DIFFERENTIAL EQUATIONS INVARIANT UNDER ONE-PARAMETER TRANSFORMATION GROUPS

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This is to certify that the

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ABSTRACT

DIFFERENTIAL EQUATIONS INVARIANT UNDER
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By

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The present thesis is concerned with differential equations which are invariant under one-parameter transformation groups. After an introduction and some background material this idea is introduced in section 3, in which a definition of invariance of general differential equations (O.D.E.'s, or, P.D.E.'s) is given. This definition is a generalization of Lie's definition of invariance of the first order ordinary differential equations. The author derives a criterion for invariance of differential equations under one-parameter transformation groups. It is shown in section 4 that this definition can be reduced to Lie's definition of invariance of linear homogeneous partial differential equations of the first order. The author also gives in section 5 a definition of invariance of systems of differential equations and obtains a criterion. Section 6 is a method of determining the one-parameter transformation groups leaving the given differential equations invariant, which utilizes the obtained criteria.

In section 7, the author gives a new proof of Lie's theorem of reduction of order of ordinary differential equations. Section 8 is the discussion of Morgan's theorem of reduction of the number of independent variables in partial differential equations. This theorem is generalized in this paper. In section 9, the author uses the groups found in section 6 together with the modified Morgan theorem to reduce independent variables in the system of equations of nonsteady rotational plane flow of incompressible fluid. The author also obtains some classes of solutions of this system. In the last section there is obtained a simplification of the form of the system of differential equations of plane flow of polytropic gas. The author starts by reducing the system to a canonical form and then finds the one-parameter transformation group leaving the canonical system invariant and finally uses the obtained group to reduce the canonical system to a system of ordinary differential equations.

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TABLE OF CONTENTS

1. INTRODUCTION

The idea of integrating differential equations with the aid of continuous transformation groups was first considered by Sophus Lie (1842-1899), the founder of the theory of continuous transformation groups. Lie discovered magnificent methods of integrating ordinary differential equations of the first and the second order and linear partial differential equations of the first order. Moreover, he discovered an important theorem, the theorem of reduction of order of ordinary differential equations. These methods and the theorem are based on one-parameter transformation groups. All of such work of Lie is given by G. Scheffers in the book entitled <u>Differentialgleichungen</u> [1]. Later, the Lie theory of differential equations was published in English by many authors namely A. Cohen [6], E. L. Ince [7], L. E. Dickson [8], and K. O. Friedrichs [9].

In 1950 G. Birkhoff [5] suggested that the continuous one-parameter transformation groups could be used to reduce the number of independent variables of some partial differential equations. At about the same time A. J. A. Morgan [3] and A. D. Michal [4] published results on reducing the number of independent variables in systems of partial differential equations, which constitute a generalization of Birkhoff's suggestion.

The present paper deals with differential equations which are invariant under one-parameter transformation groups. Section 1 is introductory, Section 2 contains a general idea of the theory of one-parameter transformation groups.

In Section 3, Lie's idea of invariance of differential equations is generalized by giving a definition of invariance of general differential equations (general O.D.E., or general P.D.E.) under one-parameter transformation groups. This definition is a generalization of Lie's definition for invariance of the first order ordinary differential equations. A theorem which gives another property of differential equations is proved. The new property is called a criterion of invariance of differential equations under one-parameter transformation groups. Our criterion is the same as Lie's criterion for ordinary differential equations of the first and the second order, even the ways of obtaining the criteria are different.

In Section 4, it is shown that the definition, which is set in Section 3, can be reduced to Lie's definition for invariance of linear homogeneous partial differential equations of the first order.

In Section 5 a definition of invariance of systems of differential equations is defined. As in Section 3, we obtain a criterion. It is shown by example that the invariance in our sense agrees with the invariance of differential equations in the sense of H. A. Lorentz; that is, agrees with the invariance of Maxwell's equations under Lorentz's transformations.

Section 6 contains a method of determining the groups for given differential equations. As one-parameter transformation groups furnish us a new tool for integrating differential equations and for simplifying the work of integrating differential equations, the methods of finding groups for given differential equations are important. Many authors developed different methods of finding such groups. Lie [1] actually started with given groups and found the class of all ordinary differential equations which are invariant under each of those groups. Thus, having a large table of classes of differential equations and their groups, one could try to find in it the groups corresponding to a given ordinary differential equation. L. V. Ovsjannikov [14] discovered an algebraic method for determining groups. M. Z. V. Krzyworblocki and H. Roth [15], who paid attention to Morgan's method of reduction of independent variables in partial differential equations, developed a method of determining groups by finite transformations. G. W. Bluman and J. D. Cole [13] found a method of finding infinitesimal transformations in their work of finding similarity solution of heat equation. The method of finding groups in this paper, enable us to find all groups leaving given differential equations (O.D.E., or P.D.E.) invariant. Our method is the utilization of the criteria of invariance of differential equations and of systems of differential equations (Theorem 3.2 and Theorem 5.1), which make use of the extended groups of one-parameter transformations.

In Section 7 a new proof of Lie's theorem of reduction of order of ordinary differential equations is given. This proof is clearer than the original proof of Lie ([1], pp. 386-387). However, the interesting matter in this section is a lemma, which is a key for proving Lie's theorem. This lemma shows an important property of one-parameter transformation groups.

Section 8 contains a discussion of Morgan's theorem of reduction of the number of independent variables in systems of partial differential equations. This theorem is slightly modified so that the more general groups (one-parameter transformation groups) can be used in place of the groups which appear in Morgan's theorem. And so the more general classes of solutions of partial differential equations can be obtained from Morgan's method.

In 1961 E. A. Müller and K. Matschat [12] applied transformation groups to the equations of one-dimensional nonsteady flow of isentropic gas, and obtained some exact solutions. In Section 9 of this paper, one-parameter transformation groups are applied to the system of equations of nonsteady rotational plane flow of incompressible fluid to reduce the number of independent variables and eventually obtained some exact solutions of the system.

The last section, Section 10, we apply one-parameter transformation group to the system of equations of steady plane flow of polytropic gas to reduce it into a system of ordinary differential equations. A reduction of independent variable in this system has been made before by P. Kucharczyk [16], who uses Lie derivatives to find coordinate system in which this system can be transformed into a system of ordinary differential equations. Our reduced system is simpler than that obtained by Kucharczyk.

Following the spirit of Lie, work in this area has focused relatively more on algebraic and geometric properties of differential equations than on analytic properties In particular, domain of definition, continuity and differentiability properties, etc. of functions which are introduced are implicitly assumed to be adequate for each stage of each argument. Usually, no explicit assumption about these properties are inserted in the course of the discussion. We shall follow this spirit.

2. ONE-PARAMETER TRANSFORMATION GROUPS

Consider parametric transformation in k-dimensional space

$$(2.1) \qquad \bar{z}^i = \Phi^i(z^i, \ldots, z^k, a) \qquad (i = 1,\ldots,k)$$

where a is the parameter. The above set of transformations is called a one-parameter transformation group if the following properties hold:

1) The Φ's are continuous functions of their arguments, and the Jacobian of the Φ's with respect to the z's is not zero, i.e.,

$$\det \left(\frac{\partial \Phi^i}{\partial z^j}\right) \neq 0,$$

which implies that we can solve (2.1) for the z's in terms of the $\bar{z}$'s in the form

$$z^i = {}^*\Phi^i(\bar{z}^1, \ldots, \bar{z}^k, a) \qquad (i = 1, \ldots, k)$$

2) For values a_1, a_2 of a such that $\bar{z}^i = \Phi^i(z, a_1)$, $\bar{\bar{z}}^i = \Phi^i(\bar{z}, a_2)$, $(i = 1, \ldots, k)$, there exists a function $f(a_1, a_2)$ such that

$$\bar{\bar{z}}^i = \Phi^i(\bar{z}, a_2) = \Phi^i(z, f(a_1, a_2)) \qquad (i = 1, \ldots, k).$$

3) There exists a value of a, say a_0, corresponding to the identity transformation, i.e.,

$$z^i = \Phi^i(z^1, \ldots, z^k; a_0) \qquad (i = 1,\ldots,k)$$

4) For any value a of the parameter which yields a transformation from a point z to a point $\bar{z}$, there exists a* corresponding to the inverse transformation from $\bar{z}$ to z, i.e., we have

$$z^i = {}^*\Phi^i(\bar{z}^1, \ldots, \bar{z}^k; a) = \Phi^i(\bar{z}^1, \ldots, \bar{z}^k; a^*).$$

L. P. Eisenhart has shown,[1] in the case of the set of transformations (2.1) form a group, that the derivative of the Φ's with respect to the parameter a can be written in the form

$$\text{(2.2)} \qquad \frac{d\bar{z}^i}{da} = \frac{\partial\Phi^i(z, a)}{\partial a} = \xi^i(\bar{z})\cdot A(a) \quad (i=1,\ldots,k)$$

for some functions $\xi^i(\bar{z})$ and $A(a)$. After defining a new parameter t by

$$t = \int_{a_0}^{a} A(a')\, da',$$

the relations (2.2) become

$$\text{(2.2')} \qquad \frac{d\bar{z}^i}{dt} = \frac{\partial\Phi^i(z, a(t))}{\partial t} = \xi^i(\bar{z}).$$

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[1] [2], p. 32.

Observe that the value $t = 0$ gives the identity transformation.

From now on, we shall assume that the set of transformations (2.1) has the mentioned group properties. For some advantage, we substitute the a in (2.1) in terms of t and denote the results by

$$(2.1') \qquad G: \quad \bar{z}^i = \phi^i(z^1, \ldots, z^k; t) \qquad (i = 1,\ldots,k),$$

where the symbol G indicates that the transformations form a group.

Let us expand the ϕ's in (2.1') as Taylor series at $t = 0$, i.e., at the value of t which yields the identity transformation:

$$\phi^i(z, t) = \phi^i(z, 0) + t\left[\frac{\partial\phi^i(z,t)}{\partial t}\right]_{t=0} + \frac{t^2}{2!}\left[\frac{\partial^2\phi^i(z, t)}{\partial t^2}\right]_{t=0} + \ldots\ldots,$$

from which we obtain

$$(2.3) \qquad \bar{z}^i = z^i + t\left[\frac{\partial\phi^i(z, t)}{\partial t}\right]_{t=0} + \frac{t^2}{2!}\left[\frac{\partial^2\phi^i(z, t)}{\partial t^2}\right]_{t=0} + \ldots\ldots .$$

For a small change of t from 0, say δt, so that the powers greater than one can be neglected, (2.3) becomes

$$\bar{z}^i = z^i + \delta t \left[\frac{\partial \phi^i(z, t)}{\partial t}\right]_{t=0}$$

With consideration of (2.2') the last result can be written as

$$\bar{z}^i = z^i + \xi^i(z) \cdot \delta t \qquad (i = 1, \dots, k) \tag{2.4}$$

This is called the infinitesimal transformations of (2.1'). The relations (2.4) tell us that the vector $\xi^i(z)$ is tangent to the path of transformation. We call such path the trajectory of the group. Thus the trajectories of the group are characterized by

$$\frac{dz^1}{\xi^1(z)} = \frac{dz^2}{\xi^2(z)} = \dots = \frac{dz^k}{\xi^k(z)} . \tag{2.5}$$

Consider a continuous function $f(\bar{z}^1, \dots, \bar{z}^k)$ which is composed with the group, i.e.,

$$f(\bar{z}^1, \dots, \bar{z}^k) = f(\phi^1(z, t), \dots, \phi^k(z,t)).$$

Expanding this function as a power series of t, we have

$$f(\bar{z}) = f(z) + t\left[\frac{df(\bar{z})}{dt}\right]_{t=0} + \frac{t^2}{2!}\left[\frac{d^2 f(\bar{z})}{dt^2}\right]_{t=0} + \dots \tag{2.6}$$

$$\dots + \frac{t^n}{n!}\left[\frac{d^n f(\bar{z})}{dt^n}\right]_{t=0} + \dots\dots .$$

Since

$$\left[\frac{df(\bar{z})}{dt}\right]_{t=0} = \left[\sum_{i=1}^{k} \frac{\partial f(\bar{z})}{\partial \bar{z}^i} \frac{d\bar{z}^i}{dt}\right]_{t=0} = \left[\sum_{i=1}^{k} \frac{\partial f(\bar{z})}{\partial \bar{z}^i} \xi^i(\bar{z})\right]_{t=0}$$

$$= \sum_{i=1}^{k} \xi^i(z) \frac{\partial f(z)}{\partial z^i} \quad .$$

and denoting

$$\bar{X} = \sum_{i=1}^{k} \xi^i(\bar{z}) \frac{\partial}{\partial \bar{z}^i} \tag{2.7}$$

or

$$X = \sum_{i=1}^{k} \xi^i(z) \frac{\partial}{\partial z^i} , \tag{2.7'}$$

we have

$$\left[\frac{df(\bar{z})}{dt}\right]_{t=0} = [\bar{X}f(z)]_{t=0} = Xf(z),$$

$$\left[\frac{d^2f(\bar{z})}{dt^2}\right]_{t=0} = \left[\frac{d}{dt} \bar{X} f(\bar{z})\right]_{t=0} = [\bar{X} (\bar{X} f(\bar{z}))]_{t=0}$$

$$= X(Xf(z)) = X^2f(z),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\left[\frac{d^nf(\bar{z})}{dt^n}\right]_{t=0} = \left[\frac{d}{dt} \bar{X}^{n-1}f(\bar{z})\right]_{t=0} = [\bar{X}^nf(\bar{z})]_{t=0} = X^nf(z).$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \ .$$

Then (2.6) becomes

$$(2.8)\quad f(\bar{z}) = f(z) + tXf(z) + \frac{t^2}{2!} X^2 f(z) + \dots$$

$$\dots + \frac{t^n}{n!} X^n f(z) + \dots\dots .$$

Setting $f(z) = z^i$ in (2.8), we get

$$(2.9)\quad \bar{z}^i = z^i + t\xi^i(z) + \frac{t^2}{2!} X\xi^i(z) + \dots$$

$$\dots + \frac{t^n}{n!} X^{n-1}\xi^i(z) + \dots \qquad (i = 1,\dots,k)$$

This is the other form of (2.1').

Absolute Invariants

Definition: A function $u(z^1,\dots,z^k)$ which is unaltered by all transformations of the group (2.1), that is, such that $u(\bar{z}^1,\dots,\bar{z}^k) = u(z^1,\dots,z^k)$, is called an absolute invariant of the group.

There is a theorem helping us to find absolute invariants for a given group of transformation, that is:

Theorem 2.1 ([2], p. 62): A necessary and sufficient condition for a function $u(z^1,\dots,z^k)$ to be an absolute invariant of a group generated by $X = \xi^1(z)\frac{\partial}{\partial z^1} + \dots$ $\dots + \xi^k(z)\frac{\partial}{\partial z^k}$ is that

$$(2.10)\quad Xu(z) = \xi^1(z)\frac{\partial u(z)}{\partial z^1} + \dots + \xi^k(z)\frac{\partial u(z)}{\partial z^k} = 0.$$

Since any integral of $\frac{dz^1}{\xi^1(z)} = \ldots = \frac{dz^k}{\xi^k(z)}$ is a solution of (2.10), it follows that the absolute invariants of the group X can be found from the system of equations

$$(2.11) \qquad \frac{dz^1}{\xi^1(z)} = \cdots\cdots = \frac{dz^k}{\xi^k(z)} \quad .$$

Note that the system (2.11) is the system of differential equations for the trajectories of the group X (cf. (2.5)). We also note that, since there are only k-1 independent solutions of the equation (2.10), there are only k-1 independent absolute invariants of the given group X (see [2], p. 62).

Extended Groups

Given a one-parameter group of transformations

$$(2.12) \qquad G: \begin{cases} \bar{x}^i = \phi^i(x^1,\ldots,x^m,\ y^1,\ldots,y^n;\ t) \\ \bar{y}^r = \psi^r(x^1,\ldots,x^m,\ y^1,\ldots,y^n;\ t) \end{cases} \qquad \begin{pmatrix} i=1,\ldots,m \\ r=1,\ldots,n \end{pmatrix}$$

Suppose the y's are considered as functions of the x's. Then these transformations induce definite transformations of the derivatives of y's with respect to the x's of the form:

$$\bar{y}^r_{\bar{i}} = \psi^{r,i}(x^j,\ y^s,\ y^s_j\ ;\ t)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\bar{y}^r_{\bar{i}_1\ldots\bar{i}_\theta} = \psi^{r,i_1\ldots i_\theta}\ (x^j,\ y^s,\ y^s_j,\ldots,y^s_{j_1\ldots j_\theta}\ ;\ t)$$

$$\left(\begin{array}{c} i,\ i_1,\ \ldots,\ i_\theta,\ j,\ j_1,\ \ldots,\ j_\theta = 1,\ldots,m; \\ r,\ s = 1,\ \ldots,\ n \end{array}\right)$$

where

$$\bar{y}^r_{\bar{i}} \equiv \frac{\partial \bar{y}^r}{\partial \bar{x}^i},\quad y^s_j \equiv \frac{\partial y^s}{\partial x^j},\quad \bar{y}^r_{\bar{i}_1 \ldots \bar{i}_\theta} \equiv \frac{\partial^\theta \bar{y}^r}{\partial \bar{x}^{i_1} \ldots \partial \bar{x}^{i_\theta}}$$

and

$$y^s_{j_1 \ldots j_\theta} \equiv \frac{\partial^\theta y^s}{\partial x^{j_1} \ldots \partial x^{j_\theta}}.$$

The method of determining the functions $\psi^{r,i},\ldots,\psi^{r,i_1\ldots i_\theta}$ is described by L. P. Eisenhart ([2], pp. 102-104). And it is shown by him that the induced transformations (2.13) have the same group properties as the given transformations (2.12). So, the set of transformations (2.12) together with (2.13) form a one-parameter transformation group and will be called <u>the extended group (of order θ) of G</u>, and so will be denoted by $G_{(\theta)}$, i.e.,

$$(2.14)\quad G_{(\theta)}: \begin{cases} \bar{x}^i = \phi^i(x^j,\ y^s;\ t) \\ \bar{y}^r = \psi^r(x^j,\ y^s;\ t) \\ \bar{y}^r_{\bar{i}} = \psi^{r,i}(x^j,\ y^s,\ y^s_j;\ t) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \bar{y}^r_{\bar{i}_1 \ldots \bar{i}_\theta} = \psi^{r,i_1\ldots i_\theta}(x^j,\ y^s,\ y^s_j,\ldots,y^s_{j_1\ldots j_\theta};\ t). \end{cases}$$

The formula for determining the functions $\psi^{r,i_1 \cdot\cdot i_{\alpha+1}}$ from $\psi^{r,i_1 \cdot\cdot i_a}$ is that

$$(2.15)\quad \frac{\partial\psi^{r,i_1\cdot\cdot i_\alpha}}{\partial x^i} + \frac{\partial\psi^{r,i_1\cdot\cdot i_\alpha}}{\partial y^s} y^s_i + \frac{\partial\psi^{r,i_1\cdot\cdot i_\alpha}}{\partial y^s_j} y^s_{ji}$$

$$+ \frac{\partial\psi^{r,i_1\cdot\cdot i_\alpha}}{\partial y^s_{j_1\cdot\cdot j_\alpha}} \cdot y^s_{j_1\cdot\cdot j_\alpha i}$$

$$- \sum_{k=1}^{m} \bar{y}^r_{\bar{i}_1\cdot\cdot\bar{i}_\alpha\bar{k}} \left(\frac{\partial\phi^k}{\partial x^i} + \frac{\partial\phi^k}{\partial y^s} \cdot y^s_i \right) = 0$$

$$(j,\ j_1,\dots,j_\alpha = i,\dots,m\ ;\ s = 1,\dots,n)$$

where summation convention is used in (2.15). This equation can be written shortly as

$$(2.15')\quad \frac{d\psi^{r,i_1\cdot\cdot i_\alpha}}{dx^i} - \sum_{k=1}^{m} \bar{y}_{\bar{i}_1\cdot\cdot\bar{i}_\alpha\bar{k}} \cdot \frac{d\phi^k}{dx^i} = 0.$$

When i runs from 1 to m, the system (2.15) gives m equations which we can solve for $\bar{y}^r_{\bar{i}_1\cdot\cdot\bar{i}_\alpha\bar{1}},\dots,\bar{y}^r_{\bar{i}_1\cdot\cdot\bar{i}_\alpha\bar{m}}$ to get the functions $\psi^{r,i_1\cdot\cdot i_\alpha 1}$, ... , $\psi^{r,i_1\cdot\cdot i_\alpha m}$.

Let the operator of the given group (2.12) be

$$(2.16)\quad X = \sum_{i=1}^{m} \xi^i(x^1,\dots,x^m,y^1,\dots,y^n)\frac{\partial}{\partial x^i}$$

$$+ \sum_{r=1}^{n} \eta^r(x^1,\dots,x^m,\ y^1,\dots,y^n)\ \frac{\partial}{\partial y^r}$$

where

$$\xi^i = [\frac{\partial}{\partial t}\, \phi^i(x^1,\dots,x^m,\ y^1,\dots,y^n;\ t)]_{t=0}$$

$$\eta^r = [\frac{\partial}{\partial t}\, \psi^r(x^1,\dots,x^m,y^1,\dots,y^n;\ t)]_{t=0}$$

are known, $t = 0$ signifies the identity transformation of the group (2.12). Denoting the operator of the group $G_{(\theta)}$ in (2.14) by

$$(2.12)\qquad X_{(\theta)} = X + \Sigma\eta^{r,i}\frac{\partial}{\partial y_i^r} + \dots + \Sigma\eta^{r,i_1\dots i_\theta}\frac{\partial}{\partial y^r_{i_1\dots i_\theta}}\ ,$$

the coefficients $\eta^{r,i},\dots,\ \eta^{r,i_1\dots i_\theta}$ can be found directly from the group $G_{(\theta)}$ if the transformation laws of $G_{(\theta)}$ are known, i.e.,

$$\eta^{r,i} = [\frac{\partial}{\partial t}\, \psi^{r,i}(x^j,\ y^s,\ y^s_j;\ t)]_{t=0}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\eta^{r,i_1\dots i_\theta} = [\frac{\partial}{\partial t}\, \psi^{r,i_1\dots i_\theta}\ (x^j,\ y^s,\ y^s_j,\dots,y^s_{j_1\dots j_\theta};t)]_{t=0}$$

We also can find $\eta^{r,i},\dots,\eta^{r,i_1\dots i_\theta}$ from the known operator X by recurrent formula of Eisenhart ([2], p. 106):

$$(2.18)\quad \eta^{r,i_1..i_\alpha i} = \frac{\partial\eta^{r,i_1..i_\alpha}}{\partial x^i} + \frac{\partial\eta^{r,i_1..i_\alpha}}{\partial y^s}\cdot y_i^s$$

$$+ \frac{\partial\eta^{r,i_1..i_\alpha}}{\partial y_j^s}\cdot y_{ji}^s + \dots + \frac{\partial\eta^{r,i_1..i_\alpha}}{\partial y_{j_1..j_\alpha}^s}\cdot y_{j_1..j_\alpha i}^s$$

$$- \sum_{k=1}^{m} y_{i_1..i_\alpha k}^r \left(\frac{\partial\xi^k}{\partial x^i} + \frac{\partial\xi^k}{\partial y^s}\cdot y_i^s\right)$$

$$(j,\ j_1,\dots,j_\alpha = 1,\dots,m\ ;\ s = 1,\dots,n)$$

where the summation convention is used in (2.18). We can write (2.18) shortly as

$$(2.18')\quad \eta^{r,i_1..i_\alpha i} = \frac{d\eta^{r,i_1..i_\alpha}}{dx^i} - \sum_{k=1}^{m} y_{i_1..i_\alpha k}^r \frac{d\xi^k}{dx^i}\ .$$

Commutators

Given two operators

$$X = \sum_{i=1}^{n} \xi^i(x^1,\dots,x^n)\frac{\partial}{\partial x^i}\ ,\quad Y = \sum_{j=1}^{n} \eta^j(x^1,\dots,x^n)\frac{\partial}{\partial x^j}.$$

we define the commutator of X and Y by

$$(2.19)\quad [X,\ Y] = XY - YX.$$

As a consequence of (2.19), we have

$$(2.20)\quad [Y,\ X] = -[X,\ Y].$$

The commutator can be written precisely in the form

$$(2.21)\quad [X, Y] = (X\eta^1 - Y\xi^1)\frac{\partial}{\partial x^1} + \ldots + (X\eta^n - Y\xi^n)\frac{\partial}{\partial x^n},$$

Since by direct calculation:

$$XY = X\left(\sum_{i=1}^{n} \eta^i \frac{\partial}{\partial x^i}\right) = \sum_{i=1}^{n} (X\eta^i)\frac{\partial}{\partial x^i} + \sum_{i=1}^{n} \eta^i \sum_{j=1}^{n} \xi^j \frac{\partial^2}{\partial x^i \partial x^j}$$

$$YX = Y\left(\sum_{i=1}^{n} \xi^i \frac{\partial}{\partial x^i}\right) = \sum_{i=1}^{n} (Y\xi^i)\frac{\partial}{\partial x^i} + \sum_{i=1}^{n} \xi^i \sum_{j=1}^{n} \eta^j \frac{\partial}{\partial x^i \partial x^j},$$

we have

$$[X, Y] = XY - YX = \sum_{i=1}^{n} (X\eta^i)\frac{\partial}{\partial x^i} - \sum_{i=1}^{n} (Y\xi^i)\frac{\partial}{\partial x^i}$$

which can be written as (2.21).

Differential Invariants

A function $f(x, y, y')$ which actually involves the derivative y' is called a differential invariant of the first order of the group of transformations:

$$G:\quad \bar{x} = \phi(x, y; t),\ \bar{y} = \psi(x, y; t);$$

if it is an absolute invariant of the group $G_{(1)}$, the first extended group of the group G. In the same way, a function $F(x, y, y', \ldots, y^{(k)})$ which actually involves

$y^{(k)}$ is called a differential invariant of order k of the group G, if it is an absolute invariant of the extended group $G_{(k)}$. We then have a theorem (cf. theorem 2.1).

<u>Theorem 2.2</u>: A necessary and sufficient condition for a function $F(x, y, y', \ldots, y^{(k)})$, which actually involves $y^{(k)}$, to be a differential invariant of order k of the group generated by $X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}$ is that

$$X_{(k)}F(x, y, y', \ldots, y^{(k)}) = 0$$

where $X_{(k)}$ is the k^{th} extended operator of X.

3. INVARIANCE OF DIFFERENTIAL EQUATIONS UNDER ONE-PARAMETER TRANSFORMATION GROUPS

The definitions for differential equations to be invariant under transformations are given by S. Lie for the first and the second order ordinary differential equations and for linear homogeneous partial differential equations of the first order. His definition for the first order ordinary differential equation is:

Definition 3.1 (Lie's definition, [1], p. 101): It is said that a differential equation

(3.1) $M(x, y)dx - N(x, y)dy = 0$

is invariant under the transformations

(3.2) $\bar{x} = \phi(x, y)$, $\bar{y} = \psi(x, y)$

if its form is unaltered, save for a factor, by the transformations, i.e., it may be written, in terms of the new variables, in the form

$$\rho(\bar{x}, \bar{y}) \; [M(\bar{x}, \bar{y})d\bar{x} - N(\bar{x}, \bar{y})d\bar{y}] = 0.$$

Lie proved a theorem which gives another property equivalent to the property in the above definition.

Theorem 3.1 (Lie's theorem, [13, p. 101): The form of the equation (3.1) is unaltered, save for a factor, by the transformations (3.2) if and only if each integral curve of (3.1) is transformed by (3.2) to some integral curve of the same equation.

For invariance of the second order ordinary differential equations, Lie made an analogous definition.

We shall now make a generalization of the Lie idea of invariance of differential equations. Let us consider a general differential equation (general O.D.E., or, general P.D.E.) of order r with independent variables $x^1,\ldots,x^n$ (for P.D.E. $n > 1$, for O.D.E. $n = 1$) and dependent variable y:

$$(3.3) \quad F(x^i, y, y_i,\ldots,y_{i_1\ldots i_r}) = 0$$

where

$$y_i \equiv \frac{\partial y}{\partial x^i}, \quad y_{i_1\ldots i_r} \equiv \frac{\partial^r y}{\partial x^{i_1}\cdots\partial x^{i_r}};$$

together with a one-parameter transformation group

$$(3.4) \quad G: \begin{cases} \bar{x}^i = \phi^i(x^1,\ldots,x^n, y; t) \\ \bar{y} = \psi(x^1,\ldots,x^n, y; t). \end{cases} \qquad (i=1,\ldots,n)$$

Let the r^{th} extended group of the group G be

$$G_{(r)}: \begin{cases} \bar{x}^i = \phi^i(x^1,\ldots,x^n,\ y;\ t) = \phi^i(x^j,\ y;\ t) \\ \bar{y} \ = \psi(x^1,\ldots,x^n,\ y;\ t) = \psi(x^j,\ y;\ t) \\ \bar{y}_{\bar{i}} = \psi^i(x^j,\ y,\ y_j;\ t) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \bar{y}_{\bar{i}_1..\bar{i}_r} = \psi^{i_1..i_r}\ (x^j,\ y,\ y_j,\ldots,y_{j_1..j_r},t) \end{cases}$$

$$(i,\ i_1,\ ..\ ,i_r,\ j,\ j_1,\ ..\ ,\ j_r = 1,\ldots,n)$$

where

$$\bar{y}_{\bar{i}} = \frac{\partial\bar{y}}{\partial\bar{x}^i}\ ,\ \bar{y}_{\bar{i}_1..\bar{i}_r} = \frac{\partial^r\bar{y}}{\partial\bar{x}^{i_1}..\partial\bar{x}^{i_r}}\ .$$

<u>Definition 3.2</u>: It will be said that the differential equation (3.3) is invariant under the transformation group G, (3.4), if and only if under the transformations of $G_{(r)}$ the following relations hold:

$$(3.5)\quad F(x^i(\bar{x}^j,\bar{y},t),y(\bar{x}^j,\bar{y},t),\ldots,y_{i_1..i_r}(\bar{x}^j,\bar{y},\ldots,\bar{y}_{\bar{j}_1..\bar{j}_r},t))$$

$$= \nu(\bar{x}^i,\ \bar{y},\ t)\ F(\bar{x}^i,\ \bar{y},\ \bar{y}_{\bar{i}},\ldots,\bar{y}_{\bar{i}_1..\bar{i}_r})\ ,$$

or,

$$(3.6)\quad F(\bar{x}^i(x^j,y,t),\bar{y}(x^j,y,t),\ldots,\bar{y}_{\bar{i}_1..\bar{i}_r}(x^j,y,\ldots,y_{j_1..j_r},t))$$

$$= \mu(x^i,\ y,\ t)\cdot F(x^i,\ y,\ y_i,\ldots,y_{i_1..i_r}),$$

where ν and μ are not zero.

Note that (3.5) and (3.6) are equivalent relations. We shall now derive a property equivalent to that given in definition 3.1. First, we assume that for the given differential equation (3.3) and the group (3.4), the relations (3.5) or (3.6) hold. Since (cf. Section 2)

$$(3.7)\quad [\frac{d}{dt} F(\bar{x}^{\bar{i}}, \bar{y}, \bar{y}_{\bar{i}}, \ldots, \bar{y}_{\bar{i}_1 \ldots \bar{i}_r})]_{t=0}$$

$$= \left[\frac{\partial F(\bar{x}^{\bar{i}}, \ldots)}{\partial \bar{x}} \frac{d\bar{x}^{\bar{j}}}{dt} + \ldots + \frac{\partial F(\bar{x}^{\bar{i}}, \ldots)}{\partial \bar{y}_{\bar{j}_1 \ldots \bar{j}_r}} \frac{d\bar{y}_{\bar{j}_1 \ldots \bar{j}_r}}{dt} \right]_{t=0}$$

$$= \frac{\partial F(x^i, \ldots)}{\partial x^j} \left[\frac{\partial \phi^j}{\partial t}\right]_{t=0} + \ldots + \frac{\partial F(x^i, \ldots)}{\partial y_{j_1 \ldots j_r}} \left[\frac{\partial \psi^{j_1 \ldots j_r}}{\partial t}\right]_{t=0}$$

$$= X_{(r)} F(x^i, y, y_i, \ldots, y_{i_1 \ldots i_r})$$

where t=0 signifies the identity transformation of the group (3.4), $X_{(r)}$ denotes the operator of the group $G_{(r)}$, the summation convention is used in (3.7); we have from (3.6) and (3.7)

$$(3.8)\quad X_{(r)} F(x^i, y, y_i, \ldots, y_{i_1 \ldots i_r})$$

$$= [\frac{\partial}{\partial t} \mu(x^i, y, t)]_{t=0} F(x^i, y, y_i, \ldots, y_{i_1 \ldots i_r})$$

$$= \lambda(x^i, y) \cdot F(x^i, y, y_i, \ldots, y_{i_1 \ldots i_r}), \text{ say.}$$

Conversely, if the differential form $F(x^i,y,y_i,\ldots,y_{i_1\ldots i_r})$ in (3.3) satisfies a relation of the form (3.8), i.e.,

$$(3.9)\quad X_{(r)}F(x^i,y,y_i,\ldots,y_{i_1\ldots i_r})$$

$$= \ell(x^i,y)F(x^i,y,y_i,\ldots,y_{i_1\ldots i_r}),$$

we claim that the differential form $F(x^i,y,y_i,\ldots,y_{i_1\ldots i_r})$ also satisfies the relation (3.6). Since (cf. Section 2)

$$(3.10)\quad F(\bar{v}) = F(v) + \frac{t}{1!}X_{(r)}F(v) + \frac{t^2}{2!}X^2_{(r)}F(v) + \ldots$$

$$\ldots + \frac{t^k}{k!}X^k_{(r)}F(v) + \ldots ,$$

where $F(\bar{v}) \equiv F(\bar{x}^i,\bar{y},\ \bar{y}_{\bar{i}},\ldots,\bar{y}_{\bar{i}_1\ldots\bar{i}_r})$,

$$F(v) \equiv F(x^i,y,y_i,\ldots,y_{i_1\ldots i_r}),$$

and since

$$(3.11)\begin{cases} X^2_{(r)}F(v) = X_{(r)}[\ell(x,y)F(v)] = [X_{(r)}\ell(x,y) \\ \qquad + \ell(x,y)\cdot\ell(x,y)]F(v) = \ell_1(x,y)F(v) \\ \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ X^k_{(r)}F(v) = X_{(r)}[\ell_{k-2}(x,y)F(v)] = [X_{(r)}\ell_{k-2}(x,y) \\ \qquad + \ell_{k-2}(x,y)\cdot\ell(x,y)]F(v) = \ell_{k-1}(x,y)F(v) \\ \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots , \end{cases}$$

we obtain by substitution from (3.9) and (3.11) into (3.10)

$$F(\bar{v}) = [1 + t\ell(x,y) + \frac{t^2}{2!}\ell_1(x,y) + \ldots + \frac{t^k}{k!}\ell_{k-1}(x,y)$$

$$+ \ldots]F(v) = m(x,y,t)F(v)$$

where $m(x,y,t)$ denotes the sum in the bracket which is not zero. Thus the $F(v)$ satisfies the relation (3.6), and the claim is proved. We now have conclusion:

Theorem 3.2: Given the differential equation $F(x^i, y, y_i, \ldots, y_{i_1 \ldots i_r}) = 0$ and the group of transformations

$$G: \quad \bar{x}^i = \phi^i(x^j, y, t), \; \bar{y} = \psi(x^j, y, t);$$

the relations

$$(3.5) \quad F(x^i(\bar{x}^j,\bar{y},t),\; y(\bar{x}^j,\bar{y},t), \ldots, y_{i_1 \ldots i_r}(\bar{x}^j,\bar{y},\ldots,\bar{y}_{\bar{j}_1 \ldots \bar{j}_r},t))$$

$$= \nu(\bar{x}^i,\bar{y},t)\cdot F(\bar{x}^i,\bar{y},\bar{y}_{\bar{i}},\ldots,\bar{y}_{\bar{i}_1 \ldots \bar{i}_r})$$

or

$$(3.6) \quad F(\bar{x}^i(x^j,y,t),\; \bar{y}(x^j,y,t), \ldots, \bar{y}_{\bar{i}_1 \ldots \bar{i}_r}(x^j,y,\ldots,y_{j_1 \ldots j_r},t))$$

$$= \mu(x^i,y,t)\cdot F(x^i,y,y_i,\ldots,y_{i_1 \ldots i_r})$$

where ν and μ are not zero, hold if and only if a relation

$$(3.8)\quad X_{(r)}F(x^i,y,\ldots,y_{i_1\ldots i_r}) = \lambda(x^i,y)\cdot F(x^i,y,\ldots,y_{i_1\ldots i_r})$$

holds for some function λ. In other words, the differential equation $F(x^i,y,y_i,\ldots,y_{i_1\ldots i_r}) = 0$ is invariant under the group G if and only if the relation of the form
$X_{(r)}F(x^i,y,y_i,\ldots,y_{i_1\ldots i_r}) = \lambda(x^i,y)F(x^i,y,y_i,\ldots,y_{i_1\ldots i_r})$
holds.

We shall call the relation (3.8) the criterion for differential equations to be invariant under one-parameter groups of transformations. Lie obtained the same criterion as (3.8) for the invariance of ordinary differential equations of the first order, although the ways of obtaining the criteria are different. Actually, Lie obtained the criterion by making use of his theorem 3.1.

Example 3.1 (from [1], p. 278): Lie found that the differential equation for a one-parameter family of circles $(x - a)^2 + y^2 = r^2$ where r is fixed, is

$$(3.12)\quad F(v) \equiv y^2(1 + y'^2) - r^2 = 0.$$

Since this family of circles is invariant[1] under the group of translations parallel to the x-axis, i.e., the group

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[1]By invariance of a family of curves under a transformation we mean that each curve of the family is transformed into some curve of the same family.

$$(3.13) \qquad X = \frac{\partial}{\partial x},$$

Lie concluded (by using theorem 3.1) that the differential equation (3.12) is invariant under the group (3.13). From (3.13) we find the following:

$$(3.14) \qquad X_{(1)} = \frac{\partial}{\partial x} = X,$$

$$(3.15) \qquad G\colon \ \bar{x} = x + t,\ \bar{y} = y,$$

$$(3.16) \qquad G_{(1)}\colon \ \bar{x} = x + t,\ \bar{y} = y,\ \frac{d\bar{y}}{d\bar{x}} = \frac{dy}{dx}.$$

Then, by substitution, we have

$$F(v(\bar{v})) = \bar{y}^2\left(1 + \left(\frac{d\bar{y}}{d\bar{x}}\right)^2\right) - r^2 = 1\cdot F(\bar{v}),$$

$$F(\bar{v}(v)) = y^2\left(1 + \left(\frac{dy}{dx}\right)^2\right) - r^2 = 1\cdot F(v).$$

We also have from (3.14) and (3.12)

$$X_{(1)}F(v) \equiv 0 = 0\cdot F(v).$$

<u>Example 3.2</u>: Consider a differential equation

$$(3.17) \quad F(v) \equiv yu_x - u_{xx} = 0,$$

and a group of transformations

$$(3.18) \quad X = \frac{\partial}{\partial x} + yu\frac{\partial}{\partial u}.$$

We obtain from (3.18) the following:

(3.19) $$X_{(2)} = X + yu_x \frac{\partial}{\partial u_x} + \ldots + yu_{xx} \frac{\partial}{\partial u_{xx}} + \ldots ,$$

(3.20) $$G: \quad \bar{x} = x + t, \; \bar{y} = y, \; \bar{u} = e^{ty}u ,$$

(3.21) $$G_{(2)}: \quad \bar{x} = x+t, \; \bar{y}=y, \; \bar{u}=e^{ty}u, \; \bar{u}_{\bar{x}}=e^{ty}u_x, \ldots,$$

$$\bar{u}_{\bar{x}\bar{x}} = e^{ty}u_{xx} .$$

We then have

$$F(v(\bar{v})) = \bar{y}\cdot e^{-t\bar{y}}\bar{u}_{\bar{x}} - e^{-t\bar{y}}\bar{u}_{\bar{x}\bar{x}} = e^{-t\bar{y}}\cdot F(\bar{v}) ,$$

$$F(\bar{v}(v)) = y\cdot e^{ty}u_x - e^{ty}u_{xx} = e^{ty}\cdot F(v) ,$$

$$X_{(2)}F(v) = y\cdot yu_x - yu_{xx} = y\cdot(yu_x - u_{xx}) = y\cdot F(v).$$

Thus the equation (3.17) is invariant under the group (3.18).

4. THE LINEAR HOMOGENEOUS PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

S. Lie gave a definition for the linear homogeneous partial differential equations of the first order to be invariant under transformations as follows:

Lie's Definition ([1], p. 311): It is said that the differential equation

$$(4.1)\quad A_y \equiv \alpha^1(x^1,\ldots,x^n)\,\frac{\partial y}{\partial x^1} + \ldots + \alpha^n(x^1,\ldots,x^n)\,\frac{\partial y}{\partial x^n} = 0$$

is invariant under the transformations

$$(4.2)\quad \bar{x}^i = \phi^i(x^1,\ldots,x^n) \qquad (i=1,\ldots,n),$$

if these transformations preserve its solutions.

Observe that the transformations in the above definition are the transformations of independent variables only, while the set of transformations in our definition 3.2 involve the dependent variables also.

It is our purpose here to verify that our definition 3.2 can be reduced to the above Lie definition. That is, for the special case where our differential equation (3.3) is of the form (4.1) and our group (3.4) is of the form

$$(4.3)\quad \begin{cases} \bar{x}^i = \phi^i(x^1,\dots,x^n,\ t) & (i=1,\dots,n) \\ \bar{y} = y, & \end{cases}$$

if differential equation is invariant under the group according to our definition, then it is invariant under the group according to Lie's definition. The definition of Lie that differential equation (4.1) is invariant under the group (4.3) is the same as the above definition. That is, <u>the differential equation (4.1) is invariant under the group (4.3) if the transformations of the group preserve its solutions</u>. Lie proved directly from this definition a theorem which gives a criterion of invariance of the equation (4.1). Before stating this theorem, let us point out some consequences of the group (4.3). The operator of the group (4.3) is in the form

$$(4.4)\quad X = \xi^1(x^1,\dots,x^n)\frac{\partial}{\partial x^1} + \dots$$

$$\dots + \xi^n(x^1,\dots,x^n)\frac{\partial}{\partial x^n} + 0\,\frac{\partial}{\partial y}$$

where

$$\xi^i(x^1,\dots,x^n) = [\partial\phi^i(x^1,\dots,x^n;\ t)/\partial t]_{t=0}.$$

Then the first extended operator $X_{(1)}$ can be found, with the help of the formula (2.18), in the form:

$$(4.5) \quad X_{(1)} = X - \left(\sum_{k=1}^{n} y_k \cdot \frac{\partial \xi^k}{\partial x^1} \right) \frac{\partial}{\partial y_1} - \cdots$$

$$- \left(\sum_{k=1}^{n} y_k \cdot \frac{\partial \xi^k}{\partial x^n} \right) \frac{\partial}{\partial y_n}$$

where $y_i \equiv \dfrac{\partial y}{\partial x^i}$.

Lie's theorem ([1], p. 316): The differential equation (4.1) is invariant under the group generated by the operator (4.4) (i.e., the group (4.2)) if and only if a relation of the form

$$(4.6) \quad [X, A]y = \rho(x^1, \ldots, x^n)Ay$$

holds identically.

If we can show that our definition 3.2 implies the property (4.6) in the Lie theorem, then we can conclude that our definition 3.2 can be reduced to the Lie definition. We shall now show that implication. From (4.5) we have

$$(4.7)\quad X_{(1)}Ay = X_{(1)}\ \{\alpha^1 y_1 + \dots + \alpha^n y_n\}$$

$$= (X_{(1)}\alpha^1)y_1 + \dots + (X_{(1)}\alpha^n)y_n$$

$$+ \alpha^1\left(-\sum_{k=1}^{n} y_k \frac{\partial\xi^k}{\partial x^1}\right) + \dots + \alpha^n\left(-\sum_{k=1}^{n} y_k \frac{\partial\xi^k}{\partial x^n}\right).$$

$$= \left(X_{(1)}\alpha^1 - \sum_{i=1}^{n} \alpha^i \frac{\partial\xi^1}{\partial x^i}\right) y_1$$

$$+ \dots + \left(X_{(1)}\alpha^n - \sum_{i=1}^{n} \alpha^i \frac{\partial\xi^n}{\partial x^i}\right) y_n\ .$$

Since α^i are functions of $x^1,\dots,x^n$ only, we have from (4.5) that $X_{(1)}\alpha^i = X\alpha^i$. Thus (4.7) becomes

$$(4.8)\quad X_{(1)}Ay = (X\alpha^1 - A\xi^1)\ \frac{\partial y}{\partial x^1} + \dots + (X\alpha^n - A\xi^n)\ \frac{\partial y}{\partial x^n}\ .$$

And thus, by (2.21), (4.8) can be written as

$$(4.9)\quad X_{(1)}Ay \ = \ [X,\ A]y.$$

Since we assume that the equation (4.1) is invariant under the group (4.3), by theorem 3.2 we have a relation of the form

$$(4.10)\quad X_{(1)}Ay \ = \ \lambda(x^1,\dots,x^n,\ y)Ay$$

for some function λ. Since from (4.8) and (4.10), we must have the relations of the form:

$$(4.11)\quad \lambda(x,\ y)\ \alpha^i(x) = X\alpha^i(x) - A\xi^i(x)$$

for $i = 1,\dots,n$. The right hand member of (4.11) is a function of $x^1,\dots,x^n$. Thus, the λ is not a function of y, i.e.,

$$(4.12)\quad \lambda = \lambda(x^1,\dots,x^n).$$

Now from (4.9), (4.10) and (4.12), we have

$$[X,\ A]y = \lambda(x^1,\dots,x^n)Ay,$$

which is a relation of the form (4.6). Therefore, our definition 3.2 can be reduced to Lie's definition.

<u>Remark</u>: We see from (4.9) that the relation $[A,\ X]y = \rho(x^1,\dots,x^n)Ay$ implies the relation $X_{(1)}Ay = \rho(x^1,\dots,x^n)Ay$.

5. INVARIANCE OF SYSTEMS OF DIFFERENTIAL EQUATIONS UNDER ONE-PARAMETER TRANSFORMATION GROUPS

Consider a system of differential equations (P.D.E.'s, or, O.D.E.'s) of order θ with independent variables $x^1,\ldots,x^m$ (for P.D.E.'s $m > 1$, for O.D.E.'s $m = 1$) and dependent variables $y^1,\ldots,y^n$:

$$(5.1)\qquad \begin{cases} F_1(x^i,\ y^r,\ y^r_i,\ \ldots\ ,\ y^r_{i_1\ldots i_\theta}) = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ F_\ell(x^i,\ y^r,\ y^r_i,\ \ldots\ ,\ y^r_{i_1\ldots i_\theta}) = 0 \end{cases}$$

where $y^r_i \equiv \partial y^r/\partial x^i$, $y^r_{i_1\ldots i_\theta} \equiv \partial^\theta y^r/\partial x^{i_1}\ldots\partial x^{i_\theta}$. We associate the system (5.1) with a one-parameter transformation group

$$(5.2)\,G\!:\begin{cases} \bar{x}^i = \phi^i(x^1,\ldots,x^m,y^1,\ldots,y^n;\ t) \equiv \phi^i(x^j,y^s;\ t) \\ \bar{y}^r = \psi^r(x^1,\ldots,x^m,y^1,\ldots,y^n;\ t) \equiv \psi^r(x^j,y^s;\ t) \end{cases}$$

$$(i,\ j = 1,\ldots,m;\ r,\ s = 1,\ldots,n)$$

and so with the extended group $G_{(\theta)}$ of G:

$$G_{(\theta)}:\begin{cases}\bar{x}^i = \phi^i(x^j, \ y^s;\ t)\\ \bar{y}^r = \psi^r(x^j,\ y^s;\ t)\\ \bar{y}^r_{\bar{i}} = \psi^{r,i}(x^j,\ y^s,\ y^s_j;\ t)\\ \dots\dots\dots\dots\dots\dots\dots\\ \bar{y}^r_{\bar{i}_1..\bar{i}_\theta} = \psi^{r,i_1..i_\theta}(x^j,y^s,y^s_j,\dots,y^s_{j_1..j_\theta};\ t)\end{cases}$$

$$(i,i_1,\dots,i_\theta,j,j_1,\dots,j_\theta = 1,\dots,m;\ r,s = 1,\dots,n).$$

Definition 5.1: It will be said that the system of differential equations (5.1) is invariant under the group G if and only if under the transformations of the group $G_{(\theta)}$, we have for $\alpha = 1,\dots,\ell$

$$(5.3)\quad F_\alpha(x^i(\bar{x}^j,\bar{y}^s,t),y^r(\bar{x}^j,\ \bar{y}^s,t),\dots$$

$$\dots,\ y^r_{i_1\dots i_\theta}(\bar{x}^j,\bar{y}^s,\dots,\bar{y}^s_{\bar{j}_1\dots\bar{j}_\theta}))$$

$$= \sum_{\beta=1}^{\ell} \nu_{\alpha\beta}(\bar{x}^i,\bar{y}^r,t)\cdot F_\beta(\bar{x}^i,\bar{y}^r,\dots,\bar{y}^r_{\bar{i}_1..\bar{i}_\theta})$$

with

$$(5.4)\qquad \det(\nu_{\alpha\beta}) \neq 0,$$

or, equivalently

$$(5.5)\quad F_\alpha(\bar{x}^i(x^j,y^s,t),\ \bar{y}^r(x^j,y^s,t),\ \ldots$$
$$\ldots,\ \bar{y}_{\bar{i}_1..\bar{i}_\theta}(x^j,y^s,\ldots,y^s_{j_1..j_\theta},\ t))$$
$$= \sum_{\beta=1}^{\ell} \mu_{\alpha\beta}(x^i,y^r,t)\cdot F(x^i,y^r,\ldots,y^r_{i_1..i_\theta})$$

with

$$(5.6)\qquad \det\,(\mu_{\alpha\beta}) \neq 0.$$

<u>Remark</u>: The conditions (5.4) and (5.6) are sufficient to imply the equivalence between the relations (5.3) and (5.5). Moreover, the condition (5.4) assures us that the transform of the system (5.1) will be a system of ℓ independent equations of the form

$$\begin{cases} \sum_{\beta=1}^{\ell} \nu_{1\beta}(\bar{x}^i,\bar{y}^r,t)\cdot F_\beta(\bar{x}^i,\bar{y}^r,\ldots,\bar{y}^r_{\bar{i}_1..\bar{i}_\theta}) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \sum_{\beta=1}^{\ell} \nu_{\ell\beta}(\bar{x}^i,\bar{y}^r,t)\cdot F_\beta(\bar{x}^i,\bar{y}^r,\ldots,\bar{y}^r_{\bar{i}_1..\bar{i}_\theta}) = 0 \end{cases}$$

where every $F_\alpha(\bar{x}^i,\bar{y}^r,\ldots,\bar{y}^r_{\bar{i}_1..\bar{i}_\theta})$ appears on the left hand side of the above system. A similar system of independent equations is obtained, under the condition (5.6), when the inverse transformation applied to the system

$$\begin{cases} F_1(\bar{x}^i, \bar{y}^r, \bar{y}^r_{\bar{i}}, \ldots, \bar{y}^r_{\bar{i}_1 \ldots \bar{i}_\theta}) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_\ell(\bar{x}^i, \bar{y}^r, \bar{y}^r_{\bar{i}}, \ldots, \bar{y}^r_{\bar{i}_1 \ldots \bar{i}_\theta}) = 0. \end{cases}$$

We shall now obtain another property of invariance of system of differential equations under one-parameter transformation group. This new property will be called a criterion of invariance of system of differential equations under one-parameter transformation group.

Theorem 5.1: The system of differential equations (5.1) is invariant under the group G if and only if the relations

$$(5.7)\quad X_{(\theta)}F_\alpha(x^i, y^r, y^r_i, \ldots, y^r_{i_1 \ldots i_\theta})$$

$$= \sum_{\beta=1}^{\ell} \lambda_{\alpha\beta}(x^i, y^r) F_\beta(x^i, y^r, y^r_i, \ldots, y^r_{i_1 \ldots i_\theta})$$

$$(\alpha = 1, \ldots, \ell)$$

hold for some functions $\lambda_{\alpha\beta}$, where $X_{(\theta)}$ is the operator of the group $G_{(\theta)}$.

Proof: We first assume that the system (5.1) is invariant under the group G, then the relations (5.3) or (5.5) hold. As in section 3 (cf. (3.7)), we have

$$(5.8)\quad X_{(\theta)}F_\alpha(x^i,y^r,\ldots,y^r_{i_1\ldots i_\theta})$$

$$= [\frac{d}{dt} F_\alpha(\bar{x}^i(x^j,y^s,t),\ldots,\bar{y}^r_{\bar{i}_1\ldots\bar{i}_\theta}(x^j,y^s,\ldots,y^s_{j_1\ldots j_\theta},t))]_{t=0}$$

$$(\alpha = 1, \ldots , \ell).$$

Then from (5.5) and (5.8) we have

$$(5.9)\quad X_{(\theta)}F_\alpha(x^i,y^r,\ldots,y^r_{i_1\ldots i_\theta})$$

$$= \sum_{\beta=1}^{\ell} [\frac{\partial}{\partial t} \mu_{\alpha\beta}(x^i,y^r,t)]_{t=0}\cdot F_\beta(x^i,y^r,\ldots,y^r_{i_1\ldots i_\theta})$$

$$(\alpha = 1, \ldots , \ell).$$

Writing

$$\lambda_{\alpha\beta}(x^i,y^r) = [\frac{\partial}{\partial t} \mu_{\alpha\beta}(x^i,y^r,t)]_{t=0}$$

in (5.9), we obtain the relation of the form (5.7).

Conversely, if the relations (5.7) hold for the given system of differential equations (5.1) and the given group G, then we claim that the relations (5.3) or (5.5) hold. Since we have from (5.7)

$$X^2_{(\theta)}F_\alpha(x^i,y^r,\ldots,y^r_{i_1\ldots i_\theta})$$

$$= X_{(\theta)}[\sum_{\beta=1}^{\ell} \lambda_{\alpha\beta}(x^i,y^r)\cdot F_\beta(x^i,y^r,\ldots,y^r_{i_1\ldots i_\theta})]$$

$$= \sum_{\beta=1}^{\ell} (X_{(\theta)}\lambda_{\alpha\beta})F_\beta + \sum_{\beta=1}^{\ell} \lambda_{\alpha\beta} \sum_{\gamma=1}^{\ell} \lambda_{\beta\gamma}F_\gamma$$

$$= \sum_{\beta=1}^{\ell} f_{\alpha\beta}F_\beta$$

here $f_{\alpha\beta} \equiv f_{\alpha\beta}(x^i,y^r)$, it follows that for positive integer k:

$$(5.10)\quad X^k_{(\theta)}F_\alpha(x^i,y^r,\ldots,y^r_{i_1\ldots i_\theta})$$

$$= \sum_{\beta=1}^{\ell} f_{k\alpha\beta}(x^i,y^r)\cdot F_\beta(x^i,y^r,\ldots,y^r_{i_1\ldots i_\theta})$$

where $f_{2\alpha\beta} \equiv f_{\alpha\beta}$, $f_{1\alpha\beta} \equiv \lambda_{\alpha\beta}$. From the fact that (cf. (2.8))

$$(5.11)\quad F_\alpha(\bar{v}) = F_\alpha(v) + \frac{t}{1!} X_{(\theta)}F_\alpha(v) + \frac{t^2}{2!} X^2_{(\theta)}F_\alpha(v) + \ldots\ldots$$

where

$$F_\alpha(\bar{v}) \equiv F_\alpha(\bar{x}^i,\bar{y}^r,\ldots,\bar{y}^r_{\bar{i}_1\ldots\bar{i}_\theta}),\ F_\alpha(v) \equiv F_\alpha(x^i,y^r,\ldots,y^r_{i_1\ldots i_\theta});$$

we have after substituting (5.10) into (5.11):

$$
(5.12)\quad \begin{cases}
F_1(\bar{v}) = \{1 + g_{11}\}F_1(v) + g_{12}F_2(v) +\dots+ g_{1\ell}F_\ell(v) \\
F_2(\bar{v}) = g_{21}F_1(v) + \{1 + g_{22}\}F_2(v) +\dots+ g_{2\ell}F_\ell(v) \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
F_\ell(\bar{v}) = g_{\ell 1}F_1(v) + g_{\ell 2}F_2(v) +\dots+ \{1 + g_{\ell\ell}\}F_\ell(v)
\end{cases}
$$

where

$$g_{\alpha\beta} = \frac{t}{1!} f_{1\alpha\beta} + \frac{t^2}{2!} f_{2\alpha\beta} + \frac{t^3}{3!} f_{3\alpha\beta} + \dots\dots .$$

The system (5.12) is of the type (5.5) in definition 5.1. Assuming that the functions $\lambda_{\alpha\beta}(x^i,y^r)$ in (5.7) are continuous, it follows that $f_{k\alpha\beta}$ are continuous functions of x's and y's. Then the $g_{\alpha\beta}$ are continuous functions of x's, y's and t. Since $g_{\alpha\beta}$ vanish at $t = 0$, by continuity $1 + g_{\alpha\beta}$ do not vanish in some neighborhood of $t = 0$, say $N_{t=0}$. Thus

$$\det \begin{pmatrix} 1+g_{11} & g_{12} & \cdots & g_{1\ell} \\ g_{21} & 1+g_{22} & \cdots & g_{2\ell} \\ \cdots & \cdots & \cdots & \cdots \\ g_{\ell 1} & g_{\ell 2} & & 1+g_{\ell\ell} \end{pmatrix} \neq 0$$

in $N_{t=0}$. This is the condition (5.6). Therefore, by definition 5.1, the system of differential equations (5.1) is invariant, in $N_{t=0}$, under the group G.

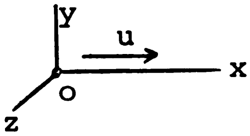
<u>Example 5.1</u>: We shall show that the system of Maxwell's equations of an electromagnetic wave is invariant (in our sense) under the group of Lorentz's transformations. So, our definition of invariance of a system of differential equations (definition 5.1) agrees with the invariance in the sense of Lorentz. The Maxwell equations of electromagnetic wave are:

$$(5.13)\quad \begin{cases} F_1(v) \equiv H^3_y - H^2_z - \frac{1}{c}E^1_t = 0, & F_2(v) \equiv H^1_z - H^3_x - \frac{1}{c}E^2_t = 0 \\ F_3(v) \equiv H^2_x - H^1_y - \frac{1}{c}E^3_t = 0, & F_4(v) \equiv E^3_y - E^2_z + \frac{1}{c}H^1_t = 0 \\ F_5(v) \equiv E^1_z - E^3_x + \frac{1}{c}H^2_t = 0, & F_6(v) \equiv E^2_x - E^3_y + \frac{1}{c}H^3_t = 0 \\ F_7(v) \equiv E^1_x + E^2_y + E^3_z = 0, & F_8(v) \equiv H^1_x + H^2_y + H^3_z = 0 \end{cases}$$

where $H = (H^1, H^2, H^3)$ and $E = (E^1, E^2, E^3)$ are magnetic and electric field intensity vectors. The Lorentz transformations, which form a one-parameter transformation group, are

$$(5.14)\quad G: \begin{cases} \bar{x} = b(x + cat),\ \bar{y} = y,\ \bar{z} = z,\ \bar{t} = b(t + \frac{1}{c}ax), \\ \bar{E}^1 = E^1,\ \bar{E}^2 = b(E^2 + aH^3),\ \bar{E}^3 = b(E^3 - aH^2), \\ \bar{H}^1 = H^1,\ \bar{H}^2 = b(H^2 - aE^3),\ \bar{H}^2 = b(H^3 + aE^2), \end{cases}$$

where $a = u/c$ is the parameter such that $a = 0$ gives identity transformation, u is the velocity of the observer (see figure), c is the velocity of light, $b = 1/\sqrt{1-a^2}$.



The observer moves with his frame of reference along x-direction with velocity u.

Using (2.15), we find from (5.14) the extended group $G_{(1)}$:

$$
(5.15)\quad G_{(1)}: \begin{cases}
G,\ \bar{E}^1_{\bar{x}} = b(E^1_x - \frac{1}{c}aE^1_t),\ \bar{E}^1_{\bar{y}} = E^1_y,\ \bar{E}^1_{\bar{z}} = E^1_z, \\
\bar{E}^1_{\bar{t}} = b(E^1_t - caE^1_x),\ \bar{E}^2_{\bar{x}} = b^2(E^2_x - \frac{1}{c}aE^2_t + aH^3_x - \frac{1}{c}a^2H^3_t), \\
\bar{E}^2_{\bar{y}} = b(E^2_y + aH^3_y),\ \bar{E}^2_{\bar{z}} = b(E^2_z + aH^3_z), \\
\bar{E}^2_{\bar{t}} = b^2(E^2_t - caE^2_x + aH^3_t - ca^2H^3_x), \\
\bar{E}^3_{\bar{x}} = b^2(E^3_x - \frac{1}{c}aE^3_t - aH^2_x + \frac{1}{c}a^2H^2_t),\ \bar{E}^3_{\bar{y}} = b(E^3_y - aH^2_y), \\
\bar{E}^3_{\bar{z}} = b(E^3_z - aH^2_z),\ \bar{E}^3_{\bar{t}} = b^2(E^3_t - caE^3_x - aH^2_t + ca^2H^2_x), \\
\bar{H}^1_{\bar{x}} = b(H^1_x - \frac{1}{c}aH^1_t),\ \bar{H}^1_{\bar{y}} = H^1_y,\ \bar{H}^1_{\bar{z}} = H^1_z, \\
\bar{H}^1_{\bar{t}} = b(H^1_t - caH^1_x),\ \bar{H}^2_{\bar{x}} = b^2(H^2_x - \frac{1}{c}aH^2_t - aE^3_x + \frac{1}{c}a^2E^3_t), \\
\bar{H}^2_{\bar{y}} = b(H^2_y - aE^3_y),\ \bar{H}^2_{\bar{z}} = b(H^2_z - aE^3_z), \\
\bar{H}^2_{\bar{t}} = b^2(H^2_t - caH^2_x - aE^3_t + ca^2E^3_x), \\
\bar{H}^3_{\bar{x}} = b^2(H^3_x - \frac{1}{c}aH^3_t + aE^2_x - \frac{1}{c}a^2E^2_t),\ \bar{H}^3_{\bar{y}} = b(H^3_y + aE^2_y), \\
\bar{H}^3_{\bar{z}} = b(H^3_z + aE^2_z),\ \bar{H}^3_{\bar{t}} = b^2(H^3_t - caH^3_x + aE^2_t - ca^2E^2_x).
\end{cases}
$$

Substituting the unbar variables in $F_1(v),\ldots,F_8(v)$ in terms of the bar variables from (5.15), we get

$$F_1(v(\bar{v})) = bF_1(\bar{v}) - ab\, F_7(\bar{v}),\quad F_2(v(\bar{v})) = F_2(\bar{v}),$$

$$F_3(v(\bar{v})) = F_3(\bar{v}),\quad F_4(v(\bar{v})) = bF_4(\bar{v}) + abF_8(\bar{v}),$$

$$F_5(v(\bar{v})) = F_5(\bar{v}),\quad F_6(v(\bar{v})) = F_6(\bar{v}),$$

$$F_7(v(\bar{v})) = bF_7(\bar{v}) - abF_1(\bar{v}),\quad F_8(v(\bar{v})) = bF_8(\bar{v}) + abF_4(\bar{v}),$$

which are relations of the type (5.3) in definition 5.1. It is easy to check that the determinant of the coefficients of $F_i(\bar{v})$ in the above system is not zero. That is, the condition (5.4) is satisfied. From (5.15) we find the operator $X_{(1)}$ to be

$$\begin{aligned}(5.16)\quad X_{(1)} = {} & ct\frac{\partial}{\partial x} + \frac{1}{c}x\frac{\partial}{\partial t} + H^3\frac{\partial}{\partial E^2} - H^2\frac{\partial}{\partial E^3} - E^3\frac{\partial}{\partial H^2} + E^2\frac{\partial}{\partial H^3} \\
& -\frac{1}{c}E^1_t\frac{\partial}{\partial E^1_x} - cE^1_x\frac{\partial}{\partial E^1_t} - (\frac{1}{c}E^2_t - H^3_x)\frac{\partial}{\partial E^2_x} + H^3_y\frac{\partial}{\partial E^2_y} \\
& + H^3_z\frac{\partial}{\partial E^2_z} - (cE^2_x - H^3_t)\frac{\partial}{\partial E^2_t} - (\frac{1}{c}E^3_t + H^2_x)\frac{\partial}{\partial E^3_x} \\
& - H^2_y\frac{\partial}{\partial E^3_y} - H^2_z\frac{\partial}{\partial E^3_z} - (cE^3_x + H^2_t)\frac{\partial}{\partial E^3_t} - \frac{1}{c}H^1_t\frac{\partial}{\partial H^1_x} - cH^1_x\frac{\partial}{\partial H^1_t} \\
& -(\frac{1}{c}H^1_t + E^3_x)\frac{\partial}{\partial H^2_x} - E^3_y\frac{\partial}{\partial H^2_y} - E^3_z\frac{\partial}{\partial H^2_z} - (cH^2_x + E^3_t)\frac{\partial}{\partial H^2_t} \\
& -(\frac{1}{c}H^3_t - E^2_x)\frac{\partial}{\partial H^3_x} + E^2_y\frac{\partial}{\partial H^3_y} + E^2_z\frac{\partial}{\partial H^3_z} - (cH^3_x - E^2_t)\frac{\partial}{\partial H^3_t}\ .\end{aligned}$$

From (5.13) and (5.16) we have

$$X_{(1)}F_1(v) = F_7(v),\ X_{(1)}F_2(v) \equiv 0,\ X_{(1)}F_3(v) \equiv 0,$$

$$X_{(1)}F_4(v) = -F_8(v),\ X_{(1)}F_5(v) \equiv 0,\ X_{(1)}F_6(v) \equiv 0,$$

$$X_{(1)}F_7(v) = F_1(v),\ X_{(1)}F_8(v) = -F_4(v)$$

which are relations of type (5.7) in the theorem 5.1.

Example 5.2: Consider the system of differential equations

$$(5.17)\quad \begin{cases} F_1(v) \equiv \psi_{xx} + \psi_{yy} + \omega = 0 \\ F_2(v) \equiv \omega_t + \psi_y\omega_x - \psi_x\omega_y = 0. \end{cases}$$

This system defines nonsteady rotational flow of incompressible fluid in (x, y)-plane, where ψ and ω are respectively the stream function and vorticity of the flow. The system (5.17) is invariant under the group

$$(5.18)\quad X = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - 2\omega\frac{\partial}{\partial \omega}$$

Derivation of this group is in the next section. Here, we only show the relationship between this group and the system (5.17). From (5.18) we find:

$$(5.19)\quad X_{(2)} = X - 2\psi_t\frac{\partial}{\partial\psi_t} - \psi_x\frac{\partial}{\partial\psi_x} - \psi_y\frac{\partial}{\partial\psi_y} - 4\omega_t\frac{\partial}{\partial\omega_t} - 3\omega_x\frac{\partial}{\partial\omega_x}$$

$$- 3\omega_y\frac{\partial}{\partial\omega_y} + \dots - 2\psi_{xx}\frac{\partial}{\partial\psi_{xx}} - 2\psi_{yy}\frac{\partial}{\partial\psi_{yy}} + \dots .$$

$$(5.20)\quad G: \begin{cases} \bar{t} = e^{2a}t,\ \bar{x} = e^{a}x,\ \bar{y} = e^{a}y \\ \bar{\psi} = \psi,\ \bar{\omega} = e^{-2a}\omega. \end{cases}$$

$$(5.21)\quad G_{(2)}: \begin{cases} G,\ \bar{\psi}_{\bar{t}} = e^{-2a}\psi_t,\ \bar{\psi}_{\bar{x}} = e^{-a}\psi_x,\ \bar{\psi}_{\bar{y}} = e^{-a}\psi_y, \\ \bar{\omega}_{\bar{t}} = e^{-4a}\omega_t,\ \bar{\omega}_{\bar{x}} = e^{-3a}\omega_x,\ \bar{\omega}_{\bar{y}} = e^{-3a}\omega_y, \\ \ldots,\ \bar{\psi}_{\bar{x}\bar{x}} = e^{-2a}\psi_{xx},\ \bar{\psi}_{\bar{y}\bar{y}} = e^{-2a}\psi_{yy},\ \ldots, \end{cases}$$

where $a = 0$ gives identity transformation. We now have

$$X_{(2)}F_1(v) = -2\psi_{xx} - 2\psi_{yy} - 2\omega = -2(\psi_{xx} + \psi_{yy} + \omega) = -2F_1(v)$$

$$X_{(2)}F_2(v) = -4\omega_t - \psi_y\omega_x + \psi_y(-3\omega_x) + \psi_x\omega_y - \psi_x(-3\omega_y)$$

$$= -4F_2(v)$$

Substituting from (5.21) in $F_1(\bar{v})$ and $F_2(\bar{v})$, we get

$$F_1(\bar{v}(v)) = e^{-2a}\psi_{xx} + e^{-2a}\psi_{yy} + e^{-2a}\omega = e^{-2a}F_1(v)$$

$$F_2(\bar{v}(v)) = e^{-4a}\omega_t + e^{-a}\psi_y \cdot e^{-3a}\omega_x - e^{-a}\psi_x \cdot e^{-3a}\omega_y = e^{-4a}F_2(v),$$

which are the relations of the type (5.5) in definition 5.1. We also have that

$$\det \begin{pmatrix} e^{-2a} & 0 \\ 0 & e^{-4a} \end{pmatrix} \neq 0,$$

i.e., the condition (5.6) is satisfied.

6. DETERMINATION OF ONE-PARAMETER TRANSFORMATION GROUPS LEAVING GIVEN DIFFERENTIAL EQUATIONS INVARIANT

One-parameter transformation groups furnish an important tool for integrating differential equations. They are used in Lie's theory for integrating first and second order ordinary differential equations, and first order linear partial differential equations. Transformation groups also may simplify the work of integrating differential equations, for example, in the reduction of the order of ordinary differential equations and in the reduction of the number of independent variables in partial differential equations. In this section we give a method of finding the groups of transformations which are required for these applications.

The method here is to find one-parameter transformation groups leaving the given differential equations invariant. That is, given a differential equation (P.D.E., or, O.D.E.) of order r with independent variables

6. DETERMINATION OF ONE-PARAMETER TRANSFORMATION GROUPS LEAVING GIVEN DIFFERENTIAL EQUATIONS INVARIANT

The one-parameter transformation groups furnish us with an important tool for integrating differential equations and for simplifying the work of integrating differential equations; the first case, according to the Lie-group-methods of integrating ordinary differential equations of the first and the second order, and linear partial differential equations of the first order; the later case, according to the theorem of reduction of the order of ordinary differential equations and the theorem of reduction of number of independent variables in partial differential equations. To achieve the group-methods of integrating differential equations and the methods of simplifying the work of integrating differential equations, we propose to find the required groups of transformations.

The method here is to find one-parameter transformation groups leaving the given differential equations invariant. That is, given a differential equation (P.D.E., or, O.D.E.) of order r with independent variables

$x^1,..,x^n$ ($n > 1$ for P.D.E., $n = 1$ for O.D.E.) and dependent variable y:

$$(6.1) \qquad F(x^i, y, y_i, \ldots, y_{i_1..i_r}) = 0$$

where $y_i \equiv \partial y/\partial x^i$, $y_{i_1..i_r} \equiv \partial^r y/\partial x^{i_1}..\partial x^{i_r}$; we look for a group of the form

$$(6.2) \quad X = \xi^1(x^1,\ldots,x^n,y)\frac{\partial}{\partial x^1} + \ldots + \xi^n(x^1,\ldots,x^n,y)\frac{\partial}{\partial x^n}$$
$$+ \eta(x^1,\ldots,x^n, y)\frac{\partial}{\partial y}$$

where $\xi^1(x^1,\ldots,x^n,y),\ldots,\xi^n(x^1,\ldots,x^n,y),\eta(x^1,\ldots,x^n,y)$ are to be determined such that (cf. theorem 3.2)

$$(6.3) \quad X_{(r)}F(x^i, y, y_i, \ldots, y_{i_1..i_r})$$
$$= \lambda(x^i, y)F(x^i, y, y_i,\ldots,y_{i_1..i_r})$$

where $\lambda(x^i, y)$ is also to be determined. The extended group $X_{(r)}$ is derived from X, i.e., if we denote $X_{(r)}$ by

$$X_{(r)} = X + \sum_i \eta^i \frac{\partial}{\partial y_i} + \ldots + \sum_{i_1,..,i_r} \eta^{i_1..i_r} \frac{\partial}{\partial y_{i_1..i_r}},$$

then the coefficients $\eta^i,\ldots,\eta^{i_1..i_r}$ are calculated, with the help of (2.18), in terms of the derivatives of ξ's and η with respect to x's and y, and the derivatives of y with respect to x's. Since the form of $F(x^i, y, y_i, \ldots, y_{i_1 \ldots i_r})$ is known, the left hand member of (6.3) is known in terms of x's, y, ξ's, η, the derivatives of ξ's and η with respect to x's and y, and the derivatives of y with respect to x's. Since λ is a function of x's and y, the equation (6.3) enables us to equate the coefficients of the derivatives of y with respect to x's. The result of equating these coefficients is a system of partial differential equations with n + 2 unknowns ξ^i, η, λ $(i = 1, \ldots, n)$; from which we solve for ξ^i, η, and λ.

In a similar manner, from a given system of differential equations

$$(6.4) \quad \begin{cases} F_1(x^i, y^r, y^r_i, \ldots, y^r_{i_1 \ldots i_\theta}) = 0 \\ \cdots\cdots\cdots\cdots \\ F_\ell(x^i, y^r, y^r_i, \ldots, y^r_{i_1 \ldots i_\theta}) = 0, \end{cases}$$

we can find groups of transformations leaving the system invariant from the relations (cf. theorem 5.1)

$$(6.5) \quad \begin{cases} X_{(\theta)} F_1(x^i, y^r, \ldots, y^r_{i_1 \ldots i_\theta}) \\ \qquad = \sum_{s=1}^{\ell} \lambda_{1s}(x^i, y^r) F_s(x^i, y^r, \ldots, y^r_{i_1 \ldots i_\theta}) \end{cases}$$

$$\left\{ \begin{aligned} & X_{(\theta)} F_\ell(x^i, y^r, \dots, y^r_{i_1 \dots i_\theta}) \\ & \qquad = \sum_{s=1}^{\ell} \lambda_{\ell s}(x^i, y^r) F_s(x^i, y^r, \dots, y^r_{i_1 \dots i_\theta}) \end{aligned} \right.$$

The following are the examples of determination of groups for some differential equations which appeared in the previous sections, and some differential equations which will appear later.

Example 6.1: Consider the differential equation in example 3.1:

$$(6.6) \qquad F \equiv y^2(1 + y'^2) - r^2 = 0, \qquad r = \text{constant},$$

which is the differential equation of a family of circles $(x - a)^2 + y^2 = r^2$. This family of circles is invariant under the group of translations parallel to the x-axis:

$$(6.7) \qquad X = \frac{\partial}{\partial x} .$$

Lie concluded, by using his theorem 3.1, that the differential equation (6.6) is invariant under the group (6.7).

Ignoring the knowledge of the origin of the equation (6.6), we propose to find the group leaving the equation invariant by our method. Let the group be

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} ,$$

and write

$$X_{(1)} = X + \eta^1 \frac{\partial}{\partial y'}$$

By the formula (2.18) we find

$$(6.8) \quad \eta^1 = \eta_x + \eta_y y' - y'(\xi_x + \xi_y y')$$

$$= \eta_x + (\eta_y - \xi_x)y' - \xi_y (y')^2.$$

We need

$$X_{(1)}F = \lambda(x, y)F$$

or

$$(6.9) \quad 2y\eta(1+y'^2) + 2y^2y'\eta_1 = \lambda(x,y)\ [y^2(1+y'^2) - r^2].$$

Substituting from (6.8) into (6.9), and equating the coefficients of 1*, y', $(y')^2$, and $(y')^3$, we get, respectively:

$$(6.10) \quad \text{(coeff. of 1):} \quad 2y\eta = \lambda(y^2 - r^2)$$

$$(6.11) \quad \text{(coeff. of } y'\text{):} \quad 2y^2\eta_x = 0$$

$$(6.12) \quad \text{(coeff. of } y'^2\text{):} \quad 2y\eta - 2y^2(\xi_x - \eta_y) = \lambda y^2$$

$$(6.13) \quad \text{(coeff. of } y'^3\text{):} \quad 2y^2\xi_y = 0.$$

The equation (6.11) implies $\eta = \eta(y)$. The equation (6.13) implies $\xi = \xi(x)$. Then, the equation (6.10) gives

---

*By the coefficient of 1, we mean the terms not including any derivative of y with respect to x.

$\lambda = \lambda(y)$. And then, from (6.12), we have

$$\xi_x = \text{constant} = k, \text{ say.}$$

From which we obtain

$$(A_1) \qquad \xi = \xi(x) = kx + c$$

where c is a constant of integration. Subtracting (6.10) from (6.12), and rearranging, we get

$$(A_2) \qquad \frac{d\eta(y)}{dy} = \frac{r^2\lambda(y)}{2y^2} + k.$$

Substituting the value of $\lambda(y)$ from (6.10), i.e.,

$$(A_3) \qquad \lambda(y) = \frac{2y\eta(y)}{y^2 - r^2} ,$$

into (A_2) to get

$$(A_4) \qquad \frac{d\eta(y)}{dy} = \frac{r^2\eta(y)}{y(y^2 - r^2)} + k$$

from which we solve for $\eta(y)$. When $\eta(y)$ is obtained from (A_4), $\lambda(y)$ is obtained from (A_3), we finally have the required group

$$(6.14) \qquad X = (kx + c)\frac{\partial}{\partial x} + \eta(y)\frac{\partial}{\partial y}$$

with the property

$$X_{(1)}F = \lambda(y)F.$$

If we choose $\eta(y) = 0$, the equations (A_3) and (A_4) imply, respectively, $\lambda(y) = 0$, $k = 0$; and the group (6.14) becomes

$$X = c\frac{\partial}{\partial x} .$$

Setting $c = 1$, we have the group (6.7).

Example 6.2: Given the differential equation

$$(6.15) \qquad F \equiv y^2y' + x^2(y')^3 - x^4(y'')^2 = 0,$$

we shall find a group

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}$$

such that

$$(6.16) \qquad X_{(2)} F - \lambda(x, y)F.$$

Using the formula (2.18) we find η^1, η^2 in $X_{(2)} = X + \eta^1\frac{\partial}{\partial y'} + \eta^2\frac{\partial}{\partial y''}$ to be

$$(6.17) \quad \begin{cases} \eta^1 = \eta_x + \eta_y y' - \xi_x y' - \xi_y(y')^2 \\ \eta^2 = \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})(y')^2 \\ \qquad - \xi_{yy}(y')^3 + (\eta_y - 2\xi_x)y'' - 3\xi_y y'y''. \end{cases}$$

The other form of (6.16) is

$$(6.16') \quad 2y\eta y' + y^2\eta^1 + 2x\xi(y')^3 + 3x^2(y')^2\eta^1$$

$$- 4x^3\xi(y'')^2 - 2x^4y''\ \eta^2$$

$$= \lambda(x,y)\{y^2y' + x^2(y')^3 - x^4(y'')^2\}.$$

Substituting from (6.17) into (6.16') and equating the coefficients of 1, y', $(y')^2$, $(y')^3$, $(y')^4$, y'', ..., $y'(y'')^2$; we get:

(6.18) (coeff. of 1): $y^2\eta_x = 0$

(6.19) (coeff. of y'): $2y\eta + y^2\eta_y - y^2\xi_x = \lambda y^2$

(6.20) (coeff. of y'^2): $-y^2\xi_y + 3x^2y^2\eta_x = 0$

(6.21) (coeff. of y'^3): $2x\xi + 3x^2\eta_y - 3x^2\xi_x = \lambda x^2$

(6.22) (coeff. of y'^4): $-3x^2\xi_y = 0$

(6.23) (coeff. of y''): $-2x^4\eta_{xx} = 0$

(6.24) (coeff. of $y'y''$): $-4x^4\eta_{xy} + 2x^4\xi_{xx} = 0$

(6.25) (coeff. of y'^2y''): $4x^4\xi_{xy} - 2x^4\eta_{yy} = 0$

(6.26) (coeff. of y'^3y''): $2x^4\xi_{yy} = 0$

(6.27) (coeff. of y''^2): $-4x^3\xi - 2x^4\eta_y + 4x^4\xi_x = -\lambda x^4$

(6.28) (coeff. of $y'y''^2$): $6x^4\xi_y = 0.$

The equation (6.18) implies that η is not a function of x, i.e.,

$$(B_1) \qquad \eta = \eta(y) .$$

Then the equations (6.23) is satisfied automatically. The equation (6.22) and (6.28) imply that η is not a function of y, i.e.,

$$(B_2) \qquad \xi = \xi(x).$$

And the equations (6.26) and (6.20) are satisfied. Now, the equations (6.24) and (6.25) imply, respectively, that ξ is a linear function of x, and η is a linear function of y. Thus,

$$(B_3) \qquad \xi = ax + b,$$

$$(B_4) \qquad \eta = cy + d,$$

where a, b, c, d are constants.

Substituting from (B_3) and (B_4) into (6.19), we get

$$3cy^2 + 2dy - ay^2 = \lambda y^2$$

which gives

$$(B_5) \qquad \lambda = 3c - a$$

$$(B_6) \qquad d = 0.$$

Substituting from (B_3) and (B_4) into (6.21), we get

$$-ax^2 + 2bx + 3cx^2 = \lambda x^2$$

which gives

$$(B'_5) \qquad \lambda = 3c - a$$

$$(B_7) \qquad b = 0.$$

Substituting from (B_3), (B_4), (B_5), (B_6) and (B_7) into (6.27) we get

$$-4ax^4 - 2cx^4 + 4ax^4 = -(3c - a)x^4$$

which implies

$$a = c.$$

Thus, we have the results:

$$\xi = ax, \eta = ay, \lambda = 2a$$

where a is arbitrary constant. And the required group is

$$(6.29) \qquad X = ax\frac{\partial}{\partial x} + ay\frac{\partial}{\partial y}$$

Such that

$$X_{(2)}F = 2aF.$$

<u>Example 6.3</u>: We shall find the group under which the two-dimensional Laplace equation is invariant. The given differential equation is

$$u_{xx} + u_{yy} = 0. \tag{6.30}$$

Let the group be

$$X = \xi^1(x,y,u)\frac{\partial}{\partial x} + \xi^2(x,y,u)\frac{\partial}{\partial y} + \eta(x,y,u)\frac{\partial}{\partial u}$$

where ξ^1, ξ^2, η are to be determined. We write

$$X_{(2)} = X + \eta^1\frac{\partial}{\partial u_x} + \eta^2\frac{\partial}{\partial u_y} + \eta^{11}\frac{\partial}{\partial u_{xx}} + \eta^{12}\frac{\partial}{\partial u_{xy}} + \eta^{22}\frac{\partial}{\partial u_{yy}} .$$

where η^1, $\eta^2,\ldots,\eta^{22}$ are found, with the help of the formula (2.18), to be

$$\eta^1 = \eta_x + \eta_u u_x - u_x(\xi^1_x + \xi^1_u u_x) - u_y(\xi^2_x + \xi^2_u u_x)$$

$$\eta^2 = \eta_y + \eta_u u_y - u_x(\xi^1_y + \xi^1_u\ u_x) - u_y(\xi^2_y + \xi^2_u u_y)$$

$$\begin{aligned}\eta^{11} = {} & \eta_{xx} + \eta_{xu}u_x - u_x(\xi^1_{xx} + \xi^1_{xu}u_x) - u_y(\xi^2_{xx} + \xi^2_{xu}u_x) \\ & + [\eta_{xu} + \eta_{uu}u_x - u_x(\xi^1_{xu} + \xi^1_{uu}u_x) \\ & - u_y(\xi^2_{xu} + \xi^2_{uu}u_x)]u_x + [\eta_u - \xi^1_x - 2u_x\xi^1_u \\ & - u_y\xi^2_u]u_{xx} - [\xi^2_x + u_x\xi^2_u]u_{xy} - u_{xx}(\xi^1_x + \xi^1_u u_x) \\ & - u_{xy}(\xi^2_x + \xi^2_u u_x).\end{aligned}$$

. . . .

$$\eta^{22} = \eta_{yy} + \eta_{yu}u_y - u_x(\xi^1_{yy} + \xi^1_{yu}u_y) - u_y(\xi^2_{yy} + \xi_{yu}u_y)$$

$$+[\eta_{yu} + \eta_{uu}u_y - u_x(\xi^1_{yu} + \xi^1_{uu}u_y)$$

$$- u_y(\xi^2_{yu} + \xi^2_{uu}u_y)]u_y - [\xi^1_y + \xi^1_u u_y]u_{xy}$$

$$+ [\eta_u - u_x\xi^1_x - \xi^2_y - 2u_y\xi^2_u]u_{yy} - u_{xy}(\xi^1_y + \xi^1_u u_y)$$

$$- u_{yy}(\xi^2_y + \xi^2_u u_y).$$

Since we require

$$X_{(2)}\ \{u_{xx} + u_{yy}\} = \lambda(x,y,u)\ \{u_{xx} + u_{yy}\},$$

or

$$\eta^{11} + \eta^{22} = \lambda(x,y,u)\ \{u_{xx} + u_{yy}\}. \tag{6.31}$$

Substituting the value of η^{11}, η^{22} into (6.31) and equating the coefficients of 1, u_x, u_y,...., we get:

(6.32) (coeff. of 1): $\eta_{xx} + \eta_{yy} = 0.$

(6.33) (coeff. of u_x): $-\xi^1_{xx} + 2\eta_{xu} - \xi^1_{yy} = 0.$

(6.34) (coeff. of u_y): $-\xi^2_{xx} + 2\eta_{yu} - \xi^2_{yy} = 0.$

(6.35) (coeff. of u_x^2): $-2\xi^1_{xu} + \eta_{uu} = 0.$

(6.36) (coeff. of u_xu_y): $-\xi^2_{xu} - \xi^1_{yu} - \xi^1_{uu} = 0.$

(6.37) (coeff. of u_y^2): $-2\xi^2_{yu} + \eta_{uu} = 0.$

(6.38) (coeff. of u_x^3): $-\xi^1_{uu} = 0.$

(6.39) (coeff. of $u_x^2 u_y$): $-\xi^2_{uu} = 0.$

(6.38') (coeff. of $u_x u_y^2$): $-\xi^1_{uu} = 0.$

(6.39') (coeff. of u_y^3): $-\xi^2_{uu} = 0.$

(6.40) (coeff. of u_{xx}): $\eta_u - 2\xi^1_x = \lambda.$

(6.41) (coeff. of u_{xy}): $-2\xi^2_x - 2\xi^1_y = 0.$

(6.42) (coeff. of u_{yy}): $\eta_u - 2\xi^2_y = \lambda.$

(6.43) (coeff. of $u_x u_{xx}$): $-3\xi^1_u = 0.$

(6.44) (coeff. of $u_y u_{xx}$): $-\xi^2_u = 0.$

(6.44') (coeff. of $u_x u_{xy}$): $-2\xi^2_u = 0.$

(6.43') (coeff. of $u_y u_{xy}$): $-2\xi^1_u = 0.$

(6.43") (coeff. of $u_x u_{yy}$): $-\xi^1_u = 0.$

(6.44") (coeff. of $u_y u_{yy}$): $-3\xi^2_u = 0.$

The equation (6.43) implies that ξ^1 is not a function of u, i.e.,

$$(C_1) \qquad \xi^1 = \xi^1(x, y).$$

Similarly, the equation (6.44) implies

$$(C_2) \qquad \xi^2 = \xi^2(x, y).$$

From the equations (6.40) and (6.42) we have

$$\xi^1_x = \xi^2_y.$$

This and (6.41) yield

$$(C_3) \qquad \begin{cases} \xi^1_{xx} + \xi^1_{yy} = 0 \\ \xi^2_{xx} + \xi^2_{yy} = 0. \end{cases}$$

The equations (C_1) and (6.35), or (C_2) and (6.37), show that η is a linear function of u, i.e.,

$$(C_4) \qquad \eta = f(x, y)u + g(x, y).$$

Substituting (C_4) in (6.33) and (6.34), and using (C_3), we get

$$f_x(x, y) = 0$$

$$f_y(x, y) = 0$$

which imply

(C_5) $\qquad f(x, y) = \text{constant} = 2m$, say.

From (C_4), (C_5) and (6.32), we have

(C_6) $\qquad g_{xx}(x, y) + g_{yy}(x, y) = 0.$

Thus

(C_7) $\qquad \eta = 2mu + g(x, y)$

where m is a constant, and $g(x, y)$ satisfies (C_6).

The equations (6.40), (6.42) and (C_7) yield

(C_8) $\qquad 2m - 2\xi^1_x = \lambda$

(C_9) $\qquad 2m - 2\xi^2_y = \lambda.$

Since from (C_1) or (C_2), the equations (C_8) or (C_9) imply

$$\lambda = \lambda(x, y).$$

This is the only restriction on λ.

If we choose $\lambda = 4x$, we get from (C_8) and (C_9)

(C_{10}) $\qquad \xi^1 = mx - x^2 + h_1(y),$

(C_{11}) $\qquad \xi^2 = my - 2xy + h_2(x).$

The equations (C_{10}), (C_{11}) and (6.41) imply

$$h_1'(y) = 2y - h_2'(x).$$

So

$$h_1'(y) - 2y = -h_2'(x) = \text{constant} = n, \text{ say.}$$

Thus

$$h_1(y) = y^2 + ny + c$$

$$h_2(x) = -nx + d$$

where c and d are arbitrary constants. We now have

$$(C_{12}) \qquad \xi^1 = c + mx + ny - x^2 + y^2$$

$$(C_{13}) \qquad \xi^2 = d - nx + my - 2xy.$$

We can check that the values of η, ξ^1, ξ^2 in (C_7), (C_{12}) and (C_{13}) satisfy all of the remaining equations in (6.32) - (6.42").

Thus the group

$$(6.43) \quad X = (c + mx + ny - x^2 + y^2)\frac{\partial}{\partial x}$$

$$+ (d - nx + my - 2xy)\frac{\partial}{\partial y} + (2mu + g(x,y))\frac{\partial}{\partial u},$$

where $g(x,y)$ satisfies (C_6), leaves the Laplace equation (6.30) invariant so that

$$X_{(2)}[u_{xx} + u_{yy}] = 4x\,[u_{xx} + u_{yy}].$$

If we set $\lambda = 0$, we get from (C_8) and (C_9):

$$\xi^1 = mx + q_1(y)$$

$$\xi^2 = my + q_2(x).$$

These and (6.41) imply

$$q_1'(y) = -q_2'(x).$$

Thus

$$q_1'(y) = -q_2'(x) = \text{constant} = k, \text{ say.}$$

Consequently,

$$q_1 = ky + a, \quad q_2 = -kx + b$$

where a, b are constants of integrations. Then we have

(C_{14}) $$\xi^1 = mx + ky + a$$

(C_{15}) $$\xi^2 = my - kx + b.$$

And we have the other group:

$$(6.44) \quad X = (mx + ky + a) \frac{\partial}{\partial x} + (my - kx + b) \frac{\partial}{\partial y} + (2mu + g(x, y)) \frac{\partial}{\partial u}$$

leaving the equation (6.30) invariant so that

$$X_{(2)} \{u_{xx} + u_{yy}\} \equiv 0.$$

<u>Example 6.4</u>: Let us consider a system of differential equations in fluid dynamics. Nonsteady rotational plane flows of incompressible fluid is governed by the system [11]

$$(6.45)\qquad \begin{cases} v_x - u_y = \omega \\ \omega_t + u\omega_x + v\omega_y = 0 \\ u = \psi_y,\ v = -\psi_x \end{cases}$$

where (u,v), ψ, ω are respectively the velocity vector, the stream function, the vorticity of the flow. Equivalently, the above system can be written in the form

$$(6.45')\qquad \begin{cases} F_1 \equiv \psi_{xx} + \psi_{yy} + \omega = 0 \\ F_2 \equiv \omega_t + \psi_y\omega_x - \psi_x\omega_y = 0. \end{cases}$$

Note that (6.45') is the system (5.18) in the example 5.2.

We shall find the group

$$X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \xi^3 \frac{\partial}{\partial y} + \eta^1 \frac{\partial}{\partial \psi} + \eta^2 \frac{\partial}{\partial \omega}$$

leaving invariant the system (6.45'). Let us restrict ourselves to find the group of the form

$$(6.46)\qquad \xi^i = \xi^i(t,\ x,\ y),\ \eta^i = \eta^i(t,\ x,\ y,\ \psi,\ \omega)$$

$$(i = 1,\ 2,\ 3;\ j = 1,\ 2).$$

Using (2.18), we find the extended group:

$$X_{(2)} = X + \eta^{1,1}\frac{\partial}{\partial\psi_t} + \eta^{1,2}\frac{\partial}{\partial\psi_x} + \eta^{1,3}\frac{\partial}{\partial\psi_y} + \eta^{2,1}\frac{\partial}{\partial\omega_t} + \eta^{2,2}\frac{\partial}{\partial\omega_x} + \eta^{2,3}\frac{\partial}{\partial\omega_y} + \ldots + \eta^{1,22}\frac{\partial}{\partial\psi_{xx}} + \eta^{1,23}\frac{\partial}{\partial\psi_{xy}} + \eta^{1,33}\frac{\partial}{\partial\psi_{yy}} + \ldots;$$

where

$$(6.47)\left\{\begin{aligned}
\eta^{1,2} &= \eta^1_x + \eta^1_\psi\cdot\psi_x + \eta^1_\omega\cdot\omega_x - \psi_t\xi^1_x - \psi_x\cdot\xi^2_x - \psi_y\cdot\xi^3_x\\
\eta^{1,3} &= \eta^1_y + \eta^1_\psi\cdot\psi_y + \eta^1_\omega\cdot\omega_y - \psi_t\cdot\xi^1_y - \psi_x\cdot\xi^2_y - \psi_y\cdot\xi^3_y\\
\eta^{2,1} &= \eta^2_t + \eta^2_\psi\cdot\psi_t - \eta^2_\omega\cdot\omega_t - \omega_t\cdot\xi^1_t - \omega_x\cdot\xi^2_t - \omega_y\cdot\xi^3_t\\
\eta^{2,2} &= \eta^2_x + \eta^2_\psi\cdot\psi_x + \eta^2_\omega\cdot\omega_x - \omega_t\cdot\xi^1_x - \omega_x\cdot\xi^2_x - \omega_y\cdot\xi^3_x\\
\eta^{2,3} &= \eta^2_y + \eta^2_\psi\cdot\psi_y + \eta^2_\omega\cdot\omega_y - \omega_t\cdot\xi^1_y - \omega_x\cdot\xi^2_y - \omega_y\cdot\xi^3_y\\
\eta^{1,22} &= \eta^1_{xx} + 2\eta^1_{x\psi}\cdot\psi_x + 2\eta^1_{x\omega}\cdot\omega_x - \psi_t\cdot\xi^1_{xx} - \psi_x\cdot\xi^2_{xx}\\
&\quad - \psi_y\cdot\xi^3_{xx} + \eta^1_{\psi\psi}\cdot\psi^2_x + 2\eta^1_{\psi\omega}\cdot\psi_x\omega_x + \eta^1_{\omega\omega}\cdot\omega^2_x\\
&\quad + \eta^1_\psi\cdot\psi_{xx} + \eta^1_\omega\cdot\omega_{xx} - 2\xi^2_x\cdot\psi_{xx} - 2\xi^1_x\cdot\psi_{xx} - 2\xi^3_x\cdot\psi_x
\end{aligned}\right.$$

$$
(6.47)\begin{cases}
\eta^{1,33} = \eta^1_{yy} + 2\eta^1_{y\psi}\cdot\psi_y + 2\eta^1_{y\omega}\cdot\omega_y - \psi_t\cdot\xi^1_{yy} - \psi_x\cdot\xi^2_{yy} \\
\qquad - \psi_y\cdot\xi^3_{yy} + \eta^1_{\psi\psi}\cdot\psi^2_y + 2\eta^1_{\psi\omega}\cdot\psi_y\omega_y + \eta^1_{\omega\omega}\cdot\omega^2_y \\
\qquad + \eta^1_{\psi}\cdot\psi_{yy} + \eta^1_{\omega}\cdot\omega_{yy} - 2\xi^3_y\cdot\psi_{yy} - 2\xi^1_y\cdot\psi_{ty} - 2\xi^2_y\cdot\psi_{xy}.
\end{cases}
$$

By the theorem 5.2, the group X must satisfy the relations

$$
(6.48)\begin{cases}
X_{(2)}F_1 = \lambda_{11}F_1 + \lambda_{12}F_2 \\
X_{(1)}F_2 = \lambda_{21}F_1 + \lambda_{22}F_2
\end{cases}
$$

or,

$$
(6.48')\begin{cases}
\eta^{1,22} + \eta^{1,33} + \eta^2 = \lambda_{11}F_1 + \lambda_{12}F_2 \\
\eta^{2,1} + \psi_y\eta^{2,2} + \omega_x\eta^{1,3} - \psi_x\eta^{2,3} - \omega_y\eta^{1,2} \\
\qquad = \lambda_{21}F_1 + \lambda_{22}F_2
\end{cases}
$$

where λ_{ij}; the functions of t, x, y, ψ, ω; are to be determined. Substituting from (6.45') and (6.47) into the first equation of (6.48') and equating the coefficients of 1, ψ_t, ψ_x, ψ_y, ω_t, ... , ψ_{xx}, ψ_{xy}, ...; we obtain:

(6.49) (coeff. of 1): $\eta^1_{xx} + \eta^1_{yy} + \eta^2 = \lambda_{11}\omega.$

(6.50) (coeff. of ψ_t): $-\xi^1_{xx} - \xi^1_{yy} = 0.$

(6.51) (coeff. of ψ_x): $2\eta^1_{x\psi} - \xi^2_{xx} - \xi^2_{yy} = 0.$

(6.52) (coeff. of ψ_y): $-\xi^2_{xx} + 2\eta^1_{y\psi} - \xi^3_{yy} = 0.$

(6.53) (coeff. of ω_t): $0 = \lambda_{12}.$

(6.54) (coeff. of ω_x): $2\eta^1_{x\omega} = 0.$

(6.55) (coeff. of ω_j): $2\eta^1_{y\omega} = 0.$

(6.56) (coeff. of ψ^2_x): $\eta^1_{\psi\psi} = 0.$

(6.56') (coeff. of ψ^2_y): $\eta^1_{\psi\psi} = 0.$

(6.57) (coeff. of $\psi_x\omega_x$): $2\eta^1_{\psi\omega} = 0.$

(6.53') (coeff. of $\psi_x\omega_y$): $0 = -\lambda_{12}.$

(6.56') (coeff. of ψ^2_y): $\eta^1_{\psi\psi} = 0.$

(6.53") (coeff. of $\psi_y\omega_x$): $0 = \lambda_{12}.$

(6.57') (coeff. of $\psi_y\omega_y$): $2\eta^1_{\psi\omega} = 0.$

(6.58) (coeff. of ω^2_x): $\eta^1_{\omega\omega} = 0.$

(6.58') (coeff. of ω^2_y): $\eta^1_{\omega\omega} = 0.$

(6.59) (coeff. of ψ_{tx}): $-2\xi^1_x = 0.$

(6.60) (coeff. of ψ_{ty}): $-2\xi^1_y = 0.$

(6.61) (coeff. of ψ_{xx}): $\eta^1_\psi - 2\xi^2_x = \lambda_{11}.$

(6.62) (coeff. of ψ_{xy}): $-2\xi^3_x - 2\xi^2_y = 0.$

(6.63) (coeff. of ψ_{yy}): $\eta^1_\psi - 2\xi^3_y = \lambda_{11}.$

(6.64) (coeff. of ω_{xx}): $\eta^1_\omega = 0.$

(6.64') (coeff. of ω_{yy}): $\eta^1_\omega = 0.$

We have immediately from (6.53):

(D_1) $$\lambda_{12} = 0.$$

The equations (6.59) and (6.60) imply

(D_2) $$\xi^1 = \xi^1(t),$$

and this satisfies the equation (6.50). The equations (6.61) and (6.63) give

(D_3) $$\xi^2_x = \xi^3_y = \frac{1}{2}(\eta^1_\psi - \lambda_{11})$$

The equation (6.62) gives

(D_4) $$\xi^2_y = -\xi^3_x .$$

It follows from (D_3) and (D_4) that ξ^2 and ξ^3 are conjugate harmonic fuctions of x and y, and so

(D_5) $$\xi^2_{xx} + \xi^2_{yy} = 0,$$

(D_6) $$\xi^3_{xx} + \xi^3_{yy} = 0.$$

Using (D_5), (D_6) in (6.51), (6.52) respectively, we obtain

$$(D_7) \qquad \eta^1_{x\psi} = 0$$

$$(D_8) \qquad \eta^1_{y\psi} = 0.$$

The equations (6.56) and (6.64) imply that η^1 is in the form

$$(D_9) \qquad \eta^1 = f(t, x, y) + g(t, x, y)\psi.$$

This satisfies the equations (6.54), (6.55), (6.57) and (6.58). The equations (D_7) and (D_8) imply that g is a function of t only. Thus,

$$(D_9') \qquad \eta^1 = f(t, x, y) + g(t)\psi.$$

Now (D_3) becomes:

$$(D_3') \qquad \xi^2_x = \xi^3_y = \frac{1}{2}(g(t) - \lambda_{11}).$$

Since ξ^2, ξ^3 are functions of t, x, y; it follows from (D_3') that λ_{11} is a function of t, x, and y; i.e.,

$$(D_{10}) \qquad \lambda_{11} = \lambda_{11}(t, x, y).$$

The equations (D_9') and (6.49) imply

$$(D_{11}) \qquad f_{xx}(t, x, y) + f_{yy}(t, x, y) + \eta^2 = \lambda_{11}\omega.$$

We now substitute from (6.45') and (6.47) into the second equation of (6.48), and equate the coefficients of 1, ψ_t, ψ_x, ... , to get

(6.65) (coeff. of 1): $\eta^2_t = 0.$

(6.66) (coeff. of ψ_t): $\eta^2_\psi = 0.$

(6.67) (coeff. of ψ_x): $-\eta^2_y = 0.$

(6.68) (coeff. of ψ_y): $\eta^2_x = 0.$

(6.69) (coeff. of ω_t): $\eta^2 - \xi^1_t = \lambda_{22}.$

(6.70) (coeff. of ω_x): $-\xi^2_t + \eta^1_y = 0.$

(6.71) (coeff. of ω_y): $-\xi^3_t - \eta^1_x = 0.$

(6.60') (coeff. of $\psi_t\omega_x$): $-\xi^1_y = 0.$

(6.59') (coeff. of $\psi_t\omega_y$): $\xi^1_x = 0.$

(6.72) (coeff. of $\psi_x\psi_y$): $\eta^2_\psi - \eta^2_\psi = 0.$

(6.60") (coeff. of $\psi_x\omega_t$): $\xi^1_y = 0.$

(6.73) (coeff. of $\psi_x\omega_x$): $\xi^2_y - \xi^2_y = 0.$

(6.74) (coeff. of $\psi_x\omega_y$): $-\eta^1_\psi + \xi^2_x - \eta^2_\omega + \xi^3_y = -\lambda_{22}.$

(6.59") (coeff. of $\psi_y\omega_t$): $-\xi^1_x = 0.$

(6.75) (coeff. of $\psi_y\omega_x$): $\eta^1_\psi - \xi^3_y + \eta^2_\omega - \xi^2_x = \lambda_{22}.$

(6.76) (coeff. of $\psi_y\omega_y$): $-\xi^3_x + \xi^3_x = 0.$

(6.77) (coeff. of $\omega_x\omega_y$): $\eta^1_\omega - \eta^1_\omega = 0.$

(6.78) (coeff. of ψ_{xx}): $0 = \lambda_{21}$.

(6.78') (coeff. of ψ_{yy}): $0 = \lambda_{21}$.

The equations (6.65), (6.66), (6.67) and (6.68) imply that η^2 is a function of ω only. This enables us to split (D_{11}) into three equations

(D_{11}') $f_{xx}(t, x, y) + f_{yy}(t, x, y) = \text{constant} = m$, say,

(D_{11}'') $\eta^2 = \lambda_{11}\omega - m$,

(D_{11}''') $\lambda_{11} = \lambda_{11}(\omega)$.

Now, we have from (D_{10}) and (D_{11}'''):

$$\lambda_{11} = \text{constant} = a, \text{ say,}$$

and then (D_{11}'') becomes

$$\eta^2 = a\omega - m.$$

From the fact that η^2 is a function of ω only and ξ^1 is a function of t only (cf. (D_2)), we get from (6.69)

$$\frac{d\eta^2}{d\omega} - \frac{d\xi^1}{dt} = \lambda_{22}.$$

From (6.74) (or (6.75)) and using (D_9') and (D_3'), we get

(D_{12}) $$\lambda_{11} + \frac{d\eta^2}{d\omega} = \lambda_{22}.$$

Thus,

$$\frac{d\xi^1}{dt} = -\lambda_{11} = -a,$$

and

$$\xi^1 = -at + \ell, \qquad \ell \text{ is a constant.}$$

Differentiate (D''_{11}) with respect to ω and compare the result with (D_{12}). We get

$$\lambda_{22} = 2\lambda_{11} = 2a.$$

From (D'_3) we have

$$\xi^2 = \frac{1}{2}(g(t) - a)x + h^2(t, y)$$

$$\xi^3 = \frac{1}{2}(g(t) - a)y + h^3(t, x)$$

for some functions h^2 and h^3. Using these in (D_4), we find

$$\frac{\partial h^2(t, y)}{\partial y} = -\frac{\partial h^3(t, x)}{\partial x}$$

which implies

$$h^2(t, y) = p(t)y + q(t)$$

$$h^3(t, x) = -p(t)x + r(t)$$

where $p(t)$, $q(t)$ and $r(t)$ are arbitrary functions. Thus

$$(D_{13}) \quad \xi^2 = \frac{1}{2}(g(t) - a)x + p(t)y + q(t)$$

(D_{14}) $\quad \xi^3 = \frac{1}{2}(g(t) - a)y - p(t)x + r(t).$

We now have from (D_{13}), (6.70) and (D_9'):

(D_{15}) $\quad f_y(t, x, y) = \frac{1}{2}g'(t)x + p'(t)y + q'(t).$

Similarly, from (D_{14}), (6.71) and (D_9') we have

(D_{16}) $\quad f_x(t, x, y) = - \frac{1}{2}g'(t)y + p'(t)x - r'(t).$

Using the property that $f_{yx}(t, x, y) = f_{xy}(t, x, y)$, we get

$$g'(t) = 0$$

Or,

$$g(t) = \text{constant} = b, \text{ say.}$$

Using (D_{15}) and (D_{16}) in (D_{11}'), we get

$$p'(t) = \frac{m}{2}$$

or

$$p(t) = \frac{m}{2}t + k, \qquad k \text{ is a constant.}$$

Now, we have from (D_{15}) and (D_{16}):

$$f(t, x, y) = \frac{m}{4}(x^2 + y^2) - r'(t)x + q'(t)y + s(t)$$

where $s(t)$ is a new arbitrary function. Finally, (D_9'), (D_{13}) and (D_{14}) become

(D_9'') $\quad \eta^1 = b\psi + \frac{m}{4}(x^2 + y^2) - r'(t)x + q'(t)y + s(t)$

(D_{13}') $\quad \xi^2 = \frac{1}{2}(b - a)x + (\frac{m}{2}t + k)y + q(t)$

(D_{14}') $\quad \xi^3 = \frac{1}{2}(b - a)y - (\frac{m}{2}t + k)x + r(t).$

We now have the required group

$$(6.79) \quad X = (\ell - at)\frac{\partial}{\partial t} + \{\frac{1}{2}(b - a)x + (\frac{m}{2}t + k)y + q(t)\}\frac{\partial}{\partial x}$$

$$+ \{\frac{1}{2}(b - a)y - (\frac{m}{2}t + k)x + r(t)\}\frac{\partial}{\partial y}$$

$$+ \{b\psi + \frac{m}{4}(x^2 + y^2) - r'(t)x + q'(t)y + s(t)\}\frac{\partial}{\partial \psi}$$

$$+ (a\omega - m)\frac{\partial}{\partial \omega} .$$

where a, b, k, ℓ, m are arbitrary constants; q(t), r(t), s(t) are arbitrary functions. This group has the properties:

$$(6.80) \quad X_{(2)}F_1 = aF_1, \quad X_{(1)}F_2 = 2aF_2;$$

since from (6.48) and $\lambda_{11} = a$, $\lambda_{12} = 0$, $\lambda_{21} = 0$, $\lambda_{22} = 2a$. Since the group X involves 5 arbitrary constants and 3 arbitrary functions, it can be decomposed into 8 smaller groups under which (6.45') is invariant

(set: a = 1, the others = 0): $X_1 = -t\frac{\partial}{\partial t} - \frac{1}{2}x\frac{\partial}{\partial x}$

$$- \frac{1}{2}y\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial \omega}$$

(set: b=1, the others = 0): $X_2 = \frac{1}{2}x\frac{\partial}{\partial x} + \frac{1}{2}y\frac{\partial}{\partial y} + \psi\frac{\partial}{\partial \psi}$

(set: k=1, the others = 0): $X_3 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$

(set: ℓ=1, the others = 0): $X_4 = \frac{\partial}{\partial t}$

(set: m=1, the others = 0): $X_5 = \frac{1}{2}ty\frac{\partial}{\partial x} - \frac{1}{2}tx\frac{\partial}{\partial y}$

$$+ \frac{1}{4}(x^1 + y^2)\frac{\partial}{\partial \psi} - \frac{\partial}{\partial \omega}$$

(for: q(t) $\nequiv$ constant, the others = 0):

$$X_6 = q(t)\frac{\partial}{\partial x} + q'(t)y\frac{\partial}{\partial \psi}$$

(for: r(t) $\nequiv$ constant, the others = 0):

$$X_7 = r(t)\frac{\partial}{\partial y} - r'(t)x\frac{\partial}{\partial \psi}$$

(for: s(t) $\nequiv$ 0, the others = 0): $X_8 = s(t)\ \frac{\partial}{\partial \psi}$.

These small groups have the properties:

$$X_{1(2)}\, F_1 = F_1, \quad X_{1(1)}F_2 = 2F_2$$

from (6.80) with a = 1;

$$X_{i(2)}F_1 = 0, \; X_{i(1)}F_2 = 0 \qquad\qquad (i = 2,\ldots,8)$$

from (6.80) with a = 0.

Note that the group $-2X_1$ is the group (5.19) in the example 5.2.

7. REDUCTION OF ORDER OF ORDINARY DIFFERENTIAL EQUATIONS

In this section, we deal with Lie's theorem of reduction of order of ordinary differential equations, by the utilization of one-parameter transformation groups. This theorem is considered as an important one, since it enables us to simplify the forms of ordinary differential equations. It is our purpose here to clarify this theorem by giving a new proof. We note here that the main result in this section is the proof of a lemma, which is a key for proving the Lie theorem. This lemma shows an important property of the extended group of transformations of two variables where one variable is regarded as a function of the other.

Lemma: Suppose $u(x, y)$ and $v(x, y, y')$ are, respectively, an absolute invariant and a differential invariant of the first order of the group generated by $X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}$. Then $v_1 = \frac{dv/dx}{du/dx} = \frac{dv}{du}$, $v_2 = \frac{dv_1/dx}{du/dx} = \frac{dv_1}{du} = \frac{d^2v}{du^2}, \ldots, v_n = \frac{dv_{n-1}/dx}{du/dx} = \frac{dv_{n-1}}{du} = \frac{d^nv}{du^n}$ are, respectively, differential invariants of the second order, ... , the $(n + 1)^{th}$ order of the group generated by X.

<u>Proof</u>: From the fact that $v(x, y, y')$ actually involves y' (cf. Sect. 2) and from $v_1 = \frac{dv/dx}{du/dx} = \dfrac{\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}y' + \frac{\partial v}{\partial y'}y''}{\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}y'}$, it follows that v_1 involves the derivative y''. The only thing we have to do is to show that $X_{(2)}(\frac{dv/dx}{du/dx})$ vanishes identically, where $X_{(2)} = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} + \eta^1(x, y, y')\frac{\partial}{\partial y'} + \eta^2(x, y, y', y'')\frac{\partial}{\partial y''}$ is the second extended operator of X. Since

$$(7.1) \qquad X_{(2)}\left(\frac{dv/dx}{du/dx}\right) = \frac{1}{du/dx} X_{(2)}\left(\frac{dv}{dx}\right) - \frac{dv/dx}{(du/dx)^2} X_{(2)}\left(\frac{du}{dx}\right),$$

and since

$$(7.2) \quad X_{(2)}\left(\frac{dv}{dx}\right) = X_{(2)}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}y' + \frac{\partial v}{\partial y'}y''\right) = X_{(2)}\left(\frac{\partial v}{\partial x}\right) + X_{(2)}\left(\frac{\partial v}{\partial y}y'\right) + X_{(2)}\left(\frac{\partial v}{\partial y'}y''\right),$$

expanding each term on the right side of (7.2), and using the formula (2.18) we get

$$X_{(2)}\left(\frac{\partial v}{\partial x}\right) = X_{(1)}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial x}(X_{(1)}v) - \left(\frac{\partial \xi}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial \eta}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial \eta^1}{\partial x}\frac{\partial v}{\partial y'}\right)$$

$$= -\left(\frac{\partial \xi}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial \eta}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial \eta^1}{\partial x}\frac{\partial v}{\partial y'}\right),$$

$$X_{(2)}\left(\frac{\partial v}{\partial y}y'\right) = y'X_{(2)}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial v}{\partial y}\eta' = y'\ X_{(1)}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial v}{\partial y}\eta'$$

$$= -y'\left(\frac{\partial \xi}{\partial y}\frac{\partial v}{\partial x} + \frac{\partial \eta}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial \eta^1}{\partial y}\frac{\partial v}{\partial y'}\right)$$

$$+ \frac{\partial v}{\partial y}\left(\frac{\partial \eta}{\partial x} + \frac{\partial \eta}{\partial y}y' - \frac{\partial \xi}{\partial x}y' - \frac{\partial \xi}{\partial y}y'^2\right),$$

$$X_{(2)}\left(\frac{\partial v}{\partial y'}y''\right) = y''X_{(2)}\left(\frac{\partial v}{\partial y'}\right) + \frac{\partial v}{\partial y'}\eta^2 = y''X_{(1)}\left(\frac{\partial v}{\partial y'}\right) + \frac{\partial v}{\partial y'}\eta^2$$

$$= -y''\left(\frac{\partial \eta^1}{\partial y'}\frac{\partial v}{\partial y'}\right) + \frac{\partial v}{\partial y'}\left(\frac{\partial \eta^1}{\partial x} + \frac{\partial \eta^1}{\partial y}y' + \frac{\partial \eta^1}{\partial y'}y''\right.$$

$$\left. - \frac{\partial \xi}{\partial x}y'' - \frac{\partial \xi}{\partial y}y'y''\right).$$

We see that

$$(7.2')\quad X_{(2)}\left(\frac{dv}{dx}\right) = -\frac{\partial v}{\partial x}\left(\frac{\partial \xi}{\partial x} + \frac{\partial \xi}{\partial y}y'\right) - \frac{\partial v}{\partial y}\left(\frac{\partial \xi}{\partial x} + \frac{\partial \xi}{\partial y}y'\right)y'$$

$$- \frac{\partial v}{\partial y'}\left(\frac{\partial \xi}{\partial x} + \frac{\partial \xi}{\partial y}y'\right)y''$$

$$= -\left(\frac{\partial \xi}{\partial x} + \frac{\partial \xi}{\partial y}y'\right)\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}y' + \frac{\partial v}{\partial y'}y''\right)$$

$$= -\frac{d\xi}{dx}\frac{dv}{dx}$$

Similarly,

$$(7.3)\quad X_{(2)}\left(\frac{du}{dx}\right) = X_{(2)}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}y'\right) = X_{(2)}\left(\frac{\partial u}{\partial x}\right) + X_{(2)}\left(\frac{\partial u}{\partial y}y'\right)$$

$$= X\left(\frac{\partial u}{\partial x}\right) + X_{(1)}\left(\frac{\partial u}{\partial y}y'\right) = -\frac{d\xi}{dx}\frac{du}{dx}.$$

Substituting from (7.2') and (7.3) into the right hand side of (7.1) we get

$$X_{(2)}\left(\frac{dv/dx}{du/dx}\right) = 0.$$

Thus $\frac{dv}{du} = \frac{dv/dx}{du/dx}$ is differential invariant of the second order of the group generated by X.

Moreover, if v_{k-1} is the differential invariant of the k^{th} order of the group generated by $X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$ then we claim that $\frac{dv_{k-1}}{du} = \frac{dv_{k-1}/dx}{du/dx}$ is differential invariant of the $(k+1)^{th}$ order of X. We have that $v_{k-1} = v_{k-1}(x,y,y^1,\ldots,y^{(k)})$ depends on $y^{(k)}$, and $X_{(k)}v_{k-1} = 0$ where $X_{(k)} = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} + \eta^1(x,y,y')\frac{\partial}{\partial y'} + \ldots + \eta^k(x,y,y',\ldots,y^{(k)})\frac{\partial}{\partial y^{(k)}}$ is the k^{th} extended operator of X. As an immediate consequence $\frac{dv_{k-1}/dx}{du/dx} = \dfrac{\frac{\partial v_{k-1}}{\partial x} + \frac{\partial v_{k-1}}{\partial y}y' + \ldots + \frac{\partial v_{k-1}}{\partial y^{(k)}}y^{(k+1)}}{\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}y'}$ contains $y^{(k+1)}$. We shall now show that $X_{(k+1)}\left(\frac{dv_{k-1}/dx}{du/dx}\right) = 0$. Since

$$X_{(k+1)}\left(\frac{\partial v_{k-1}}{\partial x}\right) = X_{(k)}\left(\frac{\partial v_{k-1}}{\partial x}\right) = \frac{\partial}{\partial x}(X_{(k)}v_{K-1}) - \left(\frac{\partial\xi}{\partial x}\frac{\partial v_{k-1}}{\partial x} + \frac{\partial\eta}{\partial x}\frac{\partial v_{k-1}}{\partial y} + \frac{\partial\eta^1}{\partial x}\frac{\partial v_{k-1}}{\partial y'} + \ldots \right.$$

$$\left. \ldots + \frac{\partial\eta^k}{\partial x}\frac{\partial v_{k-1}}{\partial y^{(k)}}\right)$$

$$= -\left(\frac{\partial \xi}{\partial x}\frac{\partial v_{k-1}}{\partial x} + \frac{\partial \eta}{\partial x}\frac{\partial v_{k-1}}{\partial y} + \frac{\partial \eta^1}{\partial x}\frac{\partial v_{k-1}}{\partial y'} + \cdots\right.$$

$$\left.\cdots + \frac{\partial \eta^k}{\partial x}\frac{\partial v_{k-1}}{\partial y^{(k)}}\right),$$

$$X_{(k+1)}\left(\frac{\partial v_{k-1}}{\partial y}y'\right) = y'X_{(k+1)}\left(\frac{\partial v_{k-1}}{\partial y}\right) + \frac{\partial v_{k-1}}{\partial y}\eta^1 = y'X_{(k)}\left(\frac{\partial v_{k-1}}{\partial y}\right)$$

$$+ \frac{\partial v_{k-1}}{\partial y}\eta^1 = -y'\left(\frac{\partial \xi}{\partial y}\frac{\partial v_{k-1}}{\partial x} + \frac{\partial \eta}{\partial y}\frac{\partial v_{k-1}}{\partial y}\right.$$

$$\left.+ \frac{\partial \eta^1}{\partial y}\frac{\partial v_{k-1}}{\partial y'} + \cdots + \frac{\partial \eta^k}{\partial y}\frac{\partial v_{k-1}}{\partial y^{(k)}}\right)$$

$$+ \frac{\partial v_{k-1}}{\partial y}\left(\frac{\partial \eta}{\partial x} + \frac{\partial \eta}{\partial y}y' - \frac{\partial \xi}{\partial x}y' - \frac{\partial \xi}{\partial y}y'^2\right),$$

$$X_{(k+1)}\left(\frac{\partial v_{k-1}}{\partial y'}y''\right) = y''X_{(k+1)}\left(\frac{\partial v_{k-1}}{\partial y'}\right) + \frac{\partial v_{k-1}}{\partial y'}\eta^2 = y''X_{(k)}\left(\frac{\partial v_{k-1}}{\partial y'}\right)$$

$$+ \frac{\partial v_{k-1}}{\partial y'}\eta^2 = -y''\left(\frac{\partial \eta^1}{\partial y'}\frac{\partial v_{k-1}}{\partial y'} + \frac{\partial \eta^2}{\partial y'}\frac{\partial v_{k-1}}{\partial y''} + \cdots\right.$$

$$\left.\cdots + \frac{\partial \eta^k}{\partial y'}\frac{\partial v_{k-1}}{\partial y^{(k)}}\right) + \frac{\partial v_{k-1}}{\partial y'}\left(\frac{\partial \eta^1}{\partial x} + \frac{\partial \eta^1}{\partial y}y'\right.$$

$$\left.+ \frac{\partial \eta^1}{\partial y'}y'' - \frac{\partial \xi}{\partial x}y'' - \frac{\partial \xi}{\partial y}y'y''\right),$$

. . . .

$$X_{(k+1)}\left(\frac{\partial v_{k-1}}{\partial y^{(k)}}y^{(k+1)}\right) = y^{(k+1)}X_{(k+1)}\left(\frac{\partial v_{k-1}}{\partial y^{(k)}}\right) + \frac{\partial v_{k-1}}{\partial y^{(k)}}\eta^{k+1}$$

$$= -y^{(k+1)}\left(\frac{\partial \eta^k}{\partial y^{(k)}}\frac{\partial v_{k-1}}{\partial y^{(k)}}\right) + \frac{\partial v_{k-1}}{\partial y^{(k)}}\left(\frac{\partial \eta^k}{\partial x}\right.$$

$$+ \frac{\partial \eta^k}{\partial y}y' + \frac{\partial \eta^k}{\partial y'}y'' + \cdots + \frac{\partial \eta^k}{\partial y^{(k)}}y^{(k+1)}$$

$$- \frac{\partial \xi}{\partial x} y^{(k+1)} - \frac{\partial \xi}{\partial y} y' y^{(k+1)} \Bigg) ,$$

we have that

$$X_{(k+1)} \left(\frac{dv_{k-1}}{dx} \right) = X_{(k+1)} \left(\frac{\partial v_{k-1}}{\partial x} \quad \frac{\partial v_{k-1}}{\partial y} y' + \ldots + \frac{\partial v_{k-1}}{\partial y^{(k)}} y^{(k+1)} \right)$$

$$= X_{(k+1)} \left(\frac{\partial v_{k-1}}{\partial x} \right) + X_{(k+1)} \left(\frac{\partial v_{k-1}}{\partial y} y' \right) + \ldots$$

$$\ldots + X_{(k+1)} \left(\frac{\partial v_{k-1}}{\partial y^{(k)}} y^{(k+1)} \right)$$

$$= - \frac{\partial v_{k-1}}{\partial x} \left(\frac{\partial \xi}{\partial x} + \frac{\partial \xi}{\partial y} y' \right) - \frac{\partial v_{k-1}}{\partial y} \left(\frac{\partial \xi}{\partial x} + \frac{\partial \xi}{\partial y} y' \right) y' - \ldots$$

$$\ldots - \frac{\partial v_{k-1}}{\partial y^{(k)}} \left(\frac{\partial \xi}{\partial x} + \frac{\partial \xi}{\partial y} y' \right) y^{(k+1)}$$

$$= - \frac{d\xi}{dx} \frac{dv_{k-1}}{dx} .$$

In a similar fashion, we can show that

$$X_{(k+1)} \left(\frac{du}{dx} \right) = X_{(k+1)} \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} y' \right) = X \left(\frac{\partial u}{\partial x} \right) + X_{(1)} \left(\frac{\partial u}{\partial y} y' \right) = - \frac{d\xi}{dx} \frac{du}{dx} .$$

Thus

$$X_{(k+1)} \left(\frac{dv_{k-1}/dx}{du/dx} \right) = \frac{1}{du/dx} X_{(k+1)} \left(\frac{dv_{k-1}}{dx} \right) - \frac{dv_{k-1}/dx}{(du/dx)^2} X_{(k+1)} \left(\frac{du}{dx} \right)$$

$$= \frac{1}{du/dx} \left(- \frac{d\xi}{dx} \frac{dv_{k-1}}{dx} \right) - \frac{dv_{k-1}/dx}{(du/dx)^2} \quad - \left(\frac{d\xi}{dx} \frac{du}{dx} \right)$$

$$= 0.$$

We now have a conclusion that $\frac{dv_{k-1}}{du} = \frac{dv_{k-1}/dx}{du/dx}$ is a differential invariant of order k+1 of the group X. By induction, we have that for positive integer n, $v_n = \frac{dv_{n-1}/dx}{du/dx}$ is a differential invariant of order n+1. The proof of the lemma is completed.

We now are in position to prove Lie's theorem on the reduction of order of ordinary differential equations.

Lie's theorem ([1], pp. 386-387): Suppose that the ordinary differential equation of order m:

$$F(x, y, y', \ldots, y^{(m)}) = 0 \tag{7.4}$$

and the group

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} \tag{7.5}$$

are such that $X_{(m)}F(x, y, y', \ldots, y^{(m)})$ vanishes whenever $F(x, y, y', \ldots, y^{(m)})$ vanishes, and that $u(x, y)$ and $v(x, y, y')$ are, respectively, an absolute invariant and a differential invariant of the first order of the group (7.5). Then the differential equation (7.4) can be written in the form

$$G(u, v, \frac{dv}{du}, \ldots, \frac{d^{m-1}v}{du^{m-1}}) = 0, \tag{7.6}$$

a differential equation of order m-1.

Proof: Since $F(x, y, y', \dots, y^{(m)})$ involves $y^{(m)}$ and since we have

$$X_{(m)}F(x, y, y', \dots, y^{(m)}) = 0$$

in the domain of definition of equation (7.4), by theorem 2.2 $F(x, y, y', \dots, y^{(m)})$ is a differential invariant of order m of the group X in the domain of definition of equation (7.4). Since u is an absolute invariant and v is a differential invariant of the first order of the group X, by the above lemma $\frac{dv}{du}, \frac{d^2v}{du^2}, \dots, \frac{d^{m-1}v}{du^{m-1}}$ are, respectively, differential invariants of the second order, ... , of the m^{th} order of X. We now have that $u, v, \frac{dv}{du}, \frac{d^2v}{du^2}, \dots, \frac{d^{m-1}v}{du^{m-1}}$ are m + 1 independent functions satisfying the equation $X_{(m)}f = 0$. We also have that $F(x, y, y', \dots, y^{(m)})$ satisfies the equation $X_{(m)}f = 0$ in the domain of definition of equation (7.4). Thus, in the domain of definition of equation (7.4), the $F(x, y, y', \dots, y^{(m)})$ can be written as a function of $u, v, \frac{dv}{du}, \dots, \frac{d^{m-1}v}{du^{m-1}}$; i.e.,

$$F(x, y, y^1, \dots, y^{(m)}) = G(u, v, \frac{dv}{du}, \dots, \frac{d^{m-1}v}{du^{m-1}}).$$

Therefore, equation (7.6) follows from (7.4).

Example: We know, from the example 6.2, that the differential equation

$$(7.7) \qquad F \equiv y^2y^1 + x^2(y')^2 - x^4(y'')^2 = 0$$

is invariant under the group

$$X = ax\frac{\partial}{\partial x} + ay\frac{\partial}{\partial y}, \quad a \neq 0 \tag{7.8}$$

so that

$$X_{(2)}F = 2aF.$$

Thus $X_{(2)}F$ vanishes whenever F vanishes. An absolute invariant of the group (7.8) is found from

$$\frac{dx}{ax} = \frac{dy}{ay}. \tag{7.9}$$

Solving (7.9), we get $\frac{y}{x}$ = constant as a solution. The first extended operator of the group (7.8) is

$$X_{(1)} = ax\frac{\partial}{\partial x} + ay\frac{\partial}{\partial y} + 0\frac{\partial}{\partial y'}.$$

We see that any function of y' satisfies the equation $X_{(1)}f = 0$, so that we can take y' as a differential invariant of the first order. We now set

$$u = y/x \quad , \quad v = y'.$$

Then

$$\frac{dv}{du} = \frac{dv/dx}{du/dx} = \frac{y''}{\frac{y'}{x} - \frac{y}{x^2}} = \frac{xy''}{v - u},$$

or,

$$y'' = \frac{v - u}{x}\cdot\frac{dv}{du}.$$

Substituting $y = xu$, $y' = v$, $y'' = \frac{v - u}{x}\cdot\frac{dv}{du}$ into (7.7), we obtain

$$x^2u^2v + x^2v^2 - x^4\left(\frac{v - u}{x}\cdot\frac{dv}{du}\right)^2 = 0,$$

or,

$$u^2v + v^2 - (v - u)^2\left(\frac{dv}{du}\right)^2 = 0 \tag{7.7'}$$

which is an equation of the first order.

8. REDUCTION OF NUMBER OF INDEPENDENT VARIABLES IN PARTIAL DIFFERENTIAL EQUATIONS

One of the methods of changing the form of partial differential equations to simplify the finding of solutions, is the reduction of the number of independent variables. Here, we shall deal with Morgan's method of reduction of the number of independent variables in partial differential equations, which is the utilization of one-parameter transformation groups.

Let

$$\Phi_{\delta}(x^1,\ldots,x^m,y^1,\ldots,y^n,\ldots,\frac{\partial^k y^1}{\partial (x^1)^k},\ldots,\frac{\partial^k y^n}{\partial (x^m)^k}) = 0 \tag{8.1}$$

be a system of partial differential equations of order k.

<u>Definition 8.1</u> (Morgan's definition, [3]): The solution $y^j = \Omega^j(x^1,\ldots,x^m)$ is called invariant solution with respect to the transformations T: $(x, y) \to (\bar{x}, \bar{y})$, if after transformation the solution becomes $\bar{y}^j = \Omega^j(\bar{x}^1,\ldots,\bar{x}^m)$; that is the $\bar{y}^j$ is exactly the same function of the $\bar{x}$'s as the y^j is of the x's.

<u>Definition 8.2</u> (Morgan's definition, [3]): The differential form $\Phi(x^1,\ldots,x^m,y^1,\ldots,y^n,\ldots,\frac{\partial^k y^1}{\partial (x^1)^k},\ldots,\frac{\partial^k y^n}{\partial (x^m)^k})$ is said to be conformally invariant under the

one-parameter transformation group G: $\bar{x}^i = \varphi^i(x^1,\ldots, x^m, y^1,\ldots,y^n; t)$, $\bar{y}^r = \psi^r(x^1,\ldots, x^m, y^1,\ldots,y^n; t)$; if it satisfies

$$(8.2)\quad \Phi(\bar{x}^1,\ldots,\bar{x}^m,\bar{y}^1,\ldots,\bar{y}^n,\ldots,\frac{\partial^k\bar{y}^1}{\partial(\bar{x}^1)^k},\ldots,\frac{\partial^k\bar{y}^n}{\partial(\bar{x}^m)^k} = H(x^1,\ldots,x^m,y^1,\ldots,y^n,\ldots,\frac{\partial^k y^1}{\partial(x^1)^k},\ldots,\frac{\partial^k y^n}{\partial(x^m)^k},\ t) \cdot\Phi(x^1,\ldots,x^m,y^1,\ldots,y^n,\ldots,\frac{\partial^k y^1}{\partial(x^1)^k},\ldots,\frac{\partial^k y^n}{\partial(x^m)^k})$$

where H is not zero (see Appendix).

Observe that the invariance of the differential equation $\Phi = 0$ under the group G is a special case of the conformal invariance of the differential form Φ under the group G. We can prove, by following the proof of theorem 3.2, that the equivalent form of (8.2) is

$$(8.2')\quad X_{(k)}\Phi(x^1,\ldots,x^m,y^1,\ldots,y^n,\ldots,\frac{\partial^k y^1}{\partial(x^1)^k},\ldots,\frac{\partial^k y^n}{\partial(x^m)^k}) = h(x^1,\ldots,x^m,y^1,\ldots,y^n,\ldots,\frac{\partial^k y^1}{\partial(x^1)^k},\ldots,\frac{\partial^k y^n}{\partial(x^m)^k}) \cdot\Phi(x^1,\ldots,x^m,y^1,\ldots,y^n,\ldots,\frac{\partial^k y^1}{\partial(x^1)^k},\ldots,\frac{\partial^k y^n}{\partial(x^m)^k})$$

where $X_{(k)}$ is the operator of $G_{(k)}$, the k^{th} extended group of the group G.

one-parameter transformation group G: $\bar{x}^i = \phi^i(x^1, \ldots, x^m, y^1, \ldots, y^n; t)$, $\bar{y}^r = \psi^r(x^1, \ldots, x^m, y^1, \ldots, y^n; t)$; if it satisfies

$$(8.2) \quad \Phi(\bar{x}^1, \ldots, \bar{x}^m, \bar{y}^1, \ldots, \bar{y}^n, \ldots, \frac{\partial^k \bar{y}^1}{\partial (\bar{x}^1)^k}, \ldots, \frac{\partial^k \bar{y}^n}{\partial (\bar{x}^m)^k}) = H(x^1, \ldots, x^m, y^1, \ldots, y^n, \ldots, \frac{\partial^k y^1}{\partial (x^1)^k}, \ldots, \frac{\partial^k y^n}{\partial (x^m)^k}, t) \cdot \Phi(x^1, \ldots, x^m, y^1, \ldots, y^n, \ldots, \frac{\partial^k y^1}{\partial (x^1)^k}, \ldots, \frac{\partial^k y^n}{\partial (x^m)^k})$$

where H is not zero.

Observe that the invariance of the differential equation $\Phi = 0$ under the group G is a special case of the conformal invariance of the differential form Φ under the group G. We can prove, by following the proof of theorem 3.2, that the equivalent form of (8.2) is

$$(8.2') \quad X_{(k)} \Phi(x^1, \ldots, x^m, y^1, \ldots, y^n, \ldots, \frac{\partial^k y^1}{\partial (x^1)^k}, \ldots, \frac{\partial^k y^n}{\partial (x^m)^k}) = h(x^1, \ldots, x^m, y^1, \ldots, y^n, \ldots, \frac{\partial^k y^1}{\partial (x^1)^k}, \ldots, \frac{\partial^k y^n}{\partial (x^m)^k}) \cdot \Phi(x^1, \ldots, x^m, y^1, \ldots, y^n, \ldots, \frac{\partial^k y^1}{\partial (x^1)^k}, \ldots, \frac{\partial^k y^n}{\partial (x^m)^k})$$

where $X_{(k)}$ is the operator of $G_{(k)}$, the k^{th} extended group of the group G.

Morgan associated with system (8.1) the group of the form

$$(8.3) \qquad G': \begin{cases} S_I: & \bar{x}^i = \phi^i(x^1,\dots,x^m;\ t) \qquad (i=1,\dots,m) \\ S_D: & \bar{y}^r = \psi^r(y^r;\ t) \qquad (r=1,\dots,n) \end{cases}$$

here S_I and S_D denote, respectively, the set of transformations of independent variables and dependent variables. Note that S_I form a group in m-dimensional space, and S_D form a group in n-dimensional space. Let $\sigma_1(x^1,\dots,x^m),\dots,\sigma_{m-1}(x^1,\dots,x^m)$ be a set of absolute invariants of S_I. These are also absolute invariants of G'. Let the other absolute invariants of G' be $g_1(x^1,\dots,x^m,y^1,\dots,y^n),\dots,g_n(x^1,\dots,x^m,y^1,\dots,y^n)$; so that $\sigma_1(x),\dots,\sigma_{m-1}(x)$, $g_1(x, y),\dots,g_n(x, y)$ form a set of absolute invariants of G'. In the method of reduction of the number of independent variables, we need the set of absolute invariants such that

$$(8.4) \qquad R\left(\frac{\partial(\sigma_1,\dots,\sigma_{m-1})}{\partial(x^1,\dots,x^m)}\right) = m - 1$$

and

$$(8.5) \qquad \frac{\partial(g_1,\dots,g_n)}{\partial(y^1,\dots,y^n)} \neq 0$$

where R indicates the rank of the Jacobian. If we make a change of variables defined by

$$(8.6) \qquad \sigma_i = \sigma_i(x^1,\ldots,x^m) \qquad (i=1,\ldots,m-1)$$

then the condition (8.4) enables us to express m-1 of the x's in terms of $\sigma_1,\ldots,\sigma_{m-1}$ and the remaining x, say x^m, in the form

$$(8.7) \qquad x^j = f_j(\sigma_1,\ldots,\sigma_{m-1}, x^m) \qquad (j=1,\ldots,m-1).$$

We now consider the y^r and $\bar{y}^r$ to be implicitly defined as functions of x^i and $\bar{x}^i$, respectively, by the equations

$$(8.8) \qquad z_k(x^1,\ldots,x^m) = g_k(x^1,\ldots,x^m,y^1,\ldots,y^n)$$

$$(8.9) \qquad \bar{z}_k(\bar{x}^1,\ldots,\bar{x}^m) = g_k(\bar{x}^1,\ldots,\bar{x}^m,\bar{y}^1,\ldots,\bar{y}^n).$$

Morgan has shown that a necessary and sufficient condition for the y^r implicitly defined as functions of $x^1,\ldots,x^m$ by the relations (8.8), to be exactly the same functions of $x^1,\ldots,x^m$ as the $\bar{y}^r$, implicitly defined as functions of $\bar{x}^1,\ldots,\bar{x}^m$ by the relations (8.9), are of the $\bar{x}^1,\ldots,\bar{x}^m$ is that

$$(8.10) \qquad z_k(x^1,\ldots,x^m) = \bar{z}_k(\bar{x}^1,\ldots,\bar{x}^m) = z_k(\bar{x}^1,\ldots,\bar{x}^m).$$

The condition (8.10) can be replaced by

$$z_k(x^1,\ldots,x^m) = F_k(\sigma_1,\ldots,\sigma_{m-1})$$

where $\sigma_1,\ldots,\sigma_{m-1}$ are absolute invariants of the group (8.3). Thus, when $y^1,\ldots,y^n$ are considered as invariant solutions of partial differential equations, we have the relations of the form

$$(8.11)\quad F_k(\sigma_1,\ldots,\sigma_{m-1}) = g_k(x^1,\ldots,x^m,y^1,\ldots,y^n)$$

$$(k=1,\ldots,n).$$

Note that the condition (8.5) enables us to express the y's in terms of the x's and the F's defined in (8.11), i.e., we have

$$y^r = H_r(\sigma_1,\ldots,\sigma_{m-1},\ x^m,\ F_1,\ldots,F_n)$$

$$(r=1,\ldots,n).$$

When $x^1,\ldots,x^{m-1}$ are substituted from (8.7), we obtain the relations of the form

$$y^r = H_r(\sigma_1,\ldots,\sigma_{m-1},\ x^m,\ F_1,\ldots,F_n)$$

$$(r=1,\ldots,n).$$

<u>Morgan's theorem</u> ([3]): If each differential form Φ_δ in (8.1) is conformally invariant under the k^{th} enlargements (the k^{th} extended group) of the group (8.3), then the invariant solutions of (8.2) can be expressed in terms of the solutions of a system of the form

$$(8.12)\quad A_\delta(\sigma_1,\ldots,\sigma_{m-1},F_1,\ldots,F_n,\ldots,\frac{\partial^k F^1}{\partial \sigma_1{}^k},\ldots,\frac{\partial^k F^n}{\partial \sigma_{m-1}{}^k}) = 0,$$

a system of the k^{th} order partial differential equations containing one-less independent variable than that in (8.2). In the above, $\sigma_1,\ldots,\sigma_{m-1}$ are those defined by (8.6), and $F_1,\ldots,F_n$ are defined by the relations (8.11).

Definition 8.3: We shall call the system (8.12) the reduced system of the system (8.1).

Remarks: (1) In practical problems of reduction of independent variables by Morgan's method, the solutions of the system of differential equations in question are unknown. So, the existence of invariant solutions of the system is also unknown. If the invariant solutions of the system (8.1) with respect to the group (8.3) exist, then the reduced system (8.12) is derivable.

(2) We can make a generalization by replacing the set S_D in the Morgan theorem by a new set of the form S_D: $\bar{y}^r = \psi^r(x^1,\ldots,x^m,y^1,\ldots,y^n;\ t)$; that is, the ψ^r in the new set are functions involving the independent variables. Every step of the proof of Morgan for his theorem is still valid for this generalization.

We now can restate the Morgan theorem as follows:

Morgan's theorem (modified): If each differential form Φ_δ in (8.1) is conformally invariant under the group

$$(8.13)\qquad G: \begin{cases} S_I: & \bar{x}^i = \phi^i(x^1,\ldots,x^m;\ t) \qquad (i=1,\ldots,m) \\ S_D: & \bar{y}^r = \psi^r(x^1,\ldots,x^m,y^1,\ldots,y^n;\ t) \\ & \qquad\qquad (j=1,\ldots,n) \end{cases}$$

and if the invariant solutions of (8.1) with respect to the group (8.13) exist, then these invariant solutions can be expressed in terms of the solutions of a system of the form (8.12).

<u>Example</u>: We have found in the example 6.3 that the Laplace equation

$$\Phi \equiv u_{xx} + u_{yy} = 0 \tag{8.14}$$

is invariant under the group

$$X = (mx + ky + a)\frac{\partial}{\partial x} + (my - kx + b)\frac{\partial}{\partial y} + (2mu + g(x, y))\frac{\partial}{\partial u} \tag{8.15}$$

where m, k, a, b are arbitrary constants, and g(x, y) satisfied the relations $g_{xx} + g_{yy} = 0$. The equation (8.14) and the group (8.15) are such that

$$X_{(2)}\Phi \equiv 0 = 0\cdot\Phi. \tag{8.16}$$

We see that the equation (8.14) and the group (8.15) satisfy the conformally invariant condition of Morgan's theorem. We set m = 0, k = 1, a = 0, b = 0, g(x, y) = 1 - y; to get a group

$$X_1 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + (1 - y)\frac{\partial}{\partial u}\,. \tag{8.17}$$

Note that X_1 still has the property that

$$(8.18) \qquad X_{1(2)}\Phi \equiv 0 = 0\cdot\Phi.$$

We observe that the finite equations of the group (8.17) are

$$(8.17') \qquad G: \begin{cases} S_I: & \begin{cases} \bar{x} = x\cos t + y\sin t \\ \bar{y} = y\cos t - x\sin t \end{cases} \\ S_D: & \bar{u} = u + x + t - x\cos t - y\sin t \end{cases}$$

(t = 0 yields the identity transformation),

that is, it is the group of the form (8.13) in the modified Morgan theorem. We now assume that the invariant solutions of (8.14) with respect to the group (8.17) exist, and we shall find the corresponding reduced equation.

The complete set of absolute invariants of the group (8.17) is found from

$$(8.19) \qquad \frac{dx}{y} = \frac{dy}{-x} = \frac{du}{1-y}.$$

An absolute invariant of S_I is found from

$$(8.20) \qquad \frac{dx}{y} = \frac{dy}{-x},$$

to be $x^2 + y^2$. We find that $u + x - \tan^{-1}(\frac{x}{y}) = \text{const.}$ is a solution of (8.19), that is, $u + x - \tan^{-1}(\frac{x}{y})$ is an absolute invariant of G. Now, we have

$$x^2 + y^2, \quad u + x - \tan^{-1}(\frac{x}{y})$$

as independent absolute invariants of the group G. We set

$$\sigma = x^2 + y^2, \; F(\sigma) = u + x - \tan^{-1}(\frac{x}{y}). \tag{8.21}$$

Then

$$\begin{cases} u = F(\sigma) - x + \tan^{-1}(\frac{x}{y}) \\ u_x = 2xF'(\sigma) - 1 + \dfrac{y}{x^2 + y^2} \\ u_{xx} = 4x^2F''(\sigma) + 2F'(\sigma) - \dfrac{2xy}{(x^2 + y^2)^2} \\ u_y = 2yF'(\sigma) - \dfrac{x}{x^2 + y^2} \\ u_{yy} = 4y^2F''(\sigma) + 2F'(\sigma) + \dfrac{2xy}{(x^2 + y^2)^2} \end{cases} \tag{8.22}$$

Substituting the values from (8.22) into (8.14) and simplifying, we obtain

$$4(x^2 + y^2)F''(\sigma) + 4F'(\sigma) = 0$$

or,

$$\sigma F''(\sigma) + F'(\sigma) = 0 \tag{8.23}$$

which is the reduced equation of (8.14).

The equation (8.23) gives

$$F(\sigma) = c\ln\sigma + k, \tag{8.24}$$

c, k are arbitrary constants. Substituting from (8.21) into (8.24), we get

$$u = c\ln(x^2 + y^2) + \tan^{-1}(\frac{x}{y}) - x + k \tag{8.25}$$

as invariant solution of (8.14) with respect to the group (8.17). We now substitute u, x, y in (8.25) in terms of $\bar{u}$, $\bar{x}$, $\bar{y}$ and t from (8.17'), to get after rearranging:

$$\bar{u} = c\ln(\bar{x}^2 + \bar{y}^2) + \tan^{-1}\left(\frac{\bar{x}}{\bar{y}}\right) - \bar{x} + k.$$

This shows the invariant property of the solution (8.25) under the group (8.17').

9. SOME SOLUTIONS OF THE SYSTEM OF DIFFERENTIAL EQUATIONS OF NONSTEADY ROTATIONAL FLOW OF INCOMPRESSIBLE FLUID

Nonsteady rotational flow of incompressible fluid is governed by

$$
(9.1) \qquad \begin{cases} F_1 \equiv \psi_{xx} + \psi_{yy} + \omega = 0 \\ F_2 \equiv \omega_t + \psi_y \omega_x - \psi_x \omega_y = 0 \end{cases}
$$

where ψ is the stream function, ω is the vorticity. We have found in the example 6.4 that the system (9.1) is invariant under the group

$$
\begin{aligned}
(9.2) \qquad X = (\ell - at)\frac{\partial}{\partial t} &+ \{\tfrac{1}{2}(b-a)x + (\tfrac{m}{2}t + \ell)y + q(t)\}\frac{\partial}{\partial x} \\
&+ \{\tfrac{1}{2}(b-a)y - (\tfrac{m}{2}t + \ell)x + r(t)\}\frac{\partial}{\partial y} \\
&+ \{b\psi + \tfrac{m}{4}(x^2 + y^2) - r'(t)x + q'(t)y \\
&+ s(t)\}\frac{\partial}{\partial \psi} + (a\omega - m)\frac{\partial}{\partial \omega}
\end{aligned}
$$

where a, b, k, ℓ, m are arbitrary constants; $q(t)$, $r(t)$, $s(t)$ are arbitrary functions. The system (9.1) and the group (9.2) are such that

$$(9.3) \qquad X_{(2)}F_1 = aF_1, \quad X_{(1)}F_2 = 2aF_2.$$

We also have shown in the example 6.4 that the group X can be decomposed into the following eight smaller groups under which the system (9.1) is invariant

$$X_1 = -t\frac{\partial}{\partial t} - \frac{1}{2}x\frac{\partial}{\partial x} - \frac{1}{2}y\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial \omega}$$

$$X_2 = \frac{1}{2}x\frac{\partial}{\partial x} + \frac{1}{2}y\frac{\partial}{\partial y} + \psi\frac{\partial}{\partial \psi}$$

$$X_3 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$$

$$X_4 = \frac{\partial}{\partial t}$$

$$X_5 = \frac{1}{2}ty\frac{\partial}{\partial x} - \frac{1}{2}tx\frac{\partial}{\partial y} + \frac{x^2 + y^2}{4}\frac{\partial}{\partial \psi} - \frac{\partial}{\partial \omega}$$

$$X_6 = q(t)\frac{\partial}{\partial x} + q'(t)y\frac{\partial}{\partial \psi}$$

$$X_7 = r(t)\frac{\partial}{\partial y} - r'(t)x\frac{\partial}{\partial \psi}$$

$$X_8 = s(t)\frac{\partial}{\partial \psi} .$$

We shall use these groups together with Morgan's theorem (cf. Section 8) to find exact solutions of the system (9.1).

1.° Consider the group

$$X_9 = X_3 + X_8 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + s(t)\frac{\partial}{\partial \psi} .$$

This group can be obtained from X by setting $a = 0$, $b = 0$, $c = 1$, $k = 0$, $q(t) = 0$, $r(t) = 0$. Thus, we obtain from (9.3) that

$$X_{9(2)}F_1 = 0, \quad X_{9(1)}F_2 = 0.$$

We shall find invariant solutions of (9.1) with respect to X_9. Since

$$X_9 = 0\frac{\partial}{\partial t} + y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + s(t)\frac{\partial}{\partial \psi} + 0\frac{\partial}{\partial \omega},$$

the absolute invariants of X_9 can be found from

$$\frac{dt}{0} = \frac{dx}{y} = \frac{dy}{-x} = \frac{d\psi}{s(t)} = \frac{d\omega}{0}.$$

We find that

$$t, \sqrt{x^2 + y^2}, \ \psi + s(t)\tan^{-1}\left(\frac{y}{x}\right), \ \omega$$

is a complete set of absolute invariants of X_9. This suggests that the invariant solutions are in the form

$$(9.4) \qquad \begin{cases} \psi = s(t)\tan^{-1}\left(\frac{y}{x}\right) + \Psi(t, \sigma) \\ \omega = W(t, \sigma) \\ \sigma = \sqrt{x^2 + y^2} \end{cases}$$

To find $\Psi(t, \sigma)$ and $W(t, \sigma)$, we substitute from (9.4) into (9.1) to obtain

$$(9.1)_1 \qquad \begin{cases} \Psi_{\sigma\sigma} + \frac{1}{\sigma}\Psi_{\sigma\sigma} + W = 0 \\ W_t - \frac{1}{\sigma}s(t)W_\sigma = 0 \end{cases}$$

The last equation of $(9.1)_1$ yields

$$W = \Omega(\sigma^2 + 2S(t))$$

where Ω is arbitrary function, $S(t) = \int s(t)dt$. Now, at $\sigma \neq 0$, the first equation of $(9.1)_1$ can be written as

$$\sigma\Psi_{\sigma\sigma} + \Psi_\sigma + \sigma\Omega = 0,$$

which can be reduced to

$$\sigma\Psi_\sigma + \int\sigma\Omega(\sigma^2 + 2S(t))d\sigma = \alpha(t)$$

where $\alpha(t)$ is an arbitrary function. Integrating again, we have

$$\Psi = -\int\frac{1}{\sigma}(\int\sigma\Omega(\sigma^2 + 2S(t))d\sigma)d\sigma + \alpha(t)\ln\sigma + \beta(t)$$

where $\beta(t)$ is also an arbitrary function. Thus, we have solution of (9.1) defined for $\sigma \neq 0$.

$$(9.4)_1 \quad \begin{cases} \psi = -s(t)\tan^{-1}(\frac{y}{x}) - \int\frac{1}{\sigma}(\int\sigma\Omega(\sigma^2 + 2S(t))d\sigma)d\sigma \\ \qquad\qquad + \alpha(t)\ln(x^2 + y^2) + \beta(t) \\ \omega = \Omega(\sigma^2 + 2S(t)) \end{cases}$$

where $\sigma = \sqrt{x^2 + y^2}$; $S(t) = \int s(t)dt$; α, β, Ω, s are abritrary functions. Note that the solution $(9.4)_1$ is an invariant solution with respect to X_8.

2°. Let us pay attention to the group X_6:

$$X_6 = q(t)\frac{\partial}{\partial x} + q'(t)y\frac{\partial}{\partial \psi}$$

which has the properties

$$X_{6(2)}F_1 = 0, \quad X_{6(1)}F_2 = 0.$$

A complete set of absolute invariants of X_6 is

$$t,\ y,\ \Psi - \frac{q'(t)}{q(t)}xy,\ \omega;$$

which suggests that the invariant solution of (9.1) with respect to the group X_6 has the form

$$(9.5) \qquad \begin{cases} \Psi = \dfrac{q'(t)}{q(t)}xy + \Psi(t, y) \\[2ex] \omega = W(t, y). \end{cases}$$

Substituting (9.5) into (9.1), we get after simplifying:

$$(9.1)_2 \qquad \begin{cases} \Psi_{yy} + W = 0 \\[2ex] W_t - \dfrac{q'(t)}{q(t)}yW_y = 0 \end{cases}$$

or,

$$(9.1)_3 \qquad \Psi_{tyy} - \frac{q'(t)}{q(t)}y\Psi_{yyy} = 0$$

The equation $(9.1)_3$ can be reduced to

$$(9.1)_4 \quad \Psi_t - \frac{q'(t)}{q(t)} y \Psi_y + 2\frac{q'(t)}{q(t)}\Psi = a(t)y + b(t)$$

where a(t), b(t) are arbitrary functions. The equation $(9.1)_4$ suggests that the solution of $(9.1)_3$ is in the form

$$\Psi = \frac{A(\lambda)}{q^2(t)} + B(t)\lambda + C(t)$$

where $\lambda = q(t)y$; A, B, C are arbitrary functions. From the first equation of $(9.1)_2$ we find

$$W = -\Psi_{yy} = -A''(\lambda).$$

Finally, we get the solution of (9.1):

$$(9.5)_1 \quad \begin{cases} \psi = \frac{q'(t)}{q(t)} xy + \frac{A(\lambda)}{q^2(t)} + B(t)\lambda + C(t) \\ \omega = -A''(\lambda) \end{cases}$$

where $\lambda = q(t)y$; A, B, C, q are arbitrary functions.

3°. We now take the group

$$X_7 = r(t)\frac{\partial}{\partial y} - r'(t)x\frac{\partial}{\partial \psi}$$

which has the properties

$$X_{7(2)}F_1 = 0, \quad X_{7(1)}F_2 = 0.$$

We find that the functions

$$t,\ x,\ \psi + \frac{r'(t)}{r(t)}xy,\ \omega$$

form a complete set of absolute invariants of X_7. Thus, the invariant solutions of (9.1) with respect to X_7 will be in the form

$$(9.6)\qquad \begin{cases} \psi = -\dfrac{r'(t)}{r(t)}xy + \Psi(t, x) \\ \omega = W(t, x). \end{cases}$$

Substituting these in (9.1), we get

$$(9.1)_5\qquad \begin{cases} \Psi_{xx} + W = 0 \\ W_t - \dfrac{r'(t)}{r(t)}xW_x = 0. \end{cases}$$

By the same procedure as the case 2°, we find a set of general solutions of (9.1) to be

$$(9.6)_1\qquad \begin{cases} \psi = -\dfrac{r'(t)}{r(t)}xy + \dfrac{A^*(\lambda^*)}{r^2(t)} + B^*(t)\lambda^* + C^*(t) \\ \omega = -A^{*''}(\lambda^*) \end{cases}$$

where A^*, B^*, C^*, r are arbitrary functions, and $\lambda^* = r(t)x$.

4°. Let us take the group

$$X_{10} = X_6 + X_7 = q(t)\frac{\partial}{\partial x} + r(t)\frac{\partial}{\partial y} + (q'(t)y - r'(t)x)\frac{\partial}{\partial \psi}$$

which has the properties

$$X_{10(2)}F_1 = 0, \quad X_{10(1)}F_2 = 0,$$

from (9.3) when a = 0. In the case where $q(t) \not\equiv r(t)$, $q(t) \not\equiv 0$, $r(t) \not\equiv 0$, $q(t)$ and $r(t)$ are not constants simultaneously, we find that

$$t,\ q(t)y - r(t)x,\ \psi - \frac{(q'(t)y - r'(t)x)^2}{2(q'(t)r(t) - r'(t)q(t))},\ \omega$$

is a complete set of absolute invariants of X_{10}. This set of absolute invariants suggests invariant solutions of the form

$$(9.7)\quad \begin{cases} \psi = \dfrac{(q'y - r'x)^2}{2(q'r - r'q)} + \Psi(t, \sigma) \\ \omega = W(t, \sigma) \\ \sigma = qy - rx\ . \end{cases}$$

Substituting from (9.7) into (9.1), we get

$$(9.1)_6\quad \begin{cases} \dfrac{q'^2 + r'^2}{q'r - r'q} + (q^2 + r^2)\Psi_{\sigma\sigma} + W = 0 \\ W_t = 0. \end{cases}$$

This yields

$$W = \Omega(\sigma)$$

$$\Psi = -\frac{1}{q^2 + r^2}\int\int\Omega(\sigma)\,d\sigma d\sigma - \frac{q'^2 + r'^2}{2(q^2+r^2)(q'r-r'q)}\sigma^2 + G(t)\sigma + H(t)$$

where $\Omega(\sigma)$, $G(t)$, $H(t)$ are arbitrary functions.

Finally, we obtain the following solutions of (9.1):

$$
(9.7)_1 \quad \begin{cases}
\omega = \Omega(\sigma) \\[2ex]
\psi = \dfrac{(q'y - r'x)^2}{2(q'r - r'q)} - \dfrac{1}{q^2 + r^2}\int\int\Omega(\sigma)\,d\sigma d\sigma \\[2ex]
\qquad - \dfrac{q'^2 + r'^2}{2(q^2 + r^2)(q'r - r'q)}(qy - rx)^2 \\[2ex]
\qquad + G(t)(qy - rx) + H(t) \\[2ex]
\sigma = q(t)y - r(t)x
\end{cases}
$$

where $q = q(t)$, $r = r(t)$, $\Omega(\sigma)$, $G(t)$, $H(t)$ are arbitrary functions; $q(t) \not\equiv r(t)$; $q(t) \not\equiv 0$; $r(t) \not\equiv 0$; $q(t)$ and $r(t)$ are not constants simultaneously.

5°. Consider the groups X_1, X_2:

$$X_1 = -t\frac{\partial}{\partial t} - \frac{1}{2}x\frac{\partial}{\partial x} - \frac{1}{2}y\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial \omega}$$

$$X_2 = \frac{1}{2}x\frac{\partial}{\partial x} + \frac{1}{2}y\frac{\partial}{\partial y} + \psi\frac{\partial}{\partial \psi} .$$

Both are groups of similarity transformations. To get a more general group of similarity transformations, we form a new group

$$X_{11} = aX_1 + bX_2 = -at\frac{\partial}{\partial t} - \frac{1}{2}(a-b)x\frac{\partial}{\partial x} - \frac{1}{2}(a-b)y\frac{\partial}{\partial y} + b\psi\frac{\partial}{\partial \psi} + a\omega\frac{\partial}{\partial \omega} .$$

Note that X_{11} can be obtained from the group (9.2) by setting $k = 0$, $q(t) = 0$, $c = 0$, $r(t) = 0$ and $s(t) = 0$. Thus, by (9.3) we have

$$X_{11(2)}F_1 = aF_1, \; X_{11(1)}F_2 = 2aF_2.$$

The system of differential equations determining absolute invariants of X_{11} is

$$\frac{dt}{-at} = \frac{dx}{-\frac{1}{2}(a - b)x} = \frac{dy}{-\frac{1}{2}(a - b)y} = \frac{d\psi}{b\psi} = \frac{d\omega}{a\omega}$$

or,

$$\frac{dt}{-t} = \frac{dx}{-\frac{1}{2}(1 - c)x} = \frac{dy}{-\frac{1}{2}(1 - c)y} = \frac{d\psi}{c\psi} = \frac{d\omega}{\omega}$$

where $c = b/a$, $a \neq 0$. We find that

$$xt^{-\frac{1}{2}(1-c)}, \quad yt^{-\frac{1}{2}(1-c)}, \quad t^c\psi, \quad t\omega \tag{9.8}$$

form a complete set of absolute invariants of X_{11}. This suggests that the invariant solutions of (9.1) with respect to X_{11} are in the form

$$\left\{ \begin{aligned} &\psi = t^{-c}\Psi(\sigma_1, \sigma_2), \qquad &&\omega = t^{-1}W(\sigma_1, \sigma_2) \\ &\sigma_1 = xt^{-\frac{1}{2}(1-c)}, \qquad &&\sigma_2 = yt^{-\frac{1}{2}(1-c)} \; . \end{aligned} \right. \tag{9.9}$$

Substituting from (9.9) into (9.1), we find

$$(9.1)_7 \quad \begin{cases} \Psi_{\sigma_1\sigma_1} + \Psi_{\sigma_2\sigma_2} + W = 0 \\ W + \frac{1-c}{2}\sigma_1 W_{\sigma_1} + \frac{1-c}{2}\sigma_2 W_{\sigma_1} - \Psi_{\sigma_2} W_{\sigma_1} + \Psi_{\sigma_1} W_{\sigma_2} = 0. \end{cases}$$

This is a reduced form of (9.1). We see that it is still difficult to solve the system $(9.1)_7$.

Let us return to the group X_1

$$X_1 = -t\frac{\partial}{\partial t} - \frac{1}{2}x\frac{\partial}{\partial x} - \frac{1}{2}y\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial \omega}$$

$$= -t\frac{\partial}{\partial t} - \frac{1}{2}x\frac{\partial}{\partial x} - \frac{1}{2}y\frac{\partial}{\partial y} + 0\frac{\partial}{\partial \psi} + \omega\frac{\partial}{\partial \omega} .$$

One can check that

$$(x + y)^2/t, \ \psi \ , \ t\omega$$

are absolute invariants of X_1. This set suggests invariant solutions of (9.1) with respect to X_1 of the form*

$$(9.10) \quad \begin{cases} \psi = \Psi(\sigma) \ , \ \omega = t^{-1}w(\sigma) \\ \sigma = (x + y)^2/t \ . \end{cases}$$

*The motivation of idea of reducing two or more independent variables at a time is due to the discussion of W. F. Ames about extending Morgan's method of reduction of independent variables ([10], pp. 141-144).

Substituting (9.10) into (9.1), we find

$$(9.1)_8 \qquad \begin{cases} 8\sigma\Psi'' + 4\Psi' + W = 0 \\ W + \sigma W' = 0 \end{cases}$$

where the prime means differentiation with respect to σ. The last equation of $(9.1)_8$ gives

$$(9.11) \qquad W = A/\sigma$$

where A is a constant. Substituting (9.11) into the first equation of $(9.1)_8$, we have

$$\Psi'' + \frac{1}{2\sigma}\Psi' = -\frac{A}{8\sigma^2}$$

This differential equation yields

$$(9.12) \qquad \Psi = \frac{A}{4}\ln\sigma + B\sqrt{\sigma} + C$$

where B, C are constants of integration. From (9.10), (9.11) and (9.12) we get solutions of (9.1)

$$(9.10)_1 \qquad \begin{cases} \omega = A/(x + y)^2 \\ \psi = \frac{A}{4}\ln\left(\frac{(x + y)^2}{t}\right) + B\frac{x + y}{\sqrt{t}} + C, \end{cases}$$

which are invariant solutions with respect to X_1.

10. REDUCTION OF INDEPENDENT VARIABLES OF THE EQUATIONS OF STEADY PLANE FLOW OF POLYTROPIC GAS

Let us consider compressible fluid having an equation of state of the form:

$$(10.1)_1 \qquad \rho = f(p) \cdot g(s)$$

where ρ, p and s denote, respectively, the density, the pressure, and the entropy of the fluid, f and g are given functions. The other equations governing the flow of compressible fluid are

$$(10.1)_2 \qquad \rho\bar{v} \cdot \nabla\bar{v} = -\nabla p \qquad \text{(equation of motion)}$$

$$(10.1)_3 \qquad \nabla \cdot \rho\bar{v} = 0 \qquad \text{(continuity equation)}$$

$$(10.1)_4 \qquad \bar{v} \cdot \nabla s = 0 \qquad \text{(entropy is constant along streamline)}$$

where $\bar{v} = (v^1, v^2, v^3)$ denote the velocity vector of the flow. We shall now change the system (10.1) into a canonical form. Let $P = f(p)$ and $S = g(s)$ so that $p = f^{-1}(P)$ and $s = g^{-1}(s)$. Then we let $\bar{W} = \bar{v}\sqrt{s}$. Define

$$(10.2) \qquad F(P) = \frac{1}{P} \frac{df^{-1}(P)}{dP}$$

so that F is a known function of P.

Theorem: The variables $\bar{W}$, P and S as functions of x, y, z satisfy the equations

$$(10.3)_1 \qquad \bar{W}\cdot\nabla\bar{W} = -F(P)\nabla P$$

$$(10.3)_2 \qquad \nabla\cdot P\bar{W} = 0$$

$$(10.3)_3 \qquad \bar{W}\cdot\nabla S = 0$$

if and only if $\bar{v}$, p, s and ρ defined by

$$(10.4) \qquad \bar{v} = \bar{W}/\sqrt{s}\ ,\ p = f^{-1}(P)\ ,\ s = g^{-1}(s)\ \text{and}\ \rho = f(p)\cdot g(s)$$

satisfy the system of equations (10.1).

Proof: First, assume that $\bar{W}$, P, s satisfy (10.3) and we shall show that $\bar{v}$, p, s, ρ satisfy (10.1).

$$\text{(i)} \qquad \bar{v}\cdot\nabla s = \frac{\bar{W}}{\sqrt{s}}\cdot\frac{dg^{-1}}{ds}\nabla s = \frac{1}{\sqrt{s}}\frac{dg^{-1}}{ds}\bar{W}\cdot\nabla s = 0,$$

so $(10.1)_4$ is satisfied.

(ii) From $(10.2)_1$ we have

$$(\bar{v}\sqrt{s})\cdot\nabla(\bar{v}\sqrt{s}) = -\frac{1}{P}\frac{df^{-1}}{dP}\nabla P\ ,$$

or,

$$(\bar{v}\sqrt{s})\sqrt{s}\cdot\nabla\bar{v} + (\bar{v}\sqrt{s})\bar{v}\cdot\nabla\sqrt{s} = -\frac{1}{P}\nabla p\ .$$

The term $(\bar{v}\sqrt{s})\bar{v}\cdot\nabla\sqrt{s}$ can be put in the form $\frac{\bar{v}}{2s}\bar{W}\cdot\nabla s$ which vanishes by $(10.2)_3$. Thus

$$S\bar{v}\cdot\nabla\bar{v} = -\frac{1}{P}\nabla p \ ,$$

or,

$$\rho\bar{v}\cdot\nabla\bar{v} = -\nabla p$$

which is $(10.1)_2$.

(iii) From $(10.2)_2$ we have

$$\nabla\cdot P\sqrt{s}\ \bar{v} = 0 \ ,$$

or,

$$\nabla\cdot\frac{\rho}{\sqrt{s}}\bar{v} = 0$$

or,

$$\frac{1}{\sqrt{s}}\nabla\cdot\rho\bar{v} + \rho\bar{v}\cdot\nabla\frac{1}{\sqrt{s}} = 0.$$

The term $\rho\bar{v}\cdot\nabla\frac{1}{\sqrt{s}} = -\frac{1}{2}\frac{1}{S^{3/2}}\rho\bar{v}\cdot\nabla s = -\frac{1}{2}\frac{1}{S^2}\rho\bar{W}\cdot\nabla s$ vanishes by $(10.2)_3$. Thus,

$$\frac{1}{\sqrt{s}}\nabla\cdot\rho\bar{v} = 0,$$

or,

$$\nabla\cdot\rho\bar{v} = 0$$

which is $(10.1)_3$. The proof of the first part is completed. By the same procedure, we can prove the converse of the theorem.

As long as the equation of state has the form $(10.1)_1$ which includes many important cases, we can reduce the usual system (10.1) of 6 equations to the system (10.3) of 5 equations. The advantage of the system (10.3) is that we can solve for $\bar{W}$ and P from $(10.3)_1$ and $(10.3)_2$, then using the known value of $\bar{W}$ in $(10.3)_3$ we obtain a linear differential equation for determining S. Once $\bar{W}$, P and S are found, we find $\bar{v}$, p, s and ρ from the relations (10.4) to get the required flow.

Let us restrict ourselves to the case of plane flow of polytropic gas which is characterized by the equation of state

$$\rho = p^{1/\gamma} e^{(s_0 - s)/c_p} \tag{10.5}$$

where $\gamma = c_p/c_v$ is the ratio of specific heats, c_p is the specific heat at constant pressure, c_v is the specific heat at constant volume, s_0 is some constant value of entropy. Note that (10.5) is in the form $(10.1)_1$. There corresponds a function F defined in (10.2) for (10.5)

$$F(P) = \frac{1}{P}\frac{dP^\gamma}{dP} = \gamma P^{\gamma-2} .$$

Then, from (10.3), we have a canonical form of equations defining polytropic gas flow

$$\left\{ \begin{aligned} \bar{W}\cdot\nabla\bar{W} &= -\gamma P^{\gamma-2}\nabla P \\ \nabla\cdot P\bar{W} &= 0 \\ \bar{W}\cdot\nabla S &= 0 \end{aligned} \right. \tag{10.6}$$

where $\bar{W} = (W^1, W^2)$ for plane flow.

The system of equations of plane flow of polytropic gas has been dealt with before by P. Kucharczyk [16], who uses Lie derivatives to reduce this system to the system of ordinary differential equations. It is our purpose here to get a more simple form of the reduced system than that obtained by Kucharczyk. For this purpose, we shall make a reduction of independent variables of canonical equations of plane flow of polytropic gas (10.6). By the reason mentioned previously, we shall only pay attention to the first three equations of (10.6) and write them in the form:

$$(10.7)\quad \begin{cases} F_1 \equiv W^1W^1_x + W^2W^1_y + \gamma P^{\gamma-2}P_x = 0 \\ F_2 \equiv W^1W^2_x + W^2W^2_y + \gamma P^{\gamma-2}P_y = 0 \\ F_3 \equiv P(W^1_x + W^2_y) + W^1P_x + W^2P_y = 0 . \end{cases}$$

To satisfy Morgan's theorem of reduction of independent variables (sect. 8), we shall utilize our method to find group X such that

$$(10.8)\quad X = \xi^1(x,y)\frac{\partial}{\partial x} + \xi^2(x,y)\frac{\partial}{\partial y} + \eta^1(x,y,W^1,W^2,P)\frac{\partial}{\partial W^1} + \eta^2(x,y,W^1,W^2,P)\frac{\partial}{\partial W^2} + \eta^3(x,y,W^1,W^2,P)\frac{\partial}{\partial P}$$

and

$$(10.9)\quad \begin{cases} X_{(1)}F_1 = h_1(x, y, w^1, w^2, P)F_1 \\ X_{(1)}F_2 = h_2(x, y, w^1, w^2, P)F_2 \\ X_{(1)}F_3 = h_3(x, y, w^1, w^2, P)F_3 \end{cases}$$

where $X_{(1)}$ is the first extended group of the group X. Note that ξ^1, ξ^2 are functions of only the independent variables and equation (10.9) is less general than equation (5.7). From (10.8) we find, with the help of (2.18), the extended group $X_{(1)}$:

$$X_{(1)} = X + \eta^{1,1}\frac{\partial}{\partial w_x^1} + \eta^{1,2}\frac{\partial}{\partial w_y^1} + \eta^{2,1}\frac{\partial}{\partial w_x^2} + \eta^{2,2}\frac{\partial}{\partial w_y^2} + \eta^{3,1}\frac{\partial}{\partial P_x} + \eta^{3,2}\frac{\partial}{\partial P_y}$$

where

$$\eta^{1,1} = \eta_x^1 + \eta_{w^1}^1 w_x^1 + \eta_{w^2}^1 w_x^2 + \eta_P^1 P_x - w_x^1 \xi_x^1 - w_y^1 \xi_x^2$$

$$\eta^{1,2} = \eta_y^1 + \eta_{w^1}^1 w_y^1 + \eta_{w^2}^1 w_y^2 + \eta_P^1 P_y - w_x^1 \xi_y^1 - w_y^1 \xi_y^2$$

$$\eta^{2,1} = \eta_x^2 + \eta_{w^1}^2 w_x^1 + \eta_{w^2}^2 w_x^2 + \eta_P^2 P_x - w_x^2 \xi_x^1 - w_y^2 \xi_x^2$$

$$\eta^{2,2} = \eta_y^2 + \eta_{w^1}^2 w_y^1 + \eta_{w^2}^2 w_y^2 + \eta_P^2 P_y - w_x^2 \xi_y^1 - w_y^2 \xi_y^2$$

$$\eta^{3,1} = \eta_x^3 + \eta_{w^1}^3 w_x^1 + \eta_{w^2}^3 w_x^2 + \eta_P^3 P_x - P_x \xi_x^1 - P_y \xi_x^2$$

$$\eta^{3,2} = \eta_y^3 + \eta_{w^1}^3 w_y^1 + \eta_{w^2}^3 w_y^2 + \eta_P^3 P_y - P_x \xi_y^1 - P_y \xi_y^2 .$$

The left hand members of (10.9) can be written as

$$(10.9) \quad \begin{cases} X_{(1)}F_1 = h_1(x, y, W^1, W^2, P)F_1 \\ X_{(1)}F_2 = h_2(x, y, W^1, W^2, P)F_2 \\ X_{(1)}F_3 = h_3(x, y, W^1, W^2, P)F_3 \end{cases}$$

where $X_{(1)}$ is the first extended group of the group X. From (10.8) we find, with the help of (2.18), the extended group $X_{(1)}$:

$$X_{(1)} = X + \eta^{1,1}\frac{\partial}{\partial W_x^1} + \eta^{1,2}\frac{\partial}{\partial W_y^1} + \eta^{2,1}\frac{\partial}{\partial W_x^2} + \eta^{2,2}\frac{\partial}{\partial W_y^2}$$

$$+ \eta^{3,1}\frac{\partial}{\partial P_x} + \eta^{3,2}\frac{\partial}{\partial P_y}$$

where

$$\eta^{1,1} = \eta_x^1 + \eta_{W^1}^1 W_x^1 + \eta_{W^2}^1 W_x^2 + \eta_P^1 P_x - W_x^1\xi_x^1 - W_y^1\xi_x^2$$

$$\eta^{1,2} = \eta_y^1 + \eta_{W^1}^1 W_y^1 + \eta_{W^2}^1 W_y^2 + \eta_P^1 P_y - W_x^1\xi_y^1 - W_y^1\xi_y^2$$

$$\eta^{2,1} = \eta_x^2 + \eta_{W^1}^2 W_x^1 + \eta_{W^2}^2 W_x^2 + \eta_P^2 P_x - W_x^2\xi_x^1 - W_y^2\xi_x^2$$

$$\eta^{2,2} = \eta_y^2 + \eta_{W^1}^2 W_y^1 + \eta_{W^2}^2 W_y^2 + \eta_P^2 P_y - W_x^2\xi_y^1 - W_y^2\xi_y^2$$

$$\eta^{3,1} = \eta_x^3 + \eta_{W^1}^3 W_x^1 + \eta_{W^2}^3 W_x^2 + \eta_P^3 P_x - P_x\xi_x^1 - P_y\xi_x^2$$

$$\eta^{3,2} = \eta_y^3 + \eta_{W^1}^3 W_y^1 + \eta_{W^2}^3 W_y^2 + \eta_P^3 P_y - P_x\xi_y^1 - P_y\xi_y^2.$$

The left hand members of (10.9) can be written as

$$X_{(1)}F_1 = \eta^1 W^1_x + W^1\eta^{1,1} + \eta^2 W^1_y + W^2\eta^{1,2} + \gamma(\gamma-2)P^{\gamma-3}\eta^3 P_x$$
$$+ \gamma P^{\gamma-2}\eta^{3,1}$$

$$X_{(1)}F_2 = \eta^1 W^2_x + W^1\eta^{2,1} + \eta^2 W^2_y + W^2\eta^{2,2}$$
$$+ \gamma(\gamma-2)P^{\gamma-3}\eta^3 P_y + \gamma P^{\gamma-2}\eta^{3,2}$$

$$X_{(1)}F_3 = \eta^3(W^1_x + W^2_y) + P(\eta^{1,1} + \eta^{2,2})$$
$$+ \eta^1 P_x + W^1\eta^{3,1} + \eta^2 P_y + W^2\eta^{3,2} \; .$$

We now can equate the coefficients of $1, W^1_x, \ldots, P_y$ in (10.9). From the first equation of (10.9) we have:

(10.10) (coeff. of 1): $W^1\eta^1_x + W^2\eta^1_y + \gamma P^{\gamma-2}\eta^3_x = 0$

(10.11) (coeff. of W^1_x): $\eta^1 + W^1\eta^1_{W^1} - W^1\xi^1_x - W^2\xi^1_y$
$$+ \gamma P^{\gamma-2}\eta^3_{W^1} = h_1 W^1$$

(10.12) (coeff. of W^1_y): $\eta^2 - W^1\xi^2_x + W^2\eta^1_{W^1} - W^2\xi^2_y = h_1 W^2$

(10.13) (coeff. of W^2_x): $W^1\eta^1_{W^2} + \gamma P^{\gamma-2}\eta^3_{W^2} = 0$

(10.14) (coeff. of W^2_y): $W^2\eta^1_{W^2} = 0$

$$(10.15)\quad (\text{coeff. of } P_x):\quad \gamma(\gamma-2)P^{\gamma-3}\eta^3 + W^1\eta^1_P + \gamma P^{\gamma-2}\eta^3_P$$

$$- \gamma P^{\gamma-2}\xi^1_x = h_1\gamma P^{\gamma-2}$$

$$(10.16)\quad (\text{coeff. of } P_y):\quad W^2\eta^1_P - \gamma P^{\gamma-2}\xi^2_x = 0.$$

From the second equation of (10.9), we have

$$(10.17)\quad (\text{coeff. of } 1):\quad W^1\eta^2_x + W^2\eta^2_y + \gamma P^{\gamma-2}\eta^3_y = 0$$

$$(10.18)\quad (\text{coeff. of } W^1_x):\quad W^1\eta^2_{W^1} = 0$$

$$(10.19)\quad (\text{coeff. of } W^1_y):\quad W^2\eta^2_{W^1} + \gamma P^{\gamma-2}\eta^3_{W^1} = 0$$

$$(10.20)\quad (\text{coeff. of } W^2_x):\ \eta^1 + W^1\eta^2_{W^2} - W^1\xi^1_x - W^2\xi^1_y = h_2W^1$$

$$(10.21)\quad (\text{coeff. of } W^2_y):\ \eta^2 - W^1\xi^2_x + W^2\eta^2_{W^2} - W^2\xi^2_y$$

$$+ \gamma P^{\gamma-2}\eta^3_{W^2} = h_2W^2$$

$$(10.22)\quad (\text{coeff. of } P_x):\quad W^1\eta^2_P - \gamma P^{\gamma-2}\xi^1_y = 0$$

$$(10.23)\quad (\text{coeff. of } P_y):\quad W^2\eta^2_P + \gamma(\gamma-2)P^{\gamma-3}\eta^3 + \gamma P^{\gamma-2}\eta^3_P$$

$$- \gamma P^{\gamma-2}\xi^2_y = h_2\gamma P^{\gamma-2}\ .$$

The third equation of (10.9) gives

$$(10.24)\quad (\text{coeff. of } 1):\quad P(\eta^1_x + \eta^2_y) + W^1\eta^3_x + W^2\eta^3_y = 0$$

(10.25) (coeff. of W^1_x): $\eta^3 + P\eta^1_{W^1} - P\xi^1_x + W^1\eta^3_{W^1} = h_3P$

(10.26) (coeff. of W^1_y): $-P\xi^2_x + P\eta^2_{W^1} + W^2\eta^3_{W^1} = 0$

(10.27) (coeff. of W^2_x): $P\eta^1_{W^2} - P\xi^1_y + W^1\eta^3_{W^2} = 0$

(10.28) (coeff. of W^2_y): $\eta^3 + P\eta^2_{W^2} - P\xi^2_y + W^2\eta^3_{W^2} = h_3P$

(10.29) (coeff. of P_x): $P\eta^1_P + \eta^1 + W^1\eta^3_P - W^1\xi^1_x - W^2\xi^1_y$

$= h_3W^1$

(10.30) (coeff. of P_y): $P\eta^2_P + \eta^2 + W^2\eta^3_P - W^2\xi^2_y - W^1\xi^2_x$

$= h_3W^2$.

From (10.14) we get $\eta^1_{W^2} = 0$. Then (10.13) gives $\eta^3_{W^2} = 0$. From the fact that $\eta^1_{W^1}$ is not a function of W^2, and ξ^2 is a function of x, y only; the equation (10.16) implies $\eta^1_P = 0$ and $\xi^2_x = 0$. Similarly, the equations (10.18), (10.19) and (10.22) give $\eta^2_{W^1} = 0$, $\eta^3_{W^1} = 0$, $\eta^2_P = 0$ and $\xi^1_y = 0$. Observe that the equations (10.26) and (10.27) are now satisfied. Since η^1 and η^3 are not functions of W^2, the equation (10.10) implies $\eta^1_y = 0$. From the fact that η^2 and η^3 are not functions of W^1, the equation (10.17) implies $\eta^2_x = 0$. Eliminating η^1_x, η^2_y from (10.10), (10.17) and (10.24), we get

$$\left(W^1 - \frac{\gamma P^{\gamma-2}}{W^1}\right)\eta^3_x + \left(W^2 - \frac{\gamma P^{\gamma-2}}{W^2}\right)\eta^3_y = 0$$

which implies that $\eta^3_x = \eta^3_y = 0$, since η^3 is not a function of W^1 and W^2. Then (10.10) gives $\eta^1_x = 0$, and (10.17) gives $\eta^2_y = 0$.

We now have that

$$(10.31)\quad \xi^1 = \xi^1(x),\ \xi^2 = \xi^2(y),\ \eta^1 = \eta^1(W^1),$$
$$\eta^2 = \eta^2(W^2),\ \eta^3 = \eta^3(P).$$

Divide (10.28) by P and (10.29) by W^1 and substract the results, we get

$$(10.32)\quad \left(\frac{\eta^3}{P} - \eta^3_P\right) + \eta^2_{W^2} - \frac{\eta^1}{W^1} - \xi^2_y + \xi^1_x = 0.$$

Similarly, we get from (10.25) and (10.30):

$$(10.33)\quad -\left(\frac{\eta^3}{P} - \eta^3_P\right) + \frac{\eta^2}{W^2} - \eta^1_{W^1} - \xi^2_y + \xi^1_x = 0.$$

The equations (10.31), (10.32) and (10.33) imply

$$(10.34)\quad \xi^1 = a_1x + b_1,\ \xi^2 = a_2y + b_2,\ \eta^1 = k_1W^1,$$
$$\eta^2 = k_2W^2,\ \eta^3 = k_3P$$

where a_1, b_1, a_2, b_2, k_1, k_2 and k_3 are constants. The equations (10.25) and (10.28) (or, (10.29) and (10.30)) imply

$$h_3 = k_3 + k_1 - a_1 = k_3 + k_2 - a_2 , \tag{10.35}$$

which gives

$$k_1 - a_1 = k_2 - a_2. \tag{10.36}$$

Eliminating ξ_x^1 and h_1 in (10.11) and (10.15), and using the values from (10.34), we find

$$k_1 = \frac{\gamma - 1}{2}k_3.$$

Similarly, from (10.21) and (10.23) we have

$$k_2 = \frac{\gamma - 1}{2}k_3.$$

Thus,

$$k_1 = k_2 = \frac{\gamma - 1}{2}k_3 = \frac{\gamma - 1}{2}k$$

where we set $k_3 = k$. Then from (10.36) we have

$$a_1 = a_2 = a, \text{ say.}$$

From (10.35) we get

$$h_3 = \frac{\gamma + 1}{2}k - a.$$

From (10.20) (or (10.21), or (10.23)) we find

$$h_2 = (\gamma - 1)k - a.$$

From (10.11) (or (10.12), or (10.15)) we find

$$h_1 = (\gamma - 1)k - a.$$

We now have the required group

$$X = (ax + b_1)\frac{\partial}{\partial x} + (ay + b_2)\frac{\partial}{\partial y} + \frac{\gamma - 1}{2}kW^1\frac{\partial}{\partial W^1} + \frac{\gamma - 1}{2}kW^2\frac{\partial}{\partial W^2} + kP\frac{\partial}{\partial P} \tag{10.37}$$

with the properties:

$$X_{(1)}F_1 = ((\gamma - 1)k - a)F_1$$

$$X_{(1)}F_2 = ((\gamma - 1)k - a)F_2$$

$$X_{(1)}F_3 = (\frac{\gamma + 1}{2}k - a)F_3 .$$

The differential equations determining absolute invariants of the group (10.37) are

$$\frac{dx}{ax + b_1} = \frac{dy}{ay + b_2} = \frac{dW^1}{(\gamma - 1)kW^1/2} = \frac{dW^2}{(\gamma - 1)kW^2/2} = \frac{dP}{kP} ,$$

or, in case $a \neq 0$

$$\frac{dx}{x + c_1} = \frac{dy}{y + c_2} = \frac{dW^1}{mW^1} = \frac{dW^2}{mW^2} = \frac{dP}{nP} \tag{10.38}$$

where $c_1 = b_1/a$, $c_2 = b_2/a$, $m = (\gamma - 1)k/(2a)$, $n = k/a$. A set of independent functions satisfying system (10.38), and so a set of independent absolute invariants of X, is

$$(10.39)\qquad \frac{y + c_2}{x + c_1}\,,\quad \frac{W^1}{(x + c_1)^m}\,,\quad \frac{W^2}{(x + c_1)^m}\,,\quad \frac{P}{(x + c_1)^n}\quad .$$

Thus the invariant solutions of (10.7) with respect to X are in the form

$$(10.40)\qquad \begin{cases} W^1 = (x + c_1)^m \omega^1(\sigma)\ , & W^2 = (x + c_1)^m \omega^2(\sigma) \\ P = (x + c_1)^n \pi(\sigma)\ , & \sigma = \dfrac{y + c_2}{x + c_1} \end{cases}$$

To obtain the differential equations determining $\omega^1(\sigma)$, $\omega^2(\sigma)$ and $\pi(\sigma)$; we substitute from (10.40) into (10.7):

$$(10.41)\qquad \begin{cases} m\omega^1\omega^1 + \gamma\pi^{\gamma-1} - (\sigma\omega^1 - \omega^2)\dfrac{d\omega^1}{d\sigma} - \gamma\sigma\pi^{\gamma-2}\dfrac{d\pi}{d\sigma} = 0 \\ m\omega^1\omega^2 - (\sigma\omega^1 - \omega^2)\dfrac{d\omega^2}{d\sigma} + \gamma\pi^{\gamma-2}\dfrac{d\pi}{d\sigma} = 0 \\ (m + n)\omega^1\pi - \sigma\pi\dfrac{d\omega^1}{d\sigma} + \pi\dfrac{d\omega^2}{d\sigma} - (\sigma\omega^1 - \omega^2)\dfrac{d\pi}{d\sigma} = 0 \end{cases}$$

where $m = (\gamma - 1)k/(2a)$, $n = k/a$. This is a reduced form of the system (10.7). We now set

$$(10.42)\qquad \theta = \pi^{\gamma-1}$$

Then (10.41) can be written as

$$(10.43)\quad \begin{cases} m\omega^1\omega^1 + n\gamma\theta - (\sigma\omega^1 - \omega^2)\dfrac{d\omega^1}{d\sigma} - \dfrac{\gamma}{\gamma - 1}\dfrac{d\theta}{d\sigma} = 0 \\[2ex] m\omega^1\omega^2 \qquad\qquad - (\sigma\omega^1 - \omega^2)\dfrac{d\omega^2}{d\sigma} + \dfrac{\gamma}{\gamma - 1}\dfrac{d\theta}{d\sigma} = 0 \\[2ex] (m + n)\omega^1\theta \quad -\sigma\theta\dfrac{d\omega^1}{d\sigma} + \theta\dfrac{d\omega^2}{d\sigma} - \dfrac{1}{\gamma - 1}(\sigma\omega^1 - \omega^2)\dfrac{d\theta}{d\sigma} = 0. \end{cases}$$

which is simpler than (10.41). For the consistency of (10.43), we must have

$$\det \begin{pmatrix} \sigma\omega^1 - \omega^2 & 0 & \dfrac{\gamma\sigma}{\gamma - 1} \\[2ex] 0 & \sigma\omega^1 - \omega^2 & \dfrac{\gamma}{\gamma - 1} \\[2ex] \sigma\theta & -\theta & \dfrac{\sigma\omega^1 - \omega^2}{\gamma - 1} \end{pmatrix} \neq 0,$$

or,

$$(10.44)\qquad \sigma\omega^1 \neq \omega^2 \quad \text{and} \quad (\sigma\omega^1 - \omega^2)^2 \neq \gamma(1 + \sigma^2)\theta\ .$$

From the relations (10.4), (10.40) and the first condition of (10.44), we have

$$\frac{v^2}{v^1} \neq \frac{y + c_2}{x + c_1}$$

This tells us that the direction of the flow is not along the ray through $(-c_1, -c_2)$. Thus, any set of solutions of (10.43) gives a flow which is not a flow from a source or a sink located at $(-c_1, -c_2)$.

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APPENDIX

APPENDIX

MORGAN'S DEFINITION OF CONFORMAL INVARIANCE

Morgan's definition of conformal invariance of a function Φ with respect to a group G seems to lack precision because of his failure to describe the function H. Actually, the basis of the problem is the failure to describe precisely the classes of functions Φ and groups G for which the definition is made. For functions Φ which never vanish, no new concept is described - for every such Φ and for all G's we have the defined property. We can take $H(x,y,\ldots, t) = \frac{\Phi(\bar{x},\bar{y},\ldots)}{\Phi(x,y,\ldots)}$. Thus we are interested in making the definition only for functions Φ which vanish on some set in their domain and which are defined in a neighborhood, N, of that set. Then, there exists a function $H(x,y,\ldots, t)$ such that

$$\Phi(\bar{x},\bar{y},\ldots) = H(x,y,\ldots,t)\cdot\Phi(x,y,\ldots)$$

$$x,y,\ldots \in N \; ; \quad |t| < \varepsilon$$

does impose a meaningful condition on Φ and G. Further, it is implicit in this definition that the x,y-domain of G is contained in the domain of Φ, otherwise the definition would not make sense.

For $x, y, \ldots$ such that $\Phi(x,y,\ldots) \neq 0$ continuity and differentiability properties of H are determined by those of Φ and G, but when $\Phi(x,y,\ldots) = 0$ no such properties are imposed on $H(x,y, \ldots)$.

A restricted (stronger) form of this condition is used in this work (1) as an hypothesis in Morgan's theorem for reducing the number of independent variables in partial differential equations, and (2) in our method for finding groups. The assumption that H has continuous first derivatives in all its arguments will suffice for the corresponding function h in (8.2′) to be continuous which, in turn, suffices to satisfy our requirements for (1) and (2) above.

Clearly, it is possible that imposing conditions on H beyond those in the definition could restrict the class of functions Φ and/or groups G which satisfy the definition.