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MARTINGALES OF BANACH-VALUED

RANDOM VARIABLES

By

Srishti Dhar

S. D. CHATTERJI

A THESIS

Submitted to the School for Advanced Graduate
Studies of Michigan State University in
partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Statistics

1960

ACKNOWLEDGMENTS

It is with pleasure that I thank Dr. G. Kallianpur for having drawn my attention to the general area of martingales of Banach-valued r.v.'s. My sincere thanks are also due to Dr. C. Kraft for his kind encouragement to me throughout the period during which the research was conducted.

The writer deeply appreciates the financial support of the National Science Foundation.

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ABSTRACT

The main purpose of the thesis is to consider conditional expectations of r.v.'s which take values in a Banach space and to study the limit properties of certain sequences of such r.v.'s. These sequences are called martingale sequences, following the terminology of Doob. We first of all demonstrate that every Bochner-integrable r.v. has a conditional expectation relative to any Borel-field and establish some of the basic properties of conditional expectations. Then we go on to study the convergence in the mean and convergence almost-everywhere of martingale sequences. This we have done by studying operators on certain generalized Lebesgue-spaces, discussed in our Chapter 11. We have established the generalizations of most of the theorems of the classical theory of martingales and have shown by a counter-example in Chapter 1V that some restrictions on the Banach space in which the r.v.'s take value, are necessary. In the last chapter, we have considered some applications of our theory.

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Introduction

The notion of a measurable function defined on an arbitrary measurable space and taking values in another measurable space is a fairly well-known one in modern mathematics. When the range space of the functions happens to have a topology also, various special concepts of measurability become important. Much research has been carried out, for instance, in the case when the functions take values in a linear topological space or more restrictedly in a Banach or a Hilbert space. A considerable body of the research is devoted to extending suitably the ordinary theory of Lebesgue integrals for real-valued functions. For functions taking values in a Banach space, there exist at least three different important concepts of measurability and integrability. This sort of extension of the theory of real-valued functions has been carried out in recent years in the study of random variables (r.v.) which after all are measurable functions on a finite measure space. Frechet (18)* considered r.v.'s taking values in a metric space and introduced notions of mean and variance for such r.v.'s. Doss (19) considered r.v.'s taking values in topological

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* Numbers in brackets refer to the bibliography at the end.

spaces with uniform structure and proved various generalizations of the classical strong law of large numbers. Many other studies have been made with r.v.'s taking values in locally compact topological groups. But it seems that one can generalize the classical results of probability theory most satisfactorily only when the range space is at least a linear topological space for then much of the usual integration theory remains valid. In this direction, pioneering work was done by Mourier (10) who considered the range space to be a Banach space and not only proved some strong laws but also studies characteristic functionals of r.v.'s taking values in Banach spaces. Since then quite a few papers have been published concerning general strong laws of Banach-valued r.v.'s, e.g. Beck & Schwartz (13), Beck (20). However, to the best of the author's knowledge, no more than one attempt has been made to define an extension of a basic concept of probability theory, namely, the concept of conditional expectation of a r.v. taking values in a Banach space. Beck & Schwartz (13) do define a notion of conditional expectation that we have used here, but they did not make any attempt to prove its existence. Dubins (21) defined a conditional expectation of a more general nature than ours but the difficulty with his definition is that it does not yield an exact analogue of the standard theory. There is a basic difficulty in the process of defining conditional expectations for r.v.'s taking values in spaces like Banach spaces. That difficulty is the non-existence of a general Radon-Nikodym theorem for set functions taking values in non-compact spaces. The definition that we have used circumvents this by considering Bochner-integrals, for which although a general Radon-Nikodym type theorem is not valid, much can be done owing to the simple structure of integrable functions.

Our main purpose here is to study this particular notion of conditional expectations for Banach-valued r.v.'s and then use this definition for considering generalizations of martingale theory for Banach-valued r.v.'s. One of the most important considerations in the study of martingale theory of scalar-valued r.v.'s is that of convergence of the martingales. We have studied this for the case of Banach-valued martingales specially from the point of view of treating conditional expectations as operators on suitable Banach spaces. For instance, our mean convergence theorems in Chapter 111 are reminiscent of the work of Lorch (22) concerning monotone sequences of projections on a reflexive Banach space. Our results on the mean convergence of martingales, specially, have been obtained by simple linear space methods which are different from Doob's (1) approach. For proving almost-everywhere convergence we have used a generalization of a theorem of Banach and thus shown how many of the properties of martingales are simply the properties of a type of sequence of operators on a Banach space.

In Chapter 1 we define our conditional expectation and prove its existence and general properties.

In Chapter 11 we prove for future work weak compactness properties of certain Lebesgue type Banach spaces, some of which at least (e.g. Th. 2.3.1 and Th. 2.4.2) are not to be found in current literature.

In Chapter 111 we consider the mean convergence of Banach-valued martingales. We prove the most general mean convergence theorem here under the assumption that the Banach space is reflexive. As shown by a counter-example in Chapter 1V, it is clear that some such restriction on the Banach space is necessary.

In Chapter 1V we consider the almost everywhere convergence of Banach-valued martingales. We prove three different types of theorems, some using a theorem of Banach, one using Doob's idea of optional stopping and one using results from standard martingale theory.

In Chapter V we consider two different applications of the theory, one to the study of the strong law of large numbers for Banach-valued independent identically distributed r.v.'s and the second to the study of derivatives of Banach-valued measures with respect to nets.

An attempt has been made to construct as far as was possible, a theory based only on linear methods. It is hoped that in the future more powerful linear space methods will make the phenomenon of convergence of martingales of Banach-valued r.v.'s quite transparent to our comprehension.

Notation, some definitions and known theorems

Let Ω be an abstract set of elements or points ω. Sub-sets of Ω will be denoted by upper case Latin letters like A, B, F etc. Given two subsets A and B we shall mean by

$A \subset B$	=	A contained in B
$B \subset A$	=	B contained in A
$A \cup B$	=	union of A and B
$A \cap B$	=	intersection of A and B
A^c	=	the complement of A
$A - B$	=	$A \cap B^c$
$A \Delta B$	=	$(A - B) \cup (B - A)$
ϕ	=	the empty set

If $A \subset B$ and $B \subset A$ then we shall write $A = B$. The symbol "ϵ" shall denote the relationship of an element belonging to a class. We shall occasionally use the symbols "$\ni$" and "$\exists$" as short-hand for the phrases "such that" and "there exist(s)" respectively.

By a field $\mathfrak{F}$ of sets in Ω we shall mean a class of sub-sets such that

i) ϕ and Ω are in $\mathfrak{F}$

ii) If $A \in \mathfrak{F}$, then $A^c \in \mathfrak{F}$

iii) If $A_i \in \mathfrak{F}$, $i = 1, 2, .. n$ where n is a finite postive integer then

$$\bigcup_{i=1}^{n} A_i \in \mathfrak{F}$$

By a Borel-field of sets in Ω we shall mean a class of subsets $\mathfrak{F}$ of Ω such that

i) ϕ and Ω are in $\mathfrak{F}$

ii) If $A \in \mathfrak{F}$ then $A^c \in \mathfrak{F}$

iii) If $A_i \in \mathfrak{F}$ $i = 1, 2, \ldots$ (a denumerable sequence of sets in $\mathfrak{F}$) then $\bigcup_i Ai \in \mathfrak{F}$

.

A "probability space" will be a triple (Ω, $\mathfrak{F}$, P) where Ω is any abstract set, $\mathfrak{F}$, a Borel-field of sets in Ω , and P is a non-negative function defined on $\mathfrak{F}$ such that

i) $P(A) \geqslant 0$ for all $A \in \mathfrak{F}$

ii) If $A_i \in \mathfrak{F}$ $i = 1, 2, \ldots$ then

$$P(\bigcup_i Ai) = \sum_{i=1}^{\infty} P(Ai)$$

iii) $P(\Omega) = 1$.

By a "Banach space" $\mathfrak{X}$ over the complex numbers (for brevity, Banach space) we shall mean a set of elements which is such that

i) It forms a vector space on the field of complex numbers

ii) There is a function $\| x \|$ defined on $\mathfrak{X}$, called norm such that

$\| \lambda . x \| = | \lambda | . \| x \|$ λ any complex number

$\| x + y \| \leqslant \| x \| + \| y \|$

$\| x \| = 0$ if and only if $x = 0$ (the zero element of the vector space)

iii) for any sequence x_i, $i = 1, 2, \ldots$ of elements of $\mathfrak{X}$ for which

$$\lim_{m,\, n \to \infty} \| x_m - x_n \| = 0$$

there exists an element $x \in \mathfrak{X}$ such that

$$\lim_{n \to \infty} \| x_n - x \| = 0$$

A complex-valued function x^* defined on $\mathfrak{X}$ such that

$$x^* (x + y) = x^* (x) + x^* (y)$$

$$x^* (\lambda . x) = \lambda . x^* (x) \qquad \lambda \text{ any complex number}$$

$$|x^* (x)| \leq A . \| x \| \quad \text{for some } A \geqslant 0 \text{ and all}$$

$x \in \mathfrak{X}$ will be called a bounded linear functional on $\mathfrak{X}$. Occasionally we shall use the notation $\langle x, x^* \rangle$ for $x^* (x)$.

With

$$\| x^* \| = \sup \left\{ |x^* (x)| \, ; \, \| x \| \leq 1 \right\}$$

the set of all bounded linear functionals on $\mathfrak{X}$ forms a Banach space $\mathfrak{X}^*$ called the "dual" or "conjugate" of $\mathfrak{X}$. We shall denote by $\mathfrak{X}^{**}$ the dual of $\mathfrak{X}^*$ i.e. $\mathfrak{X}^{**} = (\mathfrak{X}^*)^*$.

If we consider the function $x^{**} (x^*)$ on $\mathfrak{X}^*$ defined by

$$x^{**}(x^*) = x^*(x) \qquad x \in \mathfrak{X}, \ x \text{ fixed.}$$

then x^{**} is a bounded linear functional on $\mathfrak{X}^*$ with

$$\| x^{**} \| = \| x \|$$

If all the bounded linear functionals on $\mathfrak{X}^*$ are of this type then we shall write $\mathfrak{X} = \mathfrak{X}^{**}$ and call $\mathfrak{X}$ a "reflexive" Banach space.

A sequence of elements $x_n \in \mathfrak{X}$ is said to be "weakly convergent" to $x \in \mathfrak{X}$ if

$$\lim_{n \to \infty} x^*(x_n) = x^*(x)$$

for all $x^* \in \mathfrak{X}^*$.

A set of elements $S \subset \mathfrak{X}$ will be said to be "weakly compact" if for any sequence of elements $x_n \in S$ there is a subsequence of elements x_{n_j} which converges weakly to some element x which may or may not belong to S. (Actually, in standard theory, this is called conditionally, sequentially, weakly compact. But because we shall not have occasion to use any other kind of compactness, therefore we prefer this briefer expression. However, the works of Eberlein and Phillips (see Hille & Phillips (3) pp.37) show that in many cases our definition of weak compactness is the same as the notion of compactness under the weak topology of $\mathfrak{X}$ which we do not discuss here.)

The following theorem of Pettis shall be used often: (For proof, see Dunford & Schwartz (2), pp. 68-69).

A set S in a reflexive Banach space is weakly compact if and only if it is bounded i.e. $\{\|x\| : x \in S\}$ is a bounded set on the real line.

A reflexive space is weakly complete i.e. whenever a sequence x_n of elements is such that $\lim_{n \to \infty} x^*(x_n)$

exists for every $x^* \in \mathfrak{X}^*$ there exists an element x such that x_n's converge weakly to x.

A bounded linear operator T from a Banach space $\mathfrak{X}$ to a Banach space $\mathfrak{Y}$ is a function on $\mathfrak{X}$ taking values in $\mathfrak{Y}$ such that

i) $T(x + y) = T(x) + T(y)$

ii) $T(\lambda . x) = \lambda . T(x)$

iii) $\|Tx\| \leq A . \|x\| \quad A \geqslant 0$, and $x \in \mathfrak{X}$.

We define $\|T\| = \sup \left\{ \|Tx\| : \|x\| \leq 1 \right\}$

The following is sometimes called the Banach-Steinhaus theorem: "Let $\mathfrak{X}$, $\mathfrak{Y}$ be Banach-spaces and $\{T_n\}$ be a sequence of bounded linear operators on $\mathfrak{X}$ to $\mathfrak{Y}$. Then the limit

$$Tx = \lim_{n \to \infty} T_n x$$

exists for every $x \in \mathfrak{X}$ if and only if

i) the limit Tx exists for every x in a everywhere dense sub-set of $\mathfrak{X}$

ii) $\sup_n \|T_n x\| < +\infty$ for each $x \in \mathfrak{X}$

When the limit Tx exists for each $x \in \mathfrak{X}$, the operator T is linear and bounded and

$$\|T\| \leq \underline{\lim}_{n \to \infty} \|T_n\| \leq \sup_n \|T_n\| < +\infty".$$

(For proof: See (2) Dunford and Schwartz, pp. 60-61.)

If $X(\omega)$ is a function on a probability space $(\Omega, \mathcal{B}, P)$ taking values in a Banach space $\mathfrak{X}$ then $X(\omega)$ is said to be strongly measurable with respect to

$\mathcal{B}$ if

$$\lim_{n\to\infty} x_n(\omega) = X(\omega) \quad \text{a.e.} \quad \text{(everywhere except}$$

on a set of points of P - measure 0) where

$$x_n(\omega) = \sum_{i=1}^{\infty} a_i^{(n)} \chi_{E_i^{(n)}}(\omega)$$

$$a_i^{(n)} \in \mathfrak{X}$$

$$E_i^{(n)} \in \mathcal{B}, \quad E_i^{(n)} \cap E_j^{(n)} = \phi, \quad \bigcup_i E_i^{(n)} = \Omega$$

$$\chi_F(\omega) = 1 \quad \text{if} \quad \omega \in F \quad \text{for any set } F$$

in Ω .

$$= 0 \quad \text{if} \quad \omega \in F^c$$

Define $X_n(\omega)$ to be integrable if $\sum_{i=1}^{\infty} \| a_i^{(n)} \| \, P(E_i^{(n)})$ $< +\infty$ and write

$$\int X_n(\omega)\,dp = \sum_{i=1}^{\infty} P(E_i^{(n)}) \, a_i^{(n)}$$

We say that $X(\omega)$ is Bochner-integrable if there exist a sequence of integrable functions $X_n(\omega)$ as above such that

$$\lim_{n\to\infty} X_n(\omega) = X(\omega) \quad \text{a.e.}$$

and

$$\lim_{m,\,n\to\infty} \int \| X_n(\omega) - X_m(\omega) \| \, dp = 0 \quad .$$

Then it follows that

$$\lim_{m,\,n\to\infty} \left\| \int X_n(\omega)\,dp - \int X_m(\omega)\,dp \right\| = 0$$

Hence

$$\lim_{n\to\infty} \int X_n(\omega)\,dp$$

exists and this we define to be $\int X(\omega)dP$, called the Bochner-integral of $X(\omega)$, occasionally denoted also by us as $E(X(\omega))$ or $E(X)$.

It can be shown that $X(\omega)$ is Bochner-integrable if and only if

i) $X(\omega)$ is "almost everywhere separable-valued" i.e. $\exists\ N \in \mathcal{B}$, $P(N) = 0$ such that the set $S \subset \mathfrak{X}$ defined by

$$S = \{ X(\omega) : \omega \in N^c \}$$

has a denumerable dense sub-set.

ii) $E(\|X(\omega)\|) = \int \|X(\omega)\| \, dP < +\infty$

(If (i) is satisfied then $\|X(\omega)\|$ is automatically a non-negative function measurable with respect to $\mathcal{B}$.)

For a discussion of Bochner-integrals see (3) Hille & Phillips, pp. 71-89.

The notion of "uniform integrability" of a family of complex-valued measurable, integrable functions $X_t(\omega)$ $t \in T$, on $(\Omega, \mathcal{B}, P)$ shall be defined as follows:

For any $\epsilon > 0$, there is a $\delta > 0$ such that

$$\int_A |X_t(\ )| \, dP < \epsilon \quad \text{for all} \quad t \in T$$

Whenever $P(A) < \delta$.

In the text a theorem, lemma or equation numbered u.v.w. where u.v.w. are positive integers will be the w th one in v th section of u th chapter.

Chapter 1

Conditional expectation of Bochner-integrable random variables

§1. Let $(\Omega, \mathfrak{B}, P)$ be a probability space i.e. Ω be an abstract set of elements ω, $\mathfrak{B}$, a Borel-field of sub-sets of Ω, called measurable sub-sets and $P(\cdot)$ a countably additive, non-negative set-function defined on $\mathfrak{B}$ and such that $P(\Omega) = 1$. Let $\mathfrak{X}$ be a Banach-space. We shall denote by $\|x\|$ the norm of an element $x \in \mathfrak{X}$ and by $\mathfrak{X}^*$ the dual of $\mathfrak{X}$.

A function $X(\omega)$ defined on Ω and taking values in $\mathfrak{X}$ which is strongly measurable with respect to the Borel-field $\mathfrak{B}$ will be called a $\mathfrak{X}$-valued strong random variable or when there is no scope for confusion simply a random variable (r.v.).

Let $\mathfrak{F}$ be a Borel-field contained in $\mathfrak{B}$ i.e. $\mathfrak{F} \subset \mathfrak{B}$ and let $X(\omega)$ be a r.v. which is Bochner-integrable. Following Doob (1, pp. 17) we shall define the strong conditional expectation of $X(\omega)$ relative to or given $\mathfrak{F}$, in symbols $E(X \mid \mathfrak{F})$ as follows:

<u>Def</u>: 1.1.1.

$E(X \mid \mathfrak{F})$ is a $\mathfrak{X}$-valued Bochner-integrable r.v. strongly measurable with respect to the sub-Borel-field $\mathfrak{F}$ (for brevity a $\mathfrak{F}$-meas. r.v.) such that for every $A \in \mathfrak{F}$ it satisfies the equation

$$\int_A E(X \mid \mathfrak{F}) \, dP = \int_A X dP \quad . \; . \; . \; . \quad (1.1.1)$$

where the integrals are taken in the sense of Bochner.

We shall now prove the existence and uniqueness of $E(X \mid \mathfrak{F})$ for every Bochner-integrable r.v. $X(\omega)$ and Borel-field $\mathfrak{F} \subset \mathfrak{B}$.

The standard proof for scalar-valued r.v.'s $X(\omega)$ cannot be extended to cover the situation here since the analogues of the Radon-Nikodym theorems for set-functions taking values in a Banach-space are not in general valid. For counter-examples see Bochner, (7) Clarkson (5).

<u>Theorem 1.1.1</u> $E(X \mid \mathfrak{F})$ exists and is unique except for sets of measure 0 for every Bochner-integrable r.v. $X(\omega)$ and any Borel-field $\mathfrak{F} \subset \mathfrak{B}$. (Notice that no assumptions on the Banach-space $\mathfrak{X}$ are made.)

Proof:

We shall use the fact that $X(\omega)$ being Bochner-integrable is almost everywhere (a.e.) separable-valued and $E(\|X(\omega)\|) < +\infty$. (3, Hille pp. 80) Because $X(\omega)$ is almost everywhere separable-valued we might and shall without loss of generality consider $\mathfrak{X}$ to be separable; for otherwise, we can carry out the proof by restricting our attention to the separable sub-space in which the values of $X(\omega)$ lie with probability one. $\mathfrak{X}$ being separable there exists a denumerable determining set (3, Hille pp.34) i.e. linear functionals $x_i^* \in \mathfrak{X}^*$ $i = 1, 2, \ldots$ such that for any $x \in \mathfrak{X}$ we have

$$\| x \| = \sup_i | x_i^* (x) |$$

Of course it follows that

$$\| x_i^* \| \leq 1$$

From equation (1.1.1) it follows because of an elementary property of Bochner-integrals that for every $x^* \in \mathfrak{X}^*$

$$x^* \left\{ E(X | \mathfrak{F}) \right\} = E(x^* (X(\omega)) | \mathfrak{F}) \text{ a.e.}$$

We shall first show that if $E(X | \mathfrak{F})$ exists then it must be unique except for sets of measure 0 **i.e.** if

$Y_1(\omega)$, $Y_2(\omega)$, are r.v.'s which satisfy def. (1.1.1) then

$$Y_1(\omega) = Y_2(\omega) \text{ a.e.}$$

This follows because

$$\int_A Y_1(\omega) \, dP = \int_A Y_2(\omega) \, dP$$

for all $A \in \mathfrak{F}$ and hence

$$x_i^* \left(\int_A Y_1 \, dP \right) = x_i^* \left(\int_A Y_2 \, dP \right)$$

i.e. $$\int_A x_i^* (Y_1 - Y_2) \, dP = 0$$

Now Y_1, Y_2 being $\mathfrak{F}$-meas. so is $Y_1 - Y_2$ and hence $x_i^* (Y_1 - Y_2)$ is $\mathfrak{F}$-meas. It follows from a standard theorem in measure theory that

$$x_i^*(Y_1 - Y_2) = 0 \quad \text{a.e.}$$

Hence

$$x_i^* (Y_1 - Y_2) = 0 \quad \text{for all } i = 1, 2, \ldots$$

except for $\omega \in N$.

$P(N) = 0$

so that if $\omega \notin N$

$$\| Y_1(\omega) - Y_2(\omega) \| = \sup_i \left| x_i^* (Y_1(\omega) - Y_2(\omega)) \right| = 0$$

i.e. $Y_1(\omega) = Y_2(\omega)$.

This proves the uniqueness of $E(X \mid \mathfrak{F})$.

We shall now prove the existence of $E(X \mid \mathfrak{F})$. To do this we consider two different cases:

(1) $X(\omega)$ is countably-valued i.e.

$$X(\omega) = \sum_{n=1}^{\infty} a_n \chi_{E_n}(\omega) \quad \text{with} \quad E(\| X(\omega) \|) = \sum_{n=1}^{\infty} \| a_n \| P(E_n) < +\infty .$$

where

$$a_n \in \mathfrak{X}, \quad E_n \in \mathfrak{B}, \quad \chi_{E_n}(\omega) \begin{cases} = 1 & \omega \in E_n \\ = 0 & \omega \in E_n^c \end{cases} \quad n=1,2,$$

and E_n's are disjoint.

Consider $Y_N(\omega) = \sum_{n=1}^{N} a_n f_n(\omega)$ where

$$f_n(\omega) = P(E_n \mid \mathfrak{F}) = E(\chi_{E_n}(\omega) \mid \mathfrak{F})$$

i.e. a conditional probability of E_n relative to $\mathfrak{F}$,

and let

$$X_N(\omega) = \sum_{n=1}^{N} a_n \chi_{E_n}(\omega)$$

Then

$$x_i^* \; (Y_N(\omega)) = \sum_{n=1}^{N} x_i^* \; (a_n) \; . \; f_n(\omega)$$

$$= \sum_{n=1}^{N} E\left(x_i^* \; (a_n) \; \chi_{E_n}(\omega) \middle| \mathfrak{F} \right)$$

$$= E\left(\sum_{n=1}^{N} x_i^* \; (a_n) \; \chi_{E_n}(\omega) \middle| \mathfrak{F} \right)$$

Hence for $N > M$, using standard properties of conditional expectations of scalar-valued random variables we have

$$| x_i^* \; (Y_N - Y_M) | = | E(\sum_{n=1}^{N} x_i^* \; (a_n) \; \chi_{E_n}(\omega) \; | \; \mathfrak{F}) - E(\sum_{n=1}^{M} x_i^*(a_n) \chi_{E_n}(\omega) \; | \; \mathfrak{F} \;) |$$

$$\leq E(| \sum_{n=M+1}^{N} x_i^* \; (a_n) \; \chi_{E_n}(\omega) | \; | \; \mathfrak{F})$$

$$\leq E(\sum_{n=M+1}^{N} \| a_n \| . \chi_{E_n} | \mathfrak{F}) = E(\| X_N - X_M \| \; | \; \mathfrak{F} \;)$$

Hence

$$\| Y_N(\omega) - Y_M(\omega) \| = \sup_i | x_i^*(Y_N - Y_M) |$$

$$\leq E(\| X_N - X_M \| \; | \; \mathfrak{F} \;) \; \ldots \; (1.1.2)$$

Now

$$\lim_{M,\, N \to \infty} \| X_N - X_M \| = 0 \quad \text{a.e.} \quad \text{and}$$

$$\| X_N - X_M \| \leq 2 \; \| X \| \quad \text{a.e.}$$

From whence

$$\lim_{M,\, N \to \infty} E(\| X_N - X_M \| \; | \; \mathfrak{F} \;) = 0 \quad \text{a.e.}$$ (1 Doob CE_5 pp.23)

Hence the series $Y(\omega) = \sum_{n=1}^{\infty} a_n \, f_n(\omega)$ converges strongly a.e.

It is also clear that

$$\int_A Y_N dP = \int_A X_N dP \quad \text{for all} \quad A \qquad \ldots. (1.1.3)$$

and hence $Y_N = E(X_N \mid \mathfrak{F})$ according to Def. 1.1.1

Now

$$\left.\begin{array}{l} X_N(\omega) \longrightarrow X(\omega) \quad \text{a.e.} \\ \| X_N(\omega) \| \leq X(\omega) \quad \text{which is integrable} \\ Y_N(\omega) \longrightarrow Y(\omega) \quad \text{a.e.} \end{array}\right\} \cdots (1.1.4)$$

$$\| Y_N(\omega) \| \leq E(\| X_N \| \mid \mathfrak{F}) \leq E(\| X \| \mid \mathfrak{F})$$

which is integrable.

Hence applying the dominated convergence theorem for Bochner-integrals (3, Hille pp. 83) we have by passing to limits as $N \longrightarrow \infty$ on both sides of (1.1.3)

$$\int_A Y \ dP = \int_A X \ dP$$

Also $Y(\omega)$ being the a.e. limit of $Y_N(\omega)$ which are $\mathfrak{F}$ -measurable r.v.'s is $\mathfrak{F}$ -meas. Hence we have proved that

$$Y(\omega) = \sum_{n=1}^{\infty} a_n \ P(E_n \mid \mathfrak{F}) = E(X \mid \mathfrak{F}) \quad \text{a.e.}$$

It is also clear from (1.1.4) that

$$\| Y(\omega) \| \leq E(\| X \| \mid \mathfrak{F}) \quad \text{a.e.} \quad (1.1.5)$$

(ii) Let $X(\omega)$ be an arbitrary Bochner-integrable r.v. Let a_n be a denumerable dense set in $\mathfrak{X}$.

Then for any $k = 1, 2, 3, \ldots.$

$$\Omega = \bigcup_{n=1}^{\infty}\left\{\omega: \| X(\omega) - a_n \| \leq 1/k\right\}$$
$$= \bigcup_{n=1}^{\infty} S_{n,\,k}$$

Define $X_k(\omega)$ $k = 1, 2, \ldots.$ as follows

$$\begin{aligned} X_k(\omega) &= a_1 && \omega \in S_{1,\,k} \\ &= a_2 && \omega \in S_{2,\,k} \cap S^c_{1,\,k} \\ &= a_3 && \omega \in S_{3,\,k} \cap S^c_{1,\,k} \cap S^c_{2,\,k} \quad \text{etc.} \end{aligned}$$

Obviously for all $\omega \in \Omega$

$$\| X_k(\omega) - X(\omega) \| \leq 1/k \qquad \text{--- (1.1.6)}$$

$$\| X_k(\omega) \| \leq \| X(\omega) \| + 1/k$$

and hence $X_k(\omega)$ is Bochner-integrable.

Let

$$Y_k(\omega) = E(X_k | \mathfrak{F})$$

which exists according to the proof of case (i) above.

For $n > m$, and any x_i^* of the determining set we have

$$\begin{aligned} |x_i^* (Y_n - Y_m)| &= | x_i^* (E(X_n|\mathfrak{F}) - E(X_m|\mathfrak{F}))| \\ &= | E(x_i^* (X_n) | \mathfrak{F}) - E(x_i^* (X_m)|\mathfrak{F})| \text{ a.e.} \\ &= | E(x_i^* (X_n - X_m) | \mathfrak{F})| \quad \text{a.e.} \\ &\leq E(|x_i^* (X_n - X_m)| | \mathfrak{F}) \quad \text{a.e.} \\ &\leq E\left(\| X_n - X_m\| | \mathfrak{F}\right) \quad \text{a.e.} \end{aligned}$$

Hence

$$\| Y_n - Y_m \| = \sup_i | x_i^* (Y_n - Y_m)| \leq E(\| X_n - X_m\| |\mathfrak{F}) \text{ a.e.}$$
$$\leq E(\| X_n - X \| | \mathfrak{F}) + E (\| X_m - X \| | \mathfrak{F}) \quad \text{a.e.}$$

$$\leq 1/n + 1/m \qquad \text{a.e. because of (1.1.6)}$$

Hence

$$\lim_{m,\, n \to \infty} \| Y_n - Y_m \| = 0 \qquad \text{a.e. so that}$$

$$\lim_{n \to \infty} Y_n(\omega) = Y(\omega) \quad \text{exists a.e.} \qquad \ldots(1.1.7)$$

From (1.1.5), (1.1.6) and (1.1.7) we have now a.e.

$$\left.\begin{aligned} &\lim_{n \to \infty} X_n(\omega) = X(\omega) \\ &\| X_n(\omega) \| \leq \| X(\omega) \| + 1 \\ &\lim_{n \to \infty} Y_n(\omega) = Y(\omega) \end{aligned}\right\} \cdots (1.1.8)$$

$$\| Y_n(\omega) \| \leq E(\| X_n \| \mid \mathfrak{F}) \leq E(\| X(\omega) \| + 1 \mid \mathfrak{F})$$
$$= E(\| X(\omega) \| \mid \mathfrak{F}) + 1$$

Also because $Y_n = E(X_n \mid \mathfrak{F})$ we have for all $A \in \mathfrak{F}$

$$\int_A X_n \, dP = \int_A Y_n \, dP \qquad \ldots\ldots(1.1.9)$$

Because of (1.1.8) we can pass to the limit as $n \to \infty$ on both sides of (1.1.9) invoking the bounded convergence theorem of Bochner integrals (3, Hille pp. 83) thus obtaining

$$\int_A X \, dP = \int_A Y \, dP$$

This then proves that $Y(\omega) = E(X \mid \mathfrak{F})$ and completes the proof of the theorem.

§ 2. Properties of strong conditional expectations:

Almost all the properties of conditional expectations of scalar-valued r.v.'s (1, See Doob pp. 20-26) can be established for the general Banach-valued case and their proofs can either be obtained by mimicking the usual proof or can be derived from the scalar-valued case. In the following we shall mention a few of the standard properties.

Theorem 1.2.1 If $\mathcal{G}$ is a Borel-field such that $A \in \mathcal{G}$ implies that there is $B \in \mathcal{F}$ such that $P(A \Delta B) = 0$ then

$$E(X(\omega) \mid \mathcal{F}) = E(X(\omega) \mid \mathcal{G}) \qquad \text{a.e.}$$

Proof: Let $\{x_i^*\}$ be a denumerable determining set for the $X(\omega)$ values in $\mathfrak{X}$. Because of the validity of the theorem for scalar r.v.'s we have

$$\begin{aligned} x_i^*\left(E(X \mid \mathcal{F})\right) &= E\left(x_i^*(X) \mid \mathcal{F}\right) && \text{a.e.} \\ &= E\left(x_i^*(X) \mid \mathcal{G}\right) && \text{a.e.} \end{aligned}$$

Hence

$$x_i^*\left(E(X \mid \mathcal{F}) - E(X \mid \mathcal{G})\right) = 0 \quad \text{a.e. for each } i$$

so that

$$\begin{aligned} &\| E(X \mid \mathcal{F}) - E(X \mid \mathcal{G}) \| \\ &\quad = \sup_i \left| x_i^*\left(E(X \mid \mathcal{F}) - E(X \mid \mathcal{G})\right)\right| = 0 \quad \text{a.e.} \end{aligned}$$

This proves that

$$E(X \mid \mathcal{F}) = E(X \mid \mathcal{G}) \qquad \text{a.e.}$$

Theorem 1.2.2 Suppose $\mathcal{G}_1 \subset \mathcal{G}_2$ are Borel-fields and that some version (and therefore every) of $E(X(\omega) \mid \mathcal{G}_2)$ is measurable $\mathcal{G}_1$. Then

$$E(X(\omega) \mid \mathcal{G}_1) = E(X(\omega) \mid \mathcal{G}_2) \qquad \text{a.e.}$$

Proof: Let $\{x_i^*\}$ be as in the previous theorem. Then

$$x_i^*\left(E(X \mid \mathcal{G}_2) \right) = E\left(x_i^*(X) \mid \mathcal{G}_2 \right) \text{ a.e.}$$
$$= E\left(x_i^*(X) \mid \mathcal{G}_1 \right) \text{ a.e.}$$

(because of the validity of the theorem for scalar r.v.'s.)

Hence

$$x_i^*\left(E(X \mid \mathcal{G}_1) - E(X \mid \mathcal{G}_2) \right) = 0 \quad \text{a.e.}$$

for each i so that

$$\| E(X \mid \mathcal{G}_1) - E(X \mid \mathcal{G}_2) \| = \sup_i \left| x_i^*\left(E(X \mid \mathcal{G}_1) - E(X \mid \mathcal{G}_2) \right) \right| = 0$$

which proves the theorem.

Theorem 1.2.3: If $f(\omega)$ is a scalar function measurable with respect to $\mathfrak{F}$ and if both $X(\omega)$ and $f(\omega)X(\omega)$ are Bochner-integrable then

$$E\left(X(\omega) f(\omega) \mid \mathfrak{F} \right) = f(\omega)\, E\left(X(\omega) \mid \mathfrak{F} \right) \quad \text{a.e.}$$

Proof: As before, the proof can be derived from the corresponding theorem for scalar r.v.'s by the use of the determining set $\{x_i^*\}$.

Theorem 1.2.4: For any Borel-field $\mathfrak{F} \subset \mathcal{B}$

i) $E(a \mid \mathfrak{F}) = a$ a.e. for any $a \in \mathfrak{X}$

ii) $E\left(\sum_{j=1}^{n} c_j X_j \mid \mathfrak{F} \right) = \sum_{j=1}^{n} c_j E(X_j \mid \mathfrak{F})$ a.e. for any finite number n of scalars c_j and Bochner-integrable $\mathfrak{X}$-valued r.v.'s $X_j(\omega)$.

iii) $\| E(X \mid \mathfrak{F}) \| \leq E(\| X \| \mid \mathfrak{F})$

iv) If $\lim_{n \to \infty} X_n(\omega) = X(\omega)$ a.e. and if there exists $f(\omega) \geqslant 0$ integrable and such that

$$\| X_n(\omega) \| \leq f(\omega)$$

then

$$\lim_{n \to \infty} E(X_n \mid \mathfrak{F}) = E(X \mid \mathfrak{F}) \quad \text{a.e.}$$

v) For any bounded linear operator T on $\mathfrak{X}$ to another Banach-space $\mathfrak{Y}$

$$E(TX(\omega) \mid \mathfrak{F}) = TE(X(\omega) \mid \mathfrak{F}) \quad \text{a.e.}$$

<u>Proof:</u> i) and ii) follow from the very definition of conditional expectation. iii) Let $\{ x_i^* \}$ again be a determining set for $X(\omega)$ values in $\mathfrak{X}$. Then

$$\begin{aligned} \| E(X \mid \mathfrak{F}) \| &= \sup_i | x_i^*(E(X \mid \mathfrak{F})) | \\ &= \sup_i | E(x_i^*(X) \mid \mathfrak{F}) | \leq \sup_i E(| x_i^*(X) | \mid \mathfrak{F}) \\ &\leqq E(\| X \| \mid \mathfrak{F}) \end{aligned}$$

iv) $X(\omega)$ is clearly Bochner-integrable as $\| X(\omega) \| \leq f(\omega)$ a.e. and so $E(X \mid \mathfrak{F})$ exists (Th. 1.1.1)

$$\| E(X_n \mid \mathfrak{F}) - E(X \mid \mathfrak{F}) \| \leq E(\| X_n - X \| \mid \mathfrak{F}) \quad \text{by}$$

(iii) above

Now

$$\| X_n - X \| \longrightarrow 0 \quad \text{a.e.} \quad \text{and}$$

$$\| X_n - X \| \leq 2 f(\omega) \quad \text{a.e. which is integrable}$$

Hence by the corresponding theorem for the scalar r.v.'s (1, Doob pp. 23) $E(\| X_n - X \| \mid \mathfrak{F}) \longrightarrow 0$ so that we have the desired result from the preceding inequality.

v) $T(X(\omega))$ is Bochner-integrable (3, Hille pp. 84) and by a standard theorem for Bochner integrals (3, Hille pp. 83)

$$\int_A TX(\omega)\, d\,P = T\left(\int_A X(\omega)\, dP\right) \qquad A \in \mathfrak{F}$$

Hence

$$\int_A TX(\omega)\, dP = T\left(\int_A X(\omega)\, dP\right) = T\left(\int_A E(X\,|\,\mathfrak{F})\, dP\right)$$
$$= \int_A T(E(X\,|\,\mathfrak{F}))\, dP$$

Also $T\,E(X\,|\,\mathfrak{F})$ is $\mathfrak{F}$-measurable so that according to Def. 1.1.1 we have

$$T(E(X\,|\,\mathfrak{F})) = E(T(X(\omega))\,|\,\mathfrak{F}) \quad \text{a.e.}$$

Theorem 1.2.5. If $\mathfrak{G} \subset \mathfrak{F}$ are sub-Borel-fields of $\mathfrak{B}$ then

$$E(E(X(\omega)\,|\,\mathfrak{F})\,|\,\mathfrak{G}) = E(X(\omega)\,|\,\mathfrak{G}) .$$

Proof: **Follows directly from definition 1.1.1.**

Chapter 11

Weak convergence in certain special Banach spaces.

§1. For our later investigations we shall need to know some properties of weak convergence in certain Lebesgue-type Banach spaces, first introduced and systematically studied by Bochner and Taylor in 1938 in (8). We define these Banach-spaces as follows:

<u>Definition</u> 2.1.1 We define $L_p(\Omega, \mathcal{B}, P, \mathfrak{X})$ $1 \leq p < +\infty$ as the set of all equivalence classes of strontly measurable r.v.'s $X(\omega)$ defined on the probability space $(\Omega, \mathcal{B}, P)$, taking values in the Banach-sapce $\mathfrak{X}$ and such that the "norm"

$$[X(\omega)]_p = \left(\int_\Omega \|X(\omega)\|^p \, dP \right)^{1/p} < +\infty \quad \ldots\ldots(2.1.1)$$

The equivalence class $\{X(\omega)\}$ is set of all r.v.'s $Y(\omega)$ such that $Y(\omega) = X(\omega)$ a.e.

<u>Definition 2.1.2</u> $L_\infty(\Omega, \mathcal{B}, P, \mathfrak{X})$ is the set of all equivalence classes of strongly measurable r.v.'s $X(\omega)$ defined on the probability space $(\Omega, \mathcal{B}, P)$ and taking values in the Banach space $\mathfrak{X}$ and such that the norm

$$\left[X(\omega)\right]_\infty = \operatorname*{ess.\,sup.}_{\omega\in\Omega} \|X(\omega)\| < +\infty \quad \ldots\ldots (2.1.2)$$

It can be shown that with the norms defined by (2.1.1) and (2.1.2) the spaces $L_p(\Omega, \mathcal{B}, P, \mathfrak{X})$ $1 \leq p \leq \infty$ are Banach spaces (2, Dunford and Schwarz pp. 146). When there is no danger of any confusion we shall abbreviate $L_p(\Omega, \mathcal{B}, P, \mathfrak{X})$ as $L_p(\mathfrak{X})$. If $\mathfrak{X}$ is the Banach-space of ordinary complex numbers we shall simply write L_P. We shall invariably write $X(\omega)$ for the equivalence class $\{X(\omega)\}$.

Our main objective is to study the weakly compact subsets of $L_p(\mathfrak{X})$, $1 \leq p < \infty$ under the further assumption that $\mathfrak{X}$ is reflexive. This we do in three steps. In § 2 we settle the problem completely for $p > 1$ with the help of known results and give a representation for linear functionals on $L_1(\mathfrak{X})$. In § 3 we study weak convergence of sequences of $L_1(\mathfrak{X})$ to an element in $L_1(\mathfrak{X})$ and in § 4 we give one necessary condition and one sufficient condition for a set in $L_1(\mathfrak{X})$ to be weakly compact.

§2. The linear functionals of the Banach space $L_p(\mathfrak{X})$, $1 \leq p < \infty$ have been studied by Bochner and Taylor 1938 (8), Day 1941 (14), Phillips 1943 (9), Dieudonné 1951 (15), Mourier 1952 (10), Fortet and Mourier 1951 (11). Bochner and Taylor gave for an arbitrary $\mathfrak{X}$ a general representation for the linear functionals in terms of certain Stieltjes integrals with vector-valued measures.

Under some conditions on the Banach-space $\mathfrak{X}$, they and others have also given simpler integral representations. We shall mention one such result due to Phillips for the case $1 < p < \infty$.

<u>Theorem 2.2.1</u> Let $\mathfrak{X}$ be reflexive and $1 < p < \infty$. Then a linear functional $F(\cdot)$ defined on $L_p(\Omega, \mathcal{B}, P, \mathfrak{X})$ has the form

$$F(X(\omega)) = \int \langle X(\omega), X^*(\omega) \rangle \, dP \quad .... \quad (2.2.1)$$

where

$$X^*(\omega) \in L_q(\Omega, \mathcal{B}, P, \mathfrak{X}^*), \ \frac{1}{p} + \frac{1}{q} = 1$$

and $\langle x, x^* \rangle$ denotes the value of the linear functional x^* at x.

Also

$$\| F \| = \left(\int \| X^*(\omega) \|^q \, dP \right)^{1/q}$$

so that $(L_p(\Omega, \mathcal{B}, P, \mathfrak{X}))^*$ is isometrically isomorphic to the Banach space $L_q(\Omega, \mathcal{B}, P, \mathfrak{X}^*)$.

<u>Corollary 2.2.1</u> If $\mathfrak{X}$ is reflexive then the space $L_p(\Omega, \mathcal{B}, P, \mathfrak{X})$, $1 < p < \infty$ is weakly complete and a subset of it is weakly compact if and only if it is bounded in $L_p(\mathfrak{X})$ - norm.

<u>Proof:</u> It follows from theorem 2.2.1 that if $\mathfrak{X}$ is reflexive then $L_p(\mathfrak{X})$ is also reflexive (Notice that the converse is true also) and the corollary follows from standard theorems about reflexive spaces (see 2, Dunford & Schwarz pp. 68-69).

Mourier, 1952 (10), proved essentially the same result and Bochner & Taylor (8) proved the above under a condition on $\mathfrak{X}$ which is more general than reflexivity. Fortet & Mourier 1951 (11) proved a similar result for $p \geq 1$ under the assumption that $\mathfrak{X}$ is separable. We shall need an extension of theorem 2.2.1 to the case $p = 1$ for our future work and shall in the following give a simple proof for it using a theorem of Phillips 1943 (9).

<u>Theorem 2.2.2</u> If $F(\cdot)$ is a bounded linear functional on $L_1(\Omega, \mathfrak{B}, P, \mathfrak{X})$ and $\mathfrak{X}$ is reflexive then

$$F(X) = \int \langle X(\omega), \quad Y^*(\omega) \rangle \, dP \quad \ldots . \quad (2.2.2)$$

where

$$Y^*(\omega) \in L_\infty(\Omega, \mathfrak{B}, P, \mathfrak{X}^*).$$

<u>Proof:</u> For fixed $E \in \mathfrak{B}$, $a\chi_E(\omega) \in L_1(\mathfrak{X})$ for all $a \in \mathfrak{X}$. Consider $F(a.\chi_E(\omega))$. Because F is a bounded linear functional on $L_1(\mathfrak{X})$ we have

$$|F(a\chi_E(\omega))| \leq \| F \| . [a\chi_E(\omega)]_1 = \| F \| . \int_E \| a \| \, dP$$

$$= \| F \| . \| a \| . P(E)$$

$$\ldots \ldots \quad (2.2.3)$$

Hence $F(a.\chi_E(\omega))$, for a fixed $E \in \mathfrak{B}$, is a bounded linear functional on $\mathfrak{X}$ and let us write

$$F\left(a. \chi_E(\omega)\right) = x_E^*(a) \quad \text{where} \quad x_E^* \in \mathfrak{X}^* /$$

Also from (2.2.3) we have that

$$\| x^*_E \| = \sup_{\|a\| \leq 1} | x^*_E(a) | \leq \| F \| . P(E) \quad ..(2.2.4)$$

It is also clear that $x^*_{E \cup F} = x^*_E + x^*_F$, $E \cap F = \phi$

, $E, F, \in \mathcal{B}$.

In other words x^*_E is an additive $\mathfrak{X}^*$ -valued set function on $\mathcal{B}$ having the property (2.2.4). According to a theorem of Phillips (9) there is a function $Y^*(\omega) \in L_\infty(\mathfrak{X}^*)$ such that

$$x^*_E = \int_E Y^*(\omega) \, dP.$$

so that

$$F\left(\chi_E(\omega).a\right) = \int_E \langle a, \ Y^*(\omega) \rangle \, dP$$

Hence for all simple functions

$$X(\omega) = \sum_{i=1}^{n} a_i \chi_{E_i} (\omega) \qquad \begin{matrix} a_i \in \mathfrak{X} \\ E_i \in \mathcal{B} \end{matrix}$$

we have

$$F(X(\omega)) = \int \langle X(\omega), \ Y^*(\omega) \rangle \, dP \quad . . .(2.2.5)$$

If $X(\omega) \in L_1(\mathfrak{X})$ is an arbitrary function then we can construct a sequence of simple functions $X_n(\omega) \in L_1(\mathfrak{X})$ such that

$$\lim_{n \to \infty} \int \| X(\omega) - X_n(\omega) \| \, dP = 0$$

(2, Dunford & Schwartz pp. 125)

Hence from (2.2.5) we have

$$\begin{aligned} F(X(\omega)) &= \lim_{n \to \infty} F(X_n(\omega)) \\ &= \lim_{n \to \infty} \int \langle X_n(\omega), \ Y^*(\omega) \rangle \, dP \\ &= \int \langle X(\omega) \ , \ Y^*(\omega) \rangle \, dP \end{aligned}$$

as

$$\left|\int\langle X_n(\omega),\ Y^*(\omega)\rangle\ dP - \int\langle X(\omega),\ Y^*(\omega)\rangle\ dP\right|$$
$$\leq \int \| X(\omega) - X_n(\omega)\| . \| Y^*(\omega)\|\ dP$$
$$\leq \left[Y^*(\omega)\right]_\infty \cdot \int \| X(\omega) - X_n(\omega)\|\ dP \to 0$$
$$\text{as } n \to \infty .$$

This concludes the proof of the theorem.

§ 3. Conditions under which a sequence of r.v.'s $X_n(\omega) \in L_1(\mathfrak{X})$ converges weakly to a r.v. $X(\omega) \in L_1(\mathfrak{X})$ were given by Bochner & Taylor (8) when $\mathfrak{X}$ is of a special type. Our theorem 2.3.1 is of a different nature although the conditions involved are similar.

<u>Theorem 2.3.1</u> If $X_n(\omega) \in L_1(\Omega, \mathfrak{B}, P, \mathfrak{X})$, $\mathfrak{X}$ reflexive, is weakly convergent and if $\| X_n \|$ is uniformly integrable i.e. given $\epsilon > 0$, $\exists\ \delta > 0$ such that

$$\int_E \| X_n \|\ dP < \epsilon \quad \text{for all} \quad E \ni P(E) < \delta$$

then there exists $X(\omega) \in L_1(\Omega, \mathfrak{B}, P, \mathfrak{X})$ such that

$$X_n \xrightarrow{\omega} X \text{ i.e. } X_n(\omega) \text{ converges weakly to } X(\omega).$$

<u>Proof:</u> We use a Radon-Nikodym type theorem due to Dunford and Pettis, 1940 (6) which can be stated as follows in our case:

Let $\mathfrak{X}$ be the adjoint to a separable Banach space $\mathfrak{Y}$ and let $X(E)$ be defined from $\mathfrak{B}$ to $\mathfrak{X}$. Suppose that

(i) for each $y \in \mathfrak{Y}$ the set-function $X_E(y)$ is completely additive

(ii) $X_E(y) = 0$ when $P(E) = 0$ for all y

(iii) the numerical function

$$\sigma_E = \sup_{Y} \frac{1}{\|y\|} \, | x_E(y) | = \| x_E \|$$

has finite total variation on any set $E^1 \in \mathcal{B}$ then there exists $X(\omega) \in L_1(\mathfrak{X})$ such that

$$X_E = \int_E X(\omega) \, dP$$

Because $\{ X_n \}$ is weakly convergent, it follows from a general theorem that $[X_n]_1 < c$.

According to the representation theorem of linear functionals of $L_1(\mathfrak{X})$ (Th. 2.2.2) we have that

$$\lim_{n \to \infty} \int \langle X_n(\omega), \; Y^*(\omega) \rangle \, dP \quad \text{exists for all}$$

$$Y^*(\omega) \in L_\infty(\mathfrak{X}^*)$$

Take $Y^*(\omega) = \chi_E(\omega) . a^*$, $a^* \in \mathfrak{X}^*$, $E \in \mathcal{B}$. Then we have that

$$\lim_{n \to \infty} a^*\left(\int_E X_n \, dP \right) \quad \text{exists for all}$$

$$a^* \in \mathfrak{X}^*$$

and hence because $\mathfrak{X}$ is reflexive the limit is $a^*(\lambda(E))$ for some $\lambda(E) \in \mathfrak{X}$.

Now $\mathfrak{X}$ being reflexive we have $\mathfrak{X} = (\mathfrak{X}^*)^*$ and since we are concerned only with $\{ X_n(\omega) \}$ we might as well consider $\mathfrak{X}$ to be separable. Then $\mathfrak{X}^*$

would also be separable. We shall now show that under the hypotheses of the theorem $\lambda(E)$ satisfies the conditions of Dunford & Pettis theorem.

i) Let $E_i \in \mathcal{B}$. $\bigcup_{i=1}^{\infty} E_i = E$. Then

$$x^*\left(\lambda\left(\bigcup_1^{\infty} E_i\right)\right) = \lim_{n \to \infty} x^*\left(\int_{\bigcup_{i=1}^{\infty} E_i} X_n \, dP\right)$$

$$= \lim_{n \to \infty} \lim_{m \to \infty} x^*\left(\int_{\bigcup_{i=1}^{m} E_i} X_n \, dP\right)$$

We now show that

$$\lim_{m \to \infty} x^*\left(\int_{\bigcup_{i=1}^{m} E_i} X_n \, dP\right) = x^*\left(\int_{\bigcup_1^{\infty} E_i} X_n \, dP\right)$$

uniformly in n because

$$\left| x^*\left(\int_{\bigcup_{m+1}^{\infty} E_i} X_n \, dP\right)\right| \leq \|x^*\| \int_{\bigcup_{m+1}^{\infty} E_i} \| X_n \| \, dP < \epsilon$$

by choosing $m > M_\epsilon$, M_ϵ independent of n because $\| X_n \|$ are uniformly integrable. Hence we can interchange limits above and we have

$$x^*\left(\lambda\left(\bigcup_1^{\infty} E_i\right)\right) = \lim_{m \to \infty} \lim_{n \to \infty} x^*\left(\int_{\bigcup_{i=1}^{m} E_i} X_n \, dP\right)$$

$$= \lim_{m \to \infty} x^*\left(\lambda\left(\bigcup_{i=1}^{m} E_i\right)\right)$$

$$= \lim_{m \to \infty} \sum_{i=1}^{m} x^*(\lambda(E_i))$$

Verification of (ii) is quite trivial.

iii) We shall show that for any finite number of sets E_i $i = 1, \ldots\ldots N$

E_i disjoint

$$\sum_{i=1}^{N} \sigma_{E_i} < C$$

$$\sum_{i=1}^{N} \sigma_{E_i} = \sum_{i=1}^{N} \sup_{x^*} \lim_{n} \frac{\left| x^*\left(\int_{E_i} X_n \, dP \right) \right|}{\| x^* \|}$$

$$\leq \sum_{i=1}^{N} \lim_{n} \int_{E_i} \| X_n \| \, dP$$

$$= \lim_{n} \int_{\bigcup_{i=1}^{n} E_i} \| X_n \| \, dP \leq \lim_{n} [X_n]_1 < C$$

Thus, all the conditions in the theorem are satisfied and hence we have a function $X(\omega) \in L_1(\mathfrak{X})$ such that

$$\lambda(E) = \int_E X(\omega) \, dP$$

and hence

$$\int \langle X_n(\omega), Y^*(\omega) \rangle \, dP \longrightarrow \int \langle X, Y^* \rangle \, dP$$

for all simple functions $Y^*(\omega) \in L_\infty(\mathfrak{X}^*)$

Now if

$$Y^*(\omega) = \sum_{i=1}^{\infty} x_i^* \chi_{E_i}(\omega) \qquad \| x_i^* \| \leq A$$

we have

$$\lim_{n \to \infty} \int \langle X_n, Y^* \rangle \, dP = \lim_n \lim_m \sum_{i=1}^{m} x_i^*\left(\int_{E_i} X_n \, dP \right)$$

$$= \lim_m \lim_n \sum_{i=1}^{m} x_i^*\left(\int_{E_i} X_n \, dP \right)$$

$$= \lim_m \sum_{i=1}^{m} x_i^*\left(\int_{E_i} X \, dP \right)$$

$$= \int \langle X, Y \rangle \, dP$$

the interchange of limit being permissible because

$$\lim_m \sum_{i=1}^{m} x_i^*\left(\int_{E_i} X_n \, dP \right)$$

exists uniformly in n as for any $\epsilon > 0$.

$$\left| \sum_{M}^{M+p} x_i^*\left(\int_{E_i} X_n \, dP \right) \right| \leq \sum_{M}^{M+p} \| x_i^* \| \cdot \left\| \int_{E_i} X_n \, dP \right\|$$

$$\leq A \cdot \sum_{M}^{M+p} \int_{E_i} \| X_n \| \, dP$$

$$\leq A \cdot \int_{\bigcup_{M}^{M+p} E_i} \| X_n \| \, dP$$

for all n and p if $M > M_\epsilon$ because $\| X_n \|$ are uniformly integrable.

Now an arbitrary $Y^*(\omega) \in L_\infty(\mathfrak{X}^*)$ being a uniform limit of countably-valued functions in $L_\infty(\mathfrak{X}^*)$ we have in general

$$\int \langle X_n, Y^* \rangle \, dP \longrightarrow \int \langle X, Y^* \rangle \, dP$$

i.e. $\quad X_n \xrightarrow{\omega} X$.

§ 4. In this section we study the weakly compact sets of $L_1(\mathfrak{X})$. The necessary conditions for a set in $L_1(\mathfrak{X})$ to be weakly compact, given in Th. 2.4.1 in the following, are known to be both necessary and sufficient for L_1 (2, Dunford & Schwarz pp. 292). Our sufficient conditions given in Th. 2.4.2 are stronger than the necessary conditions but are equivalent to the latter in the case of complex-valued r.v.'s (2, Dunford & Schwarz, pp.293).

<u>Theorem 2.4.1.</u> Let $K \subset L_1(\Omega, \mathcal{B}, P, \mathfrak{X})$, $\mathfrak{X}$ any Banach space. If K is weakly sequentially compact then

i) it is bounded

ii) $\int_E X(\omega) \, dP$, $X \in K$ is weakly uniformly countably additive

i.e.

for any sequence $E_n \in \mathcal{B}$, $E_n \downarrow$, $\bigcap_{n=1}^{\infty} E_n = \phi$

we must have

$$\lim_{n \to \infty} x^*\left(\int_{E_n} X(\omega) \right) dP = 0 \quad \text{uniformly in}$$

$$x \in K, \quad x^* \in \mathfrak{X}^*$$

<u>Proof</u>: We shall use the following generalization for vector-valued measures of a theorem of Nikodym (2, Dunford & Schwartz, Th. IV. 6. pp. 321) namely, "Let $\{\mu_n\}$ be a sequence of vector-valued measures defined on the Borel-field $\mathcal{B}$. If $\mu(E) = \lim_{n \to \infty} \mu_n(E)$ (III) exists for each $E \in \mathcal{B}$, then μ is a vector measure on $\mathcal{B}$ and the countable additivity of μ_n is uniform in n".

If K is weakly sequentially compact, then from a general theorem it follows that K is bounded. If (ii) is not satisfied then $\exists\ \epsilon > 0$, $E_n \in \mathcal{B}$, $\downarrow$, $\bigcap_1^\infty E_n = \phi$ $x^* \in \mathfrak{X}^*$ and $x_n \in K$ such that

$$\left| x^* \left(\int_{E_n} x_n \, dP \right) \right| \geq \epsilon \quad \text{------} \quad (*)$$

We may assume $\{x_n\}$ to be weakly convergent since K is weakly sequentially compact. Hence

$\int_E x_n \, dP$ converges weakly to a limit for each $E \in \mathcal{B}$ as

$x^*\left(\int x . \chi_E \, dP\right)$ is a linear functional on $L_1(\mathfrak{X})$, for any $x^* \in \mathfrak{X}^*$

But then $(*)$ is a contradiction to Nikodym's theorem. This proves the theorem.

<u>Theorem 2.4.2</u> Let $K \subset L_1(\Omega, \mathcal{B}, P, \mathfrak{X})$,

$\mathfrak{X}$ reflexive be such that

i) $[X]_1 \leq C$ for all $X \in K$, C independent of X

ii) $\{ \|X(\omega)\| : X \in K \}$ is an uniformly integrable family then

K is weakly sequentially compact.

Proof: We shall need a lemma due to (2) Dunford & Schartz, pp. 202.

Lemma: Let $\mathcal{B}$ be a Borel-field of sets and $\mathcal{B}_1$ a field contained in $\mathcal{B}$ which generates $\mathcal{B}$. Let $\{\mu_n\}$ be a sequence of countably additive set functions on $\mathcal{B}$ with values in $\mathfrak{X}$. Suppose that the countable additivity of μ_n is uniform in n and that $\lim_{n \to \infty} \mu_n(E)$ exists for $E \in \mathcal{B}_1$. Then $\lim_{n \to \infty} \mu_n(E)$ exists for $E \in \mathcal{B}$.

Corr. If $\mu_n(E)$ is weakly convergent for $E \in \mathcal{B}_1$ then it is so for $E \in \mathcal{B}$.

To prove the sufficiency we now show that if $X_n \in K$, $[X_n]_1 \leq C$, $n \geq 1$ then there is a subsequence which converges weakly.

It is easy to see that there exists a separable subspace $\mathfrak{X}_0$ of $\mathfrak{X}$ and a Borel-field $\mathcal{B}_0$ contained in $\mathcal{B}$ generated by a denumerable number of sets $\{E_n\}$ such that

$$\{X_n\} \in L_1(\Omega, \mathcal{B}_0, P, \mathfrak{X}_0)$$

Let Σ_0 be the field generated by $\{E_n\}$. Σ_0 evidently has only a denumerable number of sets. Now for any $E \in \Sigma_0$

$$\left\| \int_E X_n \, dP \right\| \leq \int_E \| X_n \| \, dP \leq [X_n]_1 \leq C$$

and $\mathfrak{X}$ being reflexive there exists a subsequence n_j such that

$\int_E X_{n_j} \, dP$ converges weakly. Since Σ_0 has

only a denumerable number of sets we can choose a subsequence $\{n_i\}$ by Cantor's diagonalization process such that

$\int_E X_{n_i} \, dP$ converges weakly for every $E \in \Sigma_0$.

Now because $P(\Omega) < \infty$ the uniform integrability of $\| X_{n_i} \|$ implies the uniform countable additivity of the set functions $\int_E X_{n_i} \, dP$ and hence by virtue of the preceding lemma we have the weak convergence of $\int_E X_{n_i} \, dP$ for all $E \in \mathcal{B}_0$. Hence

$$\int \langle X_{n_i}, \quad \underline{Y}^* \rangle \, dP$$

converges for all simple functions $Y^*(\omega) \in L_\infty(\mathfrak{X}_0^*)$.

Let $\quad Y^*(\omega) = \sum_{i=1}^{\infty} x_i^* \chi_{E_i}(\omega) \qquad \| x_i^* \| \leq M$

Now

$$\int \langle X_{n_i} , Y^*(\omega) \rangle \, dP$$

$$= \lim_{k \to \infty} \sum_{j=1}^{k} x_j^* \left(\int_{E_j} X_{n_i} dP \right)$$

and the limit exists uniformly in "i" because

$$\left| \sum_{j=N}^{N+p} x_j^* \left(\int_{E_j} X_{n_i} dP \right) \right| \leq \sum_{j=N}^{N+p} \| x_j^* \| \int_{E_j} \| X_{n_i} \| \, dP$$

$$\leq M \int_{\bigcup_{j=N}^{N+p} E_j} \| X_{n_i} \| \, dP < \epsilon$$

for $N > N_\epsilon$, $p > 0$ independent of X_{n_i} because $\{ \| X_{n_i} \| \}$ is uniformly integrable.

Hence by a standard theorem on interchangability of repeated limits we have the existence of

$$\lim_{i \to \infty} \int \langle X_{n_i} , Y^*(\omega) \rangle \, dP$$

$$= \lim_{i} \lim_{k} \sum_{j=1}^{K} x_j^* \left(\int_{E_j} X_{n_k} \, dP \right)$$

Because an arbitrary $Y^*(\omega)$ can be uniformly approximated by a countably-valued function in $L_\infty(\mathfrak{X}^*_0)$

this proves that X_{n_i} are weakly convergent and by Theorem 2.3.1 it must converge to a function $X_0(\omega) \in L_1(\mathcal{X}_0)$. This completes the proof of the theorem.

Chapter 111

Strong martingales of Banach-valued r.v.'s and their mean-convergence.

1. Following Doob (1, pp. 294) we define a discrete parameter strong martingale of Banach-valued r.v.'s as follows:

Definition 3.1.1

Let $(\Omega, \mathcal{B}, P)$ be a probability space and $\mathfrak{X}$ an arbitrary Banach space. Let I be a subset of the set of all integers and let $X_t(\omega) \in L_1(\Omega, \mathcal{B}, P, \mathfrak{X})$ for all $t \in I$. For each $t \in I$ let there be a Borel-field $\mathfrak{F}_t \subset \mathcal{B}$ such that

$$\mathfrak{F}_s \subset \mathfrak{F}_t \quad \text{whenever} \quad s < t .$$

We shall define $\{X_t, \mathfrak{F}_t, t \in I\}$ as a martingale or in detail a strong-martingale of $\mathfrak{X}$-valued r.v.'s if whenever $s < t$, $s, t \in I$

$$X_s(\omega) = E(X_t(\omega) \mid \mathfrak{F}_s) \quad \text{a.e.} \quad \ldots.(3.1.1)$$

As in the case of complex-valued r.v.'s it can be shown that if

$$Z(\omega) \in L_1(\Omega, \mathcal{B}, P, \mathfrak{X}) \text{ and if}$$
$$X_n(\omega) = E(Z(\omega) \mid \mathfrak{F}_n) \quad n \geq 1$$

where

$$\mathfrak{F}_n \subset \mathfrak{F}_{n+1}$$

then $\left\{ X_n, \mathfrak{F}_n, n \geq 1 \right\}$ is a martingale.

By the same token, if $\mathfrak{F}_{-(n+1)} \subset \mathfrak{F}_{-n}, \ n \geq 1$,

and if

$$X_{-n}(\omega) = E(Z(\omega) \mid \mathfrak{F}_{-n})$$

then $\left\{ X_n, \mathfrak{F}_n, n \leq -1 \right\}$ is a martingale. Conversely any martingale $\left\{ X_n, \mathfrak{F}_n, n \leq -1 \right\}$ is generated in this manner by $Z(\omega) = X_{-1}(\omega)$.

We shall need the following lemma:

<u>Lemma 3.1.1</u> Given any Borel-field $\mathfrak{F} \subset \mathfrak{B}$, the conditional expectation $E(X(\omega) \mid \mathfrak{F})$ for r.v.'s $X(\omega) \in L_p(\Omega, \mathfrak{B}, P, \mathfrak{X})$ $p \geq 1$ defines a bounded linear operator on the Banach space $L_p(\Omega, \mathfrak{B}, P, \mathfrak{X})$ to the sub-Banach-space $L_p(\Omega, \mathfrak{F}, P, \mathfrak{X})$.

<u>Proof:</u> Let $TX = E(X \mid \mathfrak{F})$ $X \in L_p(\Omega, \mathfrak{B}, P, \mathfrak{X})$; surely $TX \in L_p(\Omega, \mathfrak{F}, P, \mathfrak{X})$ and

$$T(X + Y) = TX + TY$$

$$T(\lambda . X) = \lambda . (TX) \qquad \lambda \text{ any complex number.}$$

Also

$$[TX]_p = \left(\int_\Omega \| TX(\omega) \|^p \, dP \right)^{1/p}$$

$$\leq \left(\int_\Omega E(\| X(\omega) \|^p \mid \mathfrak{F}) dP \right)^{1/p}$$

because $\| X \| \leq E(\| X \| \mid \mathfrak{F})$ a.e.
and hence $\| X \|^{p} \leq E(\| X \|^{p} \mid \mathfrak{F})$
by Jensen's inequality.

$$= \left(\int_{\Omega} \| X(\omega) \|^{p} \, dP \right)^{1/p} = [X]_{p}$$

so that $\| T \| \leq 1$. Actually $\| T \| = 1$ as we can show by taking $X(\omega) \equiv a$ where $\| a \| = 1$.

According to the above lemma, we can associate with every martingale $\{ X_t, \mathfrak{F}_t, t \in I \}$, a sequence of operators T_t, $t \in I$ defined by

$T_t X = E(X \mid \mathfrak{F}_t)$

and hence mean convergence of martingales can be considered from the point of view of convergence of the sequence of operators T_t. In the following section we shall make this statement precise. In theorems 3.2.1 and 3.2.2 we show that the operators T_t converge to an operator T in the strong topology if $I = (n \geq 1)$ or $(n \leq -1)$ respectively.

2. <u>Theorem 3.2.1.</u> Let $\{ X_n, \mathfrak{F}_n, n \geq 1 \}$ be a martingale such that

$$X_n = E(Z \mid \mathfrak{F}_n) \qquad n \geq 1$$

where

$$Z(\omega) \in L_p(\Omega, \mathcal{B}, P, \mathfrak{X}) \quad p \geq 1, \quad -$$

$\mathfrak{X}$ arbitary.

Then

$$\lim_{n \to \infty} \left[X_n - X_\infty \right]_p = 0 \quad \text{where}$$

$$X_\infty = E(Z \mid \mathfrak{F}_\infty)$$

and $\mathfrak{F}_\infty$ = Borel-field generated by $\bigcup_{n=1}^{\infty} \mathfrak{F}_n$.

..............(3.2.1)

<u>Proof:</u> Let us define $T_n Z = E(Z \mid \mathfrak{F}_n)$ for any $Z \in L_p(\Omega, \mathfrak{B}, P, \mathfrak{X})$ $p \geq 1$. By lemma 3.1.1, T_n is a linear bounded operator mapping $L_p(\Omega, \mathfrak{B}, P, \mathfrak{X})$ into $L_p(\Omega, \mathfrak{F}_n, P, \mathfrak{X})$. The conclusion of the theorem then asserts that the sequence of operators T_n converges in the strong topology to the operator T_∞ on $L_p(\Omega, \mathfrak{B}, P, \mathfrak{X})$ where

$$T_\infty (Z) = E(Z \mid \mathfrak{F}_\infty) \ .$$

We shall give two different proofs of this. Our first proof applies only to the case when $\mathfrak{X}$ is reflexive and is based on an application of a very general mean ergodic theorem, due to Eberlein, 1949, (16). This method of proving mean convergence for real-valued martingales was used by Jerison 1959 (17) in the case of martingales with index set $n \leq -1$. Our second proof is elementary and is based on an application of the Banach-Steinhaus theorem and is valid for an arbitrary Banach-space $\mathfrak{X}$.

Proof I. $\mathfrak{X}$ reflexive.

We shall first state the mean ergodic theorem in the form we shall apply it. Eberlein (16) proved it more generally for linear vector spaces, his theorem being a generalization of similar theorems of Yosida and Kakutani 1941, Birkhoff and Alaoglu 1940 and Day 1942.

Eberlein's theorem: Let G be a semigroup of bounded linear transformations on a Banach space $\mathfrak{X}$. A net (T_α) of linear transformations of $\mathfrak{X}$ into itself is called a system of almost invariant integrals for G if

i) for each $x \in \mathfrak{X}$ and all α, $T_\alpha x$ belongs to the closed convex hull of $\{Tx : T \in G\}$

ii) $\| T_\alpha \| \leq C$, C independent of α.

iii) for every $x \in \mathfrak{X}$ and $T \in G$

$$\lim_\alpha (TT_\alpha x - T_\alpha x) = \lim_\alpha (T_\alpha Tx - T_\alpha x) = 0 .$$

Now, if for a given $x \in \mathfrak{X}$, the net $T_\alpha x$ has a weak cluster point Y then $Y = \lim_\alpha T_\alpha x$ in the strong topology of $\mathfrak{X}$.

We shall apply the above theorem to the Banach space

$$L_p(\Omega, \mathfrak{F}_\infty, P, \mathfrak{X}) \qquad p \geq 1$$

Define $\quad S_n X = X - E(X | \mathfrak{F}_n) \qquad X \in L_p(\Omega, \mathfrak{F}_\infty, P, \mathfrak{X}),$

$n \geq 1$

Then $\quad S_m S_n = S_{\max(m,\, n)} \qquad \ldots\ldots(3.2.2)$

as
$$S_m S_n X = S_m \left\{ X - E(X | \mathfrak{F}_n) \right\}$$
$$= X - E(X|\mathfrak{F}_n) - E\Big(X - E(X|\mathfrak{F}_n) \Big| \mathfrak{F}_m\Big)$$

$$
\begin{aligned}
&= X - E(X \mid \mathfrak{F}_n) - E(X \mid \mathfrak{F}_m) \\
&+ E\left(E(X \mid \mathfrak{F}_n) \mid \mathfrak{F}_m\right) \\
&= X - E(X \mid \mathfrak{F}_n) - E(X \mid \mathfrak{F}_m) \\
&+ E\left(X \mid \mathfrak{F}_{\min(m,\ n)}\right)
\end{aligned}
$$

by Th. 1.2.5

Hence $S_m S_n X = X - E(X \mid \mathfrak{F}_{\max(m,n)}) = S_{\max.(m,n)}$

so that

$G = (S_n, n \geq 1)$ is a semi-group and according to lemma 3.1.1 S_n's are bounded linear operators. We shall show that the sequence of operators $(S_n, n \geq 1)$ themselves form a system of almost invariant integrals for G. Condition (1) is clearly satisfied. Now $S_n = I - T_n$ where T_n is as defined in 3.2.1.

$$\therefore \quad \| S_n \| \leq \| I \| + \| T_n \| = 1 + 1 = 2 \ldots . (3.2.3)$$

by lemma 3.1.1 so that S_n's are uniformly bounded in norm.

Also, for any m $\lim_n (S_m S_n S - S_n X) = \lim_n (S_n S_m X - S_n X)$
$= 0$

because of (3.2.2) .

Thus all the conditions (1) - (iii) in Eberlein's theorem are satisfied and we can therefore conclude that whenever $S_n X$ has a weak cluster point Y, $S_n X$ actually converges strongly to Y.

Now if $p > 1$, and $X \in L_p(\Omega, \mathfrak{F}_\infty, P, \mathfrak{X})$ we have

$$\left[S_n X \right]_p \leq \| S_n \| \cdot \left[X \right]_p$$

$$\leq 2 . \left[X \right]_p$$

because of (3.2.3).
Hence $\{ S_n X, n \geq 1 \}$ is a bounded set in $L_p(\Omega, \mathfrak{F}_\infty, P, \mathfrak{X})$; $\mathfrak{X}$ being reflexive so is $L_p(\Omega, \mathfrak{F}_\infty, P, \mathfrak{X})$, (Th. 2.2.1) and hence every bounded set in $L_p(\Omega, \mathfrak{F}_\infty, P, \mathfrak{X})$ is weakly compact. Therefore, $\{ S_n X \}$ has a weak cluster point and hence according to Eberlein's theorem $\lim_{n \to \infty} S_n X$ exists in the strong topology of $L_p(\Omega, \mathfrak{F}_\infty, P, \mathfrak{X})$.
If $p = 1$, $\{ S_n X, n \geq 1 \}$ is still a weakly compact set in $L_1(\Omega, \mathfrak{F}_\infty, P, \mathfrak{X})$. This is so because $\{ \| S_n X \|, \mathfrak{F}_n, 1 \leq n \leq \infty \}$ is a semi-martingale of real-valued r.v.'s and hence from Doob (1) Th. 3.1 (iii) pp. 311 we conclude that $\| S_n X \|$ are uniformly integrable. Also $S_n X$ is bounded in norm in $L_1(\Omega, \mathfrak{F}_\infty, P, \mathfrak{X})$. Therefore, $\{ S_n X, n \geq 1 \}$ is weakly compact by our Th. 2.4.1.

Thus, we have shown that for any $X \in L_p(\Omega, \mathfrak{F}_\infty, P, \mathfrak{X})$ $p \geq 1$ $\lim_{n \to \infty} S_n X$ exists in the strong topology or in terms of operator theory, the sequence of operators S_n converge strongly.

Let $Y_\infty = \lim_{n \to \infty} S_n X_\infty$

(the limit on the right exists because $X_\infty \in L_p(\Omega, \mathfrak{F}_\infty, P, \mathfrak{X})$

(lemma 3.1.1).

Now $E(X_\infty \mid \mathfrak{F}_n) = X_n$ a.e. . . . . .(3.2.4)

and hence for any $E \in \mathfrak{F}_n$

$$\int_E X_\infty \, dP = \int_E X_n \, dP \quad . . . . . .(3.2.5)$$

Also by (3.2.4) $Y_\infty = \lim_{n \to \infty} (I - T_n) X = X - \lim_{n \to \infty} X_n$

. . .(3.2.6)

i.e.

$$\lim_{n \to \infty} \left[X_\infty - X_n - Y_\infty \right]_p = 0$$

Hence for any $E \in \mathfrak{F}_\infty$

$$\int_E (X_\infty - X_n - Y_\infty) \longrightarrow 0 \text{ strongly in } \mathfrak{X}$$

i.e.

$$\int_E (X_\infty - X_n) \, dP \xrightarrow{S} \int_E Y_\infty \, dP$$

or $$\int_E X_n dP \xrightarrow{S} \int_E (X_\infty - Y_\infty) \, dP \qquad (3.2.7)$$

From (3.2.5) and (3.2.7) we have for every $E \in \mathfrak{F}_n$

$$\int_E X_\infty \, dP = \int_E (X_\infty - Y_\infty) \, dP$$

i.e. $$\int_E Y_\infty \, dP = 0$$

Hence $\int_E Y_\infty \, dP = 0$ for all $E \in \bigcup_{n=1}^{\infty} \mathfrak{F}_n$

Now $\bigcup_{n=1}^{\infty} \mathfrak{F}_n$ being the field that generates the Borel-field $\mathfrak{F}_\infty$ we must have

$$\int_E Y_\infty \, dP = 0 \quad \text{for all} \quad E \in \mathfrak{F}_\infty$$

and hence $Y_\infty = 0$ a.e.

From (3.2.6) then it follows that

$$\lim_{n \to \infty} X_n = X_\infty$$

and this completes the first proof of the theorem.

Proof (ii) Let $\mathfrak{F}_0 = \bigcup_{n=1}^{\infty} \mathfrak{F}_n$. Because $\mathfrak{F}_n \subset \mathfrak{F}_{n+1}$, $\mathfrak{F}_0$ is a field.

We shall need the following lemma:

<u>Lemma 3.2.1</u>

The class of simple functions measurable with respect to $\mathfrak{F}_0$ (i.e. functions like $X(\omega) = \sum_{i=1}^{k} a_i \chi_{E_i}(\omega)$, $a_i \in \mathfrak{X}$, $E_i \in \mathfrak{F}_0$) is dense in $L_p(\Omega, \mathfrak{F}_\infty, P, \mathfrak{X})$ $p \geq 1$.

<u>Proof of the lemma:</u> Let $E \in \mathfrak{F}_\infty$. Be a theorem in measure theory ((4) Halmos, Th. D, pp. 56), for any $\epsilon > 0$, there exists $E_0 \in \mathfrak{F}_0$ such that

$$P(E \,\Delta\, E_0) = P(E - E_0) + P(E_0 - E) < \epsilon$$

if $X(\omega) = \chi_E(\omega).a \quad E \in \mathfrak{F}_\infty, \quad a \in \mathfrak{X}$

$Y(\omega) = \chi_{E_0}(\omega).a \quad E_0 \in \mathfrak{F}_0, \quad P(E \,\Delta\, E_0) < \epsilon$

then

$$[X - Y]_p = \left(\int_\Omega \| X - Y \|^p \, dP \right)^{1/p}$$

$$= \| a \| \left(\int_{E \Delta E_0} dP \right)^{1/p} \leq \|a\| \cdot \epsilon^{1/p}$$

ϵ being arbitrary it's clear that we can choose $Y(\omega)$ measurable with respect to $\mathfrak{F}_0$ and as close to $X(\omega)$ in $L_p(\mathfrak{X})$ norm as we please.

Hence any simple function in $L_p(\Omega, \mathfrak{F}_\infty, P, \mathfrak{X})$ can be approximated by simple functions measurable with respect to $\mathfrak{F}_0$. Since the simple functions are dense in $L_p(\mathfrak{X})$, ((2), Dunford & Schwartz, pp. 125), we can approximate an arbitrary function in $L_p(\mathfrak{X})$ by simple functions measurable with respect to $\mathfrak{F}_0$ as closely as we wish. This proves the lemma.

Consider now the following sequence of mappings from the Banach space $L_p(\Omega, \mathfrak{F}_\infty, P, \mathfrak{X})$ to itself

$$T_n : L_p(\Omega, \mathfrak{F}_\infty, P, \mathfrak{X}) \longrightarrow L_p(\Omega, \mathfrak{F}_\infty, P, \mathfrak{X})$$

$$T_n X = E(X \mid \mathfrak{F}_n)$$

Now

$$\sup_n [T_n X]_p = \sup_n \left(\int \| T_n X \|^p \, dP \right)^{1/p}$$

$$\leq [X]_p \quad \text{since} \quad \| T_n X \|^p$$

$$\leq \left\{ E(\| X \| \mid \mathfrak{F}_n) \right\}^p \leq E(\| X \|^p \mid \mathfrak{F}_n)$$

Hence the set $(T_n X, n \geq 1)$ is bounded for each $X \in L_p(\Omega, \mathfrak{F}_\infty, P, \mathfrak{X})$. If $X(\omega) = \chi_F(\omega).a$ where $a \in \mathfrak{X}$, $F \in \mathfrak{F}_o$ then since for some N, $F \in \mathfrak{F}_n$ $n \geq N$ we have

$$T_n X = X \qquad \text{a.e. for } n \geq N .$$

Hence

$$\lim_{n \to \infty} \left[T_n X - X \right]_p = 0 \quad \ldots . (3.2.8)$$

and so (3.2.8) is true for all simple functions $X(\omega)$ measurable with respect to $\mathfrak{F}_o$. Since such functions are dense in $L_p(\Omega, \mathfrak{F}_\infty, P, \mathfrak{X})$ according to lemma 3.2.1 we have by the Banach-Steinhaus Theorem (3) that

$$\lim_{n \to \infty} T_n X = TX \quad \ldots . (3.2.9)$$

exists for all $X \in L_p(\Omega, \mathfrak{F}_\infty, P, \mathfrak{X})$ and moreover that T is a bounded linear operator.

For $X(\omega)$'s which are simple functions measurable with respect to $\mathfrak{F}_o$ we have $TX = X$.

Such functions being dense in $L_p(\Omega, \mathfrak{F}_\infty, P, \mathfrak{X})$ we can obtain, given an arbitrary X, a sequence X_n of them such that

$$\lim_{n \to \infty} \left[X_n - X \right]_p = 0$$

T being continuous we have

$$TX = \lim_{n \to \infty} TX_n = \lim_{n \to \infty} X_n = X \quad ..(3.2.10)$$

Hence, we have proved ((3.2.9) and (3.2.10)) that $E(X \mid \mathfrak{F}_n)$ converges in $L_p(\Omega, \mathfrak{F}_\infty, P, \mathfrak{X})$ norm to X for every $X \in L_p(\Omega, \mathfrak{F}_\infty, P, \mathfrak{X})$. The theorem then follows by taking

$$X = E(Z \mid \mathfrak{F}_\infty).$$

Theorem 3.2.2

Let $(X_n(\omega), \mathfrak{F}_n, n \leq -1)$ be a martingale with $\mathfrak{F}_{-n} \supset \mathfrak{F}_{-(n+1)}$, $X_{-1}(\omega) \in L_p(\Omega, \mathfrak{B}, P, \mathfrak{X})$ $p \geq 1$. Then $X_{-n}(\omega)$ converges in $L_p(\mathfrak{X})$ norm to $X_{-\infty}(\omega)$ i.e.

$$\lim_{n \to \infty} \left[X_{-\infty} - X_{-n} \right]_p = 0$$

where

$$X_{-\infty}(\omega) = E(X_{-1}(\omega) \mid \mathfrak{F}_{-\infty}) \quad \text{a.e.}$$

and

$$\mathfrak{F}_{-\infty} = \bigcap_{n=1}^{\infty} \mathfrak{F}_{-n}$$

Proof: We shall again present two different proofs; the first proof uses Eberlein's mean ergodic theorem with the additional assumption that $\mathfrak{X}$ is reflexive and the second prof is based on an application of Banach-Steinhaus theorem.

Proof (1) $\mathfrak{X}$ reflexive:

Define the bounded linear operators T_n on $L_p(\Omega, \mathfrak{F}_1, P, \mathfrak{X})$ to itself as follows:

$$T_n X = E(X \mid \mathfrak{F}_{-n}) \qquad X \in L_p(\Omega, \mathfrak{F}_{-1}, P, \mathfrak{X}),\ n \geq 1.$$

Then

$$\| T_n X \| \leq E(\|X\| \mid \mathfrak{F}_{-n}) \quad \text{a.e.}$$

and by Jensen's inequality

$$\| T_n X\|^p \leq E(\|X\|^p \mid \mathfrak{F}_{-n})$$

Hence

$$\left[T_n X\right]_p = \left(\int_\Omega \| T_n X\|^p \, dP \right)^{1/p} \leq \left[X\right]_p$$

so that $T_n X \in L_p(\Omega, \mathfrak{F}_1, P, \mathfrak{X})$ $(\because \mathfrak{F}_{-n} \subset \mathfrak{F}_{-1})$

and $\quad \|T_n\| = 1 \quad$ for all $\quad n$

Also $\quad T_m \cdot T_n = T_{\max.(m,\, n)}$

Hence $\quad G = (T_n, n \geq 1)$ is a semigroup of bounded linear operators for which $(T_n, n \geq 1)$ itself is a system of almost invariant integrals for G. Hence by Eberlein's mean ergodic theorem (see Th. 3.2.1) $T_n X_{-1} = X_{-n}(\omega)$ goes to a limit in the norm of $L_p(\Omega, \mathfrak{F}_{-1}, P, \mathfrak{X})$ whenever it has a weak cluster point.

Now $L_p(\Omega, \mathfrak{F}_{-1}, P, \mathfrak{X})$ for $p > 1$ is reflexive (Th. 2.2.1) and hence every bounded set in it has a weak cluster point.

But $\left[X_{-n}\right]_p \leq \left[X_{-1}\right]_p$ so that the set $\left\{T_n X_{-1}\right\}$ or $\left\{X_{-n}\right\}$ is bounded and this, in conjunction with the previous comment, proves the assertion of the theorem when $p > 1$.

When $p = 1$ we notice that $\left\{\|X_{-n}\|, \mathfrak{F}_{-n}, n \geq 1\right\}$

is a semi-martingale and hence by Th. 3.1 (iii), pp. 311, Doob (1) we conclude that $\|X_{-n}\|$ are uniformly integrable. Also $\{X_{-n}\}$ is a bounded set in $L_1(\mathfrak{X})$ and so by our theorem 2.4.2 $\{X_{-n}\}$ is weakly compact. Hence X_{-n} converges strongly in $L_1(\mathfrak{X})$.

We shall now show that $\lim_{n \to \infty} X_{-n} = X_{-\infty}$.

Let $\lim_{n \to \infty} X_{-n} = Y$. Then $E(Y \mid \mathfrak{F}_{-m}) = Y$ for all $m \geq 1$. i.e. Y is measurable with respect to $\mathfrak{F}_{-m}$ for any $m \geq 1$ and hence measurable with respect to $\mathfrak{F}_{-\infty}$.

Also for any $A \in \mathfrak{F}_{-\infty}$.

$$\int_A Y \, dP = \lim_{n \to \infty} \int_A X_{-n} \, dP = \int_A X_{-1} \, dP$$

because $X_{-n} = E(X_{-1} \mid \mathfrak{F}_{-n})$.

This proves that $Y = X_{-\infty}$ a.e.

Proof (ii) Define
$$T_n : L_p(\Omega, \mathfrak{F}_{-1}, P, \mathfrak{X}) \longrightarrow L_p(\Omega, \mathfrak{F}_{-1}, P, \mathfrak{X})$$
$$T_n X = E(X \mid \mathfrak{F}_{-n}) \quad n \geq 1$$

T_n are bounded linear operators (lemma 3.1.1) such that

$$\sup_n \left[T_n X \right]_p \leq [X]_p \quad \ldots\ldots (3.2.11)$$

Also if $X(\omega) = \chi_E(\omega).a$, $a \in \mathfrak{X}$, $E \in \mathfrak{F}_{-1}$

then $T_n X = P(E \mid \mathfrak{F}_{-n}).a$

Now $P(E \mid \mathfrak{F}_{-n})$ converges to $P(E \mid \mathfrak{F}_{-\infty})$ in L_P

i.e.

$$\lim_{n \to \infty} \left(\int \left| P(E \mid \mathfrak{F}_{-\infty}) - P(E \mid \mathfrak{F}_{-n}) \right|^{p} dP \right)^{1/p} = 0 \qquad \ldots\ldots (3.2.12)$$

This follows either by applying Doob's Th. 4.2, pp. 328, (1), or by considering $P(E \mid \mathfrak{F}_{-n})$ as a real-valued martingale and then applying Proof (1) which is applicable since the real numbers form a reflexive Banach-space.

It follows from (3.2.12) that if $X(\omega) = \chi_E(\omega).a$ then $T_n X$ converges in $L_p(\mathfrak{X})$. Hence $T_n X$ converges for all simple functions $X(\omega)$ which form an everywhere dense set in $L_p(\mathfrak{X})$. This and (3.2.11) enables us to conclude by virtue of Banach-Steinhaus theorem (3) that

$$\lim_{n \to \infty} T_n X = TX$$

in $L_p(\mathfrak{X})$ for all $X \in L_p(\mathfrak{X})$ where T is a bounded linear operator.

This proves Th. 3.2.2 for a general Banach-space $\mathfrak{X}$.

§ 3. In this section we shall prove mean convergence theorems for arbitrary martingales of r.v.'s taking values in a reflexive Banach-space. We shall need the following lemma:

Lemma 3.3.1

Let T_n, $n = 1, 2, \ldots$ and T be bounded linear operators mapping the Banach space $\mathfrak{X}$ into itself and such that

i) $\lim_{n \to \infty} T_n x = Tx$ for all $x \in \mathfrak{X}$

and ii) $T_m \cdot T_n = T_{\min.(m,n)}$

Let $x_n \in \mathfrak{X}$ such that there is a subsequence x_{n_k} converging weakly to x_∞ and also that $T_n x_{n+1} = x_n$.

Then $\lim_{n \to \infty} x_n = x_\infty$ strongly.

<u>Proof:</u> From the conditions of the lemma we have

$$T_m x_n = x_{\min(m,\, n)} \qquad \ldots\ldots\ldots (3.2.13)$$

Also $T_m x_{n_k} \xrightarrow{\omega} T_m x_\infty$ as $k \longrightarrow \infty$ for any m.

By (3.2.13) $T_m x_{n_k} = x_m$ for large k so that

$$T_m x_\infty = x_m$$

Now by condition (i) of the lemma $T_m x_\infty \xrightarrow{S} T x_\infty$ so that it follows that

$$\lim_{m \to \infty} x_m = x_\infty \text{ strongly.}$$

<u>Theorem 3.3.1</u> Let $\mathfrak{X}$ be a reflexive Banach space and let $\{ X_n, \mathfrak{F}_n, n \geq 1 \}$ be a $\mathfrak{X}$-valued martingale such that

$$X_n \in L_p(\Omega, \mathcal{B}, P, \mathfrak{X}) \quad n \geq 1, \quad p > 1 \text{ and } \left[X_n\right]_p < C$$

C independent of n. Then there exists $X_\infty \in L_p(\Omega, \mathcal{B}, P, \mathfrak{X})$ such that

$$\lim_{n \to \infty} \left[X_n - X_\infty\right]_p = 0$$

<u>Proof:</u> Define $T_n : L_p(\Omega, \mathfrak{F}_\infty, P, \mathfrak{X})$

$$\longrightarrow L_p(\Omega, \mathfrak{F}_\infty, P, \mathfrak{X})$$

$$T_n X = E(X \mid \mathfrak{F}_n)$$

$$\mathfrak{F}_\infty = \text{Borel-field generated by } \bigcup_{n=1}^{\infty} \mathfrak{F}_n$$

We have

$$T_m \cdot T_n = T_{\min.(m,\, n)}$$

and by Th. 3.2.1 $\lim_{n \to \infty} T_n X = X$ for all $X \in L_p(\Omega, \mathfrak{F}_\infty, P, \mathfrak{X})$.

Also $T_n X_{n+1} = X_n$ because $\{X_n, \mathfrak{F}_n, n \geq 1\}$ is a martingale.

Hence

$$[X_n]_p \leq \| T_n \| . [X_{n+1}]_p = [X_{n+1}]_p < C$$

so that $\{X_n\}$ is a bounded set in $L_p(\Omega, \mathfrak{F}_\infty, P, \mathfrak{X})$ which being reflexive ($\because$ $\mathfrak{X}$ is reflexive) (Th. 2.2.1) the set $\{X_n\}$ is weakly compact i.e. there is a subsequence $\{X_{n_k}\}$ converging weakly to some element, say $X_\infty \in L_p(\Omega, \mathfrak{F}_\infty, P, \mathfrak{X})$.

Thus T_n's and X_n's satisfy all the conditions of lemma 3.3.1 which therefore guarantees the assertion in Th. 3.3.1.

<u>Theorem 3.3.2</u> Let $\mathfrak{X}$ be a reflexive Banach space and let $\{X_n, \mathfrak{F}_n, n \geq 1\}$ be a $\mathfrak{X}$-valued martingale such that

$$X_n \in L_1(\Omega, \mathcal{B}, P, \mathfrak{X})$$

Suppose that $\|X_n\|$'s are uniformly integrable.

Then there exists $X_\infty \in L_1(\Omega, \mathcal{B}, P, \mathfrak{X})$ such that

$$\lim_{n \to \infty} [X_n - X_\infty]_1 = 0$$

<u>Proof:</u> We define T_n's as in the previous proof and then the previous arguments would prove the assertion in this theorem if we could show that $\{X_n\}$ is weakly compact in $L_1(\mathfrak{X})$. This we do by first showing that $[X_n]_1 < K$, K independent of n.

Because $E(X_{n+1} \mid \mathfrak{F}_n) = X_n$ a.e.

$$\int_E X_{n+1} dP = \int_E X_n \, dP \quad \text{for all} \quad E \in \mathfrak{F}_n .$$

Hence $\int_E X_n \, dP$ converges strongly to a limit for every $E \in \bigcup_{n=1}^{\infty} \mathfrak{F}_n$. Now let $\mu_n(E) = \int_E X_n dP \quad E \in \mathfrak{F}_\infty$

$\mu_n(E)$ is uniformly countably additive on $\mathfrak{F}_\infty$ i.e. if $E_n \subset E_{n-1}$ and

$$E_n \in \mathfrak{F}_\infty , \quad \bigcap_{n=1}^{\infty} E_n = \phi$$

then $\lim_{n \to \infty} \mu_m(E_n) = 0$ uniformly in m

This is so because

$$\| \mu_m(E_n) \| \leq \int_{E_n} \| X_m \| \, dP$$

and because $\| X_n(\omega) \|$ are uniformly integrable.

Since $\bigcup_{n=1}^{\infty} \mathfrak{F}_n$ generates the Borel-field $\mathfrak{F}_\infty$, it follows from Lemma 8 pp. 292 (2, Dunford & Schwartz) that $\mu_n(E)$ converges strongly for all $E \in \mathfrak{F}_\infty$.

Hence, also for any $x^* \in \mathfrak{X}^*$ $\lim_{n \to \infty} x^*\left(\int_E X_n \, dP\right)$ exists so that

$$\lim_{n \to \infty} \int \langle X_n(\omega), Y^*(\omega) \rangle \, dP$$

converges for all $Y^*(\omega) \in L_\infty(\mathfrak{X}^*)$ which are simple. Hence, it can be shown that

$$\int \langle X_n(\omega), Y^*(\omega) \rangle \, dP$$

converges for all $Y^*(\omega) \in L_\infty(\mathfrak{X}^*)$. In other words, $X_n(\omega)$ is weakly convergent in $L_1(\mathfrak{X})$ and hence bounded ((3), pp. 36).

$\{X_n(\omega)\}$ being a bounded set such that $\|X_n(\omega)\|$'s are uniformly integrable it follows from Th. 2.4.1 that $\{X_n(\omega)\}$ is weakly compact.

This terminates the proof of the theorem.

Chapter 1V

Almost everywhere convergence of Banach-valued strong martingales.

§1. In this chapter we study the almost everywhere convergence of certain special types of martingales, namely the ones generated by taking repeated conditional expectations of a fixed r.v., and other cases which can be reduced to this case. Our proofs are quite different from the ones used by Doob (1) in the classical real or complex-valued cases. We use a theorem originally due to Banach (1926, 12) and a generalized version of which is in (2, Dunford & Schwarz, pp. 332, Th. 3). As pointed out in the foot-note of a paper by (Schwarz & Beck, 1957, 13), the theorem can be extended to Banach-valued functions without any change in proof. We shall state the theorem in a slightly restricted form in which we shall apply it here:

Let T_n be a sequence of continuous linear operators on a Banach space $\mathcal{Y}$ to $L_1(\Omega, \mathcal{B}, P, \mathfrak{X})$ such that

i) $\sup_n \| T_n Y(\omega) \| < \infty$ a.e. for each $Y \in \mathcal{Y}$

ii) $\lim_{n \to \infty} T_n Y(\omega)$ exists a.e. for $Y \in \mathcal{Y}_0 \subset \mathcal{Y}$, $\mathcal{Y}_0$ dense in $\mathcal{Y}$.

Then $\lim_{n \to \infty} T_n Y(\omega)$ exists a.e. for any $Y \in \mathcal{Y}$

Here $T_n Y(\omega)$ stands for that functional element in $L_1(\Omega, \mathcal{B}, P, \mathfrak{X})$ which corresponds to the element $Y \in \mathcal{Y}$ under the mapping T_n.

§2. Theorem 4.2.1 Let $Z(\omega) \in L_1(\Omega, \mathcal{B}, P, \mathfrak{X})$ where $\mathfrak{X}$ is arbitrary and let $X_n = E(Z \mid \mathfrak{F}_n)$ $n \geq 1$ where $\mathfrak{F}_n$ are Borel-fields such that

$$\mathfrak{F}_n \subset \mathfrak{F}_{n+1} \subset \mathcal{B} .$$

Let $\mathfrak{F}_\infty$ be the Borel-field generated by the field

$$\mathfrak{F}_0 = \bigcup_{n=1}^{\infty} \mathfrak{F}_n$$

and let

$$X_\infty(\omega) = E(Z \mid \mathfrak{F}_\infty) .$$

Then $X_n(\omega)$ converges strongly in $\mathfrak{X}$ to $X_\infty(\omega)$ for a.e. ω.

Proof: Define $T_n Z = E(Z \mid \mathfrak{F}_n)$ where $Z \in L_1(\Omega, \mathcal{B}, P, \mathfrak{X})$. Then by lemma 3.1.1, T_n is a bounded linear operator from the Banach space $L_1(\Omega, \mathcal{B}, P, \mathfrak{X})$ into $L_1(\Omega, \mathfrak{F}_\infty, P, \mathfrak{X})$.

We have

$$\| T_n Z(\omega) \| \leq E(\| Z \| \mid \mathfrak{F}_n) \quad \text{a.e.} \quad ..(4.2.1)$$

We shall first demonstrate that

$$\sup_n E(\| Z \| \mid \mathfrak{F}_n) < +\infty \quad \text{a.e.} \quad ...(4.2.2)$$

This we could do by appealing to Th. 4.3., pp. 331 in Doob (1). However, we prefer to give a simple independent proof of (4.2.2) in the following lemma:

<u>Lemma 4.2.</u> Let $\{X_n, \mathfrak{F}_n, n \geq 1\}$ be any martingale taking values in any Banach space $\mathfrak{X}$.
Then $P(\omega: \sup_n \| X_n(\omega) \| = +\infty) = 0$ if $E(\| X_n \|) < C$, independent of n.

<u>Proof of the lemma:</u>

Let
$$A = (\omega : \sup_n \| X_n(\omega) \| = +\infty)$$
$$A_M = (\omega : \sup_n \| X_n(\omega) \| > M)$$

Then
$$A = \bigcap_{M=1}^{\infty} A_M \quad \text{and} \quad A_M \supset A_{M+1}$$

Now
$$A_M = \bigcup_{i=1}^{\infty} B_i \quad \text{where}$$

$$B_i = (\omega : \| X_i(\omega) \| > M, \ \| X_1(\omega) \| \leq M, \ \|X_2(\omega)\| \leq M, \ . \ . \ . \ \|X_{i-1}(\omega)\| \leq M)$$

B_i's are disjoint and $B_i \in \mathfrak{F}_i$.

Since $B_i \in \mathfrak{F}_i$ and $\| X_i(\omega) \| \leq E(\| X_N \| \mid \mathfrak{F}_i)$ for $N \geq i$,

we have
$$\int_{B_i} \| X_N \| dP \geq \int_{B_i} \|X_i\| dP > M \, P(B_i)$$

Hence
$$\int_{\bigcup_{i=1}^{N} B_i} \| X_N \| dP = \sum_{i=1}^{N} \int_{B_i} \| X_N \| dP > M \sum_{i=1}^{N} P(B_i)$$

so that

$$P\Big(\bigcup_{i=1}^{N} B_i\Big) < \frac{1}{M} \int_{\bigcup_{i=1}^{N} B_i} \| X_N \| \, dP \leq \frac{1}{M} \int \| X_N \| \, dP \leq \frac{C}{M}$$

Hence taking $N \longrightarrow \infty$

$$P(A_M) < \frac{C}{M}$$

Hence $\lim_{M \to \infty} P(A_M) = 0$

But $\lim_{M \to \infty} P(A_M) = P(A)$

and so

$$P(A) = 0$$

This completes the proof of the lemma.

If $Z(\omega) = \chi_E(\omega).a$ where $a \in \mathfrak{X}$, $E \in \mathfrak{F}_o$ then we have $T_n Z(\omega) = Z(\omega)$ for $n \geq N$. Hence

$$\lim_{n \to \infty} T_n Z(\omega) \text{ exists for such } Z(\omega)$$

and hence for simple $Z(\omega)$ measurable with respect to $\mathfrak{F}_o$.

Now let us apply the theorem mentioned in § 1 with

$$\mathcal{Y} = L_1(\Omega, \mathfrak{F}_\infty, P, \mathfrak{X})$$

$$T_n Z(\omega) = E(Z \mid \mathfrak{F}_n)$$

According to lemma 3.2.1, the simple $Z(\omega)$ measurable with respect to $\mathfrak{F}_o$ are dense in $L_1(\Omega, \mathfrak{F}_\infty, P, \mathfrak{X})$ and hence, all the conditions stipulated in the theorem in § 1 are valid. Thus, we can conclude that

$$\lim_{n \to \infty} E(X \mid \mathfrak{F}_n) \text{ exists a.e. strongly for every}$$

$$Z(\omega) \in L_1(\Omega, \mathfrak{F}_\infty, P, \mathfrak{X}).$$

For any $Z(\omega) \in L_1(\Omega, \mathfrak{B}, P, \mathfrak{X})$ we consider

$$X_\infty(\omega) = E(Z \mid \mathfrak{F}_\infty) \in L_1(\Omega, \mathfrak{F}_\infty, P, \mathfrak{X})$$

and as

$$X_n = E(Z \mid \mathfrak{F}_n) = E(X_\infty \mid \mathfrak{F}_n)$$

it follows that

$$\lim_{n \to \infty} X_n(\omega)$$

exists a.e. for any $Z \in L_1(\Omega, \mathfrak{B}, P, \mathfrak{X})$.

To show that the limit is indeed $X_\infty(\omega)$ we simply observe that $X_n(\omega)$ converges in mean to $X_\infty(\omega)$ by Th. 3.2.1 and that a.e. convergence and convergence in mean are compatible.

This concludes the proof of the theorem.

<u>Theorem 4.2.2</u> Let $\{X_n(\omega), \mathfrak{F}_n, n \geq 1\}$ be any $\mathfrak{X}$-valued martingale where $\mathfrak{X}$ is a reflexive Banach space. Let $\|X_n(\omega)\|$, $n \geq 1$ be uniformly integrable. Then there exists a $\mathfrak{X}$-valued r.v. $X_\infty(\omega)$ such that

$$\lim_{n \to \infty} X_n(\omega) = X_\infty(\omega) \quad \text{a.e.}$$

<u>Proof:</u> According to Th. 3.3.2, there exists $X_\infty(\omega) \in L_1(\Omega, \mathfrak{F}_\infty, P, \mathfrak{X})$ such that

$$\lim_{n \to \infty} \left[X_n - X_\infty \right]_1 = 0$$

Hence $$\lim_{n \to \infty} \int_A X_n dP = \int_A X_\infty dP \qquad A \in \mathfrak{F}_\infty \quad ..(4.2.3)$$

As $E(X_{n+k} \mid \mathfrak{F}_n) = X_n$ a.e. $k \geq 0$ we have for any $B \in \mathfrak{F}_n$

$$\int_B X_n dP = \int_B X_{n+k} dP$$

Making $k \longrightarrow \infty$ we have because of (4.2.3)

$$\int_B X_n dP = \int_B X_\infty dP \quad \text{for any} \quad B \in \mathfrak{F}_n .$$

This means that

$$E(X_\infty \mid \mathfrak{F}_n) = X_n \qquad \text{a.e.} \qquad n \geq 1$$

From the preceding theorem, then, we can conclude that

$$\lim_{n \to \infty} X_n(\omega) = X_\infty(\omega) \qquad \text{a.e.}$$

Our next theorem is about the almost everywhere convergence of martingales with decreasing index set.

<u>Theorem 4.2.3</u> Let $\{ X_n, \mathfrak{F}_n, n \leq -1 \}$ be any $\mathfrak{X}$-valued martingale where the Banach space $\mathfrak{X}$ is arbitrary. Then

$$\lim_{n \to \infty} X_n(\omega) = X_{-\infty}(\omega) \qquad \text{a.e.}$$

where

$$X_{-\infty}(\omega) = E(X_{-1}(\omega) \mid \mathfrak{F}_{-\infty}), \quad \mathfrak{F}_\infty = \bigcap_{n \leq -1} \mathfrak{F}_n$$

<u>Proof:</u> We notice firstly that $X_{-n} = E(X_{-1} \mid \mathfrak{F}_{-n})$ a.e. Define T_n as the continuous linear operator from $L_1(\Omega, \mathfrak{F}_{-1}, P, \mathfrak{X})$ to itself, given by $T_n X = E(X \mid \mathfrak{F}_{-n})$. The proof will be completed by showing that for every $X \in L_1(\Omega, \mathfrak{F}_{-1}, P, \mathfrak{X})$, $T_n X$ converges a.e. That the limit is the prescribed one follows by

noticing that it is also the limit we get from the mean convergence of $T_n X$ (Th. 3.2.2).

If $X(\omega) = \chi_E(\omega).a$, $a \in \mathfrak{X}$, $E \in \mathfrak{F}_{-1}$

Because $\lim_{n \to \infty} P(E \mid \mathfrak{F}_{-n}) = P(E \mid \mathfrak{F}_{-\infty})$ a.e.
(see Doob, 1)

$$\lim_{n \to \infty} T_n X = P(E \mid \mathfrak{F}_{-\infty}).a = E(X \mid \mathfrak{F}_{-\infty}) \text{ a.e.}$$

Hence

$\lim_{n \to \infty} T_n X$ exists a.e. for any simple function

$$X(\omega) \in L_1(\Omega, \mathfrak{F}_{-1}, P, \mathfrak{X})$$

Also

$$\sup_n \| T_n X(\omega) \| = \sup_n \| E(X \mid \mathfrak{F}_{-n}) \| < \infty \text{ a.e.}$$

as one can show by a proof similar to lemma 4.2.1.

Now by an application of the theorem mentioned in § 1 we can conclude that

$\lim_{n \to \infty} T_n X$ exists a.e. for every $X \in L_1(\Omega, \mathfrak{F}_{-1}, P, \mathfrak{X})$.

This finishes our proof.

§ 3. In this section we shall prove an almost-everywhere convergence theorem by using the idea of optional stopping (Doob, pp. 300, 1).

Let $m(\omega)$ be a random-variable whose finite values are positive integers and which may be $+\infty$ with positive probability. Let $(X_n, \mathcal{F}_n, n \geq 1)$ be a $\mathcal{X}$-valued martingale and let

$$\{\omega : m(\omega) = k\} \in \mathcal{F}_k$$

We shall define the random variables $\check{X}_n(\omega)$ $n \geq 1$ as follows

$$\begin{aligned}\check{X}_j(\omega) &= X_j(\omega) & \omega \in \{\omega : j \leq m(\omega)\} \\ &= X_{m(\omega)}(\omega) & \omega \in \{\omega : j > m(\omega)\}\end{aligned}$$

<u>Lemma 4.3.1</u>

$\{\check{X}_n, \mathcal{F}_n, n \geq 1\}$ is a $\mathcal{X}$-valued martingale.

<u>Proof:</u>

Because $\{\omega : m(\omega) = k\} \in \mathcal{F}_k$ we have $\{\omega : m(\omega) \geq n\} \in \mathcal{F}_{n-1}$ and $\{\omega : m(\omega) < n\} \in \mathcal{F}_{n-1}$

Hence

$$\check{X}_n(\omega) = X_n(\omega)\, \chi_{\{m(\omega) \geq n\}}(\omega) + \sum_{k=1}^{n-1} X_k(\omega)\, \chi_{\{m(\omega) = k\}}(\omega)$$

is $\mathcal{F}_n$-measurable.

We shall now prove that $E(\check{X}_{n+1} \mid \mathcal{F}_n) = \check{X}_n$ a.e.

Let $A \in \mathcal{F}_n$. Then

$$\int_A \check{X}_{n+1}\, dP = \int_{A \cap \{m(\omega) \leq n\}} \check{X}_{n+1}\, dP + \int_{A \cap \{m(\omega) > n\}} \check{X}_{n+1}\, dP$$

Now on $\{m(\omega) \leq n\}$, $\check{X}_{n+1} = \check{X}_n$ and

$\{m(\omega) > n\} \in \mathfrak{F}_n$; also on the latter set $\check{X}_n = X_n$ and hence by martingale property

$$\int_A \check{X}_{n+1} \, dP = \int_{A \cap \{m(\omega) \leq n\}} \check{X}_n \, dP + \int_{A \cap \{m(\omega) > n\}} \check{X}_n \, dP$$

$$= \int_A \check{X}_n \, dP$$

This proves the lemma.

<u>Theorem 4.3.1</u> Let $\{X_n, \mathfrak{F}_n, n \geq 1\}$ be a $\mathfrak{X}$-valued martingale where $\mathfrak{X}$ is a reflexive Banach space.

Let $E(\sup_{n \geq 0} \| X_n^{(\omega)} - X_{n-1}^{(\omega)} \|) < +\infty$. $X_0(\omega) \equiv 0$

Then $\lim_{n \to \infty} X_n(\omega)$ exists whenever

$$\omega \in \{\omega : \sup_n \| X_n(\omega) \| < +\infty\}$$

<u>Proof:</u> Let $M > 0$ be any positive integer and define

$$m(\omega) = n \quad \omega \in \{\| X_1(\omega)\| \leq M, \ldots$$
$$\|X_{n-1}(\omega)\| \leq M, \| X_n(\omega)\| > M\}$$
$$= \infty \quad \omega \in \{\sup_n \|X_n(\omega)\| < +\infty\}$$

Of course $\{m(\omega) = n\} \in \mathfrak{F}_n$.

Let $\check{X}_{n, M}(\omega)$ be defined as follows

$$\check{X}_{n,M}(\omega) = X_n(\omega) \qquad \omega \in \{m(\omega) > n\}$$

$$= X_{m(\omega)}(\omega) \qquad \omega \in \{m(\omega) \leq n\}$$

Let $Y(\omega) = \sup_{n \leq 0} \| X_n(\omega) - X_{n-1}(\omega) \|$.

Then $\| \check{X}_{n,M}(\omega) \| \leq M + Y(\omega)$, $E(Y) < +\infty$

and so

$$\| \check{X}_{n,M}(\omega) \| \quad n \geq 1$$

are uniformly integrable. According to lemma 4.3.1

$$\{ \check{X}_{n,M}, \mathcal{F}_n, n \geq 1 \}$$

is a martingale and so Th. 4.2.2 allows us to conclude that

$$\lim_{n \to \infty} \check{X}_{n,M}(\omega)$$

exists almost everywhere.

Since $\check{X}_{n,M}(\omega) = X_n(\omega)$ if $\sup_{j \geq 1} \| X_j(\omega) \| \leq M$

the theorem is proved.

<u>Corollary</u>. Let $\{ X_n, \mathcal{F}_n, n \geq 1 \}$ be a $\mathfrak{X}$ -valued martingale, $\mathfrak{X}$ reflexive, such that

$$E(\| X_n \|) < C \quad \text{(independent of n)}$$

$$E(\sup_{n \geq 0} \| X_n - X_{n-1} \|) < +\infty$$

Then $\lim_{n \to \infty} X_n(\omega)$ exists a.e.

<u>Proof:</u>

By lemma 4.2.1 $P(\omega : \sup_n \| X_n(\omega) \| < +\infty) = 1$

and this fact combined with Th. 4.3.1 implies the statement in the corollary.

Analogues of other theorems for real or complex-valued martingales can be proved as in Th. 4.3.1 for reflexive Banach spaces. We omit them here for brevity.

§4. In this section we should like to point out what can be done by direct applications of convergence theorems about scalar valued martingales (see Doob, 1). The fundamental idea here is simply that if $\{X_n, \mathfrak{F}_n, n \geq 1\}$ is any $\mathfrak{X}$-valued martingale then

$$\{ x^*(X_n(\omega)), \mathfrak{F}_n, n \geq 1\}$$

is a complex-valued martingale for any $x^* \in \mathfrak{X}^*$.

<u>Theorem 4.4.1</u> Let $\{X_n, \mathfrak{F}_n, n \geq 1\}$ be a $\mathfrak{X}$-valued martingale where $\mathfrak{X}$ is a reflexive Banach space. If $E(\|X_n\|) < C$ independent of n, then there exists a Bochner-integrable r.v. $X_\infty(\omega)$ such that $X_n(\omega)$ converges to $X_\infty(\omega)$ a.e. weakly.

<u>Proof:</u> Because each $X_n(\omega)$ is Bochner integrable, there is a separable subspace $\mathfrak{X}_0 \subset \mathfrak{X}$ such that $X_n(\omega) \in \mathfrak{X}_0$, $n \geq 1$ except for $\omega \in N$, $P(N) = 0$. $\mathfrak{X}$ being reflexive so is $\mathfrak{X}_0$. Also, $\mathfrak{X}_0 = \mathfrak{X}_0^{**} = (\mathfrak{X}_0^*)^*$ and hence $\mathfrak{X}_0^*$ is also separable. Let $\{x_i^*\} \in \mathfrak{X}_0^*$ be a denumerable set of elements dense in $\mathfrak{X}_0^*$. Now for each x_i^*

$$\{x_i^*(X_n), \mathfrak{F}_n, n \geq 1\}$$

is a complex-valued martingale such that

$$E(|x_i^*(X_n)|) \leq \|x_i^*\| . E(\|X_n\|) \leq C . \|x_i^*\|$$

Hence

$$\lim_{n \to \infty} x_i^*(X_n)$$

exists for all i except for $\omega \in M$. $P(M) = 0$.

Let $x^* \in \mathfrak{X}_o^*$ be an arbitrary element and let

$$x_{i_k}^* \longrightarrow x^* .$$

Then if $\omega \notin M$

$$|x^*(X_n(\omega)) - x^*(X_m(\omega))|$$

$$\leq |x^*(X_n(\omega)) - x_{i_k}^*(X_n(\omega))| + |x^*(X_m(\omega)) - x_{i_k}^*(X_m(\omega))|$$

$$+ |x_{i_k}^*(X_n(\omega)) - x_{i_k}^*(X_m(\omega))|$$

$$\leq \|x^* - x_{i_k}^*\| . \|X_n(\omega)\| + \|x^* - x_{i_k}^*\| . \|X_m(\omega)\|$$

$$+ |x_{i_k}^*(X_n(\omega)) - x_{i_k}^*(X_m(\omega))|$$

As we can choose M such that for $\omega \notin M$. $\sup_n \|X_n(\omega)\|$.

$< + \infty$ also,

hence it follows from above that

$$\lim_{n \to \infty} x^*(X_n(\omega)) \quad \text{exists a.e.}$$

for any $x^* \in \mathfrak{X}^*$.

$\mathfrak{X}_o$ being reflexive is weakly complete and hence there is a r.v. $X_\infty(\omega)$ such that

$$\lim_{n \to \infty} x^*(X_n(\omega)) = x^*(X_\infty(\omega)) \qquad \text{a.e.}$$

and the exceptional set of measure 0 is independent of x^*. It is also clear that $X_\infty(\omega)$ is Bochner-integrable.

This completes the proof of our theorem.

§5. In this section we present an example of a martingale taking values in a non-reflexive Banach space which is uniformly integrable and yet converges to no r.v., either in the mean or a.e. (neither strongly nor weakly). This then shows that some restrictions on the Banach space $\mathfrak{X}$ in which the martingales considered take their values, are necessary for ensuring any kind of convergence.

We consider the probability space $(\Omega, \mathcal{B}, P)$ where

Ω = (0, 1) = the open unit interval

$\mathcal{B}$ = the Borel subsets of the open unit interval

P = Lebesgue measure

Let $\mathfrak{X}$ be the Banach space of all Lebesgue integrable functions on (0, 1) with the usual norm. Let $\epsilon_\lambda(t)$ be the following element of $\mathfrak{X}$, $(0<\lambda<1)$:

$$\begin{aligned} \epsilon_\lambda(t) &= 1 \qquad 0< t \leq \lambda \\ &= 0 \qquad \lambda< t < 1 \end{aligned}$$

Let $\mathfrak{F}_n$ be the Borel-field generated by the intervals $\left(\frac{m}{2^n}, \frac{m+1}{2^n}\right)$ $0 \leq m \leq 2^n - 1$ for $n \geq 1$.

Define $X_n(\omega)$ as follows

$$X_n(\omega) = 2^n \left\{ \epsilon_{\frac{m+1}{2^n}} - \epsilon_{\frac{m}{2^n}} \right\} \quad \omega \in \left(\frac{m}{2^n}, \frac{m+1}{2^n}\right)$$

$$= 0 \quad \text{elsewhere.}$$

It can be easily seen that $\left\{ X_n, \mathfrak{F}_n, n \geq 1 \right\}$ is a martingale and that

$$\| X_n(\omega) \| \equiv 1 \qquad \text{a.e.}$$

$$E(\| X_n(\omega) \|) = 1 \quad n \geq 1$$

$$E(\sup_{n \geq 0} \| X_n(\omega) - X_{n-1}(\omega) \|) = 1 \qquad X_0 = 0$$

But if $\omega \neq p/2^q$ then $X_n(\omega)$ does not go to any limit either weakly or strongly. Actually no subsequence $X_{n_k}(\omega)$ converges weakly or strongly if $\omega \neq p/2^q$. Hence $X_n(\omega)$ does not converge in $L_1(\mathfrak{X})$ - mean either.

<u>N. B.</u> The function $\epsilon_\lambda(t)$ from the unit interval to L_1 was given by Clarkson (5) as an example of a Banach-valued absolutely continuous function having derivative almost nowhere. Our construction of $X_n(\omega)$ is patterned after Doob's (1) method of applying martingale theory to the theory of derivatives.

Chapter V

Some applications of the general theory.

§ 1. In this chapter we shall show how martingale theory can be brought to bear upon some classical problems. We shall not aim at exhausting all possible applications of our theory of Banach-valued martingales, but shall rather indicate how our theorems can tackle the extensions to Banach spaces of some of the problems which Doob (1) has considered in the complex-valued case.

§ 2. In this section we consider two different types of strong law of large numbers for a sequence $Y_n(\omega)$, $n = 1, 2, \ldots$ of independent, identically distributed r.v.'s taking values in an arbitrary Banach space $\mathfrak{X}$. The results are stated in theorems 5.2.1 and 5.2.2; they generalize Mourier's (10) results which were proved for separable and reflexive spaces. However, Mourier's results are more general in the sense that they concern arbitrary stationary sequences.

<u>Theorem 5.2.1</u> Let $Y_n(\omega)$, $n = 1, 2, \ldots$ be a sequence of independent, identically distributed r.v.'s taking values in an arbitrary Banach space $\mathfrak{X}$ and let $Y_1(\omega)$ be Bochner-integrable.

If

$$S_n(\omega) = \sum_{i=1}^{n} Y_i(\omega)$$

then

$$\lim_{n \to \infty} \frac{S_n(\omega)}{n} = E(Y_1(\omega)) \quad \text{a.e.}$$

N. B. Two $\mathfrak{X}$-valued r.v.'s $Z_1(\omega)$, $Z_2(\omega)$ will be said to independent if for any two Borel sets B_1, B_2 of $\mathfrak{X}$

$$P(\omega : Z_1(\omega) \in B_1, \ Z_2(\omega) \in B_2)$$

$$= P(\omega : Z_1(\omega) \in B_1) \, P(\omega : Z_2(\omega) \in B_2)$$

They will be said to be identically distributed if for any Borel set B of $\mathfrak{X}$

$$P(\omega : Z_1(\omega) \in B) = P(\omega : Z_2(\omega) \in B)$$

Proof: Because the Y_i's are identically distributed and one of them, namely Y_1 is Bochner-integrable, so are the rest. Hence we can consider their conditional expectations relative to any Borel-fields.

According to Theorem 4.2.3

$$\lim_{n \to \infty} E(Y_1 | S_n, S_{n+1}, \dots) = X_{-\infty}$$

exists a.e. (in the following, the symbol

$$E(Z(\omega) \,|\, Z_t(\omega), \ t \in T)$$

shall stand for a conditional expectation of the Bochner-integrable r.v. $Z(\omega)$ relative to the smallest Borel-field with respect to which the family of r.v.'s

$Z_t(\omega)$, $t \in T$ are measurable .)

Now

$$E(Y_1 | S_n, S_{n+1}, \ldots) = E(Y_1 | S_n, Y_{n+1}, Y_{n+2}, \ldots)$$

$$= E(Y_1 | S_n)$$

since the Y_i's are mutually independent. Hence

$$\lim_{n \to \infty} E(Y_1 | S_n) = \lim_{n \to \infty} E(Y_1 | S_n, S_{n+1}, \ldots)$$

$$= X_{-\infty}$$

exists a.e.

Also, as Y_i's are identically distributed

$$E(Y_1 | S_n) = E(Y_j | S_n) \quad \text{a.e.} \quad 1 \leq j \leq n$$

so that

$$E(Y_1 | S_n) = \frac{1}{n} \sum_{j=1}^{n} E(Y_j | S_n) \quad \text{a.e.}$$

$$= \frac{1}{n} E(S_n | S_n) = \frac{S_n}{n} \quad \text{a.e.}$$

Hence $$\lim_{n \to \infty} \frac{S_n(\omega)}{n} = \lim_{n \to \infty} E(Y_1 | S_n) = X_{-\infty}$$

a.e. exists.

That $X_{-\infty}(\omega) = E(Y_1(\omega))$ a.e. follows from the fact that

$$X_{-\infty} = \text{constant}$$

$$= E(X_{-\infty}) \quad \text{a.e.}$$

because of the Zero-one law and that

$$X_{-\infty}, \ldots E(Y_1\ S_2, S_3, \ldots), E(Y_1\ S_1, S_2, \ldots)$$

is a martingale.

This completes the proof of Theorem 5.2.1.

Theorem 5.2.2 Let $Y_n(\omega)$, $S_n(\omega)$ be as in Th. 5.2.1. If

$$Y_1(\omega) \in L_p(\Omega, \mathfrak{B}, P, \mathfrak{X}) \quad 1 \leq p < \infty$$

then

$$\lim_{n \to \infty} \left[\frac{S_n}{n} - E(Y_1) \right]_p = 0$$

Proof:

According to Theorem 3.2.2, there is $X_{-\infty} \in L_p(\Omega, \mathfrak{B}, P, \mathfrak{X})$ such that

$$\lim_{n \to \infty} \left[E(Y_1 | S_n, S_{n+1}, \ldots) - X_{-\infty} \right]_p = 0$$

The conclusion of this theorem then follows by proceeding exactly as in the previous proof.

§ 3. In this section we shall consider the problem of existence of derivatives with respect to nets of a countably-additive Banach-valued set function defined on an arbitrary probability space. It would be clear from our considerations that similar results can be proved for σ-finite measure spaces. We limit ourselves only to the case when the set functions take values in a reflexive Banach space.

Examples due to Bochner (7) and Clarkson (5) clearly indicate that some restrictions on the Banach space are necessary.

We shall first state a lemma:

Lemma 5.2.1 Let $(\Omega, \mathcal{B}, P)$ be a probability space and let $\{\mathfrak{F}_n\}$, $n \geq 1$ be a sequence of Borel-fields such that

$$\mathfrak{F}_n \subset \mathcal{B} \quad \text{for} \quad n \geq 1$$

and each $\mathfrak{F}_n$ is generated by a finite or denumerable number of disjoint sets $\{M_j^{(n)}, j \geq 1\}$ i.e. $\mathfrak{F}_n$ is the smallest Borel-field containing $\{M_j^{(n)}, j \geq 1\}$. Furthermore, for any n and j let there be a $k \geq 1$ such that

$$M_j^{(n+1)} \subset M_k^{(n)}$$

Let $\varphi(\cdot)$ be a countably-additive $\mathfrak{X}$-valued set function defined for sets in

$$\bigcup_{n=1}^{\infty} \mathfrak{F}_n$$

and let $\mathfrak{X}$ be a reflexive Banach space.

Let $X_n(\omega)$ be a sequence of $\mathfrak{X}$-valued r.v.'s defined as follows:

$$X_n(\omega) = \frac{\varphi(M_j^{(n)})}{P(M_j^{(n)})} \quad \text{if} \quad \omega \in M_j^{(n)}, \; P(M_j^{(n)}) > 0$$

$$= 0 \quad \text{otherwise.}$$

Then there is a r.v. $X(\omega) \in L_1(\Omega, \mathfrak{F}_\infty, P, \mathfrak{X})$ such that

$$\varphi(A) = \int_A X(\omega)\, dP \quad \text{(where } \mathfrak{F}_\infty \text{ is}$$

the Borel-field generated by $\bigcup_{n=1}^{\infty} \mathfrak{F}_n$)

for all

$$A \in \bigcup_{n=1}^{\infty} \mathfrak{F}_n \quad \text{if and only if the real-}$$

valued r.v.'s $\| X_n(\omega) \|$ are uniformly integrable.

<u>Proof:</u> It is clear that $\mathfrak{F}_n \subset \mathfrak{F}_{n+1}$ for $n \geq 1$ and $X_n(\omega)$ is measurable with respect to $\mathfrak{F}_n$.

If $\| X_n(\omega) \|$'s are uniformly integrable, then $X_n(\omega)$'s individually are Bochner-integrable and so it is meaningful to talk about their conditional expectations. Also

$$\left\{ X_n, \ \mathfrak{F}_n, \ n \geq 1 \right\}$$

forms a martingale. According to our Theorems 4.2.2 and 3.3.2, there exists a r.v. $X(\omega) \in L_1(\Omega, \mathfrak{F}_\infty, P, \mathfrak{X})$ such that

$$\lim_{n \to \infty} X_n(\omega) = X(\omega) \qquad \text{a.e.}$$

and

$$\lim_{n \to \infty} \left[X_n - X \right]_1 = 0$$

and

$$E(X \mid \mathfrak{F}_n) = X_n \qquad \text{a.e.}$$

Hence

$$\int_A X\, dP = \int_A X_n\, dP = \varphi(A) \quad \text{for} \quad A \in \mathfrak{F}_n$$

This proves that

$$\varphi(A) = \int_A X \, dP \quad \text{for } A \in \bigcup_{n=1}^{\infty} \mathfrak{F}_n$$

Conversely, if

$$\varphi(A) = \int_A X \, dP \quad \text{for } A \in \bigcup_{n=1}^{\infty} \mathfrak{F}_n$$

then $\quad E(X \mid \mathfrak{F}_n) = X_n \quad$ a.e.

and hence $\| X_n \|$'s are uniformly integrable.

N. B. A sequence of partitions $\{ M_j^{(n)} \}$ of Ω as in the lemma is called a net. The function $X(\omega)$ is said to be the derivative of φ with respect to the probability measure P, relative to the net $\{ M_j^{(n)} \}$.

For the formulation of our next theorem we need the concept of "total variation" of a $\mathfrak{X}$-valued set function φ defined on a field $\mathfrak{F}$. We define the set function $v_\varphi(A)$, $A \in \mathfrak{F}$, which we shall call the total variation of φ on A as follows:

$$v_\varphi(A) = \left\{ \sup \sum_{i=1}^{n} \| \varphi(A_i) \| \right\}$$

where the supremum is taken over all finite disjoint sequence A_i of sets in $\mathfrak{F}$ such that $A_i \subset A$.

Clearly $\quad \| \varphi(A) \| \leq v_\varphi(A)$

If φ is countably additive on $\mathfrak{F}$, then $v_\varphi(A)$

is also countably additive on $\mathfrak{F}$.

<u>Theorem 5.3.1</u> Let $(\Omega, \mathfrak{B}, P)$ be a probability space and let $\mathfrak{F}_n$, $\mathfrak{F}_\infty$, φ, and $\mathfrak{X}$ be as in lemma 5.3.1. Then

$$\varphi(A) = \int_A X(\omega)\, dP \qquad A \in \mathfrak{F}_0 = \bigcup_{n=1}^{\infty} \mathfrak{F}_n$$

where

$$X(\omega) \in L_1(\Omega, \mathfrak{F}_\infty, P, \mathfrak{X})$$

if and only if $v_\varphi(A)$ on $\mathfrak{F}_0 = \bigcup_{n=1}^{\infty} \mathfrak{F}_n$ is finite and absolutely continuous with respect to P i.e. for any $\epsilon > 0$ there is $\delta > 0$ such that

$$v_\varphi(A) < \epsilon$$

whenever

$$P(A) < \delta \;, \quad \text{and} \quad A \in \mathfrak{F}_0 \;.$$

<u>Sufficiency:</u>

<u>Proof:</u> If $v_\varphi(A)$ is a finite, non-negative, countably additive measure on $\mathfrak{F}_0$ which is a field, then it has an unique extension $\bar{v}_\varphi$ to the Borel-field $\mathfrak{F}_\infty$ generated by $\mathfrak{F}_0$. It follows from simple considerations that $\bar{v}_\varphi$ on $\mathfrak{F}_\infty$ is absolutely continuous with respect to P if v_φ is absolutely continuous with respect to P on $\mathfrak{F}_0$. According to the Radon-Nikodym theorem, there is a non-negative function $Y(\omega)$ measurable with respect to $\mathfrak{F}_\infty$ such that

$$\bar{\nu}_{\varphi}(A) = \int_A Y(\omega) \; dP \qquad A \in \mathfrak{F}_{\infty} \quad .$$

Define $X_n(\omega)$, $Y_n(\omega)$ as in lemma 5.2.1 by means of φ and $\bar{\nu}_{\varphi}$ respectively.

Clearly $\| X_n(\omega) \| \leq Y_n(\omega)$ a.e. $n \geq 1$ $\bar{\nu}_{\varphi}$ being an integral, it follows from lemma 5.2.1 that Y_n's are uniformly integrable. Hence, $\| X_n \|$'s are uniformly integrable and this implies according to lemma 5.2.1 that φ has the integral form as stated in the theorem.

<u>Necessity:</u> If $\varphi(A) = \int_A X(\omega) \; dP$ then

$$\nu_{\varphi}(A) \leq \int_A \| X(\omega) \| \; dP$$

and this immediately proves that $\nu_{\varphi}(A)$ is finite and absolutely continuous with respect to P on $\mathfrak{F}_0 = \bigcup_{n=1}^{\infty} \mathfrak{F}_n$.

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