{r\to 1^-} \frac{1}{2\pi}\int_0^{2\pi} [f(re^{i\theta}) - p(re^{i\theta})]r^{-n}e^{-in\theta}\overline{g(e^{i\theta})}\,d\theta$$

$$= \lim_{r\to 1^-}\left[\frac{1}{2\pi}r^{-n}\int_0^{2\pi} e^{-in\theta}f(re^{i\theta})\overline{g(e^{i\theta})} - \frac{1}{2\pi}r^{-n}\int_0^{2\pi} e^{-in\theta}\, p(re^{i\theta})\overline{g(e^{i\theta})}\,d\theta\right]$$

$$= \lim_{r\to 1^-}\left[r^{-n}\hat{h}_r(n) - \frac{r^{-n}}{2\pi}\sum_{j=0}^{n-1} a_j r^j\int_0^{2\pi} e^{i(j-n)\theta}\overline{g(e^{i\theta})}\,d\theta\right]$$

$$= \lim_{r\to 1^-}\left[r^{-n}\hat{h}_r(n)\right] - 0,$$

since for $j-n < 0$,

$$\frac{1}{2\pi}\int_0^{2\pi} e^{i(j-n)\theta}\overline{g(e^{i\theta})}\,d\theta$$

is a Fourier coefficient of positive index of $\overline{g(e^{i\theta})}$ and thus is zero by Theorem 2.24. We have thus shown that $0 = \lim_{r\to 1^-} \hat{h}_r(n)$, for $n = 0,1,2,\cdots$.

We now restate the functions which we have defined and some of what we have shown about them.

$$k_r(e^{i\theta}) = f(re^{i\theta})\overline{G(e^{i\theta})}$$

$$h_r(e^{i\theta}) = f(re^{i\theta})\overline{g(e^{i\theta})}$$

For r fixed, $0 < r < 1$,

$$k_r(e^{i\theta}) = \sum_{n=-\infty}^{\infty} \hat{k}_r(n)e^{in\theta}$$

$$h_r(e^{i\theta}) = \sum_{n=-\infty}^{\infty} \hat{h}_r(n)e^{in\theta}$$

where convergence of the two series above is in the L^2 sense.

Thus, by the Riesz-Fischer Theorem [13]

$$k_r(e^{i\theta}) = \sum_{n=-\infty}^{-1} \hat{k}_r(n)e^{in\theta} + \sum_{n=0}^{\infty} \hat{k}_r(n)e^{in\theta}$$

and

$$h_r(e^{i\theta}) = \sum_{n=-\infty}^{-1} \hat{h}_r(n)e^{in\theta} + \sum_{n=0}^{\infty} \hat{h}_r(n)e^{in\theta},$$

again r is still fixed and convergence of each series is in the L^2 sense, to an L^2 function.

Now by Lemma 3.2 we have

$$\hat{h}_r(-n) = \hat{k}_r(-n), \quad \text{for} \quad n = +1,+2,+3,\cdots.$$

Thus

$$\sum_{n=1}^{\infty} \hat{h}_r(-n)e^{-in\theta} = \sum_{n=1}^{\infty} \hat{k}_r(-n)e^{-in\theta} \quad \text{a.e.}$$

Now we have shown that $\|k_r(e^{i\theta}) - k(e^{i\theta})\|_{L^1} \to 0$ as $r \to 1^-$, and by Theorem 2.17 for any fixed r, $0 < r < 1$ we have

$$\left\| \sum_{n=0}^{\infty} \hat{k}_r(n)z^n - \sum_{n=0}^{\infty} \hat{k}(n)z^n \right\|_{H^{1/2}}^{1/2} \leq \|k_r(e^{i\theta}) - k(e^{i\theta})\|_{L^1}^{1/2} .$$

Now by the Riesz-Fischer Theorem [Hoffman, p.14], $\sum_{n=0}^{\infty} |\hat{k}_r(n)|^2 < \infty$. Hence by the Riesz-Fischer Theorem [Rudin, p. 332], we have $\sum_{n=0}^{\infty} \hat{k}_r(n)z^n \in H^2(D) \subset H^{1/2}(D)$ for $0 \leq r < 1$. Since $H^{1/2}$ is a complete space we have $\sum_{n=0}^{\infty} \hat{k}(n)z^n \in H^{1/2}(D)$.

Denote the boundary function of $\sum_{n=0}^{\infty} \hat{k}(n)z^n$ by $K(e^{i\theta})$ where $K(e^{i\theta}) \in \mathfrak{H}^{1/2}(D)$.

Also, note that $\sum_{n=0}^{\infty} \hat{k}_r(n)z^n \in H^2(D)$ has boundary function $\sum_{n=0}^{\infty} \hat{k}_r(n)e^{in\theta} \in L^2\,[0,2\pi]$. Now we have just shown that

$$\Big\| \sum_{n=0}^{\infty} \hat{k}_r(n)z^n - \sum_{n=0}^{\infty} \hat{k}(n)z^n \Big\|_{H^{1/2}}^{1/2} \to 0 \quad \text{as} \quad r \to 1^-.$$

But by Theorem 2.11, convergence in the $H^{1/2}$ metric implies convergence of the corresponding boundary functions in the $L^{1/2}$ metric. Thus $\sum_{n=0}^{\infty} \hat{k}_r(n)e^{in\theta}$ converges to $K(e^{i\theta})$ in the $L^{1/2}$ metric as $r \to 1^-$.

Also $k_r(e^{i\theta})$ converges in the L^1 metric to the L^1 function $k(e^{i\theta}) = f(e^{i\theta})G(e^{i\theta})$. Since L^1 converges implies $L^{1/2}$ convergence and $L^1 \subset L^{1/2}$ we have that $\sum_{n=-\infty}^{-1} \hat{k}_r(n)e^{in\theta}$ converges in the $L^{1/2}$ metric, as $r \to 1^-$, to the $L^{1/2}$ function $k(e^{i\theta}) - K(e^{i\theta})$.

Now $\sum_{n=-\infty}^{-1} \hat{h}_r(n)e^{in\theta}$ converges in the $L^{1/2}$ metric to the $L^{1/2}$ functions $k(e^{i\theta}) - K(e^{i\theta})$. Also $h_r(e^{i\theta})$ converges to $h(e^{i\theta}) = f(e^{i\theta})\overline{g(e^{i\theta})}$, in the $L^{1/2}$ metric $r \to 1^-$, and thus $\sum_{n=0}^{\infty} \hat{h}_r(n)e^{in\theta}$ converges in the $L^{1/2}$ metric as $r \to 1^-$ to an $L^{1/2}$ function, say $E(e^{i\theta}) \equiv h(e^{i\theta}) - [k(e^{i\theta}) - K(e^{i\theta})]$.

Now for each fixed r, $0 < r < 1$, $\sum_{n=0}^{\infty} \hat{h}_r(n)e^{in\theta}$ is the boundary function of $\sum_{n=0}^{\infty} \hat{h}_r(n)z^n \in H^2(D) \subset H^{1/2}(D)$. Thus since $\mathfrak{H}^{1/2}(D)$ is a closed subspace of $L^{1/2}$, we have $E(e^{i\theta}) \in \mathfrak{H}^{1/2}(D)$. Let $E(z)$ be the $H^{1/2}$ function whose

boundary function is $E(e^{i\theta})$. Now by Theorems 2.11 and 2.13 the convergence, as $r \to 1^-$ of $\sum_{n=0}^{\infty} \hat{h}_r(n)e^{in\theta}$ to $E(e^{i\theta})$ in $\mathfrak{H}^{1/2}$ implies that $\sum_{n=0}^{\infty} \hat{h}_r(n)z^n$ converges to $E(z)$ in $H^{1/2}$ as $r \to 1^-$. But, for $n = 0,1,2,\cdots$, $\lim_{r\to 1^-} \hat{h}_r(n) = 0$, by Corollary 2.26 the Taylor coefficients of $\sum_{n=0}^{\infty} \hat{h}_r(n)z^n$ converge to those of $E(z)$. Hence $E(z)$ has all of its Taylor coefficients zero and thus $E(z) \equiv 0$. Therefore $E(e^{i\theta})$ is the zero function.

We now note that $\sum_{n=-\infty}^{-1} \hat{h}_r(n)e^{in\theta} \in \mathfrak{H}^2(D_e) \subset \mathfrak{H}^{1/2}(D_e)$ for any r, $0 \le r < 1$, and thus as $r \to 1^-$, $h_r(e^{i\theta})$ converges to

$$h(e^{i\theta}) = \lim_{r\to 1^-} \sum_{n=-\infty}^{-1} \hat{h}_r(n)e^{in\theta} \in \mathfrak{H}^{1/2}(D_e).$$

Thus $h_r(e^{i\theta}) = f(re^{i\theta})\overline{g(e^{i\theta})}$ converges in $L^{1/2}$ metric to $h(e^{i\theta}) \in \mathfrak{H}^{1/2}(D_e)$. Now $\lim_{r\to 1^-} h_r(e^{i\theta}) = f(e^{i\theta})\overline{g(e^{i\theta})}$ and therefore $f(e^{i\theta})\overline{g(e^{i\theta})} = h(e^{i\theta}) \in \mathfrak{H}^{1/2}(D_e)$ and $\overline{g(e^{i\theta})} \in \mathfrak{H}^2(D_e)$. Also, $\overline{g(e^{i\theta})}$ is not the zero function and hence is zero only on a set of measure zero. Thus $f(e^{i\theta}) = \dfrac{h(e^{i\theta})}{\overline{g(e^{i\theta})}}$ a.e. and since both $h(e^{i\theta})$ and $\overline{g(e^{i\theta})}$ are quotients of bounded functions, $f(z)$ has a pseudocontinuation $\tilde{f}(z)$ across T, $\tilde{f}(z)$ is meromorphic on D_e and of bounded Nevanlinna characteristic on D_e.

2. Applications of the Characterization Theorem.

Theorems 3.3 and 3.4 enable us to determine certain classes of cyclic and non-cyclic functions. Since Theorems 3.3 and 3.4 are extensions of the H^2 result of Douglas, Shapiro, and Shields [3], the results of [3] for H^2 whose proofs depend only on pseudocontinuations carry over to H^p, $1 \le p < \infty$ with the proofs unchanged.

<u>Theorem 3.5</u>:

[3] If $f \in H^p$, $1 \le p < \infty$ and f is analytically continuable across all points of T with the exception of an isolated winding on T, then f is cyclic for U^*.

<u>Proof</u>:

It follows from the definition of pseudocontinuations that if f has an analytic continuation across any subarc of T that this analytic continuation must be the same as any pseudocontinuation across T.

<u>Examples</u>:

$f(z) = (z-1)^{1/2}$, for $|z| < 1$ (either branch)

$g(z) = \ln(z-1)$, for $|z| < 1$ (any branch)

Both $f(z)$ and $g(z)$ defined above are in $H^1(D)$. However, it is impossible to define a pseudocontinuation across all of T onto D_e for either $f(z)$ or $g(z)$.

Theorem 3.6:

[3] All Rational functions in H^p, $1 \le p < \infty$, are non-cyclic for U^*.

Proof:

Rational functions are meromorphic on $\mathbb{C}$, and of bounded Nevanlinna characteristic on D_e.

Corollary 3.7:

The non-cyclic functions for U^* are dense in H^p, $1 \le p < \infty$.

Theorem 3.8:

[3] If f is holomorphic in $|z| < R$ for some $R > 1$, then f is either cyclic or a rational function (and hence non-cyclic).

Proof:

By Theorem 3.6 rational functions are non-cyclic. If $f(z)$ is non-cyclic and holomorphic in $|z| < R$ with $R > 1$, then the pseudocontinuation of f, $\tilde{f}$, is an analytic continuation of f across T. Since f can be continued to be meromorphic on the Riemann Sphere, f is a rational function.

Theorem 3.9:

[3] Let f and g be non-cyclic and h be cyclic for U^*. Then $f + g$ is non-cyclic and $f+h$ is cyclic for U^*. Furthermore, fg and f/g are non-cyclic while fh and f/h are cyclic for U^* insofar as any of these are in H^p, $1 \le p < \infty$.

Proof:

Theorems 3.3 and 3.4.

Theorems 3.10:

[3] The set of non-cyclic vectors is a dense linear manifold in H^p, while the set of cyclic vectors is dense in H^p, $1 \le p < \infty$.

Proof:

Since the polynomials are non-cyclic and dense, the first part of the theorem follows from Theorem 3.9. For the second part let f be a fixed cyclic vector and p an arbitrary polynomial. $\{f+p\}$ where p ranges over all polynomials is dense in H^p, $1 \le p < \infty$ and by Theorem 3.9 this set consists only of cyclic vectors.

We conclude with an observation about non-cyclic vectors in H^p, $1 \le p < \infty$.

Theorem 3.11:

Let $f \in H^p$, $1 \leq p < \infty$, have pseudocontinuation $\tilde{f}$ across T with $\tilde{f}$ meromorphic on D_e and of bounded Nevanlinna characteristic on D_e. Then the closed linear span of $\{U^{*n}f\}_{n=0}^{\infty}$ is a proper closed invariant (under U^*) subspace of H^p.

Proof:

Theorems 3.3 and 3.4.

We remark that Douglas, Shapiro and Shields [3] have shown that in H^2 every closed invariant subspace for U^* is the closed linear span of $\{U^{*n}f\}_{n=0}^{\infty}$ for some $f \in H^2$. The proof of this theorem uses the Beurling theory for the invariant subspaces of U in H^2 as well as the fact that U^* is the adjoint of U (in H^2). Since the Beurling theory is not available in the pre-dual of H^1, it is not known whether the U^* invariant subspaces of H^1, are "cyclic".

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