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ON ROBUST TRACKING IN UNCERTAIN SYSTEMS
- A VARIABLE STRUCTURE APPROACH

By

Seung-Bok Choi

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ABSTRACT

ON ROBUST TRACKING IN UNCERTAIN SYSTEMS
- A VARIABLE STRUCTURE APPROACH

By

Seung-Bok Choi

A simple tracking controller design applicable for both linear and nonlinear uncertain dynamical systems based on variable structure system(VSS) theory is given. The control design employs so called "sliding conditions" to guarantee asymptotic tracking for any arbitrary initial conditions lying off of sliding surfaces. To assure the existence of sliding modes for multi-input systems, we study the domains of attraction for sliding modes in terms of Lyapunov stability theory. It is shown that the robustness of the control scheme to input uncertainties is achieved primarily by assuming a form of matching conditions. The important properties of gradient vectors of the sliding surfaces will also be highlighted. The design methodology is straightforward and requires little computational effort. To illustrate this, several numerical examples are presented, and the method is applied to the control of a three-degrees-of-freedom manipulator subjected to variable payloads and external torque disturbances.

Dedicated to my parents

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TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER I

INTRODUCTION

In recent years, increasing attention has been given to control designs that utilize the theory of variable structure systems (VSS) as described in [2,3] : that is a class of systems with discontinuous feedback control. Young[4], Ryan[5], and Slotine and Sastry[6] proposed variable structure controllers for multi-input systems by introducing a single surface of control discontinuity for each input. Young and Ryan used the so called hierarchy of control method. The basic idea here is that the system states are forced to the surfaces of control discontinuity sequentially. Slotine and Sastry on the other hand used sliding conditions to drive the system states to all the switching surfaces simultaneously. It is interesting that the continuous controller proposed by Corless and Leitmann[9] appears to be a variable structure controller in the limit as its saturation function parameter tends to zero.

One of salient features of VSS is that the so-called sliding mode may occur on the switching surfaces independently or on the intersection of several switching surfaces. During the sliding mode motion, the system remains insensitive to disturbances and parameter variations producing perfect tracking of the desired trajectories. However, the ideal sliding mode does not occur in practice due to the ever present non-idealities such as delays, small time constants, hysteresis, etc. These non-idealities cause the trajectories to chatter along the sliding surfaces resulting in the generation of undesirable high frequency components. Thus, in real systems the discontinuous controls should be replaced by smooth approximations[6,7] so that the actual trajectories will be in the vicinity of the sliding surfaces.

The technique used in this thesis to construct variable structure controllers is based on a special way of imposing the sliding conditions. With a knowledge of Filippov's solution concept for differential equations with discontinuous right-hand sides[1], we exploit the condition for the existence of a sliding mode. For the special case of scalar control, a sliding mode exists, if at a point on a sliding surface, the directions of motion along the state trajectories on either side of the surface tend towards the surface. In the multi-input case, the sliding mode occurs, if the condition for the existence of a sliding mode is met on each of the discontinuity surfaces associated with corresponding inputs. To substantiate the domain of sliding mode on the intersections of sliding surfaces for the multi-input case, we consider the sliding mode in terms of Lyapunov stability theory. The latter approach makes it possible to realize sliding modes along individual surfaces or along the intersection of such surfaces for multi-input systems considered in this study. The design procedure based on these ideas is straightforward and extremely simple compared with [4,6]. Moreover, the methodology is quite general and may be applied to various dynamical systems having bounded uncertainties.

In the synthesis of the above mentioned controllers , it is necessary to choose appropriate values of gradient vectors of sliding surfaces to guarantee asymptotic tracking in view of the internal characteristics of a system. The approach developed so far in selecting the parameters of the gradient vectors is essentially a trial and error one and no analytical method is yet available for the proper choice of the gradient vectors. In this thesis the importance of the gradient vectors is highlighted by presenting a specific example. Suggestions are made for the resolution of this issue and will remain an interesting area for further investigation.

The layout of the thesis is as follows. In chapter 2 we review the properties of the sliding mode in discontinuous dynamic systems including the mathematical framework proposed by Filippov[1] and conditions for the existence of a sliding mode. Chapter 3 illustrates the procedures for the design of feedback controllers for both single and multi-input systems in the presence of parameter variations and disturbances. The control scheme is then applied to the control of a robotic manipulater handling variable payloads in chapter 4, followed by our conclusions in chapter 5.

CHAPTER II

SLIDING MODES IN DISCONTINUOUS DYNAMIC SYSTEMS

2.1 Mathematical description

We begin by constructing a mathematical framework to analyze differential equations with discontinuous right-hand side associated with our discontinuous control laws to be developed. A solution concept for such differential equations has been developed and compared with other solution concepts by Filippov[1].

We review this concept by considering the vector differential equation

$$\dot{x}(t)=f(x,t) \qquad (1)$$

where $f:R^n \times R \rightarrow R^n$ satisfies the following:

<u>Assumption</u>:

The function $f(x,t)$ is a real-valued measurable function, defined almost everywhere in an open domain $Q \subset R^{n+1}$. Further, for all compact $D \subset Q$ there exists integrable $M(t)$ such that $||f(x,t)|| \leq M(t)$ almost everywhere in D, where $||\cdot||$ denotes any norm.

<u>Definition</u>(solution concept for equation (1)):

The vector function x(t), defined on the interval $[t_0, t_1]$ is called a solution of (1) if it is absolutely continuous on $[t_0, t_1]$ and if for almost all $t \in [t_0, t_1]$

$$\dot{x}(t) \in \bigcap_{\delta>0} \bigcap_{\mu N=0} \text{Conv } f(B(x(t),\delta)-N) \tag{2}$$

where $\bigcap_{\mu N=0}$ denotes the intersection over all sets N of Lebesgue measure zero and Conv refers to the convex hull of a set. Geometrically, (2) means that at every point of the discontinuity surface the velocity vector characterizing the solution belongs to a minimal convex closed set containing all values of f(x) when x covers the entire δ-neighbourhood of the point under consideration (with δ tending to zero) except a zero measure set.

We now apply this solution concept to our equation (1) to determine the phase velocity along the sliding mode. Let the regions G^- and G^+ in the space $x_1, \ldots, x_n$ be separated by a switching surface(smooth function) s as shown in Figure 1. Suppose that there exist limiting values $f^-(x,t)$ and $f^+(x,t)$ of f(x,t) for any constant t when s is approached from G^- and G^+. Let f_N^- and f_N^+ be the projections of the vectors f^- and f^+ on the normal to the surface s directed from G^- and G^+. Then, when $x(t) \in s$, and $f_N^- > 0$ and $f_N^+ < 0$, we may construct the phase velocity $f^0(x,t)$ as in Figure 1. By the definition given above, at each point of the switching surface the end of the velocity vector must belong to the segment

joining the points $f^-(x,t)$ and $f^+(x,t)$, i.e., its end point is the intersection point of the plane tangential to the surface and the straight line connecting the ends of the vector $f^-(x,t)$ and $f^+(x,t)$. Thus, we define the sliding mode equation as

$$\dot{x}(t)=f^0(x,t) \tag{3a}$$

$$f^0(x,t)=\alpha f^+ + (1-\alpha) f^- \qquad (0\leq\alpha\leq 1) \tag{3b}$$

where α is a parameter depending on the directions and magnitudes of f^-, f^+ and the gradient of the switching surface s. On the other hand, the projection of the f^0 on the normal to the s must be equal zero so that the trajectory slides along s once it hits s. This fact is referred to as ideal sliding. Computing α with grad $s \cdot f^0=0$ gives the equation

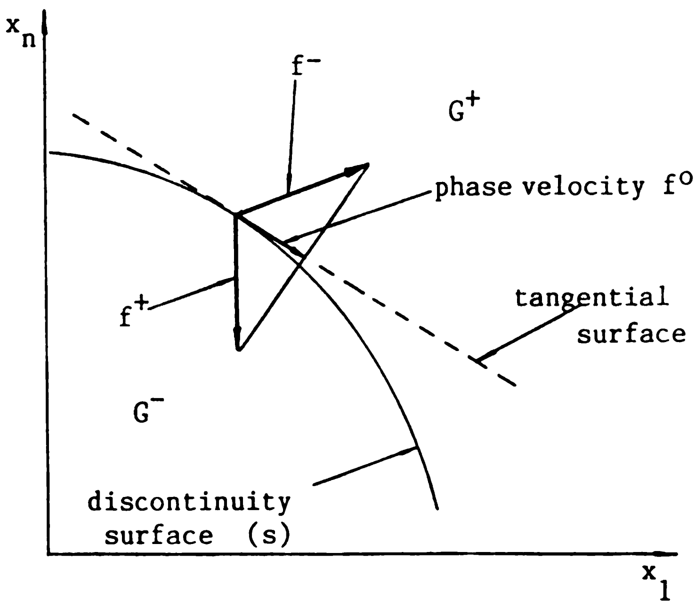

Figure 1. The construction of $f^0(x,t)$

$$\dot{x}(t)=f^{o}(x,t)=\frac{\text{grad } s\cdot f^{-}}{\text{grad } s\cdot(f^{-}-f^{+})}\, f^{+} - \frac{\text{grad } s\cdot f^{+}}{\text{grad } s\cdot(f^{-}-f^{+})}\, f^{-} \tag{4}$$

which is known as Filippov's continuation equation.

Note that to substantiate the validity of equation (4) in studies of actual control systems with discontinuous controls, various non-idealities such as delays, and hysteresis are recognized. These are discussed in detail in [2].

2.2 Conditions for the existence of a sliding mode

2.2.1 For single-input systems

With this mathematical background, we turn our attention to a dynamical system of a general type with a discontinuous scalar control described by the equation

$$\dot{x}(t)=f(x,t,u) \tag{5}$$

where $f:R^{n}\times R\times R\rightarrow R^{n}, x\epsilon R^{n}$ and $u\epsilon R$. VSS are characterized by discontinuous control which changes structure on reaching a set of switching surfaces. The control has the form

$$u(x,t)=\begin{cases} u^{+}(x,t) & , \ s(x)>0 \\ u^{-}(x,t) & , \ s(x)<0 \end{cases} \tag{6}$$

where $u^{+}(x,t)$, $u^{-}(x,t)$ and $s(x)$ are certain continuous functions and $u^{+}\neq u^{-}$. Since we intend to have a sliding mode in the system, conditions for the existence of a sliding mode should be found analytically. Such conditions for a sliding mode to occur on a switching surface may be stated in a number of ways. In this study state concepts are employed to describe such conditions for the general system (5). From the solution concept of equation (3) in 2.1, it follows that a sliding mode exists if there are domains of non-zero measure on the surface $s(x)=0$ where the projections of the vectors $f^{+}=f(x,t,u^{+})$ and $f^{-}=(x,t,u^{-})$ on the surface gradient are of opposite sign and are directed towards the surface(Figure 2). Analytically, these conditions can be expressed as

$$\lim_{s\to 0^{+}} \dot{s}<0 \quad \text{and} \quad \lim_{s\to 0^{-}} \dot{s}>0 \tag{7}$$

or equivalently

$$s\dot{s}<0. \tag{8}$$

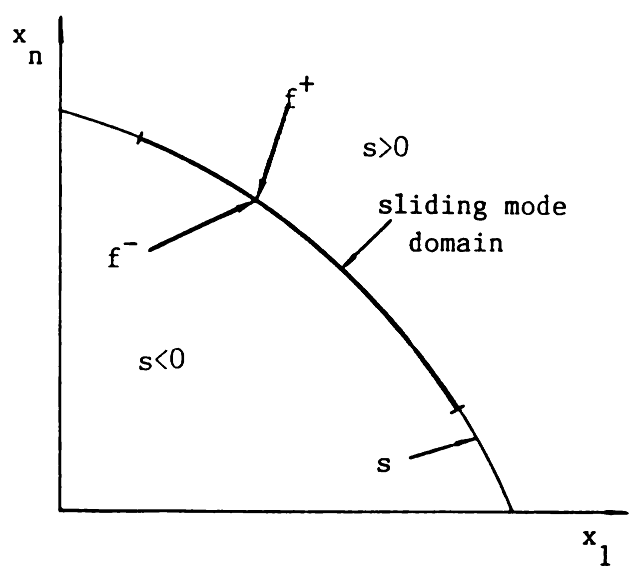

Figure 2. Sliding mode domain

Once in the domain of attraction the describing point is assumed to move to the interior of the domain until the boundaries are reached followed by motion along the boundary. This motion will be termed ideal sliding which should be regarded as the motion that results from a limiting process with all the non-idealities tending to zero.

The condition (8) is referred to as the local sliding condition, since it is sufficient to guarantee that trajectories starting from initial conditions close to s converge to s and slide along s. Without loss of generality, the global sliding condition can be extracted from (8) in view of stability, i.e.,

$$s\dot{s}<-\gamma(|s|) \tag{9}$$

where $\gamma(\cdot)$ is a continuous function of class K defined in [10]. The condition (9) implies that all initial conditions lying off s will converge to s and then slide along s. Also condition (8) and (9) guarantee that trajectories originating on s will remain on s. Note that once in the sliding mode, the system (5) satisfies the equation

$$s(x)=0 \quad \text{and} \quad \dot{s}(x)=0. \tag{10}$$

Thus, the system has invariant properties yielding a motion which is independent of diturbances and parameter variations.

To illustrate concepts described above, consider the scalar system

$$\dot{x}_1(t)=x_2(t) \tag{11a}$$

$$\dot{x}_2(t)=-a(t)x_1(t)+x_2(t)+u(t)+d(t) \tag{11b}$$

where $a(t)$ is a time-varying parameter and $d(t)$ is an external disturbance whose precise value is unknown but bounded. Defining the switching function $s(x)=cx_1+x_2$ yields the discontinuous control

$$u=\begin{cases} u^+, & cx_1+x_2>0 \\ u^-, & cx_1+x_2<0 \end{cases} \tag{12}$$

where $c>0$ and $u^+\neq u^-$. The line $s=0$ is the surface on which the control has a discontinuity. It can be easily shown that the state $x(t)$ reaches the switching line $s=0$ in a finite time if u^+ and u^- (see Figure 3) are chosen appropriately. The state $x(t)$ crosses the switching line and enter the domain $s<0$ resulting in the value of u being altered from u^+ to u^-. The proper choice

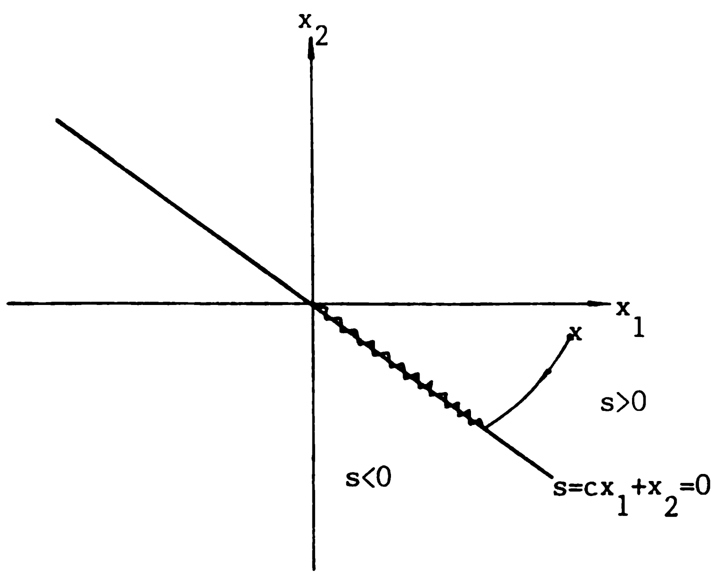

Figure 3. Sliding mode motion due to a scalar control function

of control makes the state trajectory immediately recross the switching line and enter the domain $s>0$, i.e., this yields the sliding mode motion. For infinitely fast switching logic, the state is forced to remain in a neighbourhood of a switching line by the control which oscillates between the values u^+ and u^-. Then, we have a new dynamic system during the sliding mode motion described by

$$s=cx_1+x_2=0 \tag{13}$$

which is quite independent of the original system, and stable since $c>0$. The sliding condition for this system can be established by appropriately selecting u^+ and u^- so that

$$\dot{s}=-a(t)x_1(t)+(c+1)x_2(t)+u^+(t)+d(t) < 0, \text{ for } s>0$$

$$\dot{s}=-a(t)x_1(t)+(c+1)x_2(t)+u^-(t)+d(t) > 0, \text{ for } s<0$$

for expected values of $a(t)$ and $d(t)$.

2.2.2 For multi-input systems

We now consider multi-input systems described by

$$\dot{X}(t)=F(X,t,U) \tag{14}$$

where $F:R^n \times R \times R^m \rightarrow R^n$, $X \epsilon R^n$ and $U \epsilon R^m$. Analogous to what was done for VSS with scalar control each component of control is assumed to undergo a discontinuity on an appropriate surface in the state space, i.e.,

$$u_i=\begin{cases} u_i^+(X,t) & , s_i(X)>0 \\ u_i^-(X,t) & , s_i(X)<0 \end{cases} \tag{15}$$

where s_i is a switching surface associated with a corresponding u_i. The design problem then consists of choosing the continuous functions u_i^+, u_i^- and the vector $S \epsilon R^m$ with the functions $s_i(X)$ as components. For the scalar control case, we have a transparent geometric interpretation(Figure 2) of the conditions for the existence of a sliding mode, but for the vector control case the conditions can be hardly described in geometrical terms of mutual positioning of state velocity vectors. Since introducing one switching surface associated with one control permits us to consider the sliding conditions as in the scalar case,we can construct the following sliding conditions from (8) and (9)

$$s_i \dot{s}_i < 0 \qquad (16a)$$

and

$$s_i \dot{s}_i < -\gamma(|s_i|). \qquad (16b)$$

A possible trajectory of the system (14),(15) is given in Figure 4 if the sliding conditions (16) are met. Case A implies that there exists a sliding mode on the intersection of two sliding surfaces, not necessarily separate sliding modes on the switching surfaces. This type of sliding mode is a new phenomenon and an interesting subject for investigation. Case B indicates the possibility of a sliding mode occuring on the switching surface s_1. Note that in the final stage of motion the phase point is forced to move along the intersection of the two discontinuous surfaces in both cases.

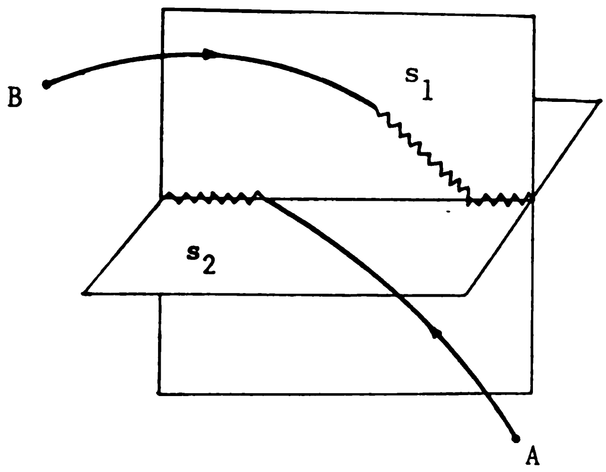

Figure 4. Sliding mode motion due to two control functions

Let us further investigate case A in terms of Lyapunov stability theory. As is usually done in stability theory, we can try to find sufficient conditions for a certain domain S of the intersection of discontinuous surfaces in the state space of the system (14),(15). Utkin[2] established these conditions in a theorem. This theorem says that for the intersection of discontinuous surfaces of the system(14),(15) to be a sliding mode, it is sufficient that for all X belonging to a certain domain S of the sliding mode there exists a continuously differentiable function V(S,X,t) such that V(S,X,t)>0 with $S \neq 0$, V(0,X,t)=0 with arbitrary X and t, and its total time derivative along the trajectories of (14) and (15) be negative everywhere, except the discontinuous surfaces on which this function is not defined. To show that the requirement of continuous differentiability is essential for finding the conditions for a sliding mode to exist, we now consider two simple examples. First consider the example of a system with two dimensional control whose motion projections on a plane s_1, s_2 are described by the equations

$$\dot{s}_1 = -2 \operatorname{sgn} s_1 + 3 \operatorname{sgn} s_2$$

$$\dot{s}_2 = -3 \operatorname{sgn} s_1 - 2 \operatorname{sgn} s_2$$

where

$$\operatorname{sgn} s_i = \begin{cases} 1, & \text{if } s_i > 0 \\ 0, & \text{if } s_i = 0. \\ -1, & \text{if } s_i < 0 \end{cases}$$

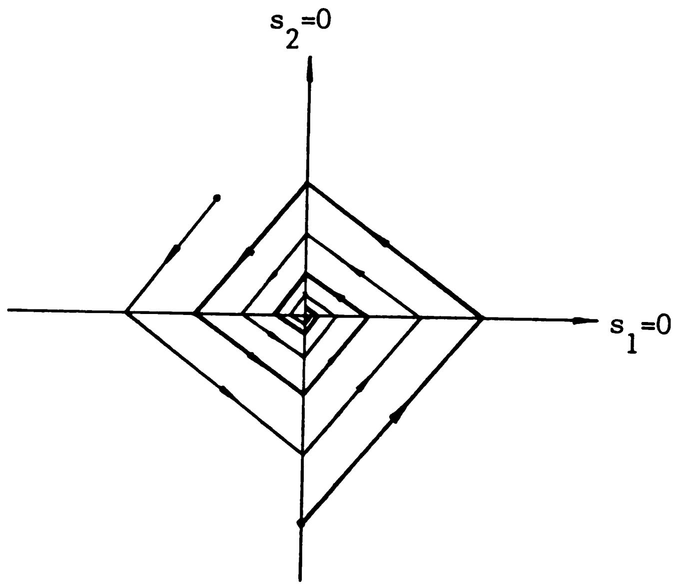

Figure 5. State portrait of two dimensional VSS(sliding mode)

Select a positive definite function V in the form

$$V = |s_1| + |s_2|.$$

Then its time derivative

$$\dot{V} = \text{sgn } s_1\dot{s}_1 + \text{sgn } s_2\dot{s}_2 = -4$$

is negative everywhere except at $s_1=0$. Consequently, the function V and $\dot{V}$ have opposite signs and from the state portrait of Figure 5 it follows that there exists a sliding mode at the origin for arbitrary initial conditions.

For the second example, consider the system whose motion projections are on a plane s_1, s_2 described by the equations

$$\dot{s}_1 = -5 \text{ sgn } s_1 - 2 \text{ sgn } s_2$$

$$\dot{s}_2 = -\text{sgn } s_1 + \text{sgn } s_2.$$

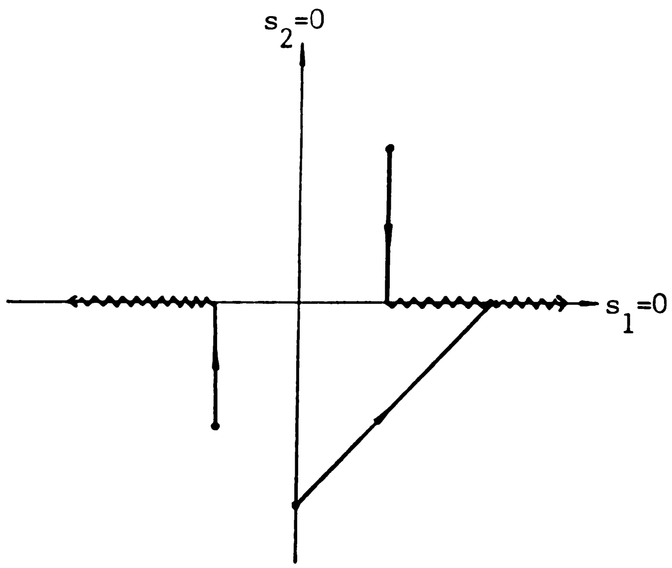

Figure 6. State portrait of two dimensional VSS(no sliding mode)

We select the positive definite function V as

$$V = |s_1| + |s_2|.$$

Its time derivative

$$\dot{V} = -4 -3 \operatorname{sgn} s_1 s_2$$

is negative everywhere except at $s_i = 0$. However, from the state portrait of Figure 6 it follows that there is no sliding mode despite the difference in signs of the functions V and $\dot{V}$, i.e., this example shows that the knowledge of signs of the piecewise differentiable function and its derivative is not sufficient to ascertain the existence of a sliding mode.

Thus, establishing the existence of a sliding mode is the most important portion in the design of a discontinuous controller by using the theory of VSS. In the next chapter, we formulate the discontinuous control laws that are based on the concepts discussed in this chapter.

CHAPTER III

CONTROLLER FORMULATIONS

3.1 A class of single-input systems

3.1.1 Linear dynamical systems

Let us start with a linear time-varying single-input control system described by

$$\dot{x}(t)=A(t)x(t)+b(t)u(t) \tag{17}$$

where $x(t)\epsilon R^n$, $u(t)\epsilon R$ and

$$A(t)=\begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \vdots & \vdots & \vdots & & & & \vdots \\ 0 & 0 & 0 & & & & 1 \\ * & * & * & \cdot & \cdot & \cdot & * \end{bmatrix} , \quad b(t)=\begin{bmatrix} 0 \\ \vdots \\ 0 \\ * \end{bmatrix}.$$

The control problem is to get $x(t)$ to track a desired trajectory $x_d(t)$ which belongs to the class of continuously differentiable(C^1) functions on $[t_0,\infty)$. In other words, the controller should force the tracking error to zero asymptotically for any given initial states. Thus we define the tracking error $e(t)$ as

$$e(t) \triangleq x(t)-x_d(t) \tag{18}$$

where $e(t)\epsilon R^n$, $x_d(t)\epsilon R^n$, and also define the sliding surface s in the error state space by

$$s(e(t)) \triangleq Ce(t) = 0 \tag{19}$$

where C is a $1\times n$ row vector with constant elements of the form $[c_1, \cdot \cdot \cdot \cdot, c_{n-1}, 1]$. We see that the tracking error $e(t)\to 0$ for any given initial conditions, provided (19) is asymptotically stable. Thus it remains only to obtain a control $u(t)$ so as to cause the trajectory $x(t)$ to slide along the surface defined by (19). This can be achieved by satisfying the sliding condition (9). It can be easily shown that the sliding condition (9) is always satisfied by letting the time derivative of s be

$$\dot{s} = -k \text{ sgn } s \tag{20}$$

where k is a suitably selected positive constant. Now, using (17)-(20) we construct a discontinuous control $u(t)$ that satisfies the sliding condition (9), with the obvious assumption $Cb(t)\neq 0$

$$u(t) = [Cb(t)]^{-1}[-k \text{ sgn } s - CA(t)x(t) + C\dot{x}_d(t)]. \tag{21}$$

The design procedure is straightforward and requires little computational effort. The discontinuous control law (21) guarantees that all state trajectories starting off the sliding surface converge to it and those starting on the surface remain on it for all future time. Therefore, the feedback control yields that $e(t)\to 0$ as $t\to\infty$ for any given initial conditions. Note that $u(t)$ is discontinuous across s and that control discontinuity increases as the value of k increases.

We next add an unknown but bounded disturbance to the system (17), namely we consider

$$\dot{x}(t) = A(t)x(t)+b(t)u(t)+h(t)d(t) \tag{22}$$

where $d(t)\epsilon R$, and $h(t)\triangleq[0, \cdots, 0, *]^T$. It is obvious that with the exact same structure of controller (21) with the value of k properly chosen in accordance with the magnitude of the disturbance assures the existence of a sliding mode of the system (22) resulting in asymptotic tracking. To illustrate the suggested technique, consider a second order single-input system in the presence of bounded external disturbance described by the equation

$$\dot{x}(t)=\begin{bmatrix}0 & 1\\ 1 & t\end{bmatrix}x(t) + \begin{bmatrix}0\\ 1\end{bmatrix}u(t) + \begin{bmatrix}0\\ 1\end{bmatrix}d(t) \tag{23}$$

where $d(t)\epsilon[\delta_1,\delta_2]$. The problem is to get $x(t)$ to track a desired trajectory $x_d(t)$ which belongs to C^1 on $[t_0,\infty)$. The sliding surface is chosen from (19) as

$$s = (x_1-x_{1_d})+(x_2-x_{2_d}). \tag{24}$$

Thus, from (21) we obtain the discontinuous controller

$$u(t) = [-k \text{ sgn } s-2x_1-(1+t)x_2+\dot{x}_{1_d}+\dot{x}_{2_d}] \tag{25}$$

which yields the sliding mode motion of system (23), so long as $k>\max(|\delta_1|,|\delta_2|)$. Of course, with a small value of k enough to compensate process noise and measurement noise we may guarantee asymptotic tracking of the system (23) in the absence of $d(t)$. Note that the control discontinuity increases with the strength of disturbance to be compensated for.

Assume now that equation (22) is replaced by

$$\dot{x}(t) = A(t)x(t)+\Delta A(t)x(t)+b(t)u(t)+h(t)d(t) \tag{26}$$

where $A(t)$, $b(t)$, $h(t)$ are as in (22) and

$$\Delta A(t) \triangleq \begin{bmatrix} & \bigcirc & \\ \delta a_1 & \cdots\cdots & \delta a_n \end{bmatrix}_{n\times n}$$

such that each entry of last row δa_i is bounded as $\hat{a}_i \leq \delta a_i \leq \tilde{a}_i < \infty$, for $i=1,\cdots,n$. Let us define that

$$A_0 \triangleq \begin{bmatrix} & \bigcirc & \\ a_{01} & \cdots\cdots & a_{0n} \end{bmatrix}_{n\times n}$$

$$\bar{A} \triangleq \begin{bmatrix} & \bigcirc & \\ \bar{a}_1 & \cdots\cdots & \bar{a}_n \end{bmatrix}_{n\times n}$$

where a_{0i} is an average value corresponding to δa_i and $\bar{a}_i = \tilde{a}_i - a_{0i}$. It can then easily be shown that the sliding condition (9) is always satisfied by letting the time derivative of s be

$$\dot{s} = C(\Delta A(t)-A_0)x(t)-(k+|C\bar{A}x(t)|)\ \mathrm{sgn}\ s + Ch(t)d(t) \tag{27}$$

where C is a row vector ensuring a stable sliding surface (19), $|\cdot|$ denotes absolute value, and k is a parameter to be tuned according to the magnitude of the disturbance. A simple calculation shows that the sliding mode motion of system (26) will occur if we use the feedback control law

$$u(t) = [Cb(t)]^{-1}[-(k+|C\bar{A}x(t)|)\ \mathrm{sgn}\ s - C(A(t)x(t) + A_0x(t)) + C\dot{x}_d(t)]. \tag{28}$$

Thus, this controller produces asymptotic tracking for arbitrary initial conditions. Note again that the control discontinuity increases in order to compensate parameter uncertainties and external disturbances.

By a minor modification of the foregoing procedure, it can be extended to systems of the form

$$\dot{x}(t) = (A(t)+\Delta A(t))x(t)+(b(x,t)+\delta b(x,t))u(t)+h(t)d(t). \quad (29)$$

In order to deal with input possessing uncertainty, we make the following assumption.

Assumption(matching condition for single input):

There exist a Caratheodory function $\delta p(x,t)\epsilon R$, and a continuous function $\phi(x,t)\epsilon R_+$ such that, for all $(x,t)\epsilon R^n \times R$,

$$\delta b(x,t)=b(x,t)\delta p(x,t), \quad |\delta p(x,t)|\leq \phi(x,t)<1 \quad (30)$$

where $|\cdot|$ denotes absolute value.

Since $|\delta p(x,t)|<1$, we have $1+\delta p(x,t)>0$ and $((1+\delta p(x,t)/(1-\phi(x,t))\geq 1$. We now propose control $u(t)$ of the form

$$u(t) = [Cb(x,t)]^{-1}[-(k+|CA(t)x(t)-C\dot{x}_d(t)|+|C\bar{A}x(t)|) \text{ sgn } s$$
$$- CA_0x(t)] \;/\; [1-\phi(x,t)]. \quad (31)$$

From which, we obtain

$$s\dot{s} = s[C\dot{x}(t)-C\dot{x}_d(t)] = s[CA(t)x(t)+C\Delta A(t)x(t)+(1+\delta p(x,t))$$
$$Cb(x,t)u(t)+Ch(t)d(t)-C\dot{x}_d(t)]$$
$$= [(CA(t)x(t)-C\dot{x}_d(t))s-\psi|CA(t)x(t)-C\dot{x}_d(t)||s|] + [(C\Delta A(t)$$
$$x(t)-\psi CA_0x(t))s-\psi|C\bar{A}x(t)||s|] + [Ch(t)d(t)s-\psi k|s|]$$

where $\psi=(1+\delta p(x,t))/(1-\phi(x,t))$. It is easy to see that the controller (31) satisfies the sliding condition if k is chosen sufficiently large to compensate for the external disturbance.

3.1.2 Nonlinear dynamical systems

We now consider a nonlinear time-varying single-input control system. The control design methodology for nonlinear systems follow along the same lines of linear systems. Therefore, we just formulate discontinuous feedback controllers which guarantee asymptotic tracking without verifying sliding conditions. Let the dynamic system be represented by

$$\dot{x}(t) = f(x,t)+b(t)u(t) \tag{32}$$

where $x(t)\epsilon R^n$, $u(t)\epsilon R$ and

$$f(x,t) \triangleq [x_2(t),\cdot \cdot \cdot \cdot \cdot,x_n(t),g(x,t)]^T$$

$$b(t) \triangleq [0,\cdot \cdot \cdot \cdot \cdot,0,*]^T.$$

Then we have

$$u(t) = [Cb(t)]^{-1}[-k \text{ sgn } s-Cf(x,t)+C\dot{x}_d(t)]. \tag{33}$$

Assume next that system equation (32) is replaced by

$$\dot{x}(t) = f(x,t)+b(x,t)u(t)+h(t)d(t) \tag{34}$$

where $d(t)$ is an unknown external disturbance bounded as $d(t)\epsilon[\delta_1,\delta_2]$ and $h(t)$ is in (22). Then we obtain

$$u(t) = [Cb(x,t)]^{-1}[-k \text{ sgn } s - Cf(x,t) + C\dot{x}_d(t)] \tag{35}$$

where $k > \max(Ch(t)|\delta_1|, Ch(t)|\delta_2|)$.

Now let us consider parameter variations of the system (34) so that the system dynamics are described by

$$\dot{x}(t) = f(x,t) + \Delta f(x,t) + b(x,t)u(t) + h(t)d(t) \tag{36}$$

where

$$\Delta f(x,t) \triangleq [0, \cdots\cdots, \sum_{i=1}^{n} \delta a_i v_i(x,t)]^T_{1\times n}$$

such that each δa_i is unknown but bounded as $\hat{a}_i \le \delta a_i \le \tilde{a}_i < \infty$. Defining

$$f_0(x,t) \triangleq [0, \cdots\cdots, \sum_{i=1}^{n} a_{0i} v_i(x,t)]^T_{1\times n}$$

$$\bar{f}(x,t) \triangleq [0, \cdots\cdots, \sum_{i=1}^{n} \bar{a}_i v_i(x,t)]^T_{1\times n}$$

where a_{0i} is an average value of the corresponding δa_i and $\bar{a}_i = \tilde{a}_i - a_{0i}$ yields the robust control law

$$u(t) = [Cb(x,t)]^{-1}[-(k + |C\bar{f}(x,t)|) \text{ sgn } s - C(f(x,t) + f_0(x,t)) + C\dot{x}_d(t)] \tag{37}$$

where k is as defined in (35) and $|\cdot|$ denotes absolute value. Now we consider an application of this controller to a second order nonlinear system given by

$$\dot{x}(t) = \begin{bmatrix} x_2 \\ 2x_2^2 \end{bmatrix} + \begin{bmatrix} 0 \\ a_1(t)x_1^3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d(t) \tag{38}$$

where $a_1(t)\epsilon[1,5]$ and $d(t)\epsilon[\delta_1,\delta_2]$. The sliding surface is chosen from (19) as

$$s = c_1(x_1-x_{1_d})+(x_2-x_{2_d}) \tag{39}$$

where $c_1>0$ and the desired trajectory $x_d\epsilon\ C^1[t_0,\infty)$. Thus, from (37)

$$u(t) = [-(k+|2x_1^3|)\ \mathrm{sgn}\ s-(c_1x_2+2x_2^2+3x_1^3)+c_1\dot{x}_{1_d}+\dot{x}_{2_d}] \tag{40}$$

where $k>\max(|\delta_1|,|\delta_2|)$.

In order to retain asymptotic tracking in the presence of input possessing uncertainty we consider the

$$\dot{x}(t) = f(x,t)+\Delta f(x,t)+(b(x,t)+\delta b(x,t))u(t)+h(t)d(t). \tag{41}$$

Assuming the matching condition (30) the control is given by

$$u(t) = [Cb(x,t)]^{-1}[-(k+|Cf(x,t)-C\dot{x}_d(t)|+|C\tilde{f}(x,t)|)\ \mathrm{sgn}\ s$$
$$- Cf_0(x,t)]\ /\ [1-\phi(x,t)] \tag{42}$$

where k is proportional to the magnitude of external disturbance proposed for system (41). Next we generalize the above design procedures to a large class of multi-input time-varying systems.

3.2 A class of multi-input systems

3.2.1 Linear dynamical systems

We discussed that in vector control cases if the conditions for the existence of a sliding mode are met on each of the sliding surfaces, the generation of the sliding mode can be represented by the cases A and B in Figure 4. Thus the problem is to design discontinuous controllers which satisfy sliding condition (16).

Let us consider a linear time-varying multi-input system in the presence of external disturbances given by

$$\dot{X}(t) = A(t)X(t)+B(t)U(t)+H(t)D(t) \tag{43}$$

where $X(t)\epsilon R^n$, $U(t)\epsilon R^m$ and disturbances $D(t)\epsilon R^{\ell}$. The $n\times n$ matrix $A(t)$ has m diagonal blocks each of which is in the phase canonical form, whereas the off-diagonal blocks have nonzero entries only in the last row of each block. $B(t)$ and $H(t)$ have m and ℓ diagonal block vectors respectively, which have only the last entries and also the off-diagonal block vectors have entries in the last row of each block. By way of notation, define

$$X(t) \triangleq [x_1(t),\cdot \cdot \cdot \cdot \cdot,x_m(t)]^T$$

$$U(t) \triangleq [u_1(t),\cdot \cdot \cdot \cdot \cdot,u_m(t)]^T$$

$$D(t) \triangleq [d_1(t),\cdot \cdot \cdot \cdot \cdot,d_{\ell}(t)]^T$$

and

$$X_d(t) \triangleq [x_{1d}(t),\cdot \cdot \cdot \cdot \cdot,x_{md}(t)]^T$$

where desired trajectories $X_d(t)\epsilon R^n$, and each $x_{id}(t) \;\epsilon\; C^1[t_0,\infty)$. Since we want to design control laws that make each $x_i(t)$ track a corresponding desired trajectory $x_{id}(t)$, define tracking errors $E(t)\epsilon R^n$ as

$$E(t) \triangleq [e_1(t),\cdot \cdot \cdot \cdot \cdot,e_m(t)]^T = X(t)-X_d(t) \tag{44}$$

where $e_i(t)=x_i(t)-x_{id}(t)$, for $i=1,\cdot\cdot\cdot\cdot,m$.

The components of $U(t)$ undergo discontinuities on m planes. Thus, the set of sliding surfaces $S\epsilon R^m$ is defined in the error state space as

$$S(t) \triangleq [s_1,\cdot\cdot\cdot\cdot\cdot,s_m]^T \triangleq GE(t)=0 \tag{45}$$

where $s_i=C_ie_i$, and G is an $m\times n$ constant matrix whose each row C_i is a $1\times n$ gradient vector of the function s_i. We are now in the same situation as we were in the single input case, that is, we have to check the sliding conditions (16) for each surface to construct control $u_i(t)$ forcing the trajectory $x_i(t)$ to slide along the surface s_i thus yielding $e_i(t)\rightarrow 0$ as $t\rightarrow\infty$ for any arbitrary initial conditions. It is easy to verify that the sliding conditions (16) are always satisfied by selecting the time derivative of S as

$$\dot{S} = -K \text{ sgn } S + GH(t)D(t) \tag{46}$$

where $K \triangleq [k_1,\cdot\cdot\cdot\cdot\cdot,k_m]^T$ such that each k_i is greater than the magnitudes of the corresponding elements of $GH(t)D(t)$. From (43)-(46) we can obtain discontinuous control laws $U(t)$ that satisfy the sliding condition (16) resulting in asymptotic tracking for all desired trajectories. These feedback controllers take the form

$$U(t) = [GB(t)]^{-1}[-K \text{ sgn } S- GA(t)X(t)+ G\dot{X}_d(t)] \tag{47}$$

where K is an appropriately chosen parameters accounting for the process noise,measurement noise and disturbances. It is obvious that $GB(t)$ should be invertible. Implications of singularities of

GB(t) can be found in [2]. To illustrate the design procedure, let us consider the system

$$\dot{X}(t)=\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1 & 0 & 0\\ -1 & 2 & 1 & 2\\ 0 & 0 & 0 & 1\\ -1 & 1 & 1 & -1\end{bmatrix}X(t)+\begin{bmatrix}0 & 0\\ 1 & 0\\ 0 & 0\\ 0 & 1\end{bmatrix}U(t)+\begin{bmatrix}0 & 0\\ 1 & 0\\ 0 & 0\\ 0 & 1\end{bmatrix}D(t) \qquad (48)$$

where disturbances are bounded as $d_1(t)\epsilon[\delta_1,\delta_2]$ and $d_2(t)\epsilon[\delta_3,\delta_4]$. The objective is to get $x_i(t)$ to track the corresponding desired trajectory $x_{id}(t)\epsilon C^1[t_0,\infty)$. From (45) we select sliding surfaces as

$$S=\begin{bmatrix}s_1\\ s_2\end{bmatrix}=\begin{bmatrix}c_{11}(x_1-x_{1d})+(\dot{x}_1-\dot{x}_{1d})\\ c_{21}(x_2-x_{2d})+(\dot{x}_2-\dot{x}_{2d})\end{bmatrix}$$

$$\triangleq\begin{bmatrix}c_{11}(X_1-X_{1_d})+(X_2-X_{2_d})\\ c_{21}(X_3-X_{3_d})+(X_4-X_{4_d})\end{bmatrix} \qquad (49)$$

where c_{i1} should be positive to make stable sliding motions. Thus, using (47) the control laws can be constructed as

$$U(t)=\begin{bmatrix}u_1(t)\\ u_2(t)\end{bmatrix}=\begin{bmatrix}-k_1\text{sgn } s_1+X_1-(c_{11}+2)X_2-X_3-2X_4+c_{11}\dot{X}_{1_d}+\dot{X}_{2_d}\\ -k_2\text{sgn } s_2+X_1-X_2-X_3-(c_{21}-1)X_4+c_{21}\dot{X}_{3_d}+\dot{X}_{4_d}\end{bmatrix} \qquad (50)$$

where $k_1>\max(|\delta_1|,|\delta_2|)$ and $k_2>\max(|\delta_3|,|\delta_4|)$. We note that $u_1(t)$ contains only s_1 and $u_2(t)$ does s_2 only. However, as we defined B(t) in (43), it is not necessary that S should be decoupled for each u_i, i. e. , u_i may include s_i and s_j $(i\neq j)$. For example, by changing matrix B in (48) as

$$B=\begin{bmatrix}0 & 0\\ 2 & 1\\ 0 & 0\\ 1 & 1\end{bmatrix}$$

we have corresponding feedback cotrollers

$$U(t) = \begin{bmatrix} \bar{u}_1(t) \\ \bar{u}_2(t) \end{bmatrix} = \begin{bmatrix} u_1(t)-u_2(t) \\ -u_1(t)+2u_2(t) \end{bmatrix} \tag{51}$$

where $u_1(t)$ and $u_2(t)$ are as in (50). Both (50) and (51) guarantee the existence of a sliding mode on the intersection of s_1 and s_2.

We now consider the case where equation (43) is replaced by

$$\dot{X}(t) = A(t)X(t)+\Delta A(t)X(t)+B(X,t)U(t)+H(t)D(t) \tag{52}$$

where parameter uncertainties $\Delta A(t)$ have m diagonal blocks which have elements only in the last row and also off-diagonal blocks have elements only in the last row of each block such that each non-zero element is unknown but bounded as $\hat{a}_{ij} \leq \delta a_{ij} \leq \tilde{a}_{ij} < \infty$. Let us define that

$A_0 \triangleq$ nxn matrix with entries such that each entry $a_{0_{ij}}$ is an average value of corresponding element of the matrix $\Delta A(t)$

$\bar{A} \triangleq$ nxn matrix with entries such that each entry $\bar{a}_{ij} = \tilde{a}_{ij} - a_{0_{ij}}$.

Similar to the scalar case, we choose the time derivative of each component of S to meet the sliding condition (16), that is,

$$\dot{S} = G(\Delta A(t)-A_0)X(t)-(K+|G\bar{A}X(t)|) \text{ sgn } S + GH(t)D(t) \tag{53}$$

where K is in (47) and $|\cdot|$ denotes absolute value of each component. The feedback control laws which will generate sliding mode motion of the system (52) are formulated as

$$U(t) = [GB(X,t)]^{-1}[-(K+|G\tilde{A}X(t)|)\ \text{sgn}\ S - G(A(t)X(t) + A_0X(t)) + G\dot{X}_d(t)]. \quad (54)$$

Note that the discontinuity of each control component increases as the magnitudes of parameter uncertainties and external disturbances increase.

We now extend the results to systems of the form

$$\dot{X}(t) = (A(t)+\Delta A(t))X(t)+(B(X,t)+\Delta B(X,t))U(t)+H(t)D(t). \quad (55)$$

We assume the following matching condition which is quite similar to the single input case.

Assumption(matching condition for multi-input case):

There exist Caratheodory functions $\Delta P(X,t)\epsilon R^{m\times m}$, and a continuous function $\phi(X,t)\epsilon R_+$ such that, for all $(X,t)\epsilon R^n\times R$,

$$\Delta B(X,t) = B(X,t)\Delta P(X,t)\ ,\quad ||\Delta P(X,t)|| \le \phi(X,t) < 1 \quad (56)$$

where $||\cdot||$ denotes any induced norm. With this assumption it can be easily shown that the sliding conditions (16) are satisfied if we construct the discontinuous control laws as

$$U(t) = [GB(X,t)]^{-1}[-(K+|GA(t)X(t)-G\dot{X}_d(t)|+|G\tilde{A}X(t)|)\ \text{sgn}\ S - GA_0X(t)] \ /\ [1-\phi(X,t)]. \quad (57)$$

3.2.2 Nonlinear dynamical systems

Since the same design concepts developed for linear multi-input systems apply to nonlinear multi-variable systems, we only do costruct the discontinuous control laws that guarantee asymptotic tracking for arbitrary initial conditions. For dynamic systems described by

$$\dot{X}(t) = F(X,t)+B(X,t)U(t)+H(t)D(t) \quad (58)$$

where $X(t)\epsilon R^n$, $U(t)\epsilon R^m$, $D(t)\epsilon R^\ell$, $B(X,t)$ and $H(t)$ are as in (52), and

$$F(X,t) \triangleq [f_1(X,t),\cdot\cdot\cdot\cdot\cdot,f_m(X,t)]^T$$

such that each $f_i(X,t)$ has the form of

$$f_i(X,t) \triangleq [X_{2i},\cdot\cdot\cdot\cdot, X_{pi},g_i(X,t)]^T_{1\times p} \quad , p < n.$$

We have

$$U(t) = [GB(X,t)]^{-1}[-K \text{ sgn } S-GF(X,t)+G\dot{X}_d(t)] \quad (59)$$

where K should be determined in accordance with the strength of disturbances and measurement noise.

Add now parameter uncertainties to the system (58), i.e.,

$$\dot{X}(t) = F(X,t)+\Delta F(X,t)+B(X,t)U(t)+H(t)D(t) \quad (60)$$

where

$$\Delta F(X,t) \triangleq [\delta f_1(X,t),\cdot\cdot\cdot\cdot\cdot,\delta f_m(X,t)]^T$$

such that each $\delta f_i(X,t)$ has the form of

$$\delta f_i(X,t) \triangleq [0,\cdot\cdot\cdot\cdot\cdot,\sum_{j=1}^{n}\delta a_j v_j(X,t)]^T_{1\times p}$$

where δa_j's are unknown but bounded as $\hat{a}_j \leq \delta a_j \leq \tilde{a}_j < \infty$. Let us define

$$F_0(X,t) \triangleq [f_{0_1}(X,t),\cdot\cdot\cdot\cdot\cdot,f_{0_m}(X,t)]^T$$

$$\bar{F}(X,t) \triangleq [\bar{f}_1(X,t),\cdot\cdot\cdot\cdot\cdot,\bar{f}_m(X,t)]^T$$

where $f_{0i}(X,t)$ has the form

$$f_{0i}(X,t) \triangleq [0, \cdot \cdot \cdot \cdot \cdot, \sum_{j=1}^{n} a_{0j} v_j(X,t)]^T_{1\times p}$$

such that a_{0j} is an average value of the corresponding δa_j of $\delta f_i(X,t)$, and $\bar{f}_i(X,t)$ is of the form

$$\bar{f}_i(X,t) \triangleq [0, \cdot \cdot \cdot \cdot \cdot, \sum_{j=1}^{n} \bar{a}_j v_j(X,t)]^T_{1\times p}$$

such taht $\bar{a}_j = \tilde{a}_j - a_{0j}$. Then, we have

$$U(t) = [GB(X,t)]^{-1}[-(K+|G\bar{F}(X,t)|) \text{ sgn } S - G(F(X,t) + F_0(X,t)) + G\dot{X}_d(t)] \tag{61}$$

where $|\cdot|$ denotes absolute value of each component. To illustrate this design technique, we consider the following example represented by

$$\dot{X}(t) = \begin{bmatrix} X_2 \\ 2X_1 + X_2^2 + 2X_1X_4\cos X_3 \\ X_4 \\ X_1^3 - 2X_2 - X_4\cos X_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2a_1(t)X_4(t) \\ 0 \\ a_2(t)X_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} U(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} D(t) \tag{62}$$

where $a_1(t)\epsilon[-2,8]$, $a_2(t)\epsilon[0,10]$, $d_1(t)\epsilon[\delta_1,\delta_2]$ and $d_2(t)\epsilon[\delta_3,\delta_4]$. Since the problem is to get $x_1(t)$ to track $x_{1d}(t)$ and $x_2(t)$ to track $x_{2d}(t)$, from (45) we select sliding surfaces as

$$S = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} c_{11}(x_1 - x_{1d}) + (\dot{x}_1 - \dot{x}_{1d}) \\ c_{21}(x_2 - x_{2d}) + (\dot{x}_2 - \dot{x}_{2d}) \end{bmatrix}$$

$$\triangleq \begin{bmatrix} c_{11}(X_1 - X_{1d}) + (X_2 - X_{2d}) \\ c_{21}(X_3 - X_{3d}) + (X_4 - X_{4d}) \end{bmatrix}$$

$$\triangleq \begin{bmatrix} c_{11} & 1 & 0 & 0 \\ 0 & 0 & c_{21} & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ \dot{e}_1 \\ e_2 \\ \dot{e}_2 \end{bmatrix} \triangleq GE(t) \tag{63}$$

where c_{i1} is in (49) and $x_{id}(t) \epsilon C^1[t_0,\infty)$. Then, from (61) the controllers become

$$u_1(t) = -(k_1+|10X_4|)\ \text{sgn}\ s_1 - c_{11}X_2 - 2X_1 - 2X_1X_4\cos X_3$$

$$-6X_4 + c_{11}\dot{X}_{1d} + \dot{X}_{2d} \tag{64a}$$

$$u_2(t) = -(k_2+|5X_3|)\ \text{sgn}\ s_2 - c_{21}X_4 - X_1^3 + 2X_2 + X_4\cos X_1$$

$$-5X_3 + c_{21}\dot{X}_{3d} + \dot{X}_{4d} \tag{64b}$$

where $k_1 > \max(|\delta_1|,|\delta_2|)$, $k_2 > \max(|\delta_3|,|\delta_4|)$ and c_{i1} should be chosen according to the qualitative characteristics of the system which will be discussed later in this chapter.

For the input possessing uncertainties of the system (60) described as

$$\dot{X}(t) = F(X,t) + \Delta F(x,t) + (B(X,t) + \Delta B(X,t))U(t) + H(t)D(t) \tag{65}$$

we have feedback controllers

$$U(t) = [GB(X,t)]^{-1}[-(K+|GF(X,t) - G\dot{X}_d(t)| + |G\bar{F}(X,t)|)\ \text{sgn}\ S$$

$$-\ GF_0(X,t)]\ /\ [1-\phi(X,t)] \tag{66}$$

which guarantee asymptotic tracking for desired trajectories belonging to C^1 on $[t_0,\infty)$.

3.3 Gradient vectors of sliding surfaces and approximations of discontinuous control laws

We now discuss the importance of gradient vectors of sliding surfaces. Since the matrix GB affects the rate of convergence to the sliding surface, the matrix G should be chosen according to the intrinsic characteristics of the system such as unmodelled high frequency of the actual system and the desired eigenvalues to be located. For linear time-invariant systems, there are some analytical methods to specify G by using geometric notions[11] and minimizing quadratic functionals[12]. However, in general it is hard to select the optimal G in nonlinear time-varying systems, especially in uncertain dynamical systems. Here we take up one simple single input system to show the importance of the gradient vector. Consider a system described by

$$\ddot{x}(t)=a(t)x^{3}(t)+2\dot{x}^{2}(t)+u(t)+d(t) \qquad (67)$$

where parameter uncertainty $a(t)\epsilon[1,5]$ and external disturbance $d(t)\epsilon[-4,4]$. We choose the desired trajectory as

$$x_d(t) = t$$

which belongs to C^1. By defining

$$[x,\ \dot{x}] \triangleq [x_1, x_2]\ ,\quad [x_d, \dot{x}_d] \triangleq [x_{1d}, x_{2d}]$$

we define the sliding surface from (19) as

$$s = c_1(x_1 - x_{1_d}) + (x_2 - x_{2_d})$$

where c_1 is an element of the gradient vector of a sliding surface. After some manipulations, following (37) we obtain the discontinuous control law

$$u(t) = [-(5+|2x_1^3|)\ \mathrm{sgn}\ s - c_1 x_2 - 2x_2^2 - 3x_1^3 + c_1 x_{2_d} + \dot{x}_{2_d}]. \qquad (68)$$

Figure 7 shows the resulting state trajectories for $c_1=0.5$, 100 and 500 respectively, on application of the controller (68). It is clear that asymptotic tracking is not produced for the value $c_1=0.5$ even though this value satisfies a sliding condition (8) for the system (67). This implies that for the given desired trajectory the value of $c_1=0.5$ is not enough to cope with the speed of convergence to the sliding surface. The speed of convergence depends on unmodelled high frequency dynamics of the actual system and the eigenvalues of the desired system. Simulations reveal that $c_1=0.5$ is enough to force the states to the sliding surface if the desired trajectory is $x_d(t)=\sin(t)$ for system (67). Another interesting observation is made from Figure 7. When we use $c_1=100$, it takes 2.3 seconds to get asymptotic tracking, whereas 4 seconds are needed to hit the sliding surface when we choose $c_1=500$. These indicate the possible existence of an optimal c_1 value that depends on the desired trajectory. A study of this aspect may be worth considering in future work.

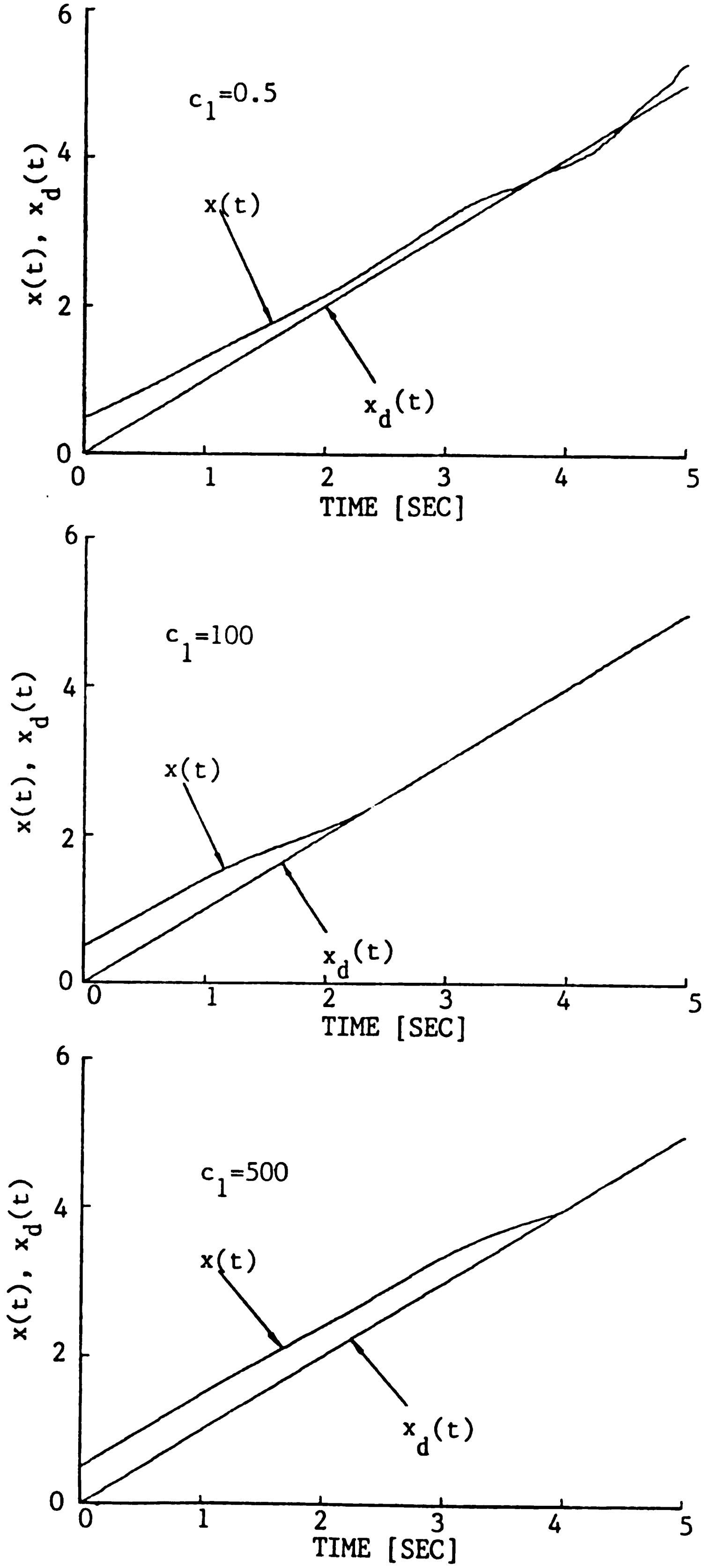

Figure 7. State trajectories for c_1=0.5, 100, 500

As we mentioned in the introduction, the application of discontinuous control laws in practice is not desirable due to chattering which is undesirable both in itself and in the fact it generates a high frequency signal component resulting in destruction of the plant. Thus, the discontinuous controller should be approximated by a smooth one which will preserve the properties of the discontinuous one, and in addition not generate undesirable high frequency signals and excessive control efforts. The basic idea is to find continuous control law within a small boundary layer neighbouring the sliding surface by smoothing out the discontinuity in the control law. Then the boundary layer becomes a modified sliding surface to which trajectories starting outside the boundary layer converge. Thus this is achieved by choosing the discontinuous control law outside the boundary layer, and then interpolating the control law inside the boundary layer. Figure 8 shows the construction of the boundary layer in the case of second order systems. By defining the sliding surface as

$$\begin{aligned} s^+ &\triangleq s - c\epsilon \\ s^- &\triangleq s + c\epsilon \end{aligned} \tag{69}$$

where ϵ is the boundary layer width, c is an element of C in (19), we may define the boundary layer by

$$\beta(t) \triangleq \{x \,|\; s^+ < 0 \quad \text{and} \quad s^- > 0 \}. \tag{70}$$

We choose control $u(t)$ outside $\beta(t)$, i.e.,

$$\{ s^+ > 0 \quad \text{or} \quad s^- < 0 \}.$$

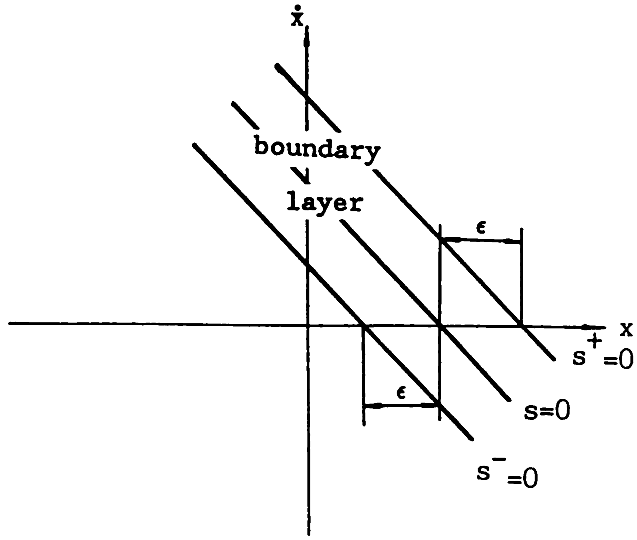

Figure 8. Construction of the boundary layer

Then, we have the sliding conditions $s^+\dot{s}^+ < 0$ and $s^-\dot{s}^- < 0$, since from (69) $\dot{s}^+ = \dot{s} = \dot{s}^-$. Now it remains to specify continuous controller inside $\beta(t)$. The Urysohn's lemma[13] says that there exists at least one continuous interpolation between $u^+(t)$ and $u^-(t)$. A simple interpolation of controller in the boundary layer is shown in Figure 9. This amounts to replacing sgn s by sat(s/cϵ) inside the boundary layer. For instance, the discontinuous control law (40) then becomes

$$u(t)=[-(k+|2x_1^3|)\ \mathrm{sat}(s/c_1\epsilon)-(c_1x_2+2x_2^2+3x_1^3)+c_1\dot{x}_{1_d}+\dot{x}_{2_d}] \tag{71}$$

where $\mathrm{sat}(\sigma)$ is defined by

$$\mathrm{sat}(\sigma) \triangleq \begin{cases} \mathrm{sat}(\sigma)=\sigma & , \ |\sigma|\le 1 \\ \mathrm{sat}(\sigma)=\mathrm{sgn}\ (\sigma) & , \ |\sigma|>1 \end{cases} . \tag{72}$$

Note that the tracking error decreases as the boundary layer width ϵ decreases. In other words, we trade off tracking accuracy against the generation of chattering in the state trajectory by approximating the discontinuous control law. The selection of an optimal ϵ is not resolved yet. Some of the important properties of this trade off are quantified in [7].

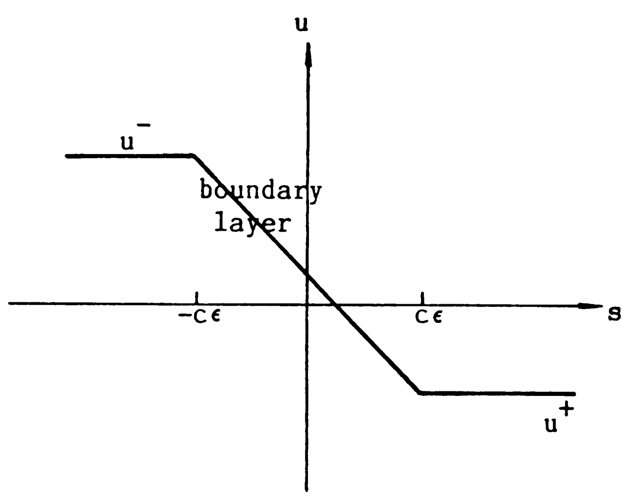

Figure 9. An interpolation of u(t) in the boundary layer

CHAPTER IV

AN APPLICATION TO A ROBOTIC MANIPULATOR

The development of modern industrial robots and manipulators calls for robustness of performance with regard to variable loads, torque disturbances, as well as other task specifications. In general the system dynamics of industrial robots can be represented by equations of the form (65). We shall illustrate the applicability of our methodology to these robots by designing controllers for a three-degrees-of-freedom manipulator. By construction, it will be shown that our sliding mode feedback controllers are robust to variable payloads and time-varying torque disturbance.

Consider the three-degrees-of-freedom manipulator of Figure 10. The manipulator has one rotational and translational joint in the (x,y) plane, and the arm can be lifted along the vertical z-axis which constitutes the third degree of freedom. The kinetic equations of this configuration follow directly from an application of Lagrange's equations(see [14]). By assuming normalized unit mass and unit length of the arm and upright column, and neglecting the gravity force, we obtain the following dynamic equations.

$$\ddot{r}(t)=r(t)\dot{\theta}^2(t)-\frac{1}{2(1+M)}\dot{\theta}^2(t)+\frac{1}{1+M}F_1(t) \tag{73a}$$

$$\ddot{\theta}(t)=\frac{-2(1+M)r(t)+1}{(5/6)-r(t)+(1+M)r^2(t)}\dot{r}(t)\dot{\theta}(t)+\frac{1}{(5/6)-r(t)+(1+M)r^2(t)}T(t)+d(t) \tag{73b}$$

$$\ddot{z}(t)=\frac{1}{1+M}F_2(t) \tag{73c}$$

where an unknown but bounded external torque disturbance $d(t)$ and a variable load M bounded as $0 \le M_{min} \le M \le M_{max}$ are imposed to demonstrate the robustness of our control scheme.

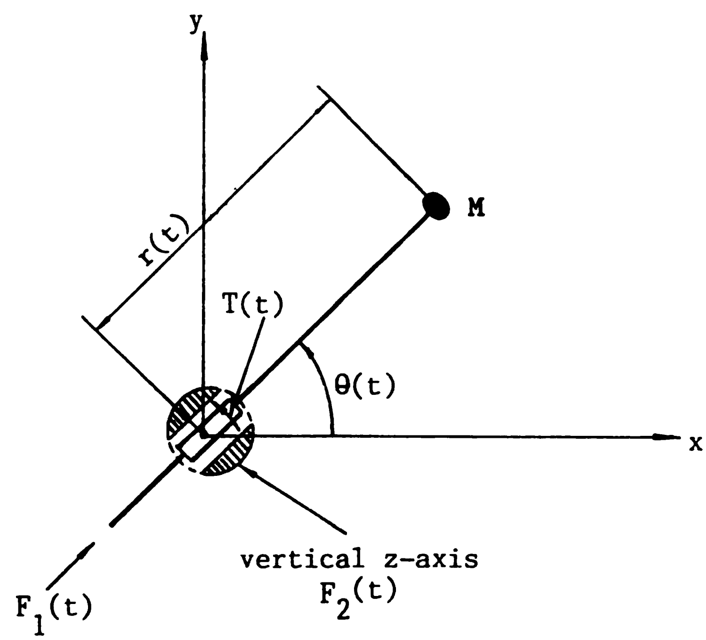

Figure 10. Three-degrees-freedom manipulator

We introduce state variables

$$X = [X_1, X_2, X_3, X_4, X_5, X_6]^T = [r(t), \dot{r}(t), \theta(t), \dot{\theta}(t), z(t), \dot{z}(t)]^T \quad (74a)$$

and inputs

$$U = [u_1, u_2, u_3]^T = [F_1(t), T(t), F_2(t)]^T \quad (74b)$$

to obtain the following problem statement which is of the form (65). Thus

$$\dot{X}(t) = \begin{bmatrix} X_2 \\ X_1 X_4^2 \\ X_4 \\ X_2 X_4 \\ X_6 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \Delta f_2(X,t) \\ 0 \\ \Delta f_4(X,t) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ \Delta b_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \Delta b_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Delta b_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} d(t) \quad (75)$$

where

$$\Delta f_2(X,t) = -\frac{1}{2(1+M)} X_4^2$$

$$\Delta f_4(X,t) = \frac{-(1+M)X_1^2 - X_1(1+2M) + (1/6)}{(5/6) - X_1 + (1+M)X_1^2} X_2 X_4 \quad (76a)$$

$$\Delta b_1 = -\frac{M}{1+M}$$

$$\Delta b_2 = \frac{1+6X_1 - 6X_1^2(1+M)}{5-6X_1+6X_1^2(1+M)} \quad (76b)$$

$$\Delta b_3 = -\frac{M}{1+M}.$$

The system (75) can be rewritten in the form

$$\dot{X}(t) = F(X,t) + \Delta F(X,t) + (B + \Delta B(X,t))U(t) + hd(t). \quad (77)$$

The objective of the design is to force $r(t)$, $\theta(t)$ and $z(t)$ to track asymptotically the desired trajectories $r_d(t)$, $\theta_d(t)$ and $z_d(t)$ respectively. We choose desired trajectories as

$$r_d(t) = 0.5+0.3\sin(t) \quad m \tag{78a}$$

$$\theta_d(t) \begin{cases} = -70^0+50^0(1-\cos(t)), & t\leq 3.14 \text{ sec} \\ = 30^0 & , \quad t>3.14 \text{ sec} \end{cases} \tag{78b}$$

$$z_d(t) = 0.4+0.3\cos(t) \quad m \tag{78c}$$

and define

$$X_d(t) \triangleq [X_{1_d}, X_{2_d}, X_{3_d}, X_{4_d}, X_{5_d}, X_{6_d}]^T$$

$$\triangleq [r_d(t), \dot{r}_d(t), \theta_d(t), \dot{\theta}_d(t), z_d(t), \dot{z}_d(t)]^T. \tag{79}$$

From (45) we select sliding surface as

$$S = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 5(X_1-X_{1_d})+(X_2-X_{2_d}) \\ 7(X_3-X_{3_d})+(X_4-X_{4_d}) \\ 5(X_5-X_{5_d})+(X_6-X_{6_d}) \end{bmatrix}. \tag{80}$$

This gives the matrix

$$G = \begin{bmatrix} 5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 7 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 1 \end{bmatrix}. \tag{81}$$

Matching condition (56) is shown to hold by considering

$$\Delta P(X,t) = \begin{bmatrix} \Delta b_1 & 0 & 0 \\ 0 & \Delta b_2 & 0 \\ 0 & 0 & \Delta b_3 \end{bmatrix} \tag{82}$$

and for which it is readily shown that

$$||\Delta P(X,t)||_1 = ||\Delta P(X,t)||_\infty = \max|\Delta b_i|.$$

Thus, from (76b) we see that if the $M_{max}-M_{min}$ is sufficiently small, there exists a $\phi(X,t)$ such that $\phi(X,t)<1$. Now we choose $M_{min}=0$ Kg, $M_{max}=0.6$ Kg, and $\phi(X,t)=\phi=0.8$. According to the design procedure given by (60), we obtain the following results after some algebraic manipulations

$$GB = I_{3\times 3} \tag{83a}$$

$$GF(X,t)-G\dot{X}_d(t) \triangleq \begin{bmatrix}\eta_1(X,t)\\ \eta_2(X,t)\\ \eta_3(X,t)\end{bmatrix} = \begin{bmatrix}5X_2+X_1X_4^2-5X_{2d}-\dot{X}_{2d}\\ 7X_4+X_2X_4-7X_{4d}-\dot{X}_{4d}\\ 5X_6-5X_{6d}-\dot{X}_{6d}\end{bmatrix} \tag{83b}$$

$$GF_0(X,t) \triangleq \begin{bmatrix}f_{01}(X,t)\\ f_{02}(X,t)\\ f_{03}(X,t)\end{bmatrix} = \begin{bmatrix} 2-(1/2.6)X_4^2 \\ \dfrac{-1.3X_1^2-1.6X_1+(1/6)}{1.3X_1^2-X_1+(5/6)}X_2X_4 \\ 0\end{bmatrix} \tag{83c}$$

$$G\bar{F}(X,t) \triangleq \begin{bmatrix}\bar{f}_1(X,t)\\ \bar{f}_2(X,t)\\ \bar{f}_3(X,t)\end{bmatrix} = \begin{bmatrix}f_{01}(X,t)\\ f_{02}(X,t)\\ f_{03}(X,t)\end{bmatrix}. \tag{83d}$$

Now, from (66) we can construct sliding mode feedback cotrollers as

$$\begin{bmatrix}u_1\\ u_2\\ u_3\end{bmatrix} = \frac{1}{1-\phi}\begin{bmatrix}-(k_1+|\eta_1(X,t)|+|\bar{f}_1(X,t)|)\ \mathrm{sgn}\ s_1-f_{01}(X,t)\\ -(k_2+|\eta_2(X,t)|+|\bar{f}_2(X,t)|)\ \mathrm{sgn}\ s_2-f_{02}(X,t)\\ -(k_3+|\eta_3(X,t)|)\ \mathrm{sgn}\ s_3\end{bmatrix}. \tag{84}$$

The value of k_i should be chosen according to the magnitude of the external disturbances, measurement noise and process noise. We assume the torque disturbance is bounded as $d(t)\epsilon[-0.3,0.3]$ N•m. For simulations k_1, k_2 and k_3 are chosen to be 1, 2 and 1 respectively. And we chose the initial conditions as

$$X(0)=[60 \text{ cm}, 0 \text{ cm/sec}, -86 \text{ deg}, 0 \text{ deg/sec}, 60 \text{ cm}, 0 \text{ cm/sec}]^T.$$

Figures 11 and 12 show the results of asymptotic tracking for the desired trajectories described by (78) under no load and full load when we use controllers (84). The converging time to the sliding surface may be decreased by increasing the value of k_i and ϕ. However, the discontinuities of the control effort increase proportionately causing the generation of higher chattering which is undesirable. Under full payload, after hitting the sliding surface, the simulation results show tracking errors to be within 0.02 cm, 0.06^{0} and 0.05 cm for $r(t)$, $\theta(t)$ and $z(t)$ respectively. Figures 13 and 14 show the discontinuous control histories under no load and full load. As we mentioned earlier, it is not desirable to use these controllers in practice. So we approximate these discontinuous feedback control laws by continuous ones inside the boundary layer. To do this, we replaced sgn s_i in (84) by $\text{sat}(\sigma_i)$. Then, the new controls become

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \frac{1}{1-\phi} \begin{bmatrix} -(k_1+|\eta_2(X,t)|+|\bar{f}_1(X,t)|) \text{ sat } (\sigma_1) - f_{0_1}(X,t) \\ -(k_2+|\eta_2(X,t)|+|\bar{f}_2(X,t)|) \text{ sat } (\sigma_2) - f_{0_2}(X,t) \\ -(k_3+|\eta_3(X,t)|) \text{ sat}(\sigma_3) \end{bmatrix} \quad (85)$$

where

$$\sigma_1 = s_1 / (5\epsilon_1)$$

$$\sigma_2 = s_2 / (7\epsilon_2)$$

$$\sigma_3 = s_3 / (5\epsilon_3).$$

The selection of ϵ_i depends on the strength of the discontinuities of control efforts. We chose $\epsilon_1=0.005$, $\epsilon_2=0.01$ and $\epsilon_3=0.01$ for this simulation. Figures 15 and 16 show asymptotic tracking under no load and full load when we employ continuous feedback controllers (85). Under full payload, after converging to the sliding surface, the simulation shows tracking precisions of 0.08 cm, 0.07^{0} and 0.07cm for $r(t)$, $\theta(t)$ and $z(t)$ respectively. As we expected the tracking errors have increased. These tracking errors may be reduced by decreasing the ϵ_i's. Figures 17 and 18 reveal that the control efforts of approximated controllers under no load and full load do not indicate explicitly the discontinuities seen in Figures 13 and 14. Simulation results also indicate that larger control efforts are needed to drive the system states to the sliding surface. However, once on this surface smaller control efforts are needed to maintain tracking.

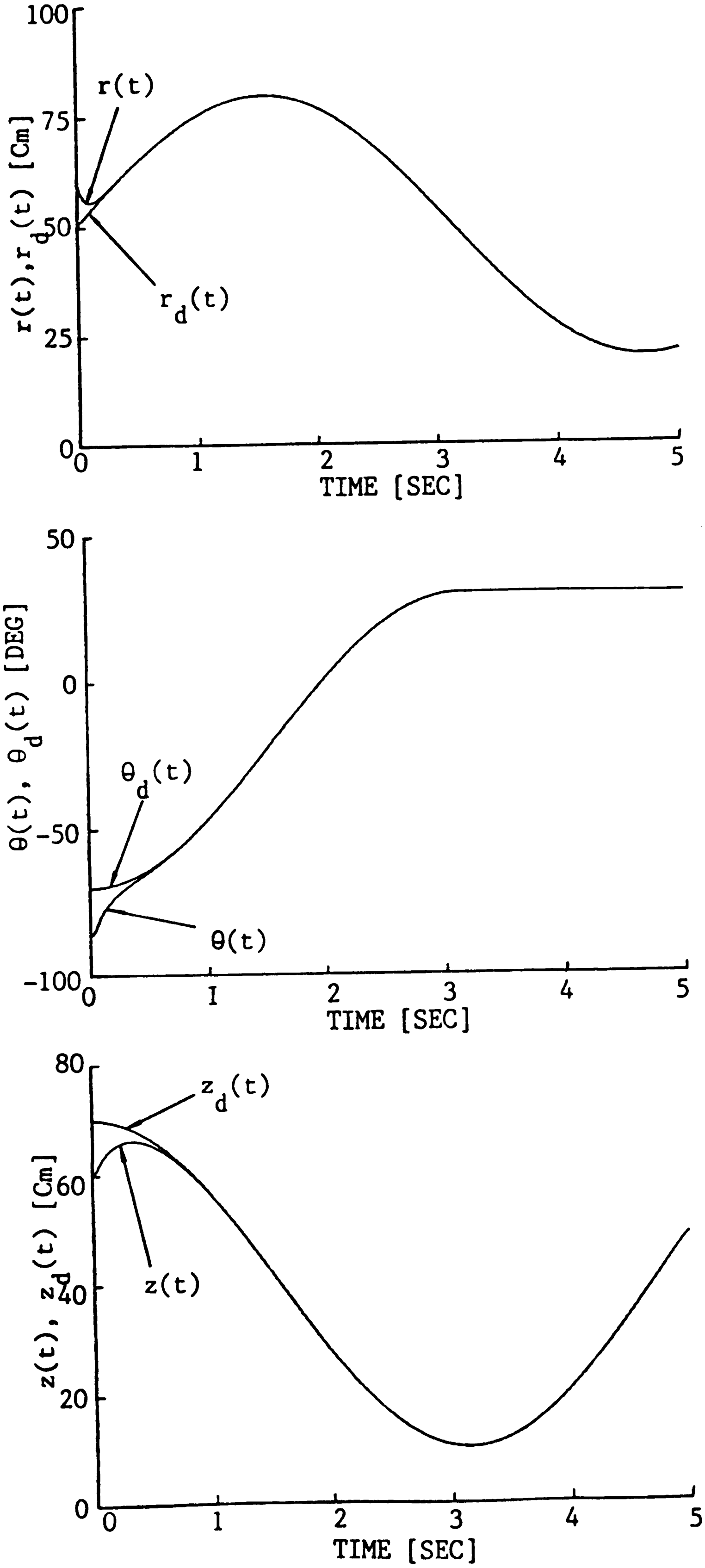

Figure 11. State trajectories under no load(discontinuous)

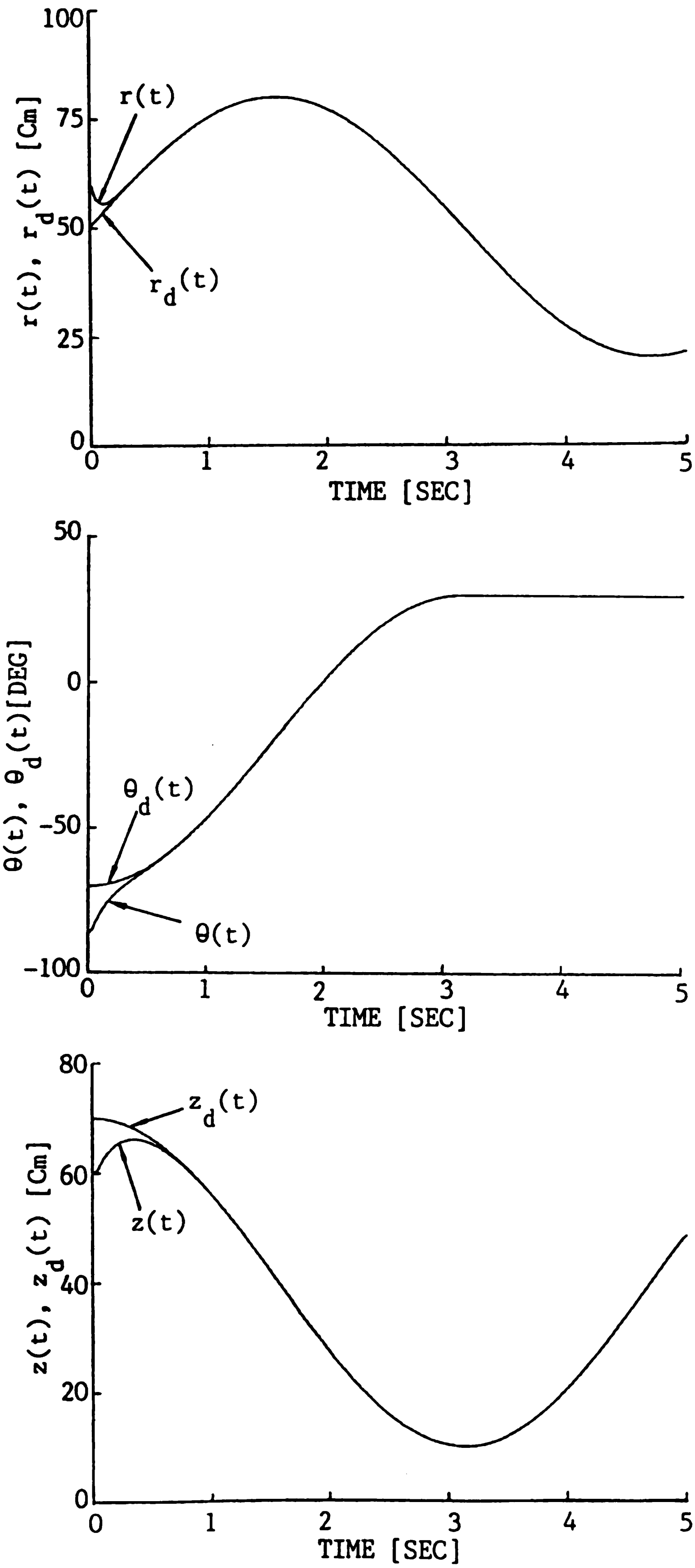

Figure 12. State trajectories under full load(discontinuous)

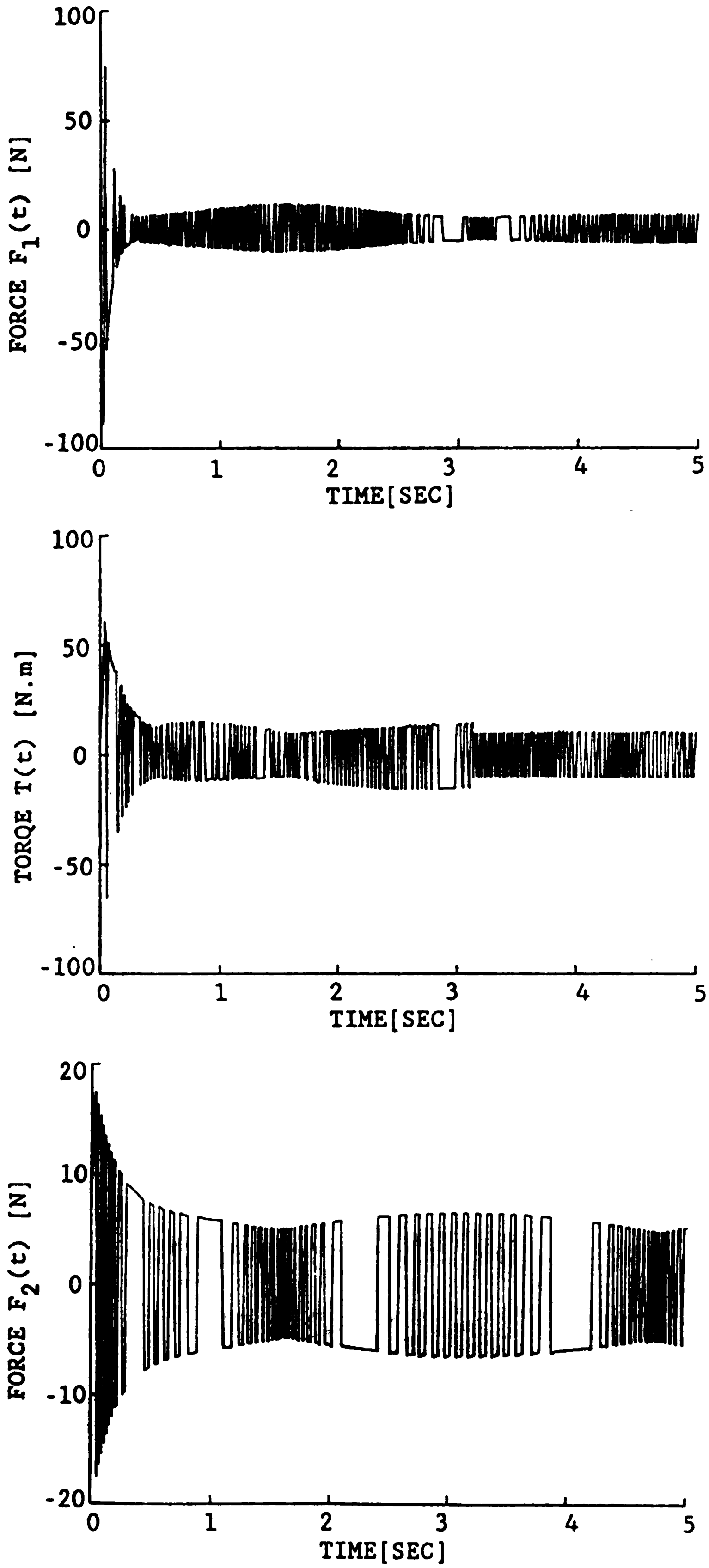

Figure 13. Discontinuous control efforts under no load

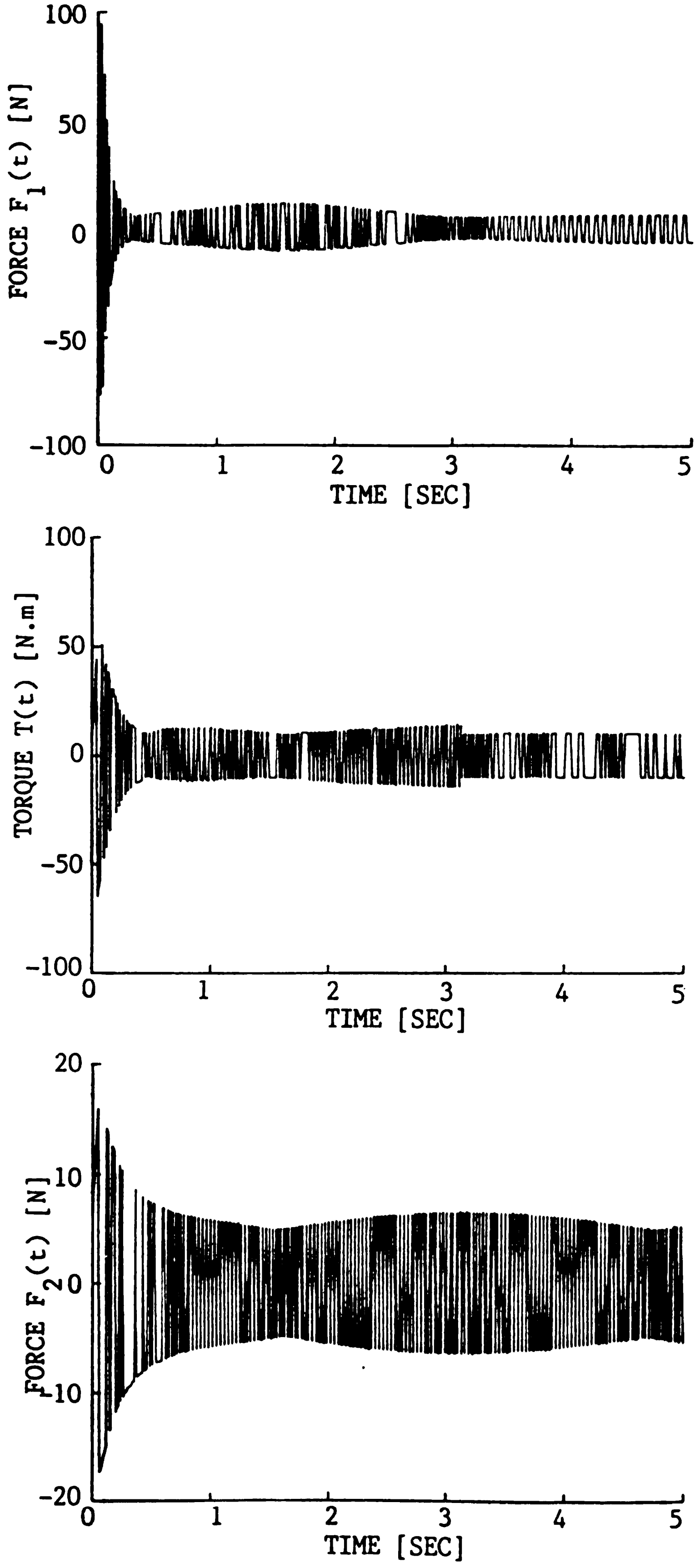

Figure 14. Discontinuous control efforts under full load

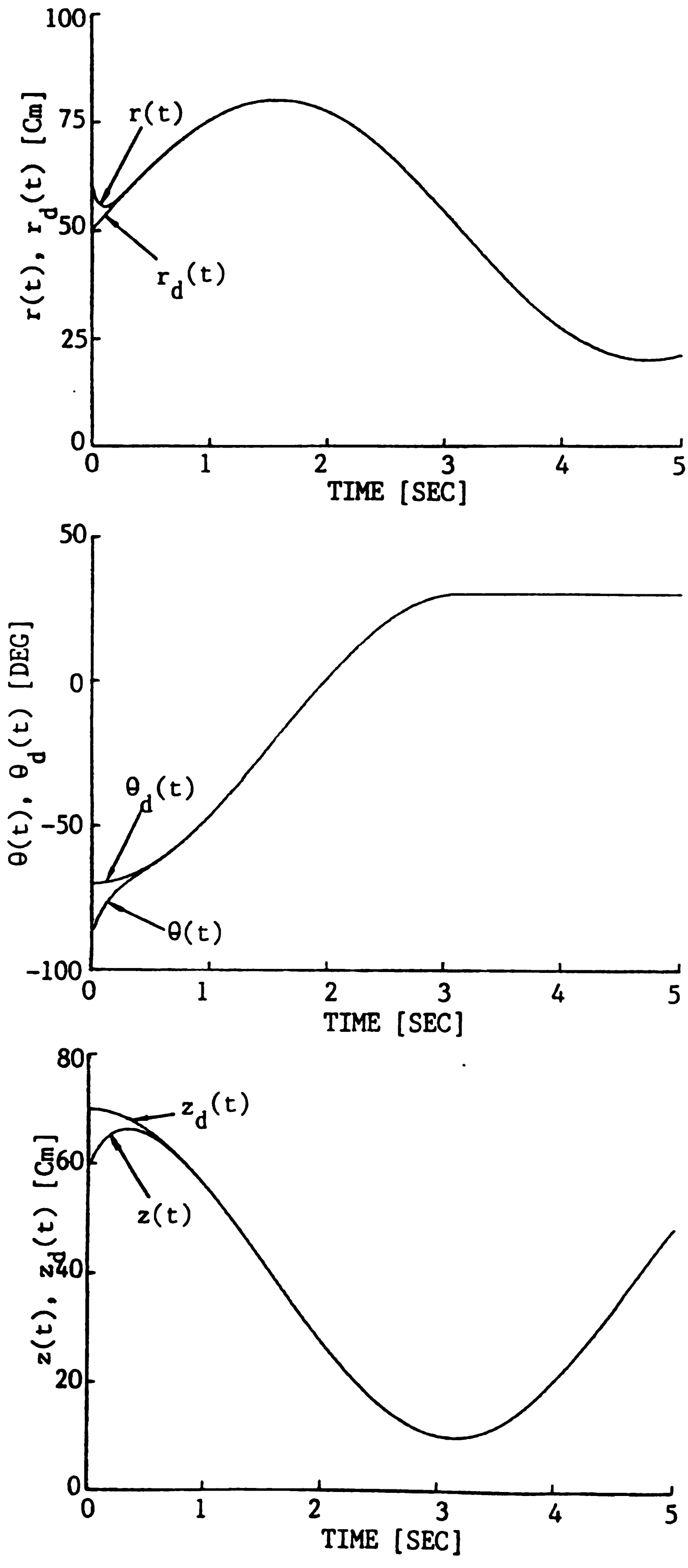

Figure 15. State trajectories under no load(continuous)

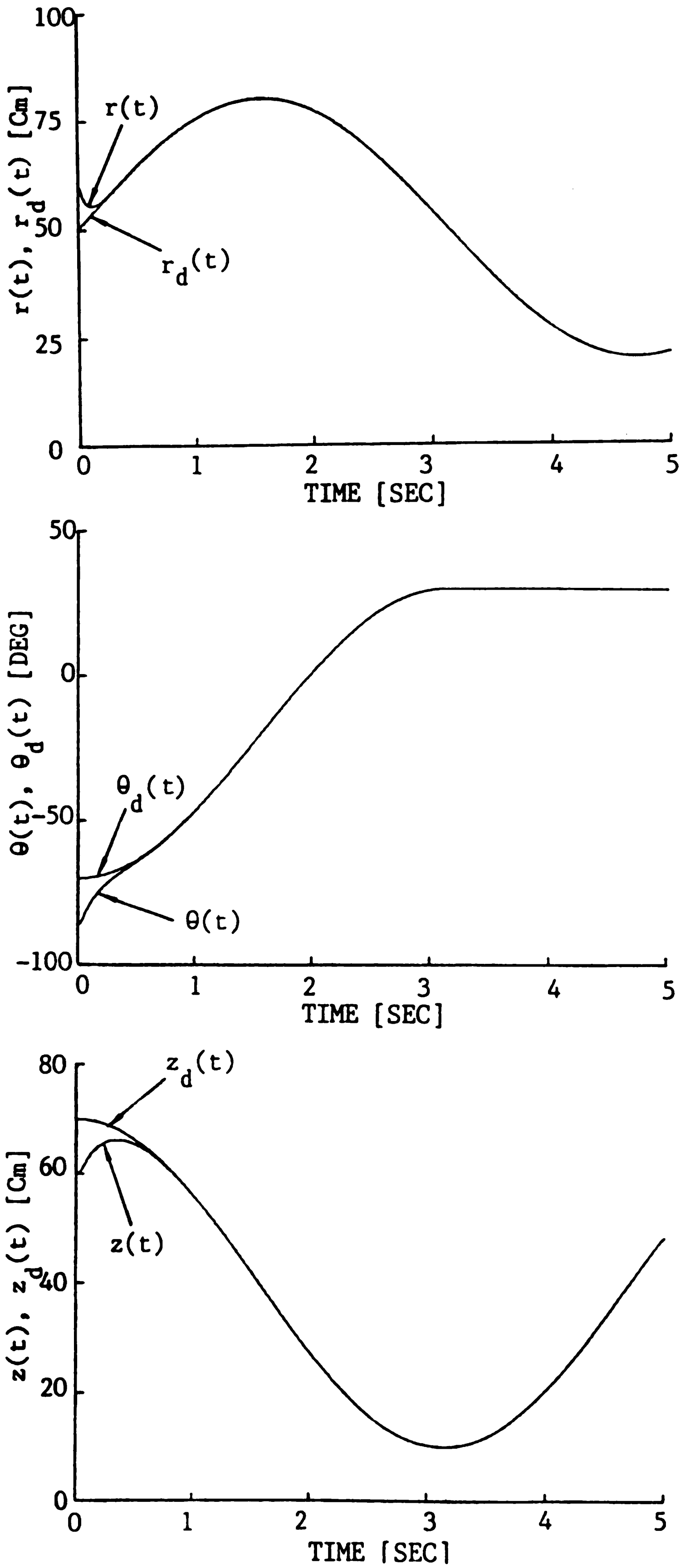

Figure 16. State trajectories under full load(continuous)

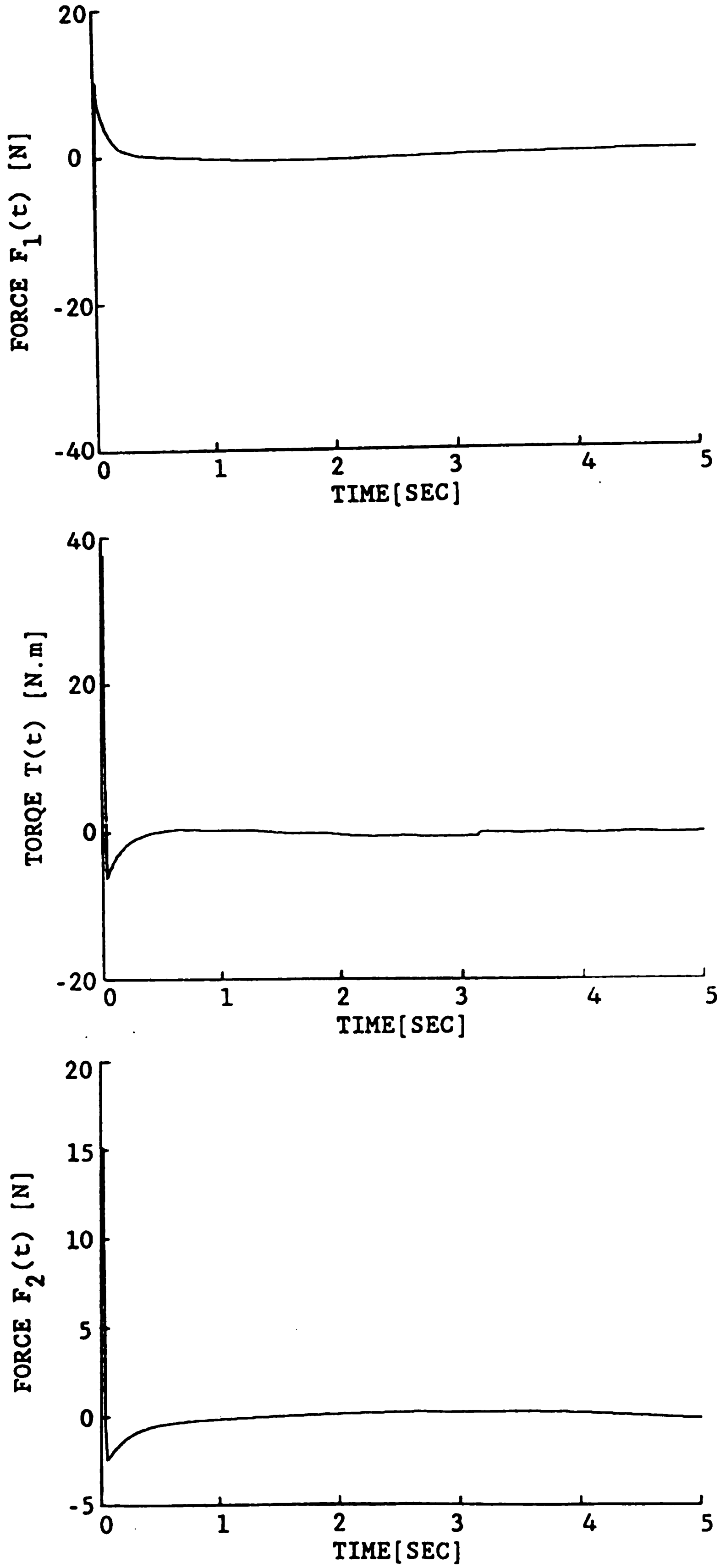

Figure 17. Continuous control efforts under no load

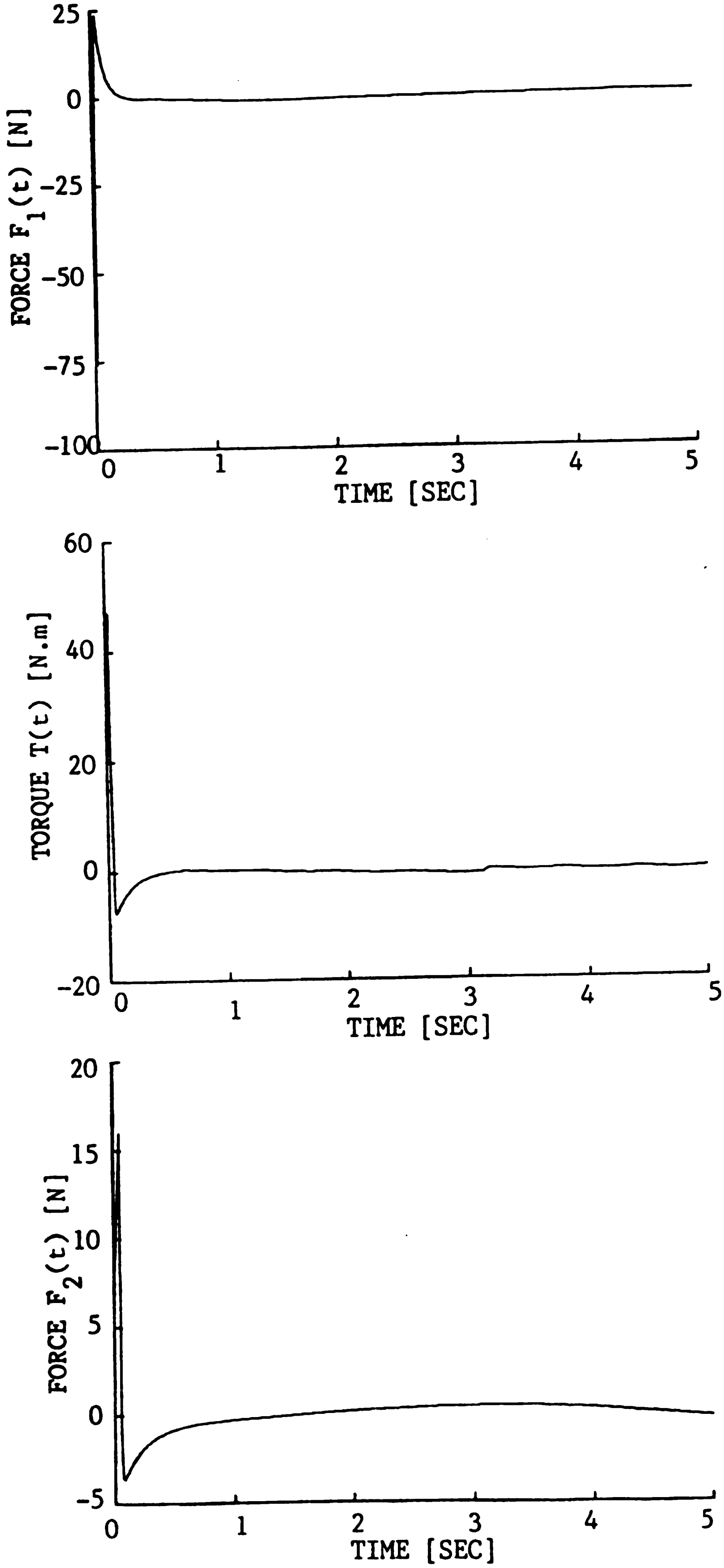

Figure 18. Continuous control efforts under full load

CHAPTER V

CONCLUSIONS

Based on sliding conditions, variable structure control laws for both single and multi-input systems in the presence of uncertainties were derived. The variable structure controllers provide compensations that eliminate dynamic interactions by introducing sliding modes. By ensuring sliding mode motion on the switching surfaces, the robustness to parameter variations and disturbances was achieved. Compared to the state of the art of involved compensations, the proposed control strategies are much easier to construct and require less knowledge of the physical parameters of the systems since only inequality of sliding conditions are to be satisfied in the design process.

It is obvious that the sliding mode controller generates a discontinuous control signal that changes sign rapidly similar to pulse amplitude signals. To eliminate adverse effects of such control efforts on physical hardware, we have approximated discontinuous controllers by smooth ones inside the boundary layer. It was shown by simulation that the proposed sliding mode feedback controllers are very effective for an industrial manipulator handling variable payloads. It will be useful to apply these control schemes to more complicated robot manipulators.

We selected gradient vectors C_i, boundary layer widths ϵ_i and gains k_i, which are vital for the success of our methodology in an ad-hoc way. As we have seen, these values affect tracking accuracy and the magnitude of control discontinuities. Formally selecting these parameters in an optimal manner remains an open research issue. Further research should include the effects of process and measurement noise on the sliding mode control laws, the sensitivity to inaccuracy in the implementation, the relaxation of matching conditions, and the design of observers arising when some of the states are not available.

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