DESIGN CALCULATION FOR FOUR BAR LINKAGE FUNCTION GENERATORS

Thesis for the Degree of Ph. D.

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Jerome Chmielewski

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THESIS

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DESIGN CALCULATION FOR FOUR BAR LINKAGE

FUNCTION GENERATORS

presented by

Jerome Chmielewski

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ABSTRACT

DESIGN CALCULATION FOR FOUR BAR LINKAGE
FUNCTION GENERATORS

by Jerome Chmielewski

Since a four bar linkage has only one degree of freedom, the output displacement angle of the linkage is an intrinsic function of the input displacement angle. Because of this fundamental property of four bar linkages, it is possible to design a four bar linkage to have its two displacement angles (input and output) satisfy an arbitrary functional relation at up to and including five precision points. That is, the intrinsic function can be made to approximate an arbitrary function over a given range of precision points, and the intrinsic functional relation and the arbitrary functional relation will each be satisfied by the pairs of displacement angles at the several precision points. Thus, the four bar linkage may be designed as a function generator of an arbitrary function.

In the same vein, it is possible to design a four bar linkage function generator that has precision derivatives at one or two of its precision points.

In the design of a four bar linkage function generator that has five precision points, the ratios of each crank link and the coupler link to the separation link, the starting position of the linkage, and the

five pairs of displacement angles which are the precision points of the generator must all be compatible with one another. The starting position of the linkage is given by a pair of starting angles. These starting angles must be determined in each generator design and they depend upon the pairs of displacement angles specified as the precision points.

After the starting angles have been determined for a function generator, the link ratios may be determined. The determination of the starting angles and link ratios completes the design of the function generator. That is, the starting angles and the link ratios completely define a function generator up to its actual physical size.

The thesis presents a design calculation for four bar linkage function generators that have five precision points. In the calculation one of a pair of starting angles of the linkage is determined from the solution of a cubic equation in its tangent. The remaining unknown starting angle of the pair is then evaluated directly. Although this design calculation is rather lengthy, it is straight forward and requires no iterations or graphical solutions. The calculation is inherently accurate.

The thesis also provides a design calculation for four bar linkage function generators that have precision derivatives at precision points. It is also implied that the design calculation can be modified to accept higher order derivatives at each of two precision points.

Further, it is shown that there is a possibility in a given generator design that three pairs of starting angles may exist. When three pairs of starting angles exist, they provide three different four bar linkages which are capable of generating the same arbitrary function with the same specified precision points.

Two problems, solved in detail, are included in the thesis. The solutions of these problems demonstrate the calculation and its accuracy. In the first solved problem it is determined that only one four bar linkage generator exists for the five specified precision points. In the second problem it is determined that three four bar linkage generators exist that are capable of satisfying the five pairs of displacement angles which are the specified precision points.

DESIGN CALCULATION FOR FOUR BAR LINKAGE FUNCTION GENERATORS

By

Jerome Chmielewski

A THESIS

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1961

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Jerome F. Chmielewski

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INTRODUCTION

A four bar linkage, as its name implies, consists of four links which are simply joined at four pinned joints to form a closed kinematic chain. When one of the links is held fast, the four bar linkage is a mechanism, and two of four links are cranks, one is a coupler, and the fixed link is a separation link. The two pinned joints at the ends of the separation link are crank pivots. The separation link is often merely a separating distance between two fixed pinned joints that are the crank pivots and are attached to a stationary frame. Since an end of each crank is attached to one of the separated crank pivots about which the crank may rotate and the cranks are coupled at the opposite ends at the pinned joints with the coupler, the mechanism is assembled in a manner that permits only one degree of freedom. Motion is generally imparted in the linkage by the rotation of one of the cranks about its crank pivot, and input and output are generally given in terms of the crank displacement angles which are measured from an initial position of the linkage.

Because of their simple construction and operation and their inherent amicability toward analysis, four bar linkages have achieved widespread popularity in the field of kinematic design. The linkages are used extensively in instruments and machines where the transformation of motion is a necessity or where a particular type of motion is to be generated as an end in itself. Needless to say, there exist different classes of four bar linkages.

One class of four bar linkages consists of those four bar linkages that are designed for coupler action. A four bar linkage in this class is designed to have a tracer point affixed to the coupler link execute a prescribed motion when motion is imparted in the linkage. Typical examples of these linkages are the Cyclodial Linkage and Robert's Linkage. Each of these linkages is designed to have the tracer point approximate rectilinear motion as one of the cranks is rotated about its crank pivot.

Another class consists of four bar linkages that are designed for coupler position. A linkage in this class is designed to have the coupler link occupy a series of prescribed positions as motion is imparted in the linkage.

The four bar linkages of yet another class are designed for the purpose of generating particular functions in terms of the input and output displacement angles of the linkage. It is to this type of four bar linkage that the following discussion applies.

Since a four bar linkage has only one degree of freedom, the output displacement angle of the linkage is an intrinsic function of the input displacement angle. This intrinsic function depends entirely upon the ratios of each crank link and the coupler link to the separation link. The displacement angles have the character of variables when motion is imparted in the linkage, thus form a discrete pair of values for each position of the linkage.

When up to and including five discrete pairs of displacement angles have been specified as precision points, one or more four bar linkages can be designed to fit together in each of the various positions of the linkage obtained by measuring the specified displacement angles from a common initial position of the linkage. This initial position or configuration depends upon the displacement angles specified as precision points, and there is no guarantee that any of the four bar linkages will move freely from one position to another. That is, there is no guarantee that the intrinsic functional relation between real values of the input and output displacement angles will be continuous for any of the four bar linkages.

Because of the property of four bar linkages stated in the above paragraph, a four bar linkage can always be designed to have its crank displacement angles satisfy an arbitrary functional relation at several specified precision points, and if the designed four bar linkage has a continuous intrinsic function, the intrinsic function will approximate the arbitrary function. The intrinsic functional relation and the arbitrary functional relation between the displacement angles will be satisfied simultaneously at the several precision points. Thus, the four bar linkage will generate the arbitrary function.

The design or synthesis of four bar linkages for use as function generators has only rather recently been put in analytical form by Freudenstein (1)* in a three, four, and five point approximation, and

*Numbers refer to references cited on page 60.

in an nth order approximation for n = 5, 6, and 7. In an nth order approximation an arbitrary function and its first (n-1) derivatives are equal to the intrinsic function and its first (n-1) derivatives at a single precision point. Freudenstein (1) has shown that the crux of the five precision point approximation is the determination of a pair of starting angles which give the initial position of the synthesized four bar linkage generator. Freudenstein (1) determines these starting angles by solving two complicated higher-order simultaneous equations graphically or by an iterative process with the aid of a digital computer.

It was felt that an investigation into the problem of designing four bar linkages for use as function generators would prove justifiable if the investigation provided a reasonably straight forward solution to the five precision point problem, a procedure for designing function generators with precision derivatives at more than one precision point, and a determination of the number of different four bar linkages that can be constructed upon the same five precision points.

CHAPTER I

ANALYSIS

1. <u>General Information</u>. The four bar linkage shown in Fig. 1 page 8 can be designed to have its two variable displacement angles satisfy an arbitrary functional relation f at five discrete precision points. That is

$$\psi_i = f(\phi_i) \quad \text{for } i = 1, 2, 3, 4, \text{ and } 5 \tag{1}$$

where

f is an arbitrary function relating ψ_i to ϕ_i

ψ_i is the angular displacement of link L_1 from a starting position as indicated by the starting angle ψ measured from the x-axis,

ϕ_i is the angular displacement of link L_3 from a starting position as indicated by the starting angle ϕ measured from the x-axis,

In order for equation (1) to hold true for each of the five precision points, there must exist a certain compatibility between the four bar link ratios L_1/L_4, L_2/L_4, and L_3/L_4, the starting angles ϕ and ψ, and the displacement angles ϕ_i and ψ_i for i = 1, 2, 3, 4, and 5. Surprisingly, there may exist more than one four bar linkage that will satisfy equation (1) for the five precision points (ϕ_i, ψ_i). The determination of the starting position of links L_1 and L_3 as indicated by ψ and ϕ rests upon the solution of a cubic equation in ψ_t or ϕ_t. Consequently, there will exist one or three solutions, as given by the real roots of the cubic equation.

2. Notation. It will be convenient to use the following notation.

$$\phi_{ic} = \cos \phi_i$$

$$\phi_{is} = \sin \phi_i$$

$$\phi_t = \tan \phi$$

$$\psi_{ic} = \cos \psi_i$$

$$\psi_{is} = \sin \psi_i$$

$$\psi_t = \tan \psi$$

$$(\phi_i - \psi_i)_c = \cos (\phi_i - \psi_i)$$

$$(\phi_i - \psi_i)_s = \sin (\phi_i - \psi_i)$$

$$(\phi - \psi)_t = \tan (\phi - \psi)$$

$$[(\phi - \psi) + (\phi_i - \psi_i)]_c = \cos [(\phi - \psi) + (\phi_i - \psi_i)]$$

An expression of the type

$$|\phi_{is}, \psi_{ic}, 1, (\phi_i - \psi_i)_c| \qquad \text{for } i = j,\ k,\ m, \text{ and } n$$
$$\text{or } i = j,\ k,\ m, \text{ and } p$$

shall represent a determinant which has elements as indicated by the angle symbol and indices. That is, the first index and Greek symbol shall indicate the angle and the second index shall indicate a trigonometric function of that angle for any element. It should be observed that in each determinant each element in the third column is equal to unity. For instance for i = j, k, m, and n

$$\left|\phi_{is},\ \psi_{ic},\ 1,\ (\phi_i-\psi_i)_c\right| = \begin{vmatrix} \phi_{js}, & \psi_{jc}, & 1, & (\phi_j-\psi_j)_c \\ \phi_{ks}, & \psi_{kc}, & 1, & (\phi_k-\psi_k)_c \\ \phi_{ms}, & \psi_{mc}, & 1, & (\phi_m-\psi_m)_c \\ \phi_{ns}, & \psi_{nc}, & 1, & (\phi_n-\psi_n)_c \end{vmatrix}$$

$$= \begin{vmatrix} \sin\phi_j, & \cos\psi_j, & 1, & \cos(\phi_j-\psi_j) \\ \sin\phi_k, & \cos\psi_k, & 1, & \cos(\phi_k-\psi_k) \\ \sin\phi_m, & \cos\psi_m, & 1, & \cos(\phi_m-\psi_n) \\ \sin\phi_n, & \cos\psi_n, & 1, & \cos(\phi_n-\psi_n) \end{vmatrix}$$

The meaning of j, k, m, n, and p will be explained later in paragraph 4.

3. Intrinsic Equation. From Fig. (1) it is seen that

$$L_2^2 = (P_1P_2)^2 = (x_2-x_1)^2 + (y_2-y_1)^2$$

where

$$x_1 = L_1(\psi+\psi_i)_c$$

$$y_1 = L_1(\psi+\psi_i)_s$$

$$x_2 = L_4+L_3(\phi+\phi_i)_c$$

$$y_2 = L_3(\phi+\phi_i)_s$$

or

$$L_2^2 = [L_4+L_3(\phi+\phi_i)_c - L_1(\psi+\psi_i)_c]^2 + [L_3(\phi+\phi_i)_s - L_1(\psi+\psi_i)_s]^2.$$

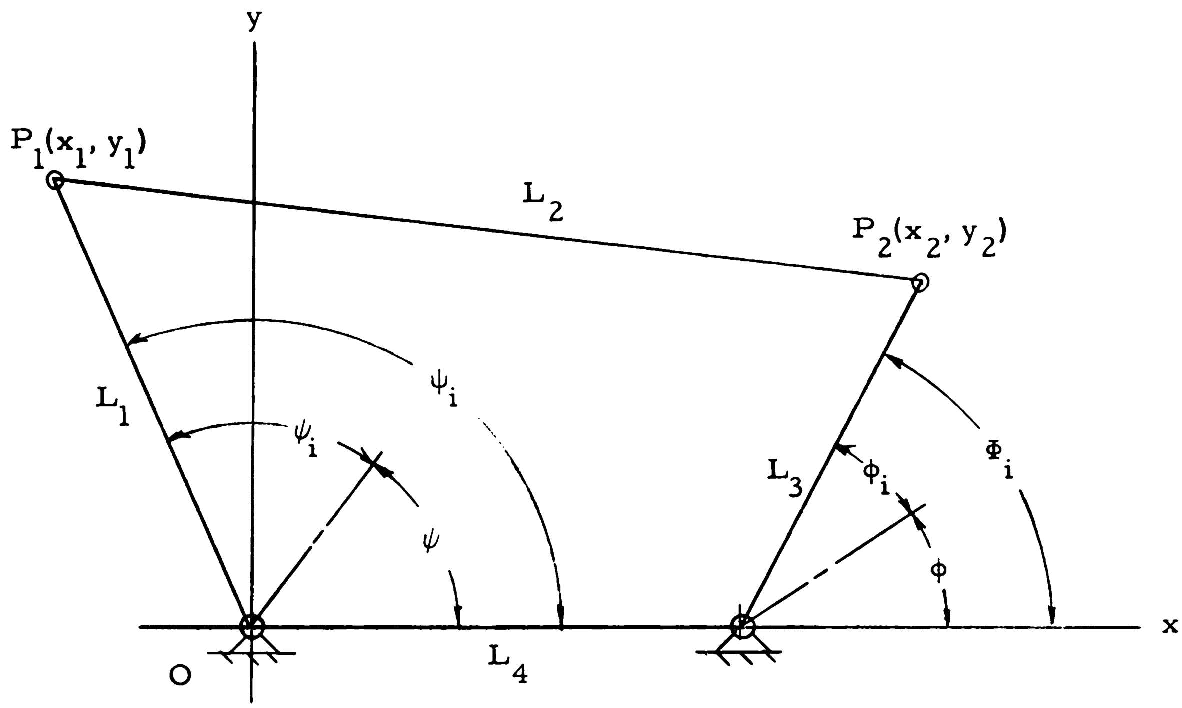

Fig. 1. Four Bar Linkage

After simplifying, the above equation may be put in the form*

$$\Phi_{ic}R_1 + \psi_{ic}R_2 + R_3 = (\Phi_i - \psi_i)_c \tag{2}$$

where

$$R_1 = L_4/L_1 \tag{2a}$$

$$R_2 = -L_4/L_3 \tag{2b}$$

$$R_3 = \left[(L_1/L_4)^2 - (L_2/L_4)^2 + (L_3/L_4)^2 + 1\right]\Big/(L_1/L_4)(L_3/L_4) \tag{2c}$$

$$\Phi_1 = \phi + \phi_1 \tag{2d}$$

$$\psi_i = \psi + \psi_i \tag{2e}$$

*This form is given by Freudenstein (1). If the values of Φ_{ic} and ψ_{ic} given in (2d) and (2e) are substituted in equation (2) and if the equation that will result from this substitution is solved for ψ_i in terms ϕ_i, then ψ_i will be defined as an intrinsic function of ϕ_i.

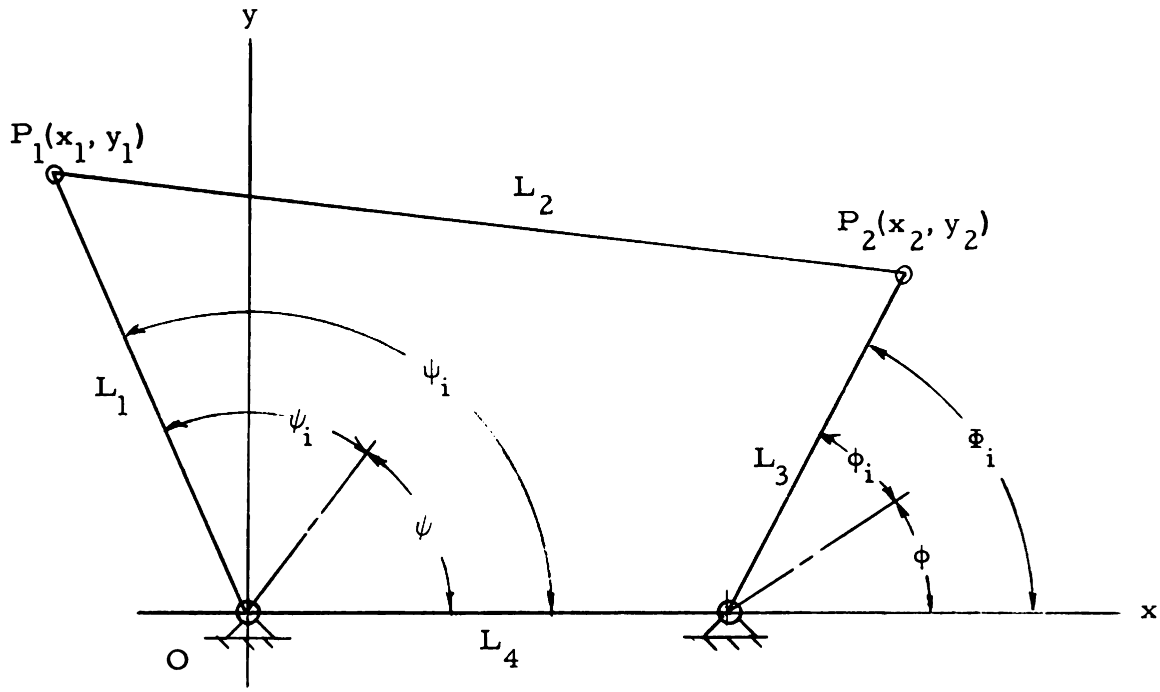

Fig. 1. Four Bar Linkage

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$$\Phi_1 = \phi + \phi_1 \tag{2d}$$

$$\psi_i = \psi + \psi_i \tag{2e}$$

*This form is given by Freudenstein (1). If the values of Φ_{ic} and ψ_{ic} given in (2d) and (2e) are substituted in equation (2) and if the equation that will result from this substitution is solved for ψ_i in terms ϕ_i, then ψ_i will be defined as an intrinsic function of ϕ_i.

4. Determination of the Starting Angles. Now equation (2) is a linear equation in R_1, R_2, and R_3, and if i takes the values 1, 2, 3, 4, and 5, there will exist a system of five equations in three unknowns which may be regarded as consistent. With this in mind consider the set of four equations produced by letting i take the values j, k, m, and n, and also consider the set of four equations where i has the values j, k, m, and p. The quantities j, k, m, n, and p are each to take a different value from the set of five points 1, 2, 3, 4, and 5. If each set of four equations is to be consistent, then the resultant or eliminant of each set must vanish. Let the resultant of the first set be Δ, and the resultant of the second set be Δ'. Then

$$\Delta = \left| (\phi+\phi_i)_c, \ (\psi+\psi_i)_c, \ 1, \ [(\phi+\phi_i) - (\psi+\psi_i)]_c \right| = 0 \qquad (3)$$

for i = j, k, m, and n

$$\Delta' = \left| (\phi+\phi_i)_c, \ (\psi+\psi_i)_c, \ 1, \ [(\phi+\phi_i) - (\psi+\psi_i)]_c \right| = 0 \qquad (4)$$

for i = j, k, m, and p

Each resultant can be expanded into a polynomial in the two variables ϕ_t and ψ_t. (Refer to Table I for the expansion of the resultant Δ.) The resulting polynomials can each be factored into a quadratic expression in ϕ_t with coefficients that are quadratic expressions in ψ_t which have constant coefficients. Thus

$$\Delta = \Big[|\phi_{is}, \psi_{is}, 1, (\phi_i - \psi_i)_c| \psi_t^2$$

$$- \big(|\phi_{is}, \psi_{ic}, 1, (\phi_i - \psi_i)_c| + |\phi_{is}, \psi_{is}, 1, (\phi_i - \psi_i)_s| \big) \psi_t$$

$$+ |\phi_{is}, \psi_{ic}, 1, (\phi_i - \psi_i)_s| \Big] \phi_t^2$$

$$+ \Big[\big(|\phi_{is}, \psi_{is}, 1, (\phi_i - \psi_i)_s| - |\phi_{ic}, \psi_{is}, 1, (\phi_i - \psi_i)_c| \big) \psi_t^2$$

$$+ \big(|\phi_{is}, \psi_{is}, 1, (\phi_i - \psi_i)_c| + |\phi_{ic}, \psi_{ic}, 1, (\phi_i - \psi_i)_c|$$

$$+ |\phi_{ic}, \psi_{is}, 1, (\phi_i - \psi_i)_s| - |\phi_{is}, \psi_{ic}, 1, (\phi_i - \psi_i)_s| \big) \psi_t$$

$$- \big(|\phi_{is}, \psi_{ic}, 1, (\phi_i - \psi_i)_c| + |\phi_{ic}, \psi_{ic}, 1, (\phi_i - \psi_i)_s| \big) \Big] \phi_t$$

$$- |\phi_{ic}, \psi_{is}, 1, (\phi_i - \psi_i)_s| \psi_t^2$$

$$+ \big(|\phi_{ic}, \psi_{ic}, 1, (\phi_i - \psi_i)_s| - |\phi_{ic}, \psi_{is}, 1, (\phi_i - \psi_i)_c| \big) \psi_t$$

$$+ |\phi_{ic}, \psi_{ic}, 1, (\phi_i - \psi_i)_c| = 0 \qquad (5)$$

for i = j, k, m, and n

If in equation (5) the following substitutions are made

$$\begin{aligned}
\delta_1 &= |\phi_{is}, \psi_{is}, 1, (\phi_i - \psi_i)_c| \\
\delta_2 &= |\phi_{is}, \psi_{ic}, 1, (\phi_i - \psi_i)_c| \\
\delta_3 &= |\phi_{is}, \psi_{is}, 1, (\phi_i - \psi_i)_s| \\
\delta_4 &= |\phi_{is}, \psi_{ic}, 1, (\phi_i - \psi_i)_s| \qquad \text{for } i = j,\ k,\ m, \text{ and } n \qquad (5a) \\
\delta_5 &= |\phi_{ic}, \psi_{is}, 1, (\phi_i - \psi_i)_s| \\
\delta_6 &= |\phi_{ic}, \psi_{ic}, 1, (\phi_i - \psi_i)_s| \\
\delta_7 &= |\phi_{ic}, \psi_{is}, 1, (\phi_i - \psi_i)_c| \\
\delta_8 &= |\phi_{ic}, \psi_{ic}, 1, (\phi_i - \psi_i)_c|
\end{aligned}$$

the result will be

$$[\delta_1\psi_t^2 - (\delta_2+\delta_3)\psi_t + \delta_4]\phi_t^2$$

$$+[(\delta_3-\delta_7)\psi_t^2 + (\delta_1+\delta_8+\delta_5-\delta_4)\psi_t - (\delta_2+\delta_6)]\phi_t$$

$$-\delta_5\psi_t^2 + (\delta_6-\delta_7)\psi_t + \delta_8 = 0 \qquad (5b)$$

If in equation (5b) the following substitutions are made

$$\begin{array}{lll} \alpha_2 = \delta_1, & \alpha_1 = \delta_3-\delta_7, & \alpha_0 = -\delta_5 \\ \beta_2 = -(\delta_2+\delta_3), & \beta_1 = \delta_1+\delta_8+\delta_5-\delta_4, & \beta_0 = \delta_6-\delta_7 \\ \gamma_2 = \delta_4, & \gamma_1 = -(\delta_2+\delta_6), & \gamma_0 = \delta_8 \end{array} \qquad (5c)$$

the result will be

$$(\alpha_2\psi_t^2 + \beta_2\psi_t + \gamma_2)\phi_t^2 + (\alpha_1\psi_t^2 + \beta_1\psi_t + \gamma_1)\phi_t + \alpha_0\psi_t^2 + \beta_0\psi_t + \gamma_0 = 0 \qquad (5d)$$

Likewise

$$\Delta' = \Big[|\phi_{is}, \psi_{is}, 1, (\phi_i-\psi_i)_c|\psi_t^2$$

$$-\big(|\phi_{is}, \psi_{ic}, 1, (\phi_i-\psi_i)_c| + |\phi_{is}, \psi_{is}, 1, (\phi_i-\psi_i)_s|\big)\psi_t$$

$$+|\phi_{is}, \psi_{ic}, 1, (\phi_i-\psi_i)_s|\Big]\phi_t^2$$

$$+\Big[\big(|\phi_{is}, \psi_{is}, 1, (\phi_i-\psi_i)_s| - |\phi_{ic}, \psi_{is}, 1, (\phi_i-\psi_i)_c|\big)\psi_t^2$$

$$+\big(|\phi_{is}, \psi_{is}, 1, (\phi_i-\psi_i)_c| + |\phi_{ic}, \psi_{ic}, 1, (\phi_i-\psi_i)_c|$$

$$+|\phi_{ic}, \psi_{ic}, 1, (\phi_i-\psi_i)_c| - |\phi_{is}, \psi_{ic}, 1, (\phi_i-\psi_i)_s|\big)\psi_t$$

$$-\big(|\phi_{is}, \psi_{is}, 1, (\phi_i-\psi_i)_s| + |\phi_{ic}, \psi_{ic}, 1, (\phi_i-\psi_i)_s|\big)\Big]\phi_t$$

$$-|\phi_{ic}, \psi_{is}, 1, (\phi_i-\psi_i)_s|\psi_t^2$$

$$+\big(|\phi_{ic}, \psi_{ic}, 1, (\phi_i-\psi_i)_s| - |\phi_{ic}, \psi_{is}, 1, (\phi_i-\psi_i)_c|\big)\psi_t$$

$$+|\phi_{ic}, \psi_{ic}, 1, (\phi_i-\psi_i)_c| = 0 \qquad (6)$$

for $i = j, k, m,$ and p

Also if in equation (6) the following substitutions are made

$$
\begin{aligned}
\delta'_1 &= |\phi_{is},\ \psi_{is},\ 1,\ (\phi_i-\psi_i)_c| \\
\delta'_2 &= |\phi_{is},\ \psi_{ic},\ 1,\ (\phi_i-\psi_i)_c| \\
\delta'_3 &= |\phi_{is},\ \psi_{is},\ 1,\ (\phi_i-\psi_i)_s| \\
\delta'_4 &= |\phi_{is},\ \psi_{ic},\ 1,\ (\phi_i-\psi_i)_s| \qquad \text{for } i = j,\ k,\ m, \text{ and } p \qquad (6a)\\
\delta'_5 &= |\phi_{ic},\ \psi_{is},\ 1,\ (\phi_i-\psi_i)_s| \\
\delta'_6 &= |\phi_{ic},\ \psi_{ic},\ 1,\ (\phi_i-\psi_i)_s| \\
\delta'_7 &= |\phi_{ic},\ \psi_{is},\ 1,\ (\phi_i-\psi_i)_c| \\
\delta'_8 &= |\phi_{ic},\ \psi_{ic},\ 1,\ (\phi_i-\psi_i)_c|
\end{aligned}
$$

the result will be

$$
\begin{aligned}
&[\delta'_1\psi_t^2 - (\delta'_2+\delta'_3)\psi_t + \delta'_4]\,\phi_t^2 \\
&\quad + [(\delta'_3-\delta'_7)\psi_t^2 + (\delta'_1+\delta'_8+\delta'_5-\delta'_4)\psi_t - (\delta'_2+\delta'_6)]\,\phi_t \\
&\quad - \delta'_5\psi_t + (\delta'_6-\delta'_7)\psi_t + \delta'_8 = 0 \qquad (6b)
\end{aligned}
$$

If in equation (6b) the following substitutions are made

$$
\begin{aligned}
\alpha'_2 &= \delta'_1, & \alpha'_1 &= \delta'_3 - \delta'_7, & \alpha'_0 &= -\delta'_5 \\
\beta'_2 &= -(\delta'_2+\delta'_3), & \beta'_1 &= \delta'_1+\delta'_8+\delta'_5-\delta'_4, & \beta'_0 &= \delta'_6-\delta'_7 \qquad (6c)\\
\gamma'_2 &= \delta'_4, & \gamma'_1 &= -(\delta'_2+\delta'_6), & \gamma'_0 &= \delta'_8
\end{aligned}
$$

the result will be

$$
(\alpha'_2\psi_t^2 + \beta'_2\psi_t + \gamma'_2)\phi_t^2 + (\alpha'_1\psi_t^2 + \beta'_1\psi_t + \gamma'_1)\phi_t + \alpha'_0\psi_t^2 + \beta'_0\psi_t + \gamma'_0 = 0 \qquad (6d)
$$

Now if in equation (5d) the following substitutions are made

$$
X_r = \alpha_r\psi_t^2 + \beta_r\psi_t + \gamma_r \qquad \text{for } r = 0,\ 1, \text{ and } 2
$$

then equation (5d) may be put in the form

$$\sum_{0}^{2} X_r \phi_t^{\,r} = 0 \tag{7}$$

And if in equation (6d) the following substitutions are made

$$X'_r = \alpha'_r \psi_t^2 + \beta'_r \psi_t + \gamma'_r \qquad \text{for } r = 0,\ 1,\ \text{and } 2$$

then equation (6d) may be put in the form

$$\sum_{0}^{2} X'_r\ \phi_t^{\,r} = 0 \tag{8}$$

The left hand members of equations (7) and (8) are quadratic expressions in the variable ϕ_t with quadratic coefficients in the variable ψ_t with constant coefficients, and if they are to have a common root for ϕ_t, then their resultant must vanish, or

$$\begin{vmatrix} X_0, & 0 & X'_0, & 0 \\ X_1, & X_0, & X'_1, & X'_0 \\ X_2, & X_1, & X'_2, & X'_1 \\ 0\ , & X_2, & 0\ , & X'_2 \end{vmatrix} = 0 \tag{9}$$

Expanding the left hand side of equation (9) into a polynomial in ψ_t leaves

$$\sum_{0}^{8} Cr\,\psi_t^{\,r} = 0 \tag{10}$$

Values of the coefficients C_r in equation (10) are given in Table II as functions of the coefficients in the quadratic terms of the determinant in equation (9). It is not necessary to generate all the coefficients C_r in equation (10) for it can be shown that five of the roots of equation (10) may be regarded as known. Therefore, equation (10) may be reduced to the following cubic equation in the unknown ψ_t.

$$\sum_{0}^{3} A_r \psi_t^r = 0 \tag{11}$$

where

$$A_0 = -C_0/\mu_1\mu_2\mu_3 C_8$$

$$A_1 = \frac{C_6}{C_8} - (\mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3) - 1$$

$$+ (\mu_1 + \mu_2 + \mu_3)\left(\frac{C_7}{C_8} + \mu_1 + \mu_2 + \mu_3\right) \tag{11a}$$

$$A_2 = C_7/C_8 + \mu_1 + \mu_2 + \mu_3$$

$$A_3 = 1$$

here

$$\mu_1 = -[(\psi_j + \psi_k)/2]_t$$

$$\mu_2 = -[(\psi_j + \psi_m)/2]_t \tag{11b}$$

$$\mu_3 = -[(\psi_k + \psi_m)/2]_t$$

Also a somewhat less accurate calculation gives

$$A_1 = -\frac{C_1}{\mu_1\mu_2\mu_3 C_8} + A_0\left(\frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3}\right) \tag{11c}$$

That is, less accurate in the sense that it tends to base the cubic equation on the known roots rather than on the coefficients of the eighth degree polynomial.

Equation (11b) gives three of the five known roots and the remaining two are

$$\mu_4 = +i$$
$$\mu_5 = -i$$

Since $\mu_5 = -\mu_4$, they do not appear in equations (11a).

Each real root of equation (11) will provide a value for the starting angle ψ. Consequently, if all the roots are real, the starting angle ψ will have three values.

When a value of ψ has been determined, the two quadratic equations (7) and (8) may be evaluated and solved simultaneously for ϕ_t, giving

$$\phi_t = \frac{X_2'X_0 - X_0'X_2}{X_1'X_2 - X_2'X_1} \tag{12}$$

Thus, a pair of starting angles ϕ and ψ are determined.

It remains to be shown here that μ_1, μ_2, μ_3, μ_4, and μ_5 actually are roots of equation (10).

The roots μ_1, μ_2, and μ_3 may be treated collectively for they are similar.

When equation (12) is evaluated for $\psi_t = \mu_1$, the result will be

$$\phi_t = -\left[\frac{(\phi_j + \phi_k)}{2}\right]_t$$

Therefore the pair of starting angles in this case is simply

$$\phi = -\frac{(\phi_j + \phi_k)}{2}$$

$$\psi = -\frac{(\psi_j + \psi_k)}{2}$$

The insertion of this pair of starting angles in each determinant of equations (3) and (4) will make the jth and kth row of each determinant identical and each determinant will vanish as required.

Thus, μ_1, μ_2, and μ_3 are roots of equation (10).

That μ_4 and μ_5 are roots of equation (10) can be verified easily by inserting the values of

$$\phi_t = i$$

$$\psi_t = i$$

or

$$\phi_t = -i$$

$$\psi_t = -i$$

in equation (5b) and (6b). Either pair of values will make the left side members of both equations (5b) and (6b) vanish.

Thus, μ_4 and μ_5 are roots of equation (10).

The coefficients α_r, β_r, and γ_r for $r = 0, 1,$ and 2 may be calculated in an alternate way. It can be shown that

$$\begin{aligned}
\alpha_2 &= -\tau_1 - \tau_2 - \tau_3 - \tau_4 \\
\beta_2 &= -2[\eta_{4t}\tau_2 - (\theta_4 - \eta_4)_t \tau_4] \\
\gamma_2 &= -\tau_1 + \tau_2 - \tau_3 + \tau_4 \\
\alpha_1 &= -2[\theta_{4t}\tau_1 + (\theta_4 - \eta_4)_t \tau_4] \\
\beta_1 &= -4\tau_4 \\
\gamma_1 &= -2[\theta_{4t}\tau_1 - (\theta_4 - \eta_4)_t \tau_4] \\
\alpha_0 &= \tau_1 - \tau_2 - \tau_3 + \tau_4 \\
\beta_0 &= -2[\eta_{4t}\tau_2 + (\theta_4 - \eta_4)_t \tau_4] \\
\gamma_0 &= \tau_1 + \tau_2 - \tau_3 - \tau_4
\end{aligned} \tag{13}$$

here

$$\tau_1 = -\theta_{4c}\Big\{(\eta_{2c}-\eta_{1c})\,[(\theta_3-\eta_3)_c - (\theta_1-\eta_1)_c]$$

$$-(\eta_{3c} - \eta_{1c})[(\theta_2-\eta_2)_c - (\theta_1-\eta_1)_c]\Big\}$$

$$\tau_2 = \eta_{4c}\Big\{(\theta_{2c}-\theta_{1c})[(\theta_3-\eta_3)_c - (\theta_1-\eta_1)_c]$$

$$-(\theta_{3c} - \theta_{1c})\,[(\theta_2-\eta_2)_c - (\theta_1-\eta_1)_c]\Big\} \tag{13a}$$

$$\tau_3 = -\Big\{(\theta_1-\eta_1)_c\,(\theta_{2c}\eta_{3c} - \theta_{3c}\eta_{2c})$$

$$-(\theta_2-\eta_2)_c\,(\theta_1\eta_{3c} - \theta_{3c}\eta_{1c})$$

$$+(\theta_3-\eta_3)_c\,(\theta_{1c}\eta_{2c} - \theta_{2c}\eta_{1c})\Big\}$$

$$\tau_4 = (\theta_4-\eta_4)_c\,[(\theta_{2c}-\theta_{1c})(\eta_{3c}-\eta_{1c}) - (\theta_{3c}-\theta_{1c})(\eta_{2c}-\eta_{1c})]$$

where

$$\begin{aligned}
\theta_1 &= \tfrac{1}{2}(\phi_j + \phi_k - \phi_m - \phi_n) & \eta_1 &= \tfrac{1}{2}(\psi_j + \psi_k - \psi_m - \psi_n)\\
\theta_2 &= \tfrac{1}{2}(\phi_j - \phi_k + \phi_m - \phi_n) & \eta_2 &= \tfrac{1}{2}(\psi_j - \psi_k + \psi_m - \psi_n)\\
\theta_3 &= \tfrac{1}{2}(\phi_j - \phi_k - \phi_m + \phi_n) & \eta_3 &= \tfrac{1}{2}(\psi_j - \psi_k - \psi_m + \psi_n)\\
\theta_4 &= \tfrac{1}{2}(\phi_j + \phi_k + \phi_m + \phi_n) & \eta_4 &= \tfrac{1}{2}(\psi_j + \psi_k + \psi_m + \psi_n)
\end{aligned} \tag{13b}$$

The coefficients α'_r, β'_r, and γ'_r for r = 0, 1, and 2 may be calculated by using (13), (13a), and a modified equation (13b). The modification of equations (13b) being the replacement of ϕ_n and ψ_n by ϕ_p and ψ_p.

5. Determination of the Link Ratios. For every pair of starting angles the position angles Φ_i and ψ_i for i = 1, 2, 3, 4, and 5 may be evaluated by equations (2d) and (2e) respectively. When these position angles are substituted in equation (2), there will exist five equations in the three unknowns R_1, R_2, and R_3. Any three of the five equations may be solved simultaneously for the values of R_1, R_2, and R_3; however, it is better to solve all five equations simultaneously in order to determine the accuracy of the calculation. When R_1, and R_2 have been determined, the link ratios L_1/L_4 and L_3/L_4 may be evaluated from equations (2a) and (2b) respectively. After these two link ratios and R_3 are known, the link ratio L_2/L_4 may be obtained from equation (2c).

CHAPTER II

CALCULATIONS

1. Solved Problems. In order to demonstrate the solution of the five point four bar linkage problem given in articles (1) through (5), two examples will be given here.

(1) Let f of equation (1),

$$\psi_i = f(\phi_i) \tag{1}$$

be defined as a continuous function of ϕ_i, and suppose that equation (1) is satisfied by the following discrete pairs of angles which are also to be the displacement angles of a four bar linkage.

$$\phi_1 = 05^\circ\ 00'\ 00'', \qquad \psi_1 = 04^\circ\ 00'\ 00''$$

$$\phi_2 = 15^\circ\ 00'\ 00'', \qquad \psi_2 = 16^\circ\ 00'\ 00''$$

$$\phi_3 = 25^\circ\ 00'\ 00'', \qquad \psi_3 = 33^\circ\ 20'\ 00''$$

$$\phi_4 = 35^\circ\ 00'\ 00'', \qquad \psi_4 = 56^\circ\ 00'\ 00''$$

$$\phi_5 = 45^\circ\ 00'\ 00'', \qquad \psi_5 = 84^\circ\ 00'\ 00''$$

The problem here is to determine the link ratios and the starting angles in the design of the four bar linkage. It follows that

$$(\phi_1 - \psi_1) = 01^\circ\ 00'\ 00''$$

$$(\phi_2 - \psi_2) = -\ 01^\circ\ 00'\ 00''$$

$$(\phi_3 - \psi_3) = -\ 08^\circ\ 20'\ 00''$$

$$(\phi_4 - \psi_4) = -\ 21^\circ\ 00'\ 00''$$

$$(\phi_5 - \psi_5) = -\ 39^\circ\ 00'\ 00''$$

and

$\phi_{1c} = .99619470$, $\psi_{1c} = .99756405$, $(\phi_1 - \psi_1)_c = .99984770$

$\phi_{2c} = .96592583$, $\psi_{2c} = .96126170$, $(\phi_2 - \psi_2)_c = .99984770$

$\phi_{3c} = .90630779$, $\psi_{3c} = .83548781$, $(\phi_3 - \psi_3)_c = .98944164$

$\phi_{4c} = .81915204$, $\psi_{4c} = .55919290$, $(\phi_4 - \psi_4)_c = .93358043$

$\phi_{5c} = .70710678$, $\psi_{5c} = .10452846$, $(\phi_5 - \psi_5)_c = .77714596$

$\phi_{1s} = .08715574$, $\psi_{1s} = .06975647$, $(\phi_1 - \psi_1)_s = .01745241$

$\phi_{2s} = .25881905$, $\psi_{2s} = .27563736$, $(\phi_2 - \psi_2)_s = -.01745241$

$\phi_{3s} = .42261826$, $\psi_{3s} = .54950898$, $(\phi_3 - \psi_3)_s = -.14493186$

$\phi_{4s} = .57357644$, $\psi_{4s} = .82903757$, $(\phi_4 - \psi_4)_s = -.35836795$

$\phi_{5s} = .70710678$, $\psi_{5s} = .99452190$, $(\phi_5 - \psi_5)_s = -.62932039$

From equations (5a) and (6a)

$$\delta_1 = -.56650938 \times 10^{-3}, \qquad \delta'_1 = -.26360800 \times 10^{-2}$$
$$\delta_2 = +.43738916 \times 10^{-3}, \qquad \delta'_2 = +.21229813 \times 10^{-2}$$
$$\delta_3 = -.83669494 \times 10^{-3}, \qquad \delta'_3 = -.39906703 \times 10^{-2}$$
$$\delta_4 = -.10916537 \times 10^{-2}, \qquad \delta'_4 = -.41711880 \times 10^{-2}$$
$$\delta_5 = +.10695348 \times 10^{-3}, \qquad \delta'_5 = +.60278368 \times 10^{-3}$$
$$\delta_6 = +.11983010 \times 10^{-3}, \qquad \delta'_6 = +.45634663 \times 10^{-3}$$
$$\delta_7 = -.12389769 \times 10^{-3}, \qquad \delta'_7 = -.55927743 \times 10^{-3}$$
$$\delta_8 = -.37663543 \times 10^{-4}, \qquad \delta'_8 = -.19376427 \times 10^{-3}$$

From equations (5c) and (6c)

$$\alpha_2 = -.56650938 \times 10^{-3}, \quad \alpha_1 = -.71279725 \times 10^{-3}, \quad \alpha_0 = -.10695348 \times 10^{-3}$$
$$\beta_2 = +.39930578 \times 10^{-3}, \quad \beta_1 = +.59443422 \times 10^{-3}, \quad \beta_0 = +.24372779 \times 10^{-3}$$
$$\gamma_2 = -.10916537 \times 10^{-2}, \quad \gamma_1 = -.55721926 \times 10^{-3}, \quad \gamma_0 = -.37663543 \times 10^{-4}$$

$$\alpha'_2 = -.26360800 \times 10^{-2}, \quad \alpha'_1 = -.34313929 \times 10^{-2}, \quad \alpha'_0 = -.60278368 \times 10^{-3}$$
$$\beta'_2 = +.18676890 \times 10^{-2}, \quad \beta'_1 = +.19441274 \times 10^{-2}, \quad \beta'_0 = +.10156241 \times 10^{-2}$$
$$\gamma'_2 = -.41711880 \times 10^{-2}, \quad \gamma'_1 = -.25793279 \times 10^{-2}, \quad \gamma'_0 = -.19376427 \times 10^{-3}$$

From Table II

$$C_0 = -.23571951 \times 10^{-14}$$
$$C_6 = -.24518283 \times 10^{-14}$$
$$C_7 = -.21154334 \times 10^{-13}$$
$$C_8 = -.52287725 \times 10^{-15}$$

From equations (11b)

$$\mu_1 = -.17632698$$

$$\mu_2 = -.33783302$$

$$\mu_3 = -.45924395$$

From equations (11a)

$$A_0 = +.16479023 \times 10^3$$

$$A_1 = -.35040734 \times 10^2$$

$$A_2 = +.39484149 \times 10^2$$

$$A_3 = +.10000000 \times 10$$

Thus, equation (11) is

$$\psi_t^3 + 39.484149\,\psi_t^2 - 35.040734\,\psi_t + 164.79023 = 0$$

The above cubic equation has only one real root which is

$$\psi_t = -40.451105$$

$$\therefore \quad \psi = -88^\circ\ 35'\ 02''$$

From equation (12)

$$\phi_t = -1.0799791$$

$$\therefore \quad \phi = -47^\circ\ 12'\ 07''$$

From equations (2d) and (2e)

$$\Phi_1 = -42^\circ\ 12'\ 07'', \qquad \psi_1 = -84^\circ\ 35'\ 02''$$

$$\Phi_2 = -32^\circ\ 12'\ 07'', \qquad \psi_2 = -72^\circ\ 35'\ 02''$$

$$\Phi_3 = -22^\circ\ 12'\ 07'', \qquad \psi_3 = -55^\circ\ 15'\ 02''$$

$$\Phi_4 = -12^\circ\ 12'\ 07'', \qquad \psi_4 = -32^\circ\ 35'\ 02''$$

$$\Phi_5 = -02^\circ\ 12'\ 07'', \qquad \psi_5 = -04^\circ\ 35'\ 02''$$

Then

$$\Phi_{1c} = .74078180, \quad \psi_{1c} = .09438825, \quad (\Phi_1 - \psi_1)_c = .73866780$$

$$\Phi_{2c} = .84617508, \quad \psi_{2c} = .29930911, \quad (\Phi_2 - \psi_2)_c = .76174251$$

$$\Phi_{3c} = .92585776, \quad \psi_{3c} = .56998880, \quad (\Phi_3 - \psi_3)_c = .83820818$$

$$\Phi_{4c} = .97740872, \quad \psi_{4c} = .84260386, \quad (\Phi_4 - \psi_4)_c = .93739179$$

$$\Phi_{5c} = .99926161, \quad \psi_{5c} = .99680139, \quad (\Phi_5 - \psi_5)_c = .99913598$$

Evaluating equation (2) for i = 1, 2, 3, 4, and 5 gives

$$\begin{aligned}
.74078180\ R_1 + .09438825\ R_2 + R_3 &= .73866780 \\
.84617508\ R_1 + .29930911\ R_2 + R_3 &= .76174251 \\
.92585776\ R_1 + .56998880\ R_2 + R_3 &= .83820818 \\
.97740872\ R_1 + .84260386\ R_2 + R_3 &= .93739179 \\
.99926161\ R_1 + .99680139\ R_2 + R_3 &= .99913598
\end{aligned} \tag{a}$$

Subtracting the first equation from the last four and dividing by the coefficient of R_1 in each of the remaining four equations leaves

$$R_1 + 1.944344648\, R_2 = .21893910$$

$$R_1 + 2.569758654\, R_2 = .53783428$$

$$R_1 + 3.162005447\, R_2 = .83981987 \qquad \text{(b)}$$

$$R_1 + 3.491232448\, R_2 = 1.00769255$$

Subtracting the first equation from the last three and dividing by the coefficient of R_2 in each of the remaining three equations leaves

$$R_2 = .50989612$$

$$R_2 = .50989633 \qquad (b_1)$$

$$R_2 = .50989700$$

Taking

$$R_2 = .5098965 \qquad \text{(c)}$$

and substituting this value in equations (b) and solving each of the four equations for R_1 determines

$$R_1 = -.772475430$$

$$R_1 = -.772475668 \qquad (c_1)$$

$$R_1 = -.772475637$$

$$R_1 = -.772474652$$

Taking

$$R_1 = -.772475 \qquad \text{(d)}$$

and substituting this value and the value of R_2 from (c) into the five equations (a), and solving each of the five equations for R_3 determines

$$R_3 = 1.26277498$$
$$R_3 = 1.26277493$$
$$R_3 = 1.26277486 \qquad (d_1)$$
$$R_3 = 1.26277483$$
$$R_3 = 1.26277504$$

Evidently

$$R_3 = 1.26277 \qquad (e)$$

Substituting the value of R_1 from (d) into equation (2a) gives

$$L_1/L_4 = -1.29454 \qquad (f)$$

Substituting the value of R_2 from (c) into equation (2b) gives

$$L_3/L_4 = -1.96118 \qquad (g)$$

Substituting the values of R_3 from (e), of L_1/L_4 from (f), and of L_3/L_4 from (g) into equation (2c), and solving for the link ratio L_2/L_4 gives

$$L_2/L_4 = .331787$$

Thus the required link ratios are

$$L_1/L_4 = 1.29454$$
$$L_2/L_4 = .331787$$
$$L_3/L_4 = 1.96118$$

where the negative signs of the values of L_1/L_4 and L_3/L_4 have been removed. The removal of a negative sign from a link ratio may always be accomplished by the addition of 180° to the starting angle associated with the link ratio.

The above link ratios and the two starting angles

$$\phi = 132^\circ\ 47'\ 53''$$

$$\psi = 91^\circ\ 24'\ 58''$$

define the four bar linkage for the given displacement angles.

The mechanism is shown in Fig. (2) in its first precision position in full lines and in each of its remaining four positions in dashed lines.

(2) As in example (1), assume that a function f has been given as

$$\psi_1 = f(\phi_i)$$

and suppose f to be satisfied by the following discrete pairs of angles which are also to be the displacement angles of a four bar linkage function generator of f

$$\phi_1 = 01^\circ\ 03'\ 54.85'', \qquad \psi_1 = 00^\circ\ 06'\ 57.21''$$

$$\phi_2 = 12^\circ\ 37'\ 45.90'', \qquad \psi_2 = 04^\circ\ 43'\ 51.63''$$

$$\phi_3 = 36^\circ\ 53'\ 38.30'', \qquad \psi_3 = 23^\circ\ 37'\ 18.26''$$

$$\phi_4 = 67^\circ\ 02'\ 48.44'', \qquad \psi_4 = 57^\circ\ 52'\ \ 8.28''$$

$$\phi_5 = 87^\circ\ 19'\ 26.95'', \qquad \psi_5 = 86^\circ\ 00'\ 58.36''$$

Again the problem is to determine the link ratios and the starting angles of the four bar linkage.*

It follows that

$$(\phi_1 - \psi_1) = 00^\circ\ 56'\ 57.64''$$

$$(\phi_2 - \psi_2) = 07^\circ\ 53'\ 54.27''$$

$$(\phi_3 - \psi_3) = 13^\circ\ 16'\ 20.04''$$

$$(\phi_4 - \psi_4) = 09^\circ\ 10'\ 40.16''$$

$$(\phi_5 - \psi_5) = 01^\circ\ 18'\ 28.59''$$

and

$\phi_{1c} = .99982717$, $\psi_{1c} = .99999796$, $(\phi_1 - \psi_1)_c = .99986274$

$\phi_{2c} = .97580464$, $\psi_{2c} = .99659290$, $(\phi_2 - \psi_2)_c = .99051328$

$\phi_{3c} = .79974782$, $\psi_{3c} = .91621077$, $(\phi_3 - \psi_3)_c = .97329025$

$\phi_{4c} = .38997930$, $\psi_{4c} = .53185733$, $(\phi_4 - \psi_4)_c = .98719808$

$\phi_{5c} = .04668573$, $\psi_{5c} = .06947422$, $(\phi_5 - \psi_5)_c = .99973946$

$\phi_{1s} = .01859080$, $\psi_{1s} = .00202269$, $(\phi_1 - \psi_1)_s = .01656843$

$\phi_{2s} = .21864427$, $\psi_{2s} = .08247787$, $(\phi_2 - \psi_2)_s = .13741703$

$\phi_{3s} = .60033609$, $\psi_{3s} = .40069668$, $(\phi_3 - \psi_3)_s = .22957809$

$\phi_{4s} = .92082362$, $\psi_{4s} = .84683397$, $(\phi_4 - \psi_4)_s = .15949908$

$\phi_{5s} = .99890965$, $\psi_{5s} = .99758375$, $(\phi_5 - \psi_5)_s = .02282591$

*These points will satisfy the requirements for a function generator of the form $y = x^{1.5}$ with suitable scale factors and were taken from Freudenstein (2).

From equation (5a) and (6a)

$$\delta_1 = +.91278590 \times 10^{-3}, \qquad \delta'_1 = +.14316076 \times 10^{-2}$$

$$\delta_2 = -.38032454 \times 10^{-3}, \qquad \delta'_2 = -.56289109 \times 10^{-3}$$

$$\delta_3 = +.95820540 \times 10^{-4}, \qquad \delta'_3 = -.26574678 \times 10^{-2}$$

$$\delta_4 = -.65941582 \times 10^{-2}, \qquad \delta'_4 = -.16643246 \times 10^{-1}$$

$$\delta_5 = +.37584876 \times 10^{-2}, \qquad \delta'_5 = +.10650043 \times 10^{-1}$$

$$\delta_6 = +.33814141 \times 10^{-2}, \qquad \delta'_6 = +.87864628 \times 10^{-2}$$

$$\delta_7 = -.10868160 \times 10^{-4}, \qquad \delta'_7 = -.28898921 \times 10^{-3}$$

$$\delta_8 = -.17119667 \times 10^{-3}, \qquad \delta'_8 = -.48642779 \times 10^{-3}$$

From equations (5c) and (6c)

$$\alpha_2 = +.91278590 \times 10^{-3}, \qquad \alpha_1 = +.10668870 \times 10^{-3}, \qquad \alpha_0 = -.37584876 \times 10^{-2}$$

$$\beta_2 = +.28450400 \times 10^{-3}, \qquad \beta_1 = +.11094235 \times 10^{-1}, \qquad \beta_0 = +.33922823 \times 10^{-2}$$

$$\gamma_2 = -.65941582 \times 10^{-2}, \qquad \gamma_1 = -.30010895 \times 10^{-2}, \qquad \gamma_0 = -.17119667 \times 10^{-3}$$

$$\alpha'_2 = +.14316076 \times 10^{-2}, \qquad \alpha'_1 = -.23684787 \times 10^{-2}, \qquad \alpha'_0 = -.10650043 \times 10^{-1}$$

$$\beta'_2 = +.32203589 \times 10^{-2}, \qquad \beta'_1 = +.28238469 \times 10^{-1}, \qquad \beta'_0 = +.90754520 \times 10^{-2}$$

$$\gamma'_2 = -.16643246 \times 10^{-1}, \qquad \gamma'_1 = -.82235717 \times 10^{-2}, \qquad \gamma'_0 = -.48642779 \times 10^{-3}$$

From Table II

$$C_0 = -.94005139 \times 10^{-13}$$

$$C_6 = +.67677143 \times 10^{-10}$$

$$C_7 = -.26412843 \times 10^{-10}$$

$$C_8 = -.43945779 \times 10^{-11}$$

From equation (5a) and (6a)

$$\delta_1 = +.91278590 \times 10^{-3}, \qquad \delta'_1 = +.14316076 \times 10^{-2}$$
$$\delta_2 = -.38032454 \times 10^{-3}, \qquad \delta'_2 = -.56289109 \times 10^{-3}$$
$$\delta_3 = +.95820540 \times 10^{-4}, \qquad \delta'_3 = -.26574678 \times 10^{-2}$$
$$\delta_4 = -.65941582 \times 10^{-2}, \qquad \delta'_4 = -.16643246 \times 10^{-1}$$
$$\delta_5 = +.37584876 \times 10^{-2}, \qquad \delta'_5 = +.10650043 \times 10^{-1}$$
$$\delta_6 = +.33814141 \times 10^{-2}, \qquad \delta'_6 = +.87864628 \times 10^{-2}$$
$$\delta_7 = -.10868160 \times 10^{-4}, \qquad \delta'_7 = -.28898921 \times 10^{-3}$$
$$\delta_8 = -.17119667 \times 10^{-3}, \qquad \delta'_8 = -.48642779 \times 10^{-3}$$

From equations (5c) and (6c)

$$\alpha_2 = +.91278590 \times 10^{-3}, \quad \alpha_1 = +.10668870 \times 10^{-3}, \quad \alpha_0 = -.37584876 \times 10^{-2}$$
$$\beta_2 = +.28450400 \times 10^{-3}, \quad \beta_1 = +.11094235 \times 10^{-1}, \quad \beta_0 = +.33922823 \times 10^{-2}$$
$$\gamma_2 = -.65941582 \times 10^{-2}, \quad \gamma_1 = -.30010895 \times 10^{-2}, \quad \gamma_0 = -.17119667 \times 10^{-3}$$

$$\alpha'_2 = +.14316076 \times 10^{-2}, \quad \alpha'_1 = -.23684787 \times 10^{-2}, \quad \alpha'_0 = -.10650043 \times 10^{-1}$$
$$\beta'_2 = +.32203589 \times 10^{-2}, \quad \beta'_1 = +.28238469 \times 10^{-1}, \quad \beta'_0 = +.90754520 \times 10^{-2}$$
$$\gamma'_2 = -.16643246 \times 10^{-1}, \quad \gamma'_1 = -.82235717 \times 10^{-2}, \quad \gamma'_0 = -.48642779 \times 10^{-3}$$

From Table II

$$C_0 = -.94005139 \times 10^{-13}$$
$$C_6 = +.67677143 \times 10^{-10}$$
$$C_7 = -.26412843 \times 10^{-10}$$
$$C_8 = -.43945779 \times 10^{-11}$$

From equations (11b)

$$\mu_1 = -.04232242$$

$$\mu_2 = -.21016469$$

$$\mu_3 = -.25260047$$

From equation (11a)

$$A_0 = +.95207252 \times 10$$

$$A_1 = -.19253445 \times 10^2$$

$$A_2 = +.55052349 \times 10$$

$$A_3 = +.10000000 \times 10$$

Thus, equation (11) is

$$\psi_t^3 + 5.5052349\, \psi_t^2 - 19.253445\, \psi_t + 9.5207252 = 0$$

The above cubic equation has the following real roots

$$\psi_t = -8.0454164$$

$$\psi_t = +1.9256499$$

$$\psi_t = +0.6145314$$

From equation (12),

$$\phi_t = -1.6229270$$

$$\phi_t = +6.6172438$$

$$\phi_t = -0.1092704$$

Therefore, the following pairs of starting angles will exist.

$$\psi = 31^{\circ}\ 34'\ 19.31''$$

$$\phi = -06^{\circ}\ 14'\ 09.58'',$$

$$\psi = 62^{\circ}\ 33'\ 24.89''$$

$$\phi = 81^{\circ}\ 24'\ 23.33'',$$

$$\psi = -82^{\circ}\ 54'\ 53.26''$$

$$\phi = -58^{\circ}\ 21'\ 35.43'',$$

Thus, there are three cases to be considered

Case I

When the first set of starting angles is inserted in equations (2d) and (2e), the result will be

$$\Phi_1 = -05^{\circ}\ 10'\ 14.73'', \quad \psi_1 = 31^{\circ}\ 41'\ 16.52''$$

$$\Phi_2 = 06^{\circ}\ 23'\ 36.32'', \quad \psi_2 = 36^{\circ}\ 18'\ 10.94''$$

$$\Phi_3 = 30^{\circ}\ 39'\ 28.72'', \quad \psi_3 = 55^{\circ}\ 11'\ 37.57''$$

$$\Phi_4 = 60^{\circ}\ 48'\ 38.86'', \quad \psi_4 = 89^{\circ}\ 26'\ 27.59''$$

$$\Phi_5 = 81^{\circ}\ 05'\ 17.37'', \quad \psi_5 = 117^{\circ}\ 35'\ 17.67''$$

Then

$$\Phi_{1c} = .99593053, \quad \psi_{1c} = .85092186, \quad (\Phi_1-\psi_1)_c = .80011745$$

$$\Phi_{2c} = .99378071, \quad \psi_{2c} = .80589688, \quad (\Phi_2-\psi_2)_c = .86681283$$

$$\Phi_{3c} = .86022648, \quad \psi_{3c} = .57080286, \quad (\Phi_3-\psi_3)_c = .90970204$$

$$\Phi_{4c} = .48769519, \quad \psi_{4c} = .00975628, \quad (\Phi_4-\psi_4)_c = .87773051$$

$$\Phi_{5c} = .15491457, \quad \psi_{5c} = -.46311416, \quad (\Phi_5-\psi_5)_c = .80385600$$

Evaluating equation (2) for i = 1, 2, 3, 4, and 5 gives

$$
\begin{aligned}
.99593053\, R_1 + .85092186\, R_2 + R_3 &= .80011745 \\
.99378071\, R_1 + .08589688\, R_2 + R_3 &= .86681283 \\
.86022648\, R_1 + .57080286\, R_2 + R_3 &= .90970204 \\
.48769519\, R_1 + .00975628\, R_2 + R_3 &= .87773051 \\
.15491457\, R_1 - .46311416\, R_2 + R_3 &= .80385600
\end{aligned}
\tag{a}
$$

Subtracting the first equation from the last four and dividing by the coefficient of R_1 in each of the remaining four equations leaves

$$
\begin{aligned}
R_1 + 20.94360458\, R_2 &= -31.02370431 \\
R_1 + 2.06419042\, R_2 &= -\ .80752630 \\
R_1 + 1.65507101\, R_2 &= -\ .15271087 \\
R_1 + 1.56243886\, R_2 &= -\ .00444528
\end{aligned}
\tag{b}
$$

Subtracting the first equation from the last three, and dividing by the coefficient of R_2 in each of the remaining three equations leaves

$$
\begin{aligned}
R_2 &= -1.60048282 \\
R_2 &= -1.60048421 \\
R_2 &= -1.60048469
\end{aligned}
\tag{b_1}
$$

Taking

$$R_2 = 1.60048 \tag{c}$$

and substituting this value in equations (b) and solving each of the four equations for R_1 determines

$$
\left.
\begin{aligned}
R_1 &= 2.4961160\\
R_1 &= 2.4961692\\
R_1 &= 2.4961972\\
R_1 &= 2.4962069
\end{aligned}
\right\} \qquad (c_1)
$$

Taking

$$R_1 = 2.49617 \qquad (d)$$

and substituting this value and the value of R_2 from (c) into equation (a), and solving each of the five equations for R_3 determines

$$
\begin{aligned}
R_3 &= -.32401103\\
R_3 &= -.32401092\\
R_3 &= -.32401093\\
R_3 &= -.32402486\\
R_3 &= -.32404205
\end{aligned}
\qquad (d_1)
$$

Evidently

$$R_3 = -.324019 \qquad (e)$$

Substituting the value of R_1 from (d) into equation (2a) gives

$$L_1/L_4 = .400614 \qquad (f)$$

Substituting the value of R_2 from (c) into equation (2b) gives

$$L_3/L_4 = .624813 \tag{g}$$

Substituting the value of R_3 from (e), of L_1/L_4 from (f), and of L_3/L_4 from (g), into equation (2c), and solving for the link ratio L_3/L_4 gives

$$L_2/L_4 = 1.30885$$

Case II

When the second set of starting angles is inserted in equation (2d) and (2e), the result will be

$$\begin{aligned}
\Phi_1 &= 82^\circ\ 28'\ 18.18'', & \psi_1 &= 62^\circ\ 40'\ 22.10'' \\
\Phi_2 &= 94^\circ\ 02'\ 09.23'', & \psi_2 &= 67^\circ\ 17'\ 16.52'' \\
\Phi_3 &= 118^\circ\ 18'\ 01.53'', & \psi_3 &= 86^\circ\ 10'\ 43.15'' \\
\Phi_4 &= 148^\circ\ 27'\ 11.77'', & \psi_4 &= 120^\circ\ 25'\ 33.17'' \\
\Phi_5 &= 168^\circ\ 43'\ 50.28'', & \psi_5 &= 148^\circ\ 34'\ 23.25''
\end{aligned}$$

Then

$$\begin{aligned}
\Phi_{1c} &= .13101559, & \psi_{1c} &= .45907127, & (\Phi_1-\psi_1)_c &= .94028721 \\
\Phi_{2c} &= -.07038146, & \psi_{2c} &= .38610051, & (\Phi_2-\psi_2)_c &= .89299485 \\
\Phi_{3c} &= -.47409474, & \psi_{3c} &= .06664565, & (\Phi_3-\psi_3)_c &= .84691967 \\
\Phi_{4c} &= -.85221373, & \psi_{4c} &= -.50642333, & (\Phi_4-\psi_4)_c &= .88272307 \\
\Phi_{5c} &= -.98071928, & \psi_{5c} &= -.85330632, & (\Phi_5-\psi_5)_c &= .93874884
\end{aligned}$$

Evaluating equation (2) for i = 1, 2, 3, 4, and 5 gives

$$
\begin{aligned}
.13101559\, R_1 + .45907127\, R_2 + R_3 &= .94088721 \\
-.07038146\, R_1 + .38610051\, R_2 + R_3 &= .89299485 \\
-.47409474\, R_1 + .06664565\, R_2 + R_3 &= .84691967 \\
-.85221373\, R_1 - .50642333\, R_2 + R_3 &= .88272307 \\
-.98071922\, R_1 - .85330632\, R_2 + R_3 &= .93874884
\end{aligned}
\qquad \text{(a')}
$$

Subtracting the first equation from the last four and dividing by the coefficient of R_1 in each of the remaining four equations leaves

$$
\begin{aligned}
R_1 + .36232288\, R_2 &= .23780070 \\
R_1 + .64851912\, R_2 &= .15528993 \\
R_1 + .98196278\, R_2 &= .05915623 \\
R_1 + 1.18047713\, R_2 &= .00192345
\end{aligned}
\qquad \text{(b')}
$$

Subtracting the first equation from the last three, and dividing by the coefficient of R_2 in each of the remaining three equations leaves

$$R_2 = -.28830138$$

$$R_2 = -.28830369 \qquad (b_1')$$

$$R_2 = -.28830413$$

Taking

$$R_2 = -.288303 \qquad (c')$$

and substituting this value in equations (b') and solving each of the four equations for R_1 determines

$$R_1 = .34225947$$

$$R_1 = .34225894$$

$$R_1 = .34225905 \qquad (c_1')$$

$$R_1 = .34225855$$

Taking

$$R_1 = .342259 \qquad (d')$$

and substituting this value and the value of R_2 from (c') into equation (a'), and solving each of the five equations for R_3 determines

$$R_3 = 1.02839757$$

$$R_3 = 1.02839748$$

$$R_3 = 1.02839700 \qquad (d_1')$$

$$R_3 = 1.02839752$$

$$R_3 = 1.02839807$$

Evidently

$$R_3 = 1.02840 \qquad (e')$$

Substituting the value of R_1 from (d') into equation (2a) gives

$$L_1/L_4 = 2.92176 \qquad (f')$$

Substituting the value of R_2 from (c') into equation (2b) gives

$$L_3/L_4 = 3.46857 \qquad (g')$$

Substituting the value of R_3 from (e'), of L_1/L_4 from (f'), and of L_3/L_4 from (g'), into equation (2c), and solving for the link ratio L_3/L_4 gives

$$L_2/L_4 = .850513$$

Case III

When the third set of starting angles is inserted in equations (2d) and (2e), the result will be

$$\Phi_1 = -57^\circ\ 17'\ 40.58'', \qquad \psi_1 = -82^\circ\ 47'\ 56.05''$$

$$\Phi_2 = -45^\circ\ 43'\ 49.53'', \qquad \psi_2 = -78^\circ\ 11'\ 01.63''$$

$$\Phi_3 = -21^\circ\ 27'\ 57.13'', \qquad \psi_3 = -59^\circ\ 17'\ 35.00''$$

$$\Phi_4 = 08^\circ\ 41'\ 13.01'', \qquad \psi_4 = -25^\circ\ 02'\ 44.98''$$

$$\Phi_5 = 28^\circ\ 57'\ 51.52'', \qquad \psi_5 = 03^\circ\ 06'\ 05.10''$$

Then

$$\Phi_{1c} = .54031954, \qquad \psi_{1c} = .12535223, \qquad (\Phi_1-\psi_1)_c = .90255299$$

$$\Phi_{2c} = .69803514, \qquad \psi_{2c} = .20477305, \qquad (\Phi_2-\psi_2)_c = .84382853$$

$$\Phi_{3c} = .93063572, \qquad \psi_{3c} = .51064713, \qquad (\Phi_3-\psi_3)_c = .78986411$$

$$\Phi_{4c} = .98852826, \qquad \psi_{4c} = .90596948, \qquad (\Phi_4-\psi_4)_c = .83163660$$

$$\Phi_{5c} = .87492152, \qquad \psi_{5c} = .99853533, \qquad (\Phi_5-\psi_5)_c = .89984047$$

Evaluating equation (2) for i = 1, 2, 3, 4, and 5 gives

$$\begin{aligned}
.54031954\ R_1 + .12535223\ R_2 + R_3 &= .90255299 \\
.69803514\ R_1 + .20477305\ R_2 + R_3 &= .84382853 \\
.93063572\ R_1 + .51064713\ R_2 + R_3 &= .78986411 \\
.98852826\ R_1 + .90596948\ R_2 + R_3 &= .83163640 \\
.87492152\ R_1 + .99853533\ R_2 + R_3 &= .89984047
\end{aligned} \qquad (a'')$$

Subtracting the first equation from the last four and dividing by the coefficient of R_1 in each of the remaining four equations leaves

$$\vdots$$

$$\left.\begin{aligned} R_1 + .50356984\,R_2 &= -.37234402 \\ R_1 + .98713535\,R_2 &= -.28871178 \\ R_1 + 1.74163780\,R_2 &= -.15822180 \\ R_1 + 2.60961725\,R_2 &= -.00810671 \end{aligned}\right\} \qquad (b'')$$

Subtracting the first equation from the three, and dividing by the coefficient of R_2 in each of the remaining three equations leaves

$$\left.\begin{aligned} R_2 &= .17294914 \\ R_2 &= .17294868 \\ R_2 &= .17294829 \end{aligned}\right\} \qquad (b_1'')$$

Taking

$$R_2 = .172949 \qquad (c'')$$

and substituting this value in equations (b'') and solving each of the four equations for R_1 determines

$$\left.\begin{aligned} R_1 &= -.45943592 \\ R_1 &= -.45943485 \\ R_1 &= -.45943632 \\ R_1 &= -.45943740 \end{aligned}\right\} \qquad (c_1'')$$

Taking

$$R_1 = -.459436 \qquad (d'')$$

and substituting this value and the value of R_2 from (c'') into equation (a''), and solving each of the five equations for R_3 determines

$$R_3 = 1.12911570$$

$$R_3 = 1.12911571$$

$$R_3 = 1.12911554 \qquad (d_1'')$$

$$R_3 = 1.12911555$$

$$R_3 = 1.12911522$$

Evidently

$$R_3 = 1.129116 \qquad (e'')$$

Substituting the value of R_1 from (d'') into equation (2a) gives

$$L_1/L_4 = -2.17658 \qquad (f'')$$

Substituting the value of R_2 from (c'') into equation (2b) gives

$$L_3/L_4 = -5.78205 \qquad (g'')$$

Substituting the value of R_3 from (e''), of L_1/L_4 from (f''), and of L_3/L_4 from (g''), into equation (2c), and solving for the value of the link ratio L_3/L_4 gives

$$L_2/L_4 = 3.27865$$

Thus, there are the following three solutions to the problem.

Case I

$$L_1/L_4 = .400614$$

$$L_2/L_4 = 1.30885$$

$$L_3/L_4 = .624813$$

$$\psi = 31^\circ\ 34'\ 19.31''$$

$$\phi = -06^\circ\ 14'\ 09.58''$$

This mechanism is shown in Fig. (3).

Case II

$$L_1/L_4 = 2.92176$$

$$L_2/L_4 = .850513$$

$$L_3/L_4 = 3.46857$$

$$\psi = 62^\circ\ 33'\ 45.51''$$

$$\phi = 81^\circ\ 24'\ 27.72''$$

This mechanism is shown in Fig. (4).

Case III

$$L_1/L_4 = 2.17658$$

$$L_2/L_4 = 3.27865$$

$$L_3/L_4 = 5.78205$$

$$\psi = 97^\circ\ 05'\ 6.74''$$

$$\phi = 121^\circ\ 38'\ 24.57''$$

This mechanism is shown in Fig. (5).

$L_1/L_4 = 1.29454$
$L_2/L_4 = .331787$
$L_3/L_4 = 1.96118$
$\phi = 132^\circ\ 47'\ 53''$
$\phi_1 = 5^\circ\ 00'\ 00''$
$\psi = 91^\circ\ 24'\ 58''$
$\psi_1 = 4^\circ\ 00'\ 00''$

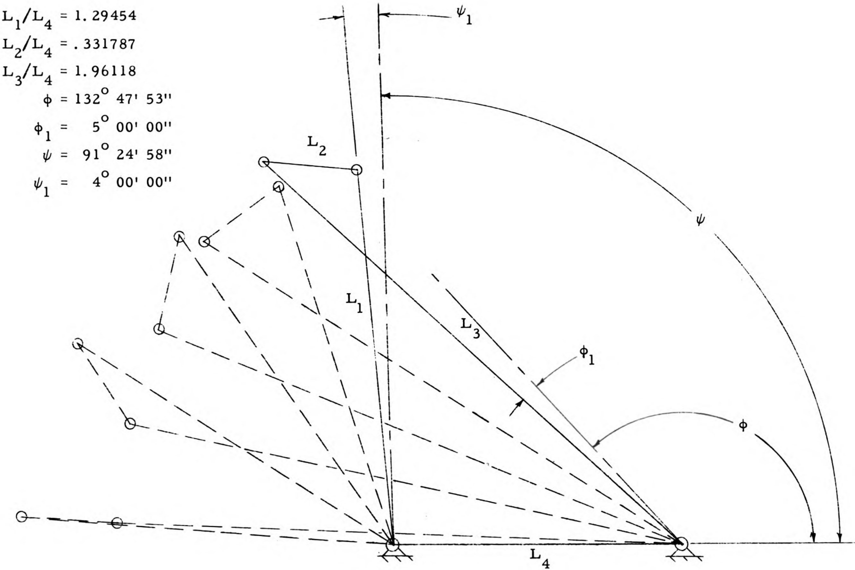

Fig. 2. Four Bar Linkage of Example (1)

$L_1/L_4 = .400614$

$L_2/L_4 = 1.30885$

$L_3/L_4 = .624813$

$\phi = 6^\circ\ 14'\ 09.58''$

$\phi_2 = 12^\circ\ 37'\ 45.90''$

$\psi = 31^\circ\ 34'\ 19.31''$

$\psi_2 = 4^\circ\ 43'\ 51.63''$

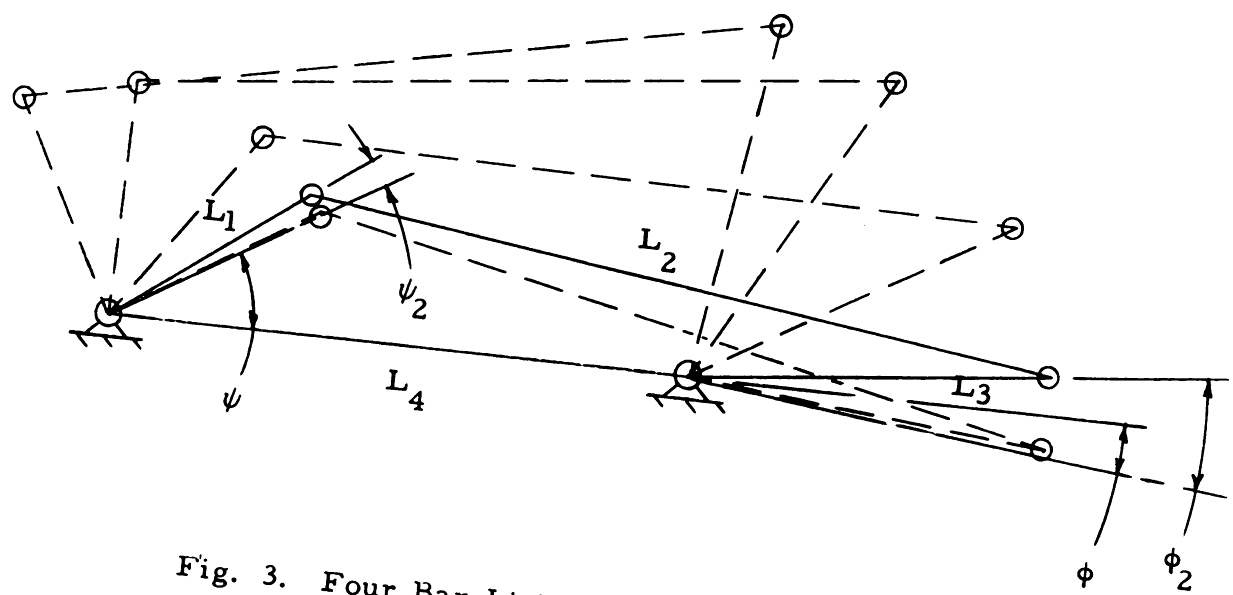

Fig. 3. Four Bar Linkage of Example (2) Case I

$L_1/L_4 = 2.92176$
$L_2/L_4 = .850513$
$L_3/L_4 = 3.46857$
$\phi = 81^\circ\ 24'\ 27.72''$
$\phi_2 = 12^\circ\ 37'\ 45.90''$
$\psi = 62^\circ\ 33'\ 45.51''$
$\psi_2 = 4^\circ\ 43'\ 51.63''$

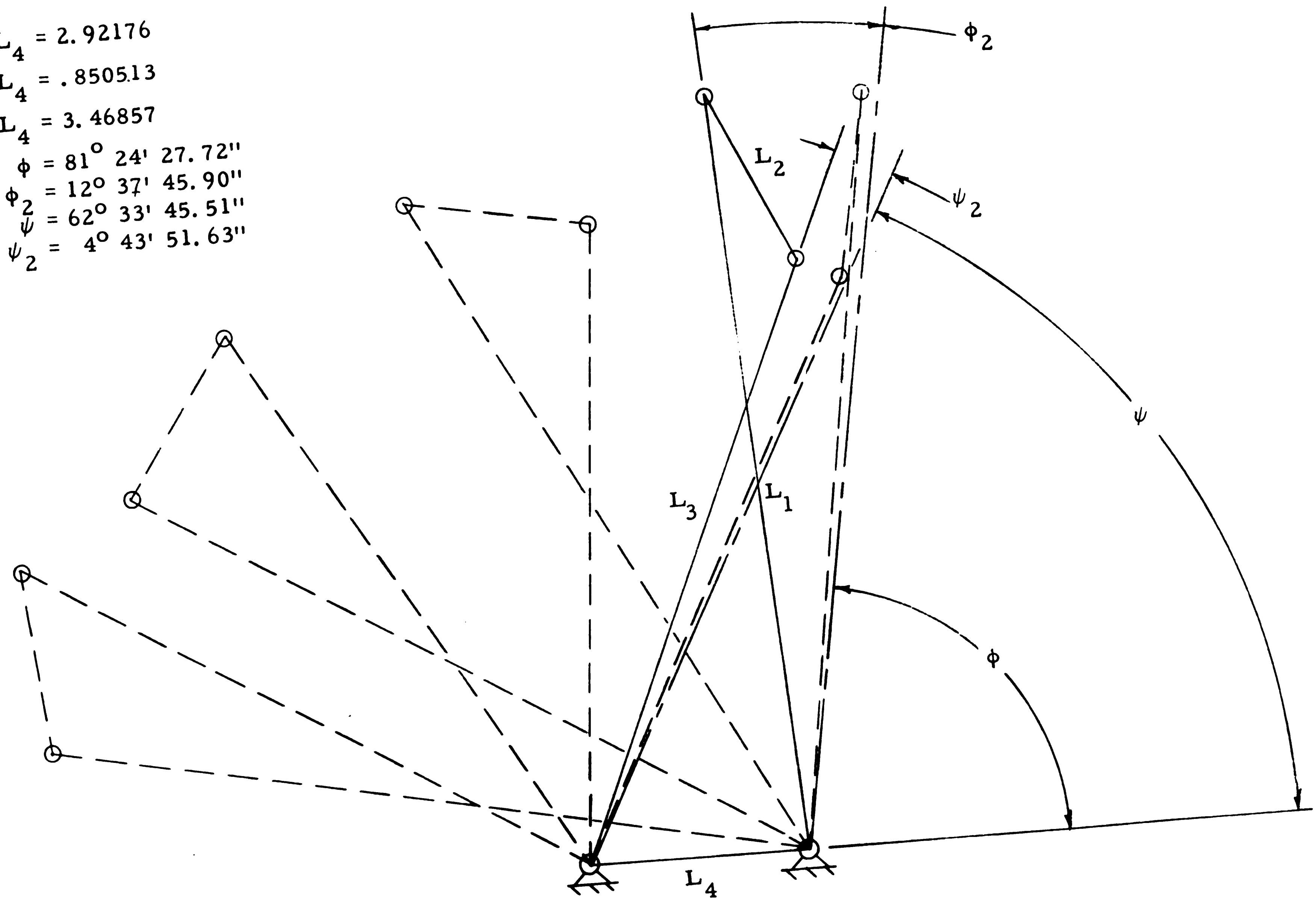

Fig. 4. Four Bar Linkage of Example (2) Case II.

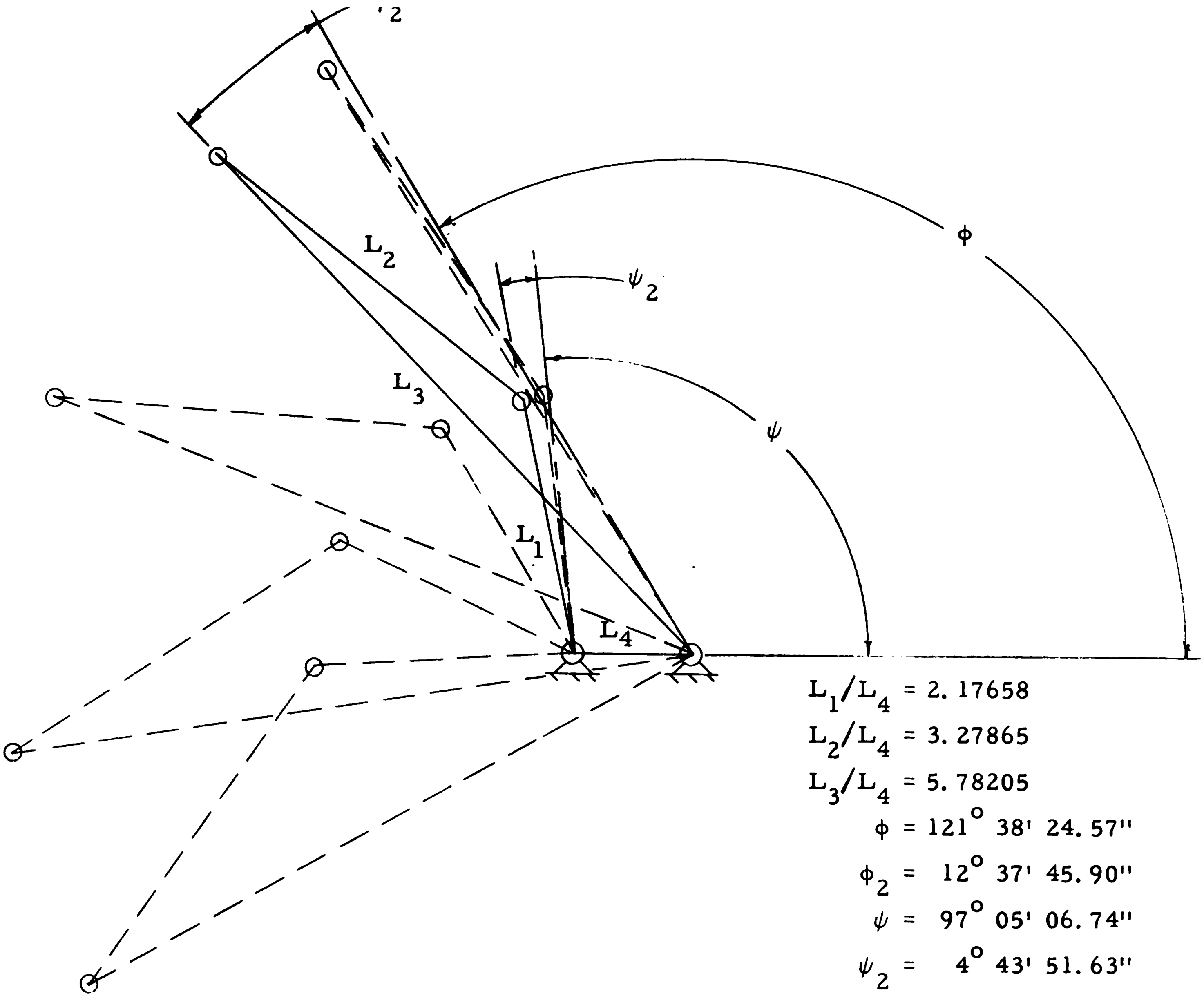

Fig. 5. Four Bar Linkage of Example (2) Case III

2. Problem Discussion. The values of the trigonometric function of the displacement angles in Example (1) and (2) were taken from Peters (3) wherein trigonometric values are given to eight places as functions of angles given in degrees, minutes, and seconds. Therefore, the trigonometric values of the displacement angles of Example (1) are accurate to the eight places given since no interpolation is required to obtain them. The values of the trigonometric functions of the displacement angles given in Example (2) are not accurate to the eight places given because they are obtained by linear interpolation in Peters (3).

The values of the starting angle ψ for each example were also determined via the generation and solution of equation (10) with the aid of a digital computer. Therefore, the following data are available.

For Example (1)

The coefficients in equation (10) are

$$C_0 = -.23571951 \times 10^{-14}$$
$$C_1 = -.24977260 \times 10^{-13}$$
$$C_2 = -.81377704 \times 10^{-13}$$
$$C_3 = -.99426682 \times 10^{-13}$$
$$C_4 = -.80949468 \times 10^{-13}$$
$$C_5 = -.95603762 \times 10^{-13}$$
$$C_6 = -.24518283 \times 10^{-14}$$
$$C_7 = -.21154334 \times 10^{-13}$$
$$C_8 = -.52287725 \times 10^{-15}$$

The roots of equation (10) are

$$\mu_1 = -.17632691$$

$$\mu_2 = -.33783343$$

$$\mu_3 = -.45924364$$

$$\mu_4 = +1.0000000i$$

$$\mu_5 = -1.0000000i$$

$$\mu_6 = -40.451105$$

$$\mu_7 = +.48347782 + 1.9596076i$$

$$\mu_8 = +.48347782 - 1.9596076i$$

For Example (2)

The coefficients in equation (10) are

$$C_0 = -.94005140 \times 10^{-13}$$

$$C_1 = -.28505340 \times 10^{-11}$$

$$C_2 = -.15132305 \times 10^{-10}$$

$$C_3 = -.37228276 \times 10^{-11}$$

$$C_4 = +.57033385 \times 10^{-10}$$

$$C_5 = -.27285139 \times 10^{-10}$$

$$C_6 = +.67677143 \times 10^{-10}$$

$$C_7 = -.26412843 \times 10^{-10}$$

$$C_8 = -.43945779 \times 10^{-11}$$

The roots of equation (10) are

$$\mu_1 = -.04232188$$

$$\mu_2 = -.21016715$$

$$\mu_3 = -.25259804$$

$$\mu_4 = +.99999999i$$

$$\mu_5 = -.99999999i$$

$$\mu_6 = +.61454099$$

$$\mu_7 = +1.9256406$$

$$\mu_8 = -.80454201$$

Although the roots of equation (10) as given by equation (11b) are theoretically correct, they become approximately correct when approximations are made for the trigonometric functions of the displacement angles. The error involved in reducing equation (10) to the cubic equation (11) is very small when the reduction is accomplished via equations (11a) and (11b). This can be seen when each solution of equation (10) given above is compared to each solution of the cubic equation (11) given in the examples. It is to be noted that the roots of equation (10) pertain to the solution of the mathematical problem established by giving certain trigonometric functions certain values, and it is tacitly assumed that all trigonometric relations involved will be satisfied by these values as they were for the trigonometric functions.

The approximations referred to above are to be understood as mathematical approximations and are of academic interest only. From a practical point of view the accuracy of the link ratios determined by the use of eight place trigonometric values is far beyond that which is normally required. If more accuracy is desired in the calculation, the values of the trigonometric functions can be calculated as accurately as desired by the use of series expansions.

The accuracy of the present calculations can be deduced from the values of the R_j's given in (b_1), (c_1), and (d_1) in Example (1), and in (b_1'), (c_1'), (d_1'), (b_1''), etc. in Example (2).

CHAPTER III

SPECIAL CASES

1. Precision Derivatives.* In the design of a four bar linkage to be used as a generator of an arbitrary function as f in equation (1), it may be desirable to have a precision derivative at a precision point. That is, it may be desirable to have the rate of displacement of angle ψ_i with respect to the rate of displacement of angle ϕ_i equal to the derivative of f with respect to ϕ_i at a precision point. The inclusion of a precision derivative at a precision point, which will be seen later, can only be accomplished at the expense of a precision point in the four bar linkage generator design.

It is not difficult to adapt the five precision point design calculation to accept precision derivatives. To this end, there are the following considerations.

If the function f has a derivative at a precision point, then

$$\frac{d\psi_i}{d\phi_i} = f'(\phi_i)$$

will exist at the point.

Equation (2) may be put in the form

$$R_1(\phi+\phi_i)_c + R_2(\psi+\psi_i)_c + R_3 = [(\phi-\psi) + (\phi_i - \psi_i)]_c$$

*Refer to Hinkle (4) for a discussion concerning a precision derivative.

Taking a derivative of the preceding equation with respect to ϕ_i, the result will be

$$R_1(\phi+\phi_i)_s + R_2(\psi+\psi_i)_s \frac{d\psi_i}{d\phi_i} = [(\phi-\psi)+(\phi_i-\psi_i)]_s \left(1-\frac{d\psi_i}{d\phi_i}\right)$$

or

$$R_1\Phi_{is} + R_2\psi_{is}\frac{d\psi_i}{d\phi_i} = (\Phi_i - \psi_i)_s\left(1-\frac{d\psi_i}{d\phi_i}\right) \tag{14}$$

where

$$\Phi_i = \phi + \phi_i$$

$$\psi_i = \psi + \psi_i$$

$$\frac{d\psi_i}{d\phi_i} = f'(\phi_i) \quad .$$

When evaluated at a precision point, equation (14) will be a linear equation in R_1 and R_2, and may replace any precision point equation (2) other than that of its associated precision point in the system of five simultaneous equations in the five precision point design calculation.

An entry of equation (14) as one of the five simultaneous equations will necessitate a modification of the determinants in equations (5a) and (6a). In order to modify the determinants, suppose that at a precision point q there is also to be a precision derivative. Then equation (2) and equation (14) can be evaluated at the precision point q providing two of the five simultaneous equations in the design calculation. Suppose further that the rth precision point in the design calculation be foregone in order to accept the slope precision equation (14). The changes that are to be made in the determinants in equations (5a) and (6a) as a result of the above modifications are in the rth rows, and they are tabulated in Table III, page 57.

If a precision derivative is to be incorporated at each of two precision points, then the procedure given above must be duplicated for the second precision point.

In order not to disturb the procedure for the reduction of the eighth degree equation (10) to the cubic equation (11), the following selection of the values of q and r should be made. If one precision derivative is to be incorporated at one precision point, select the value of q from the set j, k, and m and the value of r from the set n and p. If a precision derivative is to be incorporated at each of two precision points, select the values of the q's to have different values from the set j, k, and m and the r's to have different values from the set n and p.

When the above changes have been made, no further changes in the design calculation are necessary. Keeping in mind, of course, that the link ratios will be determined from a set of five equations of type (2) and (14) as required by the type of precision points that have been employed in the design of the linkage.

Since each derivative of equation (2) will be a linear equation in R_1 and R_2, it is obvious that the calculation may be modified to accept a number of higher order derivatives at each of two precision points.

2. Four Precision Point Design. It is obvious that a four precision point design calculation is simply a part of the five precision point design calculation regardless of the type of precision points employed in the design.

In the four precision point design calculation one of the starting angles is entirely arbitrary and the other starting angle may be determined from equation (7). The left hand member of equation (7) is a quadratic expression in ϕ_t with coefficients that are quadratic expressions in ψ_t which have constant coefficients. Therefore, if an arbitrary value is assigned to one of the starting angles and its tangent value inserted in the left hand member of equation (7), the result will be a simple quadratic expression in the tangent of the other starting angle. Thus, equation (7) will be reduced to a simple quadratic equation which can readily be solved and the starting angle determined. If the solutions of the reduced equation (7) are real values, then both values should be considered for the starting angle. If the solutions are complex roots, then another value of the arbitrary starting angle must be assigned and the above procedure repeated.

When the starting angles have been determined, the link ratios may be determined in the same manner as in the five point calculation; however, there will be only four simultaneous equations involved in the calculation.

TABLE III

PRECISION DERIVATIVE MODIFICATION

If the rth precision point equation (2) is replaced by the qth precision derivative equation (14) in the set of five equations in the design calculation, the following changes are to be made in the determinants in equations (5a) and (6a).

Column	Change	To
(1)	ϕ_{rc}	ϕ_{qs}
	ϕ_{rs}	$-\phi_{qc}$
(2)	ψ_{rc}	$\psi_{qs} [\frac{d\psi_i}{d\phi_i}]_q^*$
	ψ_{rs}	$-\psi_{qc} [\frac{d\psi_i}{d\phi_i}]_q$
(3)	1	0
(4)	$(\phi_r - \psi_r)_c$	$(\phi_q - \psi_q)_s (1 - [\frac{d\psi_i}{d\phi_i}]_q)$
	$(\phi_r - \psi_r)_s$	$-(\phi_q - \psi_q)_c (1 - [\frac{d\psi_i}{d\phi_i}]_q)$

* $[\frac{d\psi_i}{d\phi_i}]_q$ represents $\frac{d\psi_i}{d\phi_i}$ evaluated at the qth point.

CHAPTER IV

CONCLUSION

1. Five Precision Point Design Calculation. Since the determination of one of the starting angles depends upon the solution of a cubic equation, there will always exist at least one pair of starting angles, and there is a possibility of the existence of three pairs of starting angles. It is unlikely that an equation of less than the third degree exists for the determination of one of the starting angles because it is shown in Example (2) that the existence of three pairs of starting angles provides three different four bar linkages which are capable of satisfying the same five precision points.

It is obvious that any equation developed for one starting angle could likewise be developed for the other starting angle, and in general, both starting angles possess the same characteristics. Thus, the calculation is symmetrical with respect to the starting angles.

The resultant in equation (3) or (4) can be expanded into a polynomial in the two variables ϕ_t and ψ_t by the application of a double Taylor's series expansion about the origin, and the resultant in equation (9) may be expanded into equation (10) by the application of a Taylor's series expansion about the origin.

Although the design calculation is rather lengthy, it is inherently accurate because it contains no approximations, iterations, or graphical solutions.

2. Four Precision Point Design Calculation. When one of the starting angles is assigned the value of zero, the four precision point design calculation may be worked with sufficient accuracy in a reasonable time by the use of logarithmic tables. There are an infinite number of different four bar linkages that can be used to generate (approximately) an arbitrary function given by four specified precision points.

3. Precision Derivatives. The theoretical consideration of the five precision point design calculation can be used to formulate procedures whereby a function generator can be designed with precision derivatives at more than one precision point. These formulized procedures can be obtained directly by modifications to five precision point design calculation without difficulty. It is also true that the design calculation can be modified to accept higher order derivatives at different precision points.

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