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WEAK CONVERGENCE OF DISTRIBUTION-VALUED SEMIMARTINGALES AND ASSOCIATED SDE'S

by

Ravi T. Chari

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Weak Convergence of Distribution-Valued Semimartingales and Associated SDE's

ABSTRACT

by

Ravi T. Chari

We study in this work the weak convergence of solutions of certain linear stochastic differential equations taking values in $S'(\mathbb{R}^d)$. These extend the works of Holley and Stroock (1978) and Kallianpur and Wolpert (1984). We start by proving the existence and uniqueness of solutions of certain linear SDE's with $S'(\mathbb{R}^d)$-valued square integrable martingales as the driving term. The weak convergence result entails the functional central limit theorem for semimartingales taking values in $S'(\mathbb{R}^d)$. This is proved in a more general context of semimartingales taking values in the dual of a Frechét nuclear space. An example from Neurophysiology is worked out.

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INTRODUCTION

Our aim in this work is to study the weak convergence of solutions of certain linear stochastic differential equations (SDE) taking values in $S'(\mathbf{R}^d)$ --the space of tempered distributions in $\mathbf{R}^d$. We first prove the existence and uniqueness (Theorem 3.1.1) for solutions of such linear SDE's where the driving term is a $S'(\mathbf{R}^d)$-valued square integrable martingale (for a definition see Chapter I). Existence and uniqueness in particular cases of such solutions have been studied previously in [7], [13] among others. Holley and Stroock [7] have studied the existence and uniqueness in law when the martingale driving term is a Wiener process in these spaces. Kallianpur and Wolpert [13] have studied this problem in the case where the martingale is a centered Poisson process.

Next, we take up the weak convergence of the solutions of such SDE's to an Ornstein-Uhlenbeck (O-U) process in the space $D([0,\infty), S'(\mathbf{R}^d))$ ($D(S'(\mathbf{R}^d))$ for short). Similar results have occurred in [8] and [13]. Holley and Stroock [8] have studied the weak convergence result in the case where the martingale term has an increasing process which admits a density with respect to Lebesgue measure. Kallianpur and Wolpert [13] have studied the case where the martingale term--a centered Poisson process, converges to a Wiener process in $S'(\mathbf{R}^d)$. Their techniques are particular to the properties of Poisson process and cannot be generalized to our case. The basic approach to our problem is that, if the martingale term in the linear SDE converges weakly to a Wiener process in $D([0, \infty); S'(\mathbb{R}^d))$, then the corresponding solutions of such SDE's converges weakly to an O-U process.

The results of [13], in case of $S'(\mathbf{R}^d)$ valued processes, follow from our work.

In Chapter II, we take up the question of when a martingale and in general a semi-martingale taking values in $S'(\mathbb{R}^d)$ converges weakly to a Wiener process. Since $S'(\mathbb{R}^d)$ is the dual of a Frechét nuclear space, we study the problem in the context of martingales and semi-martingales taking values in the dual of Frechét nuclear spaces. In the case of real valued square integrable martingales, functional central limit theorems involve the increasing processes associated with these martingales [15]. We prove the existence and uniqueness of a positive operator-valued analog of such increasing processes corresponding to square integrable martingales in these spaces. We give sufficient conditions, analogous to [15], for the weak convergence of a semi-martingale to a Wiener process to take place in a metrisable subspace of $D([0, \infty), E')$. Here E' is the dual of a Frechét nuclear space. As a consequence of these results we prove the central limit theorem and the functional central limit theorem for E'-valued triangular arrays.

In addition to the ideas in [15], the main technique in our work, is the result on tightness for probability measures on $D(E')$ developed by Mitoma in [18]. This result, along with necessary prerequisites and notations are discussed in the next section.

CHAPTER 1

PRELIMINARIES AND NOTATION

1.1 Frechét Nuclear Spaces.

In the following a vector space is a vector space over $\mathbb{R}$.

A nonnegative function $p:E \to \mathbb{R}+$ on a vector space E is called a seminorm if it satisfies the following conditions:

a) $p(x + y) \leq p(x) + p(y)$ $\quad \forall\ x,y \in E$

b) $p(\lambda x) = |\lambda| p(x)$ $\quad \forall\ \lambda \in \mathbb{R}, \forall\ x \in E$

A seminorm p on a vector space E is called Hilbertian if

$$p(x + y)^2 + p(x - y)^2 = 2p(x)^2 + 2\, p(y)^2$$

If p is a Hilbertian seminorm then it is easily checked that

$$p(x,y) = \frac{1}{4} \{p(x + y)^2 - p(x - y)^2\}$$

defines a symmetric bilinear form on E and $p(x, x) = p(x)^2$.
$p(x, y)$ is called the inner product corresponding to $p(x)$.

For a Hilbertian seminorm p , let $N_p = \{x \in E: p(x) = 0\}$

Then $\tilde{E}_p/N_p$ is a pre-Hilbert space, where $\tilde{E}_p$ is the space E with the topology given by p . Thus E_p , the completion of $\tilde{E}_p/N_p$, is a Hilbert space.

We shall say p is a separable Hilbertian seminorm if E_p is a separable Hilbert space.

Let $(H, \langle \cdot, \cdot \rangle)$ be separable Hilbert space and let $\{h_j\}_{j=1}^{\infty}$ be an orthonormal basis (ONB) in H .

Let $L(H)$ denote the set of bounded linear operators from H into H .

An operator T in $L(H)$ is said to be positive if $\langle Tx, x\rangle \geq 0 \quad \forall\ x \in H$.

Every positive operator $T \in L(H)$ is known to have a unique positive square root $S \in L(H)$ (cf. [20] Theorem 12.33) i.e., a positive operator S such that $T = S^2$. We write $S = T^{\frac{1}{2}}$.

Let $\sigma_1(H) = \{T \in L(H) : T$ is positive and $\sum_{j=1}^{\infty} \langle h_j, Th_j\rangle < \infty\}$.

σ_1 is called the class of positive, nuclear (or trace class) operators.

For $T \in \sigma_1(H)$, define $\|T\|_1 = \sum_{j=1}^{\infty} \langle h_j, Th_j\rangle$. $\|\ \|_1$ is independent of the ONB $\{h_j\}$.

An operator T in $L(H)$ is said to be a Hilbert-Schmidt operator iff $T^*T \in \sigma_1(H)$, where T^* denotes the adjoint of T .

A topological vector space (TVS) E , is said to be countably Hilbertian, if there exists a countable family $\{p_k\}_{k=1}^{\infty}$ of separable Hilbertian seminorms determining the topology of E .

We can and do assume that these seminorms are a directed family. This is without loss of generality since the norms $\{\|\ \|_k\}_{k=1}^{\infty}$ defined by

$$\|\cdot\|_k = \left(\sum_{j=1}^{k} p^2(\cdot)\right)^{\frac{1}{2}} \qquad K = 1, 2, \ldots$$

are such that $\|\ \|_1, \leq \|\ \|_2 \leq \ldots$

and the topologies generated by these countable seminorms is the same as those generated by $\{p_k\}_{k=1}^{\infty}$. A neighborhood basis of 0 in this topology is given by sets of the form

$$U_k = \{f \in E : \|f\|_k < 1\} \qquad k = 1, 2, \ldots$$

Clearly such a topology is locally convex. Moreover E is metrisable, and the metric is translation invariant. Indeed set,

$$d(f, g) = \sum_{j=1}^{\infty} 2^{-j} \frac{\|f - g\|_j}{1+\|f-g\|_j}$$

Then it is easily checked that the topology generated by d is the topology generated by the norms $\{\| \ \|_k, k \geq 1\}$

A complete, locally convex, metric space, whose metric is translation invariant is known as a Frechét space.

Thus a complete, countably Hilbertian TVS E is a Frechét space.

Define E_k $(= E_{\| \ \|_k})$ as before. Then

$$E_1 \supset E_2 \supset \ldots \qquad \supset E$$

It can be shown that E is complete if and only if $E = \bigcap_{p=1}^{\infty} E_p$.

As the norms $\| \ \|_k$ are increasing in k , the identity mapping

$$i_{mn} : E_n \to E_m$$

defines a bounded linear operator for $m < n$.

A complete, countably Hilbertian TVS E , is said to be a Frechét nuclear space if to every $m \geq 1$ there exists an $n \geq m$ such that the map $T = i_{mn}^* \, i_{mn} : E_n \to E_n$ is a nuclear operator.

In such a case $T^{\frac{1}{2}} : E_n \to E_n$ will be a Hilbert-Schmidt operator.

For a TVS F, let F' denote its topological dual, i.e., the set of all continuous linear functionals on F .

For a Frechét nuclear space E , with E_k, $k=1, 2, \ldots$ as before, we have

$$E_1' \subset E_2' \subset \ldots \qquad \text{and} \quad E' = \bigcup_{k=1}^{\infty} E_k'$$

For x in E' and f in E we will denote the duality between them by (f, x) . Thus, by definition the map $f \ni E \to (f, x) \in \mathbb{R}$ is continuous.

If $x \in E'_p$, then define $\|x\|_{-k} = \sup_{\|f\|_k=1} |(f, x)|$.

We have $\|x\|_o \geq \|x\|_{-1} \geq \quad \geq \|x\|_{-2} \geq \cdots$

A strong neighborhood of 0 in E' is given by sets of the form

$$V(A) = \{x \in E' : \sup_{f \in A} |(f, x)| < 1\}$$

where A is an arbitrary bounded set in E. (Recall that a set A in E is bounded iff $\|f\|_k < c$ for all f in A, for some $K \geq 1$, $c > 0$).

The topology defined on E' with the strong neighborhoods as a neighborhood basis at 0 is called the strong dual topology.

We shall always consider the strong dual topology on E'.

The classical example of a Frechét nuclear space is the space $S(\mathbf{R}^d)$ of rapidly decreasing functions on $\mathbf{R}^d$[23]. To describe this space, we first set up some standard notation.

We will denote a $x \in \mathbf{R}^d$ by $(x_1,\ldots,x_d)$. A multiindex α on $\mathbf{R}^d$ is d-tuple of nonnegative integers $(\alpha_1,\ldots,\alpha_d)$. By $|\alpha|$ we will mean $\alpha_1 + \alpha_2 + \ldots + \alpha_d$.

D^α will denote $\dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}}$ for any multiindex α.

The space $S(\mathbf{R}^d)$ can now be described as the space of all real, infinitely differentiable functions f, satisfying

$$|f|_N = \sup_{|\alpha| \leq N} \sup_{x \in \mathbf{R}^d} (1 + |x|^2)^N |(D^\alpha f)(x)| < \infty$$

for $N = 0, 1, 2, \ldots$ where $|x|^2 = x_1^2 + \ldots + x_d^2$

The collection of norms $| \ |_N$ define a locally convex topology which makes $S(\mathbf{R}^d)$ into a Frechét space (cf. [20]).

In our work it is more convenient to work with an equivalent collection of norms $\{\| \ \|_n, n \geq 0\}$ which generate the same topolgoy on $S(\mathbf{R}^d)$. These

have previously occurred in other works, such as in [7], [10] or [21].

Given $k \geq 0$ define the Hermite polynomials $h_k : \mathbb{R} \to \mathbb{R}$ by

$$h_k(x) = (\pi^{\frac{1}{2}} 2^k k!)^{-\frac{1}{2}}(-1)^k \exp(\frac{x^2}{2})(\frac{d}{dx})^k(\exp(\frac{-x^2}{2}))$$

Then $\{h_k, k \geq 0\}$ form an ONB in $L^2(\mathbb{R})$. For a multiindex α on $\mathbb{R}^d$ define h_α on $\mathbb{R}^d$ by $h_\alpha(x) = h_{\alpha_1}(x_1)\ldots \cdot h_{\alpha_\alpha}(x_\alpha)$.

Clearly $\{h_\alpha\}$ form an ONB in $L^2(\mathbb{R}^d)$. Let

$$\|f\|^2 = \int_{\mathbb{R}^d} |f(x)|^2 dx \quad \text{and} \quad \langle f, g\rangle = \int_{\mathbb{R}^d} f(x)g(x)dx$$

denote the norm and inner product on $L^2(\mathbb{R}^d)$.

The k^{th} norm $\| \ \|_k$ in $L^2(\mathbb{R}^d)$ is defined as follows

$$(1.1.1) \qquad \|f\|_k^2 = \sum_{|\alpha|=0}^{\infty} (2|\alpha|+d)^k \langle f, h_\alpha\rangle^2 \quad \text{for} \quad f \text{ in } L^2(\mathbb{R}^d).$$

Define the space $(E_k, \langle \cdot, \cdot \rangle_k)$ by

$$(1.1.2) \quad E_k = \{f \in L^2(\mathbb{R}^d) : \|f\|_k < \infty\}. \qquad k = 0, 1, 2, \ldots$$

$$\text{with } \langle f, g\rangle_k = \sum_\alpha \langle f, h_\alpha\rangle \langle g, h_\alpha\rangle (2|\alpha|+d)^k \quad f, g \in E_k.$$

Here $E_o = L^2(R^d)$, E_k is a Hilbert space, and $S(\mathbb{R}^d) = \bigcap_{k=0}^{\infty} E_k$

Further,

$$S(\mathbb{R}^d) \subset \ldots\ldots \subset E_k \subset \ldots\ldots \subset E_1 \subset E_o .$$

It can be checked from equation (1.1.1) that the identity map $i_{n,n+d+1} : E_{n+d+1} \to E_n$ is such that $i^*_{n,n+d+1}\, i_{n,n+d+1} : E_{n+d+1} \to E_{n+d+1}$ is nuclear.

The dual, $S'(\mathbb{R}^d)$ is known as the space of tempered distributions.

We define $\| \ \|_{-k}$ by

$$\|x\|_{-k} = \sum_\alpha (2|\alpha|+d)^{-p} (h_\alpha, x)^2 \qquad \text{for} \quad x \in S'(\mathbb{R}^d)$$

and henceforth we shall denote (h_α, x) by $x(h_\alpha)$ $\forall\ \alpha$.

and $$E'_k = \{x \in S'(\mathbb{R}^d) : \|x\|_{-k} < \infty\} \qquad k = 0, 1, \ldots$$

Then E'_k is a Hilbert space with inner product

$$\langle x, y\rangle_{-k} = \sum_\alpha x_\alpha(h_\alpha)y(h_\alpha)(2|\alpha| + d)^{-p} \qquad x, y \in E'_k$$

$(E'_k\ \langle\cdot, \cdot\rangle_{-k})$ is the dual of $(E_k, \langle\cdot, \cdot\rangle_k)$.

$$S(\mathbb{R}^d) \subset \ldots \quad \subset E_1 \subset E_0 \subset E'_1 \subset \ldots \quad \subset S'(\mathbb{R}^d)$$

If $x \in E'_p$ and $f \in E_p$, the duality takes the form

$$(f, x) = \sum_\alpha \langle f, h_\alpha\rangle\, x(h_\alpha)$$

and $$\|x\|_{-k} = \sup_{f \in S(\mathbb{R}^d)\sim\{0\}} \frac{|(f, x)|}{\|f\|_k} .$$

1.2 E'-valued processes and semi-martingales

Throughout, we will be dealing with "weak" semi-martingales taking values in the dual of a Frechét nuclear space. Before defining it we need the following.

Let (Ω,F,P) be a complete probability space and $(F_t, t\geq 0)$ be a non-decreasing family of sub-σ-algebras of F satisfying the usual conditions [12], i.e., $F_{t+} = F_t$ and F_0 contains all P-null sets of F . The filtration $(F_t, t\geq 0)$ will be denoted by $\underline{F}$.

Let E be a Frechét nuclear space and E' its dual. Let $\mathcal{B}(E')$ denote the cylinder σ-algebra on E', i.e., the smallest σ-algebra on E' which makes all the maps $x \ni E' \to (f, x) \in \mathbb{R}$ measurable for f in E .

A measurable map $X : (\Omega, F) \to (E', \mathcal{B}(E'))$ will be called a E'-valued random variable. Thus X is a E'-valued random variable if for each f in E , (f, X) is a real random variable.

A collection $X = \{X(t), t \geq 0\}$ of E'-valued random variables on (Ω, F) is called an E'-valued stochastic process.

An E'-valued process X on $(\Omega, F, \underline{F}, P)$ is said to be $\underline{F}$-adapted if $(f, X(t))$ is F_t adapted for each f in E and for all $t \geq 0$.

Let $Y = (Y(t), t \geq 0)$ and $X = (X(t), t \geq 0)$ be two E'-valued processes on $(\Omega, F, \underline{F}, P)$.

Y is said to be a modification of X if $\forall\, t \geq 0$ $P[X(t) = Y(t)] = 1$.

An E' or E'_k-valued process $X = (X(t), t \geq 0)$ is said to be a cadlag process if in the topology of E' or E'_k the process X has left limits at each $t \geq 0$ and is right continuous.

Recall that a real valued process X $(= X(t), t \geq 0)$ is said to be a semi-martingale if X admits a decomposition of the form $X = M + A$ where $M = (M(t), t \geq 0)$ is a cadlag local martingale [12] and $A = (A(t); t \geq 0)$ is a cadlag adapted process of finite variation on compacts.

We will extend this definition for E'-valued processes.

1.2.1 An E'-valued process $X = (X(t), t \geq 0)$ is said to be a semi-martingale (martingale) on $(\Omega, F, \underline{F}, P)$ if

a) X is $\underline{F}$-adapted and

b) the real process $t \mapsto (f, X(t))$ is a semi-martingale (martingale) for each f in E.

In the theory of real semi-martingales two processes play an important role--the increasing process associated with a square integrable martingale and the dual predictable projection of the integer-valued random measure associated with the jumps of a semi-martingale. We describe these briefly. For more on this subject see [12] (Chapters II and III).

Let $X = (X(t), t \geq 0)$ be a real martingale on $(\Omega, F, \underline{F}, P)$ satisfying $EX^2(t) < \infty$ for each $t \geq 0$. Then the Doob-Meyer decomposition shows that there exists an increasing predictable process, denoted by $\langle X \rangle$. Satisfying $t \mapsto X^2(t) - \langle X \rangle_t$ is an $\underline{F}$-martingale. $\langle X \rangle$ is unique up to P-equivalence and is called the increasing process associated with X.

Let $X = (X(t); t \geq 0)$ be a real semi-martingale on $(\Omega, F, \underline{F}, P)$. By definition, it is a cadlag adapted process. Define the integer-valued random measure associated with the jumps of X by

$$N((0, t], A) = \sum_{0<s\leq t} [\Delta X(s) \in A]$$

where $\Delta X(s) = X(s) - X(s-)$ is the jump at s and A is a Borel set of $\mathbf{R}$. The dual predictable projection (compensator) of the random measure $N(ds, dx)$ denoted $\hat{N}(ds, dx)$ is a predictable random measure such that for every Borel set A of $\mathbf{R}$, we have $t \mapsto N((0, t], A) - \hat{N}((0, t], A)$ is $\underline{F}$ - P martingale.

For an E'-valued semi-martingale $X = (X(t), t \geq 0)$ on $(\Omega, F, \underline{F}, P)$ the integer valued random measure associated with the jumps of (f, X) is denoted by $\hat{N}(f, ds, dx)$ and its compensator is denoted by $N(f, ds, dx)$.

Similarly, if $X = (X(t), t \geq 0)$ is an $\underline{F}$-P, E'-valued martingale satisfying $E(f, X(t))^2 < \infty \ \forall\ f \in E$, then the increasing process associated with (f, X) is denoted by $\langle (f, X) \rangle$. Next we discuss an example of an E-valued martingale without jumps with the explicit form for $\langle (f, X) \rangle$.

1.3 The Wiener Martingale

One of the most important examples of an E-valued martingale is the Wiener martingale, the analog of Brownian motion on these spaces.

Let $Q : E \times E \to \mathbf{R}$ be a continuous, positive definite bilinear form.

An adapted E'-valued process $W = (W(t), t \geq 0)$ on $(\Omega, F, \underline{F}, P)$ is said to be an $\underline{F}$ - P Wiener martingale if

a) $W(0) \equiv 0$ a.s. P

b) the map $t \mapsto W(t)$ is continuous.

c) $\{(f, W(t))\ t \geq 0\ ,\ f \in E\}$ is a centered Gaussian system with covariance function

$$E(f, W(t))(g, W(s)) = (t \wedge s)\ Q(f, g) \quad \forall\ f, g \in E$$

d) $W(t) - W(s)$ is independent of F_s $\forall$ $s < t$.

Observe that these conditions imply that W is an E'-valued $\underline{F}$- P martingale satisfying $E(f, W(t))^2 < \infty$ for all f in E and $t \geq 0$. By (a), (c), (d) it follows that the increasing process associated with W is given by $tQ(f,f)$ i.e., $t \mapsto (f, W(t))^2 - tQ(f,f)$ is a martingale for each f in E

If $Q(f,f) \leq c\|f\|_k$ $\forall$ $f \in E$, for some $k \geq 1$ and some $c > 0$, then by the Bochner-Minlos theorem ([6] Theorem 3.1) there exists a $m > k$ such that we can find a version of W taking values in E'_m . In such a case we will call W an E'_m-Wiener martingale.

Typically, the kinds of processes which we shall be considering will not be continuous E'-valued process, but those which are cadlag in E' . We describe these spaces now, along with the result of Mitoma [18] on tightness of probability measures on such spaces.

1.4 The spaces $D(E')$, $D(E'_p)$.

Recall that we only consider the strong dual topology on E' . With this topology let $D(E') = D(\mathbf{R}_+; E')$ denote the space of all cadlag mappings from $\mathbf{R}_+$ to E' . Let $D(E'_p) = D(\mathbf{R}_+; E'_p)$ denote the space of all cadlag mappings from $\mathbf{R}_+$ to E'_p . The topology on these spaces is given by the Lindvall metrics

described in [19]. $\mathcal{B}(D(E'))$ is the smallest σ-algebra on $D(E')$ which makes all the maps

$$x(\cdot) \in D(E') \mapsto ((f, x(t_1)), (f_2, x(t_2)), \ldots (f_k, x(t_k))) \in \mathbb{R}^k$$

measurable for $0 \leq t_1 < t_2 < \ldots < t_k$, $f_1, \ldots, f_k$ in E; $k \geq 1$. $\mathcal{B}(D(E'_p))$ is the Borel σ-algebra on $D(E'_p)$, given by the Skorohod topology in the metric space $D(E'_p)$.

We quote below the main results of [19] in the form of a theorem. Before we do that we need a definition.

Let $\{P_n\}\ n \geq 1$ be a sequence of probability measures on $(D(E'), \mathcal{B}(D(E')))$. We say that P_n is uniformly k-continuous if for every $\varepsilon > 0$, $\rho > 0$ and $T > 0$ there exists a $\delta > 0$ such that whenever $\|f\|_k \leq \delta$, then

$$P_n\{x(\cdot) \in D(E') : \sup_{t \leq T} |(f, x(t))| > \varepsilon\} \leq \rho \quad \text{for all} \quad n \geq 1$$

Let $D = D(\mathbb{R}_+; \mathbb{R})$. Define the map

$$\Pi(f) : D(E') \to D \quad \text{by}$$

$$\Pi(f)(x) = \langle f, x(\cdot) \rangle \qquad f \text{ in } E .$$

1.4.1 Theorem ([18]): Let $\{P_n\}\ n \geq 1$ be a sequence of probability measures on $(D(E'), \mathcal{B}(D(E')))$.

a) If $\{P_n \circ \Pi(f)^{-1}\}_{n \geq 1}$ is tight in D for every f in E, then $\{P_n\}$ is tight in $D(E')$.

b) If $\{P_n\}\ n \geq 1$ is uniformly k-continuous and $\{P_n\}\ n \geq 1$ is tight in $D(E')$, then $\{P_n\}\ n \geq 1$ is tight in $D(E'_p)$ for some $p \geq k$.

c) Let $\Pi^{f_1, \ldots, f_m}_{t_1, \ldots, t_m}$ be the mapping that carries the point x of $D(E')$ to the point $((f_1\ x(t_1)), \ldots, (f_m\ x(t_m)))$.

If $\{P_n\}$ is tight in $D(E')$ and for any finite number of elements $f_1,\ldots,f_m$ in E and time points $t_1,\ldots,t_m$ we have

$$P_n \circ \pi\left(\begin{matrix} f_1,\ldots,f_m \\ t_1,\ldots,t_m \end{matrix}\right)^{-1} \Rightarrow Q^{f_1,\ldots,f_m}_{t_1,\ldots,t_m} \qquad \text{where}$$

$Q^{f_1,\ldots,f_m}_{t_1,\ldots,t_m}$ is a probability measure on $\mathbb{R}^m$, then there exists a unique probability measure P on $D(E')$ such that P_n converges weakly to P. If moreover $\{P_n\}$ is uniformly k-continuous, then $P_n \Rightarrow P$ on $D(E'_p)$ for some $p \geq k$.

CHAPTER II

FUNCTIONAL CLT FOR SEMIMARTINGALES TAKING VALUES IN THE DUAL OF A FRECHÉT NUCLEAR SPACE

Recall that, corresponding to every locally square integrable martingale, M, there exists a predictable increasing process A, such that $M^2 - A$ is a local martingale. This is the famous Doob-Meyer decomposition of the submartingale M^2. We generalize this result to the case where M is a E'-valued locally square integrable martingale, E being a nuclear Frechét space. We then go on to prove the functional central limit theorem for E'-valued semi-martingales. Although very useful, this turns out to be rather straightforward to prove, given Theorem 1.4.1 and the results of [15]. In fact, the only point where one encounters some difficulty is, in proving the convergence of the finite dimensional distributions. This is handled by a variation of the results of [15] and is stated in Proposition 2.1.6 below.

As a consequence of this theorem, we prove the central limit theorem and the invariance principle for E' - valued triangular arrays.

An example which hints at the applicability of the functional central limit theorem, follows.

We start by proving some auxilliary results needed for our main theorems.

2.1 Auxilliary results.

The proposition below is a regularization theorem due essentially to Ito [10]. Our proof follows closely, Theorem 3.1 in [10].

2.1.1 Proposition. Let $Y(= Y(f),\ f \text{ in } E)$ be a family of real random variables such that

(i) $$Y(c_1 f + c_2 g) = c_1 Y(f) + c_2 Y(g) \qquad \text{a.s.}$$

where the exceptional set may depend c_1, c_2, f, g

(ii) $$EY(f)^2 \leq c\|f\|_p^2 \quad \text{for some } c > 0,\ p \geq 1$$

then Y has a version in E'_q for some $q > p$ and there exists a constant ℓ such that

$$E\|Y\|^2_{-q} \leq \ell c$$

Proof: The map $Y : E_p \to L^2(\Omega, F, P)$ is clearly a bounded linear operator. Let $\{h_i\}$ $i \geq 1$ be a countable dense set in E. Let $\{e^n_i\}$ $i \geq 1$ be for each fixed n a complete orthonormal system in E_n, obtained from $\{h_i\}$ by the Schmidt orthogonalization. Since E is nuclear there exists a $q > p$ such that $\sum_\alpha \|e^q_j\|^2_p = \ell < \infty$. Let $\{\varepsilon^q_j\}$ $j \geq 1$ be the ONB in E'_q dual to $\{e^q_j\}$.

Set $Y(\varepsilon^q_j) = X_j$. Then

$$E(\sum_j X^2_j) \leq c \sum_j \|e^q_j\|^2_p = c\ell$$

Hence $\sum X^2_j(\omega) < \infty$ for all ω in Ω_1, with $P(\Omega_1) = 1$.

Let $X(\omega) = \begin{cases} \sum_j X_j(\omega) e^q_j & \text{for } \omega \text{ in } \Omega, \\ 0 & \text{otherwise} \end{cases}$

Then $X(\omega) \in E'_q$ for each ω and $X(\omega)$ is measurable in $F/\mathcal{B}(E'_q)$ in ω. X is thus an E'_q variable and

$$E(\|X\|^2_{-q}) = E(\sum X^2_j) \leq \ell c$$

As in [10] it is easy to show that $X(f) = Y(f)$ a.s. for all f in E.

Q.E.D.

The following proposition helps us in formulating the analog of an increasing process associated with square integrable E'-valued martingales. The proposition is easily proved by mimicking the proof given in [25] for continuous Hilbert space valued martingales. So we shall only present a sketch of the proof here.

2.1.2 Proposition. Let $(H, \|\cdot\|, <\cdot, \cdot>)$ be a real, separable Hilbert space. Let $(M, \underline{F}, P)$ be a H-valued process which is cadlag in H, with

$E\|M(t)\|^2 < \infty$ for each $t \geq 0$. Assume that $(M, \underline{F}, P)$ is a martingale in the sense of definition 1.2.1. Then there exists a unique increasing process $A(t)$ with values in the positive operators $\sigma_1(H) \subset L(H)$, right continuous for the norm $\|\ \|_1$ of $\sigma_1(H)$, such that for each y, z in H, the real process $t \mapsto \langle y, A(t)z\rangle$ is a predictable process of finite variation and

$$t \mapsto \langle y, M(t)\rangle \langle z, M(t)\rangle - \langle y, A(t)z\rangle$$

is an $\underline{F}$ martingale.

Proof: Since $\langle y, M\rangle$ is a real, locally square integrable martingale for each y in H, it follows that there exists a unique (up to P-equivalence) increasing, predictable process $A(t, y)$ (a unique, upto P-equivalence, predictable process of finite variation $A(t, y, z)$) such that $\langle y, M(\cdot)\rangle^2 - A(\cdot, y)$. $(\langle y, M(\cdot)\rangle \langle z, M(\cdot)\rangle - A(\cdot, y, z))$ are $\underline{F}$- martingales. If $\{e_n\}$ is an ONB for H, the argument in [25] shows that under the condition that $E\|M(t)\|^2 < \infty$, the series $\sigma_t(y) = \sum_{n,m} \langle y, e_n\rangle \langle y, e_m\rangle A(t, e_n, e_m)$ is convergent and in fact $\langle y, M(\cdot)\rangle^2 - \sigma_\cdot(y)$ is an $\underline{F}$ martingale. Thus $\sigma_\cdot(y)$ and $A(\cdot, y)$ are P-equivalent. Similarly $A(\cdot, y, z)$ is P-equivalent to the process $t \mapsto \sum_{n,m} \langle y\ e_n\rangle \langle z\ e_m\rangle A(t, e_n, e_m)$ and hence there exists a process $A(\cdot)$ which is positive operator-valued, such that

$$A(\cdot, y, z) = \langle y, A(\cdot)z\rangle \text{ with } E(\text{Trace } A(t)) < \infty$$

Since $A(\cdot, y, z)$ is predictable so is the process $t \mapsto \langle y, A(t)z\rangle$. Since $A(\cdot, y)$ is increasing, we have, for each y in H and $h > 0$

$$\langle y, A(t+h)y\rangle \geq \langle y, A(t)y\rangle .$$

Thus $A(t+h) - A(t)$ is a positive operator.

Finally, for $h > 0$, we have

$$\|A(t+h) - A(t)\|_{L(H)} \leq \|A(t+h) - A(t)\|_1$$

$$= \sum_n \langle e_n \quad (A(t+h) - A(t))e_n \rangle$$

$$\leq \sum_n \langle e_n \quad A(T)e_n \rangle$$

for all t , $t+h \leq T$

Since $t \mapsto A(t,y)$ is right continuous, it follows by dominated convergence theorem that the map $t \mapsto A(t)$ is right continuous for the norm of $L(H)$ and $\sigma_1(H)$

Q.E.D.

We will have occassion to use the continuous mapping theorem (Theorem 5.1 of [2]) for mappings from $D(E')$ to E' . However, since $D(E')$ is not, in general, a metric space, a proof is needed for the theorem.

2.1.3 Proposition: Let Q_n, Q be probability measures on $(D(E'), \mathcal{B}(D(E')))$ and let h be a continuous map from $D(E')$ to E' . If $Q_n \Rightarrow Q$ then $Q_n h^{-1} \Rightarrow Q h^{-1}$.

Proof. As in [24] $Q_n \Rightarrow Q$ iff for any closed set G

$$\limsup_n Q_n(G) \leq Q(G)$$

Applying this result to the closed set $G = h^{-1}(F)$ we have the desired result.

Q.E.D.

2.1.4 Remark: This result can be extended to maps h , which are discontinuous, provided the set D_h , of discontinuities of h , has Q measure zero. However, unlike the case where $E = \mathbf{R}^d$, it is not always true that D_h is measurable!

The following is an easy but technical lemma, needed in the proof of our main theorem.

2.1.5 Lemma: Let $\{Y_j^n(\varepsilon)\}_{n=1}^{\infty}$ be a sequence of random variables for each $\varepsilon > 0$ and $j = 1,\ldots,k$. Suppose $Y_j^n(\varepsilon) \xrightarrow{P} 0$ as $n \to \infty$ for $j = 1,\ldots,k$. Then there exists a sequence ε_n, decreasing to zero such that $Y_j^n(\varepsilon_n) \xrightarrow{P} 0$ as $n \to \infty$ for $j = 1,\ldots,k$.

Proof. This is the analog of Lemma 3 of [15]. However the proof given there has a minor error.

Set $m_j^n(\varepsilon) = E[|Y_j^n(\varepsilon)|/(1 + |Y_j^n(\varepsilon)|]$

Clearly, $Y_j^n(\varepsilon) \xrightarrow{P} 0$ as $n \to \infty$ iff $m_j^n(\varepsilon) \to 0$ as $n \to \infty$ for $j = 1,\ldots,k$. So let

$$n_1 = \min \{n_0 : \max_{1\le j\le k} m_j^n(1) \le 1 \quad \forall\ n \ge n_0\}$$

and for $i \ge 2$ $\quad n_i = \min \{n_0 > n_{i-1} : \max_{1\le j\le k} m_j^n(\tfrac{1}{i}) \le \tfrac{1}{i} \ \forall\ n \ge n_0\}$

$\{n_i\}$ is obviously an increasing sequence. Let

$$\varepsilon_n = \tfrac{1}{i} \quad \text{if} \quad n_i < n \le n_{i+1}$$

Then ε_n decreases to zero. Further, for $j = 1,\ldots,k$

$$\begin{aligned} m_j^n(\varepsilon_n) &= m_j^n(\tfrac{1}{i}) && \text{if } n_i \le n \le n_{i+1} \\ &\le \tfrac{1}{i} && \text{if } n_i < n \le n_{i+1} \\ &= \varepsilon_n \to 0 && \text{as } n \to \infty \end{aligned}$$

Q.E.D.

Finally, we need a proposition, which is based on the argument in [15] to show convergence of the finite dimensional distributions of martingales. We recast their technique there so that it fits in our context.

2.1.6 Proposition: Let $(M^n, \underline{F}^n)$ be a sequence of real valued, locally square integrable martingales satisfying

(a) $\quad \sup_{t\le T} |\Delta M^n(t)| \le d_n \quad$ for all (n,T) in $\mathbf{N} \times \mathbf{R}_+$

where $0 \leq d_n \to 0$ as $n \to \infty$

(b) $\langle M^n \rangle_t \xrightarrow{P} \sigma^2(t)$ for all t in $[0, T]$

where $t \mapsto \sigma^2(t)$ is a positive, non-decreasing function on $[0, \infty)$.

Then

(c) $E[\exp\ iM^n(T)] \to \exp\left[-\frac{1}{2}\sigma^2(T)\right]$

Proof: We will give an outline of the proof, and refer the reader to [15] for the details.

First, consider the case where

(d) $\langle M^n \rangle_T \leq a$ for all n where $\sigma^2(T) + 1 \leq a$.

To prove (c) it suffices to show that

(e) $$E\{\exp\ (iM^n(T) + \tfrac{1}{2}\sigma^2(T)\} \to 1 \text{ as } n \to \infty .$$

Set $$A^n(t) = \tfrac{-1}{2}\langle M^{nc} \rangle_t + \int_0^t \int_{|x| \leq d_n} (e^{ix} - 1 - ix)\hat{N}^n(ds, dx)$$

$$= \tfrac{-1}{2}\langle M^{nc} \rangle_t + b^n(t)$$

where $\hat{N}^n(ds, dx)$ is the compensator of the random measure $N^n(ds, dx)$ corresponding to the jumps of the process M^n and M^{nc} denotes the continuous martingale part of M^n.

Define $$Z^n(t) = \exp\ (iM^n(t))/\mathbb{E}(t, A^n)$$

where

$$\mathbb{E}(t, A^n) = \exp(A^n(t)) \prod_{0<s\leq t} (1 + b^n(s))e^{-b^n(s)}$$

Then it can be shown that $|\mathbb{E}(t, A^n)^{-1}|$ is bounded and $(Z^n, \underline{F}^n)$ is a uniformly integrable martingale under (d). Moreover $E\ Z^n(t) = 1$ for all t.

Thus (e) is equivalent to the statement

$$E\{\exp\ (iM^n(T) + \frac{\sigma^2(T)}{2})\} - E\ Z^n(T) \to 0 \qquad \text{as } n \to \infty$$

which, since $|\mathcal{E}(T, A^n)^{-1}|$ is bounded amounts to proving that

(f) $$\mathcal{E}(T, A^n) \xrightarrow{P} \exp\left(-\tfrac{1}{2}\sigma^2(T)\right)$$

It is well known that $\prod_{0<s\leq T} (1 + b^n(s)) \exp(-b^n(s)) \to 1$

as $n \to \infty$, under (a) and (d). Thus (f) follows if we show that

$$-\tfrac{1}{2}\langle M^{nc}\rangle_T + b^n(T) \xrightarrow{P} \tfrac{1}{2}\sigma^2(T)$$

which under (b) is equivalent to showing that

$$\int_0^T \int_{|x|\leq d_n} (e^{ix} - 1 - ix + \tfrac{x^2}{2})\hat{N}^n(ds, dx) \xrightarrow{P} 0$$

But $\left|\int_0^T \int_{|x|\leq d_n} (e^{ix} - 1 - ix + \tfrac{x^2}{2})\hat{N}^n(ds, dx)\right| \leq \tfrac{d_n}{6}\langle M^n\rangle_T \to 0$.

Assumption (d) can be disposed of, by a standard argument using stopping times as in [15].

Q.E.D.

2.2 The increasing process and the functional CLT.

We first show that for every E'-valued martingale satisfying $E(f, M(t))^2 < \infty$ for each f in E and all t positive, we can associate an increasing process of the type described in Proposition 2.1.2.

2.2.1 Theorem: Let $(M(\cdot), \underline{F}, P)$ be an E'-valued martingale satisfying $E(f, M(t))^2 < \infty$ for all (t, f) in $\mathbb{R}_+ \quad E$. Then for any $T > 0$, there exists a unique, increasing positive-operator valued process $A(t)$ of the type described in Proposition 2.1.2, taking values in E'_p for some $p(= p_T)$ such that

$$t \mapsto (f, M(t\wedge T))(g, M(t\wedge T)) - \langle f, A(t)g\rangle_{-p}$$

is an $\underline{F}$ martingale for all f, g in E.

Proof: Let $T > 0$ be arbitrary and define $X(f) = \sup_{0\leq t\leq T} (f, M(t))^2$ and $V_T(f) = E\, X(f)$.

By Doob's inequality $V_T(f) \leq 4E(f \quad M(T))^2 < \infty$ for all f in E.

The map $f \mapsto V_T(f)$ is lower semi-continuous since if $f_n \to f$ in E then by Fatou's lemma

$$\begin{aligned} \liminf_{n \to \infty} V_T(f_n) &\geq E \liminf_{n \to \infty} X(f_n) \\ &\geq E\, X(f) \\ &= V_T(f) \end{aligned}$$

Moreover $\{f : V_T(f) < \infty\} = E$

Since E is complete it follows by the Baire Category theorem (cf. Problem C, Chapter 3, 9 [14]) that there exists a $p \geq 1$, C, $k > 0$ such that

$$V_T(f) \leq k \quad \text{for all } f \text{ in } E \text{ satisfying } \|f\|_p \leq c$$

Thus $$V_T\left(\frac{c}{\|f\|_p} f\right) \leq k \quad \text{for all } f \text{ in } E$$

But $$V_T(a\, f) = a^2\, V_T(f) \quad \text{for } a \text{ in } \mathbb{R}. \text{ Hence}$$

$$V_T(f) \leq \frac{k}{c^2} \|f\|_p^2$$

By Propositon 2.1.1 $M(\cdot)$ has E_q'- regularization for some $q > p$ and $E\|M(t)\|_{-q}^2 \leq L$ for some constant L, for all $0 \leq t \leq T$.

Proposition 2.1.2 now gives the desired result.

Q.E.D.

The use of the Baire category theorem in this context is suggested by [17]

2.2.2 Remark: We call $A(t)$ the increasing process associated with M.

In the main theorem, stated below, the process $\Delta(t, \varepsilon, f)$ will play a crucial role. We desribe it in the next paragraph.

Let $(X(\cdot), \underline{F})$ be an E'-valued semi-martingale. Then, for any $\varepsilon > 0$, $X(t, f)$ has a decomposition of the form (cf. [15] equation 45).

$$(f, X(t)) = \Big[\sum_{0<s\le t}\int_{\varepsilon<|x|\le 1} x\,\hat{N}(f,\{s\},dx)\Big] + \Big[B^c(t,f) + \sum_{0<s\le t}\int_{|x|\le\varepsilon}\hat{N}(f,\{s\},dx)\Big]$$
$$+ \Big[\int_0^t\int_{|x|>1} x\,N(f,ds,dx) + \int_0^t\int_{\varepsilon<|x|\le 1} x\,(N-\hat{N})(f,ds,dx)\Big]$$
$$+ \Big[X^c(t,f) + \int_0^t\int_{|x|\le\varepsilon} x(\hat{N}-N)(f,ds,dx)\Big]$$

Let this be denoted by

(2.2.3) $\alpha(t,\varepsilon,f) + \beta(t,\varepsilon,f) + \gamma(t,\varepsilon,f) + \Delta(t,\varepsilon,f)$ for every f in E.

Moreover $(\Delta(\cdot,\varepsilon,f), \underline{F})$ is a locally square integrable martingale.

Let F^n $(= \underline{F}^n_t,\ t\ge 0)$ and $\underline{F}(=F_t,\ t\ge o)$ be increasing families of σ-algebras satisfying the usual conditions.

2.2.4 Theorem: Let $(X^n,\underline{F}^n,P^n)$ $n \ge 1$ be a sequence of E'-valued semi-martingales having a decomposition of the form

$$(f, X^n(t)) = \alpha^n(t,\ ,f) + \beta^n(t,\ ,f) + \gamma^n(t,\varepsilon,f) + \Delta^n(t,\varepsilon,f)$$

as described in (2.2.3) for any $\varepsilon > 0$, and all f in E. Assume $X^n(0) \equiv 0$ a.s. Let $(W(\cdot),F)$ be a Wiener martingale, with $Q(\cdot,\cdot)$ as its covariance operator. Suppose for any (t,ε,f,g) in $\mathbb{R}_+ \times (0,1] \times E \times E$, we have

(A) $\int_0^t\int_{|x|>\varepsilon}\hat{N}^n(f,ds,dx) \xrightarrow{P} 0$

(B) $\sup_{0<s\le t}|B^n(s,\varepsilon,f)| \xrightarrow{P} 0$

(C) $\langle\Delta^n(\cdot,\varepsilon,f)\,\Delta^n(\cdot,\varepsilon,g)\rangle_t \xrightarrow{P} t\,Q(f,g)$

then $X^n \xrightarrow{L} W$ in $D(E')$.

2.2.5 Corollary: Let (M^n, F^n, P^n) be a sequence of E'-valued martingales satisfying $M^n(0) \equiv 0$ a.s. for all n and $E(f, M^n(t))^2 < \infty$ for all (n, t, f) in $\mathbf{N} \times \mathbf{R} + \times E$. Suppose for each $(t, T, \varepsilon, f, g)$ in $\mathbb{R}_+ \times \mathbb{R}_+ \times (0, 1] \times E \times E$ the following is true.

(2.2.6) $$\int_0^t \int_{|x|>\varepsilon} x^2 \, \hat{N}^n(f,ds,dx) \xrightarrow{P} 0$$

(2.2.7) $$\langle f \; A^n(t)g \rangle_{-p_n} \xrightarrow{P} tQ(f,g) \qquad \text{for } 0 \le t \le T$$

where A^n is the increasing process associated with M^n and the p_n is as in Theorem 2.2.1 then

$$M^n \xrightarrow{L} W \qquad \text{in } D(E')$$

Proof of Theorem 2: By Theorem 1.4.1, it suffices to show that (a) $(f\, M^n(\cdot))$ is tight in D for each f in E and (b) for any $0 \le t_1 < t_2 < \ldots < t_k = T$, any $f_1,\ldots f_k$ in E and any $k \ge 1$, we have

$$((f_1, X^n(t_1)),\ldots,(f_k, X^n(t_k)) \xrightarrow{L} ((f_1, W(t_1)),\ldots,(f_k, W(t_k)) \text{ in } R^k$$

Since $((f, X^n(\cdot)), \underline{F}^n)$ is a real semi-martingale, Theorem 1 of [15] assures us that $(f, X^n(\cdot))$ is tight in D for each f in E.

In order to prove (b), write, for any $\varepsilon > 0$

$$\begin{aligned}[(f_1, X^n(t_1)),\ldots,(f_k, X^n(t_k))] = {} & [\alpha^n(t_1,\varepsilon,f_1),\ldots,\alpha^n(T,\varepsilon,f_k)] \\ & + [\beta^n(t_1,\varepsilon,f_1),\ldots,\beta^n(T,\varepsilon,f_k)] \\ & + [\gamma^n(t_1,\varepsilon,f_1),\ldots,\gamma^n(T,\varepsilon,f_k)] \\ & + [\Delta^n(t_1,\varepsilon,f_1),\ldots,\Delta^n(T,\varepsilon,f_k)]\end{aligned}$$

By equation (46),(47) and (50) of [15], the first three terms on the right converge in law, in $\mathbb{R}^k$, to the zero vector. Thus, to prove (b) it suffices to show that, for a choice of ε (possibly depending on n)

$$Y^n(\varepsilon) = [\Delta^n(t_1,\varepsilon,f_1),\ldots\Delta^n(T,\varepsilon,f_k)] \xrightarrow{L} Y = [(f_1 \; W(t_1)),\ldots(f_k \; W(T_k)]$$

Since $t \mapsto t\,Q(f,g)$ is continuous for all f, g in E, (C) implies that

$$\sup_{t\le T} |\langle \Delta^n(\cdot,\varepsilon,f) \quad \Delta^n(\cdot,\varepsilon,g)\rangle_t - tQ(f,g)| \xrightarrow{P} 0$$

Applying Lemma 2.1.5, we can get a sequence ε_n decreasing to zero such that for all $f_1,\ldots f_k$ in E

$$\sup_{t\le T} | \; \langle \Delta^n(\cdot,\varepsilon_n,f_i) \quad \Delta^n(\cdot,\varepsilon_n,f_j)\rangle_t - t\,Q(f_i, f_j)| \xrightarrow{P} 0$$

Thus to complete the proof of our theorem it would suffice to show that $Y^n(\varepsilon_n) \xrightarrow{L} Y$

Let $a = (a_1,\ldots,a_k)$ be an arbitrary but fixed vector in $\mathbf{R}^k$. We need to show that

$$E\{\exp(i\, a\cdot Y^n(\varepsilon_n))\} \to E\{\exp(i\, a\cdot Y) = \exp\left(-\frac{\sigma^2(T)}{2}\right),$$

where $$\sigma^2(T) = \sum_{i=1}^{k}\sum_{j=1}^{k} a_i\, a_j\, (t_i \wedge t_j)\, Q\,(f_i,f_j)$$

Set $$N^n(t) = \sum_{i=1}^{k} a_i\ \Delta^n(t_i\wedge t, \varepsilon_n, f_i)$$

Then $(N^n, \underline{F}^n)$ is a locally square integrable martingale with

$$\langle N^n\rangle_t = \sum_{i=1}^{k}\sum_{j=1}^{k} a_i\, a_j \langle\Delta^n(\cdot,\varepsilon_n,f_i)\,\Delta^n(\cdot,\varepsilon_n,f_{j>t_i\wedge t_j\wedge t}$$ and by (C) $\langle N^n\rangle_t \xrightarrow{P} \sigma^2(t)$

for all $0 \le t \le T$. Moreover $\Delta N^n(t) \le (\sum|a_j|)\varepsilon_n \to 0$. Thus, by Proposition 2.1.6 $N^n(T) = a \cdot Y^n(\varepsilon_n) \xrightarrow{L} a \cdot Y$.

Q.E.D.

Proof of Corollary: As stated in [15], (2.2.6) implies that (A) and (sup B) hold. So it suffices to show that 2.2.7 implies (C). To show that, write

$$(f, M^n(t)) = [(f, M^n(t))^c + \int_0^t \int_{|x|\le\varepsilon} x\,(N^n-\hat{N}^n)(f,ds,dx)] + [\int_0^t \int_{|x|>\varepsilon} x(N^n-\hat{N}^n)(f,ds,dx)]$$

$$= \Delta^n(t,\varepsilon,f) + B^n(t,\varepsilon,f)\ .$$

Observe that $B^n(\cdot,\varepsilon,f)$ is a square integrable martingale of finite variation with $\langle B^n(\cdot,\varepsilon,f)\rangle_t = \int_0^t \int_{|x|>\varepsilon} x^2\, \hat{N}^n(f,ds,dx)$.

Thus by (2.2.6) $\langle B^n(\cdot,\varepsilon,f)\rangle_t \xrightarrow{P} 0$.

Also

$$\langle f, A^n(t)g\rangle_{-p_n} = \langle (f\, M^n(\cdot))(g\ M^n(\cdot)\rangle_t$$

$$= \langle\Delta^n(\cdot,\varepsilon,f)\Delta^n(\cdot,\varepsilon,g)\rangle_t + \langle\Delta^n(\cdot,\varepsilon,f)B^n(\cdot,\varepsilon,g)\rangle_t$$

$$+ \langle B^n(\cdot,\varepsilon,f)\Delta^n(\cdot,\varepsilon,g)\rangle_t + \langle B^n(\cdot,\varepsilon,f)B^n(\cdot,\varepsilon,g)\rangle_t$$

It follows by (2.2.6) and the Kunita-Watanabe inequality that the last three terms converge to zero in probability showing that (C) holds.

Q.E.D.

We now study the weak convergence of semi-martingales taking values in the Hilbert space E'_q, to a E'_q Wiener martingale. Under an additional condition on the semi-martingales $X^n(\cdot)$, it is shown that conditions (A), (sup B) and (C) of Theorem 2.2.4 suffice to prove weak convergence in the Polish space $D(E'_q)$.

2.2.8 Theorem: Let $(X^n(\cdot), \underline{F}^n)$ be a sequence of E'-valued semi-martingales satisfying $X^n(0) \equiv 0$ a.s. and

$$P[\sup_{0\le t\le T} |(f, X^n(t))| > \varepsilon] \le C_T \frac{\|f\|_k}{\varepsilon}$$

for some $k \ge 1$, $C_T > 0$ and for all f in E, $\varepsilon > 0$, $T > 0$.
Let $(W(\cdot), \underline{F})$ be a E'_k-Wiener martingale, with covariance operator Q. If X^n satisfies conditions (A), (sup B) and (C) of Theorem 2, then $X^n \xrightarrow{L} W$ in $D(E'_q)$ for some $q > k$.

Proof: By Theorem 1.4.1 the X^n's are uniformly p-continuous. Hence conditions (A), (sup B) and (C) suffice to show that $X^n \xrightarrow{L} W$ in $D(E'_q)$ for some $q > k$.

Q.E.D.

2.2.9 Corollary: Let $(M^n(\cdot), \underline{F}^n)$ be a sequence of locally square integrable E'-valued martingales satisfying $E(f, M^n(t))^2 \le C_t \|f\|_k$ for some $k > 0$, $C_t > 0$ and for all $t > 0$, f in E, $n \ge 1$. Let $(W, \underline{F})$ be an E'_k-Wiener martingale. If M^n satisfies (2.2.6) and (2.2.7) of Corollary 2.2.5, then $M^n \xrightarrow{L} W$ in $D(E'_q)$ for some $q > k$.

Proof: By Doob's inequality

$$P[\sup_{0\le t\le T} |(f, M^n(t))| > \varepsilon] \le \frac{4}{\varepsilon^2} E(f, M^n(T))^2 \le \frac{4C_T}{\varepsilon^2} \|f\|_k$$

Thus M^n is uniformly k-continuous and the result follows by Theorem 2.2.8.

Q.E.D.

We now prove a few results as applications of Theorems 2 and 3.

2.3 CLT and functional CLT for E'-valued triangular arrays.

2.3.1 Theorem: Let $\{X_{nk},\ k=1,\ldots k_n,\ n=1, 2,\ldots\}$ be a triangular array of E'-valued random variables defined on a probability space (Ω, F, P) such that for each f in E, $\{(f\ X_{nk}),\ k=1,\ldots,k_n\}$ is an i.i.d. sequence for each n. Let $(W(\cdot), \underline{F})\ 0 \le t \le 1)$ be a E'-Wiener martingale with Q as its covariance operator.

Suppose for every f, g in E, we have

$$k_n P[|(f, X_{n1})| > \varepsilon] \to 0 \quad \text{for every} \quad \varepsilon > 0$$

$$k_n |E[(f,\ X_{n1})[|(f, X_{n1})| \le \varepsilon]| \to 0 \qquad \text{for every} \quad \varepsilon > 0$$

$$k_n \operatorname{Cov}\left((f, X_{n1})[|(f, X_{n1})| \le \varepsilon],\ (g, X_{n1})[|(g, X_{n1})| \le \varepsilon]\right) \to Q(f, g)$$

Then,

$$S_n = \sum_{j=1}^{k_n} X_{nj} \xrightarrow{L} W(1, \cdot) \qquad \text{in} \quad E'$$

and the process X^n given by

$$X^n(t) = \sum_{j=1}^{[k_n t]} X_{nj} \qquad 0 \le t \le 1$$

satisfies $X^n \xrightarrow{L} W$ in $D([0\ 1]; E')$

Proof: Let $F_t^n = \sigma\{X^n(s, \cdot),\ s \le t\}$. $\underline{F}^n\ (= F_t^n \quad 0 \le t \le 1)$

Then $(X^n, \underline{F}^n)$ is a E'-valued semi-martingale with a canonical decomposition, in which $B^{nc}(t, f) \equiv 0$, $(f, X^n(t))^c \equiv 0$ for all t in $[0\ 1]$, f in E.

$$\tilde{N}^n(f, (0\ t], A) = [k_n t]\, P[(f\ X_{n1}) \in A]$$

By Theorem 2.2.4, it follows now, that $X^n \xrightarrow{L} W$ in $D([0\ 1]; E')$. The map $h : D([0\ 1]; E') \to E'$ given by $h(x) = x(1)$ is obviously a continuous map. Thus by Proposition 2.1.3,

$$X^n(1) = S_n \xrightarrow{L} W(1) \quad \text{in } E'$$

2.4 Example. Let $(\Omega, F, \underline{F}, P)$ be a complete probability space with a filtration satisfying the usual conditions. In what follows we will use the notation and results developed in [9] (Chapters 1 and 3).

Let $s \mapsto Y(s)$ be a $\underline{F}$-adapted process on (Ω, F, P) taking values in $\mathbb{R} \times \mathbb{R}^d$, which is quasi-left continuous (cf. definition 3.1 in [9].)

Let $N(ds, da, dx)$ denote the counting measure associated with p i.e.

$$N(t, A \times B) = N((0\ t] \times A \times B) = \sum_{s \le t} I_{A \times B}(Y(s))$$

for $A \in \mathcal{B}(\mathbb{R})$ and $B \in \mathcal{B}(\mathbb{R}^d)$.

Let $\hat{N}(ds, da, dx)$ denote the compensator of N.

We shall make the following assumption on $\hat{N}$.

(2.4.1) For every $t > 0$, $E \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}^d} a^2 f^2(x)\, \hat{N}(ds, da, dx) \le C\|f\|_k$ for all $f \in S(\mathbb{R}^d)$, where $\| \ \|_k$ is as defined in 1.1.1. C - a positive constant, for some $k \ge 1$.

Then (see page 63 of [9]) the real valued process defined by

$$X(t, f) = \int_{(0\ t] \times R \times R^d} a\, f(x)\, [N(ds,da,dx) - \hat{N}(ds,da,dx]$$

is a $\underline{F}$-P square integrable martingale with

$$EX^2(t, f) = E \int_{(0\ t] \times R \times R^d} a^2\, f^2(x)\, \hat{N}(ds,da,dx) \le C\|f\|_k .$$

By Proposition 2.1.1, we can find a E_q-version of the stochastic process X, denoted, X again for a $q \geq k$ such that for all $t \geq 0$ $X(t, f) = (f\ X(t))$.

In [13], the interpretation for $N(ds, da, dx)$ is that $N(t, AxB)$ is the number of voltage pulses of sizes $a \in A \subset \mathbb{R}$, arriving at sites $x \in B \subset \mathbb{R}^d$ at times $s \leq t$.

Let $(\Omega^n, F^n, \underline{F}^n, P^n)$ be a sequence of complete probability spaces with filtrations $\underline{F}^n$ satisfying the usual conditions.

Let $N^n(ds, da, dx)$ be a sequence of point processes on $(\Omega^n, F^n, \underline{F}^n, P^n)$ of the type described earlier, with their compensators $\hat{N}^n(ds, da, dx)$ satisfying assumption 2.4.1. Construct the E_q'-valued processes X^n as before by

$$(f\ X^n(t)) = \int_0^t \int_{\mathbb{R}x\mathbb{R}^d} a\ f(x)\ (N^n - \hat{N}^n)(ds, da, dx)\ .$$

(2.4.2) Theorem. Suppose that for all $(t, f, g) \in (\mathbb{R} \times S(\mathbb{R}^d) \times S(\mathbb{R}^d))$ we have

(2.4.3) $$\int_0^t \int_{\mathbb{R}x\mathbb{R}^d} |a\ f(x)|\hat{N}^n(ds, da, dx) \xrightarrow{P} 0$$

2.4.4 $$\int_0^t \int_{\mathbb{R}x\mathbb{R}^d} a^2\ f(x)\ g(x)\ \hat{N}^n(ds, da, dx) \xrightarrow{P} t\ \langle f\ g\rangle$$

with $\langle f\ g\rangle = \langle f\ g\rangle_0$ as in 1.1.2.

Then $X^n \Rightarrow W$ where W is a Wiener martingale in $S(\mathbb{R}^d)$.

Remark: Observe that the above conditions are in terms of $\hat{N}^n$ and not in terms of the compensator of the jump measure corresponding to X^n, as are the conditions of Theorem 2.2.4.

Proof: Observe that by 2.4.4 the sequence $\{(f\ X^n)\ n \geq 1\}$ is tight in D and hence by Theorem 1.4.1, X^n is tight in $D(S'(\mathbb{R}^d))$

Note that

$$\Delta(f, X^n(s)) = \int a\ f(x)\ N^n(\{s\}, da, dx)$$

Thus
$$\sup_{t \leq T} |\Delta(f, X^n(s))| \leq \sup_{s \leq T} \int |a\ f(x)|\ N^n(\{s\}, da, dx)$$
$$\leq \int_0^T \int_{\mathbb{R}x\mathbb{R}^d} |a\ f(x)| N^n(s, da, dx)$$

Clearly $\int_0 \int_{\mathbf{R}\times\mathbf{R}^d} |a\, f(x)|(N^n - \tilde{N}^n)(s, da, dx)$ is $\underline{F}^n$- P^n martingale.

Therefore by 2.4.3 and Lenglart's inequality (cf. Corollary to Lemma 1 in [5]), we have

$$\sup_{s\leq T} |\Delta(f, X^n(s))| \xrightarrow{P} 0 \tag{2.4.4}$$

Let $\nu^n(f, ds, dx)$ denote the compensator of $\mu^n(f, ds, dx)$--the integer valued random measure corresponding to the jumps of (f, X^n). As in [15], (2.4.4) implies that

$$\int_0^T \int_{|x|>\varepsilon} \nu^n(f, ds, dx) \xrightarrow{P} 0 \quad \text{for any } \varepsilon > 0 \tag{2.4.5}$$

and

$$\int_0^T \int_{|x|>1} |x|\, \mu^n(f, ds, dx) \xrightarrow{P} 0$$

Thus once again by Lenglart's inequality we have

$$\int_0^T \int_{|x|>1} |x|\, \nu^n(f, ds, dx) \xrightarrow{P} 0 \tag{2.4.6}$$

From the proof of Corollary 2 in [15], we have

$$\beta^n(t, \varepsilon, f) = -\int_0^t \int_{|x|>1} x\, \nu^n(f, ds, dx)$$

where β^n is as in (2.2.3). Equation (2.4.6) now implies (sup B) of Theorem 2.2.4, finishing off the proof.

Q.E.D.

CHAPTER III

DISTRIBUTION VALUED SDE'S AND THEIR WEAK CONVERGENCE

We will prove here the existence and uniqueness (upto a modification) of a cadlag adapted process, taking values in $S'(\mathbb{R}^d)$--the space of tempered distributions on $\mathbb{R}^d$. These processes could be thought of as generalizations of linear stochastic differential equations. We then take up the weak convergence of such processes to a generalized Ornstein-Uhlenbeck (O-U) process.

The analogy to keep in mind is the following. Let $(\Omega, F, \underline{F}, P)$ be a probability with a filtration $\underline{F}$ satisfying the usual conditions. Let b be a constant dxd matrix and $B(\cdot)$ a d-dimensional $\underline{F}$-P Brownian motion. Then there exists a unique (upto P equivalence) $\mathbb{R}^d$-valued cadlag adapted process $X(\cdot)$ called the O-U process satisfying

$$X(t) = X(0) + \int_0^t b \cdot X(s)ds + B(t) \quad \text{a.s. } P \tag{3.0.1}$$

In fact if we define $e^{tb} = \sum_{k=0}^{\infty} \frac{t^k}{k!} b^k$ then

the solution is given by

$$X(t) = e^{bt} \cdot X(0) + e^{bt} \cdot \int_0^t e^{-bs}\, dB(s) \tag{3.0.2}$$

as can be easily seen by Ito's formula.

Observe also that if we define a semi-group of operators T_t from $\mathbb{R}^d$ to $\mathbb{R}^d$ by $T(t)x = e^{bt}x$ for $x \in \mathbb{R}^d$, then its infinitesimal generator A, is simply the matrix b. Thus equation 3.0.1 in terms of this A would read

$$X(t) = X(0) + \int_0^t A \cdot X(s)ds + B(t) \tag{3.0.3}$$

Our first aim is to generalize this result to processes, $X(\cdot)$, taking values in $S'(\mathbb{R}^d)$ with more general semi-groups T_t and the Brownian term replaced by $S'(\mathbb{R}^d)$-valued square integrable martingales, i.e., equations of the form

$$(f\ X(t)) = (f\ X(0)) + \int_0^t (Af\ X(u-))du + (f\ M(t)) \quad \text{a.s. } P \tag{3.0.4}$$

for all f in $S(\mathbb{R}^d)$, $\forall\ t \geq 0$.

In the case when the martingale $M(\cdot)$ is a Wiener martingale in $S'(\mathbb{R}^d)$, the existence and uniqueness in law of a process satisfying (3.0.4) was proved using a martingale problem approach in [7]. They called such a process a generalized O-U process. When $M(\cdot)$ is a generalized, centered Poisson process, and the semi-groups T_t satisfy certain assumptions, then the existence and uniqueness of process of the form 3.0.4 was proved in [13], for general nuclear spaces. In case of $S'(\mathbb{R}^d)$-valued processes, their assumptions on T_t are not necessary as we shall show. At a later stage we shall explain how each of these methods is connected to our method.

If M^n's are a sequence of locally square integrable $S'(\mathbb{R}^d)$-valued martingales converging weakly to a Wiener martingale, then one expects the corresponding solutions $X^n(\cdot)$ to converge weakly to a generalized O-U process. We prove this. It must be pointed out that in [8] Holley and Stroock have also studied similar limit theorems. However their methods are different from ours. Roughly speaking, they study the weak convergence of processes of the form (4), where the real square integrable martingales $(f, M^n(\cdot))$ have associated increasing processes which admit a density with respect to Lebesgue measure, for every f in $S(\mathbb{R}^d)$. In [13], weak convergence of such processes is studied where M^n's are centered Poisson processes. Again, their methods are particular to the properties of Poisson processes, and do not carry over to the general case.

The generalized O-U process turns out to be a useful limiting case in the study of infinite particle systems ([8]). More recently, these processes have played an important role in the development of the Malliavin calculus ([22]). Processes of the form (3.0.4) have been used to model neuronal behavior in the nervous system ([13].

Throughout we shall make the following assumptions. We will call the space $S(\mathbb{R}^d)$ by E and its dual by E'. Thus E_0 is the space $L^2(\mathbb{R}^d)$.

Let $A : E \to E$ be a bounded linear operator admitting a dissipative extension $\bar{A}$ (i.e., $\langle \bar{A} x\ x \rangle \leq 0$ for all $x \in E_0$). Assume there is a strongly continuous semi-group of bounded linear operators $\{T_t, t \geq 0\}$ on E into itself such that

$$x(T_t f) - x(f) = \int_0^t x(AT_s f)ds \qquad \forall\ t \geq 0$$

x in E' and f in E. As mentioned in [7], it is easily checked that $T_t f = e^{t\bar{A}} f$ a.e. for f in E (where $e^{t\bar{A}}$ is the semi-group of self-adjoint contractions on E_0, generated by $\bar{A}$) and $AT_t f = T_t Af$.

Let $(\Omega, F, \underline{F}, P)$ be a complete probability space, with a filtration satisfying the usual conditions. Let $(M(\cdot), \underline{F}, P)$ be a E_p'-valued locally square integrable martingale for some $p \geq 1$ and satisfying moreover $M(0) \equiv 0$, $E\|M(t)\|_{-p}^2 < c < \infty$ for all $t \geq 0$, and some $c > 0$.
Let N be a E_p'-valued random variable satisfying $E\|N\|_{-p}^2 < c$.

3.1.1 Theorem. There exists a unique (upto a modification) adapted E_p'-valued cadlag process $X(\cdot)$ satisfying

$$(3.1.2)\quad (\cdot\ X(t)) = (\cdot\ N) + \int_0^t (A\cdot\ X(u-))du + (\cdot, M(t)) \quad \text{a.s. } P\ \forall\ t \geq 0$$
$$\text{with } E\|X(t)\|_{-p}^2 \leq 2c \qquad \forall\ t \geq 0$$

Proof: Let $D_n = \text{card}\{\alpha : |\alpha| \leq n\}$ and $b_{\alpha\beta} = \langle Ah_\alpha\ h_\beta \rangle$. Consider the system of D_n real-valued stochastic differential equations given by

$$X_\alpha^n(t) = N(h_\alpha) + \int_0^t \Big(\sum_{|\beta| \leq n} b_{\alpha\beta}\ X_\beta^n(u-)\Big)du + M(t)(h_\alpha) \quad \text{a.s. } P$$

for $|\alpha| \leq n$. Set $X_\alpha^n(t) \equiv 0 \quad \forall\ |\alpha| > n$.

Since the coefficient of the drift term-- $\sum_{|\beta| \leq n} b_{\alpha\beta}\ x_\beta$ is clearly Lipschitz in x, independent of t and $\sum_{|\beta| \leq n} b_{\alpha\beta}\ X_\beta^n(u-)$ is F_u measurable, we have by the theorem of Doléans-Dade [4], that the systems of equations has a unique cadlag adapted solution.

Let $n > m$. Consider

$$X^n_\alpha(t) - X^m_\alpha(t) = \begin{cases} \int_0^t \sum_\beta b_{\alpha\beta}(X^n_\beta(u-) - X^m_\beta(u-))du & \text{if } |\alpha| \le m \\ N(h_\alpha) + \int_0^t \sum_\beta b_{\alpha\beta} X^n_\beta(u-)du + M(t)(h_\alpha) & \text{if } m < |\alpha| \le n \end{cases}$$

and the function $F \in C^2(\mathbb{R})$ given by $F(x) = x^2C(\alpha)$

where $C(\alpha) = (2|\alpha| + d)^{-p}$

Then by Ito's formula([12]), we have for $|\alpha| \le m$,

$$F(X^n_\alpha(t) - X^m_\alpha(t)) = 2C(\alpha) \int_0^t (X^n_\alpha(s-) - X^m_\alpha(s-)) \sum_\beta b_{\alpha\beta} (X^n_\beta(s-) - X^m_\beta(s-))ds .$$

For $m < |\alpha| \le n$ we get

$$F(X^n_\alpha(t) - X^m_\alpha(t)) = C(\alpha)N^2(h_\alpha) + 2C(\alpha) \int_0^t X^n_\alpha(s-) \sum_\beta b_{\alpha\beta} X^n_\alpha(s-)ds$$

$$+ 2C(\alpha) \int_0^t X^n_\alpha(s-)dM(s)(h_\alpha) + C(\alpha) \langle M^c(\cdot)(h_\alpha)\rangle_t +$$

$$C(\alpha) \sum_{0<s\le t} [X^n_\alpha(s)^2 - X^{n^2}_\alpha(s-) - 2X^n_\alpha(s-)\Delta X^n_\alpha(s)]$$

Since, $\Delta X^n_\alpha(s) = \Delta M(s)(h_\alpha)$ the sum of the last two terms is simply

$C(\alpha)[\langle M^c(\alpha)(h_\alpha)\rangle_t + \sum_{s\le t} \Delta M^2(s)(h_\alpha)]$. This equals

$$C(\alpha)[M(\cdot)(h_\alpha)]_t \quad (\text{where } [M(\cdot)(h_\alpha)]_t = \langle M^c(\cdot)(h_\alpha)\rangle_t + \sum_{s\le t} \Delta M^2(s)(h_\alpha)) .$$

Thus if we define $X^n(t) = \sum_\alpha X^n_\alpha(t)h_\alpha$, then

$$\|X^n(t) - X^m(t)\|^2_{-p} = \sum_\alpha F(X^n_\alpha(t) - X^m_\alpha(t)$$

$$= \sum_{|\alpha|=m+1}^n C(\alpha) N^2(h_\alpha) + 2 \int^t \langle A(\tilde{X}^n(s-) - \tilde{X}^m(s-)) \quad \tilde{X}^n(s-) - \tilde{X}^m(s-)\rangle ds$$

$$+ \sum_{|\alpha|=m+1}^n C(\alpha) \int_0^t X^n_\alpha(s-)dM(s)(h_\alpha) + 2 \sum_{|\alpha|=m+1}^n C(\alpha)[M(\cdot)(h_\alpha)]_t .$$

By the dissipativity of A , we have on taking expectations

$$E\|X^n(t) - X^m(t)\|^2_{-p} \le E \sum_{|\alpha|=m+1}^n C(\alpha)N^2(h_\alpha) + E \sum_{|\alpha|=m+1}^n C(\alpha)M(t)^2(h_\alpha) .$$

We have used here the fact([12]) that

$$M^2(\cdot)(h_\alpha) - [M(\cdot)(h_\alpha)]_t \text{ is a } \underline{F}\text{-P martingale}$$

As $E\|N\|_{-p}^2 = E \sum_{\alpha} C(\alpha)N^2(h_\alpha) < C$ and $E\|M(t)\|_{-p}^2 < C$,

we have

$$\lim_{m\to\infty} \sup_{n>m} E \int_0^T \|X^n(t) - X^m(t)\|_{-p}^2 \, dt = 0 \quad \forall \ T > 0 .$$

Thus by the Riesz-Fischer theorem there exists an adapted process $\tilde{X}(\cdot)$ in E_p' such that

$$\lim_{n\to\infty} E \int_0^T \| \tilde{X}(t) - X^n(t) \|_{-p}^2 \, dt = 0$$

Set $X(t) = N + \int_0^t (A \quad \tilde{X}(u-))du + M(t)$

Clearly, X is cadlag, adapted. Observe that $X^n(t)$ converges in norm to $\tilde{X}(t)$ and hence converges weakly to $\tilde{X}(t)$ in $H = L^2((\Omega, F, P); E_p')$ in the sense that for every f in E_p we have

$$E(f, X^n(t)) \to E(f, \tilde{X}(t))$$

We will now show that $X^n(t)$ also converges weakly to $X(t)$ in H . Since weak limits in Hilbert spaces are unique, this will imply that $X(t) = \tilde{X}(t)$ a.s. P for each $t \geq 0$ and thus

$$X(t) = N + \int_0^t (A, \quad X(u-))du + M(t) \qquad \text{a.s. } P \cdot$$

$\forall \ t \geq 0$.

Observe that N converges weakly to N and $M(t)$ converges weakly to $M(t)$ in H ! Thus to show that $X^n(t)$ converges weakly to $X(t)$, it suffices to show that $E \int_0^t (Af, X^n(u-))du$ converges to $E \int_0^t (Af \ \tilde{X}(u-))du$. But $X^n(\cdot)$ converges in norm to $X(\cdot)$ in the Hilbert space $K = L^2([0\ T] \times \Omega \mapsto E_p')$ $= \{f : [0\ T] \to E_p' \text{ such that } \int_0^t E \|f(t)\|_{-p}^2 \, dt < \infty\}$. Therefore X^n converges weakly to $\tilde{X}(\cdot)$ in K . This by definition implies that $E \int_0^t (Af, X^n(u-))du$ converges to $E \int_0^t (Af, \tilde{X}(u-))du$ for any $t \geq 0$.

This finishes the proof of the existence.

To prove the uniqueness, let $X(\cdot)$, $Y(\cdot)$ be two cadlag adapted E_p'-valued processes satisfying (1). We will show that $X(t) = Y(t)$ a.s. P $\forall$ $t \geq 0$. To do this it suffices to show that $X(t)(h_\alpha) = Y(t)(h_\alpha)$, since the action of $X(t)$ or $Y(t)$ on E_p' is given by

$$(f, X(t)) = \sum_\alpha \langle f, h_\alpha \rangle \; X(t)(h_\alpha) .$$

For any $n \geq 1$, define the operator $P_n : E_0 \to E$ by

$$P_n f = \sum_{|\alpha| \leq n} \langle f, h_\alpha \rangle h_\alpha$$

Observe that for $t \geq 0$ and $|\alpha| \leq n$.

$$X(t)(h_\alpha) - Y(t)(h_\alpha) = \int_0^t (Ah_\alpha, X(u-) - Y(u-))du .$$

Set

$$Z(t) = \sum_{|\alpha| \leq n} (X(t) - Y(t))(h_\alpha)h_\alpha$$

then $P_n Z(t) \in E_0$ $\forall$ $n \geq 1$. By applying Ito's formula to the function $x \mapsto \sum_{|\alpha| \leq n} x_\alpha^2$ for $x \in \mathbf{R}^{D_n}$, we have

$$\|P_n Z(t)\|^2$$

$$= \sum_{|\alpha| \leq n} [(X(t) - Y(t))(h_\alpha)]^2 = 2 \int_0^t \langle AP_n Z(u-) \; P_n Z(u-) \rangle du$$

By the dissipativity of A , and the above equation, we have

$$\|P_n Z(t)\|^2 \leq 0 \qquad \text{a.s.} \quad P .$$

Since n was arbitrary this gives us the uniqueness.

Lastly, since $E\|X^n(t) - X(t)\|_{-p}^2 \to 0$ and $E\|X^n(t)\|_{-p}^2 \leq 2C$ we get $E\|X(t)\|_{-p}^2 < 2C$ $\forall$ $t \geq 0$.

Q.E.D.

3.1.3 Remark: We have not shown here that $E \sup_{t \leq T} \|X(t)\|_{-p}^2 < \infty$ for any $T > 0$ as has been shown in [7], [13]. Although, we will not go into this we can, using the Ito formula for Hilbert-valued semi-martingales (see [16]), prove a similar result.

As promised earlier, we will compare Theorem 1, with the corresponding theorems of [7], [13].

Let $(W(\cdot), \underline{F})$ be a P-Wiener martingale in E_p'. Theorem 3.1.1, then tells us that there is a continuous adapted process $X(\cdot)$ in E_p' such that

$$(\cdot, X(t)) = N + \int_0^t (A\cdot, X(u))du + (\cdot, W(t)) \quad \text{a.s. } P$$

Assume that $N \equiv 0$. Then for every f in E, we have

$$(f, X(t)) - \int_0^t (Af, X(u))du = (f, W(t)) \quad \text{a.s. } P$$

So if the covariance operator Q associated with $W(\ )$ is $Q(f, g) = \langle Bf\ Bg\rangle$ for a bounded linear operator B from E_0 to E_0, then by Ito's formula

$$(3.1.4)\quad F((f,X(t)) - \int_0^t (Af, X(u))du) - \frac{\|Bf\|^2}{2} \int_0^t F''((f,X(u)) - \int_0^u (Af, X(s))ds)du$$

is a $\underline{F}$-P martingale for $F \in C_o^2(\mathbb{R})$.

Conversely if (3.1.4) is $\underline{F}$-P martingale for every $F \in C_o^2(\mathbb{R})$, then by choosing $F(x) = x$ and $F(x) = x^2$, we have

$$t \mapsto (f, X(t)) - \int_0^t (Af, X(u))du = Y(t, f)$$

$$t \mapsto [(f,X(t)) - \int_0^t (Af, X(u))du]^2 - t\|Bf\|^2 .$$

are both continuous local martingales. By Lévy's characterization of Brownian motion, it is seen that $Y(\cdot, f)$ is a Brownian motion with covariance operator Q defined earlier. Holley and Stroock make (3.1.4) their defining condition for the O-U process and then proceed to show the existence and uniqueness of a measure P, satisfying (3.1.4). They explicitly find the conditional distribution of $(f, X(t))$ given $\mathcal{F}_s$ for $s < t$. Their method however, will fail if we replace the Wiener martingale by a general locally square integrable martingale, for then the Markovian structure of the process $X(\cdot)$ is destroyed.

To see how the method of Kallianpur and Wolpert [13] works assume that the matrix of $A(= ((b_{\alpha\beta})))$ is diagonal with respect to the orthonormal basis (h_α) i.e., $\langle Ah_\alpha, h_\beta\rangle = b_{\alpha\beta} = 0$ if $\alpha \neq \beta$. By Theorem 3.1.1 we get the

existence of a unique E_p'-process $X(\cdot)$ satisfying

$$(h_\alpha, X(t)) = N(h_\alpha) + \int_0^t b_{\alpha\alpha} X(u-)du + M(t)(h_\alpha) \quad \text{a.s. } P$$

By Ito's formula we get

$$(3.1.5) \qquad (h_\alpha, X(t)) = e^{tb_{\alpha\alpha}} N(h_\alpha) + \int_0^t e^{(t-s)b_{\alpha\alpha}} d(M(s)(h_\alpha)$$

Thus $X(\cdot)$ can be explicitly solved in terms of M, N. In [13], instead of assuming that the matrix of A is diagonal, they impose conditions on $\bar{A}$ which ensure that $-\bar{A}$ has countably many eigen-vectors $f_1, f_2, f_3, \ldots$ and corresponding eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots$ satisfying a growth condition of the form $\sum(1 + \lambda_j)^{-2r_1} < \infty$ for some $r_1 > 0$. An ONB for $L^2(\mathbb{R}^d)$ is constructed using these f_j's, which diagonalize the matrix of $\bar{A}$. The growth condition on the eigenvalues enables them to construct a nuclear space $E \subset L^2(\mathbb{R}^d)$. If a process is now defined by (3.1.5), they show that it has E_q'-version for some $q \geq 1$ and solves (3.1.2). Clearly such a program can be carried out for more general L^2 spaces than $L^2(\mathbb{R}^d)$ and this is exactly what they do. But as our theorem shows, in the case of $L^2(\mathbb{R}^d)$ their assumptions are not necessary.

3.2 Weak convergence to an O-U process.

We turn next to the study of weak convergence of E'-valued processes of the form (1). Suppose $M^n(\cdot)$ is a sequence of E_p'-valued locally square integrable martingales satisfying $E\|M^n(t)\|_{-p}^2 < C$ say, on a filtration $(\Omega^n, F^n, \underline{F}^n, P^n)$. Suppose also that N^n is a sequence of E_p'-valued random variables satisfying $E\|N^n\|_{-p}^2 < C$. Let $X^n(\cdot)$ be the unique E_p'-valued cadlag adapted process satisfying

$$(\cdot, X^n(t)) = (\cdot, N^n) + \int_0^t (A\cdot, X^n(u-))du + (\cdot, M^n(t)) \quad \text{a.s. } P^n.$$

It seems intuitively clear that such equations should be robust in the sense that if M^n converges weakly to a Wiener martingale in $D(E_p')$, then the corresponding solutions should converge weakly, too. This is what we will set

out to do now. We shall put conditions on M^n that ensure its weak convergence to W and prove under these conditions the weak convergence of X^n to an O-U process.

We start by making the setting of our problem more precise.

Let $(\Omega^n, F^n, \underline{F}^n, P^n)_{n\geq 1}$, $(\Omega, F, \underline{F}, P_N)$ be a sequence of filtrations satisfying the usual conditions.

We shall say that a continuous E'-valued process $X(\cdot)$ is a $\underline{F}$ - P_N O-U process starting at N in E' and characteristic A if

$$P_N\,[X(0) = N] = 1 \qquad \text{and}$$

$\forall\ F \in C_o^\infty(\mathbf{R})$ we have

$$(3.2.1)\quad F((f, X(t)) - \int_0^t (Af, X(u))du) - \frac{\|f\|^2}{2}\int_0^t F''((f, X(u)) - \int_0^u (Af, X(s))ds)du$$

is $\underline{F}$ - P_N martingale.

Let N^n be a sequence of E_p'-valued random variables satisfying $E\|N^n\|_{-p}^2 \leq C$ for some $p \geq 1$, $C > 0$.

Let $(M^n, \underline{F}^n, P^n)$ be a sequence of locally square integrable E_p' martingales satisfying $M^n(0) \equiv 0$ $\forall$ $n \geq 1$ and $E\|M^n\|_{-p}^2 \leq C$.

Let $\tilde{N}^n(f, ds, dx)$ denote the dual predictable projection of the integer valued measure corresponding to the jumps of $(f\ M^n(\cdot))$.

Let $X^n(\cdot)$ denote the unique, cadlag, adapted, E_p'-valued process satisfying

$$(3.2.2)\quad (f, X^n(t)) = (f, N^n) + \int_0^t (Af, X^n(u-))du + (f, M^n(t)) \quad \text{a.s. } P^n$$

with $E\|X^n(t)\|_{-p}^2 \leq 2C$.

3.2.3 Theorem. Suppose for all (ε, f, t) in $(0\ 1] \times E \times \mathbf{R}_+$ we have

$$(3.2.4)\quad \int_0^t \int_{|x|>\varepsilon} x^2\ \tilde{N}^n(f, ds, dx) \xrightarrow{P} 0 \qquad \text{as } n \to \infty$$

$$(3.2.5)\quad \langle (f\ M^n)\ (g\ M^n)\rangle_t \xrightarrow{P} t\,\langle f\ g\rangle \qquad \text{as } n \to \infty \quad \forall\ g \text{ in } E$$

(3.2.6) $N^n(f) \Rightarrow N(f)$ in $\mathbb{R}$, for a deterministic N in E.

Then $X^n \Rightarrow X$ in $D(E'_q)$ for a $q \geq p$ where X is an O-U process starting at N with characteristic A.

Proof: Our first task will be to show that the X^n's are uniformly p_o-continuous for a $p_o \geq 1$ i.e. $\forall\ \varepsilon > 0$, $\forall\ \rho > 0$, $T > 0$ $\delta > 0$, such that

$$P^n[\sup_{t \leq T} |(f, X^n(t))| > \varepsilon] \leq \rho \text{ if } \|f\|_{p_o} \leq \delta \text{ for all } n \geq 1.$$

To this end, observe that

$$|(f, X^n(t))|^2 \leq 3|(f, N^n)|^2 + 3t \int_0^t |(Af, X^n(u-)|^2 du + |(f, M^n(t))|^2$$

Thus

$$E \sup_{t \leq T} |(f\, X^n(t))|^2 \leq 3E(f\, N^n)^2 + 3T \int_0^T E(Af\, X^n(u-))^2 du + 3E \sup_{t \leq T} (f\, M^n(t))^2$$

$$\leq 3C\|f\|_p^2 + 3T \int_0^T E\|X^n(u-)\|_{-p}^2 \|Af\|_p\, du + 3E(f\, M_T^n)^2$$

$$\leq 3C\|f\|_p^2 + 3T^2 \cdot 2C \cdot \|Af\|_p + 3C\|f\|_p^2$$

Observe that since A is a bounded linear operator from E to E, there exist $p_o \geq p$ such that $\|Af\|_p \leq K\|f\|_{p_o}$ for all f in E. So we get

$$E \sup_{t \leq T} (f, X^n(t))^2 \leq \|f\|_{p_o}^2 (3C + 6CKT^2 + 3C)$$

proving that X^n's are uniformly p_o-continuous.

According to Theorem 1.4.1, if X^n's are uniformly p_o-continuous, and tight in $D(E')$, then they are tight in $D(E'_q)$ for some $q \geq p_o$.

To show that X^n's are tight in $D(E')$ it suffices by Theorem 1.4.1 to show that for any f in E, $(f, X^n(\cdot))$ is tight in D. To do this it suffices to show that $N^n(f)$ is tight in $\mathbb{R}$, $Y^n(\cdot, f) = \int_0^{\cdot} (Af, X^n(u-))du$ is tight in D and $(f, M^n(\cdot))$ is tight in D.

By (3.2.6), $N^n(f)$ is tight in $\mathbb{R}$. By 3.2.5 and Theorem 1 of [15],

$(f, M^n(\cdot))$ is tight in D.

We are left to show that $\{Y^n(\cdot, f)\}_{n\geq 1}$ is tight in D.

Let T_ℓ^n be the family of all stopping times with respect to the natural σ field of Y^n and which are bounded by ℓ.

Let $0 < \delta < 1$ and note that for $T \in T_\ell^n$ we have

$$E(Y^n(T+\delta,f) - Y^n(T,f))^2 = E\left(\int_T^{T+\delta} (Af, X^n(u-))du\right)^2 \leq \delta \int_T^{T+\delta} E(Af, X^n(u-))^2 du$$

$$\leq \delta^2 \cdot 2C \cdot \|Af\|_p$$

Thus if T_n is a sequence of stopping times, with T_n in T_ℓ^n and $0 < \delta_n < 1$ is any sequence of numbers with $\delta_n \downarrow 0$ we have

$$E(Y^n(T_n + \delta_n, f) - Y^n(T_n, f))^2 \to 0$$

So by the theorem of Aldous [1], $\{Y^n(\cdot,f)\}_{n\geq 1}$ is tight in D for each f in E.

Consequently $X^n(\cdot)$ is tight in $D(E')$.

To prove the uniqueness of the limit point, notice that N^n converges weakly to N in E', so we can, without loss of generality assume that $X^n(0) \equiv 0 \quad \forall\ n \geq 1$ and show convergence to an O-U process with characteristic A, starting at 0. By conditions 3.2.4 and 3.2.5 and Corollary 2.2.9, $M^n \Rightarrow W$ in $D(E_q')$, where W is a E_q'-Wiener martingale on $(\Omega, F, \underline{F}, P)$.
By Theorem 3.1.1 there exists a unique cadlag adapted E_q' process X such that

$$(f, X(t)) = \int_0^t (Af, X(u))du + (f, W(t)) \quad \text{a.s. } P$$

So we have shown that if X^{n_k} converges weakly then

$$(X^{n_k}(\cdot) - \int_0^\cdot (A\cdot, X^{n_k}(u-))du) \ (= M^n(\cdot)) \Rightarrow (X(\cdot) - \int_0^\cdot (A\cdot, X(u))du)$$

Ito's formula now shows that every weak subsequential limit satisfies

$$F((f, X(t)) - \int_0^t (Af, X(u))du) - \frac{\|f\|^2}{2} \int_0^t F''((f, X(s)) - \int_0^s (Af, X(u))du)ds$$

is an $\underline{F}$-P martingale. By Theorem 1.2.3 of [7] we get $X^n \Rightarrow X$ in $D(E_q')$.

Q.E.D.

3.3 Example. Consider the set up of example 2.4. We shall use the notation of that example.

Thus X^n defined by

$$(f, X^n(t)) = \int_0^t \int_{\mathbb{R}\times\mathbb{R}^d} a\ f(x)(N^n - \hat{N}^n)(ds, da, dx)$$

is a martingale for each f in $S(\mathbb{R}^d)$.

By Theorem 3.1.1 there exists a unique process Y^n in E_q' such that

$$(f, Y^n(t)) = \int_0^t (Af, Y^n(u-))du + (f, X^n(t))$$

for any dissipative operator A satisfying the conditions of Theorem 3.1.1.

By Example 2.4 and Theorem 3.2.3 Y^n converges in law to an O-U process starting at 0 in $D(E_q')$.

NOTATION

E	Frechét Nuclear space
$\|\cdot\|_k$	Hilbertian semi-norms determining the topology of E, $k = 1, 2, \ldots$.
E_k	completion of E with respect to $\|\ \|_k$
E_k'	topological dual of E_k
$\|\ \|_{-k}$, $\langle\cdot, \cdot\rangle_{-k}$	norm and inner product on E_k'
E'	topological dual of E, with the strong dual topology
f, g	generic elements of E
$\mathcal{B}(E')$	the smallest σ-algebra on E' that makes all the maps $x \ni E' \mapsto x(f)$ measurable, for f in E
$\mathcal{B}(E_k')$	the Borel σ-algebra of E_k'
cadlag	right continuous with left limits
$D(\mathbb{R}_+, Z) = D(Z)$	the space of all cadlag maps from $\mathbb{R}_+$ to a topological space Z. Also, $D = D(\mathbb{R}_+, \mathbb{R})$
$\mathcal{B}(D(E'))$	the smallest σ-algebra on $D(E')$ that makes all the maps $z \ni D(E') \mapsto (z(t_1, f_1), \ldots, z(t_k, f_k)) \in \mathbb{R}^k$, measurable for $0 \leq t_1 < t_2 \ldots < t_k$, $f_1, \ldots, f_k$ in E and $k \geq 1$
$\mathcal{B}(D(E_k'))$	the Borel σ-algebra on $D(E_k')$ given by the Skorohod topology in the metric space $D(E_k')$
$(\Omega, F, (F_t), P)$	(Ω, F, P) is a complete probability space with F_t an increasing family of sub-σ-fields of F satisfying the usual conditions, i.e., $F_{t+} = F_t$ and F_0 contains all P-null sets of F
$\underline{F}$	denotes the filtration $(F_t)_{t \geq 0}$.
$\xrightarrow{L}$	convergence in law
$\xrightarrow{p}$	convergence in probability

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