THE INFLUENCE OF HEAT TRANSFER ON THE STABILITY OF PARALLEL FLOW BETWEEN CONCENTRIC CYLINDERS

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ABSTRACT

THE INFLUENCE OF HEAT TRANSFER ON THE STABILITY OF PARALLEL FLOW BETWEEN CONCENTRIC CYLINDERS

By

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The stability of parallel flow between concentric cylinders at different temperatures is investigated for infinitesimal velocity and pressure disturbances. Primary interest is in the effect of heat transfer and the radius ratio of the inner to outer cylinder on the critical point of the neutral stability curve. The modified Orr-Sommerfeld equation includes perturbations for the velocity and viscosity as well as its gradient terms, and results in a fourth order ordinary differential equation. This equation and boundary conditions represents an eigenvalue problem that is transformed to an equivalent initial value problem and solved numerically using a fourth order Runge-Kutta integration scheme.

The results indicate a strong dependence of the critical eigenvalues on both the heat transfer and the radius ratio a/b of inner to outer cylinder. Heating the inner cylinder destabilizes the flow. The critical Reynolds number of an isothermal flow monotonically decreases supporting the accepted conclusion that the critical Reynolds number becomes infinitely large as a/b

approaches zero. The critical Reynolds number of the non-isothermal flow seems to approach a finite value by showing an inflection point on the curve R_{crit} vs a/b near the radius ratio a/b=0.4. For the radius ratio a/b=0.4 the upper branch of the neutral stability curve (α vs. Reynolds number) of the non-isothermal flow appears to have a vertical, or near vertical, asymptote rather than the horizontal asymptote of the isothermal flow.

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NOMENCLATURE

a	Radius of inner cylinder, cm
a_1, a_2	Coefficients used to obtain combined eigenfunctions
b	Radius of outer cylinder, cm
$c = c_r + ic_i$	Complex propagation speed of wave
c_r	Wave speed
c_i	Amplification rate
C_i	Constant coefficients used in the theoretical expression of viscosity where $i = 1, 2, \ldots, 10$
c_p	Specific heat at constant pressure, cal/g degC
D_i	Constant coefficients used in the theoretical expression of conductivity where $i = 1, 2, \ldots, 10$
E_{ck}	Eckert number = $\dfrac{W_m^2}{c_p\|T_i - T_o\|}$
F	Test function
i	$\sqrt{-1}$
k	Thermal conductivity (Dimensionless)
k^*	Thermal conductivity, cal/s cm degC
k_o	Thermal conductivity at a selected base temperature, cal/s cm degC
p	Pressure
P_r	Prandtl number = $\mu_o c_p/k_o$

r	Radius (Dimensionless)
r^*	Radius, cm
R	Reynolds number = $\rho(b-a)W_m/\mu_o$
ΔR	Finite change in R
t	Time (Dimensionless)
t^*	Time, s
T	Temperature (Dimensionless)
T^*	Temperature, °C
T(I,1)	Temperature at node I when both dk/dr and Φ are neglected
T(I,2)	Temperature at node I when dk/dr term is kept, but Φ is neglected
T(I,3)	Temperature at node I when dk/dr term is neglected, but Φ is kept
T(I,4)	Temperature at node I when both dk/dr and Φ are kept
u	Radial velocity
v	Circumferential velocity
v_r	Dimensional radial velocity, cm/s
v_z	Dimensional axial velocity, cm/s
v_θ	Dimensional circumferential, velocity, cm/s
w	Axial velocity
W	Velocity of main flow
W(I,1)	Velocity of main flow at node I when both dk/dr and Φ are neglected
W(I,2)	Velocity of main flow at node I when both dk/dr is kept, but Φ is neglected

W(I,3)	Velocity of main flow at node I when dk/dr is neglected, but Φ is kept
W(I,4)	Velocity of main flow at node I when both dk/dr and Φ are kept
W_m	Mean velocity, cm/s
z	Coordinate in axial direction (Dimensionless)
z*	Coordinate in axial direction, cm
α	Wave number in flow direction
$\Delta\alpha$	Finite change in
λ	A constant representing R $\frac{dP}{dz}$
μ	Viscosity (Dimensionless)
μ^*	Viscosity, g/cm s
μ_o	Viscosity at a selected base temperature, g/cm s
ρ	Density, g/cm^3
ϕ	Complex eigenfunction of velocity perturbations
Φ	Viscous dissipation function
Ψ	Stream function

Superscripts

( )'	Differentiation with respect to r or perturbation quantities

Subscripts

$(\ )_i$	Quantities on inner cylinder
$(\ )_o$	Quantities on outer cylinder
$(\ )_{crit}$	Critical eigenvalues

CHAPTER I

INTRODUCTION

1.1 Review of Literature

The phenomenon of transition from laminar to turbulent flow which is fundamental for the science of fluid mechanics was first investigated experimentally by Reynolds (1883). He discovered in his pipe flow experiments that transition from laminar to turbulent flow always occured at nearly the same Reynolds number (critical Reynolds number).

Efforts to clarify and to explain theoretically the process of transition were done notably by Lord Rayleigh (1880, 1887). Based upon this work, independent studies by Orr (1907) and Sommerfeld (1908) in which disturbances of the form $\phi(y)\exp[i\alpha(x-ct)]$ were imposed on the main flow led to the now well known Orr-Sommerfeld stability equation for two-dimensional plane flow, which is given by

$$(W-c)\ (\phi''-\alpha^2\ ) - W''\phi\ = -\frac{i}{\alpha R}\ (\phi^{iv} - 2\alpha^2\phi'' + \alpha^4\phi\ )$$

where W = Velocity of the main flow

$c = c_r+ic_i$

= Complex wave propagation speed (c_r is the velocity of propagation of the imposed disturbance and c_i is the amplification rate of the disturbance)

α = Wave number of the imposed disturbance

ϕ = Eigenfuntion (amplitude funtion)

R = Reynolds number.

Theoretical investigations are based on the assumption that laminar flows may be unstable to certain small disturbances. The behavior of such disturbances is followed in time after they are superimposed on the main flow which remains to be determined in particular cases. If the disturbances decay with time (c_i<0), the main flow is considered stable. On the other hand, if the disturbances increase with time (c_i>0) the flow is considered unstable, and there exists the possibility of transition to either a secondary laminar flow or a turbulent flow.

Neglecting the effects of viscosity, Lord Rayleigh (1914) was able to show that any velocity profile that possesses an inflection point is unstable. Much later, Tollmein (1935) proved that this was not only a necessary but sufficient condition for the occurrence of instability.

Prandtl (1914) postulated the existence of a viscous boundary layer and was able to define transition, separation and drag coefficients on bodies. Incorporating the viscous boundary layer into stability theory, Prandtl (1912) considered flow over a flat plate and included the effects of the largest viscous terms near the wall. This work along with calculations performed by Tietjens (1925) gave the startling result that the introduction of viscosity into the equations did not produce damping as was presumed but amplification for sufficiently large Reynolds numbers for particular wavelengths of the disturbances. This result was obtained not only for unstable velocity profiles but also the profiles which are stable when viscosity was neglected.

Tollmein (1929) demonstrated that the effect of viscosity must be taken into account not only near the wall but also in the critical layer, a narrow region surrounding the critical point at which the main flow velocity and the wave propagation velocity are equal, that is, $W = c_r$. In addition, he also showed that the influence of viscosity leads to instability only if the main flow velocity profile is other than a straight line.

The method developed by Tollmein, based on asymptotic theory, provided the mathematical basis for later progress in the stability area. Lin (1945, 1946, 1955) was able to provide a firm mathematical basis for the asymptotic expansion theory and was able to explain the nature of the functions near the critical point. He discussed what he called the inner viscous layer which includes the critical point, and the outer viscous layer, a wall layer.

The asymtotic expansion method was used almost exclusively until the advent of modern high-speed digital computers which permit the use of more accurate numerical techniques.

The stability of plane Poiseuille flow was investigated by Thomas (1953) with a numerical scheme. He obtained a value for the minimum Reynolds number, R_{crit}, for neutral stability of 5780, which is based on maximum channel velocity and the half-width. This value has been shown to be more accurate than Lin's (1945) value of 5300 or Stuart's (1954) value of 5100 based on asymptotic techniques.

Potter (1965) studied the stability of plane Couette-Poiseuille flow by asymptotic expansions and later (1967) performed numerical calculations for symmetrical parabolic flows. The values obtained for R_{crit} were in close agreement with those of Thomas.

The point of instability as determined theoretically and the point of transition, as observed in experiment, to turbulent flow often differ considerably. An explanation for these differences was thought by some to be due to the fact that the derivation of the Orr-Sommerfeld equation is based on the assumption of two-dimensional disturbances only. Squire (1933) showed that if three-dimensional disturbances are considered the flow is more stable, thus attention is typically limited to two-dimensional disturbances.

The distance between the point of instability and the point of actual transition depends upon the degree of amplification present and the intensity of fluctuations present in the main flow. But the actual mechanism of amplification can be obtained from the study of the magnitudes of the parameters in the interior of the curve of neutral stability on which $c_i = 0$. Calculations of this kind were first performed by Schlichting (1933) for the flat plate. More recently, Shen (1954) repeated Schichting's calculations, and Stuart (1956) investigated the amplification of unstable disturbances by accounting for the first order effect of the non-linear terms in the equations.

Emmons (1951) observed that any disturbance which triggers transition may be "local in time" and once initiated, the turbulent spot moves downstream growing steadily in all directions. This phenomenon was studied by Schubauer and Klebanoff (1955, 1956). They indicated that there is no well defined point of transition but that the process of transition from laminar to fully developed turbulent flow extends over a finite distance.

Transition from laminar to turbulent flow in a boundary layer is now believed to take place within 4 stages. At the first stage, infinitesimal two-dimensional waves called Tollmein-Schlichting waves, begin to amplify and become unstable. The two-dimensional waves become three dimensional and result in hairpin eddies at the second stage. In the third stage low speed turbulent streaks or bursts (Emmons' spots) originate near the wall, and finally in the fourth stage the burst rate becomes constant and the transition to fully turbulent motion is completed. Morkovin (1958) reviews some of the recent advances in the study of transition and discussed the mechanisms involved in the above mentioned stages.

Stability theory yields a critical Reynolds number that corresponds to stage one. Since the third stage is the first point at which large scale variations take place, this is often considered to be transition by engineers. These differences along with the slower response times of earlier instrumentation serve to explain some of the discrepancies between theory and experiment.

Stability predictions in channel flow yield critical Reynolds numbers that also correspond to infinitesimal disturbances but the stages of transition are not as apparent as in boundary layer flow. Free stream disturbances or disturbances which result from wall roughness amplify and lead to the transition described above but the effect is propagated throughout the flow and the entire channel becomes turbulent.

Barnes and Coker (1905) confirmed experimentally that the critical Reynolds number for a pipe flow increases as the disturbances in the flow before the pipe are decreased. Ekman (1910) succeeded in maintaining laminar pipe flow up to a critical Reynolds number of 40000 by providing an inlet which was made exceptionally free from disturbances. The upper limit to which the critical Reynolds number can be driven with extreme care is not known at present.

Transition process of a pipe flow was investigated in detail by Rotta (1956). His measurements reveal that in a certain range of Reynolds numbers around the critical the flow becomes intermittent which means that it alternates in time between being laminar and turbulent. Recently Gill (1965) performed an exhaustive analysis of the stability problem of a pipe flow and found the flow to be always stable to axisymmetric disturbances. Davey and Drazin (1969) solved the equations for the axisymmetric case and concluded that the flow is stable to all such disturbances.

The stability of a flow over heated and cooled flat plates were investigated by Wazzan, Okamura, and Smith (1960). They found that the stability of water flows to Tollmien-Schlichting waves is extremely sensitive to initial heating or cooling. They also indicated that in the case of a heated wall, terms involving the first and second derivatives of viscosity in the modified Orr-Sommerfeld equation are found to have a considerable destabilizing effect.

Mott and Joseph (1968) investigated the linear stability of parallel flow in concentric cylinders by solving the stability equation in which the viscous gradient terms were neglected. The critical Reynolds number of an isothermal flow was found to be a monotone function of the radius ratio of the outer to inner cylinder, increasing from the plane Poiseuille flow limit to the Hagen-Poiseuille flow limit. The neutral curves for skewed profiles caused by heating the inner cylinder were found to have a second minimum, which for sufficiently skewed profiles, gave the lowest value of the Reynolds number. They also found that heating resulted in a more stable flow. Potter and Graber (1972) investigated the stability of plane Poiseuille flow with heat transfer. The study indicated that inclusion of viscosity gradient terms, even though they are small, is very important. It also indicated that a double critical point does not appear in the neutral stability curve when the viscosity gradient terms are included in the equation in contrast with the result of Mott and Joseph (1968). Finally, their results indicate that heating one of the plates results in a more unstable flow.

1.2 Purpose of the Present Study

The purpose of the present study is to investigate the stability of a parallel flow of water between concentric cylinders to infinitesimal axially symmetric disturbances under the influence of heat transfer. The main flow is affected by the heat transfer between cylinders of different temperatures through the variation in viscosity and conductivity with temperature since the viscosity and conductivity of water are quite sensitive to temperature. The viscosity gradient subsequently causes additional terms to appear in the stability equation whose solution will be sought in the present study. The effect of radius ratios will also be investigated by changing the inner radius while the outer radius is fixed.

The main flow (fully developed laminar flow) is examined, using the energy and momentum equations with the values for water given by Poots and Rogers (1965) for the viscosity and conductivity. They are taken from the experimental data in tables compiled by Mayhew and Rogers (1964).

The differences between this study and previous work done by Mott and Joseph (1968) are that in this investigation all the viscous gradient terms which appear due to the heat transfer are included in the modified Orr-Sommerfeld equation and the complete energy and momentum equations are solved to determine the main flow.

Non-linear stability theories typically involve methods which use the eigenvalues from the linear theory as an imput. Such theories have not been applied to pipe flow since for the isothermal case the flow is always stable. Heat transfer has been shown to destabilize the flow. Thus, it is of interest to investigate the flow between concentric cylinders as the inner cylinder is heated and reduced in diameter. As the diameter of the inner cylinder is decreased to zero, the flow may be unstable at a finite Reynolds number if the heating is sufficient. If it is, then eigenvalues can be found for a pipe flow.

CHAPTER II

FORMULATION OF THE PROBLEM

2.1 Governing Equations

For incompressible axially symmetrical flow (i.e. all variables are independent of θ and $v_\theta = 0$) with variable properties the equations of motion in cylindrical coordinates, using dimensional velocity components v_r and v_z and other dimensional variables indicated by astericks, are

$$\rho \frac{Dv_r}{Dt^*} = -\frac{\partial p^*}{\partial r^*} + \frac{\partial}{\partial r^*}\left[\mu^*\left(2\frac{\partial v_r}{\partial r^*}\right)\right] + \frac{\partial}{\partial z^*}\left[\mu^*\left(\frac{\partial v_r}{\partial z^*} + \frac{\partial v_z}{\partial r^*}\right)\right] + \frac{2\mu^*}{r^*}\left(\frac{\partial v_r}{\partial r^*} - \frac{v_r}{r^*}\right) \tag{2.1.1}$$

$$\rho \frac{Dv_z}{Dt^*} = -\frac{\partial p^*}{\partial z^*} + \frac{\partial}{\partial z^*}\left[\mu^*\left(2\frac{\partial v_z}{\partial z^*}\right)\right] + \frac{1}{r^*}\frac{\partial}{\partial r^*}\left[\mu^* r^*\left(\frac{\partial v_r}{\partial z^*} + \frac{\partial v_z}{\partial r^*}\right)\right] \tag{2.1.2}$$

where $\frac{D}{Dt^*} = \frac{\partial}{\partial t^*} + v_r\frac{\partial}{\partial r^*} + v_z\frac{\partial}{\partial z^*}$.

The equation of continuity for the same flow is

$$\frac{1}{r^*}\frac{\partial}{\partial r^*}(v_r\, r^*) + \frac{\partial}{\partial z^*}(v_z) = 0. \qquad (2.1.3)$$

For incompressible, steady, parallel flow ($v_z = v_z(r^*)$, $v_r = v_\theta = 0$), and constant boundary temperature the energy equation can be written as

$$\frac{1}{r^*}\frac{d}{dr^*}\left(r^* k^* \frac{dT^*}{dr^*}\right) + \mu^* \left(\frac{dv_z}{dr^*}\right)^2 = 0. \qquad (2.1.4)$$

To write the equations in non-dimensional form, choose the average velocity W_m, the difference between outer and inner radii (b-a), the difference between the temperatures on the inner and outer cylinders (T_o-T_i), and viscosity (μ_o) and conductivity (k_o) at a constant temperature, as reference quantities. The dimensionless variables will then be

$$r = \frac{r^*-a}{b-a} \qquad z = \frac{z^*}{b-a}$$

$$T = \frac{T^*-T_i}{T_o-T_i} \qquad p = \frac{p^*}{\rho W_m^{\;2}} \qquad (2.1.5)$$

$$\mu = \frac{\mu^*}{\mu_o} \qquad k = \frac{k^*}{k_o}$$

$$u = \frac{v_r}{W} \qquad w = \frac{v_z}{W_m}$$

$$t = \frac{t^* W_m}{b-a}$$

Introducing these quantities in the governing equation yields

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z}$$

$$= - \frac{\partial p}{\partial r} + \frac{1}{R}\left[\frac{\partial}{\partial r}\left(2\mu\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial z}\left\{\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)\right\}\right.$$

$$\left. + \frac{2\mu}{a_1+r}\left(\frac{\partial u}{\partial r} - \frac{u}{a_1+r}\right)\right] \qquad (2.1.6)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z}$$

$$= - \frac{\partial p}{\partial z} + \frac{1}{R}\left[\frac{\partial}{\partial z}\left(2\mu\frac{\partial w}{\partial z}\right) + \frac{1}{a_1+r}\frac{\partial}{\partial r}\left\{\mu(a_1+r)\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)\right\}\right] \qquad (2.1.7)$$

$$\frac{1}{a_1+r}\frac{\partial}{\partial r}\left[u\{a_1+r\}\right] + \frac{\partial w}{\partial z} = 0 \qquad (2.1.8)$$

$$\frac{1}{a_1+r}\frac{d}{dr}\left[\{a_1+r\}\,k\frac{dT}{dr}\right] + \mathrm{sgn}\,(T_o-T_1)P_rE_{ck}\mu\left(\frac{dw}{dr}\right)^2 = 0 \qquad (2.1.9)$$

where R = Reynolds number = $\dfrac{\rho(b-a)W_m}{\mu_o}$

P_r = Prandtl number = $\dfrac{\mu_o\ c_p}{k_o}$

E_{ck} = Eckert number = $\dfrac{W_m{}^2}{c_p|T_i-T_o|}$

$a_1 = \dfrac{a}{b-a}$

For the variable viscosity μ and conductivity k in the equations the theoretical expressions of Poots and Rogers (1965) are employed:

$$\mu = \prod_{i=0}^{10} \exp\,(C_i \Theta^i) \qquad (2.1.20)$$

$$= \prod_{i=0}^{10} \exp\,(D_i \Theta^i) \qquad (2.1.11)$$

where $\Theta = \dfrac{(T_i - 5\Omega) + (T_o - T_i)T}{5\Omega}$

and C_i and D_i are constants displayed in Table 42. These values agree with experimental data within 0.2%.

2.2 The Governing Equations of the Main Flow

The problem shall be simplified by stipulating that the main flow velocity W depends only on r, whereas the remaining two components are assumed to be zero everywhere. The equation of motion for such a parallel flow is

$$\frac{1}{a_1+r}\frac{d}{dr}\left[\mu\,\{a_1+r\}\frac{dW}{dr}\right] = R\frac{dP}{dz} \qquad (2.2.1)$$

where $R\frac{dP}{dz} = \lambda$ is a constant, P being the pressure in the main flow. The boundary conditions for equation (2.2.1) are

$$W = 0 \text{ @ } r = 0 \text{ and } r = 1. \qquad (2.2.2)$$

The energy equation for the main flow with constant temperature on the boundary is as given by equation (2.1.9). The boundary conditions are

$$T = 0 \quad @ \quad r = o$$

$$T = 1 \quad @ \quad r = 1. \tag{2.2.3}$$

The mean velocity, by which the velocities are non-dimensionalized, of the main flow is defined as

$$W_m \equiv \frac{\int_a^b v_z 2\pi r^* dr^*}{\pi(b^2 - a^2)} \tag{2.2.4}$$

Expressing v_z and r^* in terms of the non-dimensional variables defined in equation (2.1.5) and then substituting them in equation (2.2.4) yields the following constraining equation:

$$\int_o^1 W\,[a_1 + r\,]\,dr = a_1 + \frac{1}{2} \tag{2.2.5}$$

These equations with boundary conditions will be numerically solved by iteration. Iteration is necessary since the viscosity in equation (2.2.1) is not known until the temperature is known. Conversely, the temperature cannot be determined from equation (2.1.9) until the velocity $W(r)$ is known. A brief description of the iterative process is presented in Section 3.1.

2.3 The Linearized Equations for Small Disturbances

It was assumed that the main flow is a parallel flow. Hence, the perturbed dependent variables can be written as

$$
\begin{aligned}
u &= u' \\
v &= v' \\
w &= W + w' \\
p &= P + p'
\end{aligned}
\tag{2.3.1}
$$

neglecting temperature, conductivity and viscosity perturbation. W and P are the velocity and pressure of the undisturbed main flow, respectively.

Substitution of (2.3.1) in the equations of motion yields the first order equations for the small disturbances (u', v', w', and p' are assumed small in the sense that all the quadratic terms in the fluctuating components may be neglected with respect to the linear terms):

$$
\frac{\partial u'}{\partial t} + W\frac{\partial u'}{\partial z} = \frac{-\partial p'}{\partial r} + \frac{1}{R}\{\frac{\partial}{\partial r}(2\mu\frac{\partial u'}{\partial r})
+ \frac{\partial}{\partial z}[\mu(\frac{\partial u'}{\partial z} + \frac{\partial w'}{\partial r})] + \frac{2\mu}{r}(\frac{\partial u'}{\partial r} - \frac{u'}{r})\}
\tag{2.3.2}
$$

$$
\frac{\partial w'}{\partial t} + \frac{dW}{dr}u' + W\frac{\partial w'}{\partial z} = -\frac{\partial p'}{\partial z} + \frac{1}{R}\{\frac{\partial}{\partial z}(2\mu\frac{\partial w'}{\partial z})
+ \frac{1}{r}\frac{\partial}{\partial r}[\mu r(\frac{\partial u'}{\partial z} + \frac{\partial w'}{\partial r})]\}
\tag{2.3.3}
$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ru') + \frac{\partial w'}{\partial z} = 0 \tag{2.3.4}$$

where $r = \frac{r^*}{b-a}$.

With axisymmetric disturbances the motion may be represented in terms of a stream function such that

$$\begin{aligned} u' &= -\frac{1}{r}\frac{\partial}{\partial z}(r\Psi) \\ w' &= \frac{1}{r}\frac{\partial}{\partial r}(r\Psi) \end{aligned} \tag{2.3.5}$$

with which the equation of continuity is automatically satisfied.

Consider further that the stream function takes the separated form

$$\Psi = -i\phi(r) \exp [\ i\alpha\ (z-ct)] \tag{2.3.6}$$

With this expression we assume the disturbance to be a sinusoidal function of time and position. The quantity α is a dimensionless wave number and is a real. The complex wave speed c is given by

$$c = c_r + i\ c_i \tag{2.3.7}$$

where c_i is the amplification rate. A positive or negative c_i implies growth or decay respectively of the perturbation. This study is concerned with neutral stability, that is, $c_i = 0$. The function $\phi(r)$ may be complex.

Eliminating the terms containing p' in equations (2.3.2) and (2.3.3) by cross differentiation, then introducing the assumed form of the disturbances into equations (2.3.2) - (2.3.4) to separate the variables, yields the following modified Orr-Sommerfeld equation, and ordinary differential equation,

$$(W-c)L\phi \quad - \quad r\frac{d}{dr}\left(\frac{1}{r}\frac{dW}{dr}\right)\phi$$

$$= -\frac{i}{\alpha R}\Big[\ \mu(L^2\phi) \ + \ \frac{d\mu}{dr}\Big\{2\frac{d^3\phi}{dr^3} \ + \ \frac{3}{r}\frac{d^2\phi}{dr^2} \ - \left(\frac{3}{r^2} + 2\alpha^2\right)\frac{d\phi}{dr}$$

$$+ \left(\frac{3}{r^3} - \frac{\alpha^2}{r}\right)\phi\Big\} + \ \frac{d^2\mu}{dr^2}\left(L\phi + 2\alpha^2\phi\right)\ \Big] \qquad (2.3.8)$$

where

$$L = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2} - \alpha^2$$

$$L^2 = \frac{d^4}{dr^4} + \frac{2}{r}\frac{d^3}{dr^3} + \left(-\frac{3}{r^2} - 2\alpha^2\right)\frac{d^2}{dr^2}$$

$$+ \left(\frac{3}{r^3} - \frac{2\alpha^2}{r}\right)\frac{d}{dr} + \left(-\frac{3}{r^4} + \frac{2\alpha^2}{r^2} + \alpha^4\right)$$

and W is the solution of the governing equations presented in Section 2.2.

The necessary boundary conditions result from the no-slip velocity conditions on the inner and outer cylinders and from continuity. They are

$$\phi = \phi' = 0 \ @ \ r = a_1 \text{ and } a_1+1 \qquad (2.3.9)$$

2.4 Eigenvalue Problem

The differential equation and boundary conditions derived in the previous section represent an eigenvalue problem with the eigenvalues α, c, and R. The wave propagation speed, c as stated earlier is complex, with the imaginary part being the exponential growth or decay rate of the assumed disturbances. Since we are interested in studying, neutral stability, c_i is set to zero.

To solve for the eigenvalues it is necessary to specify one of them, say c_r and solve for the remaining two, namely α and R from the complex equation (2.3.8). Setting $c_i = 0$, the eigenvalues are found and then plotted as shown in Fig. 1. The resulting curve is called a neutral stability curve.

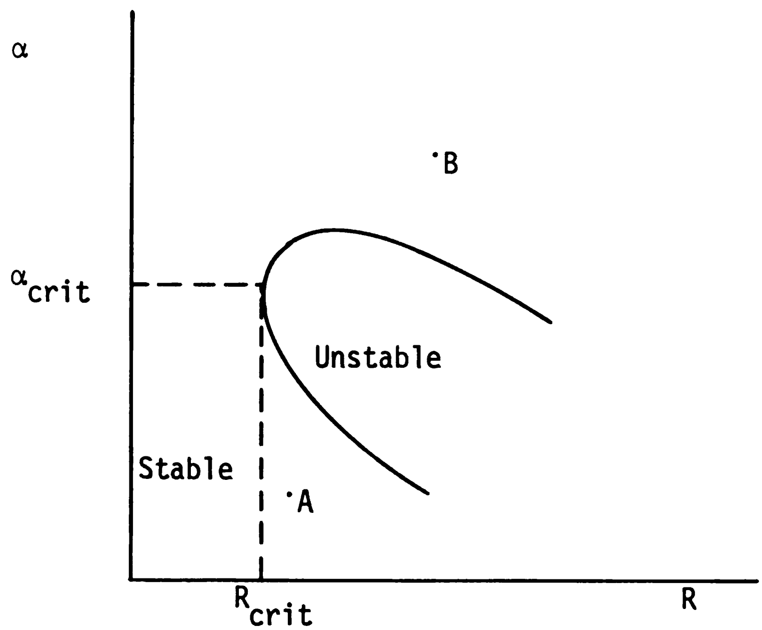

Figure 1.--A Neutral Stability Curve

The neutral stability curve will have a minimum R, called the critical Reynolds number R_{crit}, for which a flow is neutrally stable. Reynolds numbers greater than R_{crit} result in growth for that particular flow and Reynolds numbers smaller than R_{crit} result in decay. It is assumed that all wave numbers are present in any real flow. Thus, it is assumed that a stable flow at point A or point B is not possible, It is only possible to have a stable (laminar) flow below R_{crit}. Associated with R_{crit} there is also a critical wave speed $c_{r_{crit}}$, and a critical wave number α_{crit}.

CHAPTER III

NUMERICAL METHODS

3.1 Numerical Solution of the Governing Equations of the Main Flow

The energy equation (2.1.9) and the equation of motion (2.2.1), which is subject to the constraint (2.2.5), with boundary conditions (2.2.2) and (2.2.3) are solved by an iterative method. The numerical integration of the above equations are performed by using the fourth order Runge-Kutta method.

The iteration procedure which is used to solve the governing equations of the main flow through the concentric cylinders with variable $\mu(T)$ and $k(T)$ is as follow:

(1) Find an initial guess of the temperature distribution from $T(r) = \frac{1}{\ln(b/a)} \ln \left[1 + \frac{r}{a_1} \right]$ which is the solution of the energy equation when the conductivity is constant and the viscous dissipation is negligible.

(2) Find an approximation to viscosity μ and conductivity k using the following expression given by Poots and Rogers (1965):

$$\mu(r) = \prod_{i=0}^{10} \exp (C_i \Theta^i)$$

$$k(r) = \prod_{i=0}^{10} \exp (D_i \Theta^i)$$

where T(r) is included in the expression

$$\Theta(r) = \frac{(T_i - 50) + (T_o - T_i)\ T(r)}{50}$$

and C_i and D_i are given in Table 42.

(3) Find an approximation to the velocity of the main flow W by solving the equation of motion and the constraining equation:

(3a) Pick a value for λ and find W by solving the equation of motion

$$\frac{1}{a_1+r}\ \frac{d}{dr}\ [\mu\ (a_1+r)\ \ \frac{dW}{dr}\ ] = \lambda$$

with boundary conditions,

$$W = 0 \text{ @ } r = 0 \text{ and } r = 1$$

(3b) Check whether W satisfies the following constraint:

$$\int_0^1 W\ [a_1+r]dr - (a_1 + \tfrac{1}{2}) < \varepsilon$$

where ε is a preselected small number. If W does not satisfy the constraint, then pick a new value for λ and repeat the steps (3a), and (3b).

(4) Find a new approximation to the temperature, T(r), by solving the energy equation,

$$\frac{1}{a_1+r}\ \frac{d}{dr}\left[\ \{a_1+r\}\ k\,\frac{dT}{dr}\ \right]$$

$$+\ \mathrm{sgn}\ (T_o-T_i)\ \cdot\ P_r\ E_{ck}\ \mu\left(\frac{dW}{dr}\right)^2 = 0$$

with boundary conditions,

$T = 0$ @ $r = 0$

$T = 1$ @ $r = 1$

(5) Repeat steps (2) and (3) using the new $T_{(r)}$ of step (4).

(6) Check whether the velocity and temperature satisfy prescribed convergent criteria (e.g., max $|T_{new} - T_{old}| \leq 10^{-5}$). If they do they are the desired solutions. If they do not, then repeat the steps (2) through (6) until convergence is obtained.

3.2 Numerical Solution of the Stability Equation

The numerical solution to equations (2.3.8) will generate two independent solutions because the solutions are started at the inner cylinder with two boundary conditions. These solutions will not each satisfy the remaining boundary conditions at the outer cylinder. However, a proper linear combination of these functions will yield the eigenfunction which must satisfy the two boundary conditions at the outer cylinder.

The integration of the equations is begun at the inner cylinder and proceeds step by step across the annulus to the outer cylinder. A fourth order Runge-Kutta technique, detailed in Appendix A, is used to solve the equation. The two independent solutions are each initialized at the inner cylinder and integrated simultaneously step by step across the annulus.

To use the Runge-Kutta integration scheme to solve a differential equation of order n the problem must be transformed to an equivalent initial value problem where the function and its n-1 derivatives are initially specified. For equation (2.3.8), the boundary conditions (2.3.9) provide starting values for ϕ_i and ϕ_i' (i = 1 and 2 represent the independent solutions) with values for the remaining derivatives ϕ_i'' and ϕ_i''' arbitrarily chosen. The highest order derivatives, namely ϕ_i^{iv}, does not require initialization since they are determined in terms of the lower order derivatives from the describing differential equation. The arbitrary starting conditions mentioned above must be chosen so as to insure independent functions, at least at the start of the integration. Assigning a non-zero value to one of the two unspecified derivatives for each of the two solutions generally helps keep the solutions linearly independent.

The coefficient of the highest order derivative of the stability equation (2.3.8) is very small since it contains $(\alpha R)^{-1}$ which is of the order of 10^{-4} in the present problem. Hence, the equation is highly singular. There will be two linearly independent solutions satisfying the two boundary conditions on the inner cylinder, and an appropriate linear combination of these solutions will produce a solution satisfying the boundary conditions on both inner and outer cylinders.

It is well known that during the numerical integration one of the two independent functions for the fourth order problem grows much more rapidly than the other solution away from the starting boundary. Kaplan (1964) referred to the growing function as the "growing solution" and called the other the "well-behaved solution." The growing solution stems from the singular portion of the stability equations and the well behaved function originates from the inviscid portion.

The governing stability equations can be written as

$$(W-c)L\phi - r\frac{d}{dr}\left(\frac{1}{r}\frac{dW}{dr}\right)\phi = -\frac{i}{\alpha R}\Big[\mu(L^2\phi) + \frac{d\mu}{dr}\left\{2\phi''' + \frac{3}{r}\phi'' - \left(\frac{3}{r^2} + 2\alpha^2\right)\phi' + \left(\frac{3}{r^3} - \frac{\alpha 2}{r}\right)\phi\right\} + \frac{d^2\mu}{dr^2}\left(L\phi + 2\alpha^2\phi\right)\Big] \quad (3.2.1)$$

where L is a second-order operator and L^2 is a fourth-order operator defined in Section 2.3. The left hand side represents the inviscid part and the right hand side represents the viscous part which accounts for the singular behavior.

In this particular problem, the two solutions exhibited two distinct growth rates, the larger one corresponding to the growing solution, and the smaller one corresponding to the well-behaved solution. The fourth order differential equation results in only one growing solution which exhibited a growth on the order of 10^{18} across the annulus, which is in agreement with the observations reported by Reynolds and Potter (1967).

It is clear that generation of a second solution which is independent from the rapidly growing solution will be difficult. Any slight round-off error will in effect throw in some small multiple of the growing solution, which will likely dominate the second solution by the time the outer cylinder is reached.

The problem is to generate numerically a second solution which is not a multiple of the growing solution. Two approaches may be considered.

First, one might use double precision, extending the accuracy of the digital computations (to say, 24 digits). To reduce the truncation error associated with any numerical scheme, an obvious answer is to choose a step size that is as small as possible, consistent with the particular limitations of machine storage and speed. Performing all arithmetic operations in double precision should greatly reduce the error. This is in fact, found to be true. However, since all the functions are complex, adding double precision to the program significantly increases machine computation time and storage requirements. Therefore, this approach was not used for the present study.

A second approach, first used by Kaplan (1964), involves suppression of the growing solution during the calculation of the well-behaved solution. His method, that does not require double precision, consists of subtracting from the well-behaved solution a portion of the growing solution at every step of the integration. This procedure called "filtering" prevents the growing solution from ever dominating the well-behaved solution and thus maintains the needed functional independence.

Kaplan's scheme was implemented in the computer program for this problem. The CDC 6500 computer nominally carries 14 significant digits in single precision, which is equivalent to double precision on IBM equipment and in particular the IBM 7090 used by Kaplan. These computations were in effect then performed in "double precision."

The filter used consisted of a ratio of the inviscid solutions, the left hand side of equation (3.2.1), namely,

$$\text{Filter} = \frac{\text{Inviscid part of the well-behaved solution}}{\text{Inviscid part of the growing solution}} \tag{3.2.2}$$

where, $\text{Inviscid part} = i\alpha R\left[(W-c)L\phi - r\frac{d}{dr}\left(\frac{1}{r}\frac{dW}{dr}\right)\phi\right]$ (3.2.3)

The above ratio determines the fraction of the growing solution to be "extracted" from the behaved solution, so the amount subtracted off is the product of this ratio and the value of the growing solution at the particular integration step.

3.3 Iteration Scheme for the Eigenvalues

Integrating across the annulies with assmumed eigenvalues, two independent solutions are generated at each step. Upon reaching the opposite cylinder, the functions are linearly combined to form the total eigenfunction that must satisfy the boundary conditions. The two conditions that must be satisfied at the outer cylinder are $\phi = \phi' = 0$. Consider the following set of combined functions at the outer cylinder:

$$a_1\phi_{1o} + a_2\phi_{2o} = \phi_o \tag{3.3.1}$$

$$a_1\phi_{1o} + a_2\phi'_{2o} = \phi'_o \tag{3.3.2}$$

$$a_1\phi''_{1o} + a_2\phi''_{2o} = \phi''_o \tag{3.3.3}$$

Equations (3.3.1-3) are used to solve for the coefficients a_i. Note that there is no specific boundary condition for ϕ''_o, hence the choice here is arbitrary. This system can now be written as

$$\begin{bmatrix} \phi'_{1o} & \phi'_{2o} \\ \phi''_{1o} & \phi''_{2o} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

For functions that are linearly independent the determinant of the matrix containing known values of ϕ_{io} and ϕ''_{io} will be non-zero. When the suppression scheme was not used this determinant was effectively zero and indicated the functions to be linearly dependent, that is, one column is a multiple of the other.

For independent functions, the a_i can be determined by finding the inverse of this matrix or as was done for the computer program by simply writing out the solution.

The iteration scheme for eigenvalues has several options (i.e., (1) find R, α for c_r fixed, (2) find R, c_r for α fixed, (3) find α, c_r for R fixed). The first option of finding R, α for c_r fixed is further explained below.

The a_i, once determined can now be substituted into equation (3.2.4). With the correct eigenvalues ϕ_o will be zero; hence ϕ_o will serve as the test function (F). If $F\tilde{F}$, where $\tilde{F}$ is the conjugate of F, is less than 10^{-12} the convergence criteria is satisfied and the eigenvalues used to generate the functions are assumed to be the correct ones. If convergence is not attained let $F_1 = F$. Increase α by 1%, recalculate F and let $F_2 = F$. After setting α to its original value, increase R by 1%, recalculate F and let $F_3 = F$. The finite difference approximations for the change in F with respect to α and R are

$$\frac{\partial F}{\partial \alpha} \simeq \frac{F_2 - F_1}{\Delta\alpha} \tag{3.3.5}$$

$$\frac{\partial F}{\partial R} \simeq \frac{F_3 - F_1}{\Delta R} \tag{3.3.6}$$

These are substituted into the complex equation

$$\frac{\partial F}{\partial \alpha}\Delta\alpha + \frac{\partial F}{\partial R}\Delta R + F_1 = 0 \tag{3.3.7}$$

from which $\Delta\alpha$ and ΔR can be calculated. The new values for the eigenvalues are

$$\begin{aligned} \alpha_{new} &= \alpha_{old} + \Delta\alpha \\ R_{new} &= R_{old} + \Delta R \end{aligned} \tag{3.3.8}$$

It was found that if the initial guesses are relatively good, convergence is obtained in about three iterations.

The initial estimates for the eigenvalues of the near parallel plate case (i.e. when the radius ratio is near unity) were obtained from the data presented by Potter and Graber, (1972). The initial estimates of eigenvalues for the isothermal case and radius ratios a/b with 0.8 or less were obtained from the data presented by Mott and Joseph (1968).

For nonisothermal case with radius ratios of less than 0.99 guesses had to be made referring to both results of Potter and Graber, and Joseph and Mott. After a few critical points for different radius ratios and temperatures on the boundary were obtained and plotted, the initial point on the next stability curve for smaller radius ratio could be made by extrapolation. Completing a neutral stability curve for a fixed radius ratio, a/b, and boundary temperatures (T_i, T_o) was also done by extrapolation after a few points were obtained and plotted.

CHAPTER IV

RESULTS, CONCLUSIONS, AND RECOMMENDATIONS FOR FURTHER STUDY

4.1 Numerical Results

(a) Governing Equations of the Main Flow

The velocity and temperature profiles of the main flow were calculated using the iteration technique discussed in Section 3.1. For the heated inner cylinder several cases were considered. The temperature of the outer cylinder was taken at 0°C with the inner cylinder temperature at 20°C, 40°C, 60°C, 80°C, and 100°C. The radius ratio a/b, inner cylinder to outer cylinder, was varied between 0.99 and 0.1.

The energy equation (2.1.9) can be written as

$$\frac{k}{a_1+r}\ \frac{d}{dr}\left[\ \{a_1+r\}\ \frac{dT}{dr}\ \right] + \frac{dk}{dr}\ \frac{dT}{dr} + \Phi = 0$$

where $\Phi = \mathrm{sgn}\ (T_o - T_i)\ P_r\ E_{ck}\ \mu(\frac{dW}{dr})^2$. To determine the significance of the dk/dr term and viscous dissipation Φ the governing equations of the main flow were solved for the following four cases:

(1) Neglect both dk/dr and Φ.

(2) Keep dk/dr, but neglect Φ.

(3) Neglect dk/dr, but keep Φ.

(4) Keep both dk/dr and Φ.

The velocity and temperature profiles of these cases for the selected boundary temperatures and radius ratios are presented in Tables 1 through 10.

The significance of the dk/dr term and viscous dissipation are presented in Tables 11 through 20. For the near parallel plate case (a/b = 0.99) with T_i = 20°C, T_o = 0°C and R = 3900, the following indicates the maximum deviation between the temperature T(I,1) of case (1) and the temperatures T(I,k) of cases (2), (3), and (4):

$$\max \left| \frac{T(I,2) - T(I,1)}{T(I,1)} \right| \times 100 = 3.9\%$$

$$\max \left| \frac{T(I,3) - T(I,1)}{T(I,1)} \right| \times 100 = 1.0\%$$

$$\max \left| \frac{T(I,4) - T(I,1)}{T(I,1)} \right| \times 100 = 4.9\%$$

Since the temperature T(I,1) of case (1) contains the highest error, the case with the smallest error will have the largest ratio. That is, the T(I,4) values of case (4) are the most accurate. The value 3.9% of case (2) is larger than the 1.0% value of case (3). This indicates that the dk/dr term has a greater effect on temperature than Φ has. For a/b = 0.99, T_i = 100°C, and R = 13370 the deviation was

$$\max \left| \frac{T(I,2) - T(I,1)}{T(I,1)} \right| \times 100 = 7.03\%$$

$$\max \left| \frac{T(I,3) - T(I,1)}{T(I,1)} \right| \times 100 = 1.0\%$$

$$\max \left| \frac{T(I,4) - T(I,1)}{T(I,1)} \right| \times 100 = 7.84\%$$

This indicates that the dk/dr term has a significantly greater effect on the temperature distribution than the Φ term, with the effect increasing with increasing R and temperature difference. Similar results were found for a/b = 0.1.

The significance of the dk/dr term when T_i = 100°C and T_o = 0°C is shown graphically on Figure 6 for a/b = 0.1 and on Figure 7 for a/b = 0.99. The effect of variable properties on temperature is relatively small, but may be important in certain situations. The temperature T with the dk/dr term is always lower than that without the dk/dr term at any position r except on the boundary. Some representative velocity profiles are presented in Figures 2, 3, 4, and 5. They clearly show the significant effect of temperature, acting through the viscosity, on the velocity profile.

Based on the result of this analysis the dk/dr term has always been kept in the energy equation for the present study. But, based on the small error introduced, the viscous dissipation has been neglected, as is customary.

(b) Stability Equation

Numerical calculations of the eigenvalues were performed using the technique discussed in Sections 3.2 and 3.3 for T_i = 20°C, 60°C, and 100°C with T_o = 20°C, a/b = 0.99, 0.8, 0.6, 0.4, 0.3 and 0.25. The resulting eigenvalues for these cases are presented in Tables 22 through 41, and the associated neutral stability curves are presented in Figures 8 through 13. Critical eigenvalues are summarized in

Table 21 and are plotted in Figure 13 to demonstrate the effect of heat transfer and radius ratio a/b on the critical eigenvalues.

From Figure 13 there are three observations that are of particular interest:

(1) For any fixed value of a/b between the values of 0.99 and 0.25, with T_o = 20°C, the critical Reynolds number decreases with increased temperature of the inner cylinder.

(2) For the isothermal flow with $T_i = T_o$ = 20°C the critical Reynolds number increases with decreasing radius ratio. For a/b = 0.99, R_{crit} = 7687. The value of R_{crit} increases to the value of 105370 for a/b = 0.25. The isothermal curve supports the conclusion that the critical Reynolds number of an annulus approaches infinity as a/b approaches 0.

(3) The critical Reynolds number $R_{crit}(a/b)$ of non-isothermal flow also increases as a/b decreases for a/b between 0.99 and 0.25. But, non-isothermal R_{crit}-curves in this case show an inflection point near a/b = 0.4 giving the possibility of

$$\lim_{a/b \to 0} R_{crit} = \text{a finite value.}$$

The effect of heat transfer on the stability was further investigated for a/b = 0.4 with T_o = 20°C, T_i = 20°C, 30°C, 40°C, 60°C and 100°C. The resulting neutral stability curves for these cases are shown in Figure 14. When T_i = 20°C (isothermal case) the neutral stability curve (α vs. R) has the shape of a typical open loop which is shown in Figure 1. However, the neutral stability curve of the non-isothermal flow for $a/b \leq 0.4$ exhibits a different shape as shown in Figure 14.

In case of $a/b \leq 0.4$, R is of the order of 10^4 and α of the upper branch keeps increasing for non-isothermal flow. This results in small $(\alpha R)^{-1}$ which is the coefficient of the highest order derivative of the stability equation. The differential equation for these small $(\alpha R)^{-1}$ is highly singular and its solution is very difficult to obtain. Some eigenfunctions are plotted in Figures 15, 16, 17.

4.2 Conclusions

Based on the results obtained and discussion of the previous section it can be concluded that:

(1) Neglecting dk/dr results in up to 6.8% error in the temperature distribution and up to 2.7% error in the velocity distribution for high R and T = 100°C. The effect of neglecting Φ influences the temperature and velocity distributions by less than 1%.

(2) Neglecting dk/dr results in an error of up to 13.3% on the heat transfer and up to 3% on the shear stress. The effect of neglecting Φ influences the heat transfer and shear stress by less than 1%.

(3) Heating the inner cylinder destabilizes the parallel water flow through the concentric cylinders.

(4) The critical Reynolds number of an isothermal flow monotonically decreases supporting the accepted conclusion that R_{crit} becomes infinitely large as a/b approaches zero.

(5) The critical Reynolds number of a non-isothermal flow increases as a/b decreases between the values of 0.99 and 0.25. The results indicate that R_{crit} may indeed approach a finite value when $a/b \to 0$.

(6) For a/b = 0.4 the upper branch of the neutral stability curve of the non-isothermal flow appears to have a vertical, or near vertical, asymptote rather than the horizontal asymptote of the isothermal flow. But, convergence of the iteration process toward an acceptable set of eigenvalues was extremely difficult at the high (αR) values. Hence, the existance of the upper vertical asymptote, especially for small $(T_o - T_i)$, is not certain.

4.3 Recommendations for Further Study

Finite-amplitude instability of parallel flow of a liquid between concentric cylinders would be investigated using the eigenvalues found in this study as an input. This study could be further extended to find critical eigenvalues of non-isothermal flow for $a/b \leq 0.2$ by using a better method which could solve the highly singular differential equation. It could also be extended to include non-Newtonian fluid.

ILLUSTRATIONS
(Tables and Figures)

Table 1.--Velocity and Temperature Distribution of Main Flow
a/b = 0.99, T_i = 20°C, T_o = 0°C, R = 3900

I	r(I)	W(I,1)	T(I,1)	W(I,2)	T(I,2)	W(I,3)	T(I,3)	W(I,4)	T(I,4)
1	0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.05	0.3365	0.0502	0.3359	0.0483	0.3364	0.0498	0.3358	0.0479
11	0.10	0.6262	0.1005	0.6254	0.0968	0.6260	0.0997	0.6253	0.0961
16	0.15	0.8711	0.1506	0.8704	0.1455	0.8710	0.1497	0.8703	0.1445
21	0.20	1.0734	0.2008	1.0729	0.1943	1.0733	0.1997	1.0728	0.1932
26	0.25	1.2349	0.2509	1.2347	0.2433	1.2348	0.2498	1.2346	0.2421
31	0.30	1.3575	0.3011	1.3576	0.2924	1.3575	0.2998	1.3576	0.2912
36	0.35	1.4432	0.3511	1.4437	0.3418	1.4433	0.3499	1.4437	0.3405
41	0.40	1.4939	0.4012	1.4956	0.3913	1.4940	0.3999	1.4946	0.3900
46	0.45	1.5113	0.4512	1.5122	0.4410	1.5114	0.4499	1.5123	0.4397
51	0.50	1.4973	0.5013	1.4982	0.4909	1.4974	0.4999	1.4983	0.4896
56	0.55	1.4537	0.5512	1.4545	0.5410	1.4538	0.5499	1.4546	0.5396
61	0.60	1.3821	0.6012	1.3828	0.5913	1.3822	0.5998	1.3829	0.5899
66	0.65	1.2843	0.6511	1.2848	0.6417	1.2844	0.6497	1.2849	0.6403
71	0.70	1.1620	0.7011	1.1621	0.6923	1.1620	0.6997	1.1622	0.6909
76	0.75	1.0167	0.7509	1.0165	0.7431	1.0167	0.7496	1.0166	0.7418
81	0.80	0.8500	0.8008	0.8496	0.7944	0.8500	0.7996	0.8496	0.7929
86	0.85	0.6636	0.8506	0.6630	0.8453	0.6635	0.8496	0.6629	0.8442
91	0.90	0.4588	0.9005	0.4581	0.8966	0.4587	0.8996	0.4581	0.8958
96	0.95	0.2371	0.9502	0.2367	0.9482	0.2371	0.9498	0.2366	0.9477
101	1.00	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000

Table 2.--Velocity and Temperature Distribution of Main Flow
a/b = 0.99, T_i = 40°C, T_o = 0°C, R = 6390

I	r(I)	W(I,1)	T(I,1)	W(I,2)	T(I,2)	W(I,3)	T(I,3)	W(I,4)	T(I,4)
1	0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.05	0.3760	0.0502	0.3742	0.0474	0.3758	0.0498	0.3740	0.0470
11	0.10	0.6919	0.1005	0.6894	0.0950	0.6917	0.0997	0.6891	0.0942
16	0.15	0.9515	0.1506	0.9491	0.1427	0.9513	0.1497	0.9488	0.1418
21	0.20	1.1585	0.2008	1.1566	0.1907	1.1583	0.1997	1.1565	0.1897
26	0.25	1.3165	0.2509	1.3155	0.2390	1.3164	0.2498	1.3154	0.2378
31	0.30	1.4291	0.3011	1.4291	0.2874	1.4290	0.2998	1.4291	0.2862
36	0.35	1.4996	0.3511	1.5007	0.3361	1.4997	0.3498	1.5007	0.3348
41	0.40	1.5315	0.4012	1.5334	0.3850	1.5317	0.3998	1.5335	0.3837
46	0.45	1.5280	0.4512	1.5306	0.4342	1.5283	0.4498	1.5308	0.4329
51	0.50	1.4925	0.5013	1.4953	0.4837	1.4927	0.4998	1.4955	0.4823
56	0.55	1.4279	0.5512	1.4306	0.5336	1.4281	0.5497	1.4308	0.5321
61	0.60	1.3371	0.6012	1.3395	0.5838	1.3374	0.5997	1.3397	0.5822
66	0.65	1.2233	0.6511	1.2249	0.6343	1.2235	0.6496	1.2251	0.6328
71	0.70	1.0891	0.7011	1.0899	0.6853	1.0892	0.6995	1.0900	0.6838
76	0.75	0.9372	0.7509	0.9371	0.7367	0.9373	0.7495	0.9372	0.7352
81	0.80	0.7703	0.8008	0.7693	0.7885	0.7703	0.7994	0.7693	0.7871
86	0.85	0.5908	0.8506	0.5892	0.8407	0.5907	0.8495	0.5891	0.8396
91	0.90	0.4012	0.9005	0.3994	0.8934	0.4010	0.8996	0.3993	0.8925
96	0.95	0.2035	0.9502	0.2022	0.9465	0.2034	0.9497	0.2021	0.9459
101	1.00	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000

Table 3.--Velocity and Temperature Distribution of Main Flow
a/b = 0.99, T_i = 60°C, T_o = 0°C, R = 8760

I	r(I)	W(I,1)	T(I,1)	W(I,2)	T(I,2)	W(I,3)	T(I,3)	W(I,4)	T(I,4)
1	0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.05	0.4076	0.0502	0.4043	0.0469	0.4073	0.0498	0.4040	0.0465
11	0.10	0.7443	0.1005	0.7396	0.0940	0.7439	0.0997	0.7392	0.0932
21	0.20	1.2257	0.2008	1.2219	0.1887	1.2254	0.1997	1.2216	0.1877
26	0.25	1.3806	0.2509	1.3783	0.2365	1.3804	0.2497	1.3780	0.2353
31	0.30	1.4848	0.3011	1.4843	0.2845	1.4847	0.2998	1.4842	0.2833
36	0.35	1.5430	0.3511	1.5443	0.3328	1.5431	0.3498	1.5444	0.3315
41	0.40	1.5599	0.4012	1.5628	0.3814	1.5601	0.3998	1.5029	0.3800
46	0.45	1.5399	0.4512	1.5439	0.4303	1.5402	0.4498	1.5442	0.4289
51	0.50	1.4873	0.5013	1.4920	0.4796	1.4876	0.4997	1.4923	0.4782
56	0.55	1.4063	0.5512	1.4111	0.5293	1.4067	0.5496	1.4114	0.5278
61	0.60	1.3008	0.6012	1.3052	0.5794	1.3012	0.5996	1.3055	0.5778
66	0.65	1.1748	0.6511	1.1781	0.6299	1.1751	0.6495	1.1785	0.6282
71	0.70	1.0318	0.7011	1.0338	0.6808	1.0320	0.6994	1.0341	0.6792
76	0.75	0.8754	0.7509	0.8758	0.7324	0.8755	0.7494	0.8760	0.7308
81	0.80	0.7988	0.8008	0.7078	0.7845	0.7088	0.7994	0.7078	0.7831
86	0.85	0.5352	0.8506	0.5330	0.8374	0.5351	0.8494	0.5329	0.8361
91	0.90	0.3574	0.9005	0.3548	0.8909	0.3573	0.8995	0.3546	0.8900
96	0.95	0.1783	0.9502	0.1762	0.9451	0.1781	0.9497	0.1761	0.9446
101	1.00	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000

Table 4.--Velocity and Temperature Distribution of Main Flow
a/b = 0.99, T_i = 80°C, T_o = 0°C, R = 11050

I	r(I)	W(I,1)	T(I,1)	W(I,2)	T(I,2)	W(I,3)	T(I,3)	W(I,4)	T(I,4)
1	0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.05	0.4335	0.0502	0.4286	0.0468	0.4331	0.0498	0.4283	0.0464
11	0.10	0.7871	0.1005	0.7800	0.0938	0.7865	0.0997	0.7794	0.0931
16	0.15	1.0672	0.1506	1.0599	0.1408	1.0667	0.1497	1.0543	0.1400
21	0.20	1.2803	0.2008	1.2741	0.1881	1.2798	0.1997	1.2736	0.1871
26	0.25	1.4324	0.2509	1.4283	0.2356	1.4321	0.2498	1.4280	0.2345
31	0.30	1.5297	0.3011	1.5281	0.2834	1.5295	0.2998	1.5279	0.2822
36	0.35	1.5777	0.3511	1.5788	0.3314	1.5777	0.3498	1.5788	0.3301
41	0.40	1.5823	0.4012	1.5857	0.3797	1.5825	0.3998	1.5858	0.3784
46	0.45	1.5487	0.4512	1.5540	0.4284	1.5491	0.4497	1.5543	0.4269
51	0.50	1.4823	0.5013	1.4888	0.4774	1.4828	0.4997	1.4892	0.4759
56	0.55	1.3881	0.5512	1.3950	0.5269	1.3886	0.5496	1.3954	0.5253
61	0.60	1.2709	0.6012	1.2773	0.5768	1.2714	0.5995	1.2778	0.5751
66	0.65	1.1351	0.6511	1.1405	0.6272	1.1356	0.6494	1.1410	0.6255
71	0.70	0.9854	0.7011	0.9890	0.6781	0.9857	0.6993	0.9894	0.6765
76	0.75	0.8256	0.7509	0.8272	0.7297	0.8258	0.7493	0.8274	0.7281
81	0.80	0.6596	0.8008	0.6591	0.7819	0.6596	0.7993	0.6592	0.7805
86	0.85	0.4909	0.8506	0.4888	0.8350	0.4908	0.8494	0.4887	0.8338
91	0.90	0.3229	0.9005	0.3199	0.8891	0.3227	0.8995	0.3197	0.8881
96	0.95	0.1584	0.9502	0.1560	0.9441	0.1583	0.9497	0.1558	0.9435
101	1.00	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000

Table 5.--Velocity and Temperature Distribution of Main Flow
$a/b = 0.99$, $T_i = 100°C$, $T_o = 0°C$, $R = 13370$

I	r(I)	W(I,1)	T(I,1)	W(I,2)	T(I,2)	W(I,3)	T(I,3)	W(I,4)	T(I,4)
1	0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.05	0.4551	0.0502	0.4487	0.0467	0.4546	0.0498	0.4482	0.0463
11	0.10	0.8227	0.1005	0.8133	0.0936	0.8220	0.0997	0.8126	0.0929
16	0.15	1.1105	0.1506	1.1006	0.1408	1.1098	0.1497	1.0999	0.1399
21	0.20	1.3257	0.2008	1.3170	0.1882	1.3251	0.1997	1.3164	0.1871
26	0.25	1.4754	0.2509	1.4693	0.2357	1.4750	0.2497	1.4688	0.2345
31	0.30	1.5666	0.3011	1.5637	0.2833	1.5664	0.2998	1.5634	0.2821
36	0.35	1.6061	0.3511	1.6065	0.3312	1.6062	0.3498	1.6065	0.3299
41	0.40	1.6004	0.4012	1.6039	0.3793	1.6006	0.3997	1.6040	0.3780
46	0.45	1.5556	0.4512	1.5616	0.4278	1.5560	0.4497	1.5620	0.4263
51	0.50	1.4777	0.5013	1.4856	0.4766	1.4783	0.4996	1.4861	0.4751
56	0.55	1.3726	0.5512	1.3813	0.5258	1.3732	0.5495	1.3819	0.5242
61	0.60	1.2457	0.6012	1.2542	0.5755	1.2463	0.5994	1.2548	0.5738
66	0.65	1.1022	0.6511	1.1096	0.6258	1.1027	0.6493	1.1101	0.6240
71	0.70	0.9469	0.7011	0.9524	0.6766	0.9474	0.6993	0.9529	0.6749
76	0.75	0.7845	0.7509	0.7876	0.7281	0.7848	0.7493	0.7879	0.7265
81	0.80	0.6192	0.8008	0.6198	0.7803	0.6193	0.7993	0.6199	0.7788
86	0.85	0.4548	0.8506	0.4532	0.8335	0.4547	0.8494	0.4531	0.8322
91	0.90	0.2949	0.9005	0.2919	0.8878	0.2947	0.8995	0.2918	0.8868
96	0.95	0.1424	0.9502	0.1398	0.9433	0.1423	0.9497	0.1396	0.9427
101	1.00	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000

Table 6.--Velocity and Temperature Distribution of Main Flow
$a/b = 0.1$, $T_i = 20°C$, $T_o = 0°C$, $R = 401460$

I	r(I)	W(I,1)	T(I,1)	W(I,2)	T(I,2)	W(I,3)	T(I,3)	W(I,4)	T(I,4)
1	0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.05	0.6245	0.1614	0.6236	0.1558	0.6242	0.1600	0.6233	0.1545
11	0.10	1.0131	0.2788	1.0126	0.2704	1.0128	0.2769	1.0124	0.2686
16	0.15	1.2692	0.3711	1.2692	0.3613	1.2690	0.3690	1.2692	0.3593
21	0.20	1.4389	0.4472	1.4394	0.4368	1.4389	0.4450	1.4395	0.4346
26	0.25	1.5467	0.5119	1.5476	0.5014	1.5468	0.5097	1.5477	0.4992
31	0.30	1.6072	0.5682	1.6083	0.5579	1.6074	0.5660	1.6085	0.5557
36	0.35	1.6298	0.6180	1.6309	0.6081	1.6300	0.6158	1.6311	0.6059
41	0.40	1.6207	0.6628	1.6218	0.6533	1.6210	0.6605	1.6220	0.6511
46	0.45	1.5846	0.7033	1.5855	0.6945	1.5849	0.7010	1.5857	0.6922
51	0.50	1.5249	0.7404	1.5255	0.7322	1.5251	0.7381	1.5257	0.7300
56	0.55	1.4440	0.7745	1.4444	0.7661	1.4441	0.7723	1.4445	0.7649
61	0.60	1.3439	0.8062	1.3441	0.7995	1.3441	0.8040	1.3442	0.7973
66	0.65	1.2264	0.8357	1.2263	0.8298	1.2265	0.8335	1.2263	0.8277
71	0.70	1.0927	0.8633	1.0923	0.8583	1.0927	0.8613	1.0923	0.8562
76	0.75	0.9438	0.8893	0.9433	0.8851	0.9438	0.8874	0.9432	0.8832
81	0.80	0.7808	0.9138	0.7801	0.9104	0.7806	0.9121	0.7800	0.9087
86	0.85	0.6042	0.9370	0.6036	0.9344	0.6041	0.9356	0.6034	0.9330
91	0.90	0.4149	0.9590	0.4143	0.9573	0.4147	0.9580	0.4141	0.9563
96	0.95	0.2133	0.9800	0.2130	0.9791	0.2132	0.9794	0.2128	0.9786
101	1.00	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000

Table 7.--Velocity and Temperature Distribution of Main Flow
a/b = 0.1, T_i = 40°C, T_o = 0°C, R = 626120

I	r(I)	W(I,1)	T(I,1)	W(I,2)	T(I,2)	W(I,3)	T(I,3)	W(I,4)	T(I,4)
1	0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.05	0.7147	0.1614	0.7120	0.1529	0.7140	0.1601	0.7116	0.1516
11	0.10	1.1348	0.2788	1.1333	0.2656	1.1343	0.2769	1.1329	0.2638
16	0.15	1.3958	0.3711	1.3961	0.3553	1.3955	0.3690	1.3959	0.3533
21	0.20	1.5569	0.4472	1.5587	0.4300	1.5568	0.4449	1.5587	0.4278
26	0.25	1.6489	0.5119	1.6518	0.4941	1.6490	0.5096	1.6520	0.4918
31	0.30	1.6901	0.5682	1.6935	0.5503	1.6903	0.5658	1.6938	0.5480
36	0.35	1.6918	0.6180	1.6955	0.6006	1.6922	0.6156	1.6959	0.5982
41	0.40	1.6621	0.6628	1.6655	0.6460	1.6626	0.6603	1.6660	0.6435
46	0.45	1.6064	0.7033	1.6093	0.6874	1.6069	0.7008	1.6097	0.6849
51	0.50	1.5288	0.7404	1.5309	0.7256	1.5292	0.7378	1.5313	0.7231
56	0.55	1.4323	0.7745	1.4336	0.7610	1.4327	0.7720	1.4340	0.7585
61	0.60	1.3195	0.8062	1.3199	0.7940	1.3198	0.8037	1.3202	0.7915
66	0.65	1.1922	0.8357	1.1918	0.8249	1.1924	0.8333	1.1919	0.8225
71	0.70	1.0520	0.8633	1.0509	0.8540	1.0521	0.8610	1.0509	0.8517
76	0.75	0.9003	0.8893	0.8985	0.8815	0.9001	0.8872	0.8984	0.8793
81	0.80	0.7380	0.9138	0.7359	0.9075	0.7377	0.9119	0.7357	0.9056
86	0.85	0.5661	0.9370	0.5640	0.9323	0.5658	0.9355	0.5636	0.9307
91	0.90	0.3854	0.9590	0.3836	0.9559	0.3850	0.9579	0.3832	0.9547
96	0.95	0.1965	0.9800	0.1954	0.9784	0.1962	0.9794	0.1951	0.9778
101	1.00	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000

Table 8.--Velocity and Temperature Distribution of Main Flow
$a/b = 0.1$, $T_i = 60°C$, $T_o = 0°C$, $R = 826850$

I	r(I)	W(I,1)	T(I,1)	W(I,2)	T(I,2)	W(I,3)	T(I,3)	W(I,4)	T(I,4)
1	0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.05	0.7905	0.1614	0.7861	0.1512	0.7895	0.1601	0.7854	0.1500
11	0.10	1.2372	0.2788	1.2346	0.2629	1.2363	0.2769	1.2340	0.2611
16	0.15	1.5023	0.3711	1.5027	0.3519	1.5017	0.3689	1.5023	0.3498
21	0.20	1.6559	0.4472	1.6590	0.4261	1.6557	0.4449	1.6589	0.4239
26	0.25	1.7344	0.5119	1.7395	0.4899	1.7346	0.5095	1.7396	0.4876
31	0.30	1.7591	0.5682	1.7652	0.5460	1.7595	0.5657	1.7656	0.5436
36	0.35	1.7434	0.6180	1.7498	0.5961	1.7439	0.6155	1.7503	0.5936
41	0.40	1.6962	0.6628	1.7023	0.6414	1.6969	0.6601	1.7029	0.6389
46	0.45	1.6240	0.7033	1.6292	0.6829	1.6247	0.7506	1.6298	0.6803
51	0.50	1.5315	0.7404	1.5354	0.7212	1.5322	0.7376	1.5360	0.7185
56	0.55	1.4221	0.7745	1.4245	0.7568	1.4227	0.7718	1.4251	0.7541
61	0.60	1.2987	0.8062	1.2995	0.7900	1.2991	0.8035	1.2999	0.7873
66	0.65	1.1634	0.8357	1.1627	0.8213	1.1637	0.8331	1.1629	0.8187
71	0.70	1.0180	0.8633	1.0160	0.8508	1.0181	0.8608	1.0160	0.8483
76	0.75	0.8640	0.8893	0.8609	0.8787	0.8638	0.8870	0.8607	0.8764
81	0.80	0.7025	0.9138	0.6988	0.9053	0.7021	0.9118	0.6984	0.9033
86	0.85	0.5346	0.9370	0.5308	0.9306	0.5341	0.9353	0.5303	0.9289
91	0.90	0.3610	0.9590	0.3578	0.9547	0.3605	0.9578	0.3573	0.9535
96	0.95	0.1826	0.9800	0.1806	0.9778	0.1823	0.9793	0.1803	0.9771
101	1.00	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000

Table 9.--Velocity and Temperature Distribution of Main Flow
$a/b = 0.1$, $T_i = 80°C$, $T_o = 0°C$, $R = 1013510$

I	r(I)	W(I,1)	T(I,1)	W(I,2)	T(I,2)	W(I,3)	T(I,3)	W(I,4)	T(I,4)
1	0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.05	0.8564	0.1614	0.8502	0.1508	0.8551	0.1601	0.8492	0.1496
11	0.10	1.3262	0.2788	1.3223	0.2619	1.3250	0.2769	1.3213	0.2602
16	0.15	1.5948	0.3711	1.5949	0.3504	1.5941	0.3689	1.5943	0.3483
21	0.20	1.7419	0.4472	1.7457	0.4241	1.7416	0.4448	1.7455	0.4219
26	0.25	1.8086	0.5119	1.8153	0.4876	1.8088	0.5094	1.8154	0.4853
31	0.30	1.8189	0.5682	1.8272	0.5435	1.8194	0.5656	1.8276	0.5410
36	0.35	1.7878	0.6180	1.7966	0.5935	1.7886	0.6153	1.7973	0.5909
41	0.40	1.7255	0.6628	1.7339	0.6387	1.7263	0.6600	1.7347	0.6361
46	0.45	1.6389	0.7033	1.6462	0.6802	1.6399	0.7004	1.6471	0.6774
51	0.50	1.5335	0.7404	1.5391	0.7185	1.5344	0.7374	1.5399	0.7157
56	0.55	1.4130	0.7745	1.4165	0.7541	1.4137	0.7716	1.4172	0.7512
61	0.60	1.2805	0.8062	1.2818	0.7874	1.2810	0.8033	1.2823	0.7846
66	0.65	1.1383	0.8357	1.1375	0.8188	1.1387	0.8329	1.1378	0.8161
71	0.70	0.9885	0.8633	0.9858	0.8485	0.9886	0.8607	0.9559	0.8459
76	0.75	0.8326	0.8893	0.8284	0.8767	0.8324	0.8869	0.8282	0.8743
81	0.80	0.6719	0.9138	0.6668	0.9036	0.6714	0.9117	0.6663	0.9015
86	0.85	0.5075	0.9370	0.5022	0.9293	0.5069	0.9353	0.5016	0.9275
91	0.90	0.3402	0.9590	0.3357	0.9538	0.3396	0.9578	0.3350	0.9525
96	0.95	0.1709	0.9800	0.1680	0.9774	0.1704	0.9793	0.1676	0.9767
101	1.00	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000

Table 10.--Velocity and Temperature Distribution of Main Flow
$a/b = 0.1$, $T_i = 100°C$, $T_o = 0°C$, $R = 1190270$

I	r(I)	W(I,1)	T(I,1)	W(I,2)	T(I,2)	W(I,3)	T(I,3)	W(I,4)	T(I,4)
1	0.00								
6	0.05	0.9153	0.1614	0.9072	0.1508	0.9136	0.1600	0.9058	0.1496
11	0.10	1.4056	0.2788	1.4000	0.2619	1.4040	0.2769	1.3987	0.2602
16	0.15	1.6773	0.3711	1.6764	0.3501	1.6762	0.3689	1.6755	0.3481
21	0.20	1.8185	0.4472	1.8223	0.4236	1.8180	0.4447	1.8218	0.4214
26	0.25	1.8746	0.5119	1.8820	0.4868	1.8747	0.5093	1.8820	0.4844
31	0.30	1.8720	0.5682	1.8816	0.5424	1.8725	0.5655	1.8820	0.5399
36	0.35	1.8271	0.6180	1.8377	0.5921	1.8280	0.6152	1.8384	0.5895
41	0.40	1.7512	0.6628	1.7615	0.6373	1.7522	0.6598	1.7624	0.6345
46	0.45	1.6519	0.7033	1.6609	0.6786	1.6530	0.7003	1.6620	0.6758
51	0.50	1.5349	0.7404	1.5420	0.7169	1.5360	0.7373	1.5430	0.7139
56	0.55	1.4045	0.7745	1.4092	0.7524	1.4055	0.7715	1.4101	0.7495
61	0.60	1.2640	0.8062	1.2660	0.7858	1.2647	0.8032	1.2667	0.7829
66	0.65	1.1159	0.8357	1.1152	0.8172	1.1163	0.8328	1.1157	0.8144
71	0.70	0.9623	0.8633	0.9592	0.8470	0.9624	0.8606	0.9593	0.8443
76	0.75	0.8049	0.8893	0.7999	0.8753	0.8046	0.8868	0.7996	0.8728
81	0.80	0.6450	0.9138	0.6388	0.9024	0.6444	0.9116	0.6382	0.9002
86	0.85	0.4837	0.9370	0.4772	0.9283	0.4829	0.9352	0.4765	0.9265
91	0.90	0.3220	0.9590	0.3163	0.9531	0.3212	0.9577	0.3156	0.9518
96	0.95	0.1605	0.9800	0.1570	0.9770	0.1600	0.9793	0.1565	0.9763
101	1.00	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000

Table 11.--Significance of dk/dr and Viscous Dissipation Terms
a/b = 0.99, T_i = 20°C, T_o = 0°C, R = 3900.

		First Derivatives of Velocity & Temperature							
I	r(I)	dW(I,1)/dr	dT(I,1)/dr	dW(I,2)/dr	dT(I,2)/dr	dW(I,3)/dr	dT(I,3)/dr	dW(I,4)/dr	dT(I,4)/dr
1	0.00	7.2099	1.0050	7.1941	0.9654	7.2069	0.9947	7.1911	0.9552
101	1.00	-4.8886	0.9949	-4.8758	1.0388	-4.8863	1.0058	-4.8735	1.0498

Max. Ratios from Table 1

$$\underset{I}{\text{Max.}} \left| \frac{T(I,2) - T(I,1)}{T(I,1)} \right| = 0.0391 \qquad \underset{I}{\text{Max.}} \left| \frac{W(I,2) - W(I,1)}{W(I,1)} \right| = 0.0025$$

$$\underset{I}{\text{Max.}} \left| \frac{T(I,3) - T(I,1)}{T(I,1)} \right| = 0.0100 \qquad \underset{I}{\text{Max.}} \left| \frac{W(I,3) - W(I,1)}{W(I,1)} \right| = 0.0004$$

$$\underset{I}{\text{Max.}} \left| \frac{T(I,4) - T(I,1)}{T(I,1)} \right| = 0.0490 \qquad \underset{I}{\text{Max.}} \left| \frac{W(I,4) - W(I,1)}{W(I,1)} \right| = 0.0029$$

Table 12.--Significance of dk/dr and Viscous Dissipation Terms
a/b = 0.99, T_i = 40°C, T_o = 0°C, R = 6390

		First Derivatives of Velocity & Temperature							
I	r(I)	dW(I,1)/dr	dT(I,1)/dr	dW(I,2)/dr	dT(I,2)/dr	dW(I,3)/dr	dT(I,3)/dr	dW(I,4)/dr	dT(I,4)/dr
1	0.00	8.1457	1.0050	8.0970	0.9457	8.1398	0.9946	8.0912	0.9355
101	1.00	-4.1177	0.9949	-4.0812	1.0764	-4.1138	1.0065	-4.0733	1.0881

Max. Ratios from Table 2

$$\max_{I} \left| \frac{T(I,2) - T(I,1)}{T(I,1)} \right| = 0.0586 \qquad \max_{I} \left| \frac{W(I,2) - W(I,1)}{W(I,1)} \right| = 0.0084$$

$$\max_{I} \left| \frac{T(I,3) - T(I,1)}{T(I,1)} \right| = 0.0101 \qquad \max_{I} \left| \frac{W(I,3) - W(I,1)}{W(I,1)} \right| = 0.0009$$

$$\max_{I} \left| \frac{T(I,4) - T(I,1)}{T(I,1)} \right| = 0.0684 \qquad \max_{I} \left| \frac{W(I,4) - W(I,1)}{W(I,1)} \right| = 0.0093$$

Table 13.--Significance of dk/dr and Viscous Dissipation Terms
$a/b = 0.99$, $T_i = 60°C$, $T_o = 0°C$, $R = 8760$

		First Derivatives of Velocity & Temperature							
I	r(I)	dW(I,1)/dr	dT(I,1)/dr	dW(I,2)/dr	dT(I,2)/dr	dW(I,3)/dr	dT(I,3)/dr	dW(I,4)/dr	dT(I,4)/dr
1	0.00	8.8962	1.0050	8.8079	0.9360	8.8875	0.9946	8.7994	0.9259
101	1.00	-3.5424	0.9949	-3.4847	1.1064	-3.5373	1.0067	-3.4799	1.1135

Max. Ratios from Table 3

$$\max_{I} \left| \frac{T(I,2) - T(I,1)}{T(I,1)} \right| = 0.0683 \qquad \max_{I} \left| \frac{W(I,2) - W(I,1)}{W(I,1)} \right| = 0.0153$$

$$\max_{I} \left| \frac{T(I,3) - T(I,1)}{T(I,1)} \right| = 0.0101 \qquad \max_{I} \left| \frac{W(I,3) - W(I,1)}{W(I,1)} \right| = 0.0013$$

$$\max_{I} \left| \frac{T(I,4) - T(I,1)}{T(I,1)} \right| = 0.0780 \qquad \max_{I} \left| \frac{W(I,4) - W(I,1)}{W(I,1)} \right| = 0.0165$$

Table 14.--Significance of dk/dr and Viscous Dissipation Terms
a/b = 0.99, T_i = 80°C, T_o = 0°C, R = 11050

		First Derivatives of Velocity & Temperature							
I	r(I)	dW(I,1)/dr	dT(I,1)/dr	dW(I,2)/dr	dT(I,2)/dr	dW(I,3)/dr	dT(I,3)/dr	dW(I,4)/dr	dT(I,4)/dr
1	0.00	9.5122	1.0050	9.3848	0.9357	9.5008	0.9946	9.3739	0.9258
101	1.00	-3.0947	0.9949	-3.0228	1.1300	-3.0890	1.0067	-3.0174	1.1421

Max. Ratios from Table 4

$$\max_{I} \left| \frac{T(I,2) - T(I,1)}{T(I,1)} \right| = 0.0688 \qquad \max_{I} \left| \frac{W(I,2) - W(I,1)}{W(I,1)} \right| = 0.0216$$

$$\max_{I} \left| \frac{T(I,3) - T(I,1)}{T(I,1)} \right| = 0.0100 \qquad \max_{I} \left| \frac{W(I,3) - W(I,1)}{W(I,1)} \right| = 0.0017$$

$$\max_{I} \left| \frac{T(I,4) - T(I,1)}{T(I,1)} \right| = 0.0784 \qquad \max_{I} \left| \frac{W(I,4) - W(I,1)}{W(I,1)} \right| = 0.0232$$

Table 15.--Significance of dk/dr and Viscous Dissipation Terms
$a/b = 0.99$, $T_i = 100°C$, $T_o = 0°C$, $R = 13370$

		First Derivatives of Velocity & Temperature							
I	r(I)	dW(I,1)/dr	dT(I,1)/dr	dW(I,2)/dr	dT(I,2)/dr	dW(I,3)/dr	dT(I,3)/dr	dW(I,4)/dr	dT(I,4)/dr
1	0.00	10.0312	1.0050	9.8655	0.9409	10.0172	0.9946	9.8523	0.9310
101	1.00	-2.7360	0.9949	-2.6563	1.1484	-2.7300	1.0067	-2.6507	1.1606

Max. Ratios from Table 5

$$\underset{I}{\text{Max.}} \left| \frac{T(I,2) - T(I,1)}{T(I,1)} \right| = 0.0703 \qquad \underset{I}{\text{Max.}} \left| \frac{W(I,2) - W(I,1)}{W(I,1)} \right| = 0.0269$$

$$\underset{I}{\text{Max.}} \left| \frac{T(I,3) - T(I,1)}{T(I,1)} \right| = 0.0100 \qquad \underset{I}{\text{Max.}} \left| \frac{W(I,3) - W(I,1)}{W(I,1)} \right| = 0.0020$$

$$\underset{I}{\text{Max.}} \left| \frac{T(I,4) - T(I,1)}{T(I,1)} \right| = 0.0784 \qquad \underset{I}{\text{Max.}} \left| \frac{W(I,4) - W(I,1)}{W(I,1)} \right| = 0.0288$$

Table 16.--Significance of dk/dr and Viscous Dissipation Terms
a/b = 0.1, T_i = 20°C, T_o = 0°C, R = 401460

		First Derivatives of Velocity & Temperature							
I	r(I)	dW(I,1)/dr	dT(I,1)/dr	dW(I,2)/dr	dT(I,2)/dr	dW(I,3)/dr	dT(I,3)/dr	dW(I,4)/dr	dT(I,4)/dr
1	0.00	16.1190	3.9087	16.0691	3.7353	16.1020	3.8504	16.0575	3.6957
101	1.00	-4.3805	0.3908	-4.3721	0.4083	-4.3772	0.4036	-4.3687	0.4211

Max. Ratios from Table 6

$$\underset{I}{\text{Max.}} \left| \frac{T(I,2) - T(I,1)}{T(I,1)} \right| = 0.0389 \qquad \underset{I}{\text{Max.}} \left| \frac{W(I,2) - W(I,1)}{W(I,1)} \right| = 0.0028$$

$$\underset{I}{\text{Max.}} \left| \frac{T(I,3) - T(I,1)}{T(I,1)} \right| = 0.0100 \qquad \underset{I}{\text{Max.}} \left| \frac{W(I,3) - W(I,1)}{W(I,1)} \right| = 0.0009$$

$$\underset{I}{\text{Max.}} \left| \frac{T(I,4) - T(I,1)}{T(I,1)} \right| = 0.0487 \qquad \underset{I}{\text{Max.}} \left| \frac{W(I,4) - W(I,1)}{W(I,1)} \right| = 0.0034$$

Table 17.--Significance of dk/dr and Viscous Dissipation Terms
a/b = 0.1, T_i = 40°C, T_o = 0°C, R = 626120

		First Derivatives of Velocity & Temperature							
I	r(I)	dW(I,1)/dr	dT(I,1)/dr	dW(I,2)/dr	dT(I,2)/dr	dW(I,3)/dr	dT(I,3)/dr	dW(I,4)/dr	dT(I,4)/dr
1	0.00	18.9251	3.9087	18.7875	3.6583	18.8925	3.8503	18.7650	3.6195
101	1.00	-4.0019	0.3908	-3.9750	0.4233	-3.9953	0.4047	-3.9684	0.43372

Max. Ratios from Table 7

$$\max_{I}\left|\frac{T(I,2)-T(I,1)}{T(I,1)}\right| = 0.0584 \qquad \max_{I}\left|\frac{W(I,2)-W(I,1)}{W(I,1)}\right| = 0.0065$$

$$\max_{I}\left|\frac{T(I,3)-T(I,1)}{T(I,1)}\right| = 0.0100 \qquad \max_{I}\left|\frac{W(I,3)-W(I,1)}{W(I,1)}\right| = 0.0016$$

$$\max_{I}\left|\frac{T(I,4)-T(I,1)}{T(I,1)}\right| = 0.0689 \qquad \max_{I}\left|\frac{W(I,4)-W(I,1)}{W(I,1)}\right| = 0.0081$$

Table 18.--Significance of dk/dr and Viscous Dissipation Terms
a/b = 0.1, T_i = 60°C, T_o = 0°C, R = 826850

		First Derivatives of Velocity & Temperature							
I	r(I)	dW(I,1)/dr	dT(I,1)/dr	dW(I,2)/dr	dT(I,2)/dr	dW(I,3)/dr	dT(I,3)dr	dW(I,4)/dr	dT(I,4)/dr
1	0.00	21.2859	3.9087	21.0569	3.6209	21.2385	3.8503	21.0234	3.5825
101	1.00	-3.6921	0.3908	-3.6437	0.4353	-3.6824	0.4054	-3.6342	0.4500

Max. Ratios from Table 8

$$\max_{I} \left| \frac{T(I,2) - T(I,1)}{T(I,1)} \right| = 0.0682 \qquad \max_{I} \left| \frac{W(I,2) - W(I,1)}{W(I,1)} \right| = 0.0127$$

$$\max_{I} \left| \frac{T(I,3) - T(I,1)}{T(I,1)} \right| = 0.0100 \qquad \max_{I} \left| \frac{W(I,3) - W(I,1)}{W(I,1)} \right| = 0.0025$$

$$\max_{I} \left| \frac{T(I,4) - T(I,1)}{T(I,1)} \right| = 0.0775 \qquad \max_{I} \left| \frac{W(I,4) - W(I,1)}{W(I,1)} \right| = 0.0151$$

Table 19.--Significance of dk/dr and Viscous Dissipation Terms
a/b = 0.1, T_i = 80°C, T_o = 0°C, R = 1013510

		First Derivatives of Velocity & Temperature							
I	r(I)	dW(I,1)/dr	dT(I,1)/dr	dW(I,2)/dr	dT(I,2)/dr	dW(I,3)/dr	dT(I,3)/dr	dW(I,4)/dr	dT(I,4)/dr
1	0.00	23.3363	3.9087	23.0250	3.6205	23.2742	3.8503	22.9799	3.5824
101	1.00	-3.4285	0.3908	-3.3602	0.4448	-3.4163	0.4058	-3.3483	0.4599

Max. Ratios from Table 9

$$\max_{I} \left| \frac{T(I,2) - T(I,1)}{T(I,1)} \right| = 0.0688 \qquad \max_{I} \left| \frac{W(I,2) - W(I,1)}{W(I,1)} \right| = 0.0193$$

$$\max_{I} \left| \frac{T(I,3) - T(I,1)}{T(I,1)} \right| = 0.0100 \qquad \max_{I} \left| \frac{W(I,3) - W(I,1)}{W(I,1)} \right| = 0.0034$$

$$\max_{I} \left| \frac{T(I,4) - T(I,1)}{T(I,1)} \right| = 0.0782 \qquad \max_{I} \left| \frac{W(I,4) - W(I,1)}{W(I,1)} \right| = 0.0226$$

Table 20.--Significance of dk/dr and Viscous Dissipation Terms
a/b = 0.1, T_i = 100°C, T_o = 0°C, R = 1190270

		First Derivatives of Velocity & Temperature							
I	r(I)	dW(I,1)/dr	dT(I,1)/dr	dW(I,2)/dr	dT(I,2)/dr	dW(I,3)/dr	dT(I,3)/dr	dW(I,4)/dr	dT(I,4)/dr
1	0.00	25.1795	3.9087	24.7852	3.6334	25.1021	3.8503	24.7276	3.5954
101	1.00	-3.1988	0.3908	-3.1136	0.4522	-3.1844	0.4060	-3.0998	0.4676

Max. Ratios from Table 10

$$\max_{I} \left| \frac{T(I,2) - T(I,1)}{T(I,1)} \right| = 0.0680 \qquad \max_{I} \left| \frac{W(I,2) - W(I,1)}{W(I,1)} \right| = 0.0257$$

$$\max_{I} \left| \frac{T(I,3) - T(I,1)}{T(I,1)} \right| = 0.0100 \qquad \max_{I} \left| \frac{W(I,3) - W(I,1)}{W(I,1)} \right| = 0.0043$$

$$\max_{I} \left| \frac{T(I,4) - T(I,1)}{T(I,1)} \right| = 0.0773 \qquad \max_{I} \left| \frac{W(I,4) - W(I,1)}{W(I,1)} \right| = 0.0298$$

Table 21.--Critical Eigenvalues for Various Radius Ratios of Inner to Outer Cylinders

a	a/b[1]	T_i[2]	c_r	α	R
9.9 Cm	0.99	20°C	0.395	2.0292	7,687
		60°C	0.406	2.0914	4,707
		100°C	0.422	2.1618	3,041
8.0 Cm	0.80	20°C	0.384	1.9976	8,540
		60°C	0.378	2.0316	6,428
		100°C	0.386	2.0913	4,562
6.0 Cm	0.60	20°C	0.348	1.9465	12,978
		60°C	0.333	1.9931	11,341
		100°C	0.340	2.0305	8,313
4.0 Cm	0.40	20°C	0.273	1.8453	34,509
		60°C	0.274	2.0366	26,625
		100°C	0.293	2.2553	17,087
3.0 Cm	0.30	20°C	0.227	1.8613	73,321
		60°C	0.251	2.2906	40,564
		100°C	0.279	2.6563	22,467
2.5 Cm	0.25	20°C	0.210	2.0143	105,370
		60°C	0.2465	2.75847	45,847
		100°C	0.27873	3.5593	23,154

1/ b = 10 Cm

2/ T_o = 20°C

Table 22.--Eigenvalues for $c_i = 0$, $a/b = 0.99$, $T_i = T_o = 20°C$

α	R	c_r	Remarks
1.7936	9366	0.36	
1.9118	8188	0.36	
1.9849	7800	0.39	
2.0197	7701	0.394	
2.0292	7687	0.395	Critical Point
2.0420	7696	0.396	
2.1043	7843	0.400	
2.1780	9140	0.395	
2.1949	11513	0.38	
2.1796	15227	0.36	

Table 23.--Eigenvalues for $c_i = 0$, $a/b = 0.99$, $T_i = 60°C$, $T_o = 20°C$

α	R	c_r	Remarks
1.8918	5286	0.380	
2.0311	4757	0.400	
2.0689	4712	0.404	
2.0914	4707	0.406	Critical Point
2.1437	4800	0.409	
2.2415	6023	0.900	
2.2488	6500	0.395	

Table 24.--Eigenvalues for $c_i = 0$, a/b = 0.99, $T_i = 100°C$, $T_o = 20°C$

α	R	c_r	Remarks
1.9801	3305	0.400	
2.0987	3067	0.416	
2.1618	3041	0.422	Critical Point
2.1923	3062	0.424	
2.2657	3262	0.425	
2.3244	3886	0.416	

Table 25.--Eigenvalues for $c_i = 0$, a/b = 0.8, $T_i = T_o = 20°C$

α	R	c_r	Remarks
1.8895	9115	0.37	
1.9631	8666	0.38	
1.9898	8582	0.383	
1.9976	8540	0.384	Critical Point
2.0101	8550	0.385	
2.0338	8553	0.387	
2.0679	8657	0.389	
2.1367	9493	0.388	
2.1675	10994	0.383	

Table 26.--Eigenvalues for c_i = 0, a/b = 0.8, T_i = 60°C, T_o = 20°C

α	R	c_r	Remarks
1.9542	6589	0.37	
1.9986	6459	0.375	
2.0201	6436	0.377	
2.0316	6428	0.378	Critical Point
2.0449	6432	0.379	
2.0601	6450	0.38	
2.0766	6474	0.381	
2.1636	7.87	0.38	
2.2036	8605	0.37	

Table 27.--Eigenvalues for c_i = 0, a/b = 0.8, T_i = 100°C, T_o = 20°C

α	R	c_r	Remarks
1.9479	4877	0.37	
2.0102	4673	0.378	
2.0279	4633	0.38	
2.0677	4574	0.384	
2.0790	4566	0.385	
2.0913	4562	0.386	Critical Point
2.1680	4665	0.39	
2.2402	5127	0.388	
2.2828	5932	0.38	
2.3004	6910	0.37	

Table 28.--Eigenvalues for $c_i = 0$, $a/b = 0.6$, $T_i = T_o = 20°C$

α	R	c_r	Remarks
1.8696	13376	0.34	
1.9046	13126	0.344	
1.9152	13094	0.345	
1.9257	13055	0.346	
1.9361	13018	0.347	
1.9465	12978	0.348	Critical Point
1.9639	13045	0.349	
1.9821	13120	0.350	
2.0553	14208	0.35	
2.0997	17301	0.34	

Table 29.--Eigenvalues for $c_i = 0$, $a/b = 0.6$, $T_i = 60°C$, $T_o = 20°C$

α	R	c_r	Remarks
1.9519	11385	0.33	
1.9647	11363	0.331	
1.9781	11344	0.332	
1.9931	11341	0.333	Critical Point
2.0126	11391	0.334	
2.1447	13664	0.33	

Table 30.--Eigenvalues for $c_i = 0$, $a/b = 0.6$, $T_i = 100°C$, $T_o = 20°C$

α	R	c_r	Remarks
1.9608	8620	0.33	
2.0497	8326	0.338	
2.0641	8313	0.339	
2.0805	8312.8	0.34	Critical Point
2.0996	8332	0.341	
2.1244	8389	0.342	
2.2425	9479	0.34	
2.2737	10205	0.3364	
2.2956	10965	0.3324	
2.3157	11999	0.327	
2.3298	13074	0.3216	
2.3421	14423	0.3152	
2.3466	15069	0.3123	
2.353	16161	0.3076	
2.3585	17293	0.303	

Table 31.--Eigenvalues for $c_i = 0$, $a/b = 0.4$, $T_i = T_o = 20°C$

α	R	c_r	Remarks
1.5597	46188	0.24	
1.6268	41294	0.25	
1.7039	37452	0.26	
1.7484	35934	0.265	
1.7788	35171	0.268	
1.8020	34785	0.270	
1.8303	34590	0.272	
1.8453	34509	0.273	Critical Point
1.8664	34646	0.274	
1.8931	34953	0.275	
2.0365	52082	0.26	
2.0508	63245	0.25	
2.0563	76399	0.24	
2.0593	92111	0.23	

Table 32.--Eigenvalues for $c_i = 0$, $a/b = 0.4$, $T_i = 40°C$, $T_o = 20°C$

α	R	c_r	Remarks
1.85	32844	0.26336	
1.90	32343	0.26683	
1.925	32321	0.26814	Critical Point
1.95	32414	0.26922	
1.9782	32689	0.27009	
2.05	34700	0.26992	
2.20	50298	0.25354	
2.30	73576	0.23197	
2.40	90442	0.21875	
2.50	99862	0.21134	

Table 33.--Eigenvalues for $c_i = 0$, $a/b = 0.4$, $T_i = 60°C$, $T_o = 20°C$

α	R	c_r	Remarks
1.9095	27601	0.266	
1.9635	26933	0.27	
1.9959	26702	0.272	
2.0151	26645	0.273	
2.0366	26625	0.274	Critical Point
2.0637	26705	0.275	
2.1048	27021	0.276	
2.15	27708	0.27625	
2.2366	30162	0.274	
2.3	33220	0.26971	
2.4	40249	0.25926	
2.5	48107	0.24813	
2.6	54347	0.23956	
2.7	58605	0.23349	
2.8	61338	0.22913	

Table 34.--Eigenvalues for $c_i = 0$, $a/b = 0.4$, $T_i = 100°C$, $T_o = 20°C$

α	R	c_r	Remarks
1.7249	25950	0.245	
1.8945	20706	0.266	
1.9821	19063	0.275	
2.0627	18027	0.282	
2.1175	17549	0.286	
2.1661	17265	0.289	
2.1849	17193	0.290	
2.2052	17133	0.291	
2.2285	17096	0.292	
2.2553	17087	0.293	Critical Point
2.2897	17133	0.294	
2.3513	17382	0.295	
2.4169	17891	0.29487	
2.6842	22326	0.2835	
2.7741	24151	0.278	
2.8825	26068	0.272	
3.0451	28101	0.265	
3.1039	28607	0.263	
3.2081	29264	0.26	
3.2915	29613	0.258	

Table 35.--Eigenvalues for $c_i = 0$, $a/b = 0.3$, $T_i = T_o = 20°C$

α	R	c_r	Remarks
1.8232	73900	0.225	
1.8415	73574	0.226	
1.8613	73321	0.227	Critical Point
1.8901	73649	0.228	
1.9293	74475	0.229	

Table 36.--Eigenvalues for $c_i = 0$, $a/b = 0.3$, $T_i = 60°C$, $T_o = 20°C$

α	R	c_r	Remarks
1.8884	44727	0.2275	
1.9169	48369	0.23	
2.0493	43717	0.24	
2.1999	41085	0.248	
2.2553	40678	0.25	
2.2906	40564	0.251	Critical Point
2.3365	40565	0.252	
2.4675	41431	0.253	

Table 37.--Eigenvalues for $c_i = 0$, $a/b = 0.3$, $T_i = 100°C$, $T_o = 20°C$

α	R	c_r	Remarks
1.8744	36401	0.23	
2.0823	28777	0.25	
2.1437	27324	0.255	
2.2111	25996	0.26	
2.2405	25504	0.262	
2.2714	25035	0.264	
2.2794	24921	0.2645	
2.3785	23770	0.270	
2.4980	22912	0.275	
2.6039	22537	0.278	
2.6563	22467	0.279	Critical Point
2.7360	22472	0.280	
2.7492	22485	0.2801	

Table 38.--Eigenvalues for $c_i = 0$, $a/b = 0.25$, $T_i = T_o = 20°C$

α	R	c_r	Remarks
1.8089	113580	0.2	
1.9565	106150	0.208	
1.9831	105660	0.209	
2.0143	105370	0.21	Critical Point
2.0594	105740	0.211	
2.1588	108180	0.212	

Table 39.--Eigenvalues for $c_i = 0$, $a/b = 0.25$, $T_i = 60°C$, $T_o = 20°C$

α	R	c_r	Remarks
2.4094	47947	0.24	
2.4736	47134	0.242	
2.5565	46416	0.244	
2.6920	45897	0.246	
2.75847	45847	0.2465	Critical Point
2.8711	45971	0.24678	

Table 40.--Eigenvalues for $c_i = 0$, $a/b = 0.25$, $T_i = 100°C$, $T_o = 20°C$

α	R	c_r	Remarks
2.853	24548	0.273	
2.94492	24122	0.275	
3.0105	23916	0.276	
3.0887	23710	0.277	
3.1505	23584	0.2776	
3.2039	23495	0.278	
3.2374	23448	0.2782	
3.2786	23396	0.2784	
3.3363	23334	0.2786	
3.3806	23292	0.2787	
3.5593	23154	0.27873	Critical Point
3.5501	23161	0.27874	
3.5371	23170	0.278752	

Table 41.--Eigenvalues for $c_i = 0$, $a/b = 0.4$, $T_i = 30°C$, $T_o = 20°C$

α	R	c_r	Remarks
1.86	34348	0.26736	
1.885	34288	0.2688	Critical Point
1.9	34385	0.26943	
2.0	37315	0.26983	
2.1	50543	0.25687	
2.15	72537	0.23782	
2.17	85069	0.22895	
2.185	93088	0.22382	
2.195	97876	0.22092	

Table 42.--Viscosity and Conductivity of Water

$$\mu^* = \prod_{i=0}^{10} \exp(C_i \Theta^i),\quad k^* = \prod_{i=0}^{10} \exp(D_i \Theta^i)$$

i	C_i	D_i
0	-5.208294	-6.470986
1	-0.836625	0.095773
2	0.228220	-0.069060
3	-0.072689	0.027692
4	0.036071	0.204396
5	-0.025843	-0.145549
6	-0.041654	-0.793563
7	0.018216	0.309234
8	0.072355	1.126945
9	-0.006439	-0.179732
10	-0.037178	-0.523533

$\Theta = (T^* - 50)/50$

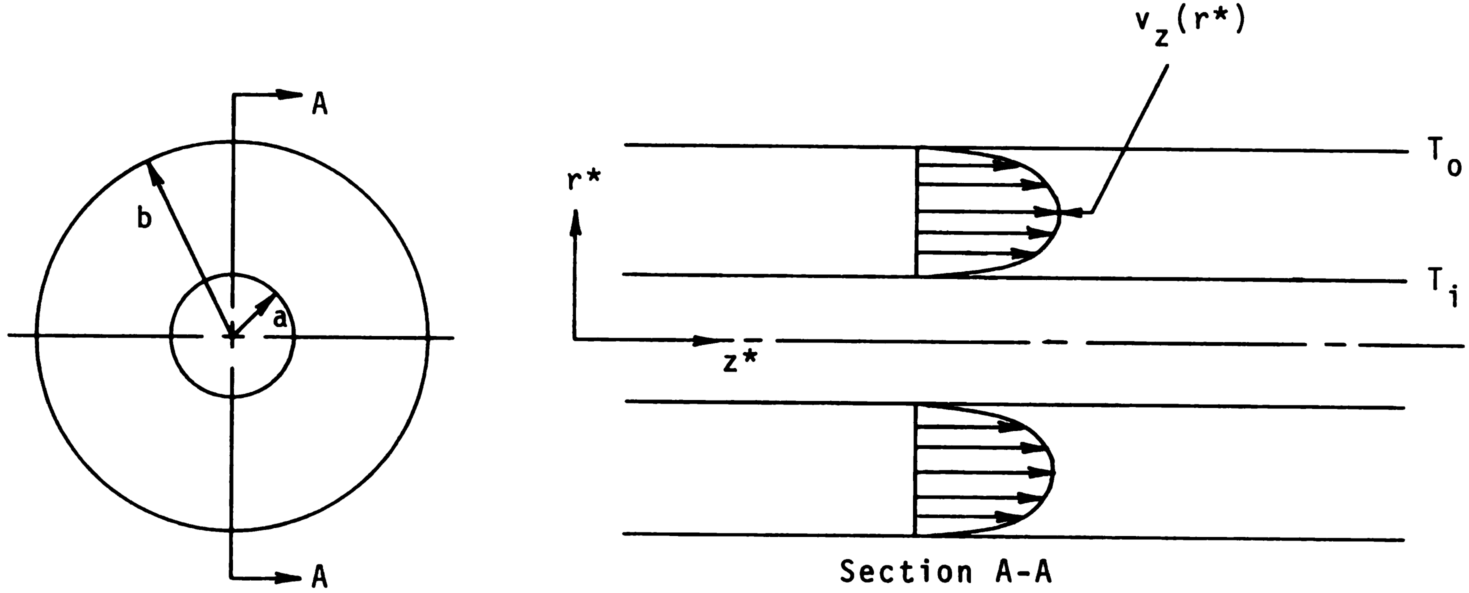

Figure 2.--Main Flow Configuration

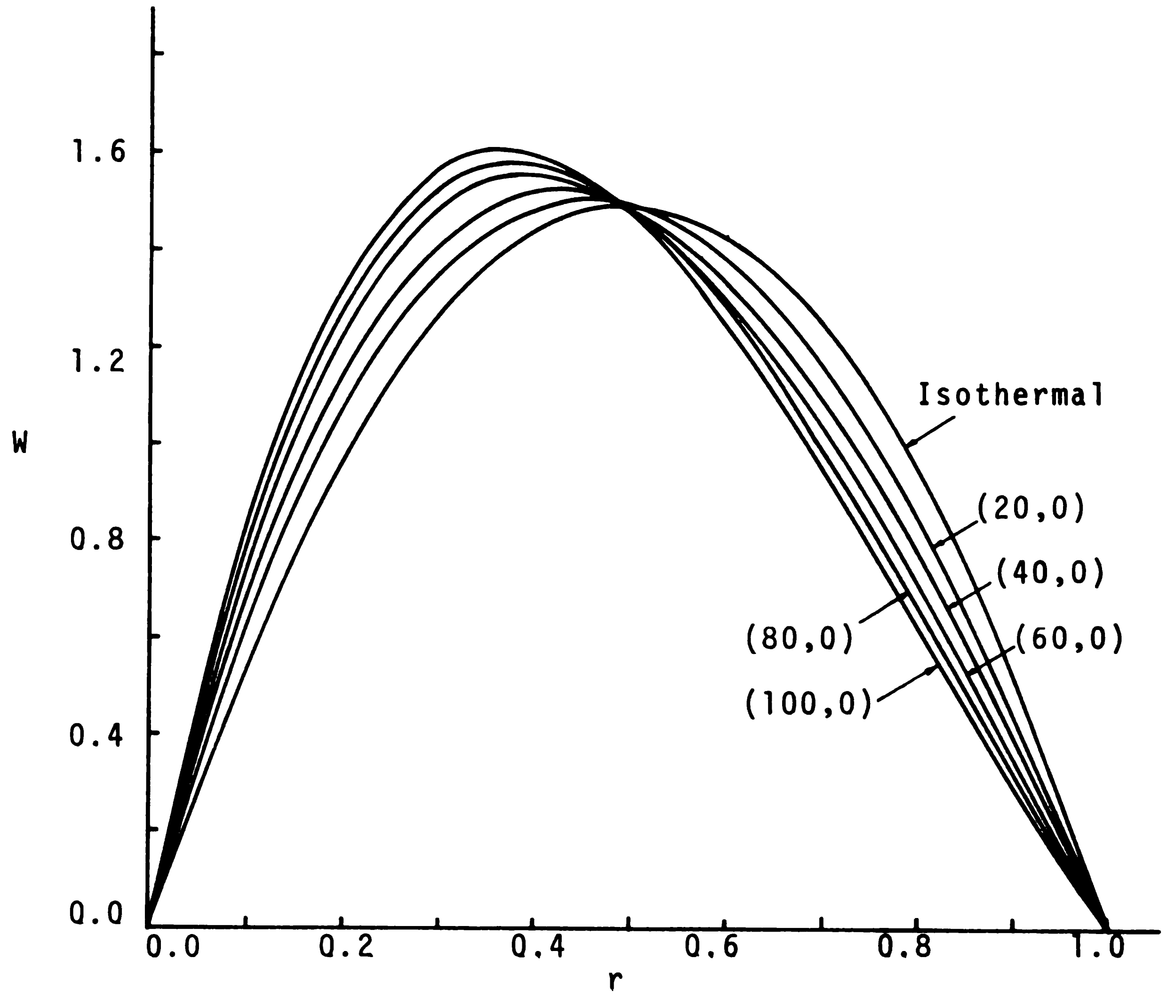

Figure 3.--Dimensionless Main Velocity Profiles for a/b = 0.99 and Various Boundary Temperatures (T_i, T_o)

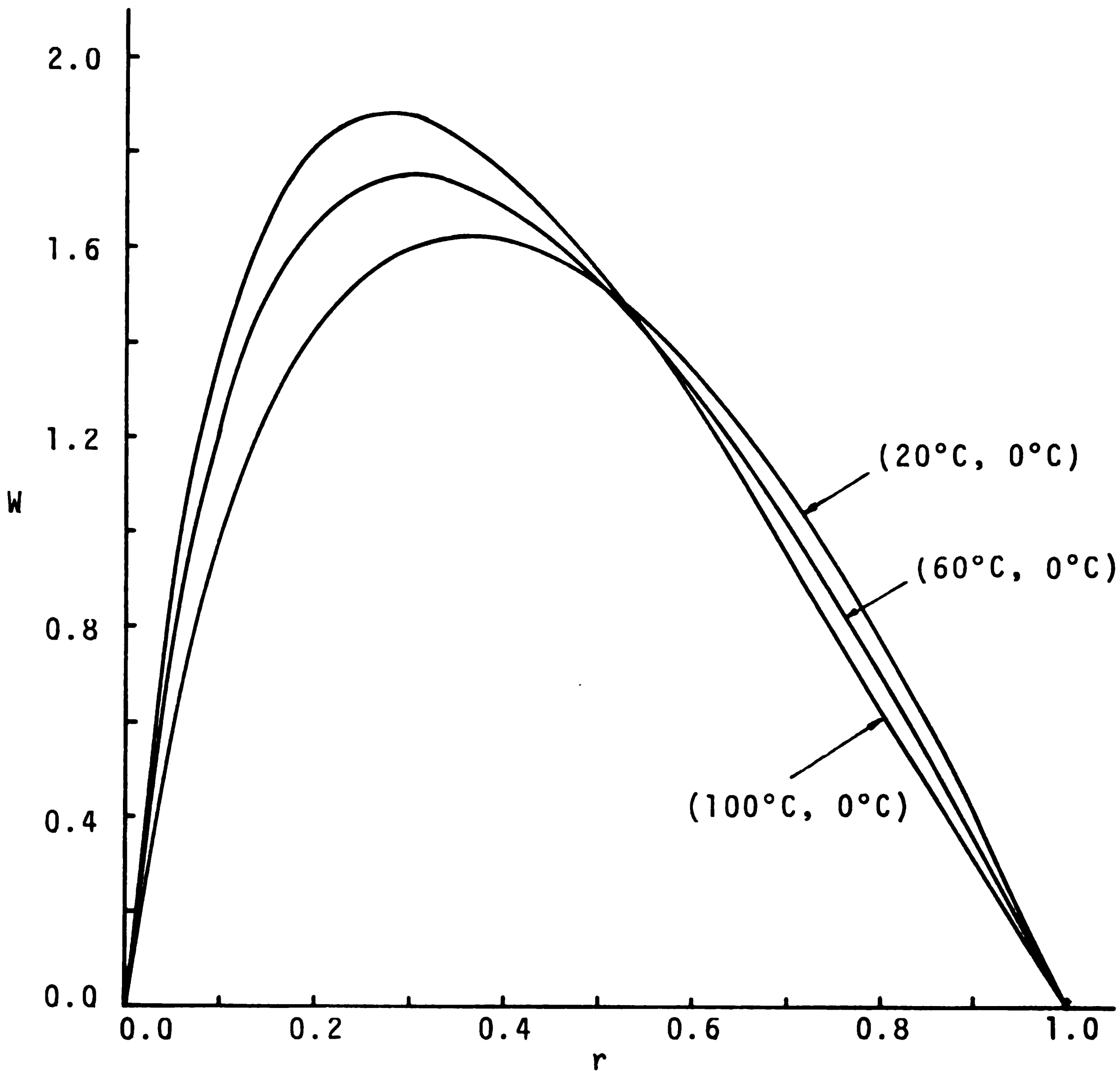

Figure 4.--Dimensionless Main Velocity Profiles for a/b = 0.1 and Various Boundary Temperatures (T_i, T_o)

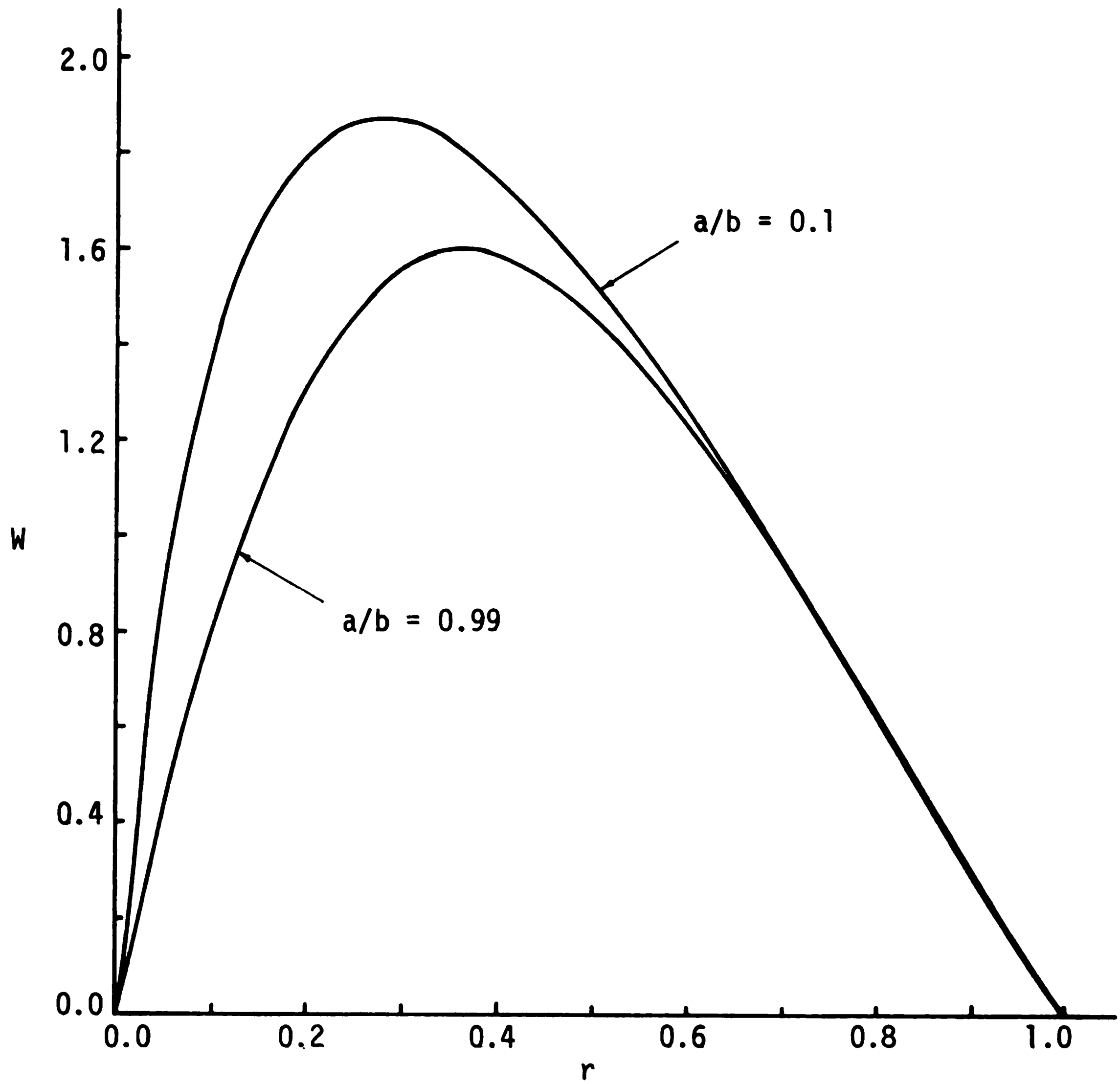

Figure 5.--Dimensionless Main Velocity Profiles for (T_i, T_o) = (100°C, 0°C), and a/b = 0.1 and 0.99

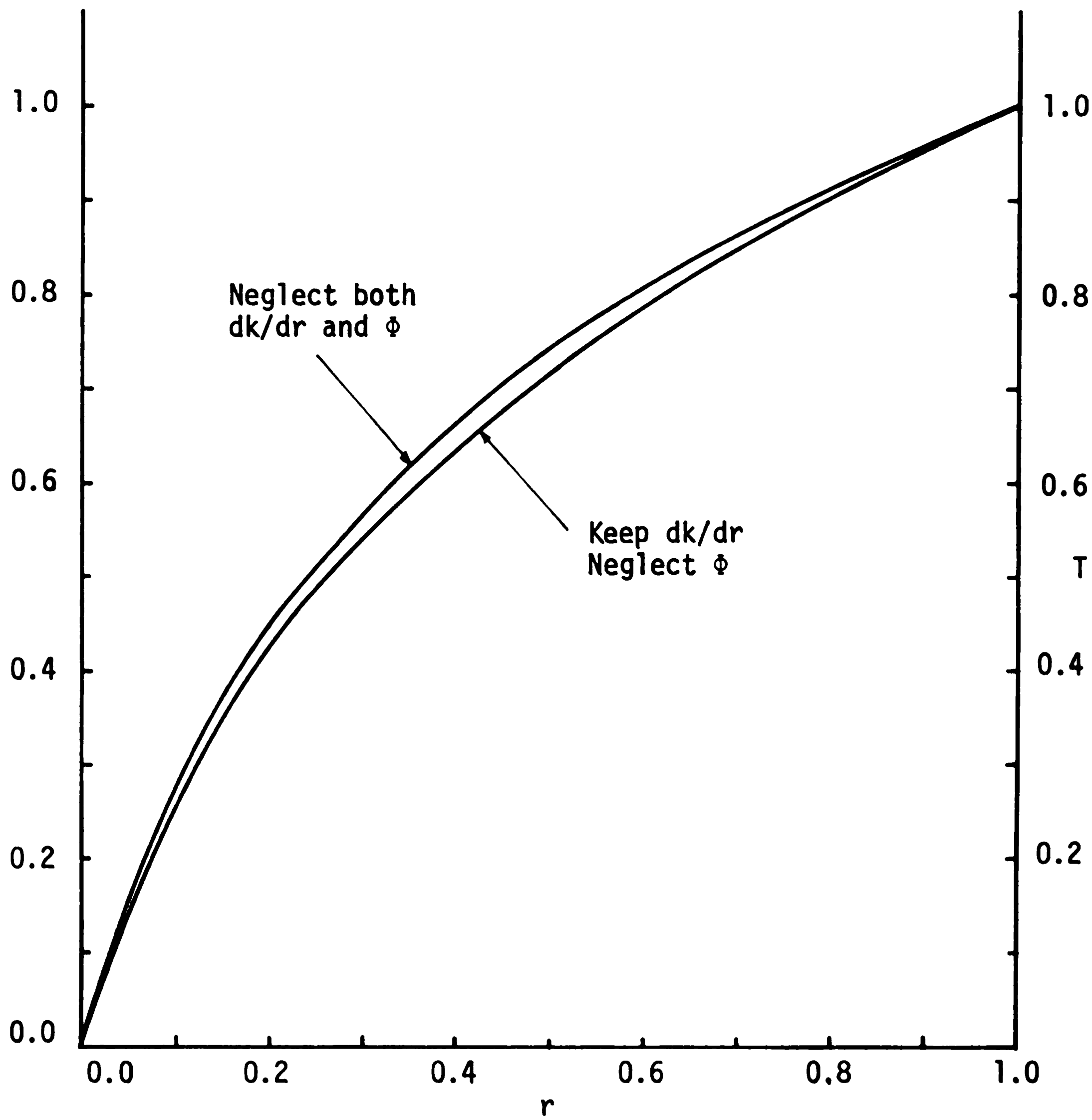

Figure 6.--Significance of dk/dr term for a/b = 0.1 and (T_i, T_o) = (100°C, 0°C)

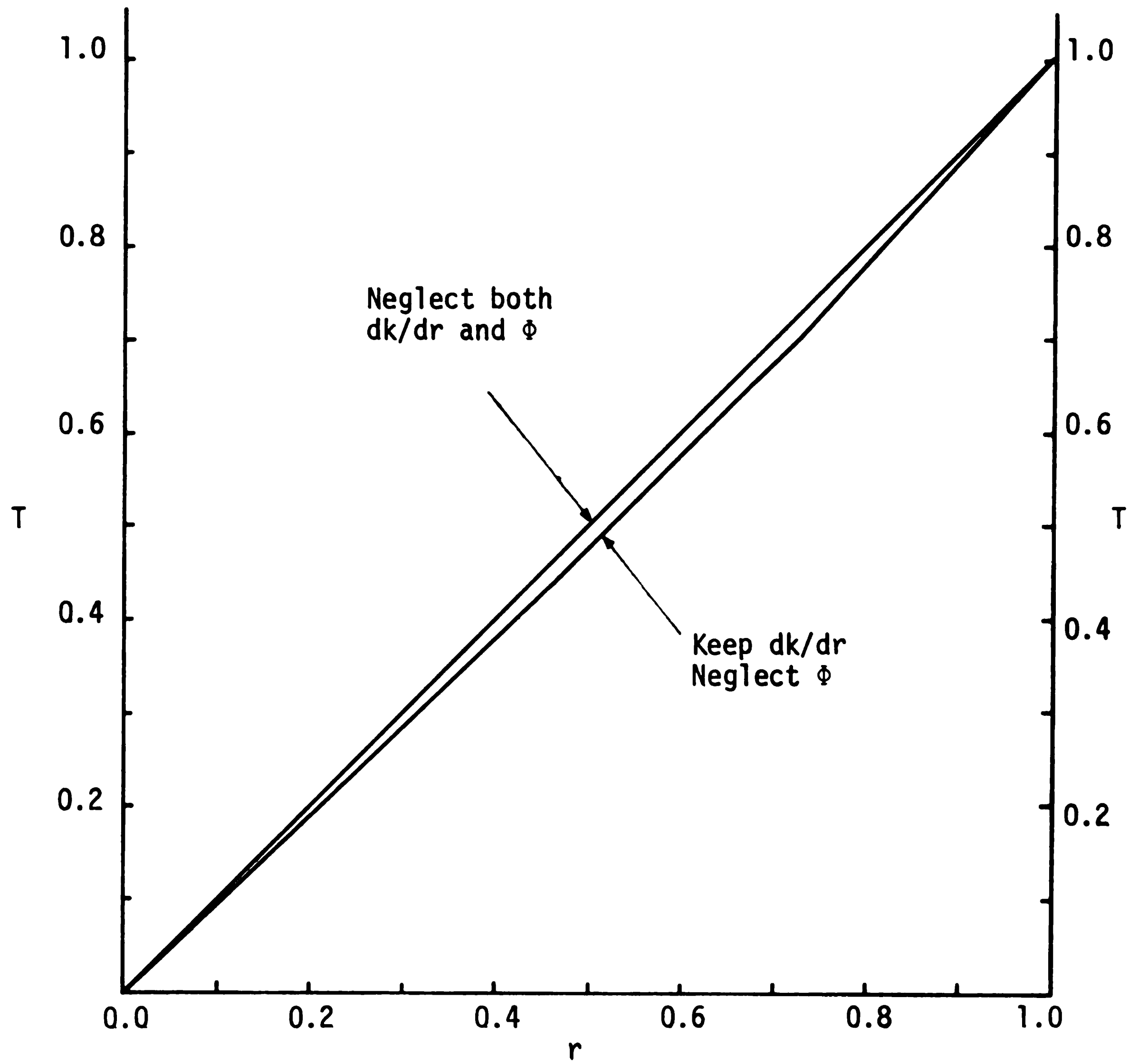

Figure 7.--Significance of dk/dr term for a/b = 0.99 and (T_i, T_o) = (100°C, 0°C)

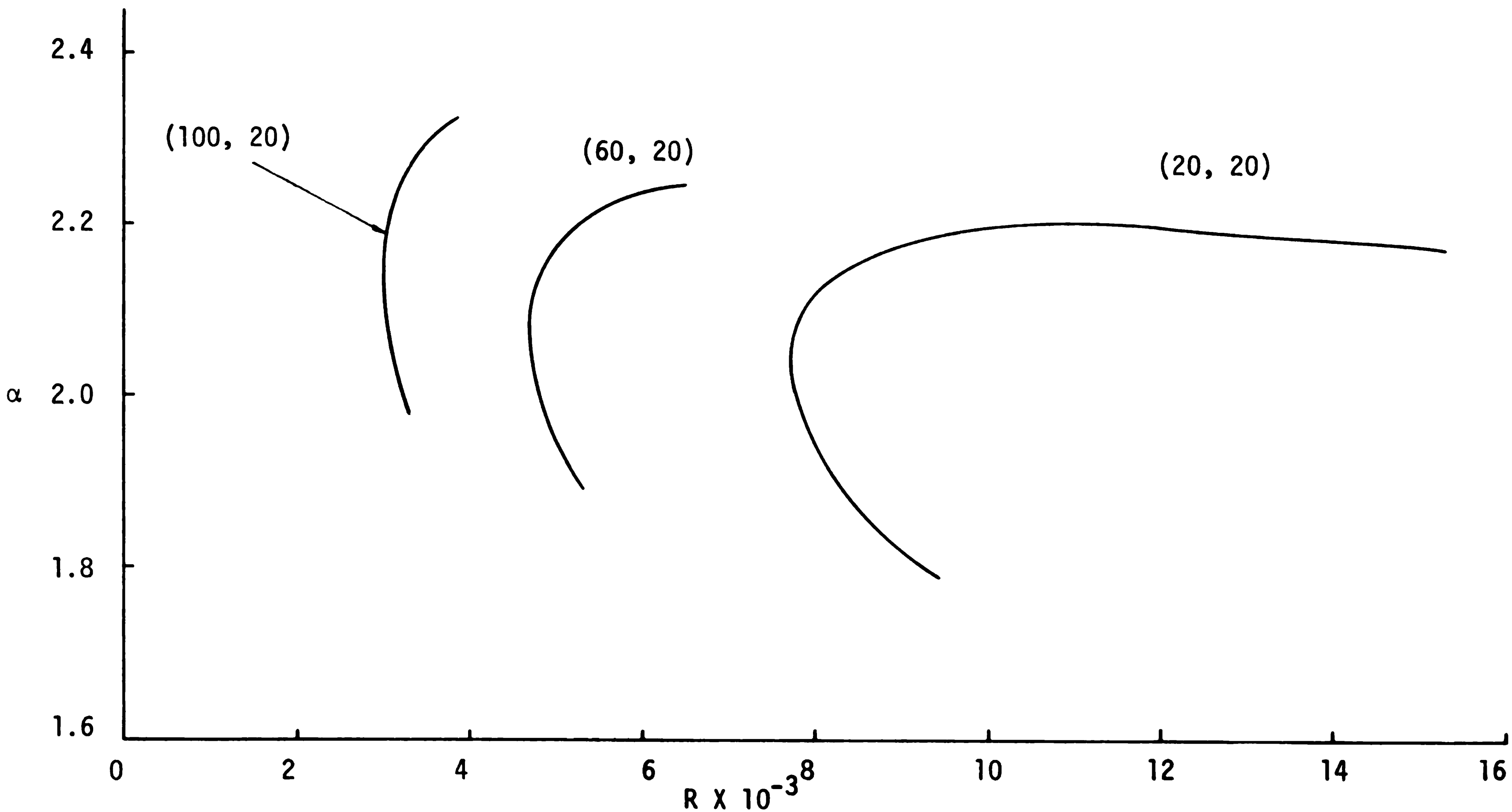

Figure 8.--Wave Number α Versus Reynolds Number R for c_i=0, a/b = 0.99 and Various Boundary Temperatures (T_i, T_o)

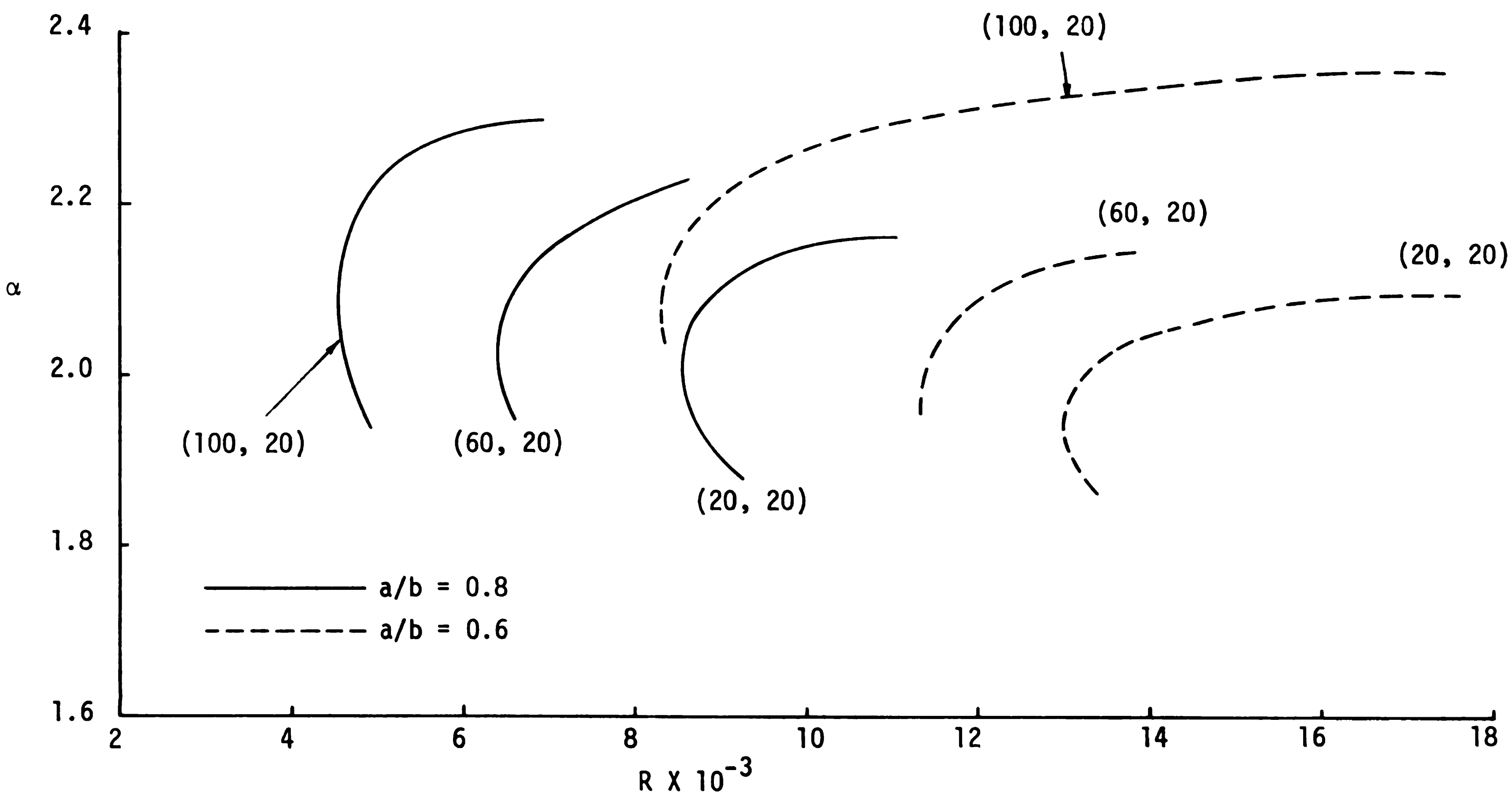

Figure 9.--Wave Number α Versus Reynolds Number R for $c_i = 0$, a/b = 0.8 and 0.6, and Various Boundary Temperatures (T_i, T_o)

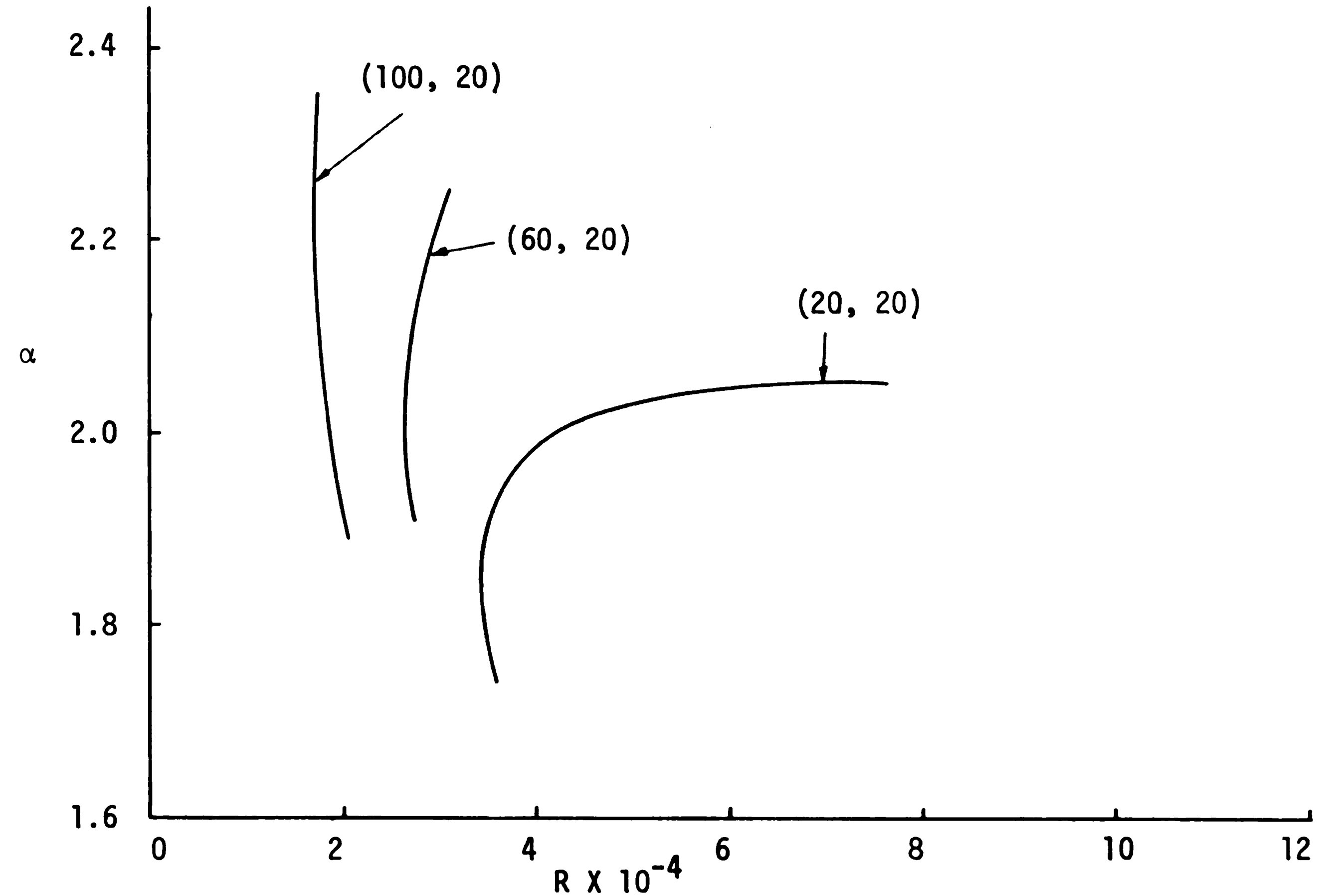

Figure 10.--Wave Number α Versus Reynolds Number R for $c_i = 0$, $a/b = 0.4$, and Various Boundary Temperatures (T_i, T_o)

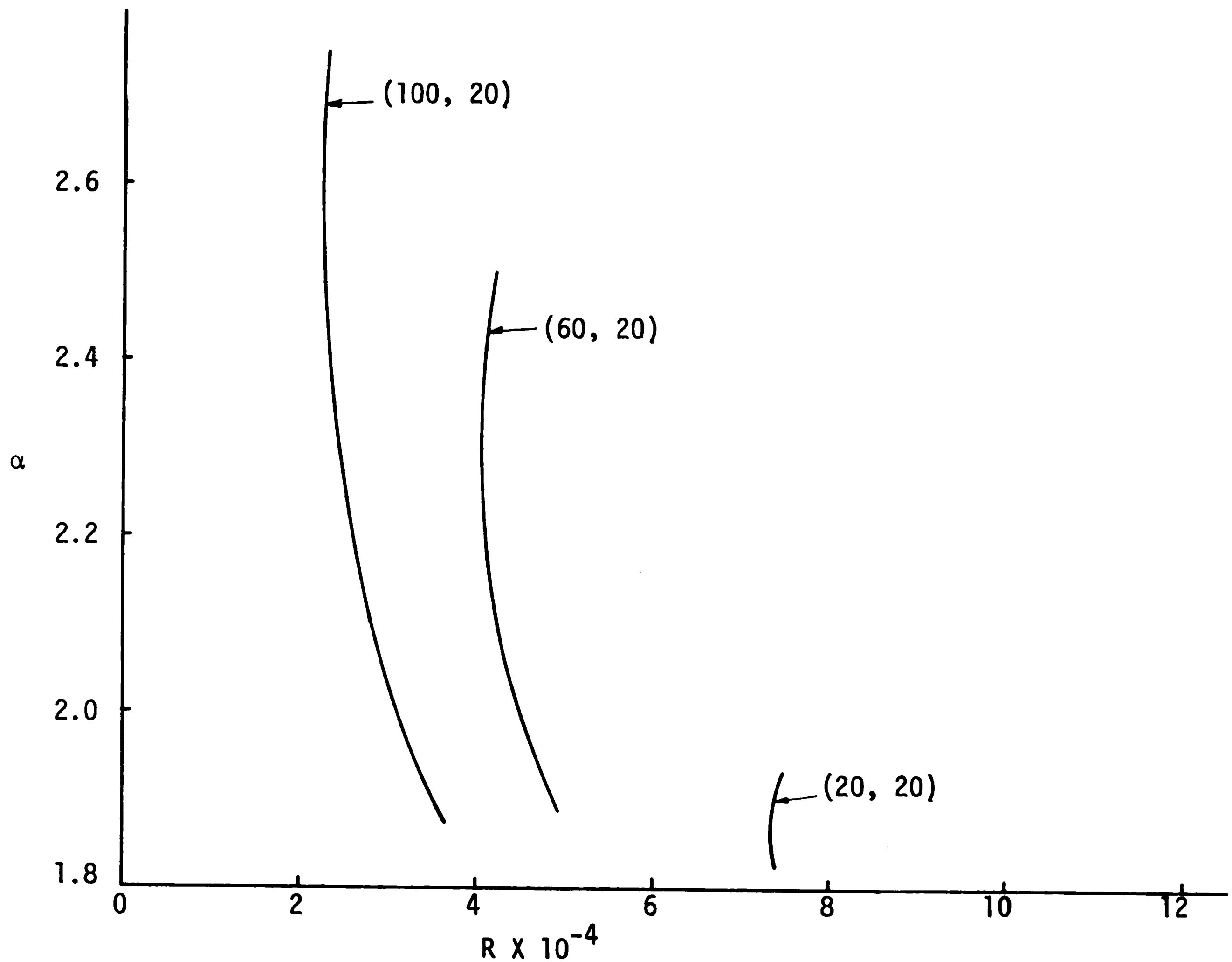

Figure 11.--Wave Number α Versus Reynolds Number R for $c_i = 0$, $a/b = 0.3$, and Various Boundary Temperatures (T_i, T_o)

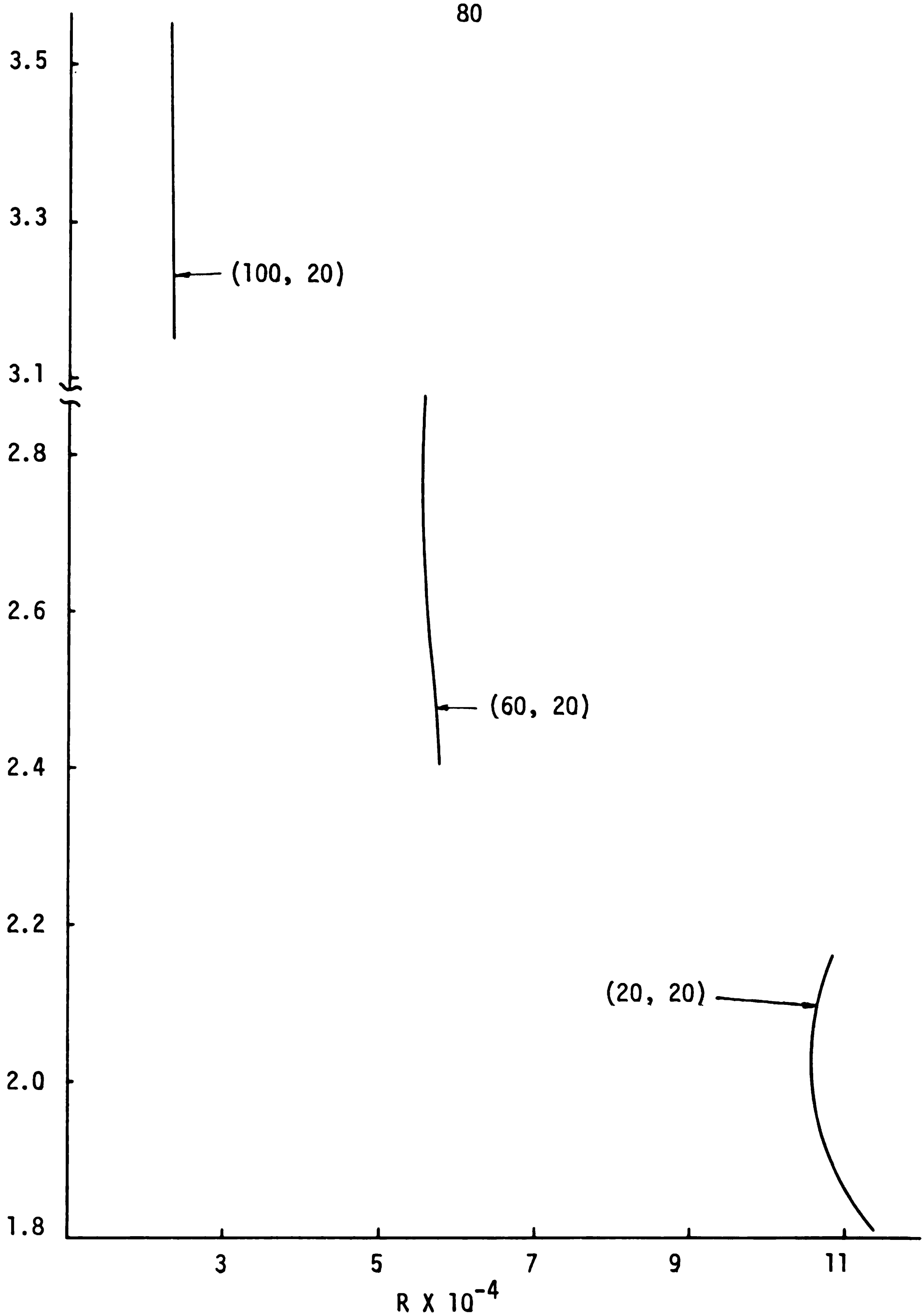

Figure 12.--Wave Number α Versus Reynolds Number R for $c_i = 0$, $a/b = 0.25$, and Various Boundary Temperatures (T_i, T_o)

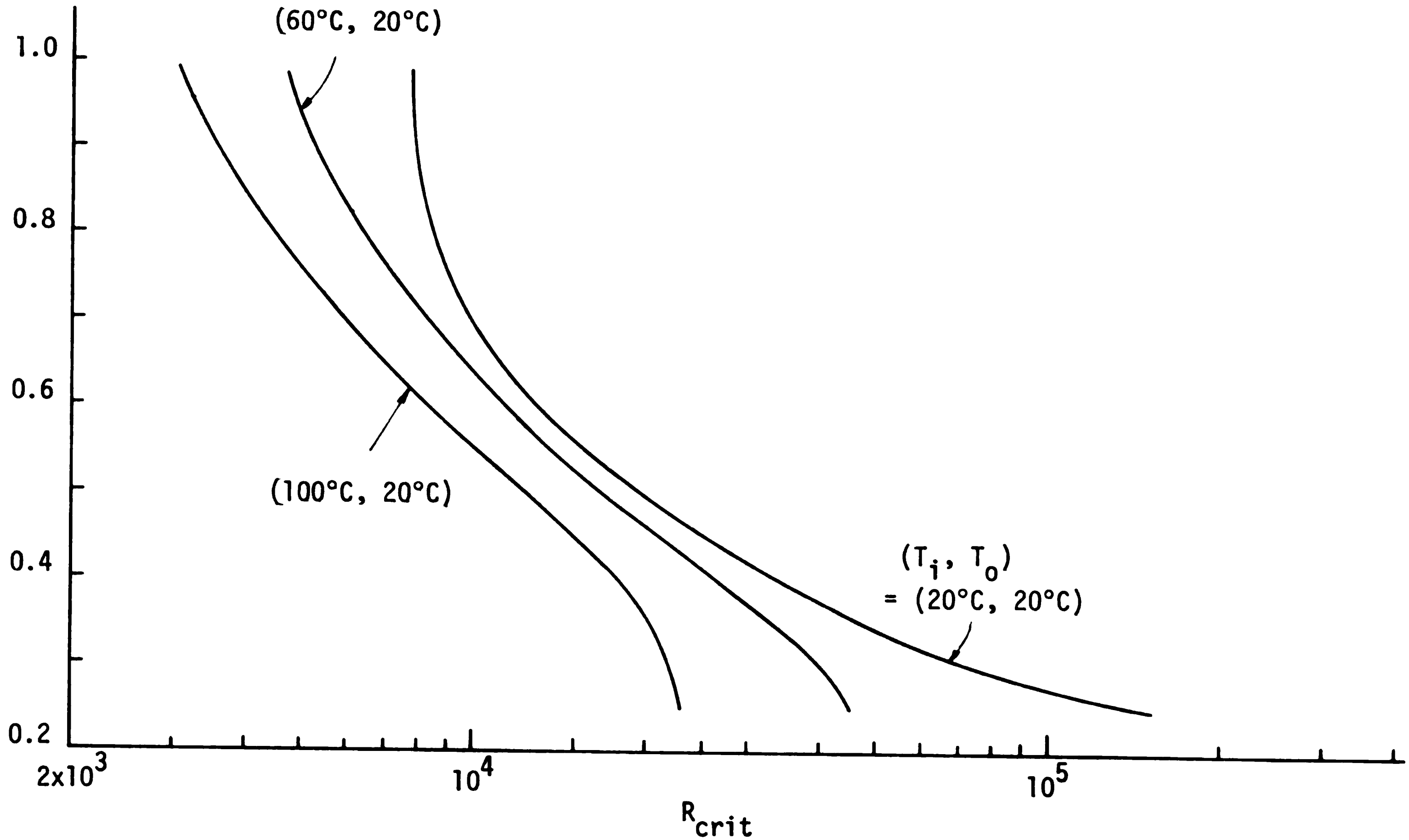

Figure 13.--Critical Reynolds Number R_{crit} Versus Radius Ratio a/b

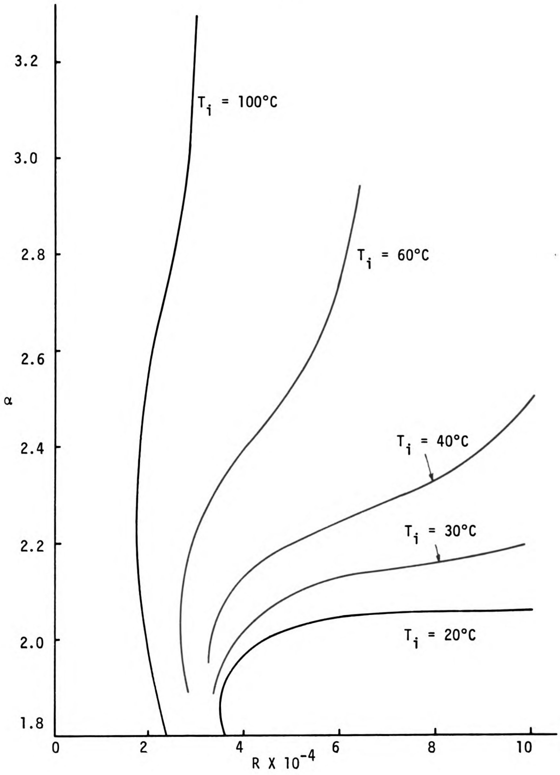

Figure 14.--Effect of Heat Transfer on the Neutral Stability Curve for a/b = 0.4, T_o = 20°C, and Various T_i

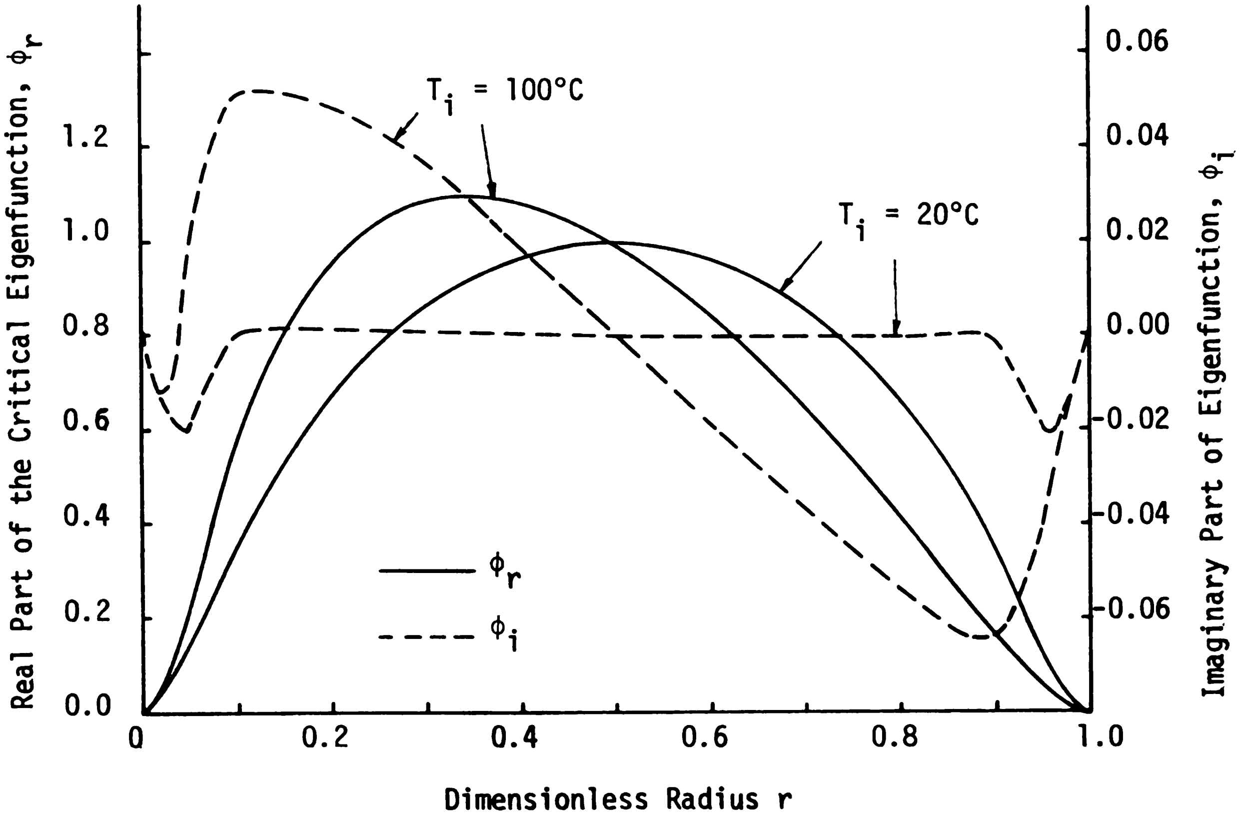

Figure 15.--Eigenfunction at the Critical Point for a/b = 0.99, T_o = 20°C.

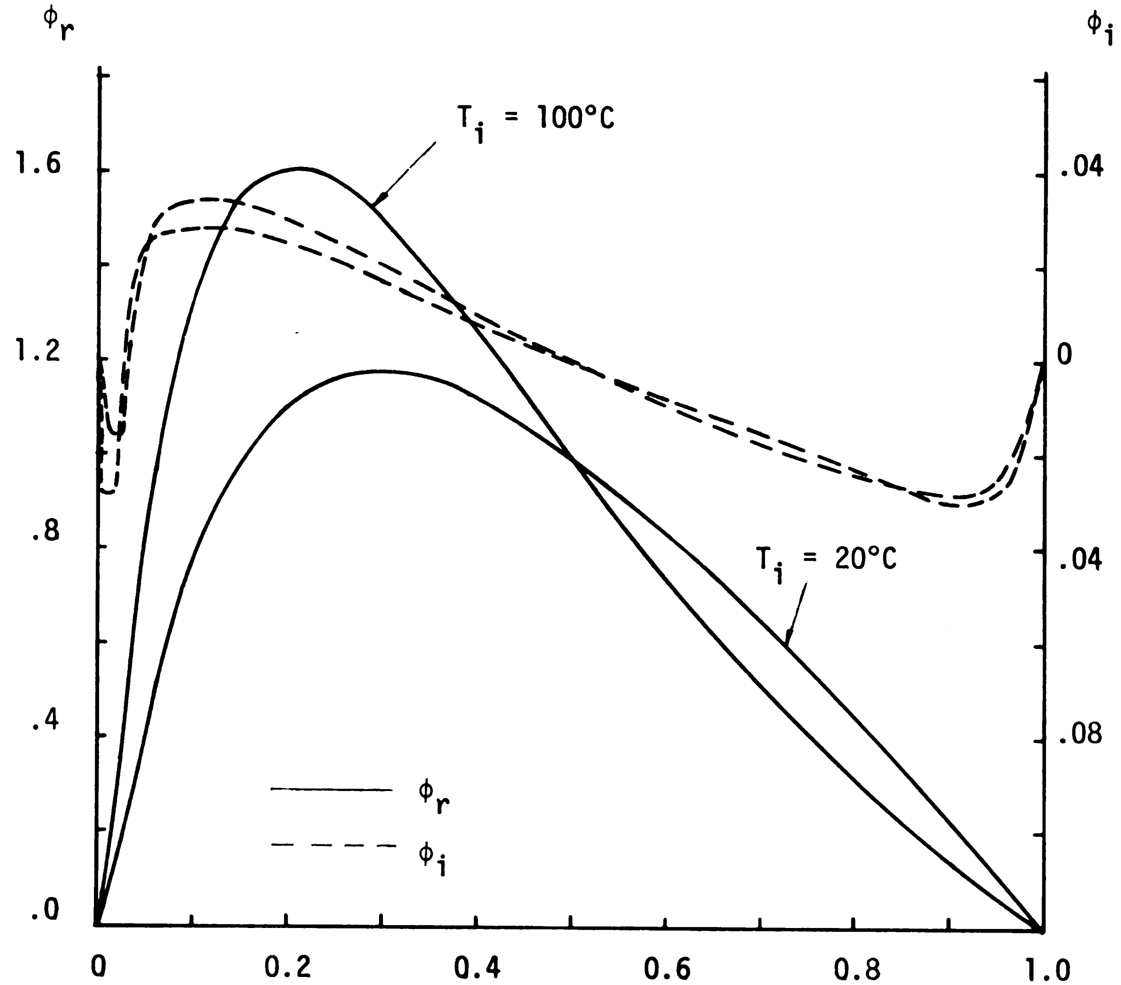

Figure 16.--Eigenfunction at the Critical Points for a/b = 0.4, T_o = 20°C

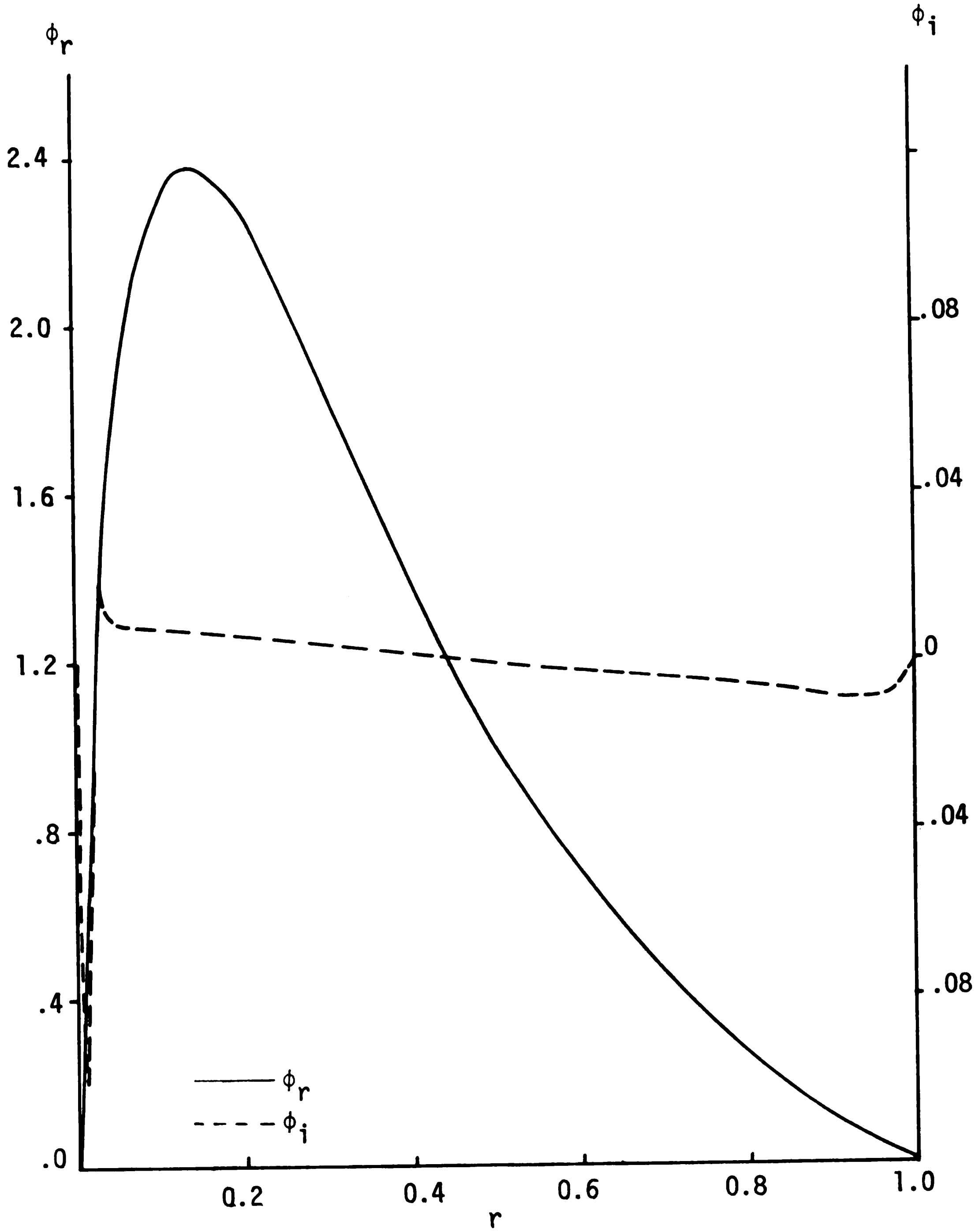

Figure 17.--Eigenfunction at α = 3.2915, R = 29613, C_r = 0.258 for a/b = 0.4, T_i = 100°C, T_o = 20°C

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APPENDIX A

RUNGE-KUTTA SCHEME FOR THE FOURTH ORDER DIFFERENTIAL EQUATION

APPENDIX A

Runge-Kutta Scheme for the Fourth Order Differential Equation

The Runge-Kutta scheme for differential equations of the fourth order which is illustrated by Collatz (1960) is used to integrate the stability equation. It is outlined below:

At $r = r_0$

$$V_{00} = \phi(r_0)$$

$$V_{10} = \Delta r \, \phi'(r_0)$$

$$V_{20} = \frac{(\Delta r)^2}{2} \phi''(r_0)$$

$$V_{30} = \frac{(\Delta r)^3}{6} \phi'''(r_0)$$

where Δr = step size. Using the equation (2.3.8)

$$G_1 = \frac{(\Delta r)^4}{24} f \left(\frac{6V_{30}}{(\Delta r)^3}, \frac{2V_{20}}{(\Delta r)^2}, \frac{V_{10}}{\Delta r}, V_{00}, r\right)$$

$$= \frac{(\Delta r)^4}{24} \{ \frac{i\alpha R}{\mu} [(W - c_r) (\phi'' + \frac{1}{r} \phi' - \frac{1}{r^2} \phi - \alpha^2 \phi)$$

$$- (\frac{d^2W}{dr^2} - \frac{1}{r} \frac{dW}{dr}) \phi]$$

$$- \frac{1}{\mu} \frac{d^2\mu}{dr^2} [\phi'' + \frac{1}{r} \phi' - (\frac{1}{r^2} - \alpha^2) \phi]$$

$$- \frac{1}{\mu} \frac{d\mu}{dr} [2\phi''' + \frac{3}{r} \phi'' - (\frac{3}{r^2} + 2\alpha^2)\phi'$$

$$+ (\frac{3}{r^3} - \frac{\alpha^2}{r}) \phi]$$

$$+ [- \frac{2}{r}\phi''' + (\frac{3}{r^2} + 2\alpha^2)\phi'' - (\frac{3}{r^3} - \frac{2\alpha^2}{r})\phi'$$

$$- (- \frac{3}{r^4} + \frac{2\alpha^2}{r^2} + \alpha^4)\phi]$$

Values at $r = r_0 + \frac{\Delta r}{2}$

$$V_{01} = V_{00} + \frac{1}{2}V_{10} + \frac{1}{4}V_{20} + \frac{1}{8}V_{30} + \frac{1}{16}G_1$$

$$V_{11} = V_{10} + V_{20} + \frac{3}{4}V_{30} + \frac{1}{2}G_1$$

$$V_{21} = V_{20} + \frac{3}{2}V_{30} + \frac{3}{2}G_1$$

$$V_{31} = V_{30} + 2G_1$$

As before

$$G_2 = \frac{(\Delta r)^4}{24} f \left(\frac{6V_{31}}{(\Delta r)^3}, \frac{2V_{21}}{(\Delta r)^2}, \frac{V_{11}}{\Delta r}, V_{11}, r\right)$$

Values at $r = r_0 + \frac{r}{2}$

$$V_{02} = V_{00} + \frac{1}{2}V_{10} + \frac{1}{4}V_{20} + \frac{1}{8}V_{30} + \frac{1}{16}G_1$$

$$V_{12} = V_{10} + V_{20} + \frac{3}{4}V_{30} + \frac{1}{2}G_1$$

$$V_{32} = V_{20} + \frac{3}{2}V_{30} + \frac{3}{2}G_1$$

$$V_{32} = V_{30} + 2G_2$$

Again as before

$$G_3 = \frac{(\Delta r)^4}{24} f \left(\frac{6V_{32}}{(\Delta r)^3}, \frac{2V_{22}}{(\Delta r)^2}, \frac{V_{12}}{\Delta r}, V_{02}, r\right)$$

Values at $r = r_0 + \Delta r$

$$V_{03} = V_{00} + V_{10} + V_{30} + G_3$$

$$V_{13} = V_{10} + 2V_{20} + 3V_{30} + 4G_3$$

$$V_{23} = V_{20} + 3V_{30} + 6G_3$$

$$V_{33} = V_{30} + 4G_3$$

Again

$$G_4 = \frac{(\Delta r)^4}{24} f \quad \left(\frac{6V_{33}}{(\Delta r)^3}, \frac{2V_{23}}{(\Delta r)^2}, \frac{V_{13}}{\Delta r}, V_{03}, r\right)$$

The eigenfuction ϕ and its derivatives at $r = r_0 + \Delta r$ are

$$\phi(r_0 + \Delta r) = V_{00} + V_{10} + V_{20} + V_{30} + G$$

$$\phi' (r_0 + \Delta r) = \frac{1}{\Delta r} (V_{10} + 2V_{20} + 3V_{30} + G')$$

$$\phi'' (r_0 + \Delta r) = \frac{2}{(\Delta r)^2} (V_{20} + 3V_{30} + G'')$$

$$\phi''' (r_0 + \Delta r) = \frac{6}{(\Delta r)^3} (V_{30} + G''')$$

where $G = \frac{1}{15} (8G_1 + 4G_2 + 4G_3 - G_4)$

$$G' = \frac{1}{5} (9G_1 + 6G_2 + 6G_3 - G_4)$$

$$G'' = 2 (G_1 + G_2 + G_3)$$

$$G''' = \frac{2}{3} (G_1 + 2G_2 + 2G_3 + G_4)$$

APPENDIX B

COMPUTER PROGRAM

APPENDIX B

COMPUTER PROGRAM

B - 1 Description of the Computer Program

The main program initially reads the required input data such as control codes, number of integration steps across the annulus, radii and temperatures of inner and outer cylinders, and initial conditions to solve the governing equations of the main flow. It finds an initial guess of the temperature distribution assuming constant conductivity and negligible viscous dissipation. First approximation to viscosity and conductivity is found using this approximate temperature distribution. Pick a value for λ, which is an unknown constant in the equation of motion, and then call subroutine RKUTTA1.

Subroutine RKUTTA1 integrates the equation of motion for two linearly independent initial conditions using the fourth order Runge-Kutta integration scheme. Returning to the main program the two solutions are combined in order to have a new solution which satisfies the boundary conditions at both cylinders. The resulting velocity W is checked whether it satisfies the continuous constraint. If it does not, then subroutine PLAM is called to find a new value for λ.

Subroutine RKUTTA2 is called from the main program to integrate the complete and/or simplified energy equation as desired. Again, two independent solutions are produced and combined so as to satisfy the boundary conditions on both inner and outer cylinders. Using this new temperature distribution repeat some of the above steps to find new main velocity distribution. Check whether the main velocity and temperature satisfy a prescribed convergent criteria. If they do not, then repeat the required previous steps until convergence is obtained.

Subroutine YUNG analyzes the effect of dk/dr and Φ terms on the velocity and temperature distributions of the main flow.

Once the temperature distribution and the main velocity distribution across the annulus is found, the subroutine EIGENVP is called. The filtering scheme and the interation scheme for the eigenvalues discussed in Section 3.2 and 3.3 are implemented in this subprogram.

The integration of the stability equation is done by subroutine RK4 which uses the scheme illustrated in Appendix A.

B - 2 Listing of the Computer Program

The program developed to compute the velocity and temperature distributions of the main flow, and the eigenvalues of the stability equation is listed below.

PROGRAM YCHOO TRACE

```
      PROGRAM YCHOO(INPUT,OUTPUT=65)
      DIMENSION R(201),W1(201), W2(201), W0(201), T1(201), T2(201),T0(201)
     11),C(11),D(11), W3(201),VIS(201), K(201), DKDR(201),T3(201),
     1 CODE(10)
      COMMON /A/ IT, ERR, H, LAMBDA, DEL, IPOS
      COMMON/B/L, DR, DT, SUM, RI
      COMMON/C/ PR,ECK, TI50,TI050, VIS20, K20
      COMMON/D/ NONLIN,RO
      COMMON/E/ SIGN, TI, TO, N, MON1, MON2
      COMMON/F/ REY, MONITOR, IC6 , IC9
      COMMON R,W1,W2,T1,T2,C,D , VIS, K, DKDR
      TYPE REAL LAMBDA ,K20, K
      TYPE INTEGER CODE
C
C                       MODIFIED ON 9-27-1974.
C READ AND PRINT DATA    VIS=VIS(C,T)    K=K(D,T)  W1=VELOCITY  T1=TEMPERATURE
C
      READ 175, (CODE(I), I=1,10)
  175 FORMAT(10I5)
C IF CODE(1)=1, THEN BOTH DK/DR AND VISCOUS DISSIPATION TERMS WILL BE NEGLECTED.
C IF CODE(2)=1, DK/DR TERM IS KEPT AND VISCOUS DISSIP. TERM=0.
C IF CODE(3)=1, THEN DK/DR=0, ANDVISCOUS DISSIP. TERM WILL BE KEPT.
C IF CODE(4)=1, THEN BOTH DK/DR AND VISCOUS DISSIP. TERMS WILL BE KEPT.
C  IF CODE(1)=CODE(2)=CODE(3)=CODE(4)=0, THEN ALL THE ABOVE CASES WILL BE CONSID
C ERED.
C IF CODE(5)=1, THEN SUBROUTINE YUNG WILL NOT BE CALLED.
C CODE(6)=0 FOR DETAILED PRINTOUTS, =1 FOR LIMITED PRINTOUTS, =2 FOR MIN. PRINTO
C PRINTOUTS.
C CODE(7)=1FOR I=1,2,3,......   =10 FOR I=1,6,11,......  IN YUNG.
C IF CODE(8)=0 TO CALL SUBROUTINE EIGENVP,  =1 TO SKIP SUBROUTINE EIGENVP, AND
C =2 FOR REY=REY(CRITICAL) AND THEN RECALCULATE THE WHOLE C THING.
        IC9=0
      IF(CODE(1) .EQ.1 .OR. CODE(2).EQ.1 .OR. CODE(3).EQ.1 .OR. CODE(4).EQ.1) CC
     1EQ.1) IC9=1
      IC6= CODE(7)
      READ 150, REY
  150 FORMAT( F10.0)
      READ 1, (C(I), I=1,11)
      READ 1, (D(I), I=1,11)
      READ 2,    W1(1), SLOPE1, SLOPE2
      READ 3, N, PC
    1 FORMAT(8F10.7)
    2 FORMAT(3F10.5)
    3 FORMAT(I5, F10.5)
C
C          DEFINITION AND CALCULATION     PR=PRANDTL NUMBER  CP=SPECIFIC HEAT
C                      H= STEP SIZE   EPS = MEASURE OF CONVERGENCE
C NCR COUNTS NUMBER OF CALCULATIONS WITH REYNOLDS NUMBERS WHEN THE VISCOUS
C DISSIPATION TERM IS KEPT.
      NCR=1
      H=1.0/N
      CP=1.0
      M = N + 1
       FH= .004*H
       EPS= 0.0004 * H
      RHO=1.0
      W3W1 = 1.0
      SIGN =1.
          RATIO = 1.0
      T20 = (20.0 - 50.0) / 50.
      VIS20= 0.01002
      K20= 0.0014307
```

```
      PR = VIS20 * CP / K20
C
C
C
C          READ ADDITIONAL DATA
      READ 2, RI, TI, TO
       IF(TI-TO) 9100,9200,9100
 9200  CODE(2)=0
      MON1=0
       IF(CODE(1).NE.1) MON2=1
      IF(CODE(1).NE.1) CODE(3)=1
      IF(CODE(1) .EQ. 1)  CODE(3)=0
C          ECK = ECKERT NUMBER
 9100 DR = RO - RI
      DT = TO - TI
      IF(DT.LT.0.) SIGN = -1.
  PROGRAM     YCHOD     TRACE

      G=41867240.71
      VV = ( (VIS20*REY   )/DR )**2
      ADT = ABS(DT)
      ECK = VV/(CP*ADT) / G
C
C
C          READ THE FIRST TRIAL VALUE OF LAMBDA
      READ 2, LAMBDA, DEL
C
C          PRINT DATA AND OTHER CONSTANTS DEFINED ABOVE
      IF( CODE(6)-1) 6210,6220,6230
 6210 PRINT 1090, REY
      PRINT 6, (C(I), I=1,11)
      PRINT 7, (D(I), I=1,11)
      PRINT 8, W1(1), SLOPE1,           W1(1), SLOPE2
 6220 PRINT 9, RI, RO, TI, TO
      PRINT 10, N, H, CP, RHO
      PRINT 11, VIS20, K20, PR, ECK
    6   FORMAT(* * 8F12.6)
    7   FORMAT(* CONSTANTS,D, IN K(D,T) ARE*,/, 3X,8F12.6)
    8   FORMAT(*0INITIAL CONDITIONS, VELOCITY AND ITS FIRST DERIVATIVE*,/,
     1 10X, 2F10.2, 10X, 2F10.2 )
    9   FORMAT(*0RI = *F6.2  ,5X,*RO = *F6.2,5X*TI= *F6.2,5X,*TO = *F6.2)
   10   FORMAT(*0N = *I5,5X,*H=*F7.3,5X, *CP=*F5.1,5X,*RHO=*F5.0)
   11   FORMAT(*0 VIS20=* F12.7 ,8X, *K20=* F12.7 ,8X, *PR=*F12.7,8X,
     1 *ECK=* F12.7)
 1090 FORMAT(*-REYNOLDS NUMBER = * F15.0)
C
C
 6230  TI50 = (TI-50)/50
      TIOLO = (TO-TI)/50
C
C
C                 NONLIN COUNTS   NUMBER OF ITERATIONS.
C ONE ITERATION IN THIS CASE INVOLVES THE SOLUTION OF BOTH EQ. OF MOTION AND
C ENERGY EQ.
C                 IT COUNTS NUMBER OF TRIALS OF LAMBDA VALUES TO GET THE SOLUTION
C                 OF THE EQ. OF MOTION.
C
      MONITOR=1 $ MON1=0  $  MON2=0
 1010  NONLIN=0
      IPOS=0
   12  IT = 1
```

```
      IIT=0
   13 CONTINUE
C              ICOUNT COUNTS THE NUMBER OF IVP SOLVED FOR A CHOSEN LAMBDA.
      ICOUNT = 0
   14 CONTINUE
      IF( ICOUNT - 1 )       15,16,16
   15 W2(1) = SLOPE1
      GO TO 17
   16 W2(1) = SLOPE2
   17 ICOUNT = ICOUNT + 1
C
C
C                   SOLVE EQ. OF MOTION WITH THE ASSUMPTION THAT DK/DR=PHI=0.
C                   FIND W1(L+1) FROM W1(L) BY RUNGE - KUTTA.
C
      DO 18 L = 1,M
       IF(MONITOR .GT. 1) GO TO 20
      IF( NONLIN - 1 ) 19,20,20
   19 IF(ICOUNT.GT.1) GO TO 20
      R(L) = H* (L-1)
      AG = ( RI + DR*R(L) ) / RI
      T1(L) = 1.0/ ALOG(RO/RI) * ALOG(AG)
C
                AR3=0.
      DO 700 J=1,11
      JJ=J-1
       AR3= AR3 + D(J)*(TI50+TI050*T1(L))**JJ
  700 CONTINUE
       K(L)= EXP(AR3)/K20
C
   20 CONTINUE
C
      ARG = 0.0
      DO 21 I = 1,11
      II = I-1
PROGRAM    YCH00    TRACE

      ARG = ARG + C(I) * (TI50 + TI050*T1(L))**II
   21 CONTINUE
      VIS(L) = EXP(ARG) / VIS20
C
      SUM = 0.0
      DO 22 J = 1, 10
      JJ = J - 1
      J2 = J+1
      SUM = SUM + J *C(J2)* ( TI50 + TI050* T1(L) )**JJ
   22 CONTINUE
C                   D(VIS)/DR=VIS*SUM*DT/50.*T2(L)
      CALL RKUTTA1(M)
C
   18 CONTINUE
C
C
C
C
      IF(ICOUNT -1) 23,23,24
   23 DO 25 I = 1,M
      W0(I) = W1(I)
   25 CONTINUE
      GO TO 14
C              W1(M) = 0 = W0(M) + X*W1(M)    SUPERPOSITION
```

```
   24  X = -W0(M) /W1(M)
       DO 26 I = 1, M
       W1(I) = W0(I) + X* V1(I)
   26  CONTINUE
C
C
C              CHECK THE CONSTRAINT CONDITION
C
       SIMSON = 0.0
       N1 = N - 1
       DO 27 I = 1, N1, 2
       SIMSON = SIMSON + W1(I) * (PI+DR*R(I))
      1                + 4*W1(I+1) * (PI+DR*R(I+1))
      1                + W1(I+2) * (PI+DR*R(I+2))
   27  CONTINUE
       SIMSON = H/3.0 * SIMSON
       ER = SIMSON - (PO+RI) /2.0
       ERR = ABS(ER)
       IF(CODE(6)-1) 6610,6610,6620
 6610  PRINT 53,ER,LAMBDA
  53   FORMAT(* ER = *E15.7,10X,*AT LAMBDA =*F12.6)
 6620  IF(ERR .LT.HH) GO TO 54
       IIT=IIT+1
       IF(IIT.GT.12) STOP 5
C
C                 FIND A NEW LAMBDA BY CALLING SUBROUTINE  PLAM IF NECESSARY.
       CALL PLAM
C
C
       GO TO 13
C
C
C
  54   PRINT 20, NONLIM, ERR, LAMBDA
C
C             FIND W2 AND DKDR=1.0/K*DK/DR FROM W1 AND K.
C             THEY WILL BE USED IN RKUTTA2.
C
       DO 50 I = 1,M
       IF(I.GE.N ) GO TO 51
       W2(I)=1./(2*H)*(-3*W1(I) + 4*W1(I+1) - W1(I+2) )
C                      DKDR= 1.0/K * DK/DR
       DKDR(I)=1.0/(2*H)*(-3*K(I) + 4*K(I+1) -K(I+2))/K(I)
       GO TO 50
    51 W2(I) = 1.0/(2*H)*(3*W1(I) - 4*W1(I-1) + W1(I-2) )
       DKDR(I)=1.0/(2*H) * (3*K(I)-4*K(I-1)+K(I-2)) / K(I)
  50   CONTINUE
C
C
       TDIF=0.
       VELDIF = 0.0
       IF(CODE(6)-1) 6250,6250,870
C          PRINT THE SOLUTION OF THE EQ. OF MOTION AND CHECK ITS CONVERGENCE
 6250  IF(NONLIM.EQ.0) GO TO 850
  PROGRAM    YS100     TRACE                                 v3.0-P380 OPT=0  04/12/75 .00.39.03.

       PRINT 860
       GO TO 870
 850    PRINT 859
 870   CONTINUE
 860   FORMAT(*-*6X*R*12X*W1*13X*W2*12X*T1*12X*T2*12X*VIS*12X* K *12X*DK/
```

```
          1DP*)
C
C
      DO 29 I = 1, M
      IF( NONLIN - 1 )130,31,31
 31   W3W1 = W3(I)-W1(I)
      T3T1=T3(I)-T1(I)
      DIF=ABS(W3W1)
      AT=ABS(T3T1)
      IF( DIF.GT. VELDIF ) VELDIF = DIF
      IF(AT.GT. TDIF) TDIF=AT
                IF(I-1) 910,910,920
 910            RATIO=0. $ TRATIO=0.
                GO TO 930
 920  RATIO= DIF/ W1(I)
      TRATIO=AT/T1(I)
930    CONTINUE
 130  W3(I) = W1(I)
       T3(I)= T1(I)
  29  CONTINUE
       IF(CODE(6)-1) 6270,8500,8500
6270   DO 8510 I=1,M,IC6
      PRINT 32, R(I), W1(I), W2(I), T1(I) , T2(I),VIS(I),K(I),DKDR(I)
8510     CONTINUE
8500   CONTINUE
C
 32   FORMAT(1X, F9.3, F15.8,F15.6, 5E15.5)
 28   FORMAT( *- NONLIN =* I5,10X*ERR =*E15.8 ,10X,*AT LAMBDA =*F12.6 )
C
      IF(CODE(1).EQ.1) GO TO 43
      IF(NONLIN.EQ.0) GO TO 120
      IF( VELDIF.LT.EPS .AND. TDIF.LT. EPS .AND. MONITOR.EQ.2) GO TO 41
      IF( VELDIF.LT.EPS .AND. TDIF.LT. EPS .AND. MONITOR.EQ.3) GO TO 42
      IF( VELDIF.LT.EPS .AND. TDIF.LT. EPS .AND. MONITOR.EQ.4) GO TO 43
                PRINT 950, VELDIF, TDIF
 950            FORMAT(*0 VELDIF=*E15.5,10X* TDIF=*E15.5      )
 120   CONTINUE
C
      IF(MON1.NE.0 .OR.  MON2.NE.0) GO TO 1080
      PRINT 1020, MON1, MON2
      IF( CODE(5).EQ.1) GO TO 3000
      CALL YUNG
3000  IF(TI-T0)9300,9400,9300
9300  MON1=1 $ MON2=0
      MONITOR = MONITOR+1
1020  FORMAT(* MON1=*I2,5X,*MON2=*I2)
1021  FORMAT(2I5)
1023  FORMAT(2F15.10)
      GO TO 1080
9400   MONITOR = 3
      MON1=0  $  MON2=1
C
C
C               SOLVE   ENERGY EQ. WHICH INCLUDES VISCOUS DISSIPATION
C               AND DK/DR TERMS.
C
 1080  NUM=1
C
C
      IF(CODE(3)-1) 5510,5520,5520
5520  MONITOR=3$ MON1=0 $ MON2=1 $ GO TO 34
```

```
5310  IF(CODE(4)-1) 34,5001,5001
5001   MONITOR = 4 $ MON1=1 $ MON2=1
 34   CALL RKUTTA2
      IF(NUM-1) 36,36,37
  36  DO 38 I = 1,M
      T0(I) = T1(I)
  38  CONTINUE
      NUM = NUM + 1
      T2(1) = T2(1) - .5
      GO TO 34
C
C              FIND NEW T1 BY SUPERPOSITION.
PROGRAM     YCHOO     TRACE

C               CALCULATE K AND DKDR FOR THIS NEW T1.
C
C               T1(M)= 1 = TO(M) + X*T1(M)          SUPERPOSITION
 37            PRINT 510
510   FORMAT(*-*10X*R*12X*T1*12X*VIS*12X*K*12X*DKDR*)
      DO 40 I = 1,M
      T1(I) = T0(I) + ( 1 - TO(M) ) / T1(M)   * T1(I)
       AR2=0.
      DO 606 J=1,11
       JJ=J-1
      AR2=AR2+D(J)*(TI50 + TI050*T1(I))**JJ
 606   CONTINUE
      K(I)= EXP(AR2)/K20
C
C
  40  CONTINUE
C
C
C
      NONLIN = NONLIN + 1
                                 IF(NONLIN.GT.5) STOP 6
C
C              CALCULATE NEW T2 WHICH WILL BE USED IN RKUTTA1.
C
      DO 800 I=1,M
      IF(I.GE. M ) GO TO 810
      T2(I)=1./(2*H)*(-3*T1(I)+ 4*T1(I+1)-T1(I+2))
      GO TO 800
 810  T2(I)=1.0/(2*H)*(3*T1(I)-4*T1(I-1)+T1(I-2))
 800   CONTINUE
      GO TO 12
C
C
C
C
41    IF(CODE(6)-1) 6310,6310,6320
6310  DO 2010 I=1,M , 10
      PRINT 32, R(I), W1(I), W2(I), T1(I) , T2(I),VIS(I),K(I),DKDR(I)
 2010   CONTINUE
 3005 PRINT 125,VELDIF , TDIF , MON1, MON2
 125  FORMAT(* ITERATION CONVERGES. VELDIF = * E15.5,10X*TDIF=*E15.5,
     15X,*MON1=*I2,5X,*MON2=*I2)
6320  CONTINUE
      IF(CODE(5).EQ.1) GO TO 5002
      CALL YUNG
5002  IF(CODE(2) .EQ. 1) GO TO 6000
 3001 MON1=0 $ MON2=1
```

```
                  MONITOR= MONITOR+1
          42      IF(CODE(6)-1)  6330,6330,6340
          6330     DO 2020 I=1,M, 10
                  PRINT 32, R(I), W1(I), W2(I), T1(I) , T2(I),VIS(I),K(I),DKDR(I)
           2020    CONTINUE
          3006    PRINT 125,VELDIF , TDIF , MON1, MON2
          6340    CONTINUE
                  IF(CODE(5).EQ.1) GO TO 3002
                  CALL YUNG
                  IF(CODE(3).EQ.1) GO TO 6000
           3002 MON1=1 $ MON2=1
                  MONITOR= MONITOR+1
                  GO TO 1010
          43      IF(CODE(6)-1)6350,6350,6000
          6350     DO 2030 I=1,M , 10
                  PRINT 32, R(I), W1(I), W2(I), T1(I) , T2(I),VIS(I),K(I),DKDR(I)
           2030     CONTINUE
          3007    PRINT 125,VELDIF , TDIF , MON1, MON2
          6000     CONTINUE
          6010    IF(CODE(5).EQ.1)  GO TO 2040
                  CALL YUNG
          2040       CONTINUE
                  IF(CODE(8)-1) 8810,8820,8810
          8810     CALL EIGENVF
          8820     CONTINUE
                  NCR=NCR+1
                  IF(NCR .LT. 3  .AND. CODE(8).EQ.2) GO TO 1010
                  END
 SUBROUTINE  RKUTTA1   TRACE

                  SUBROUTINE RKUTTA1(M)
                  TYPE REAL  K11, K21, K12, K22, K13, K23, K14, K24, LAMBDA , K
                  COMMON /A/ IT, ERR, H, LAMBDA, DEL, IPOS
                  COMMON/B/L, DR, DT, SUM, RI
                  COMMON/D/ NONLIN,RO
                  COMMON R,W1,W2,T1,T2,C,D , VIS, K, DKDR
                  DIMENSION C(11), D(11)
                  DIMENSION R(201), W1(201), W2(201), T1(201) , T2(201), VIS(201),
                 1  K(201), DKDR(201)
          C
          C
                   VI = VIS(L)
                  IF(NONLIN.EQ.0) T2(L)=1./ALOG(RO/RI)*DR/(RI+DR*R(L))
                  IF(L.EQ. M) GO TO 100
                  K11 = H * W2(L)
                  K21 = H * ( LAMBDA - (VI *SUM*DT/50.*T2(L) + VI *DR/(RI+DR*R(L)))*W2(L))/V
                 1W2(L))/VI
                  K12 = H * ( W2(L) + 1./2. * K21 )
                  K22 = H * ( LAMBDA - (VI *SUM*DT/50.*T2(L) + VI *DR/(RI + DR*(R(L)+1/2.*H)
                 1+1./2*H)))*(W2(L) +1./2*K21)) /VI
                  K13 = H * (W2(L) + 1.0/2 *K22)
                  K23 = H * ( LAMBDA - (VI *SUM*DT/50.*T2(L)+VI *DR/(RI+DR*(R(L)+1/2.
                 1.*H)))*( W2(L) +1./2*K22)) /VI
                  K14 = H * (W2(L) + K23)
                  K24=H * (LAMBDA-(VI *SUM*DT/50.*T2(L) +VI *DR/(RI+DR*(R(L)+H)))*
                 1 (W2(L)+K23)) /VI
          C
          C
                  W1(L+1) = W1(L) + 1.0/6 * ( K11 + 2*K12 + 2*K13 + K14 )
                  W2(L+1) = W2(L) + 1.0/6 * ( K21 + 2*K22 + 2*K23 + K24 )
```

```
 100    CONTINUE
        RETURN
        END
SUBROUTINE  PLAM       TRACE

        SUBROUTINE PLAM
        TYPE REAL LAMBDA, LAM1, LAM2, LAM3
        COMMON /A/ IT, ERR, N, LAMBDA, DEL, IPOS
C
C
    2   IF(IT-2) 3,4,5
C
    3   E1 = ERR
        LAM1 = LAMBDA
        LAM2 = LAM1 + DEL
        LAMBDA = LAM2
        IT = 2
        GO TO 15
C
    4   E2 = ERR
        LAM3 = LAM1 - E1 * (LAM2-LAM1) / (E2-E1)
        IF(LAM3.LT.0.0) GO TO 6
        LAM3 = LAM2 + DEL
    6   LAMBDA = LAM3
        IT = 3
        GO TO 15
C
C
C
    5   E3 = ERR
        B1 = ABS(LAM3 - LAM1)
        B2 = ABS(LAM3 - LAM2)
        IF(B1-B2) 7,8,8
C           LAM1 IS CLOSER TO LAM3.
    7   LAM2 = LAM3
        E2   = E3
        LAM3 = LAM1 - E1*(LAM2-LAM1) / (E2-E1)
        IF( LAM3.GE.0.0) GO TO 9
        LAMBDA = LAM3
        IT = IT + 1
        IF( IT.GT.7) GO TO 10
        GO TO 15
    9   LAM3 = -1.0
        IPOS = IPOS + 1
   10   IT = 1
        LAM1 = LAM3 + 10.*DEL  * IPOS
        LAMBDA = LAM1
        GO TO 15
C           LAM2 IS CLOSER TO LAM3.
    8   LAM1 = LAM3
        E1 = E3
        LAM3 = LAM1 - E1*(LAM2-LAM1) / (E2-E1)
        IF( LAM3 .GE. 0.0) GO TO 12
        LAMBDA = LAM3
        IT = IT + 1
        IF( IT.GT.7) GO TO 13
        GO TO 15
   12   LAM3 = -1.0
        IPOS = IPOS + 1
   13   IT = 1
        LAM1 = LAM3 + 10.*DEL  * IPOS
```

```
      LAMBDA = LAM1
   15 RETURN
      END
SUBROUTINE  RKUTTA2    TRACE

      SUBROUTINE RKUTTA2
      TYPE REAL K11, K21, K12, K22, K13, K23, K14, K24, K, LAMBDA
      COMMON /A/ IT, ERR, H, LAMBDA, DEL, IPOS
      COMMON/B/L, DR, DT, SUM, RI
      COMMON/C/ PR,ECK, TI50,TI050, VIS20, K20
      COMMON/E/ SIGN,TI, TO , N, MON1, MON2
      TYPE REAL K20
      COMMON R,W1,W2,T1,T2,C,D , VIS, K, DKDR
      DIMENSION R(201), W1(201), W2(201), T1(201), T2(201) , VIS(201),K(105), DK
     1DR(201), DKDR(201)
      DIMENSION C(11),D(11)
C
C                 DKDR(L)= 1./K(I) * DK(L)/DR
C
      DO 35 L = 1,N
      K11 = H*T2(L)
      K21 =(-DR/(RI+DR*R(L))* T2(L) - DKDR(L) *T2(L) * MON1
     1-1./K(L)*PR*ECK*VIS(L)    *W2(L)**2 * SIGN * MON2 ) * H
C
C
      K12=H*(T2(L)+ 1/2.*K21)
      K22 =(-DR/(RI + DR*(R(L)+1./2*H))*(T2(L)+1./2*K21)
     1  -DKDR(L)        * (T2(L)+1./2*K21) * MON1
     1-1./K(L)*PR*ECK*VIS(L)*W2(L)**2* SIGN * MON2 ) * H
C
C
      K13= H*(T2(L)+1./2*K22)
      K23 =(-DR/(RI+ DR*(R(L) + 1./2*H))*(T2(L)+1./2*K22)
     1 -DKDR(L)       * (T2(L) +1./2*K22) * MON1
     1-1./K(L)*PR*ECK*VIS(L)*W2(L)**2      * SIGN * MON2 ) * H
C
C
      K14 = H*(T2(L) +          K23)
      K24 =(-DR/(RI + DR*(R(L)+      H))*(T2(L)+      K23)
     1      -  DKDR(L)  * (T2(L)+      K23) * MON1
     1-1./K(L)*PR*ECK*VIS(L) *W2(L)**2 *SIGN * MON2 ) * H
C
C
      T1(L+1) =1./6.*(K11+2*K12+2*K13+K14)  + T1(L)
      T2(L+1) = 1./6.*(K21+2*K22+2*K23+K24) + T2(L)
   35 CONTINUE
      RETURN
      END
SUBROUTINE  YUNG       TRACE

      SUBROUTINE YUNG
      DIMENSION S(201), V1(201), V2(201), F1(201), F2(201),C(11),D(11),
     1 VIS(201), K(201), DKDR(201)
      COMMON/B/L, DR, DT, SUM, RI
      COMMON/C/ PR,ECK, TI50,TI050, VIS20, K20
      COMMON/D/ NONLIN,RO
      COMMON/E/ SIGN, TI, TO, N, MON1, MON2
      COMMON/F/ REY, MONITOR, IC6 , IC7
      COMMON S, V1, V2, F1, F2, C, D, VIS, K, DKDR
      DIMENSION W1(201,4), T1(201,4), W2(201,4), T2(201,4)
       TYPE REAL K
```

```
C YUNG ANALYZES THE SIGNIFICANCE OF DKUR TERM AND VISCOUS DISSIPATION TERM
C FOR VARIOUS REYNOLS NUMBERS)
C
C
C
            V = REY* VIS20 / (RO-RI) / 1.0
       J= MONITOR
       M=N+1
C
       DO 90 I=1,M
       W1(I,J)=V1(I)
       T1(I,J)= F1(I)
       W2(I,J) =V2(I)
       T2(I,J) = F2(I)
 90     CONTINUE
        IF(J.EQ.4 .AND. IC9.EQ.1) GO TO 202
       IF(MONITOR .EQ. 4) GO TO 100
        GO TO 202
C
C           PRINT DATA
C
 100   PRINT 6, REY
 6     FORMAT(*1HHEN REYNOLDS NUMBER=*F15.0)
       PRINT 80, V
 80     FORMAT(*     V = * F12.5 * CM/S*)
       PRINT 7, RI, RO, TI, TO, PR, ECK
 7      FORMAT(* RI=*F6.2,5X,*RO=*F6.2,5X,*TI=*F6.2,5X,*TO=*F6.2,5X,*PR=*
      1F10.5,5X,*ECK=*F12.8)
       PRINT 8
 8      FORMAT(*-*15X*MON1=MON2=0*10X*MON1=1 $ MON2=0*10X*MON1=0 $ MON2=1*10X*MON
      1*10X*MON1 = MON2 = 1*)
       PRINT 9
 9     FORMAT(* R*12X*W(I,1)*5X *T(I,1)*7X *W(I,2)*5X*T(I,2)*7X *W(I,3)*
      15X*T(I,3)*6X *W(I,4)*5X*T(I,4)*)
       DO 10 I= 1,M    , IC6
       PRINT 11, S(I), W1(I,1),T1(I,1), W1(I,2),T1(I,2), W1(I,3),T1(I,3),
      1 W1(I,4),T1(I,4)
 10    CONTINUE
 11    FORMAT(* *F7.3, 4(3X,2F10.6))
       PRINT 301
 301    FORMAT(*- W2(I,K), T2(I,K) FOR I=1,101 AND K=1,4 *)
       DO 300 I=1,M    , IC6
       PRINT 11, S(I), W2(I,1),T2(I,1),W2(I,2),T2(I,2),W2(I,3),T2(I,3),
      1W2(I,4),T2(I,4)
 300    CONTINUE
C
C               CALCULATE DIFFERENCE AND RATIO OF VELOCITIES.
C
       PRINT 12
 12    FORMAT(*- W(I,2)-W(I,1)*5X*W(I,3)-W(I,1)*5X*W(I,4)-W(I,1)*10X*RATIO21*5X*R
      1021*9X*RATIO31*9X*RATIO41*)
C
       VP2=VP4=0. $ MP2=MP4=0
       VPEAK=0. $ MPEAK=0
       DO 13 I=1,M
       VEL21=W1(I,2)-W1(I,1)
       VEL31=W1(I,3)-W1(I,1)
       VEL41=W1(I,4)-W1(I,1)
       AW = ABS(W1(I,1))
       IF(W1(I,1).EQ.0. .OR. AW .LT. 1.0E-10) GO TO 15
       RAT21=VEL21/W1(I,1)
```

```
                RAT31= VEL31/W1(I,1)
                ARAT31=ABS(RAT31)
                IF(ARAT31.GT.VPEAK) VPEAK=ARAT31
                IF(ARAT31.EQ.VPEAK) MPEAK=I
                RAT41=VEL41/W1(I,1)
                AR2=ABS(RAT21)
SUBROUTINE   YUNG       TRACE

                AR4=ABS(RAT41)
                IF(AR2.GT.VP2)VP2=AR2
                IF(AR2.GT.VP2)MP2=I
                IF(AR4.GT.VP4) VP4=AR4
                IF(AR4.GT.VP4) MP4=I
                GO TO 30
          15    RAT21=0. $ RAT31=0. $ RAT41=0.
          30    CONTINUE
                IF(IC6.NE.1) GO TO 13
                PRINT 14, VEL21,VEL31,VEL41,RAT21,RAT31,RAT41
          13    CONTINUE
          14    FORMAT(* *3(E11.4,7X),5X,3(E11.4,5X))
                PRINT 310, VPEAK, S(MPEAK)
         310    FORMAT(*- MAX. RAT31 = *E15.5*  AT R = *F7.3)
          C
          C              CALCULATE DIFFERENCE AND RATIO OF TWO DIFFERENT TEMPERATURES.
          C
                PRINT 350, VP2, S(MP2), VP4, S(MP4)
         350    FORMAT(* MAX. RAT21=*E15.5*AT R=*F7.3,5X,*MAX RT41=*E15.5*AT R=*
               1F7.3)
                PRINT 50
          50    FORMAT(*- T(I,2)-T(I,1)*5X*T(I,3)-T(I,1)*5X*T(I,4)-T(I,1)*10X*RT21*8X*RT3
               11*12X*RT31*12X*RT41*)
          C
                TP2=TP4=0. $ IP2=IP4=0
                TPEAK=0. $ IPEAK=0
                DO 40 I=1,M
                T21=T1(I,2)-T1(I,1)
                T31=T1(I,3)-T1(I,1)
                T41=T1(I,4)-T1(I,1)
                IF(T1(I,1).EQ.0.) GO TO 20
                RT21= T21/T1(I,1)
                RT31= T31/T1(I,1)
                ART31=ABS(RT31)
                IF(ART31.GT.TPEAK) TPEAK=ART31
                IF(ART31.EQ.TPEAK) IPEAK=I
                RT41= T41/T1(I,1)
                ART2=ABS(RT21)  $    ART4= ABS(RT41)
                IF(ART2.GT.TP2) TP2=ART2
                IF(ART2.GT.TP2) IP2=I
                IF(ART4.GT.TP4) TP4=ART4
                IF(ART4.GT.TP4) IP4=I
                GO TO 41
          20    RT21=0. $ RT31=0. $ RT41=0.
          41    CONTINUE
                IF(IC6.NE.1) GO TO 40
                PRINT 42, T21,T31,T41, RT21,RT31,RT41
          40    CONTINUE
          42    FORMAT(* *3(E11.4,7X),5X,3(E11.4,5X))
                PRINT 302, TPEAK, S(IPEAK)
         302    FORMAT(* MAX. RT31 = *E15.5*  AT R= *F7.3)
                  PRINT 360, TP2, S(IP2), TP4, S(IP4)
         360    FORMAT(* MAX. TP2=*E15.5*AT R=*F7.3,5X,*MAX. TP4=*E15.5* AT R=*
```

```
              1F7.3 )
       C
        202  CONTINUE
              RETURN
             END
 SUBROUTINE  EIGENVP   TRACE

             SUBROUTINE EIGENVP
             COMMON R, W1, W2, T1, T2, C, D, VIS, K, DKDR
             COMPLEX P1, P2, P3, P4, CW, TF, B1                    ,DT1,DT2
             COMPLEX U0, A1, A2, A4, A5, RR,RS,C1,C2  , F1 ,A6
             COMPLEX BE , EF1,EF2,EF3,EF4 ,BC
             COMMON/F/ REY, MONITOR, IC6 , IC7
             COMMON/Y/ VIS1, VIS2, P1, P2, P3, P4, W3
             DIMENSION       U0(2), RS(2) , RR(101)     ,G(101,2)
             DIMENSION R(201), W1(201), W2(201),T1(201),T2(201),C(11),D(11),
            1 VIS(201),VIS1(201), VIS2(201),P1(105,2), P2(105,2),P3(105,2),
            1P4(105,2), TF(5), TEST(5) , W3(201)
             DIMENSION K(201),DKDR(201) , BC(2)
             REAL  K
             COMMON/B/ L,DP,DT,SUM,RI
             COMMON/D/ NONLIN,RO
             COMMON /E/ SIGN,TI,TO,N,MON1,MON2
       C        READ IN PARAMETERS AND INITIAL GUESSES ON ALPHA AND REY.
       C        PAL=PERCENT CHANGE IN ALPHA
       C        PREY= PERCENT CHANGE IN REY
       C        PCH= MAX. CHANGE IN CR AND REY
       C        DELTA= CONVERGENCE CRITERIA
       C        ICODE= PRINTOUT CODE
       C        MAXIT=MAX. NUMBER OF ITERATION
             READ 1, PAL, PREY, PCH,DELTA
             READ 2, ICODE,MAXIT,NSET , IBUG
       2     FORMAT(5I5)
             B1=(0.0,1.0)
       C        CALCULATE THE FIRST AND SECOND DERIVATIVES OF VISCOSITY WRT R,
              M=N+1
             H= 1./N
             H2=2. * H
             MM=M/2
             J= MM+1
             DO 4 I=1,M
             R(I)= RI/(RO-RI) + (I-1)*H
             IF(I.GE.J) GO TO 5
             VIS1(I)=1./(2*H)*(-3*VIS(I)+4*VIS(I+1)- VIS(I+2))
             GO TO 4
       5     VIS1(I)=1./(2*H)*(3*VIS(I)-4*VIS(I-1)+ VIS(I-2))
       4     CONTINUE
       C VIS2=DVIS1/DR
       C W3=DW2/DR
             DO 6 I=1,M
             IF(I .GE. J-1) GO TO 7
             VIS2(I)=1./H**2 *(2*VIS(I)- 5*VIS(I+1)+ 4*VIS(I+2) -VIS(I+3))
             W3(I)=1./H**2*(2*W1(I)-5*W1(I+1)+4*W1(I+2)-W1(I+3))
             GO TO 6
       7     VIS2(I)=1./H**2 *(2*VIS(I) -5*VIS(I-1)+ 4*VIS(I-2)-VIS(I-3))
             W3(I)=1./H**2*(2*W1(I)-5*W1(I-1) +4*W1(I-2)-W1(I-3))
       6     CONTINUE
       C
             PRINT 8, PAL, PREY, PCH, DELTA
             PRINT 9, ICODE, MAXIT
       C
```

```
                NEWPAR=0
          200   IVP=0
                IP=0
                NEWPAR=NEWPAR+1
                IF(NEWPAR.GT. NSET) GO TO 300
                READ 3, CW, REY, ALPHA
                CR= REAL(CW)
                PRINT 10, CW, REY, ALPHA
          C BOUNDARY CONDITIONS... P1=P2=(0.0,0.0)
          C SET AN INITIAL CONDITIONS FOR THE RELATED INITIAL VALUE PROBLEM AND THEN
          C SOLVE THE IVP BY 4TH ORDER R-K METHOD.
                P1(1,1)=P1(1,2)=(0.0,0.0)
                P2(1,1)=P2(1,2)=(0.0,0.0)
                P3(1,2)=(1.0,0.0)$P3(1,1)=(0.0,0.0)
                P4(1,1)  =(0.0,1.E-8)          $   P4(1,2)=(0.0,0.0)
                IC=1
                PRINT 901,IC,P3(1,1),P3(1,2)    ,P4(1,1),P4(1,2)
          901   FORMAT(* *I5,4(2E15.5))
          100   CONTINUE
                E1= B1*ALPHA*REY
                PHA=ALPHA**2
           400  DO 500 III=1,MM
                    CALL RK4(H2,REY,E1,CR,PHA,III,IBUG)
          C             CALCULATE THE SOLUTION TO INVISID EQUATION TO BE USED +0- FILTERIN
 SUBROUTINE  EIGENVP   TRACE

          C             -G AND DETERMINE GROWING SOLUTION.
                JJ=(III*2)+1
                II=III+1
                DO 710 I=1,2
                U0(I)=((W1(JJ)-CR)*(P3(II,I)+1/R(II)*P2(II,I)-(1/(R(II)**2
               1) + PHA)*P1(II,I)  ) -(W3(JJ)-1/R(II)*W2(JJ))*P1(II,I))
               1 *B1*ALPHA*REY
                AA=REAL(P2(II,I)*CONJG(P2(II,I)))
                BB= REAL(P1(II,I)*CONJG(P1(II,I)))
                AABB = AA/BB
          710    G(II,I)=SQRT(AABB)
                G1=AMAX1( G(II,1),G(II,2))
          C             EXTRACT THE PORTION OF GROWING SOLUTION(U0(1)) FROM U0(2).
          730   U0 = ABS(REAL(U0(1)))
           750   RR(II)  =U0(2)/U0(1)
           780    P1(II,2)=P1(II,2) - RR(II)  *P1(II,1)
                  P2(II,2)=P2(II,2) - RR(II)  *P2(II,1)
                  P3(II,2)=P3(II,2) - RR(II)  *P3(II,1)
                  P4(II,2)=P4(II,2) - RR(II)  *P4(II,1)
          500   CONTINUE
                IF(ICODE.EQ.1) GO TO 900
          C             REPAIR THE EXTRACTION.
                I=J
                NN=J-2
                RS(2)=RR(J)
                DO 800 M5=1,NN
                I=I-1
                P1(I,2)=P1(I,2)-RS(2)*P1(I,1)
                P2(I,2)=P2(I,2)-RS(2)*P2(I,1)
                P3(I,2)= P3(I,2)-RS(2)*P3(I,1)
                P4(I,2)=P4(I,2)-RS(2)*P4(I,1)
          800   RS(2)=RS(2)+RR(I)
                PRINT 443
          443    FORMAT(*- SOLUTION 1*)
                DO 940 I=1,J
```

```
      PRINT 16, I, P1(I,1), P2(I,1),P3(I,1),P4(I,1)
940    CONTINUE
      PRINT 445
445   FORMAT(*- SOLUTION 2*)
      DO 444 I=1,J
      PRINT 16, I, P1(I,2), P2(I,2),P3(I,2),P3(I,2)
444    CONTINUE
C             MODIFY THE GROWING SOLUTION.
990   CONTINUE
      I=J
810   A6=P1(J,2)/P1(J,1)
830   DO 840 KJ=1,MM
      P1(I,1)=P1(I,2)-A6*P1(I,1)
      P2(I,1)=P2(I,2)-A6*P2(I,1)
      P3(I,1)=P3(I,2)-A6*P3(I,1)
      P4(I,1)=P4(I,2)-A6*P4(I,1)
840   I=I-1
      IF(ICODE.EQ.1) GO TO 950
      DO 650  I=1,J
      PRINT 16, I, P1(I,1),P2(I,1),P3(I,1),P4(I,1)
650   CONTINUE
 500   FORMAT(* *I5, 4(2E12.4))
C SUPERPOSITION OF THE TWO SOLUTIONS.
950   BE= P2(J,1)*P3(J,2) - P2(J,2)*P3(J,1)
      C2= P2(J,1)/ BE
      C1=-P2(J,2)/ BE
C CHECK THE BOUNDARY CONDITIONS
      BC(1)= C1*P2(J,1) + C2*P2(J,2)
      BC(2)=C1*P3(J,1) + C2*P3(J,2) -(1.,0.0)
      TBC = ABS(REAL(BC(1))) + ABS(AIMAG(BC(1))) +ABS(REAL(BC(2)))
     1 +ABS(AIMAG(BC(2)))
      IP=IP+1
      TF(IP)=C1*P1(J,1)+C2*P1(J,2)
      TEST(IP)=TF(IP)*CONJG(TF(IP))
      PRINT 20, TF(IP), TEST(IP), ALPHA, REY, CR
      IF( TEST(IP) - DELTA) 11,11,12
C
12     CONTINUE
      GO TO (13,14,15)IP
13    DAL=PAL*ALPHA  $  ALPHA=ALPHA+DAL
      GO TO 100
14    ALPHA=ALPHA-DAL $ DREY=PREY*REY  $  REY=REY+DREY $ GO TO 100
15     DT1=(TF(2)- TF(1))/DAL  $  REY= REY-DREY
SUBROUTINE  EIGENVP   TRACE

      DT2=(TF(3)-TF(1))/DREY  $  DEN=AIMAG((CONJG(DT1))*DT2)
      DAL = AIMAG(TF(1)*CONJG(DT2))/DEN
      DREY= AIMAG( CONJG(TF(1))*DT1)/DEN
      IF(ABS(DAL) .GE. (PCH*ALPHA)) DAL=PCH*ALPHA*DAL/ABS(DAL)
      IF(ABS(DREY).GE.(PCH*REY))DREY=PCH*REY*DREY/ABS(DREY)
      REY= REY+ DREY
      ALPHA = ALPHA +DAL
      IP=0
      IVP=IVP+1
      IF(IVP.GE.6) STOP 6
      GO TO 100
CC
 11   CONTINUE
      PRINT 21, RI,RO, TI, TO
21    FORMAT(*0RI,RO,TI,TO ARE *4F10.3)
C  PRINT OUT THE FINAL EIGEN FUNCTIONS IF DESIRED.
```

```
          IF(ICODE.EQ.1) GO TO 200
          PRINT 333
    333   FORMAT(*-COMBINED EIGEN FUNCTIONS*,//,5X, *I*20X*P*20X*DP*20X*D2P*
         120X*D3P*)
          DO 22 I= 1,J
          EF1= C1*P1(I,1) + C2*P1(I,2)
          EF2=C1*P2(I,1) + C2*P2(I,2)
          EF3=C1*P3(I,1) + C2*P3(I,2)
          EF4=C1*P4(I,1) + C2*P4(I,2)
          EF1 = EF1 / (C1*P1(51,1)+C2*P1(51,2))
          PRINT 16, I, EF1   ,EF2,EF3,EF4
     22   CONTINUE
     16   FORMAT(* *I5,2X,4(2F15.5) )
          GO TO 200
      1   FORMAT(3F10.5,E8.2)
      3   FORMAT(4F10.5)
      8   FORMAT(* *3F10.5,E12.5)
      9   FORMAT(* *2I5)
     10   FORMAT(* *4F15.5)
     20   FORMAT(* TF(R=B/DR) =*    2E12.5,5X,*TEST FN=*       E12.5*    ALPHA, 9VYP
         1HA, REY, CR = *3E12.5)
    300   CONTINUE $RETURN $ END
SUBROUTINE  RK4        TRACE

          SUBROUTINE RK4(H,REY,E1,CR,PHA,I,IBUG)
          COMPLEX V00,V10,V20,V30,G1,  V01,V11,V21,V31,G2,
         1        V02,V12,V22,V32,G3,  TERM1,TERM2,TERM3,TERM4,
         1        G,DG,D2G,D3G,  P,DP,D2P,D3P,  E1,EVIS
         1     ,  V03,V13,V23,V33,G4
          COMPLEX  SUB1,SUB2,SUB3, SUB4
          COMMON/Y/ VIS1,VIS2,P,DP,D2P,D3P,D2W
          COMMON R,W,DW,T1,T2,C,D,VIS,K,DKDR
          DIMENSION R(201),W(201),DW(201),T1(201),T2(201),C(11),D(11),
         1 VIS(201),VIS1(201), VIS2(201),P(105,2),DP(105,2),D2P(105,2),
         1D3P(105,2),TF(5), TEST(5),  D2W(201)
          DIMENSION K(201),DKDR(201)
          REAL K
          IF(IBUG .NE. 1) GO TO 2
          PRINT 3, H,REY,E1,CR,PHA,I,IBUG
      3   FORMAT(* *F7.3, F7.0, 2E12.3, 2F10.4,2I5)
      2   DO 1 J = 1,2
C FUNCTIONS AT R(I)
          V00 = P(I,J)
          V10 = H * DP(I,J)
          V20 = H**2 * D2P(I,J) /2
          V30 = H**3 * D3P(I,J) /6
          N=(I*2) - 1
          EVIS=E1/VIS(N)
          VISV1=1. /VIS(N) * VIS1(N)
          VISV2= 1. /VIS(N) *VIS2(N)
          SUB1 =                         (D2W(N)  -1/R(N)*DW(N)) *V00
          SUB2  =  (1/R(N)**2 -PHA) * V00
          SUB3 =           -(3/R(N)**2 +2*PHA)*V10/H + (3/R(N)**3-PHA/R(N))
         1 *V00
          SUB4  =
         1  (-3/R(N)**4 + 2*PHA/R(N)**2 + PHA*PHA) * V00
          TERM1 = EVIS*(   (W(N) -CR) *(V20*2/H**2 + 1/R(N)*V10/H
         1 -(1/R(N)**2 + PHA) * V00)    - SUB1  )
          TERM2 = -VISV2*(     V20*2/H**2 + 1/R(N)*V10/H  - SUB2  )
          TERM3 = -VISV1*(     2*V30*6/H**3 + 3/R(N)*V20*2/H**2  + SUB3 )
          TERM4 = -2/R(N)*V30*6/H**3  +  (3/R(N)**2 + 2*PHA)*V20*2/H**2
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```
          1 -(3/R(N)**3 - 2*PHA/R(N))*V10/H    - SUB4
           G1  = H**4/24 * ( TERM1 + TERM2 + TERM3 + TERM4 )
           IF(IBUG.NE.1) GO TO 4
           PRINT 5, R(N), W(N), VIS(N), TERM1,TERM2,TERM3,TERM4
        5  FORMAT(* *3F10.3,4(2E10.3) )
C  FUNCTIONS AT R + H/2
        4  V01= V00 + 0.5*V10 + 0.25*V20 + 0.125*V30 + 0.0625*G1
           V11= V10 + V20 + 0.75*V30 + 0.5 * G1
           V21= V20 + 1.5*V30 + 1.5*G1
           V31= V30 + 2.*G1
           N= I*2
           EVIS=E1/VIS(N)
           VISV1=1. /VIS(N) * VIS1(N)
           VISV2= 1. /VIS(N) *VIS2(N)
           SUB1 =          (D2W(N)-1./R(N)*DW(N)) * V01
           SUB2 =                                     (1./R(N)**2-PHA)*V01
           SUB3 =  -(3/R(N)**2 + 2*PHA)*V11/H + (3/R(N)**3 -PHA/R(N))*V01
           SUB4=     (-3./R(N)**4 + 2. *PHA/R(N)**2 + PHA**2)*V01
           TERM1 = EVIS*(   (W(N)-CR)*(V21*2/H**2 + 1./R(N)*V11/H
          1                             -(1./R(N)**2 + PHA)*V01) -SUB1 )
           TERM2= -VISV2 * (V21*2/H**2 + 1./R(N)*V11/H - SUB2 )
           TERM3 = -VISV1*(2*V31*6/H**3 + 3./R(N)*V21*2/H**2 +  SUB3  )
           TERM4 = -2./R(N)*V31*6/H**3 + (3./R(N)**2 + 2*PHA)*V21/H**2  *2.
          1        -(3./R(N)**3 - 2.*PHA/R(N))*V11/H  -  SUB4
           G2= H**4/24. *(TERM1 + TERM2+ TERM3 + TERM4)
           IF(IBUG.NE.1) GO TO 6
           PRINT 5, R(N), W(N), VIS(N), TERM1,TERM2,TERM3,TERM4
C          V02= V00 + 0.5*V10 + 0.25*V20 + 0.125*V30 + 0.0625*G1
C          V12= V10 + V20 + 0.75*V30 + 0.5 * G1
C          V22= V20 + 1.5*V30 + 1.5*G1
        6  V32=V30 + 2.*G2
C  TERM1 AND TERM2 ARE THE MOST RECENT ONES.
           TERM3 = -VISV1*(2*V32*6/H**3 + 3./R(N)*V21*2/H**2 + SUB3 )
           TERM4 = -2./R(N)*V32*6/H**3 + (3./R(N)**2 + 2*PHA)*V21/H**2 *2.
          1        -(3./R(N)**3 - 2.*PHA/R(N))*V11/H  -  SUB4
           G3=H**4. /24. *(TERM1+TERM2+TERM3+TERM4)
C  FUNCTIONS AT R(I) + H
           V03 = V00 + V10 + V20 + V30 + G3
           V13 = V10 + 2*V20 + 3*V30 + 4*G3
           V23 = V20 + 3*V30 + 6*G3
           V33 = V30 + 4*G3
SUBROUTINE   RK4        TRACE

           N= (I*2) +1
           EVIS=E1/VIS(N)
           VISV1=1. /VIS(N) * VIS1(N)
           VISV2= 1. /VIS(N) *VIS2(N)
           SUB1 =           (D2W(N)    - 1/(R(N)   ) * DW(N)   ) * V03
           SUB2 =            (1/(R(N)   )**2 -PHA) * V03
           SUB3 =         -  (3/(R(N)   )**2 +2*PHA)*V13/H + (3/(R(N)   )**3
          1                                            -PHA/(R(N)  )) *V03
           SUB4 =    (-3/(R(N)   )**4 + 2*PHA/(R(N)   )**2 + PHA*PHA) *V03
           TERM1 = EVIS*(    (W(N)  -CR) *(V23*2/H**2 + 1/(R(N)   )*V13/H
          1                                    -(1/(R(N)   )**2 + PHA) * V03)
          1  -  SUB1  )
           TERM2 = -VISV2*(    V23*2/H**2 + 1/(R(N)   ) *V13/H  - SUB2 )
           TERM3 = -VISV1*(     2*V33*6/H**3 + 3/(R(N)   ) *V23*2/H**2 + SUB3)
           TERM4 = -2/(R(N)   )*V33*6/H**3  + (3/(R(N)   )**2 + 2*PHA)*V23*2 /
          1        H**2      -(3/(R(N)   )**3 - 2*PHA/(R(N)   )) *V13/H - SUB4
           G4  = H**4/24 *( TERM1 + TERM2 + TERM3 + TERM4 )
           IF(IBUG.NE.1) GO TO 8
```

```
      PRINT 5, R(N), W(N), VIS(N), TERM1,TERM2,TERM3,TERM4
    9 G = 1.0/15 * ( 8.*G1 + 4.*G2 +4.*G3 - G4)
      DG =  1.0/5 * ( 9.*G1 +  6.*G2 + 6.*G3 - G4)
      D2G = 2.0 * ( G1 + G2 +G3)
      D3G = 2.0/3 * ( G1 + 2.*G2 +2.*G3 +G4)
      P(I+1,J) = V00 + V10 + V20 + V30 + G
      DP(I+1, J) = 1./H * ( V10 + 2.*V20 + 3.*V30 + DG )
      D2P(I+1, J) = 2./H**2 * ( V20 + 3.*V30 + D2G )
      D3P(I+1,J) = 6./H**3 * ( V30 + D3G )
      IF(IBUG .NE. 1) GO TO 1
      PRINT 10, P(I+1,J),DP(I+1,J),D2P(I+1,J), D3P(I+1,J)
   10 FORMAT(* *4(2E15.5) )
      PRINT 11, G1,G2,G3,G4
   11 FORMAT(* G1,G2,G3,G4* 4(2E11.3) )
    1 CONTINUE
      RETURN
      END
```