ABSTRACT

STRUCTURE SPACES IN VECTOR LATTICES

By

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This thesis is concerned with the relationships between some topological properties on structure spaces of an archimedean vector lattice and vector lattice properties. In chapter 1, the representation theorem of Johnson and Kist is introduced and is used to obtain some relationships between structure spaces of one vector lattice embedded in another. It is shown that if an archimedean vector lattice L is embedded in a space of continuous functions over a topological space X, then the structure space of maximal ideals of L is homeomorphic to $X/\mathcal{R}$, where $\mathcal{R}$ is the equivalence relation generated by the stationary sets of L. It is then shown that if L has a strong order unit, its Dedekind completion $\hat{L}$ is the space of continuous functions over the minimal projective extension of the structure space of maximal ideals of L. In chapter 2, the interplay between projection properties and disconnectedness of structure spaces is studied. Some results of Masterson on discrete vector lattices and work by Luxemburg and Moore on vector lattices with archimedean quotient vector lattices are tied

together and the Luxemburg and Moore results are obtained more handily. Chapter 3 finds necessary and sufficient conditions for the space of extended functions to form an algebra.

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INTRODUCTION

A vector space over the reals which has an order relation compatible with the algebraic structure is called an ordered vector space. If the ordering is a lattice ordering, the space is called a vector lattice or Riesz space or K-lineal. The only spaces considered in this thesis are vector lattices.

In 1942, K. Yosida [16] showed that an archimedean vector lattice can be represented as a vector lattice of continuous functions with compact support over a locally compact space. H. Nakano [11], B. Z. Vulikh [15], and W. A. J. Luxemburg and A. C. Zaanen [8] also give representations of archimedean vector lattices as continuous functions over a suitable topological space. D. G. Johnson and J. E. Kist [6] present a representation theorem and show that it includes the other representations. They show that the topological spaces for the representations are just structure spaces of prime ideals in the vector lattice. The Johnson and Kist representation is used in chapter 1 to obtain theorems on embedding vector lattices. In particular, it is shown that for a vector lattice L with a strong order unit, the Dedekind completion $\hat{L}$ of L is precisely $C(X)$ where X is the minimal projective extension of the space of maximal ideals. If L has only a weak order unit, then $\hat{L}$ is a foundation in $D(X)$, the space of extended functions on the minimal projective extension of the structure space of prime ideals maximal with respect to not containing the unit.

In 1928, F. Riesz [13] initiated the study of vector lattices by showing that in a Dedekind complete vector lattice every order closed ideal is a direct summand of the vector lattice; i.e., every band is a projection band. H. Nakano [11] shows that for a $C(X)$, X is extremally disconnected if and only if $C(X)$ is Dedekind complete. D. G. Johnson and J. E. Kist [5] extend this result to vector lattices, but with weaker conditions; i.e., X is a suitable structure space of prime ideals, and Dedekind completeness is replaced by the projection property. In chapter 2, it is shown that a vector lattice L has the projection property if and only if the space of maximal ideals of L is homeomorphic to the space of maximal ideals of $\hat{L}$. Thus the uniform completion of L is $\hat{L}$, by Stone-Weierstrass. It is pointed out that the analogous result for the principal projection property and the Dedekind σ-completion is not true. In some sense, this extends an investigation of structure spaces in the Dedekind completion by J. J. Masterson [10]. In that paper, Masterson also studies some properties of discrete vector lattices. These results are linked to some results of Luxemburg and Moore [7] and Luxemburg and Zaanen [8] on vector lattices where every quotient vector lattice is archimedean. The Luxemburg and Moore results are obtained more easily. A structure space characterization for sufficiently many projections is also obtained in this chapter.

M. Henriksen and D. G. Johnson [4] give necessary and sufficient conditions for $D(X)$, the space of extended functions, to form an algebra. In chapter 3, it is shown that this condition characterizes an F-space as defined by L. Gillman [2]. Vector lattices, whose structure space of maximal ideals form an F-space, are then studied.

Chapter 0. Preliminaries

In this section some elementary concepts are listed. For a more complete exposition, the reader is referred to Peressini [17], Luxemburg and Zaanen [8] or Vulikh [15].

DEFINITION 0.1: A *vector lattice* L is a real vector space with a lattice ordering that is compatible with the vector space structure; i.e.,

i) $x \le y \Rightarrow x + z \le y + z$ for every $z \in L$,

ii) $x \ge \theta \Rightarrow \lambda x \ge \theta$ for $\lambda \ge 0$, where θ is the identity element of the vector space, and

iii) for every $x, y \in L$, $\sup\{x,y\} = x \vee y$, and $\inf\{x,y\} = x \wedge y$ exist.

DEFINITION 0.2: $L^+ = \{x \in L: x \ge \theta\} \subset L$ is called the *positive cone* of L.

DEFINITION 0.3: The *positive part* of an element $x \in L$ is $x^+ = x \vee \theta$. The *negative part* of x is $x^- = (-x) \vee \theta$. The *absolute value* of x is $|x| = x^+ + x^-$.

DEFINITION 0.4: Let $x, y \in L$. x and y are said to be disjoint if $|x| \wedge |y| = \theta$, and is written $x \perp y$.

<u>DEFINITION 0.5</u>: If L and L' are vector lattices, then a mapping $\Psi: L \to L'$ is a <u>homeomorphism</u> if for $a \in \mathcal{R}$, $x,y \in L$;

i) $\Psi(ax) = a\,\Psi(x)$,

ii) $\Psi(x+y) = \Psi(x) + \Psi(y)$, and

iii) $\Psi(x \vee y) = \Psi(x) \vee \Psi(y)$.

<u>DEFINITION 0.6</u>: $L' \subset L$ is said to be <u>order dense</u> in L if for every $x \in L$ there is a set $\{x_\alpha: \alpha \in \mathfrak{A}\} \subset L'$ with $\sup\{x_\alpha: \alpha \in \mathfrak{A}\} = x$. L' is <u>quasi-order dense</u> if for each $x \in L^+$, there is $x' \in (L')^+$ such that $\theta \le x' \le x$.

<u>DEFINITION 0.7</u>: L is said to be <u>archimedean</u> if for every $x \in L^+$ $\inf\{\frac{1}{n}x: n = 1,2,\cdots\} = \theta$. Alternatively, if $x,y \in L^+$ and $\lambda x \le y$ for all $\lambda \in \mathfrak{R}^+$, then $x = \theta$.

In an archimedean vector lattice, the notions of order dense and quasi-order dense coincide. In the next chapter the Johnson and Kist version [5] of the result that a vector lattice is archimedean if and only if it is isomorphic to a vector lattice of continuous functions into the two-point compactification of the reals is presented.

<u>DEFINITION 0.8</u>: A linear subspace M of a vector lattice L is said to be a <u>vector sublattice</u> of L if for each $x,y \in M$, $x \vee y \in M$. A linear subspace $M \subset L$ is

an ideal of L if it is a vector sublattice and whenever for $x \in M$, $y \in L$, $|y| \le |x|$, $y \in M$.

<u>DEFINITION 0.9</u>: A vector lattice L is said to be <u>Dedekind complete</u> (Dedekind σ-complete) if every bounded subset (countable bounded subset) has a supremum in L.

<u>THEOREM 0.10</u>: If L is a vector lattice, then there exists a minimal Dedekind complete vector lattice $\hat{L}$ such that L can be embedded as an order dense subvector lattice of $\hat{L}$ if and only if L is archimedean. $\hat{L}$ is called the Dedekind completion of L.

<u>DEFINITION 0.11</u>: A Dedekind complete vector lattice in which every set of pairwise disjoint elements is bounded is said to be <u>universally complete</u>.

<u>THEOREM 0.12</u>: For a vector lattice L, there exists a minimal universally complete vector lattice $L^{\#}$ containing L as an order dense subvector lattice if and only if L is archimedean. $L^{\#}$ is called the universal completion of L.

If $\{x_\alpha\}_{\alpha\in\mathfrak{A}}$ is a monotone increasing (decreasing) net, in a vector lattice L, with supremum (infimum) x in L, we write $x_\alpha \uparrow x$ ($x_\alpha \downarrow x$).

<u>DEFINITION 0.13</u>: A net $\{x_\alpha\}_{\alpha\in\mathfrak{A}}$ is said to <u>order converge</u> to $x \in L$ if it is order bounded and there exists a net $\{y_\alpha\}_{\alpha\in\mathfrak{A}} \subset L^+$ such that

$$|x_\alpha - x| \le y_\alpha \downarrow \theta$$

We denote this by $x_\alpha \to x$. A net $\{x_\alpha\}_{\alpha\in\mathfrak{A}}$ is said to converge <u>e-relatively uniformly</u> to x if for arbitrary $\epsilon > 0$ there exists $\alpha_0 \in \mathfrak{A}$ such that for $\alpha > \alpha_0$, $|x_\alpha - x| < \epsilon\, e$.

<u>DEFINITION 0.14</u>: A <u>principal ideal</u> is an ideal generated by a single element; i.e., I is a principal ideal if there exists $x_0 \in I$ such that $I = \{x\in L\colon |x| \le n|x_0|$ for some integer $n\}$.

<u>DEFINITION 0.15</u>: A <u>band</u> is an order closed ideal.

<u>DEFINITION 0.16</u>: Let A be a subset of L. Then $A^\perp = \{x \in L\colon x \perp a$ for every $a \in A\}$.

<u>THEOREM 0.17</u>: L is archimedean if and only if $A^{\perp\perp} = A$ for every band A.

<u>THEOREM 0.18</u>: $A^\perp$ is a band for every subset A of L.

<u>DEFINITION 0.19</u>: B is a projection band if $L = B \oplus B^\perp$.

DEFINITION 0.20: L has the projection property (P.P.) if every band is a projection band.

DEFINITION 0.21: L has the principal projection property (P.P.P.) if every principal band is a projection band.

DEFINITION 0.22: An ideal $P \subset L$ is prime if for $x, y \in L$, $x \wedge y \in P$ implies that $x \in P$ or $y \in P$.

THEOREM 0.23: (Johnson and Kist [5]) The following are equivalent:

i) P is a prime ideal

ii) if $x \wedge y = \theta$, then $x \in P$ or $y \in P$

iii) the quotient vector lattice L/P is totally ordered

iv) if $A \cap B \subset P$ for ideals A, B, then either $A \subset P$ or $B \subset P$.

DEFINITION 0.24: Let $\mathcal{B}$ be a collection of prime ideals. Then the kernel of $\mathcal{B}$, $k(\mathcal{B})$, is the intersection of the prime ideals in $\mathcal{B}$;

$$k(\mathcal{B}) = \cap\{P : P \in \mathcal{B}\}.$$

DEFINITION 0.25: Let A be a subset of L. Then the hull of A, $h(A)$, is the set of prime ideals which contain A.

For any collection of prime ideals $\mathfrak{M}$, it is well known that $\overline{A} = h(k(A))$ for any subset $A \subset \mathfrak{M}$ defines a closure operation on $\mathfrak{M}$ and hence a topology (see [2], section 4.9). This topology is known as the hull-kernel topology, Stone topology, or Zariski topology. It is readily seen that $\mathfrak{M}_a = \{P \in \mathfrak{M}: a \notin P\}$ forms a basis for this topology.

Chapter 1. Structure Spaces

In this chapter, the representation theorem of Johnson and Kist [5] is presented. This becomes the principal tool of the section. Conditions for a structure space to be Hausdorff and for a structure space to be separated by the vector lattice are found. Gleason [3] showed that extremally disconnected spaces are projective in the category of compact sets and continuous functions. Mack and Johnson [9] observe that this is also true for the category of completely regular spaces and fitting maps. They further show the existence of an essentially unique space, the minimal projective extension, which lies above a given space. We show that there is essentially only one space for which a vector lattice L is a separating family of functions, and thus, by Theorem 1.16, if L has a strong order unit, the Dedekind completion of L is $C(X)$ where X is the minimal projective extension of the maximal ideal space.

DEFINITION 1.1: If L is an archimedean vector lattice, then a structure space for L is a collection of prime ideals of L such that the collection has θ intersection and is endowed with the hull-kernel topology.

Let $\underline{3}$ denote the three point space $\{\underline{-\infty}, \underline{0}, \underline{+\infty}\}$ which is ordered by $\underline{-\infty} < \underline{0} < \underline{+\infty}$. Define addition by

$$(\underline{+\infty}) + (\underline{+\infty}) = \underline{+\infty}$$
$$(\underline{-\infty}) + (\underline{-\infty}) = \underline{-\infty}$$
$$(\underline{+\infty}) + (\underline{0}) = \underline{+\infty}, \quad (\underline{-\infty}) + (\underline{0}) = \underline{-\infty}$$

$(\underline{-\infty}) + (\underline{+\infty})$ is undefined. Scalar multiplication for $\underline{3}$ is defined by

$$\alpha\underline{0} = \underline{0}, \quad 0(\underline{\pm\infty}) = \underline{0}$$
$$\alpha(\underline{+\infty}) = \underline{+\infty}, \quad \alpha(\underline{-\infty}) = \underline{-\infty} \quad \text{for} \quad \alpha > 0$$
$$\alpha(\underline{+\infty}) = \underline{-\infty}, \quad \alpha(\underline{-\infty}) = \underline{+\infty} \quad \text{for} \quad \alpha < 0$$

<u>DEFINITION 1.2</u>: A function f from a vector lattice L to $\underline{3}$ is said to be a <u>spectral function</u> if

i) $f(x) \neq \underline{0}$ for at least one $x \in L$,

ii) $f(\alpha x) = \alpha f(x)$ for each $x \in L$, $\alpha \in \mathfrak{R}$, and

iii) $f(x \vee y) = f(x) \vee f(y)$ for $x, y \in L$.

D. G. Johnson and J. E. Kist [5] show that the space of all prime ideals in a vector lattice L is homeomorphic and order anti-isomorphic to the space of all spectral functions on L, when the ideals are given the hull-kernel topology. If $P \subset L$ is a prime ideal of L, P defines a spectral function by

$$P(f) = \begin{cases} \underline{0} & f \in P \\ \underline{+\infty} & f^{+} \notin P \\ \underline{-\infty} & f^{-} \notin P \end{cases}$$

DEFINITION 1.3: An extended function on a topological space X is a continuous map from X into the two-point compactification of the reals, which is real-valued on a dense subset of X. The set of all extended functions on X is denoted by $D(X)$.

Let X be a topological space and $f, g \in D(X)$, $\alpha \in R$. Then αf, $f \wedge g$, and $f \vee g$ are defined pointwise. If there is a function $h \in D(X)$ such that $h(x) = f(x) + g(x)$ whenever $f(x)$ and $g(x)$ are finite, then h is called the sum of f and g. h is unique since where f and g are finite is an open dense set in X. In general, the sum does not exist. When the sum always exists is studied in Chapter 3. For this investigation, the Johnson and Kist representation will be used. So, for each $x \in L^+$, L being an archimedean vector lattice with weak order unit $\underline{1}$, define for each prime ideal P,

$$\begin{aligned} x(P) &= \inf\{\alpha \in \mathfrak{R}: P(x) \leq \alpha P(\underline{1})\} \\ &= \inf\{\alpha \in \mathfrak{R}: (\alpha\underline{1}-x)^- \in P\} \end{aligned}$$

where the infimum of the empty set is $+\infty$. Then $x \in L$ defines an extended function on certain structure spaces. The representation theorem will now be developed. L will always be an archimedean vector lattice.

LEMMA 1.4: If $b \in L$, and if $\{y_\lambda : \lambda \in \Lambda\}$ is a collection of elements in $b^\perp$ such that $y = \sup\{y_\lambda : \lambda \in \Lambda\}$ exists, then $y \in b^\perp$.

<u>Proof</u>:

$I^{\perp}$ is a band for any $I \subset L$.

<u>LEMMA 1.5</u>: If $\mathcal{B}$ is a structure space, then $b^{\perp} = k(\mathcal{B}_b)$, where $\mathcal{B}_b = \{P \in \mathcal{B}: b \notin P\}$, for every $b \in L$.

<u>Proof</u>:

If $|y| \wedge |b| = \theta$ and $P \in \mathcal{B}_b$, then $y \in P$, since P is prime. Hence, $y \in k(\mathcal{B}_b)$; i.e. $b^{\perp} \subseteq k(\mathcal{B}_b)$.

Now consider $y \in k(\mathcal{B}_b)$. If $P \in \mathcal{B}_b$, $|y| \wedge |b| \in P$. If $P \notin \mathcal{B}_b$, then $|y| \wedge |b| \in P$. So $|y| \wedge |b| \in \cap\{P: P \in \mathcal{B}\}$. But then $|y| \wedge |b| = \theta$. Hence, $y \in b^{\perp}$ implying $k(\mathcal{B}_b) \subset b^{\perp}$. Thus, $b^{\perp} = k(\mathcal{B}_b)$.

We are now ready to present the Johnson and Kist representation theorem [5]. Johnson and Kist show that this theorem includes the representations given by K. Yosida [16], H. Nakano [11], and W.A.J. Luxemburg and A.C. Zaanen [8] by showing that the topological spaces obtained by these authors are homeomorphic to structure spaces. For more details see [5].

<u>THEOREM 1.6</u>: (Johnson and Kist [5]) Let L be a vector lattice with weak order unit $\underline{1}$. Let $\mathfrak{M}$ be the structure space of all prime ideals not containing $\underline{1}$. Then L is isomorphic to a vector lattice of extended functions on $\mathfrak{M}$.

Proof:

Let $x \in L^+$. Then define

$$\bar{x}(P) = \inf\{\alpha \in \mathfrak{R}: P(x) \le \alpha P(\underline{1})\}$$

(For arbitrary x, set $\bar{x}(P) = \overline{x^+}(P) - \overline{x^-}(P)$. Since $x^+ \wedge x^- = \theta$, $\bar{x}$ is well-defined.) To show $\bar{x}$ is continuous, a) let $\bar{x}(P) = +\infty$. If $\alpha \in R$, then $(\alpha\underline{1} - x^-) \notin P$. Let $c = \underline{1} \wedge (\alpha\underline{1}-x)^-$. Then $c \notin P$, $P \in \mathcal{B}_c \subseteq \mathfrak{m}$. For any $Q \in \mathcal{B}_c$, $(\alpha\underline{1} - x)^- \notin Q$; hence, $\bar{x}(Q) \ge \alpha$. Thus, $\bar{x}$ is continuous at P; b) now consider $\bar{x}(P)$ finite valued; i.e. $\bar{x}(P) = \alpha_0$. For $\varepsilon > 0$, $((\alpha_0 + \varepsilon)\underline{1} - x)^+ \notin P$ and $((\alpha_0 - \varepsilon)\underline{1} - x)^- \notin P$. Set

$$c = \underline{1} \wedge ((\alpha_0+\varepsilon)\underline{1}-x)^+ \wedge ((\alpha_0-\varepsilon)\underline{1}-x)^-$$

Then, $P \notin \mathcal{B}_c \subseteq \mathfrak{m}$. If $Q \in \mathcal{B}_c$

$$((\alpha_0+\varepsilon)\underline{1}-x)^+ \notin Q \quad \text{and}$$

$$((\alpha_0-\varepsilon)\underline{1}-x)^- \notin Q$$

Thus, $\bar{x}(Q) \le \alpha_0 + \varepsilon$ and $\bar{x}(Q) \ge \alpha_0 - \varepsilon$. Hence, $\bar{x}$ is continuous at each point.

To show that $\bar{x}$ is an extended function on $\mathfrak{m}$, it is necessary to show that where $\bar{x}$ is finite valued is an open dense subset of $\mathfrak{m}$. We show that the infinity set of $\bar{x}$ is nowhere dense. Suppose not. Then there is a basic open set $\mathcal{B}_b$, $\theta \ne b \in L^+$, which is contained in the infinity set of $\bar{x}$. So for every $P \in \mathcal{B}_b$, $\bar{x}(P) = +\infty$. Then

$$(n\underline{1}-x)^+ \in P \quad \text{or} \quad (\underline{1} - (x/n))^+ \in P, \; n = 1,2,\cdots$$

Hence, by lemma 1.5,

$$(\underline{1} - \frac{x}{n})^+ \in b^\perp, \quad n = 1,2,\cdots$$

Let $y \geq (\underline{1} - \frac{x}{n})^+$ for all n. Then

$$\underline{1} - y \leq \frac{x}{n}, \quad n = 1,2,\cdots$$

Since L is archimedean, $\underline{1} \leq y$. But $\underline{1} \geq (\underline{1} - \frac{x}{n})^+$, $n = 1,2,\cdots$ Thus, $\underline{1} = \sup_n \{(\underline{1} - \frac{x}{n})^+\}$. So by lemma 1.4, $\underline{1} \in b^\perp$. But $\underline{1}$ is a weak order unit. Thus, $b = \theta . \Rightarrow\Leftarrow$ So $\overline{x}$ is an extended function.

It is now necessary to show that the mapping $x \to \overline{x}$ is a vector lattice homomorphism: Let P be a prime ideal in $\mathfrak{M}$. Then

$$\overline{x+y}\ (P) = \inf \{\alpha: (\alpha\underline{1} - (x+y))^- \in P\}$$

$$= \inf \{\alpha + \beta: (\alpha\underline{1} - x + \beta\underline{1} - y)^- \in P\}.$$

Since for arbitrary $f,g \in L$

$$(f+g)^- \leq f^- + g^- \quad \text{and} \quad (f+g)^+ \leq f^+ + g^+$$

we have

$$(\alpha\underline{1} - x)^-, (\beta\underline{1} - y)^- \in P \Rightarrow (\alpha\underline{1} - x + \beta\underline{1} - y)^- \in P$$

Hence,

$$\overline{x+y}\ (P) \leq \overline{x}(P) + \overline{y}(P).$$

Conversely, let $\alpha < \overline{x}(P)$, $\beta < \overline{y}(P)$. Then $(\alpha\underline{1} - x)^- \notin P$ and $(\beta\underline{1} - y)^- \notin P$. Thus, since

$$(\alpha\underline{1}-x)^+ + (\beta\underline{1}-y)^+ \geq (\alpha\underline{1}-x + \beta\underline{1}-y)^+$$

we have $(\alpha\underline{1}-x + \beta\underline{1}-y)^- \notin P$. Hence,

$$\overline{x}(P) + \overline{y}(P) \leq \overline{x+y}\ (P)$$

So addition is preserved. Now let $\alpha \in \mathfrak{R}$, then

$$\overline{\alpha x}(P) = \inf\{\beta\colon (\beta \underline{1} - \alpha x)^- \in P\}$$

$$= \inf\{\beta\colon (\tfrac{\beta}{\alpha}\underline{1} - x)^- \in P\}$$

$$= \inf\{\alpha\beta\colon (\beta\underline{1} - x)^- \in P\}$$

$$= \alpha\overline{x}(P).$$

So scalar multiplication is preserved. Finally

$$\overline{x \vee y}(P) = \inf\{\alpha\colon (\alpha\underline{1} - x \vee y)^- \in P\}$$

$$= \inf\{\alpha\colon [(\alpha\underline{1} - x) \vee (\alpha\underline{1} - y)]^- \in P\}$$

$$= \inf\{\alpha\colon [(\alpha\underline{1} - x)^- \wedge (\alpha\underline{1} - y)^-] \in P\}$$

$$= \inf\{\alpha\colon (\alpha\underline{1} - x)^- \in P\} \vee \inf\{\alpha\colon (\alpha\underline{1} - y)^- \in P\}$$

$$= \overline{x}(P) \vee \overline{y}(P).$$

Thus, $x \to \overline{x}$ is a vector lattice homomorphism. To see that this is an isomorphism, let $\overline{x} \equiv 0$. Then $(\frac{1}{n}\underline{1} - x)^- \in P$ for $n = 1, 2, \cdots$ By lemma 1.5,

$$((\tfrac{1}{n}\underline{1}) - x)^- \in \underline{1}^\perp \quad \text{for all } n.$$

Since L is archimedean

$$x = \sup\{(\tfrac{1}{n}\underline{1} - x)^-\colon n = 1, 2, \cdots\}$$

Thus, by lemma 1.4, $x \in \underline{1}^\perp$. Hence, $x = \theta$.

<u>DEFINITION 1.7</u>: A set of functions A on a topological space X is said to <u>separate</u> <u>points</u> of X (or separates X)

if for every pair of points $x, y \in X$, there is a function $f \in A$ such that $f(x) = 0$ and $f(y) = 1$.

A vector lattice need not separate points of a structure space. In general structure spaces are not Hausdorff. The following theorem may be known, but does not seem to be in the literature. It points out that for structure spaces, T_1 and Hausdorff are equivalent.

<u>THEOREM 1.8</u>: Let L be an archimedean vector lattice. Let $\mathfrak{M}$ be a structure space for L. Then $\mathfrak{M}$ is Hausdorff if and only if for every pair of distinct prime ideals $P, Q \in \mathfrak{M}$, $P \not\subset Q$.

<u>Proof</u>:

($\Rightarrow$) Let $\mathfrak{M}$ be Hausdorff, P, Q distinct elements of $\mathfrak{M}$. Then there exist basic open neighborhoods, $\mathfrak{M}_x = \{I \in \mathfrak{M} : x \notin I\}$, $\mathfrak{M}_y = \{I \in \mathfrak{M} : y \notin I\}$, with $x, y \in L^+$ such that $P \in \mathfrak{M}_y$, $Q \in \mathfrak{M}_x$ and $\mathfrak{M}_y \cap \mathfrak{M}_x = \emptyset$. Then $y \notin P$, $x \notin Q$ and $x \wedge y = \theta$ (since $\{I \in \mathfrak{M} : y \notin I\} \cap \{I \in \mathfrak{M} : x \notin I\} = \emptyset$, either x or y belongs to each $I \in \mathfrak{M}$. Thus, $x \wedge y = \theta$.) Since P, Q are prime, $x \in P$, $y \in Q$ and so $P \not\subset Q$.

($\Leftarrow$) Let $\mathfrak{M}$ be such that $P \not\subset Q$ for arbitrary $P, Q \in M$. Then there exist $x' \in P$, $x' \notin Q$ and $y' \in Q$, $y' \in P$, with $x', y' \in L^+$. Set $x = x' - (x' \wedge y')$, $y = y' - (x' \wedge y')$. Then

$$x \wedge y = \theta, \quad x \in P, \; x \notin Q, \; y \in Q, \; y \notin P.$$

Thus, $\mathcal{M}_x, \mathcal{M}_y$ are disjoint open sets in $\mathcal{M}$ separating P and Q.

<u>COROLLARY 1.9</u>: The structure space of all prime ideals is Hausdorff if and only if all prime ideals are maximal.

Vector lattices satisfying corollary 1.9 will be studied in chapter 2. We now turn to the question of what spaces are separated by L. It is clear that L separates the structure space of maximal ideals. We now characterize what structure spaces are separated by L. Let us restrict our study to structure spaces of prime ideals not containing $\underline{1}$.

<u>THEOREM 1.10</u>: Let L be an archimedean vector lattice with strong order unit $\underline{1}$. Let $\mathcal{M}$ be a structure space for L and P,Q be distinct elements of $\mathcal{M}$. Then P and Q are separated by L if and only if P,Q are contained in distinct maximal ideals (with respect to not containing $\underline{1}$).

<u>Proof</u>:

Let P be a non-maximal prime ideal in L. Then P is contained in a unique maximal ideal M. M is the relative uniform closure of P. (The only relatively uniformly closed prime ideals are maximal since L/P is totally ordered for prime P and the only totally ordered archimedean vector lattice is $\mathfrak{R}$. A maximal ideal is relatively uniformly closed since L/M is archimedean if and only if M is relatively uniformly closed [7]). We show that for $f \in L^+$

$$f(P) = 0 \text{ if and only if } f(M) = 0.$$

($\Leftarrow$) Suppose $f(M) = 0$. Then $\inf\{\alpha: (\alpha\underline{1} - f)^- \in M\} = 0$;

$$\text{i.e., } \inf\{\alpha: (f-\alpha\underline{1})^+ \in M\} = 0$$

Hence,

$$(f - \alpha\underline{1})^+ \in M \text{ for all } \alpha > 0.$$

In particular,

$$(f - \tfrac{1}{n}\underline{1})^+ \in M, \quad n = 1,2,\cdots$$

But

$$f = \lim_{n\to\infty} (f - \tfrac{1}{n}\underline{1})^+$$

Since M is relatively uniformly closed, $f \in M$. Now consider the quotient map, $\tau: L \to L/P$. Since P is prime, L/P is totally ordered. L/P is not archimedean since P is not maximal. L/P contains only one maximal ideal, M/P, since L/P is totally ordered. Note that L/P contains the real numbers $(\{\alpha\tau(1): \alpha \in \mathfrak{R}\})$, and that M/P is the set of "infinitesimal" elements in L/P; i.e., for $x \in M$,

$$|\tau(x)| \leq \alpha\tau(\underline{1}) \text{ for every positive } \alpha.$$

We will denote $\tau(x)$ by $\overline{x}$. For $\theta \leq f \in M$,

$$0 = \inf\{\alpha: (\alpha\underline{1} - f)^- \in M\}$$

$$= \inf\{\alpha: (\alpha\overline{\underline{1}} - \overline{f})^- \in M/P\}$$

If $(\alpha\overline{\underline{1}} - \overline{f})^- = \overline{\theta}$, $(\alpha\overline{\underline{1}} - f) \geq \overline{\theta}$ and if $(\alpha\underline{1}-\underline{\overline{f}})^- = -\alpha\overline{\underline{1}} + \overline{f}$, then $\alpha\overline{\underline{1}} - \overline{f} \notin M/P$, Thus:

$$0 = \inf\{\alpha: \alpha\overline{\underline{1}} - \overline{f} \geq \overline{\theta}\}$$

Since, $\alpha\overline{\underline{1}} \notin M/P$, $a \neq 0$, and L/P is totally ordered,

$$\alpha\overline{\underline{1}} - \overline{f} > \overline{\theta} \quad \text{for} \quad \alpha > 0$$

$$\alpha\overline{\underline{1}} - \overline{f} < \overline{\theta} \quad \text{for} \quad \alpha \leq 0$$

Hence,

$$0 = \inf\{\alpha: (\alpha\overline{\underline{1}} - \overline{f}) \geq \overline{\theta}\}$$

$$= \inf\{\alpha: (\alpha\overline{\underline{1}} - \overline{f}) = \overline{\theta}\}$$

$$= \inf\{\alpha: (\alpha\underline{1} - f)^- \in P\} = f(P)$$

Thus, if $f(M) = 0$, then $f(P) = 0$.

$(\Rightarrow)$ Suppose now that $f(P) = 0$. Then

$$f(P) = \inf\{\alpha: (\alpha\underline{1} - f)^- \in P\}$$

$$\geq \inf\{\alpha: (\alpha\underline{1} - f^-) \in M\} = f(M)$$

Thus, if $f(P) = 0$, then $f(M) = 0$.

Hence, if P and Q are in distinct maximal ideals they are separated.

<u>DEFINITION 1.11</u>: Let X and Y be topological spaces. A continuous map $\rho: X \to Y$ is said to be <u>tight</u> if for

each open set $U \subset X$, there exists an open set $V \subset Y$ so that $\rho^{-1}(V) \subset U$.

<u>PROPOSITION 1.12</u>: Let $\mathcal{M}$ be the structure space of maximal (with respect to not containing $\underline{1}$) ideals of an archimedean vector lattice L with strong order unit $\underline{1}$. Let $\mathcal{B}$ be any structure space of prime ideals. Then there is a continuous tight map from $\mathcal{B}$ onto $\mathcal{M}$.

<u>Proof</u>:

Define $\rho\colon \mathcal{B} \to \mathcal{M}$ by $\rho(P) = M$, where M is the unique maximal ideal containing $P \in \mathcal{B}$. Let $\{M \in \mathcal{M}: f \notin M\}$ be a basic open set in $\mathcal{M}$. Then

$$\{M \in \mathcal{M}: f \notin M\} = \{M \in \mathcal{M}: f(M) \neq 0\}.$$

$$\rho^{-1}(\{M \in \mathcal{M}: f(M) \neq 0\}) = \{P \in \mathcal{B}: P \subset M,\ f(M) \neq 0\}$$
$$= \{P \in \mathcal{B}: f(P) \neq 0\},$$

which is an open set in $\mathcal{B}$. Hence, ρ is continuous. ρ is tight since for each basic open set in $\mathcal{B}$, $\{P \in \mathcal{B}: x \notin P\}$,

$$\{P \in \mathcal{B}: x(P) \neq 0\} \subset \{P \in \mathcal{B}: x \notin P\}.$$

So structure spaces bear some relationship to the maximal ideal space of a vector lattice. The extent of this relationship is not clear. But we now show that there is essentially only one compact space for which an archimedean vector lattice is a separating family of functions.

<u>THEOREM 1.13</u>: Let L be an archimedean vector lattice with strong order unit $\underline{1}$. Let L be contained in a $C(X)$, where X is a compact topological space. Let $\mathcal{R}$ be the equivalence relation defined by $x \sim y$ if $f(x) = f(y)$ for every $f \in L$. Then $X/\mathcal{R}$ is homeomorphic to $\mathfrak{m}(L)$, the structure space of all maximal ideals in L.

<u>Proof</u>:

Let $Y = X/\mathcal{R}$ and $\rho\colon X \to Y$ be the natural projection. Let M_x be the maximal ideal in L associated with x; i.e.,

$$M_x = \{f \in L\colon f(x) = 0\}.$$

Since, $M_x = M_y$ if and only if $x \sim y$, $\tau\colon X/\mathcal{R} \to \mathfrak{m}(L)$ defined by

$$\tau(\rho(x)) = M_x$$

is a bijection between Y and $\mathfrak{m}(L)$ is quasi-compact (it need not be Hausdorff) since X is compact. Let

$$\mathfrak{m}(a) = \{M \in \mathfrak{m}(L)\colon a \in M\}$$

be a basic closed set in $\mathfrak{m}(L)$. Then

$$\begin{aligned}
\rho^{-1} \circ \tau^{-1}(\mathfrak{m}(a)) &= \rho^{-1}(\{\rho(x)\colon a \in M_x\}) \\
&= \{x \in X\colon a \in M_x\} \\
&= \bigcap_{\substack{f \in M_x \\ M_x \in \mathfrak{m}(a)}} \{x\colon f(x) = 0 \text{ for } f \in M_x\}
\end{aligned}$$

But this is a closed set. Hence, τ is continuous. Since $\mathfrak{M}(L)$ is Hausdorff and Y is quasicompact, τ is a homeomorphism.

Throughout this thesis we will use $\mathfrak{M}(L)$ to denote the structure space of maximal ideals if L has a strong order unit or the structure space of ideals maximal with respect to not containing $\underline{1}$, if $\underline{1}$ is a weak order unit. We now show that if L is a vector lattice embedded in L' (another vector lattice) and the embedding is order dense and $\underline{1} \in L$ goes to $\underline{1}' \in L'$, then $\mathfrak{M}(L')$ bears the "same" relation to $\mathfrak{M}(L)$ as a structure space in L, i.e., there exists a tight map from $\mathfrak{M}(L')$ to $\mathfrak{M}(L)$.

<u>THEOREM 1.14</u>: Let L', L be archimedean vector lattices, $L \subset L'$ is a subvector lattice and L, L' share a weak order unit $\underline{1}$. Then there exists a continuous onto function $\rho\colon \mathfrak{M}(L') \to \mathfrak{M}(L)$. If L is order dense in L', the mapping is tight.

<u>Proof</u>:

Let $\mathcal{R}$ be the equivalence relation on $\mathfrak{M}(L')$ defined by $P \sim Q$ if $f(P) = f(Q)$ for every $f \in L$. Then by theorem 1.13, $\mathfrak{M}(L')/\mathcal{R}$ is homeomorphic to $\mathfrak{M}(L)$. Then the projection $\rho\colon \mathfrak{M}(L') \to \mathfrak{M}(L)$ is the continuous onto map.

Now let $\{M \in \mathfrak{M}(L')\colon a \notin M\}$ be a basic open set of $\mathfrak{M}(L')$. Let $x \in L$ such that $\theta < x < |a|$.

Then, if $M \in \rho^{-1}(\{I \in \mathfrak{M}(L): x \notin I\})$, $M \in \rho^{-1}(\{I \in \mathfrak{M}(L): x(I) \neq 0\})$. Hence, $x(M) \neq 0$ and $x \notin M$ and $a \notin M$. Since $x \neq \theta$, $\{I \in \mathfrak{M}(L): x \notin I\}$ is not empty. Thus if L is order dense in L', ρ is a tight map.

This is really an embedding theorem. If L can be embedded in another vector lattice L', then there is a map $\rho: \mathfrak{M}(L') \to \mathfrak{M}(L)$, via the theorem to the image vector lattice; and the homeomorphism of the image structure space and the structure space. Then the embedding of L into L' is given by

$$f' = f \circ \rho, \quad \text{for all} \quad f \in L.$$

This guarantees the embedding into a space of continuous functions. Then using the following result of J. E. Mack and D. G. Johnson [9], which is an extension of a result of A. J. Gleason [3], we can find the Dedekind completion of a vector lattice.

<u>THEOREM 1.15</u>: (Mack and Johnson) Every completely regular topological space Y is the continuous image of an extremally disconnected space, Y_∞, under a tight fitting map, τ (a map is fitting if it is closed, onto, and the inverse image of a point is compact). If Z is another extremally disconnected space and σ is a tight fitting map onto Y, then there is a homeomorphism ρ

of Y_∞ onto Z such that $\sigma \circ \rho = \tau$. Y_∞ is called the minimal projective extension of Y.

<u>THEOREM 1.16</u>: Let L be an archimedean vector lattice with strong order unit $\underline{1}$. Then the Dedekind completion, $\hat{L}$, of L is $C(\mathfrak{M}(L)_\infty)$, the continuous real-valued functions on the minimal projective extension of $\mathfrak{M}(L)$.

<u>Proof</u>:

By theorem 1.15, $\mathfrak{M}(L)_\infty$ is unique. Let $\rho\colon \mathfrak{M}(L)_\infty \to \mathfrak{M}(L)$ be the projection map. ρ is tight fitting and embeds L into $C(\mathfrak{M}(L)_\infty)$. But $\hat{L} \equiv C(\mathfrak{M}(\hat{L}))$. Hence, $\mathfrak{M}(\hat{L})$ is embedded in $\mathfrak{M}(L)_\infty$ as a closed subspace. Then by the uniqueness of $\mathfrak{M}(L)_\infty$, $\mathfrak{M}(\hat{L})$ is homeomorphic to $\mathfrak{M}(L)_\infty$.

If L has only a weak order unit, by restricting ourselves to the bounded elements, it is easy to see that $\hat{L}$ is an order dense ideal in $D(\mathfrak{M}(L)_\infty)$. Thus, $L^\#$, the universal completion of L, is $D(\mathfrak{M}(L)_\infty)$.

In view of the tight map from structure spaces to $\mathfrak{M}(L)$ in an arbitrary vector lattice, a natural question, still unresolved, is whether the minimal projective extension is a structure space.

Chapter 2. Disconnectivity of Structure Spaces

We now wish to study the relationship of certain properties of a vector lattice with disconnectivity in the structure spaces. We obtain a necessary and sufficient condition for a vector lattice L to have the projection property and show that the analogous conjecture for the principal projection property is not true. We then study a property stronger than P.P.P.; i.e, when every principal ideal is a projection band. This provides a connection between results of J. J. Masterson [10] and W.A.J. Luxemburg and L. C. Moore [7]. We obtain the Luxemburg and Moore results more easily. Finally we characterize "sufficiently many projections" in terms of disconnectivity of structure spaces.

DEFINITION 2.1: A topological space X is extremally disconnected if the closure of every open subset of X is open. X is basically disconnected if the closure of every open F_σ set is open.

H. Nakano [11] has shown that a topological space X is extremally disconnected if and only if $C(X)$ is a Dedekind complete vector lattice. It is shown, in the same paper, that X is basically disconnected if and only if

$C(X)$ is Dedekind σ-complete. Since for compact X, the structure space of all maximal (algebraic or vector lattice) ideals in $C(X)$ is homeomorphic to X, these results are relations between properties on a vector lattice and topological properties on structure spaces. Johnson and Kist [6] have extended these results to $\emptyset$-algebras (archimedean lattice-ordered algebras which have for a weak order unit the identity element). Before the study of these relationships can begin, we must establish some relations between bands and structure spaces. For the following, let L be an archimedean vector lattice (L need not have an order unit), $\mathcal{M}$ be some structure space of L. Then, if I is a subset of L, $h(I) = \{P \in \mathcal{M}: I \subset P\}$ is the hull of I in $\mathcal{M}$. If $\mathcal{J}$ is a subset of $\mathcal{M}$, then $k(\mathcal{J}) = \cap\{P: P \in \mathcal{J}\}$ is the kernel of $\mathcal{J}$. γ' will denote the set complement of γ in $\mathcal{M}$. The following can be found in [6].

<u>PROPOSITION 2.2</u>: If I is a subset of L, then $I^{\perp} = k(h(I)')$. Thus $h(I^{\perp}) = \overline{h(I)'}$.

<u>PROPOSITION 2.3</u>: If $\mathcal{L}$ is a closed subset of $\mathcal{M}$, then $k(\mathcal{L})^{\perp} = k(\mathcal{L}')$. Hence, $k(\mathcal{U})$ is a band whenever $\mathcal{U}$ is an open set.

<u>PROPOSITION 2.4</u>: If I is a projection band in L, then $h(I)$ is open in $\mathcal{M}$.

The converse to 2.4 is not true. If L is the vector lattice of continuous functions on the one-point compactification of the natural numbers $\mathfrak{N}$, $\alpha\mathfrak{N}$, then L has a structure space of maximal ideals homeomorphic to $\mathfrak{N}$. Then the ideal

$$I = \{f \in L: f^{-1}(0) \subseteq 2\mathfrak{N}\}$$

is not a projection band, but it is a band. The hull of I is $2\mathfrak{N}$ which is open and closed in $\mathfrak{N}$. Thus the open and closed sets of an arbitrary structure space are not necessarily associated with a projection band. However, if the following condition is imposed on a structure space a correspondence can be established:

(*) if $\mathcal{L}, \mathcal{J} \subseteq \mathfrak{M}$ and $k(\mathcal{L}) + k(\mathcal{J}) \neq L$, then $k(\mathcal{L}) + k(\mathcal{J}) \subseteq P$ for some $P \in \mathfrak{M}$, or equivalently,

(*) two subsets $\mathcal{L}, \mathcal{J}$ of $\mathfrak{M}$ have disjoint closures in $\mathfrak{M}$ if and only if $k(\mathcal{L}) + k(\mathcal{J}) = L$.

<u>PROPOSITION 2.5</u>: Let $\mathfrak{M}$ be a structure space that satisfies (*). Then $\mathfrak{U} \subseteq \mathfrak{M}$ is open and closed if and only if $k(\mathfrak{U})$ is a projection band.

<u>PROPOSITION 2.6</u>: For a vector lattice L, the following are equivalent:

i) L has P.P.; i.e., every band is a projection band,

ii) every structure space of L is extremally disconnected,

iii) L has an extremally disconnected structure space that satisfies (*).

So Nakano's result on the Dedekind completeness of $C(X)$ and extremal disconnectivity of $X = (\mathcal{M}(C(X)))$ cannot be extended to a vector lattice and $\mathcal{M}(L)$. In particular, we have that $\mathcal{M}(L)$ may be extremally disconnected but L not be Dedekind complete. An example of this is the vector lattice of real valued sequences with finite range. However, after making the following observation, it is easy to see that the projection property is very close to Dedekind completeness.

<u>PROPOSITION 2.7</u>: Let L be an archimedean vector lattice with strong order unit $\underline{1}$. Then the structure space of all maximal ideals, $\mathcal{M}(L)$, satisfies (*).

<u>Proof</u>:

Let $\mathcal{L}, \mathcal{T}$ be subsets of $\mathcal{M}(L)$. Then

$$k(\mathcal{L}) + k(\mathcal{T}) = \{f \in L: f(P) = 0 \text{ where } P \in \overline{\mathcal{L}} \cap \overline{\mathcal{T}}\}$$

If $\overline{\mathcal{L}} \cap \overline{\mathcal{T}} \neq \emptyset$, choose $M \in \overline{\mathcal{L}} \cap \overline{\mathcal{T}}$. Then

$$k(\mathcal{L}) + k(\mathcal{T}) \subset M$$

If $\overline{\mathcal{L}} \cap \overline{\mathcal{T}} = \emptyset$, then

$$k(\mathcal{L}) + k(\mathcal{T}) = L.$$

In view of proposition 1.17, the compactness of $\mathcal{M}(L)$, and the fact that L separates points of $\mathcal{M}(L)$, we have that the minimal projective extension of $\mathfrak{M}(L)$ is $\mathfrak{M}(L)$ via the Stone-Weierstrass theorem; i.e.,

<u>THEOREM 2.8</u>: Let L be a vector lattice with weak order unit $\underline{1}$. Then L has P.P. if and only if $\mathfrak{M}(L)$ is homeomorphic to $\mathfrak{M}(\hat{L})$.

This observation raises an interesting question when taken with some results of J. J. Masterson. In [10], Masterson studies the existence of homeomorphisms between structure spaces of a vector lattice and structure spaces of its Dedekind completion. He shows that if L has P.P., the space of minimal prime ideals of L is homeomorphic to the space of minimal prime ideals of $\hat{L}$. So we have homeomorphisms between the respective maximal ideal spaces and minimal prime ideal spaces. But, in general as Masterson has shown, the natural mapping $P \to \hat{P}$ of prime ideals does not yield a homeomorphism. So we now raise the question of what happens when we collapse the maximal ideals to minimal prime ideals, i.e., when the structure space of all prime ideals is Hausdorff. In [6], Johnson and Kist show

<u>PROPOSITION 2.9</u>: Every ideal is the intersection of all prime ideals containing it.

In [10], Masterson shows

<u>PROPOSITION 2.10</u>: Let $\mathcal{M}$ be a structure space for an archimedean vector lattice L. Then $\mathcal{M}_x = \{P \in M: x \notin P\}$ is closed for all $x \in L$ if and only if $\mathcal{M}$ is a structure space of minimal prime ideals.

Thus in order for every element in L to have an open hull in every structure space, it is necessary and sufficient that every prime ideal be minimal. We can now prove:

<u>PROPOSITION 2.11</u>: Let L be an archimedean vector lattice. Every prime is minimal if and only if $(x) = \langle x \rangle$, where (x) is the ideal generated by x and $\langle x \rangle$ is the band generated by x, for every $x \in L$ and L has P.P.P.

<u>Proof</u>:

($\Rightarrow$) Let $\mathcal{M}$ = {all prime ideals in L}. Then by 2.9, $(x) = \cap\{P \in \mathcal{M}: x \in P\}$

Since $\mathcal{M}$ has topological dimension 0 with $\mathcal{M}_x$ as a basis of open and closed sets, (x) is the kernel of an open set, its hull. Then by proposition 2.3, $(x) = \langle x \rangle$. Furthermore, $\mathcal{M}$ has property (*). Thus, since $\{P \in \mathcal{M}: x \in P\} = h(x)$ is an open and closed set, $\langle x \rangle = (x)$ is a projection band by proposition 2.5.

($\Leftarrow$) Suppose $(x) = \langle x \rangle$ and L has P.P.P.

Then $h(x) = h((x)) = h(\langle x \rangle)$

$\langle x\rangle$ is a projection band; hence $h(\langle x\rangle) = h(x)$ is open by proposition 2.4. But $h(\langle x\rangle)' = \mathfrak{m}_x$. So by proposition 2.10, all prime ideals are minimal.

W. A. J. Luxemburg and L. C. Moore [7] have shown that the conditions $(x) = \langle x\rangle$ and P.P.P are equivalent to L having the property that every quotient vector lattice is archimedean. Using this and some information about P-spaces, we will link some of Masterson's results with those of Luxemburg and Moore.

<u>DEFINITION 2.12</u>: A topological space X is a <u>P-space</u> if for $C(X)$, $M_\rho = \{f \in C(X): f(\rho) = 0\}$ is precisely $O_\rho = \{f \in C(X): f(U) = 0$ for some neighborhood u of $\rho\}$ for each point $\rho \in X$. A point $\rho \in X$ for an arbitrary topological space X is a <u>P-point</u> if $M_\rho = O_\rho$; i.e., every zero set containing ρ contains a neighborhood of ρ.

A P-space is a topological space for which every prime ideal in $C(X)$ is maximal. So this is the class of spaces for which we wish to investigate $C(X)$. It is well-known that minimal prime vector lattice ideals coincide with the minimal prime algebraic ideals for $C(X)$ ([2], 14.7). For a compact X, the maximal ideals also coincide. Now notice that for a function f to be continuous at a P-point,

f must be constant on a neighborhood of that point. This immediately yields that a compact P-space is finite. It also says that if X is a P-space, $C(X) \equiv D(X)$. For more detailed information on P-spaces, we refer the reader to Gillman and Jerison [2]. We now characterize vector lattices for which every prime ideal is maximal and which are uniformly complete by using the proposition developed above and this information on P-spaces.

<u>THEOREM 2.13</u>: Let L be an archimedean vector lattice with weak order unit $\underline{1}$. Let L, further, be uniformly complete. Then every structure space $\mathfrak{M}$ is a P-space if and only if for each $x \in L^*$, the bounded elements of L, $(x) = \langle x \rangle$ and L^* has P.P.P.

<u>Proof</u>:

It is sufficient to show that the space of all maximal ideals with respect to not containing $\underline{1}$ is a P-space, since there will be no other prime ideals. So, let $\mathfrak{M}$ be this space of maximal ideals.

($\Rightarrow$) Assume $\mathfrak{M}$ is a P-space. Then, by the representation theorem, $L \subset D(\mathfrak{M})$ and $L^* = C(\mathfrak{M})$, since L^* is uniformly complete and separates points on $\mathfrak{M}$. Since $\mathfrak{M}$ is a P-space all algebraic prime ideals are maximal in $C(\mathfrak{M})$. Then by the foregoing remarks, the prime vector lattice ideals are maximal. Thus, in $L^* = C(X)$, $(x) = \langle x \rangle$ and L^* has P.P.P. by 2.11.

($\Leftarrow$) Assume $(x) = \langle x\rangle$ and L^* has P.P.P. Then prime vector lattice ideals in $L^* = C(X)$ are minimal. Hence, again by the above remarks, all prime algebraic ideals are minimal, hence maximal. Thus, $\mathfrak{M}$ is a P-space.

The condition that L have a unit is not necessary. However, if L does not have a unit, then we must consider structure spaces with respect to a complete orthogonal set $\{x_\alpha \in L^+ : \alpha \in \mathfrak{A}\}$. The structure spaces then investigated are subsets of $\bigcup_{\alpha\in\mathfrak{A}} \{P: x_\alpha \notin P\}$. The structure space of maximal ideals with respect to $\underline{1}$ becomes $\bigcup_{\alpha\in\mathfrak{A}} \{M: x_\alpha \notin M\}$ where M is maximal with respect to not containing x_α for each fixed x_α. Each set $\mathfrak{M}_\alpha = \{M: x_\alpha \notin M\}$ is compact; the $\mathfrak{M}_\alpha$ are pairwise disjoint since $x_\alpha \wedge x_\beta = \theta$ if $\alpha \neq \beta$. So the structure space $\mathfrak{M} = \cup\mathfrak{M}_\alpha$ is locally compact. Using the representation of the component of $f \in L$ on each compact set $\mathfrak{M}_\alpha$ we obtain Yosida's representation of L as a vector lattice of continuous extended functions with compact support on a locally compact space (see [5]). With this remark 2.13 becomes:

<u>THEOREM 2.13'</u>: Let L be a uniformly complete vector lattice; $\{x_\alpha : \alpha \in \mathfrak{A}\}$ be a complete orthogonal set of positive elements. Then the structure space $\mathfrak{M} = \cup\mathfrak{M}_\alpha$ is a P-space if and only if for each $x \in L$, $(x) = \langle x\rangle$ and L has P.P.P.

Thus, it follows from 2.11 that a vector lattice L has every quotient vector lattice archimedean if and only if all prime ideals are maximal. Moreover, if $\langle x \rangle$ is a principal band, then $\langle x \rangle$ has compact support. But x is an order unit for $\langle x \rangle$. So if L is uniformly complete, then, by 2.13, the support of $\langle x \rangle$ is a P-space and thus finite. If we take $x = x_\alpha$ from the complete orthogonal set, we see that $\mathfrak{M}$ is a discrete space. So we have the following result relating vector lattices with archimedean quotient spaces to discrete vector lattices.

<u>PROPOSITION 2.14</u>: Let L be a vector lattice. Then the following are equivalent:

i) every prime ideal is maximal,

ii) every quotient vector lattice of L is archimedean,

iii) every ideal is relatively-uniformly closed,

iv) every principal ideal is a projection band.

We are now prepared to link Masterson's result with Luxemburg and Moore's.

<u>THEOREM 2.15</u>: Let L be a uniformly complete vector lattice, $\{x_\alpha : \alpha \in \mathfrak{A}\}$ a complete, maximal orthogonal set, $\mathfrak{M} = \cup \mathfrak{M}_\alpha$ as above. Then the following are equivalent.

i) $\mathfrak{M}$ is a discrete topological space,

ii) $\mathfrak{M}$ is a set of closed prime ideals in L,

iii) $\mathfrak{M} \setminus \{P\}$ is not a structure space for any $P \in \mathfrak{M}$,

iv) every prime ideal is maximal,

v) every quotient vector lattice is archimedean,

vi) every principal ideal is finite dimensional,

vii) L is isomorphic to the vector lattice of all real functions on a discrete space which vanish outside an appropriate finite subset,

viii) L is super-Dedekind complete and every quotient vector lattice is archimedean.

Proof:

The equivalence of i), ii), and iii) was obtained by Masterson [10]. The equivalence of i), iv), v), vi), vii) are easy to see from 2.13, 2.13′. The equivalence of viii) follows since $\mathfrak{M}$ is discrete, it is extremally disconnected and so L has P.P., by 2.6. Since L is uniformly complete, L is Dedekind complete. L is super Dedekind complete since each $x \in L$ has finite support. Thus i) $\Leftrightarrow$ viii).

The equivalence of v), vi), vii), and viii) was previously done by W.A.J. Luxemburg and L. C. Moore [7]. The key theorem for the proof is 2.13. We noted that the unit was not necessary for the hypothesis. The following example will show that uniform completeness is necessary for theorem 2.13 and hence for 2.15. The example will point out something else. In theorem 2.8, we noted that, since P.P. and uniform completeness imply Dedekind completeness, a vector lattice with

P.P. is close enough to its Dedekind completion to yield a homeomorphism between the structure spaces of maximal ideals. It is therefore natural, knowing that P.P.P. with uniform completeness imply Dedekind σ-complete, to ask the same question about vector lattices with P.P.P. The answer, however, is no homeomorphism necessarily exists. So Dedekind σ-complete is in some sense farther from P.P.P. than Dedekind complete is from P.P. The example will show that a vector lattice may have P.P.P. and a unit, but the structure space of maximal ideals is not basically disconnected.

<u>EXAMPLE 2.16</u>: Let L be the vector lattice of finitely non-constant sequences. L has P.P.P. and is not Dedekind σ-complete. So it is not uniformly complete. L has as a strong order unit the constant sequence $\underline{1}$. The structure space of maximal ideals $\mathfrak{M}$ is a $\alpha\mathfrak{N}$, the one point compactification of the natural numbers. $\alpha\mathfrak{N}$ is not basically disconnected since $\overline{2\mathfrak{N}}$ is not an open set.

The converse to the question is true, however:

<u>PROPOSITION 2.17</u>: Let L be an archimedean vector lattice with weak order unit $\underline{1}$. Then, if $\mathfrak{M}(L)$, the structure space of ideals maximal with respect to not containing $\underline{1}$, is basically disconnected, L has P.P.P.

Proof:

Let $x \in L$. Consider $F = \{M \in \mathfrak{M}(L) : x \in M\}$. F is a closed G_δ set since $F = \{M \in \mathfrak{M}(L) : x(M) = 0\}$. F is the hull of (x); i.e., $F = h(x)$. Then F' is an open F_σ set. Hence,

$$h(x^{\perp}) = \overline{h(x)'} = \overline{F'}$$

is the closure of an open F_σ. Hence, $h(x^{\perp})$ is open and closed. But then by 2.5, $x^{\perp}$ is a projection band. Thus, $\langle x \rangle = x^{\perp\perp}$ is a projection band.

It is not known whether there are conditions weaker than Dedekind σ-complete which will guarantee basically disconnected structure spaces. It is easy to see that if L has P.P.P., then $\mathfrak{M}(L)$, indeed any structure space, has topological dimension 0. If L has a strong order unit and L has the property that for every closed G_δ set F of $\mathfrak{M}(L)$, there exists an $f \in L$ such that $f^{-1}(0) = F$, then L has P.P.P. if and only if $\mathfrak{M}(L)$ is basically disconnected. But this essentially hypothesizes basically disconnected structure spaces. If P.P.P. does not characterize basic disconnectivity in structure spaces, but does imply topological dimension 0, does sufficiently many projections imply 0-dimension? That SMP (every band contains a projection band) does not imply 0-dimension is seen from

EXAMPLE 2.18: Let $X = (\{0\} \times [0,1]) \cup \{(x,y): x,y$ are rational, $0 \le x, y \le 1\}$. X, with the relative topology from the plane, is not 0-dimensional, but has the property that every open set contains an open and closed set. We show that $C^*(X)$ has SMP. Let $\mathcal{B}$ be a band in $C^*(X)$. Then there exists a closed set F in βX, such that $f \in \mathcal{B}$ if and only if $f(F) = 0$, where βX is the Stone-Cech compactification of X and f is identified with its extension to βX. $\beta X \setminus F$ is open. Hence, $(\beta X \setminus F) \cap X$ is open in X. Then if $(\beta X \setminus F) \cap X$ is non-void, it contains an open and closed set, θ. Then $cl_{\beta X}(\theta)$, the closure of θ in βX, is an open and closed set and is contained in the support of $\mathcal{B}$. But then $\{f \in C^*(X): \text{supp } f \subset cl_{\beta X}(\theta)\}$ is a projection band contained in $\mathcal{B}$. Hence, while $C^*(X)$ has SMP, X is not 0-dimensional. Thus a vector lattice L can have SMP, but $\mathcal{M}(L)$ not be 0-dimensional, even when L is uniformly complete. The following theorem does give the condition which characterizes SMP.

THEOREM 2.19: Let L be an archimedean vector lattice. Then L has SMP if and only if $\mathcal{M}(L)$ has the property that every open set contains an open and closed set.

Proof:

($\Rightarrow$) Suppose L has SMP. Then we must show that for $f \in L$, $\{P: f \notin P\}$ contains an open and closed set. Consider $<f>$. There exists a projection band

$\mathcal{B} \subset \langle f\rangle$ since L has SMP. Then

$$\text{supp } \mathcal{B} = \{P \in \mathcal{M}(L): \mathcal{B} \not\subset P\}$$

is an open and closed set by 2.5. Since $\mathcal{B} \subset \langle f\rangle$, supp $\mathcal{B} \subset$ supp f. Then since supp $\mathcal{B}$ is open,

$$\text{supp } \mathcal{B} \subset \text{Interior (supp f)} = \{P \in \mathcal{M}(L): f \notin P\}$$

The second equality is because if $f \in P$, $P \in \overline{(\text{supp } f)}'$. So $\mathcal{M}(L)$ has the property.

($\Leftarrow$) Now let $\mathcal{M}(L)$ have the given property. We show that every principal band contains a projection band. Let $f \in L$. Then

$$\langle f\rangle = \{x \in L: \text{supp } x \subset \text{supp } f\}$$

supp $f \subset \mathfrak{U}$, where $\mathfrak{U}$ is an open and closed set by the property. Then

$$\{P: f \notin P\} \subset \text{supp } f \subset \mathfrak{U}.$$

Hence,

$$\langle f\rangle = \cap\{\sigma: f \notin P\} \supset k(\mathfrak{U}).$$

$k(\mathfrak{U})$ is a projection band since $\mathfrak{U}$ is open and closed. Since every band contains a principal band, L has SMP.

Chapter 3. Some Remarks on D(X).

In general, although an archimedean vector lattice with weak order unit can be represented as a vector lattice of extended functions on a space X, D(X) does not possess a linear structure. In this chapter, we discuss a property which will allow the extension of the vector lattice structure to D(X). For this class of vector lattices it is then possible to extend the vector lattice to a $\emptyset$-algebra without passing to the universal completion as has been done in other extensions.

<u>DEFINITION 3.1</u>: A subset s of a topological space X is said to be <u>C^*-embedded</u> in X if every continuous function which is bounded on s can be extended to a continuous function on X. s is <u>C-embedded</u> in X if every continuous function on s can be extended to X.

<u>DEFINITION 3.2</u>: A topological space X is an <u>F-space</u> if every two open disjoint F_σ-sets have disjoint closures; i.e., every pair of open disjoint F_σ sets is completely separated.

A $\emptyset$-algebra is a lattice ordered algebra with a weak order unit that is the identity element. These objects have been extensively studied by M. Henriksen and D. G. Johnson [4] and P. Nanzetta [12]. Henriksen and Johnson have established

THEOREM 3.3: Let X be a compact space. Then $D(X)$ is an algebra of extended functions if and only if each open, everywhere dense F_σ set in X is C^*-embedded in X.

THEOREM 3.4: (Tietze's Extension Theorem) X is a normal space if and only if every closed set is both C- and C^*-embedded in X.

THEOREM 3.5: (Urysohn's Extension Theorem) A subspace s of X is C^*-embedded if and only if any two completely separated sets in s are completely separated in X.

We are now ready to show that F-spaces are precisely those which have $D(X)$ as an algebra.

THEOREM 3.6: Let X be a compact space. Then $D(X)$ is an algebra if and only if X is an F-space.

Proof:

($\Rightarrow$) Let $D(X)$ be an algebra. Let A, B be disjoint open F_σ-sets in X. If A and B are completely separated, then X is an F-space.

$\overline{A \cup B}$ is a compact set in X. By Tietze's extension theorem $\overline{A \cup B}$ is C^*-embedded in X. Since every open dense F_σ-set, $\mathcal{U}$, in $\overline{A \cup B}$ is the intersection of an open dense F_σ-set, V, in X with $\overline{A \cup B}$, $D(\overline{A \cup B})$ is an algebra. (We obtain multiplication by extending continuous bounded functions from $\mathcal{U}$ in $\overline{A \cup B}$ to V by Tietze's theorem, then extending to X, multiplying in $D(X)$ and restricting down). $A \cup B$ is an open dense set in $\overline{A \cup B}$. A, B are completely separated in $A \cup B$; hence, they are completely separated in $\overline{A \cup B}$. Since $\overline{A \cup B}$ is C^*-embedded in X, A and B are completely separated in X by Urysohn's theorem. So X is an F-space.

($\Leftarrow$) Let X be an F-space. By the Henriksen and Johnson result, it is sufficient to show that every open dense F_σ set is C^*-embedded. So let $\mathcal{U}$ be an open dense F_σ-set. Then $X \setminus \mathcal{U}$ is a closed G_δ. Then, since every closed G_δ is the zero set for some continuous function, there is an $h \in C(X)$ with $X \setminus \mathcal{U} = \{x \in X: h(x) = 0\}$. Let A and B be completely separated in $\mathcal{U}$. Then there is some $k \in C^*(\mathcal{U})$ with $k(A) > 0$ and $k(B) < 0$. Then $k^{-1}(\mathbb{R}^+), k^{-1}(\mathbb{R}^-)$ are open disjoint F_σ sets, and thus completely separated. Define

$$g(x) = \begin{cases} 0, & x \in X \setminus \mathcal{U} \\ k(x)\,|h(x)|, & x \in \mathcal{U} \end{cases}$$

Since k is bounded, g is continuous on all of X. Since A, B are contained in completely separated sets, they are completely separated. So $\mathfrak{U}$ is C^*-embedded by Urysohn's theorem.

<u>DEFINITION 3.7</u>: A vector lattice L has the <u>countable interpolation property</u> (CIP) if for every pair of countable sets A and B with the property that $x \le y$ for arbitrary $x \in A$, $y \in B$, there exists $x_0 \in L$ such that $x \le x_0 \le y$ for every $x \in A$, $y \in B$.

<u>PROPOSITION 3.8</u>: If L is order separable and has CIP, then L is Dedekind complete.

<u>Proof</u>:

Let A be a bounded set in L. Then there is some $M \in L^+$ such that $|x| \le M$ for all $x \in A$. Since L is order separable, let $\{x_n\}$ be a countable set such that every $f \in L$, $f \ge A$ if and only if $f \ge x_n$, $n = 1,2,\cdots$ Let $\beta = \{x \in L : x \ge A\}$. Again, there is a countable set $\{y_n\}$ such that $f \le B$ if and only if $f \le y_n$, $n = 1,2,\cdots$ Now since L has CIP there is an x_0 such that $x_n \le x_0 \le y_m$, $n,m = 1,2,3,\cdots$ So x_0 is an upper bound for A. However, $x_0 \le y_m$, $m = 1,2,\cdots$ Hence, x_0 is a lower bound for B. Thus, $x_0 = \sup A$.

PROPOSITION 3.9: (G. L. Seever [14]) Let X be a compact Hausdorff space. Then X is an F-space if and only if $C(X)$ has CIP.

PROPOSITION 3.10: Let $D(X)$ be an order separable $\emptyset$-algebra. Then $D(X)$ is universally complete.

Proof:

X is an F-space by 3.6. Then by 3.9, $C(X)$ has CIP. So by 3.8, $C(X)$ is Dedekind complete. Thus, X is extremally disconnected by 2.6. But then $D(X)$ is universally complete.

COROLLARY 3.11: Let L be an archimedean vector lattice with weak order unit $\underline{1}$. If L has CIP, then $\mathfrak{M}(L)$ is an F-space.

PROPOSITION 3.12: Let L be an archimedean vector lattice with CIP. Then L is uniformly complete.

Proof:

Let f_n be a monotone increasing sequence which is relatively uniformly Cauchy; i.e., for $\varepsilon > 0$, there is N such that for $n, m \geq N$,

$$|f_n - f_m| < \varepsilon x.$$

Let $f = \sup_n f_n$ in the uniform completion of L. f_n converges to f x-relatively uniformly. Define $\{g_n\}_{n=1}^{\infty}$ by

$$g_n = f_N + \frac{1}{n} x$$

where $|f_N - f| < \frac{1}{n} x$. $g_n \in L$ and f_N can be chosen so that $g_n \downarrow f$ x-relatively uniformly. Hence, $g_n \geq f$. Now by CIP, there is $f_O \in L$ such that

$$f_n \leq f_O \leq g_m, \quad n,m = 1,2,\cdots$$

Hence, $f_O = f$ and L is uniformly complete.

<u>COROLLARY 3.13</u>: If L has P.P.P., then L is uniformly complete if and only if L has CIP if and only if L is Dedekind σ-complete.

<u>COROLLARY 3.14</u>: If L has P.P., then L is Dedekind complete if and only if L has CIP if and only if L is uniformly complete.

It can easily be seen that multiplication can be defined on a vector lattice with CIP by extending the structure to $D(\mathfrak{M}(L))$ and using the multiplication there. This says that is is not necessary, for an arbitrary archimedean vector lattice, to go to the universal completion to define multiplication. It is sufficient to obtain the Dedekind σ-completion and use the associated $D(X)$ for that. The countable interpolation property is not as strong as Dedekind σ-complete. In terms of structure spaces, it is quite far from Dedekind σ-completeness. If a vector lattice is Dedekind σ-complete, it possesses basically disconnected structure spaces. The CIP guarantees nothing.

EXAMPLE 3.15: Let $X = \beta\mathfrak{R}^+ \setminus \mathfrak{R}^+$. This is an F-space since for any locally compact, σ-compact space X, $\beta X \setminus X$ is a compact F-space. We show that $\beta\mathfrak{R}^+ \setminus \mathfrak{R}^+$ is connected. Suppose not. Then there is a continuous function $f: X \to \{0,1\}$. This has an extension to $C(\beta(\mathfrak{R}^+))$ since $\beta\mathfrak{R}^+ \setminus \mathfrak{R}^+$ is compact and f must have values near 0 and near 1 at arbitrarily large $x \in \mathfrak{R}^+$. Since $\mathfrak{R}^+$ is connected, f must assume the value $\frac{1}{2}$ on an unbounded set in $\mathfrak{R}^+$, and hence at some point of $\beta\mathfrak{R}^+ \setminus \mathfrak{R}^+$. $\Rightarrow\Leftarrow$ Hence $\beta\mathfrak{R}^+ \setminus \mathfrak{R}^+$ is connected. Since X is an F-space, $C(X)$ has CIP. X is homeomorphic to $\mathfrak{M}(C(X))$ and is a far cry from being basically disconnected. So CIP is quite different from Dedekind σ-complete and is also quite different from the projection properties.

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