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A STUDY OF VALUE DISTRIBUTION, ZERO SETS, INEQUALITIES
AND COEFFICIENT CONDITIONS FOR ANALYTIC FUNCTIONS
WITH SLOW GROWTH AND LACUNARY SERIES WITH HADAMARD GAPS

presented by

PETER TIEN-YU CHERN

has been accepted towards fulfillment

of the requirements for

PH.D. degree in MATHEMATICS

A handwritten signature in cursive script, reading "Peter A. Zenger".

Major professor

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**A STUDY OF VALUE DISTRIBUTION, ZERO SETS, INEQUALITIES,
AND COEFFICIENT CONDITIONS FOR ANALYTIC FUNCTIONS
WITH SLOW GROWTH AND LACUNARY SERIES WITH HADAMARD GAPS**

By

Peter Tien-Yu Chern

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ABSTRACT

A STUDY OF VALUE DISTRIBUTION, ZERO SETS, INEQUALITIES, AND COEFFICIENT CONDITIONS FOR ANALYTIC FUNCTIONS WITH SLOW GROWTH AND LACUNARY SERIES WITH HADAMARD GAPS

By

Peter Tien-Yu Chern

Let $n(r)$ be the number of zeros of $f \in \mathcal{B}$ in $|z| \leq r$, where $\mathcal{B}$ is the space of Bloch functions. Anderson, Clunie, and Pommerenke [J. Reine Angew Math. 279 (1974), p. 36] asked: is it true that $n(r) = o\left(\frac{1}{1-r} \log \log \frac{1}{1-r}\right)$ as $r \rightarrow 1^-$? The attempt to answer this problem led to the following main result concerning with the lower bound of the Nevanlinna counting function $N(r, f, a) = N(r, a)$. There exists a Bloch function f in $|z| < 1$ such that for each $a \in \mathbb{C}$

$$\liminf_{r \rightarrow 1^-} \frac{N(r, f, a)}{\log \log \frac{1}{1-r}} \geq K,$$

where K is a positive constant.

Let $\mathcal{B}_0$ be the little Bloch space. We show that $\frac{1}{2} \log^+ [o(\log \frac{1}{1-r}) + 1]$ is an upper bound of the Nevanlinna characteristic function $T(r, f)$ as $r \rightarrow 1^-$ for $f \in \mathcal{B}_0$, and we show by example that this upper bound is sharp.

We next establish inequalities for lacunary series with Hadamard gaps along small circular arcs in $|z| < 1$. Applications to the value distribution of $\mathcal{B}$ and $\mathcal{B}^\alpha$ are given.

Let $\mathcal{B}^\alpha$ ($\alpha > 1$), $L_a^p(D)$ ($1 \leq p < \infty$), and $\mathcal{S}^n$ ($0 < n < \infty$), be the α -Bloch space, the Bergman p -space in $|z| < 1$, and the Shapiro-Shields class; respectively. We show that for each $a \in \mathbb{C}$, the least upper bound and the greatest lower bound of $N(r, a)$ for $\mathcal{B}^\alpha$, L_a^p and $\mathcal{S}^n$ all have the same order $\log \frac{1}{1-r}$.

We obtain some results about the zero sets of α -Bloch functions ($\alpha > 1$). We show that these " α -Bloch zero sets" are quite different from Blaschke sequences.

Some sufficient and some necessary conditions on the coefficients for $\mathcal{B}^\alpha$ and $\mathcal{B}_0^\alpha$ are shown. We also give conditions on blocks of coefficients for $\mathcal{B}^\alpha$ and $\mathcal{B}_0^\alpha$.

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INTRODUCTION

Let D denote the unit disc $\{z: |z| < 1\}$, $\mathbb{C}$ the complex plane, and $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. A function f in D is called an α -Bloch function ($\alpha > 0$) if f is analytic in D and satisfies

$$(0.1) \quad \sup_{z \in D} (1 - |z|)^\alpha |f'(z)| < \infty.$$

The set of all α -Bloch functions is denoted by $\mathcal{B}^\alpha$. A Bloch function is precisely a 1-Bloch function. Let $\mathcal{B}_0^\alpha$ ($\alpha > 0$) be the set of functions f analytic in D such that

$$(0.2) \quad \lim_{|z| \rightarrow 1^-} (1 - |z|)^\alpha |f'(z)| = 0.$$

$\mathcal{B}_0^\alpha$ is called the space of little α -Bloch functions. A lacunary series with Hadamard gaps means a function f analytic in D which can be expressed as a power series in the form

$$(0.3) \quad f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}, \quad n_{k+1}/n_k \geq q > 1.$$

The purpose of this paper is to study $\mathcal{B}^\alpha$ and some lacunary series with Hadamard gaps. We obtain some results concerning the value distribution (Chapter 1, 3), the zero sets (Chapter 4), the coefficient conditions for α -Bloch functions in $|z| < 1$ (Chapter 5), and some useful inequalities for lacunary series with Hadamard gaps along circles in $|z| = r < 1$ (Chapter 2).

Let us begin with some background in value distribution theory.

Picard (1880) proved that if $f(z)$ is a non-rational function meromorphic in $\mathbb{C}$, then f takes every $a \in \hat{\mathbb{C}}$ infinitely often, with at most two exceptional values.

Let f be a meromorphic function in $\mathbb{C}$. If $a \in \hat{\mathbb{C}}$, we denote by $n(r, a)$ the number of solutions of $f(z) = a$ in $|z| \leq r$. We set $M(r, f) = \max_{|z|=r} |f(z)|$. The order of f is defined by

$$(0.4) \quad \rho = \limsup_{r \rightarrow \infty} \frac{\log \log M(r, f)}{\log r},$$

and the order of the roots of $f(z) = a$ is defined by

$$(0.5) \quad \rho(a) = \limsup_{r \rightarrow \infty} \frac{\log n(r, a)}{\log r}.$$

Hadamard (1893) showed that $\rho(a) \leq \rho$ for all $a \in \mathbb{C}$.

Borel (1897) extended Picard's result by proving that $\rho(a) = \rho$ for all a , with at most two exceptions.

Hadamard's theory lacked precision and did not work well for functions of infinite order or meromorphic functions. In the case when f is meromorphic in $\hat{\mathbb{C}}$, the maximum modulus $M(r, f)$ does not behave satisfactorily as an indicator of growth, since $M(r, f)$ is infinite whenever $f(z)$ has a pole on $|z| = r$.

For $a \in \hat{\mathbb{C}}$, the Nevanlinna counting function is defined by

$$(0.6) \quad N(r, a) = \int_0^r n(t, a) \frac{dt}{t}.$$

We put

$$(0.7) \quad m(r, \infty) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta, \quad \text{and}$$

$$(0.8) \quad m(r, a) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{1}{f(re^{i\theta}) - a} \right| d\theta, \quad a \in \mathbb{C}.$$

The Nevanlinna characteristic of f is defined by

$$(0.9) \quad T(r, f) = m(r, \infty) + N(r, \infty).$$

Nevanlinna (1925) stated his First Fundamental Theorem as follows: If $a \in \hat{\mathbb{C}}$ and $f(0) \neq a$, f is meromorphic in $\mathbb{C}$, then

$$(0.10) \quad m(r, a) + N(r, a) = T(r, f) + o(1).$$

The function $T(r)$ gives an excellent description of the growth of any meromorphic function in a finite disk or in the whole plane. $N(r, a)$ measures the number of roots of the equation $f(z) = a$ in $|z| \leq r$, while $m(r, a)$ measures the average closeness of $f(z)$ to a on $|z| = r$. We deduce at once Hadamard's inequality, namely $\rho(a) \leq \rho$, for every a . However Nevanlinna's aim was to obtain a sharper version of Borel's inequality. He did this by showing that in general it is the term $N(r, a)$ which dominates in (0.10). He showed that if $q \geq 3$, and $a_1, \dots, a_q$ are q distinct complex values then

$$(0.11) \quad (q - 2)T(r, f) \leq \sum_{v=1}^q N(r, a_v) - N_1(r) + S(r),$$

where $N_1(r)$ measures the zeros of f' and the multiple poles of f and $S(r)$ is a term in general much smaller than $T(r)$. This is the Second Fundamental Theorem. The Second Fundamental Theorem gives

an extension both of Picard's result and Borel's result. The deficiency of a is defined as follows:

$$(0.12) \quad \delta(a) = \liminf_{r \rightarrow \infty} \frac{m(r, a)}{T(r)} = 1 - \limsup_{r \rightarrow \infty} \frac{N(r, a)}{T(r)} .$$

Applying (0.12), Nevanlinna obtained the deficiency relation

$$(0.13) \quad \sum \delta(a) \leq 2 .$$

The concepts defined above are for meromorphic functions in $\mathbb{C}$. For functions defined in the disc D , we can get similar results by replacing $r \rightarrow \infty$ by $r \rightarrow 1^-$ and by making a few other minor changes.

The first significant result concerning with the value distribution of Bloch functions was obtained by Anderson, Clunie, and Pommerenke [ACP]. They proved that if $f \in \mathcal{B}$, then

$$(0.14) \quad \limsup_{r \rightarrow 1^-} \frac{N(r, f, 0)}{\log \log \frac{1}{1-r}} \leq \limsup_{r \rightarrow 1^-} \frac{T(r, f)}{\log \log \frac{1}{1-r}} \leq 1/2 .$$

They also asked ([ACP] tenth prob., p. 36) if it is true that

$$n(r, 0) = o\left(\frac{1}{1-r} \log \log \frac{1}{1-r}\right) \quad \text{as } r \rightarrow 1^-$$

for $f \in \mathcal{B}$?

A non-constant meromorphic function f in D is called admissible [Hay] if

$$\limsup_{r \rightarrow 1^-} \frac{T(r, f)}{\log \frac{1}{1-r}} = +\infty .$$

Although the Second Fundamental Theorem provides a lower bound for $N(r, f)$ in terms of $T(r)$, $N_1(r)$, and $S(r)$, it doesn't provide us with a practical tool to evaluate $\liminf_{r \rightarrow 1^-} N(r, f)$. The problem is that

$$(0.15) \quad S(r, f) = O\left(\log T(r, f) + \log \frac{1}{1-r}\right) \quad \text{as } r \rightarrow 1^- .$$

(see Hayman [Hay]).

By (0.14)

$$T(r, f) = O\left(\log \log \frac{1}{1-r}\right) \quad \text{as } r \rightarrow 1^-, \quad \text{and hence}$$

$$(0.16) \quad S(r, f) = O\left(\log \frac{1}{1-r}\right) \quad \text{as } r \rightarrow 1^- .$$

Thus $S(r, f)$ has order $\log \frac{1}{1-r}$ which is larger than that of $T(r, f)$. Therefore the Second Fundamental Theorem does not work on this problem. The Second Fundamental Theorem works for the admissible functions only. Bloch functions are not admissible. In Chapter 1, we develop a method to evaluate the lower bound of $N(r, f)$ as $r \rightarrow 1^-$ for analytic functions in D with slow growth, especially for Bloch functions. The attempt to answer the tenth problem led to the main results in Chapter 1.

Theorem 1.1.1. If $f(z)$ is a lacunary series with Hadamard gaps in D , and if

$$(1.1.1) \quad \liminf_{r \rightarrow 1^-} (1-r)^2 \int_0^{2\pi} |f'(re^{i\theta})|^2 d\theta \geq \text{constant} > 0$$

then

$$\liminf_{r \rightarrow 1^-} \frac{T(r, f)}{\log \log \frac{1}{1-r}} \geq C_q > 0$$

where C_q is a constant depending only on q .

Theorem 1.1.2. If $f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}$ with $a_k \neq 0$, $n_{k+1}/n_k \geq q > 1$, $\limsup_{k \rightarrow \infty} |a_k| > 0$, and $\lim_{r \rightarrow 1^-} T(r, f) = +\infty$, then for each $a \in \mathbb{C}$,

$$(1.1.7) \quad N(r, f, a) = T(r, f) + o(1) \quad \text{as } r \rightarrow 1^-.$$

Theorem 1.1.3. There exists a Bloch function f in D which is not little Bloch such that for each $a \in \mathbb{C}$

$$(1.1.8) \quad \liminf_{r \rightarrow 1^-} \frac{N(r, f, a)}{\log \log \frac{1}{1-r}} \geq K$$

where K is a positive constant.

Theorem 1.2.1. If $f \in B_0$, then $T(r, f) \leq \frac{1}{2} \log^+ [o(\log \frac{1}{1-r}) + 1]$ as $r \rightarrow 1^-$.

Theorem 1.2.2. If $\varphi(r)$ is a non-negative strictly decreasing continuous function in $(0, 1)$ with $\varphi(1^-) = 0$, then there exists a little Bloch function f in D such that

$$T(r, f) \geq K \log^+ [\varphi(r) \log \frac{1}{1-r}] \quad \text{as } r \rightarrow 1^-,$$

where K is a positive constant.

Theorem 1.2.3. If $\varphi(r)$ is a non-negative strictly decreasing continuous function in $(0, 1)$ with $\varphi(1^-) = 0$ and $\sum_{k=2}^{\infty} \varphi\left(1 - \frac{1}{2^{k-1}}\right) = +\infty$, then there exists a little Bloch function g in D such that

$$N(r, g, a) \geq K \log^+ [\varphi(r) \log \frac{1}{1-r}] \quad \text{as } r \rightarrow 1^-, \quad \text{for each } a \in \mathbb{C},$$

where K is a positive constant.

Theorem 1.2.4. There exists a little Bloch function g in D such that

$$(1.1.8) \quad \liminf_{r \rightarrow 1^-} \frac{N(r, g, a)}{\log \log \frac{1}{1-r}} \geq K \quad \text{for each } a \in \mathbb{C},$$

where K is a positive constant.

In Chapter 2 we deal with some inequalities for gap series with Hadamard gaps along small circular arcs in $|z| = r < 1$. Zygmund [Zy1], Waterman [Wa], Binmore [Bil], and Gnuschke [Gn] established some inequalities between certain sums and integrals of series with Hadamard gaps along paths leading from inside of D to the boundary of D . By way of contrast, we obtain some results along circles $|z| = r < 1$. We begin with a Lemma of Binmore and then consider Dirichlet series with Hadamard gaps and bounded coefficients. We establish some inequalities along separated circular arcs in D (see Theorem 2.1.2). Then we use a kind of interpolation method for the sequences of exponents to establish the inequalities along small circular arcs close to $|z| = 1$ (see Theorem 2.2.1). We deal also with the case with unbounded coefficients. Some applications giving examples for exhibiting the value distribution for Bloch functions and α -Bloch functions ($\alpha > 1$) are given.

In Chapter 3, we deal with counting functions of analytic functions with slow growth. A function f in D is said to be in the Shapiro-Shields class S^{-k} ($0 < k < \infty$) if f is analytic in D and satisfies

$$\sup_{z \in D} (1 - |z|)^k |f(z)| < \infty.$$

Let $L_a^p(D)$ ($1 \leq p < \infty$) be the Bergman space of analytic function f such that $|f|^p$ is integrable on $|z| < 1$. The main purpose of Chapter 3 is to show that the least upper bound and the greatest lower bound of $N(r, a)$ for $\mathcal{B}^\alpha$ ($\alpha > 1$), $L_a^p(D)$ ($1 \leq p < \infty$), and S^{-k} ($0 < k < \infty$) all have the same order $\log \frac{1}{1-r}$.

In Chapter 4, we deal with the zero sets for α -Bloch functions ($\alpha > 1$). It is well known that the zeros $\{z_k\}$ of nontrivial H^p (Hardy space $0 < p \leq \infty$) functions are completely characterized by the Blaschke condition

$$\sum_{k=1}^{\infty} (1 - |z_k|) < \infty.$$

In particular, all of the H^p ($0 < p \leq \infty$) spaces admit the same zero sets [Du].

C. Horowitz [Ho] studied the zeros of functions in the Bergman space $H_a^p(D)$ ($1 \leq p < \infty$), and obtained some important results. In Chapter 4, we obtain some analogous results for α -Bloch zero sets. We show that these " α -Bloch zero set" ($\alpha > 1$) are quite different from the Blaschke sequences. By constructing a Horowitz-type infinite product [Ho], adopting Beller's technique [Bel], and using a result on the coefficients of the α -Bloch functions [Ya] we obtain the following main results.

Theorem 4.1. If $\alpha > 1$, then there is an α -Bloch zero set which is not a Bloch zero set.

Corollary 4.1. For each $\alpha > 1$, there is an α -Bloch zero set which does not satisfy the Blaschke condition.

Theorem 4.2. If $1 < \alpha < \beta$ and if $4\alpha - 3 < \beta$ then there exists a β -Bloch zero set which is not an α -Bloch zero set.

In Chapter 5, we study the coefficients of α -Bloch functions. Coefficients of Bloch functions have been studied by Anderson, Clunie, and Pommerenke [ACP], Mathews [Ma], Neitzke [Ne], and Fernandez [Fe], among others. In this chapter we investigate coefficients of α -Bloch functions. In section 2 following Bennett, Stegenga, and Timoney [BST] we give two sufficient conditions on coefficients for $\mathcal{B}^\alpha$. In section 3 we obtain one sufficient condition and three necessary conditions on coefficients of little α -Bloch functions. In section 4, we establish some sufficient conditions and some necessary conditions on blocks of coefficients for the classes $\mathcal{B}^\alpha$ and $\mathcal{B}_0^\alpha$. The results we obtain in this chapter are related to results of Hardy [Ha], Titchmarsh [Ti], Mathews [Ma] and Neitzke [Ne].

CHAPTER I

THE VALUE DISTRIBUTION OF BLOCH FUNCTIONS AND LITTLE BLOCH FUNCTIONS

Anderson, Clunie, and Pommerenke [ACP] have listed twelve open problems for Bloch functions. The tenth problem says the following. Let $n(r)$ be the number of zeros of $f \in \mathcal{B}$ in $|z| \leq r$. Is it true that

$$n(r) = o\left(\frac{1}{1-r} \log \log \frac{1}{1-r}\right) \quad (r \rightarrow 1^-) ?$$

The attempt to answer this problem led to the results of this chapter.

§1.1 The value distribution of Bloch functions

Theorem 1.1.1. If $f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}$, with $a_k \neq 0$, $n_{k+1}/n_k \geq q > 1$, and if

$$(1.1.1) \quad \liminf_{r \rightarrow 1^-} (1-r)^2 \int_0^{2\pi} |f'(re^{i\theta})|^2 d\theta \geq K_* > 0 ,$$

then $\liminf_{r \rightarrow 1^-} \frac{T(r,f)}{\log \log \frac{1}{1-r}} \geq C_q$, where C_q is a positive constant depending only on q .

To prove Theorem 1.1.1 we need the following two lemmas.

Lemma 1.1.1. ([Zy2, Vol. 1, p. 215]).

If $h(x) = \sum_{k=1}^{\infty} (a_k \cos n_k x + b_k \sin n_k x)$, $n_{k+1}/n_k \geq q > 1$, and if $\sum_{k=1}^{\infty} (a_k^2 + b_k^2) < \infty$, then $\|h\|_1 \geq A_q \|h\|_2$, where A_q is a positive constant depending only on q .

Lemma 1.1.2. ([Zy2, Vol. 1, p. 216]) If $g(x)$ is a nonnegative measurable function defined on a set E in the real line $\mathbb{R}$ with positive linear measure $|E|$ such that

$$\frac{1}{|E|} \int_E g(x) dx \geq A > 0 \quad \text{and} \quad \frac{1}{|E|} \int_E g^2(x) dx \leq B,$$

then for $0 < \delta < 1$, we have $|E_\delta| \geq |E| (1 - \delta)^2 A^2/B$, where $E_\delta = \{x \in E \mid g(x) \geq \delta A\}$.

Proof of Theorem 1.1.1. For any $0 < r < 1$, if $f_r(z) = f(rz)$, $|z| < 1$, then $f_r \in L^2$ ($|z| = 1$). It follows from Lemma 1.1.1 that we have $\|f_r\|_1 \geq A_q \|f_r\|_2$. Putting $E_{r,\delta} = \{\theta \in [0, 2\pi] \mid |f_r(e^{i\theta})| \geq A_q \|f_r\|_2/2\}$, taking $\delta = 1/2$, $A = A_q \|f_r\|_2$, $B = \|f_r\|_2^2$, $g = |f_r|$, $E = [0, 2\pi]$ and applying Lemma 1.1.2, we have

$$(1.1.2) \quad |E_{r,\delta}| \geq 2\pi \left(1 - \frac{1}{2}\right)^2 A_q^2 = \frac{\pi}{2} A_q^2.$$

According to the definition of Nevanlinna characteristic and (1.1.2) we obtain

$$\begin{aligned}
T(r, f) &= \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta \\
&\geq \frac{1}{4\pi} \int_{E_{r,\delta}} \log^+ |f(re^{i\theta})|^2 \, d\theta \\
&\geq \frac{|E_{r,\delta}|}{4\pi} \log^+ \left(\frac{A_q^2 \|f_r\|_2^2}{4} \right) \\
&\geq \frac{A_q^2}{8} \log^+ \left(\frac{A_q^2 \|f_r\|_2^2}{4} \right).
\end{aligned}$$

Thus

$$(1.1.3) \quad T(r, f) \geq C_q \log^+ \|f_r\|_2^2 + B_q,$$

where C_q and B_q are positive constants depending only on q .

Applying Parseval's identity, we have

$$\begin{aligned}
(1.1.4) \quad \sum_{k=0}^{\infty} n_k^2 |a_k|^2 r^{2n_k-2} &= \frac{1}{2\pi} \int_0^{2\pi} |f'(re^{i\theta})|^2 \, d\theta \\
&\geq (K_*/2) \frac{1}{(1-r)^2} \quad \text{for } r \text{ near } 1,
\end{aligned}$$

where the last inequality follows from the hypothesis in Theorem 1.1.1.

Integrating twice in (1.1.4) with respect to r , we have

$$\sum_{k=0}^{\infty} \frac{n_k^2}{2n_k(2n_k-1)} |a_k|^2 r^{2n_k} \geq (K_*/2) \log \frac{1}{1-r} \quad \text{for } r \text{ near } 1,$$

and it follows that

$$\begin{aligned}
(1.1.5) \quad \|f_r\|_2^2 &= \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \\
&= \sum_{k=1}^{\infty} |a_k|^2 r^{2n_k} \\
&\geq (K_*/2) \log \frac{1}{1-r} \quad \text{for } r \text{ near } 1.
\end{aligned}$$

Combining (1.1.3) and (1.1.5), we obtain

$$(1.1.6) \quad \liminf_{r \rightarrow 1^-} \frac{T(r, f)}{\log \log \frac{1}{1-r}} \geq C_q.$$

This completes the proof of Theorem 1.1.1.

Theorem 1.1.2. If $f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}$ with $a_k \neq 0$, $n_{k+1}/n_k \geq q > 1$, $\limsup_{k \rightarrow \infty} |a_k| > 0$, and $\lim_{r \rightarrow 1^-} T(r, f) = +\infty$, then for each $a \in \mathbb{C}$,

$$(1.1.7) \quad N(r, f, a) = T(r, f) + o(1) \quad \text{as } r \rightarrow 1^-.$$

Proof of Theorem 1.1.2. Since f is a lacunary series with Hadamard gaps and $\limsup_{k \rightarrow \infty} |a_k| > 0$, by a result of Murai [Mu2], $\liminf_{r \rightarrow 1^-} m(r, f, a)$ is finite, for each $a \in \mathbb{C}$. Since for each $a \in \mathbb{C}$, $T(r, f) = N(r, f, a) + m(r, f, a) + o(1)$ and $N(r, f, a)$ is an increasing function with respect to r , hence $N(r, f, a) = T(r, f) + o(1)$ as $r \rightarrow 1^-$. This completes the proof of Theorem 1.1.2.

Theorem 1.1.3. There exists a Bloch function f in D which is not little Bloch such that

$$(1.1.8) \quad \liminf_{r \rightarrow 1^-} \frac{N(r, f, a)}{\log \log \frac{1}{1-r}} \geq K, \quad \text{for each } a \in \mathbb{C},$$

where K is a positive constant.

Proof of Theorem 1.1.3. If $f(z) = \sum_{k=0}^{\infty} z^{n_k}$ with $n_k = 2^k$, then f is Bloch but not little Bloch [Ya]. We want to show that (1.1.8) is true for this f . According to Theorem 1.1.1 and Theorem 1.1.2, it suffices to show (1.1.1). By Parseval's formula, we have

$$\frac{(1-r)^2}{2\pi} \int_0^{2\pi} |f'(rei\theta)|^2 d\theta = \sum_{k=0}^{\infty} n_k^2 r^{2n_k-2} (1-r)^2.$$

For any $0 < r < 1$, there exists n_ℓ such that $n_{\ell-1} \leq \frac{1}{1-r} < n_\ell$, hence $\frac{1}{n_{\ell-1}} \geq 1-r > \frac{1}{n_\ell}$ and $1 - \frac{1}{n_{\ell-1}} \leq r < 1 - \frac{1}{n_\ell}$. If we fix r , we obtain

$$\begin{aligned} \sum_{k=0}^{\infty} n_k^2 r^{2n_k-2} (1-r)^2 &\geq n_\ell^2 r^{2n_\ell-2} (1-r)^2 \\ &\geq n_\ell^2 r^{2n_\ell-2} \left(\frac{1}{n_\ell} \right)^2 \\ &\geq r^{2n_\ell} \\ &\geq \left(1 - \frac{1}{n_{\ell-1}} \right)^{4n_{\ell-1}} \\ &\geq e^{-4}. \end{aligned}$$

It follows that

$$\liminf_{r \rightarrow 1^-} (1-r)^2 \int_0^{2\pi} |f'(re^{i\theta})|^2 d\theta \geq \frac{2\pi}{e^4}$$

which completes the proof of Theorem 1.1.3.

A non-constant meromorphic function f in D is called admissible [Hay1] if $\limsup_{r \rightarrow 1^-} \frac{T(r,f)}{\log \frac{1}{1-r}} = +\infty$.

Theorem 1.1.4. There exists a non-admissible analytic function f in D such that

$$\delta(a,f) = \liminf_{r \rightarrow 1^-} \frac{n(r,a)}{T(r,f)} = 0 \quad \text{for each complex number } a.$$

Proof of Theorem 1.1.4. Let f be as that in the proof of Theorem 1.1.3. Then f is Bloch, and by (0.14) we have that $\limsup_{r \rightarrow 1^-} \frac{T(r,f)}{\log \frac{1}{1-r}} = 0$. Since $T(r,f) = +\infty$ as $r \rightarrow 1^-$ and $a_k = 1$, by Murai's result [Mu2] we have $\delta(a,f) = 0$, for each complex number a . This completes the proof of Theorem 1.1.4.

§1.2 The value distribution of little Bloch functions

It is natural to ask the following: Can we find a little Bloch function f such that for each $a \in \mathbb{C}$ (1.1.8) holds. The answer is affirmative and will be given in Theorem 1.2.4. We first show that $\frac{1}{2} \log^+ [o(\log \frac{1}{1-r}) + 1]$ is an upper bound of $T(r,f)$ as $r \rightarrow 1^-$, for $f \in \mathcal{B}_0$ (Theorem 1.2.1). We next obtain a result about the lower bound of $T(r,f)$ for some little Bloch functions in D (Theorem 1.2.2). The result of Theorem 1.2.1 is sharp in the sense of Theorem 1.2.2. Furthermore, in Theorem 1.2.3, we obtain a result about the lower

bound of the counting function of some little Bloch functions. By applying Theorem 1.2.3, we show that there exists a little Bloch function f in D such that for each $a \in \mathbb{C}$, (1.1.8) holds (Theorem 1.2.4).

The main results in this section are in the following.

Theorem 1.2.1. If $f \in \mathcal{B}_0$, then $T(r, f) \leq \frac{1}{2} \log^+ [o(\log \frac{1}{1-r}) + 1]$ as $r \rightarrow 1^-$.

Theorem 1.2.2. If $\varphi(r)$ is a non-negative strictly decreasing continuous function in $(0, 1)$ with $\varphi(1^-) = 0$, then there exists a little Bloch function f in D such that

$$T(r, f) \geq K \log^+ [\varphi(r) \log \frac{1}{1-r}] \quad \text{as } r \rightarrow 1^-,$$

where K is a positive constant.

Theorem 1.2.3. If $\varphi(r)$ is a non-negative strictly decreasing continuous function in $(0, 1)$ with $\varphi(1^-) = 0$ and $\sum_{k=2}^{\infty} \varphi\left(1 - \frac{1}{2^{k-1}}\right) = +\infty$, then there exists a little Bloch function g in D such that

$$N(r, g, a) \geq K \log^+ [\varphi(r) \log \frac{1}{1-r}] \quad \text{as } r \rightarrow 1^-, \quad \text{for each } a \in \mathbb{C},$$

where K is a positive constant.

Theorem 1.2.4. There exists a little Bloch function g in D such that

$$(1.1.8) \quad \liminf_{r \rightarrow 1^-} \frac{N(r, g, a)}{\log \log \frac{1}{1-r}} \geq K \quad \text{for each } a \in \mathbb{C},$$

where K is a positive constant.

Theorem 1.2.1 provides an upper bound of $T(r, f)$ for little Bloch functions. The proof is due in essence to Anderson, Clunie and Pommerenke [ACP].

Theorem 1.2.1. If $f \in \mathcal{B}_0$, then $T(r, f) \leq \frac{1}{2} \log^+ [o(\log \frac{1}{1-r}) + 1]$ as $r \rightarrow 1^-$.

Proof of Theorem 1.2.1. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a little Bloch function in D . According to Parseval's identity, we have

$$\begin{aligned} (1.2.1) \quad \sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n-2} &= \frac{1}{2\pi} \int_0^{2\pi} |f'(re^{i\theta})|^2 d\theta \\ &= o\left(\frac{1}{(1-r)^2}\right) \text{ as } r \rightarrow 1^-, \text{ since } f \in \mathcal{B}_0. \end{aligned}$$

Integrating twice in (1.2.1) with respect to r , we have

$$(1.2.2) \quad \sum_{n=1}^{\infty} \frac{n^2 |a_n|^2 r^{2n}}{2n(2n-1)} = o\left(\log \frac{1}{1-r}\right) \text{ as } r \rightarrow 1^-.$$

Since f is in $\mathcal{B}_0$, we obtain

$$\begin{aligned} (1.2.3) \quad T(r, f) &= \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta \\ &= \frac{1}{2} \cdot \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})|^2 d\theta \\ &\leq \frac{1}{2} \cdot \frac{1}{2\pi} \int_0^{2\pi} \log^+ [|f(re^{i\theta})|^2 + 1] d\theta \\ &\leq \frac{1}{2} \log \left[\frac{1}{2\pi} \int_0^{2\pi} (|f(re^{i\theta})|^2 + 1) d\theta \right] \\ &= \frac{1}{2} \log^+ [o(\log \frac{1}{1-r}) + 1] \text{ as } r \rightarrow 1^-. \end{aligned}$$

This completes the proof of Theorem 1.2.1.

Corollary 1.2.2. If $f \in \mathcal{B}_0$, then $N(r, f, a) \leq \frac{1}{2} \log^+ [o(\log \frac{1}{1-r}) + 1] + 0(1)$ as $r \rightarrow 1^-$.

Proof of Corollary 1.2.2. According to the First Fundamental Theorem of Nevanlinna, we have

$$m(r, a) + N(r, a) = T(r, f) + 0(1) \quad \text{for each } a \in \mathbb{C}.$$

By Theorem 1.2.1,

$$T(r, f) \leq \frac{1}{2} \log^+ [o(\log \frac{1}{1-r}) + 1] \quad \text{as } r \rightarrow 1^-.$$

This implies that

$$N(r, f, a) \leq \frac{1}{2} \log^+ [o(\log \frac{1}{1-r}) + 1] + 0(1) \quad \text{as } r \rightarrow 1^-.$$

Theorem 1.2.2. If $\varphi(r)$ is a non-negative strictly decreasing continuous function in $(0, 1)$ with $\varphi(1^-) = 0$. Then there is a little Bloch function f in D such that

$$T(r, f) \geq K \log^+ [\varphi(r) \log \frac{1}{1-r}] \quad \text{as } r \rightarrow 1^-,$$

where K is a positive constant.

Proof of Theorem 1.2.2. If $f(z) = \sum_{k=1}^{\infty} a_k z^{2^k}$, with $a_k = \frac{1}{\sqrt{\varphi(1 - \frac{1}{2^{k-1}})}}$, then $f \in \mathcal{B}_0$, since f is a lacunary series with Hadamard gaps with $\lim_{n \rightarrow \infty} a_n = 0$ [Ya].

According to Parseval's identity, we have

$$\begin{aligned}
 (1.2.4) \quad & \frac{(1-r)^2}{\varphi(r)} \cdot \frac{1}{2\pi} \int_0^{2\pi} |f'(rei\theta)|^2 d\theta \\
 & = \sum_{k=2}^{\infty} a_k^2 2^{2k} r^{2(2^k-1)} \frac{(1-r)^2}{\varphi(r)}.
 \end{aligned}$$

For $0 < r < 1$, there is an integer ℓ such that $2^{\ell-1} < \frac{1}{1-r} \leq 2^\ell$, $2^{-(\ell-1)} > 1-r \geq 2^{-\ell}$, $1 - 2^{-(\ell-1)} < r \leq 1 - 2^{-\ell}$. If we fix r , (1.2.4) yields

$$\begin{aligned}
 (1.2.5) \quad & \frac{(1-r)^2}{\varphi(r)} \cdot \frac{1}{2\pi} \int_0^{2\pi} |f'(rei\theta)|^2 d\theta \\
 & \geq a_\ell^2 2^{2\ell} r^{2(2^\ell-1)} \cdot \frac{(1-r)^2}{\varphi(r)} \\
 & \geq \frac{a_\ell^2}{a_\ell^2} 2^{2\ell} r^{2(2^\ell-1)} (2^{-\ell})^2 \\
 & = r^{2(2^\ell-1)} > r^{2 \cdot 2^\ell} \\
 & \geq \left[1 - \frac{1}{2^{\ell-1}}\right]^{4 \cdot 2^{\ell-1}} \geq e^{-4}.
 \end{aligned}$$

Combining (1.2.4) and (1.2.5), we obtain

$$\begin{aligned}
 (1.2.6) \quad & \frac{1}{2\pi} \int_0^{2\pi} |f'(rei\theta)|^2 d\theta = \sum_{k=2}^{\infty} 2^{2k} a_k^2 r^{2(2^k-1)} \\
 & \geq \frac{\varphi(r)e^{-4}}{(1-r)^2}.
 \end{aligned}$$

Since $\varphi(t)$ is decreasing in $(0,1)$, we have

$$\begin{aligned}
 (1.2.7) \quad \int_0^r \int_0^s \varphi(t) \frac{1}{(1-t)^2} dt ds &\geq \varphi(r) \int_0^r \int_0^s \frac{1}{(1-t)^2} dt ds \\
 &= \varphi(r) \left[\log \frac{1}{1-r} + r \right], \text{ for } 0 < r < 1.
 \end{aligned}$$

Integrating twice in (1.2.6) with respect to r , using (1.2.7), we have

$$(1.2.8) \quad \sum_{k=2}^{\infty} \frac{2^{2k} a_k^2}{2n_k(2n_k-1)} r^{2n_k} \geq e^{-4} \varphi(r) \left[\log \frac{1}{1-r} + r \right],$$

where $n_k = 2^k$.

It follows from (1.2.8) that

$$(1.2.9) \quad \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \|f_r\|_2^2 \geq \varphi(r) \log \frac{1}{1-r},$$

as $r \rightarrow 1^-$.

Since f is a lacunary series with Hadamard gaps, we have

$$(1.1.3) \quad T(r, f) \geq C_q \log^+ \|f_r\|_2^2 + B_q.$$

Combining (1.2.9) and (1.1.3), we obtain

$$(1.2.10) \quad T(r, f) \geq K \log^+ \left(\varphi(r) \log \frac{1}{1-r} \right), \text{ as } r \rightarrow 1^-,$$

where K is a positive constant. This completes the proof of Theorem 1.2.2.

Theorem 1.2.3. If $\varphi(r)$ is a nonnegative strictly decreasing continuous function in $(0, 1)$ with $\lim_{r \rightarrow 1^-} \varphi(r) = 0$ and

$\sum_{k=2}^{\infty} \psi \left(1 - \frac{1}{2^{k-1}} \right) = +\infty$, then there exists a little Bloch function g such that

$$N(r, g, a) \leq K \log^+ [\psi(r) \log \frac{1}{1-r}] \quad \text{as } r \rightarrow 1^-, \quad \text{for each } a \in \mathbb{C},$$

where K is a positive constant.

To prove Theorem 1.2.3 we need the following lemma.

Lemma 1.2.1. [Mul] Let f be analytic in D and of unbounded type and let $p(z)$ be a non-constant polynomial. For each $\theta \in [0, 2\pi]$, let $g_{\theta}(z) = f(z) + e^{i\theta} p(z)$. Then $\text{meas}\{\theta \in [0, 2\pi] : \delta(a, g_{\theta}) = 0 \text{ for each } a \in \mathbb{C}\} = 2\pi$, where meas is the linear Lebesgue measure.

Proof of Theorem 1.2.3. Let $f(z) = \sum_{k=2}^{\infty} a_k z^{2^k}$ be the function constructed in the proof of Theorem 1.2.2. According to Theorem 1.2.2, we have

$$(1.2.10) \quad T(r, f) \leq K \log^+ (\psi(r) \log \frac{1}{1-r}) \quad \text{as } r \rightarrow 1^-.$$

Since f is a lacunary series with Hadamard gaps and $\sum_{k=2}^{\infty} a_k^2 = +\infty$, according to ([Zy2, vol 1, 8.12 Theorem, p. 214]), the set $\{e^{i\theta} \mid \lim_{r \rightarrow 1^-} f(re^{i\theta}) \text{ exists and finite}\}$ has linear Lebesgue measure zero, and hence f is of unbounded type. Let $g_{\theta}(z) = f(z) + e^{i\theta} z$. According to Lemma 1.2.1, there exists a $\theta \in [0, 2\pi]$ such that $\delta(a, g_{\theta}) = 0$ for each $a \in \mathbb{C}$. If $g = g_{\theta}$, then $g \in \mathcal{B}_0$ since $f \in \mathcal{B}_0$. Also,

$$\begin{aligned}
 (1.2.11) \quad T(r, f) &= T(r, g(z) - e^{i\theta} z) \\
 &\leq T(r, g) + T(r, -e^{i\theta} z) + O(1) \\
 &= T(r, g) + O(1) .
 \end{aligned}$$

Combining (1.2.10) and (1.2.11), we obtain

$$T(r, g) \leq K \log^+(\varphi(r) \log \frac{1}{1-r}) \quad \text{as } r \rightarrow 1^- .$$

Since $\delta(a, g) = 0$, for each $a \in \mathbb{C}$, and g is of unbounded type, we obtain

$$N(r, g, a) \leq K \log^+(\varphi(r) \log \frac{1}{1-r}) \quad \text{as } r \rightarrow 1^- .$$

This completes the proof of Theorem 1.2.3.

Theorem 1.2.4. There exists a little Bloch function g in D such that

$$(1.1.8) \quad \liminf_{r \rightarrow 1^-} \frac{N(r, g, a)}{\log \log \frac{1}{1-r}} \geq K \quad \text{for each } a \in \mathbb{C} ,$$

where K is a positive constant.

Proof of Theorem 1.2.4. Let $\varphi(r) = (\log \frac{1}{1-r})^{-1/2}$, for $0 < r < 1$.

$\varphi(r)$ is a non-negative strictly decreasing continuous function in

$$\begin{aligned}
 (0, 1) \quad \text{with } \varphi(1^-) &= 0 \quad \text{and} \quad \sum_{k=2}^{\infty} \varphi\left(1 - \frac{1}{2^{k-1}}\right) = \sum_{k=2}^{\infty} ((k-1) \log 2)^{-1/2} = \\
 (\log 2)^{-1/2} \sum_{k=2}^{\infty} \frac{1}{(k-1)^{1/2}} &= +\infty .
 \end{aligned}$$

Applying Theorem 1.2.3, there exists a little Bloch function g in D such that

$$N(r, g, a) \leq K \log^+ \left[\varphi(r) \log \frac{1}{1-r} \right] \quad \text{as } r \rightarrow 1^- .$$

$$\begin{aligned} \text{Since } \log^+ \left[\varphi(r) \log \frac{1}{1-r} \right] &= \log^+ \left[\left(\log \frac{1}{1-r} \right)^{1/2} \right] = \frac{1}{2} \log^+ \left[\log \frac{1}{1-r} \right] \\ &= \frac{1}{2} \log \log \frac{1}{1-r} \quad \text{as } r \rightarrow 1^- . \end{aligned}$$

This completes the proof of Theorem 1.2.4.

CHAPTER II

SOME INEQUALITIES OF LACUNARY SERIES WITH HADAMARD GAPS ALONG CIRCLES $|z| = r < 1$

§2.0 Introduction

The motivation of this chapter is to find more Bloch functions f , in addition to the one found in Chapter 1, such that for each $a \in \mathbb{C}$

$$(1.1.8) \quad \liminf_{r \rightarrow 1^-} \frac{N(r, f, a)}{\log \log \frac{1}{1-r}} \geq K > 0 .$$

It is natural to look at a lacunary series with Hadamard gaps

$$f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}, \quad |z| < 1, \quad n_{k+1}/n_k \geq \lambda > 1 .$$

According to Murai's result [Mu2], if $\limsup_{k \rightarrow \infty} |a_k| > 0$, then $\delta(a, f)$, the deficiency of f at a satisfies $\delta(a, f) = 0$. Therefore, according to Theorem 1.1.1 and Theorem 1.1.2, in order to show (1.1.8), it suffices to show that (1.1.1) is true.

Throughout this chapter, we consider f to be an analytic function in D with Hadamard gaps. We will explore an inequality concerning the radial variation of f due to Zygmund and some of its extensions. In 1944, Zygmund [Zy1] showed that

$$\int_0^1 |f(x)| dx < \infty \Rightarrow \sum_{k=0}^{\infty} |a_k| n_k^{-1} < \infty .$$

It is clear that, with the obvious modification, the radius 1 can be replaced by any radius and that the converse is valid for arbitrary power series. In 1950, Waterman [Wal] showed that

$$\int_0^1 (1-x)^\alpha |f(x)|^\beta dx < \infty \Rightarrow \sum_{k=0}^{\infty} |a_k|^\beta n_k^{-(\alpha+1)} < \infty$$

for $\beta > 1$, $0 \leq \alpha \leq \beta - 1$. In 1967, Halasz [Hal] gave a new and simpler proof for Zygmund's result and also showed that

$$\int_0^1 (1-x)^{-1} |f(x)| dx < \infty \Rightarrow \sum_{k=0}^{\infty} |a_k| < \infty.$$

In 1972, Binmore [Bi2] proved Waterman's result for $\alpha = \beta = 2$. We say that the curve $C \subset D$ goes to the boundary of D if $C = \{z(t) : 0 \leq t < 1\}$ where $|z(t)| \rightarrow 1$ as $t \rightarrow 1^-$. In 1983, Zygmund's result was generalized by Gnuschke and Pommenerke [GP], where the radius was replaced by an arbitrary curve which goes to the boundary of D . In 1984, Gnuschke proved that if C is a half-open curve C in D with $0 \in C$, $\bar{C} \cap \partial D \neq \emptyset$ then

$$\sum_{k=0}^{\infty} |a_k|^\beta n_k^{-(\alpha+1)} \leq M_1 \int_C (1-|z|)^\alpha |f(z)|^\beta |dz|$$

for $\alpha \in \mathbb{R}$, $\beta > 0$, where M_1 depends only on α , β and λ . In Gnuschke's result, if we choose $\alpha = 2$, $\beta = 1$ and let f' replace f , we obtain

$$\begin{aligned} \int_C (1-|z|)^2 |f'(z)| |dz| &\geq \frac{1}{M_1} \sum_{k=0}^{\infty} |n_k a_k| n_k^{-3} \\ &= \frac{1}{M} \sum_{k=0}^{\infty} \frac{|a_k|}{n_k^2}. \end{aligned}$$

This result resembles (1.1.1) except for the fact that C is not a circle in D . It would be desirable if we could replace C by circles of the form $|z| = r < 1$ in the above.

Gnuschke's result was based on the following lemma which was essentially proved by Binmore [Bil].

Binmore's Lemma. Let $p \in \mathbb{N}$ and let $w_0, \dots, w_p \in \mathbb{C}$ such that

- (i) $\min_{0 \leq k \leq p} \operatorname{Re} w_k = q$, and
- (ii) $0 < \alpha_0 \leq |w_j - w_k| \leq 1/2$ for $j \neq k$.

Then there exist $c_0, \dots, c_p \in \mathbb{C}$ such that

$$(2.0.1) \quad h(t) = \sum_{j=0}^p c_j e^{-w_j t}$$

satisfies $h(1) = 1$ and both

$$(2.0.2) \quad |h(t)| \leq 2t^p e^{q(1-t)} \quad \text{for } 0 \leq t < \infty,$$

and

$$(2.0.3) \quad |c_j| \leq 2p! e^q / \alpha_0^p \quad \text{for } j = 0, \dots, p.$$

In section 1, we establish inequalities for Dirichlet series with Hadamard gaps and bounded coefficients along separated small circular arcs in $|z| = r < 1$ (see Theorem 2.1.2). In section 2, we use a kind of interpolation method, when the radius r is sufficiently near to 1, to establish inequalities along small circular arcs in $|z| = r < 1$ (see Theorem 2.2.1). We also extend the inequalities along circles $|z| = r < 1$. In section 3 and 4 we deal also with the case with

unbounded coefficients. In section 5 we present some applications, giving examples for exhibiting the value distribution for Bloch functions and α -Bloch functions ($\alpha > 1$).

§2.1 Inequalities of Dirichlet series with Hadamard gaps and bounded coefficients along small circular arcs in $|z| = r < 1$.

Theorem 2.1.1. Let $\{a_k\}$ be a sequence of complex numbers such that

$$(2.1.1) \quad |a_k/a_\ell| < M < \infty, \text{ for any } k, \ell, \text{ where } M \text{ is a constant.}$$

Let $\{y_k\}$ be a sequence of positive numbers such that

$$(2.1.2) \quad 1 < \lambda \leq y_{k+1}/y_k \leq \lambda_0, \quad \lambda_0^k/e^{\lambda k} - 1 < \text{constant} < 1,$$

for $k = 1, 2, \dots$, where λ_0, λ are constants. If

$$f(e^{-s}) = \sum_{k=0}^{\infty} a_k e^{-y_k s}, \quad \operatorname{Re} s > 0, \text{ and } \beta > 0,$$

then there exists a positive integer p which depends on λ, λ_0 , and M such that if w_j ($j = 0, \dots, p$) satisfy the hypotheses of the Binmore Lemma with $p+1 \leq q \leq p+2$, where $q = \min_{0 \leq j \leq p} \operatorname{Re} w_j$, then we have

$$(2.1.3) \quad |a_v|^\beta \leq K \sum_{j=0}^p |f(e^{-w_j/y_v})|^\beta$$

for $v = 1, 2, \dots$, where K is a constant depending on $\lambda, \lambda_0, \beta$, and M .

Proof of Theorem 2.1.1. For $j = 0, \dots, p$, let c_j be the coefficients of the function h in the Binmore Lemma. We have, for a fixed v ,

$$\begin{aligned}
 (2.1.4) \quad \sum_{j=0}^p c_j f(e^{-w_j/y_v}) &= \sum_{j=0}^p c_j \sum_{k=0}^{\infty} a_k e^{-y_k w_j/y_v} \\
 &= \sum_{k=0}^{\infty} a_k \sum_{j=0}^p c_j e^{-w_j(y_k/y_v)} \\
 &= \sum_{k=0}^{\infty} a_k h(y_k/y_v) .
 \end{aligned}$$

Hence it follows from (2.0.3), (2.1.4), $h(1) = 1$, and triangle inequality that

$$\begin{aligned}
 (2.1.5) \quad \frac{2p!e^q}{\alpha_0^p} \sum_{j=0}^p |f(e^{-w_j/y_v})| &\geq \sum_{j=0}^p |c_j| |f(e^{-w_j/y_v})| \\
 &\geq \left| \sum_{k=0}^{\infty} a_k h(y_k/y_v) \right| \\
 &\geq |a_v| \left\{ 1 - \sum_{k \neq v} \left| \frac{a_k}{a_v} \right| |h(y_k/y_v)| \right\} ,
 \end{aligned}$$

where $\alpha_0 \leq |w_j - w_k| \leq 1/2$ for $j \neq k$. Because $p+1 \leq q \leq p+2$, $\{y_k\}$ is increasing, and $|h(t)| \leq 2t^p e^{q(1-t)}$ for $t \geq 0$, we have

$$(2.1.6) \quad |h(y_k/y_v)| \leq 2e^2 \left(\frac{y_k}{y_v} e^{1-y_k/y_v} \right)^p, \text{ if } k < v, \text{ and}$$

$$(2.1.7) \quad |h(y_k/y_v)| \leq 2e \left(\frac{y_k}{y_v} e^{1-y_k/y_v} \right)^p \text{ if } k > v .$$

Because of (2.1.5), (2.1.6), and (2.1.7), we obtain

$$(2.1.8) \quad \sum_{j=0}^p |f(e^{-w_j/y_v})| \\ \geq \frac{\alpha_0^p |a_v|}{2p! e^q} \left\{ 1 - 2Me^2 \sum_{k < v} \left(\frac{y_k}{y_v} e^{1 - \frac{y_k}{y_v}} \right)^p \right. \\ \left. - 2Me \sum_{k > v} \left(\frac{y_k}{y_v} e^{1 - \frac{y_k}{y_v}} \right)^p \right\}.$$

We shall show that the expression in curly brackets on the right of (2.1.8) exceeds 1/2. Since $y_{k+1}/y_k \geq \lambda$, we have

$$(2.1.9) \quad \sum_{k=1}^{v-1} \left(\frac{y_k}{y_v} e^{1 - y_k/y_v} \right)^p \leq \sum_{k=1}^{v-1} \left(\lambda^{k-v} e^{1 - \lambda_0^{k-v}} \right)^p \\ < \sum_{j=1}^{\infty} \left(\lambda^{-j} e^{1 - \lambda_0^{-j}} \right)^p.$$

Since $e^{1 - \lambda_0^{-j}/\lambda^j} < \text{constant} < 1$ for each j , we can choose p sufficiently large such that

$$(2.1.10) \quad \sum_{k=1}^{v-1} \left(\frac{y_k}{y_v} e^{1 - y_k/y_v} \right)^p < \frac{1}{8Me^2}.$$

Since $y_{k+1}/y_k < \lambda_0$, we have

$$(2.1.11) \quad \sum_{k=v+1}^{\infty} \left(\frac{y_k}{y_v} e^{1 - y_k/y_v} \right)^p \leq \sum_{j=1}^{\infty} \left(\lambda_0^j e^{1 - \lambda^j} \right)^p.$$

By (2.1.2) and (2.1.11), we can choose p sufficiently large such that

$$(2.1.12) \quad \sum_{k=v+1}^{\infty} \left(\frac{y_k}{y_v} e^{1 - y_k/y_v} \right)^p < \frac{1}{8Me}.$$

Combining (2.1.10) and (2.1.12), we can choose p sufficiently large (depending on λ , λ_0 , and M) such that the expression in curly brackets of (2.1.8) exceeds $1/2$. Thus we obtain

$$(2.1.13) \quad |a_v| \leq K_1 \sum_{j=0}^p |f(e^{-w_j/y_v})|,$$

where p and K_1 depends on λ , λ_0 and M . It follows that

$$(2.1.14) \quad |a_v|^\beta \leq K_2 \sum_{j=0}^p |f(e^{-w_j/y_v})|^\beta,$$

where $K_2 = K_1^\beta$ if $0 < \beta \leq 1$, and $K_2 = (p+1)^{\beta-1} K_1^\beta$ if $\beta > 1$.

This completes the proof of Theorem 1.2.1.

Theorem 2.1.2. If f is the same as in Theorem 2.1.1 and if $\alpha \in \mathbb{R}$, then there exists a positive integer ν_0 depending on $\{y_k\}$, λ , λ_0 , and M such that, for $\beta > 0$,

$$(2.1.15) \quad \frac{|a_v|^\beta}{y_v^{\alpha+1}} \leq K \int_c^{c + \frac{1}{4y_v}} (1-r)^\alpha |f(re^{i\theta})|^\beta d\theta$$

for $v \geq \nu_0$, $c \in [0, 2\pi]$, where $r = e^{-\frac{p+1}{y_v}}$, with p the number given by Theorem 2.1.1, and K is a constant depending on λ , λ_0 , M , and β .

Proof of Theorem 2.1.2. We first prove the case $c = 0$. Let p be the integer resulting from Theorem 2.1.1. For $v = 1, 2, \dots$, $j = 0, 1, \dots, p$, and $\tau \in [0, 1]$ define functions

$$(2.1.16) \quad z_{v,j}(\tau) = \frac{p+1}{y_v} + \frac{j+\tau}{4(p+1)y_v} i \quad \text{and} \quad \zeta_{v,j}(z) = e^{-z_{v,j}(\tau)}.$$

For fixed v and τ , let $w_j = y_v z_{v,j}(\tau)$, $j = 0, \dots, p$. Then we have all of the following:

$$\operatorname{Re} w_j = p + 1 \quad \text{for} \quad j = 0, \dots, p,$$

$$|w_j - w_k| \leq \frac{1}{4} \quad \text{for} \quad j \neq k,$$

$$|w_j - w_k| = \frac{j-k}{4p+4} \geq \frac{1}{4p+4} \quad \text{if} \quad j > k.$$

If we put $q = p + 1$, then w_j ($j = 0, \dots, p$) satisfies the hypotheses of the Binmore Lemma with $\alpha_0 = 1/(4p + 4)$. By applying Theorem 2.1.1, we obtain

$$(2.1.17) \quad |a_v|^\beta \leq K_1 \sum_{j=0}^p |f(e^{-w_j/y_v})|^\beta = K_1 \sum |f(\zeta_{v,j}(\tau))|^\beta, \quad \beta > 0,$$

where K_1 is a constant depending on λ , λ_0 , M , and β .

Let Γ_v be the arc described by the union of the images of the functions $\zeta_{v,j}$, $j = 0, \dots, p$. Because of (2.1.16) we have

$$(2.1.18) \quad |\zeta'_{v,j}(z)| = |z'_{v,j}(\tau)| e^{-\operatorname{Re} z_{v,j}(\tau)} = \frac{1}{4(p+1)y_v} e^{-\frac{p+1}{y_v}} \quad \text{and}$$

$$1 - |\zeta_{v,j}(0)| = 1 - e^{-(p+1)/y_v}.$$

By (2.1.17) and (2.1.18), we have

$$\begin{aligned}
& \sum_{j=0}^p |f(\zeta_{v,j}(z))|^\beta |\zeta'_{v,j}(\tau)| (1 - |\zeta_{v,j}(\tau)|)^\alpha \\
& \geq \frac{|a_v|^\beta e^{-(p+1)/y_v}}{K_1 4(p+1)y_v} (1 - e^{-(p+1)/y_v})^\alpha \\
& = \frac{|a_v|^\beta e^{-(p+1)/y_v}}{4K_1 (p+1)y_v^{\alpha+1}} [y_v (1 - e^{-(p+1)/y_v})]^\alpha.
\end{aligned}$$

Since $e^{-(p+1)/y_v} \rightarrow 1$ and $y_v(1 - e^{-(p+1)/y_v}) \rightarrow p+1$ as $v \rightarrow \infty$, there exists ν_0 , depending on $\{y_k\}$, λ , λ_0 , and M , such that $e^{-(p+1)/y_v} \geq \frac{1}{2}$ and $[y_v(1 - e^{-(p+1)/y_v})]^\alpha \geq \left(\frac{p+1}{2}\right)^\alpha$ for $v \geq \nu_0$, and hence

$$(2.1.19) \quad \sum_{j=0}^p |f(\zeta_{v,j}(\tau))|^\beta |\zeta'_{v,j}(\tau)| (1 - |\zeta_{v,j}(\tau)|)^\alpha \geq K_2 \frac{|a_v|^\beta}{y_v^{\alpha+1}}$$

for $v \geq \nu_0$, $\tau \in [0,1]$ where K_2 is a constant depending only on λ_0 , λ , M , and β . If we integrate (2.1.19) with respect to τ from 0 to 1, then we obtain

$$\begin{aligned}
(2.1.20) \quad \frac{|a_v|^\beta}{y_v^{\alpha+1}} & \leq \frac{1}{K_2} \sum_{j=0}^p \int_0^1 |f(\zeta_{v,j}(\tau))|^\beta |\zeta'_{v,j}(\tau)| (1 - |\zeta_{v,j}(\tau)|)^\alpha d\tau \\
& = \frac{1}{K_2} \sum_{j=0}^p \int_{\zeta_{v,j}(0)}^{\zeta_{v,j}(1)} |f(\zeta_{v,j})|^\beta |\zeta'_{v,j}| (1 - |\zeta_{v,j}|)^\alpha d\zeta_{v,j} \\
& = \frac{1}{K_2} \int_0^{\frac{1}{4y_v}} (1 - r)^\alpha |f(re^{i\theta})|^\beta d\theta
\end{aligned}$$

where $r = e^{-(p+1)/y_v}$ for $v \geq \nu_0$, with p the number given by Theorem 2.1.1. Thus the proof of Theorem 2.1.2 is complete for $c = 0$.

For the case $c \neq 0$, replace (2.1.16) by

$$z_{v,j}(\tau) = \frac{p+1}{y_v} + ci + \frac{j+\tau}{4(p+1)y_v} i, \quad \zeta_{v,j}(\tau) = e^{-z_{v,j}(\tau)},$$

and repeat the previous procedure. This gives us

$$\frac{|a_v|^\beta}{y_v^{\alpha+1}} \leq K \int_c^{c + \frac{1}{4y_v}} (1-r)^\alpha |f(re^{i\theta})|^\beta d\theta,$$

where $r = e^{-(p+1)/y_v}$ for $v \geq v_0$, with p the number given by Theorem 2.1.1 and K is a constant depending on λ , λ_0 , M , and β . This completes the proof of Theorem 2.1.2.

§2.2 Inequalities of Lacunary series with Hadamard gaps and bounded coefficients along circles $|z| = r < 1$.

Theorem 2.2.1. Let $\{a_k\}$ be a sequence of complex numbers satisfying (2.1.1). Let $\{n_k\}$ be a sequence of positive integers satisfying (2.1.2) with $\lambda = \lambda_0$. If

$$f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}, \quad |z| < 1,$$

then there exists a positive integer v_0 , depending on $\{n_k\}$, λ , and M , such that, for any $s \geq n_{v_0}$, if $n_v \leq s < n_{v+1}$, then

$$(2.2.1) \quad \frac{|a_v|^\beta}{s^{\alpha+1}} \leq K \int_c^{c + \frac{1}{4s}} (1-r)^\alpha |f(re^{i\theta})|^\beta d\theta$$

for any $c \in [0, 2\pi]$, $\beta > 0$, and $\alpha \in \mathbb{R}$, where $r = e^{-(p+1)/s}$ with p given by Theorem 2.1.1, and K is a constant depending on λ , β , and M .

Proof of Theorem 2.2.1. Let $\{y_k\}$ be a sequence of positive numbers satisfying (2.1.2) with $\lambda = \lambda_0$. Because $y_{k+1}/y_k = n_{k+1}/n_k = \lambda$, we can put $n_k = dy_k$ for any $k = 1, 2, \dots$, where d is a positive constant. For $\operatorname{Re} t > 0$, if

$$(2.2.2) \quad f(e^{-t}) = \sum_{k=0}^{\infty} a_k e^{-n_k t} = \sum_{k=0}^{\infty} a_k e^{-y_k dt},$$

then f is a Dirichlet series with exponents $\{-y_k dt\}$ and coefficients $\{a_k\}$. Because $\{y_k\}$ and $\{a_k\}$ satisfy (2.1.2) and (2.1.1) respectively, it follows from Theorem 2.1.2 there exists ν_0 depending on $\{y_k\}$, λ , and M such that

$$(2.2.3) \quad \frac{|a_v|^\beta}{y_v^{\alpha+1}} \leq K \int_c^{c + \frac{1}{4y_v}} (1-r)^\alpha |f(re^{i\theta})|^\beta d\theta$$

for $v \geq \nu_0$, $c \in [0, 2\pi]$, $\alpha \in \mathbb{R}$, $\beta > 0$, where $r = e^{-(p+1)/y_v}$ and p is a positive integer depending only on λ and M , where K is a constant depending on λ , β , and M .

Note that the inequality (2.2.3) holds for $v \geq \nu_0$, where ν_0 is dependent only on $\{n_k\}$, λ , and M , and ν_0 is independent of the particular sequence $\{y_k\}$ satisfying (2.1.2) with $\lambda = \lambda_0$.

If $s \geq n_{\nu_0}$, there exists a n_v such that $n_v \leq s < n_{v+1}$. We can construct a sequence $\{y_k\}$ of positive numbers satisfying (2.1.2), with $\lambda = \lambda_0$, and $y_v = s$. From our previous argument, we obtain

$$(2.2.4) \quad \frac{|a_v|^\beta}{s^{\alpha+1}} \leq K \int_c^{c + \frac{1}{4s}} (1-r)^\alpha |f(re^{i\theta})|^\beta d\theta$$

Since s is arbitrary, we have completed the proof of Theorem 2.2.1.

Theorem 2.2.2. If f , $\{a_k\}$, $\{n_k\}$ are the same in Theorem 2.2.1 then there exists a positive integer ν_0 , depending on $\{n_k\}$, λ , and M , such that, for any $s \geq n_{\nu_0}$, if $n_\nu \leq s < n_{\nu+1}$, then

$$(2.2.5) \quad \frac{|a_\nu|^\beta}{n_\nu^\alpha} \leq K \int_0^{2\pi} (1-r)^\alpha |f(re^{i\theta})|^\beta d\theta$$

where $\alpha \in \mathbb{R}$, $\beta > 0$, $r = e^{-(p+1)/s}$ with p a constant depending on λ and M , and K is a constant depending on λ , β , α , and M .

Proof of Theorem 2.2.2. Since

$$\frac{2\pi}{1/4s} = 8\pi s \quad \text{and} \quad \frac{|a_\nu|^\beta}{\lambda^\alpha s^\alpha} \leq \frac{|a_\nu|^\beta}{\lambda^\alpha n_\nu^\alpha} = \frac{|a_\nu|^\beta}{n_{\nu+1}^\alpha} < \frac{|a_\nu|^\beta}{s^\alpha},$$

the desired result (2.2.5) follows from Theorem 2.2.1.

§2.3 Inequalities of Dirichlet series with Hadamard gaps and unbounded coefficients along small circular arcs **$|z| = r < 1$.**

Theorem 2.3.1. Let $\{a_k\}$ be a sequence of complex numbers such that

$$(2.3.1) \quad 1 < |a_{k+1}/a_k| < M < \infty \quad \text{for } k = 0, 1, \dots,$$

where M is a constant.

Let $\{y_k\}$ be a sequence of positive numbers satisfying

$$(2.3.2) \quad 1 < \lambda \leq y_{k+1}/y_k \leq \lambda_0 \quad \text{and} \quad (M\lambda)^k / e^{\lambda k-1} < \text{constant} < 1,$$

for $k = 0, 1, \dots$, where λ and λ_0 are constants. If

$$f(e^{-s}) = \sum_{k=0}^{\infty} a_k e^{-y_k s}, \quad \operatorname{Re} s > 0 \quad \text{and} \quad \beta > 0,$$

then there exists a positive integer p , which depends on λ , λ_0 , and M , such that if $w_j (j = 0, \dots, p)$ satisfy the hypotheses of the Binmore Lemma with $p+1 \leq q \leq p+2$, where $q = \min_{0 \leq j \leq p} \operatorname{Re} w_j$, then we have

$$(2.3.3) \quad |a_v|^\beta \leq K \sum_{j=0}^p |f(e^{-w_j/y_v})|^\beta$$

for $v = 1, 2, \dots$, where K is a constant depending on λ , λ_0 , β , and M .

Proof of Theorem 2.3.1. According to (2.0.3), (2.1.4), $h(1) = 1$, and triangle inequality, for a fixed v , we have

$$(2.3.4) \quad \frac{2p!e^q}{\alpha_0^p} \sum_{j=0}^p |f(e^{-w_j/y_v})| \geq |a_v| \left\{ 1 - \sum_{k \neq v} \left| \frac{a_k}{a_v} \right| |h(y_k/y_v)| \right\},$$

where α_0 and h are given in the Binmore Lemma. By (2.3.4), (2.1.6) and (2.1.7) we obtain

$$(2.3.5) \quad \sum_{j=0}^p |f(e^{-w_j/y_v})| \geq \frac{\alpha_0^p |a_v|}{2p!e^q} \left\{ 1 - 2e^2 \sum_{k=0}^{v-1} \left(\frac{y_k}{y_v} e^{1 - y_k/y_v} \right)^p - 2e \sum_{k=v+1}^{\infty} \left| \frac{a_k}{a_v} \right| \left[\frac{y_k}{y_v} e^{1 - \frac{y_k}{y_v}} \right]^p \right\}.$$

We shall show that the expression in curly brackets on the right of (2.3.5) exceeds $1/2$. Since $y_{k+1}/y_k \geq \lambda$, we have

$$(2.3.6) \quad \sum_{k=0}^{v-1} \left[\frac{y_k}{y_v} e^{1 - y_k/y_v} \right]^p \leq \sum_{k=0}^{v-1} \left[\lambda^{k-v} e^{1 - \lambda_0^{k-v}} \right]^p \\ < \sum_{j=1}^{\infty} \left[\lambda^{-j} e^{1 - \lambda_0^{-j}} \right]^p.$$

Because $e^{1 - \lambda_0^{-j}}/\lambda^j < \text{constant} < 1$ for each j , we can choose p sufficiently large such that

$$(2.3.7) \quad \sum_{k=0}^{v-1} \left[\frac{y_k}{y_v} e^{1 - y_k/y_v} \right]^p < \frac{1}{8e^2}.$$

Since $y_{k+1}/y_k < \lambda_0$, we have

$$(2.3.8) \quad \sum_{k=v+1}^{\infty} \left| \frac{a_k}{a_v} \right| \left[\frac{y_k}{y_v} e^{1 - y_k/y_v} \right]^p \\ < \sum_{j=1}^{\infty} M^j \left[\lambda_0^j e^{1 - \lambda^j} \right]^p < \sum_{j=1}^{\infty} \left[(M \lambda_0)^j e^{1 - \lambda^j} \right]^p.$$

Since $(M\lambda)^j/e^{\lambda^j-1} < \text{constant} < 1$ for any $j = 1, 2, \dots$, we can choose p sufficiently large such that

$$(2.3.9) \quad \sum_{k=v+1}^{\infty} \left| \frac{a_k}{a_v} \right| \left[\frac{y_k}{y_v} e^{1 - y_k/y_v} \right]^p < \frac{1}{8e}$$

Combining (2.3.7) and (2.3.9), we can choose p sufficiently large (depending on λ , λ_0 , and M) such that the expression in curly

brackets of (2.3.5) exceeds $1/2$. Thus we obtain

$$(2.3.10) \quad |a_v| \leq K_1 \sum_{j=0}^p |f(e^{-w_j/y_v})|$$

where p and K_1 depend on λ , λ_0 , and M . It follows that

$$(2.3.3) \quad |a_v|^\beta \leq K_2 \sum_{j=0}^p |f(e^{-w_j/y_v})|^\beta$$

where $K_2 = K_1^\beta$ if $0 \leq \beta < 1$ and $K_2 = (p+1)^{\beta-1} K_1^\beta$ if $\beta > 1$.

This completes the proof of Theorem 2.3.1.

Theorem 2.3.2. If f is the same as in Theorem 2.3.1 then there exists a positive integer ν_0 (depending on $\{y_k\}$, λ , λ_0 , and M) such that, for $\beta > 0$,

$$(2.3.10) \quad \frac{|a_v|^\beta}{y_v^{\alpha+1}} \leq K \int_c^{c + \frac{1}{4y_v}} (1-r)^\alpha |f(re^{i\theta})|^\beta d\theta$$

for any $v \geq \nu_0$, $c \in [0, 2\pi]$, where $r = e^{-(p+1)/y_v}$, with p a constant integer given in Theorem 2.3.1, and K a constant depending on λ , λ_0 , M and β .

Proof of Theorem 2.3.2. Inequality (2.3.3) follows from Theorem 2.3.1. Repeating the same procedure as in the proof of Theorem 2.1.2, the result is obtained.

§2.4 Inequalities of Lacunary series with Hadamard gaps and unbounded coefficients along circles $|z| = r < 1$.

Theorem 2.4.1. Let $\{a_k\}$ be a sequence of complex numbers satisfying (2.3.1) and let $\{n_k\}$ be a sequence of positive integers satisfying (2.3.2) with $\lambda = \lambda_0$. If $f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}$ then there exists a positive integer ν_0 , depending on $\{n_k\}$, λ , and M , such that, if $s \geq n_{\nu_0}$ with $n_{\nu} \leq s < n_{\nu+1}$, then

$$(2.4.1) \quad \frac{|a_{\nu}|^{\beta}}{s^{\alpha+1}} \leq K \int_c^c + \frac{1}{4s} (1-r)^{\alpha} |f(re^{i\theta})|^{\beta} d\theta$$

for $c \in [0, 2\pi]$, $\alpha \in \mathbb{R}$, $\beta > 0$, where $r = e^{-(p+1)/s}$, p depends on λ , and M and K are constants depending on λ , β , and M .

Proof of Theorem 2.4.1. Since $\{a_k\}$ satisfies (2.3.1), and $\{n_k\}$ satisfies (2.1.2) and (2.3.2) with $\lambda = \lambda_0$, we can apply Theorem 2.3.2 to the procedure of the proof of Theorem 2.2.1 to obtain the desired result.

Theorem 2.4.2. If $f(z)$, $\{a_k\}$, and $\{n_k\}$ are the same in Theorem 2.4.1, then there exists a positive integer ν_0 such that, for any $s \geq n_{\nu_0}$, if $n_{\nu} \leq s < n_{\nu+1}$ then

$$(2.4.2) \quad \frac{|a_{\nu}|^{\beta}}{s^{\alpha}} \leq K \int_0^{2\pi} (1-r)^{\alpha} |f(re^{i\theta})|^{\beta} d\theta$$

for $\alpha \in \mathbb{R}$, $\beta > 0$ where $r = e^{-(p+1)/s}$, p depends on λ , and M and K are constants depending on λ , β , and M .

Proof of Theorem 2.4.2. Since

$$\frac{2\pi}{1/4s} = 8\pi s \quad \text{and} \quad \frac{|a_v|^\beta}{\lambda^\alpha s^\alpha} \leq \frac{|a_v|^\beta}{\lambda^\alpha n_v^\alpha} = \frac{|a_v|^\beta}{n_{v+1}^\alpha} < \frac{|a_v|^\beta}{s^\alpha},$$

(2.4.2) follows from Theorem 2.4.1 by repeating the arguments used in the previous Section §2.3.

§2.5 Applications

Theorem 2.5.1. Let $\{a_k\}$ be a sequence of complex numbers and $\{n_k\}$ be a sequence of positive integers satisfying

$$(2.5.1) \quad \limsup_{k \rightarrow \infty} |a_k| > 0, \quad \left| \frac{n_{k+1} a_{k+1}}{n_k a_k} \right| < M < \infty, \quad \text{and}$$

$$(2.5.2) \quad \frac{n_{k+1} - 1}{n_k - 1} = \lambda > 1, \quad \frac{(M\lambda)^k}{e^{\lambda^k - 1}} < \text{constant} < 1,$$

for $k = 0, 1, 2, \dots$, where M, λ are positive constants. If

$$f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}$$

then

$$(2.5.3) \quad \liminf_{r \rightarrow 1^-} \frac{N(r, f, a)}{\log \log \frac{1}{1-r}} \geq K_\lambda \quad \text{for each } a \in \mathbb{C},$$

where K_λ is a positive constant depending on λ .

Proof of Theorem 2.5.1. Since f is a lacunary series with Hadamard gaps and $\limsup_{k \rightarrow \infty} |a_k| > 0$, according to Theorem 1.1.1 and

Theorem 1.1.2 we need only show that

$$(1.1) \quad \liminf_{r \rightarrow 1^-} (1-r)^2 \int_0^{2\pi} |f'(re^{i\theta})|^2 d\theta \geq \text{positive constant.}$$

Because $f'(z) = \sum_{k=0}^{\infty} n_k a_k z^{n_k-1}$ with $\{n_k a_k\}$ satisfying (2.3.1) and $\{n_k - 1\}$ satisfying (2.3.2), we can apply Theorem 2.4.2 for $\alpha = \beta = 2$ to obtain the desired result.

Theorem 2.5.2. If ε is a positive rational number and if

$$f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}, \text{ where } \{n_k\} \text{ satisfies (2.5.2) and } a_k = n_k^{\varepsilon/2},$$

then

$$(2.5.4) \quad \liminf_{r \rightarrow 1^-} \frac{N(r, f, a)}{\log \frac{1}{1-r}} \geq K_{\lambda} \text{ for each } a \in \mathbb{C},$$

where K_{λ} is a positive constant depending on λ .

Proof of Theorem 2.5.2. Since f is a lacunary series with Hadamard gaps and unbounded coefficients, we have $\lim_{r \rightarrow 1^-} T(r, f) = +\infty$, and it follows from Theorem 1.1.2 that

$$(2.5.5) \quad N(r, f, a) = T(r, f) + o(1) \text{ as } r \rightarrow 1^- \text{ for each } a \in \mathbb{C}.$$

Thus, to prove (2.5.4) it suffices to show that

$$(2.5.6) \quad \liminf_{r \rightarrow 1^-} \frac{T(r, f)}{\log \frac{1}{1-r}} \geq K \text{ as } r \rightarrow 1^-, \text{ where}$$

K is a positive constant.

Now $f'(z) = \sum_{k=0}^{\infty} n_k a_k z^{n_k-1}$, and since $\{n_k a_k\}$ satisfies (2.3.1), the sequence $\{n_k - 1\}$ satisfies (2.3.2), thus, we can apply Theorem

2.4.2 for $\alpha = \varepsilon + 2$, $\beta = 2$, to obtain

$$(2.5.7) \quad 1 = \frac{|n_v a_v|^2}{n_v^{\varepsilon+2}} \leq K \int_0^{2\pi} (1-r)^{2+\varepsilon} |f'(re^{i\theta})|^2 d\theta$$

for n_v is sufficiently large, $r = e^{-(p+1)/n_v}$, and p and K are constants depending on λ and M . Thus we have

$$(2.5.8) \quad \int_0^{2\pi} |f'(re^{i\theta})|^2 d\theta \geq \frac{1/K}{(1-r)^{2+\varepsilon}} \quad \text{as } r \rightarrow 1^-.$$

Inequality (1.1.3) says that $T(r, f) \geq C_\lambda \log^+ \|f_r\|_2^2$, where C_λ is a positive constant depending on λ . Applying Parseval's identity, we have

$$(2.5.9) \quad \sum_{k=0}^{\infty} n_k^2 |a_k|^2 r^{2n_k-2} = \frac{1}{2\pi} \int_0^{2\pi} |f'(re^{i\theta})|^2 d\theta \\ \geq \frac{1}{2\pi K(1-r)^{2+\varepsilon}} \quad \text{as } r \rightarrow 1^- ,$$

where the last inequality follows from (2.5.7).

Integrating (2.5.9) twice with respect to r , we have

$$\sum_{k=0}^{\infty} \frac{n_k^2}{2n_k(2n_k-1)} |a_k|^2 r^{2n_k} \geq K_1 \frac{1}{(1-r)^\varepsilon} \quad \text{as } r \rightarrow 1^- ,$$

where K_1 is a positive constant, and it follows that

$$(2.5.10) \quad \|f_r\|_2^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \\ = \sum_{k=0}^{\infty} |a_k|^2 r^{2n_k} \geq K_2 \frac{1}{(1-r)^\varepsilon} \quad \text{as } r \rightarrow 1^- .$$

Combining (1.1.3), (2.5.10) we obtain the desired result (2.5.6).

CHAPTER III

COUNTING FUNCTIONS OF ANALYTIC FUNCTIONS WITH SLOW GROWTH

§3.1 Introduction

Value distribution theory of analytic functions with slow growth have been a subject of recent interest, in particular with respect to bounds on the counting function. A function f in D is said to be in the Shapiro-Shields class S^{-k} ($0 < k < \infty$) if f is analytic in D and satisfies

$$\sup_{z \in D} (1 - |z|)^k |f(z)| < \infty.$$

Let $L_a^p(D)$ ($1 \leq p < \infty$) be the Bergman space of analytic functions that are p -th power integrable in $|z| < 1$ and let B^α ($\alpha > 0$) be the α -Bloch space.

Let f be a meromorphic function in D . If $a \in \hat{\mathbb{C}}$, we denote by $n(r, a)$ the number of solutions of $f(z) = a$ in $|z| \leq r$. For $a \in \hat{\mathbb{C}}$, the Nevanlinna counting function is defined by

$$N(r, a) = \int_0^r n(t, a) \frac{dt}{t}.$$

The main purpose of this chapter is to show that the least upper bound and the greatest lower bound of $N(r, a)$ for B^α ($\alpha > 1$), $L_a^p(D)$ $1 \leq p < \infty$, and S^{-n} ($0 < n < \infty$) all have the same order $\log \frac{1}{1-r}$.

§3.2 An upper bound.

Theorem 3.1. If $f \in S^{-k}$, $k > 0$, then $T(r, f) = O(\log \frac{1}{1-r})$ as $r \rightarrow 1^-$ and for each $a \in \mathbb{C}$, $N(r, f, a) = O(\log \frac{1}{1-r})$ as $r \rightarrow 1^-$.

Proof of Theorem 3.1. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Since f is analytic in D , we obtain

$$\begin{aligned} T(r, f) &= \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta \\ &= \frac{1}{4\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})|^2 d\theta \\ &\leq \frac{1}{4\pi} \int_0^{2\pi} \log[|f(re^{i\theta})|^2 + 1] d\theta \\ &\leq \frac{1}{2} \log \left\{ \frac{1}{2\pi} \int_0^{2\pi} (|f(re^{i\theta})|^2 + 1) d\theta \right\} \\ &= \frac{1}{2} \log \left\{ 0 \left[\frac{1}{(1-r)^{2k}} \right] + O(1) \right\}, \quad (f \in S^{-k}) \\ &= O(\log \frac{1}{1-r}) \quad \text{as } r \rightarrow 1^-, \end{aligned}$$

and hence $T(r, f) = O(\log \frac{1}{1-r})$ as $r \rightarrow 1^-$. Therefore for each $a \in \mathbb{C}$, $N(r, f, a) = O(\log \frac{1}{1-r})$.

Corollary 3.1.1. If $f \in \mathcal{B}^\alpha$ ($\alpha > 1$), then $T(r, f) = O(\log \frac{1}{1-r})$ and for each $a \in \mathbb{C}$, $N(r, f, a) = O(\log \frac{1}{1-r})$ as $r \rightarrow 1^-$.

Proof. Since $\mathcal{B}^\alpha \subset S^{-2(\alpha-1)}$ [Be2], the results follow from Theorem 3.1.

Corollary 3.1.2. If $f \in L_a^p(D)$, then $T(r, f) = O(\log \frac{1}{1-r})$ and for each $a \in \mathbb{C}$, $N(r, f, a) = O(\log \frac{1}{1-r})$ as $r \rightarrow 1^-$.

Proof. Since $L_a^p(D) \subset S^{-2/p}$ [Be2], the results follow from Theorem 3.1.

§3. A lower bound

It is natural to ask the following: For each $\alpha > 1$, can we find a function $f \in \mathcal{B}^\alpha$ such that for each $a \in \mathbb{C}$, $\liminf_{r \rightarrow 1^-} \frac{N(r, f, a)}{\log \frac{1}{1-r}} \geq K > 0$, where K is a constant. The answer is affirmative and will be given in Theorem 3.2. We need the following lemma.

Lemma 3.3.1. [Ya] Let $f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}$ be analytic in $|z| < 1$, with $n_{k+1}/n_k \geq q > 1$. Then $f \in \mathcal{B}^\alpha$ ($\alpha > 0$) iff $\limsup_{k \rightarrow \infty} |a_k| n_k^{1-\alpha} < \infty$.

Theorem 3.2. If $\alpha > 1$, then there exists a function $f \in \mathcal{B}^\alpha$ such that for each $a \in \mathbb{C}$, $\liminf_{r \rightarrow 1^-} \frac{N(r, f, a)}{\log \frac{1}{1-r}} \geq K > 0$, where K is a constant.

Proof. Since $\alpha > 1$, there exists a positive integer m such that $1 + \frac{1}{2^m} < \alpha$. Let $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$ with $n_{k+1}/n_k = 2$, $n_1 = 1$, and $a_k = n_k^{2^{-m}} \rightarrow \infty$. We have $a_k n_k^{1-\alpha} = n_k^{1-\alpha+\frac{1}{2^m}} < n_k^0 = 1$, and thus $\limsup_{k \rightarrow \infty} a_k n_k^{1-\alpha} < 1$. According to Lemma 3.3.1, we have $f \in \mathcal{B}^\alpha$.

We first show that

$$\liminf_{r \rightarrow 1^-} \frac{T(r, f)}{\log \frac{1}{1-r}} \geq K > 0, \quad \text{where } K \text{ is a constant.}$$

Applying Parseval's identity, we have

$$(3.3.1) \quad \frac{(1-r)^{b_m}}{2\pi} \int_0^{2\pi} |f'(rei\theta)|^2 d\theta \\ = \sum_{k=1}^{\infty} a_k^2 n_k^2 r^{2n_k-2} (1-r)^{b_m} .$$

where $b_m = 2 + 2^{-(m-1)}$.

For each $0 < r < 1$, there exists a n_ℓ such that $1 - \frac{1}{n_{\ell-1}} \leq r < 1 - \frac{1}{n_\ell}$. Fix r , and let ℓ be given by the last inequality. Then

$$(3.3.2) \quad \sum_{k=1}^{\infty} n_k^{b_m} r^{2n_k-2} (1-r)^{b_m} \\ \geq n_\ell^{b_m} r^{2n_\ell-2} (1-r)^{b_m} \\ > n_\ell^{b_m} r^{2n_\ell-2} \left(\frac{1}{n_\ell}\right)^{b_m} \\ = r^{2n_\ell-2} > r^{2n_\ell} \\ \geq \left(1 - \frac{1}{n_{\ell-1}}\right)^{4n_{\ell-1}} \geq e^{-4} .$$

Combining (3.3.1) and (3.3.2), we obtain

$$(3.3.3) \quad \sum_{k=1}^{\infty} a_k^2 n_k^2 r^{2n_k-2} \geq \frac{e^{-4}}{(1-r)^{b_m}} .$$

Integrating twice in (3.3.3) with respect to r , we have

$$(3.3.4) \quad \sum_{k=1}^{\infty} \frac{n_k^2 a_k^2 r^{2n_k}}{2n_k(2n_k-1)} \geq \frac{L}{(1-r)^{c_m}} ,$$

where $c_m = 2^{-(m-1)}$,

and it follows that

$$\begin{aligned}
 (3.3.5) \quad \|f_r\|_2^2 &= \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \\
 &= \sum_{k=1}^{\infty} |a_k|^2 r^{2n_k} \\
 &\geq K_1 \frac{1}{(1-r)^{c_m}}, \quad \text{where } K_1 \text{ is a positive constant.}
 \end{aligned}$$

According to (1.1.3) and (3.3.5), we obtain

$$(3.3.6) \quad \liminf_{r \rightarrow 1^-} \frac{T(r, f)}{\log \frac{1}{1-r}} \geq K,$$

where K is a positive constant.

Since f is a lacunary series with Hadamard gaps, $\limsup_{n \rightarrow \infty} |a_n| > 0$, and $\lim_{r \rightarrow 1^-} T(r, f) = +\infty$, by Theorem 1.1.2 (Chapter 1), we obtain (1.1.7) for each $a \in \mathbb{C}$, $T(r, f) = N(r, f, a) + O(1)$ as $r \rightarrow 1^-$. Combining (3.3.6) and (1.1.7), we have, for each $a \in \mathbb{C}$

$$\liminf_{r \rightarrow 1^-} \frac{N(r, f, a)}{\log \frac{1}{1-r}} \geq K \quad \text{as } r \rightarrow 1^-.$$

This completes the proof.

Corollary 3.2.1. If $k > 0$, then there exists a $f \in S^{-k}$ such that for each $a \in \mathbb{C}$, $\liminf_{r \rightarrow 1^-} \frac{N(r, f, a)}{\log \frac{1}{1-r}} \geq L$, where L is a positive constant.

Proof. Setting $\alpha = \frac{k}{2} + 1$, we have $k = 2(\alpha - 1)$. Since $B^\alpha \subset S^{-2(\alpha-1)} = S^{-k}$ [Be2], the result follows from Theorem 3.2.1.

Corollary 3.2.2. If $p > 0$, then there exists a function $f \in L_a^p(D)$ such that for each $a \in \mathbb{C}$, $\liminf_{r \rightarrow 1^-} \frac{N(r, f, a)}{\log \frac{1}{1-r}} \geq K$, where K is a positive constant.

Proof. Since $S^{-k} \subset L_a^{\frac{1}{p} + \varepsilon}(D)$, for $\varepsilon > 0$, the result follows from Corollary 3.2.1.

Remark. Combining results in §3.2 and §3.3, we obtain that the least upper bound and the greatest lower bound of $N(r, a)$, $a \in \mathbb{C}$, for B^α ($\alpha > 1$), $L_a^p(D)$ ($1 \leq p < \infty$), S^{-k} ($0 < k < \infty$) all have the same order $\log \frac{1}{1-r}$.

CHAPTER IV

ZEROS OF FUNCTIONS IN THE α -BLOCH SPACES

§4.0 Introduction

A function $f(z)$ analytic in the unit disc D is called an α -Bloch function ($\alpha > 0$) if f satisfies

$$\sup_{z \in D} (1 - |z|)^{\alpha} |f'(z)| < \infty$$

The set of all α -Bloch functions is denoted by $\mathcal{B}^{\alpha}$. A Bloch function is precisely a 1-Bloch function. It is known that the zeros $\{z_k\}$ of nontrivial H^p (Hardy space $0 < p \leq \infty$) functions are completely characterized by the Blaschke condition

$$\sum_{k=1}^{\infty} (1 - |z_k|) < \infty$$

In particular, all of the H^p spaces admit the same zero sets [Du].

In this chapter, we obtain some results on the zero sets of α -Bloch functions. We show that these " α -Bloch zero sets" ($\alpha > 1$) are quite different from the Blaschke sequences. Our main results are as follows.

Theorem 4.1. If $\alpha > 1$, then there is an α -Bloch zero set which is not a Bloch zero set.

Corollary 4.1. For each $\alpha > 1$, there is an α -Bloch zero set which does not satisfy the Blaschke condition.

Theorem 4.2. If $1 < \alpha < \beta$ and if $\alpha < \frac{1}{4}(\beta + 3)$ then there is a β -Bloch zero set which is not an α -Bloch zero set.

The first and the second theorems are proved by constructing a Horowitz-type infinite product [Bel], [Ho], adopting Beller's technique [Bel], and using a result on the coefficients of the α -Bloch functions (see Theorem 5.1 of this dissertation).

Corollary 4.1 is an immediate consequence of Theorem 4.1.

§4.1 Zero sets of α -Bloch functions

Beller's Lemma. [Bel] If $f \in L_a^p(D)$, ($0 < p < \infty$) and if $\{z_n\}$ is the ordered zero set of f , then $\limsup_{n \rightarrow \infty} \frac{1 - |z_n|}{n^{-1} \log n} \leq \frac{1}{p}$.

Corollary 4.1. If $f \in B$, $\{z_n\}$ is the ordered zero set of f , then

$$\limsup_{n \rightarrow \infty} \frac{1 - |z_n|}{n^{-1} \log n} = 0.$$

Proof. Since $B \subset L_a^p$ for each $0 < p < \infty$, the required result follows from Beller's Lemma.

Theorem 4.1. If $\alpha > 1$, then there is an α -Bloch zero set which is not a Bloch zero set.

Proof of Theorem 4.1. Since $\alpha > 1$, there exists a positive number q such that $1 + \frac{2}{q} < \alpha$. Let

$$(4.1.1) \quad f(z) = \prod_{k=0}^{\infty} \left[1 + b_k z^{(n_k - n_{k-1})} \right]$$

where $n_k = 2^{2^k}$, $n_{-1} = 0$, $b_k = n_k^{-\frac{1}{q}}$. Since the radius of convergence of $\sum_{k=0}^{\infty} b_k z^{(n_k - n_{k-1})}$ is 1, it follows that f is analytic in $|z| < 1$, and its zeros are precisely the zeros of the factors of the right side of (4.1.1). Let $\{z_n\}_{n=1}^{\infty}$ be the ordered zero set of f . Clearly, for $n_{k-1} \leq n \leq n_k$, we have

$$|z_n| = b_k^{-(n_k - n_{k-1})^{-1}} = n_k^{\frac{1}{q(n_k - n_{k-1})}}.$$

In particular,

$$\begin{aligned} 1 - |z_{n_k}| &= 1 - \exp \left\{ \frac{-\log n_k}{q(n_k - n_{k-1})} \right\} \\ &> \frac{\log n_k}{q(n_k - n_{k-1})} > \frac{\log n_k}{q(n_k)} \quad \text{for sufficiently large } k. \end{aligned}$$

Thus we obtain

$$(4.1.2) \quad \limsup_{n \rightarrow \infty} \frac{1 - |z_n|}{n^{-1} \log n} \geq \frac{1}{q}.$$

It follows from Corollary 4.1 that $f \notin \mathcal{B}$, and hence $\{z_n\}$ is not a Bloch zero set.

We next shall show that $f \in \mathcal{B}^{\alpha}$, for each $\alpha > 1$. Put $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_k \geq 0$. According to [Theorem 5.1], it suffices to show that $\sum_{k=1}^n k a_k = O(n^{\alpha})$. Since $\alpha > 1 + \frac{2}{q}$ and $\sum_{k=1}^n k a_k \leq n \sum_{k=1}^n a_k$, it suffices to show that

$$(4.1.3) \quad \sum_{k=1}^n a_k = o(n^{2/q}) .$$

Let $S_n = \sum_{k=1}^n |a_k|$, it follows from [Bel] that $S_{n_m} = \prod_{k=0}^m (1 + b_k)$.

Since $\sum_{k=1}^{\infty} b_k^{-1} = \sum_{k=1}^{\infty} \frac{1}{n_k^{1/q}} = \sum_{k=1}^{\infty} \left(\frac{1}{2^{2^k}} \right)^{1/q} < \infty$, we obtain

$$(4.1.4) \quad S_{n_m} < \prod_{k=1}^{\infty} (1 + b_k^{-1}) \prod_{k=0}^m b_k < K \prod_{k=0}^m b_k ,$$

for some positive constant K .

In addition

$$(4.1.5) \quad \prod_{k=0}^m b_k = \left(\prod_{k=0}^m 2^{2^k} \right)^{\frac{1}{q}} = 2^{\frac{1}{q} \left\{ \sum_{k=0}^m 2^k \right\}} = 2^{\frac{1}{q} \left(\frac{2^{m+1} - 1}{2 - 1} \right)} =$$

$$= 2^{\frac{1}{q}(2^{m+1} - 1)} < 2^{\frac{1}{q}(2^{m+1})} = \left(2^{2^m} \right)^{2/q} = n_m^{\frac{2}{q}} .$$

(4.1.4) and (4.1.5) implies

$$(4.1.6) \quad S_{n_m} < K(n_m)^{2/q} .$$

If $n_{m-1} < n < n_m - n_{m-1}$, we have

$$(4.1.7) \quad S_n = S_{n_{m-1}} < K n_{m-1}^{2/q} < K n_m^{2/q} .$$

If $n_m - n_{m-1} \leq n < n_m$, we have

$$(4.1.8) \quad S_n = S_{n_m} < K \left(n_m^{2/q} \right) < K \left(\frac{n_m}{n_m - n_{m-1}} \right)^{2/q} n^{2/q}$$

$$< K 2^{2/q} n^{2/q} .$$

Combining (4.1.6), (4.1.7) and (4.1.8), we obtain (4.1.3). Thus $f \in B^\alpha$. This completes the proof.

Corollary 4.2. For each $\alpha > 1$, there exists an α -Bloch function f such that f does not satisfy the Blaschke condition.

This is an immediate consequence of Theorem 4.1.

Theorem 4.2. If $1 < \alpha < \beta$ and if $4\alpha - 3 < \beta$ then there exists a β -Bloch ordered zero set which is not an α -Bloch ordered zero set.

Proof of Theorem 4.2. Let $\varepsilon > 0$, $p = \frac{1}{2(\alpha-1)} + \varepsilon$, and

$$f(z) = \prod_{k=0}^{\infty} \left[1 + b_k z^{(n_k - n_{k-1})} \right]$$

where $n_k = 2^{2^k}$, $b_k = n_k^{1/p}$.

Let $\{z_n\}_{n=1}^{\infty}$ be the ordered zero set of f . For $n_{k-1} < n \leq n_k$,

$$|z_n| = b_k^{-(n_k - n_{k-1})^{-1}} = n_k^{-\frac{1}{p(n_k - n_{k-1})}}.$$

Do the same procedure to reaching (4.1.2), we obtain

$$(4.1.9) \quad \limsup_{n \rightarrow \infty} \frac{1 - |z_n|}{n^{-1} \log n} \geq \frac{1}{p} = \frac{1}{\frac{1}{2(\alpha-1)} + \varepsilon} > \frac{1}{\frac{1}{2(\alpha-1)} + 2\varepsilon}.$$

It follows from Beller's Lemma and (4.1.9) that $f \notin L_a^{\frac{1}{2(\alpha-1)} + 2\varepsilon}$.

Since $B^\alpha \subset S^{-2(\alpha-1)} \subseteq L_a^{\frac{1}{2(\alpha-1)} + \eta}$ for each $\eta > 0$ [Be2], it follows that $f \notin B^\alpha$.

We next shall show that $f \in \mathcal{B}^\beta$. Put $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_k \geq 0$.

According to [Thm. 5.1], to show $f \in \mathcal{B}^\beta$ it is only necessary to show that $\sum_{k=1}^n k a_k = O(n^\beta)$. Let $S_n = \sum_{k=0}^n a_k$. As in the proof of Theorem 4.1, we obtain $S_n \leq K 2^{2/p} n^{2/p}$ for $n = 1, 2, 3, \dots$. Also,

$$\sum_{k=0}^n k a_k \leq n \sum_{k=0}^n a_k = n S_n = O(n^{1 + \frac{2}{p}})$$

and

$$\begin{aligned} 1 + \frac{2}{p} &= 1 + 2 \left(\frac{1}{\frac{1}{2(\alpha-1)} + \varepsilon} \right) = 1 + \frac{4(\alpha-1)}{1 + 2\varepsilon(\alpha-1)} < 1 + 4\alpha - 4 = \\ &= 4\alpha - 3 < \beta. \end{aligned}$$

Thus $\sum_{k=0}^n k a_k = O(n^\beta)$. This completes the proof of Theorem 4.2.

CHAPTER V

COEFFICIENTS OF α -BLOCH FUNCTIONS

§5.1 Introduction

Coefficients of Bloch functions have been studied by Anderson, Clunie, and Pommerenke [ACP], Mathews [Ma], Neitzke [Ne], and Fernandez [Fe], among others. In this chapter we investigate coefficients of α -Bloch functions. In section 2 following Bennett, Stegenga, and Timoney [BST] we give two sufficient conditions on coefficients for $\mathcal{B}^\alpha$. In section 3 we obtain one sufficient condition and three necessary conditions on coefficients of little α -Bloch functions. In section 4, we establish some sufficient conditions and some necessary conditions on blocks of coefficients for the classes $\mathcal{B}^\alpha$ and $\mathcal{B}_0^\alpha$. The results we obtain in this chapter are related to results of Hardy [Ha], Titchmarsh [Ti], Mathews [Ma] and Neitzke [Ne].

§5.2 An extension of a theorem of Mathews

Theorem 5.0 [BST] If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is holomorphic in D and $a_n \geq 0$. Then $f \in \mathcal{B}^\alpha$ iff $\sum_{k=1}^n k a_k = O(n^\alpha)$, and then $f \in \mathcal{B}_0^\alpha$ iff $\sum_{k=1}^n k a_k = o(n^\alpha)$.

Theorem 5.1. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$. If $\sum_{k=J}^n k^J |a_k| = O(n^{J+\alpha-1})$ for some $J = 1, 2, \dots$, then $f \in \mathcal{B}^\alpha$.

Proof of Theorem 5.1. For $J = 1$, put $g(z) = \sum_{n=0}^{\infty} |a_n| z^n$, since $\sum_{k=1}^n k |a_k| = O(n^\alpha)$, it follows from Theorem 5.0, that $g \in B^\alpha$. Now

$$|z| |f'(z)| \leq \sum_{k=1}^{\infty} k |a_k| |z|^k = g'(|z|) \leq \text{constant} \frac{1}{(1-|z|)^\alpha},$$

and hence $f \in B^\alpha$.

For $J > 1$, it is easy to verify that $|f^{(J)}(z)| \leq \frac{C}{(1-|z|)^{\alpha+J-1}}$, and the result follows by successive integration.

Corollary 5.1. [Ma] Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be holomorphic in D , if there is a fixed integer $J \geq 0$ such that $\sum_{k=J}^n k^J |a_k| \leq C n^J$ for each n , then $f \in B$.

Proof. Setting $\alpha = 1$, the result follows from Theorem 5.1.

Theorem 5.2. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be holomorphic in D . If $\alpha \geq 1$ and $\sum_{k=1}^n k^{q-1} |a_k|^q = O(n^{\alpha-1})$ for some q , $1 < q < \infty$, then $f \in B^\alpha$.

Proof of Theorem 5.2. Applying Theorem 5.1, it suffices to show that $\sum_{k=1}^n k |a_k| = O(n^\alpha)$.

$$\begin{aligned}
\sum_{k=1}^n k |a_k| &= \sum_{k=1}^n k^{\frac{p-1}{p}} k^{\frac{1}{p}} |a_k| \\
&\leq \left(\sum_{k=1}^n k^{p-1} \right)^{1/p} \left(\sum_{k=1}^n k^{\frac{q}{p}} |a_k|^q \right)^{1/q} \quad (\text{Hölder inequality}) \\
&= O(n) O\left(n^{\frac{\alpha-1}{q}}\right) \\
&= O\left(n^{1 + \frac{\alpha-1}{q}}\right) = O(n^\alpha) .
\end{aligned}$$

Corollary 5.2. [Ma] Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$, if $\sum_{n=1}^{\infty} n |a_n|^2 < \infty$ then $f \in \mathcal{B}$.

Proof. Setting $\alpha = 1$ and $q = 2$, the result follows from Theorem 5.2.

Corollary 5.3. ([Ne, Theorem 1, p. 11]).

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$, if $\sum_{n=1}^{\infty} k^{q-1} |a_k|^q < \infty$, for some $q > 1$, then $f \in \mathcal{B}$.

Proof. Putting $\alpha = 1$, the result follows from Theorem 5.2.

§5.3 Conditions on coefficients for the little α -Bloch space $\mathcal{B}_0^\alpha$

Theorem 5.3. Let $f(z)$ be a function in $\mathcal{B}_0^\alpha$ where $f(z) = \sum_{k=0}^{\infty} a_k z^k$ then $k^{1-\alpha} a_k = o(1)$ as $k \rightarrow \infty$.

Proof of Theorem 5.3. From Cauchy's formula

$$\begin{aligned}
 |a_k| &= \left| \frac{1}{2\pi i k} \int_{|z|=r} \frac{f'(z)}{z^k} dz \right| \\
 &\leq \frac{1}{2\pi k} \int_0^{2\pi} \left| \frac{f'(re^{i\theta})}{r^k e^{ik\theta}} r i e^{i\theta} \right| d\theta \\
 &= \frac{r^{1-k}}{2\pi k} o\left(\frac{1}{(1-r)^\alpha}\right) \\
 &= o(k^{-1} r^{1-k} (1-r)^{-\alpha}) \quad \text{as } r \rightarrow 1^-.
 \end{aligned}$$

The minimum value of the last term occurs for $r = 1 - \alpha/(k-1)$.

Evaluating for this r , we obtain

$$\begin{aligned}
 k^{-1} r^{1-k} (1-r)^{-\alpha} &= k^{-1} \left(1 - \frac{\alpha}{k-1}\right)^{1-k} \left(\frac{\alpha}{k-1}\right)^{-\alpha} = o(k^{\alpha-1}) \\
 &\quad \text{as } k \rightarrow \infty.
 \end{aligned}$$

Thus $k^{1-\alpha} a_k = o(1)$ as $k \rightarrow \infty$.

Corollary 5.4. [Ne] Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$. If $f \in \mathcal{B}_0$ then $a_n \rightarrow 0$ as $n \rightarrow \infty$.

Proof. The result follows from Theorem 5.3, by setting $\alpha = 1$.

Theorem 5.4. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$. If

$$\sum_{k=J}^n \frac{k!}{(k-J)!} |a_k| = o\left(\sum_{k=1}^n k^{J+\alpha-2}\right) = o(n^{J+\alpha-1}) \quad \text{as } n \rightarrow \infty,$$

for some fixed $J = 1, 2, \dots$, then $f \in \mathcal{B}_0^\alpha$.

To prove Theorem 5.4. We need the following Lemma 5.1.

Lemma 5.1. ([Ne, p. 16] and [Ti, p. 224])

Let $h(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k$ both have radius of convergence 1, let $s_n = \sum_{k=0}^n a_k$, $t_n = \sum_{k=0}^n b_k$, $s_n \geq 0$, $t_n \geq 0$ for each n , and suppose $\sum s_n$ and $\sum t_n$ are both divergent, and $s_n = o(t_n)$. Then $h(x) = o(g(x))$ as $x \rightarrow 1^-$.

Proof of Theorem 5.4. If $J = 1$, let $h(x) = \sum_{k=0}^{\infty} k|a_k|x^k$, $g(x) = \sum_{k=0}^{\infty} b_k x^k$ where $b_k = \frac{k^{\alpha-1}}{\Gamma(\alpha)}$; $s_n = \sum_{k=1}^n k|a_k|$ and $t_n = \sum_{k=1}^n b_k$, by hypothesis, $s_n = o(t_n)$ as $n \rightarrow \infty$. Applying Lemma 5.1, we obtain

$$h(x) = o(g(x)) \quad \text{as} \quad x \rightarrow 1^-.$$

Since $t_n \sim n^{\alpha}$ as $n \rightarrow \infty$, it follows from Theorem 5.0 that $g(|z|) = o\left(\frac{1}{(1-|z|)^{\alpha}}\right)$ as $|z| \rightarrow 1^-$. This implies that

$$|f'(z)| \leq h(|z|) = o(g(z)) = o\left(\frac{1}{(1-|z|)^{\alpha}}\right) \quad \text{as} \quad |z| \rightarrow 1^-,$$

thus $f \in \mathcal{B}_0^{\alpha}$.

For $J > 1$ we have $f^{(J)}(z) = o\left(\frac{1}{(1-|z|)^{\alpha}}\right)$ as $|z| \rightarrow 1^-$, and

the results follow by repeating the above procedure.

Corollary 5.5. ([Ne, Thm. 3(i), p. 15]) Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$. If there exists a fixed integer J , $J \geq 1$ such that

$$\sum_{k=1}^n k^J |a_k| = o(n^J) \quad \text{then} \quad f \in \mathcal{B}_0.$$

Proof: The result follows from Theorem 5.4, by setting $\alpha = 1$.

We obtain $f \in \mathcal{B}_0^1 = \mathcal{B}_0$, since $\sum_{k=1}^n k^{J-1} \sim n^J$ as $n \rightarrow \infty$.

Theorem 5.5. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be in $\mathcal{B}_0^\alpha$. If there is a θ such that $\theta \leq \arg a_k \leq \theta + \pi/2$ for each k . Then $\sum_{k=1}^n k|a_k| = o(n^\alpha)$ as $n \rightarrow \infty$.

We need a lemma for the proof of Theorem 5.5. This Lemma is an immediate consequence of ([BST, Thm 1.4 and Thm 1.10a]).

Lemma 5.2. [BST] If $g(x) = \sum_{k=0}^{\infty} c_k x^k = o\left(\frac{1}{(1-x)^\alpha}\right)$ as $x \rightarrow 1^-$, as $x \rightarrow 1^-$, where $c_k \geq 0$ then $\sum_{k=0}^n c_k = o(n^\alpha)$ as $n \rightarrow \infty$.

Proof of Theorem 5.5. There is no loss of generality in assuming that $\theta = 0$. Since $f(z) \in \mathcal{B}_0^\alpha$, we have

$$\left| \sum_{k=1}^{\infty} k a_k z^{k-1} \right| = |f'(z)| = o((1-|z|)^{-\alpha})$$

and

$$\left| \sum_{k=1}^{\infty} k \bar{a}_k z^{k-1} \right| = \overline{(f'(z))} = o((1-|z|)^{-\alpha}) \text{ as } |z| \rightarrow 1^-.$$

Therefore

$$\sum k \operatorname{Re}(a_k) z^{k-1} = o((1-|z|)^{-\alpha}) \text{ as } |z| \rightarrow 1^-$$

where $0 \leq \operatorname{Re} a_k$. Similarly

$$\sum k \operatorname{Im}(a_k) z^{k-1} = o((1-|z|)^{-\alpha}) \text{ as } |z| \rightarrow 1^-$$

where $0 \leq \operatorname{Im}(a_k)$. Thus

$$\left| \sum_{k=1}^{\infty} k |a_k| z^{k-1} \right| = o((1-|z|)^{-\alpha}) .$$

In applying the lemma above, put $c_{k-1} = k|a_k|$. We obtain $g(|z|) = o((1-|z|)^{-\alpha})$ as $|z| \rightarrow 1^-$. By Lemma 5.2, we have $\sum_{k=1}^n k|a_k| = o(n^\alpha)$ as $n \rightarrow \infty$. This completes the proof.

Theorem 5.6. If $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is in B_0^α then for any fixed integer $p = 1, 2, \dots$, $\sum_{k=1}^n k^p |a_k| = o\left(n^{p + \alpha - \frac{1}{2}}\right)$ as $n \rightarrow \infty$.

Proof of Theorem 5.6. Let $g(z) = zf'(z) = \sum_{n=1}^{\infty} n a_n z^n$. By a result which is an extension of Hardy's result ([Harl p. 45]), we have

$$\sum k^p |a_k| |z|^k = o\left((1-|z|)^{-(p + \alpha - \frac{1}{2})}\right) \text{ as } |z| \rightarrow 1^- .$$

But

$$|z|^n \sum_{k=1}^n k^p |a_k| \leq \sum_{k=1}^n k^p |a_k| |z|^k \leq \sum_{k=1}^{\infty} k^p |a_k| |z|^k \text{ for all } z, |z| < 1$$

for each $n = 1, 2, 3, \dots$. Taking $|z| = e^{-\frac{1}{n}}$,

$$e^{-1} \sum_{k=1}^n k^p |a_k| = o\left(\left(1 - e^{-\frac{1}{n}}\right)^{-(p + \alpha - \frac{1}{2})}\right) = o\left(n^{p + \alpha - \frac{1}{2}}\right) \text{ as } n \rightarrow \infty;$$

since $\frac{1}{2n} \leq 1 - e^{-\frac{1}{n}} \leq \frac{1}{n}$ for n sufficiently large. This completes the proof of Theorem 5.6.

§5.4 Conditions on blocks of coefficients in the class B^α

Let $f(z) = \sum_{k=1}^{\infty} a_k z^k$. In this section we shall give several results concerning "blocks" of coefficients for α -Bloch and little α -Bloch functions. We will consider the portion of the power series for $f(z)$ between $k = m$ and $k = \delta m$, where $\delta =$ the gauss integer of $\alpha + 1$. The integer " δ " of the upper bound is not critical to the following analysis; simple revisions will accomodate any number exceeding 1.

Define $\beta_m = \sum_{k=m}^{\delta m} |a_k|$. We will first show that for functions whose power series have real nonnegative coefficients, we can characterize B^α and B_0^α by conditions on β_m .

Theorem 5.7. Suppose that $f(z) = \sum_{k=1}^{\infty} a_k z^k$, with $a_k \geq 0$. Then $f \in B^\alpha$ if and only if $\beta_m \leq \text{constant } m^{\alpha-1}$ for each $m = 1, 2, 3, \dots$.

Proof of Theorem 5.7: If $f \in B^\alpha$, applying Theorem 5.0, we have

$$m \beta_m = m \sum_{k=m}^{\delta m} a_k \leq \sum_{k=m}^{\delta m} k a_k \leq \sum_{k=1}^{\delta m} k a_k \leq \text{constant} (\delta m)^\alpha \leq \text{constant } m^\alpha.$$

If $\beta_m \leq \text{constant} \cdot m^{\alpha-1}$ for each $m = 1, 2, 3, \dots$. Fix n , and suppose that p is an integer such that $\delta^p \leq n < \delta^{p+1}$. To show $f \in B^\alpha$, by Theorem 5.1, that it suffices to show that $\sum_{k=1}^n k a_k = o(n^\alpha)$.

$$\begin{aligned}
\sum_{k=1}^n k a_k &< \sum_{k=1}^{\delta} k a_k + \sum_{k=\delta}^{\delta^2} k a_k + \cdots + \sum_{k=\delta^p}^{\delta^{p+1}} k a_k \\
&< \delta \sum_{k=1}^{\delta} a_k + \delta^2 \sum_{k=\delta}^{\delta^2} a_k + \cdots + \delta^{p+1} \sum_{k=\delta^p}^{\delta^{p+1}} a_k \\
&\leq \delta \cdot 1^{\alpha-1} + \delta^2 \cdot \delta^{\alpha-1} + \cdots + \delta^{p+1} \cdot \delta^{p(\alpha-1)} \\
&\leq \{\delta + \delta^2 + \cdots + \delta^{p+1}\} \delta^{p(\alpha-1)} \\
&\leq \text{constant } \delta^{p+2} \cdot n^{\alpha-1} \leq \text{constant } n^{\alpha}.
\end{aligned}$$

This completes the proof of Theorem 5.7.

Theorem 5.8. Suppose $f(z) = \sum_{k=1}^{\infty} a_k z^k$ with $a_k \geq 0$ for each k .

Let $\alpha \geq 1$. If $f \in \mathcal{B}_0^{\alpha}$ then $\beta_m = o(m^{\alpha-1})$ as $m \rightarrow \infty$.

Proof of Theorem 5.8. If $f \in \mathcal{B}_0^{\alpha}$

$$\frac{1}{\delta} \beta_m = \frac{m}{\delta m} \sum_{k=1}^{\delta m} a_k \leq \frac{1}{\delta m} \sum_{k=1}^{\delta m} k a_k = \frac{1}{\delta m} o((\delta m)^{\alpha}) = o((\delta m)^{\alpha-1}).$$

Theorem 5.9. If $\sum_{k=m}^{\delta m} k^{J-1} |a_k| \leq M^* m^{J+\alpha-2}$, for each positive

integer J, m and for a positive constant M^* , then $f(z) =$

$$\sum_{k=1}^{\infty} a_k z^k \in \mathcal{B}^{\alpha}.$$

Proof of Theorem 5.9. By Theorem 5.1, it suffices to show that

$$\sum_{k=1}^n k^J |a_k| = o(n^{J+\alpha-1}) \text{ as } n \rightarrow \infty. \text{ If } \delta^s \leq n < \delta^{s+1}, \text{ then}$$

$$\begin{aligned}
\sum_{k=1}^n k^J |a_k| &\leq \sum_{k=1}^{\delta} k^J |a_k| + \sum_{\delta}^{\delta^2} k^J |a_k| + \cdots + \sum_{\delta^s}^{\delta^{s+1}} k^J |a_k| \\
&\leq \delta \sum_1^{\delta} k^{J-1} |a_k| + \delta^2 \sum_{\delta}^{\delta^2} k^{J-1} |a_k| + \cdots + \delta^{s+1} \sum_{\delta^s}^{\delta^{s+1}} k^{J-1} |a_k| \\
&\leq \delta M^* + \delta^2 M^*(\delta^{J+\alpha-2}) + \cdots + \delta^{s+1} M^*(\delta^{s(J+\alpha-2)}) \\
&\quad \text{(by hypothesis)} \\
&= \delta M^* [1 + \delta^{J+\alpha-1} + \delta^{2(J+\alpha-1)} + \cdots + \delta^{s(J+\alpha-1)}] \\
&= \frac{\delta M^* [(\delta^{J+\alpha-1})^{s+1} - 1]}{\delta^{J+\alpha-1} - 1} \leq \frac{\delta M^* (\delta^{J+\alpha-1})^{s+1}}{\delta^{J+\alpha-1} - 1} \\
&\leq 2\delta M^* (\delta^s)^{J+\alpha-1} = O(n^{J+\alpha-1}).
\end{aligned}$$

This completes the proof of Theorem 5.9.

Theorem 5.10. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be in $\mathcal{B}^{\alpha}$. If there is a θ such that $\theta < \arg a_k \leq \theta + \frac{\pi}{2}$, then $\sum_{k=1}^n k |a_k| = O(n^{\alpha})$ as $n \rightarrow \infty$.

Proof of Theorem 5.10. The result follows from repeating the procedure of the proof of Theorem 5.5, and applying ([BST, Theorem 1.10(a)]) putting $\varphi(n) = n^{\alpha}$.

Theorem 5.11. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be in $\mathcal{B}^{\alpha}$. If there is an θ such that $\theta \leq \arg a_k \leq \theta + \pi/2$ for each k , then there is a positive constant C^* such that $\sum_{k=\delta m}^{(\delta+1)m} |a_k| \leq C^*(m^{\alpha-1})$.



Proof of Theorem 5.11. It follows from Theorem 5.10 that

$\sum_{k=1}^n k|a_k| = O(n^\alpha)$ as $n \rightarrow \infty$. Then

$$\sum_{\delta m}^{(\delta+1)m} |a_k| \leq \sum_{\delta m}^{(\delta+1)m} k|a_k| \leq \text{constant}((\delta+1)m)^\alpha \leq \text{constant } m^\alpha$$

and thus $\sum_{k=\delta m}^{(\delta+1)m} |a_k| \leq C^* m^{\alpha-1}$ where C^* is a positive constant.

Theorem 5.12. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an α -Bloch function. Then for any integer $p \geq 0$, we have $\sum_{k=1}^n k^p |a_k| \leq C n^{p+\alpha-\frac{1}{2}}$.

Proof of Theorem 5.12. Hardy has shown [Hal] that if $g(z) = \sum_{k=0}^{\infty} c_n z^n$ is analytic in D and $g(z) = O\left(\frac{1}{(1-|z|)^\alpha}\right)$, $\alpha > 0$ then for $-\infty < \gamma < \alpha + \frac{1}{2}$

$$\sum_{n=1}^{\infty} n^{-\gamma} |c_n| |z|^n = O\left(\frac{1}{(1-|z|)^{\alpha-\gamma+1/2}}\right).$$

Setting $f'(z) = zg(z)$, $\gamma = -p+1$. We obtain

$$\sum_{n=1}^{\infty} n^p |c_n| |z|^n = O\left(\frac{1}{(1-|z|)^{p+\alpha-1/2}}\right).$$

Estimating $\frac{1}{(1-|z|)^{p+\alpha-1/2}}$, we have

$$\sum_{n=1}^{\infty} n^p |a_n| |z|^n \leq C \sum_{n=1}^{\infty} n^{p+\alpha-\frac{3}{2}} |z|^n$$

Titchmarsh has shown [Ti, p. 224] that the above inequality implies

that $\sum_{k=1}^n k^p |a_k| \leq C n^{p+\alpha-1/2}$, for some positive constant C .

Theorem 5.13. If $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{B}^{\alpha}$ then

$$\sum_{k=\delta m}^{(\delta+1)m} k^{p-1} |a_k| \leq \text{constant } m^{p+\alpha-3/2} \quad \text{for each positive integer } p$$

Proof of Theorem 5.13. It follows from Theorem 5.12 that

$$\delta m \sum_{\delta m}^{(\delta+1)m} k^{p-1} |a_k| \leq \sum_{\delta m}^{(\delta+1)m} k^p |a_k| \leq \sum_1^{(\delta+1)m} k^p |a_k| \leq$$

$$\leq \text{constant } ((\delta+1)m)^{p+\alpha-1/2} \Rightarrow \sum_{\delta m}^{(\delta+1)m} k^{p-1} |a_k| = O(m^{p+\alpha-3/2}) .$$

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