STUDIES ON VOID FORMATION AND GROWTH FOR INCOMPRESSIBLE NONLINEARLY ELASTIC MATERIALS

By

MIAO-SZE (OLIVIA) CHOU-WANG

A DISSERTATION

Submitted to
MICHIGAN STATE UNIVERSITY
in Partial Fulfillment of the Requirements
for the Degree of

DOCTOR OF PHILOSOPHY

in

Mechanics

Department of Metallurgy, Mechanics and Materials Science

1988

ABSTRACT

STUDIES ON VOID FORMATION AND GROWTH FOR INCOMPRESSIBLE NONLINEARLY ELASTIC MATERIALS

By

Miao-Sze (Olivia) Chou-Wang

In this study, a class of bifurcation problems for a solid sphere subjected to uniform tensile dead-loading p_0 at its boundary are examined within the framework of finite elastostatics and elastodynamics. The sphere is composed of a particular class of homogeneous isotropic incompressible nonlinearly elastic materials, namely those of power-law type. First of all, we carry out an investigation of the elastostatic problem. One solution to this problem, for all values of p_0, corresponds to a homogeneous state in which the sphere remains undeformed while stressed. However, for sufficiently large values of p_0, there is in addition a second possible configuration involving an internal traction-free spherical cavity. The dependence on constitutive parameters of the critical load at which bifurcation occurs is examined as well as the subsequent void growth. The stress distribution after cavitation occurs is also described. The results are obtained in closed analytic form.

Secondly, we study the elastodynamic version of the foregoing problem for the special case of a neo-Hookean material. The sphere is set into motion by a suddenly applied uniform radial tensile dead-load p_0. One solution to the dynamic problem, for all values of p_0, corresponds to a trivial homogeneous static state in which the sphere remains undeformed while stressed. However, for sufficiently large values of p_0, one has in addition another possible radially symmetric motion involving an internal traction-free cavity. The "critical load" at which an internal void may be initiated in the dynamic problem is shown to coincide with that for the static problem.

To the Lord my Heavenly Father

who endows the heart with wisdom

and

gives understanding to the mind.

ACKNOWLEDGEMENTS

I would like to express my sincere appreciation to my advisor, Professor Cornelius O. Horgan for his guidance and support throughout this work. I would also like to thank my colleagues for their understanding and generous advice. Grateful thanks are extended to the other members of the guidance committee, Professor Dahsin Liu, Thomas J. Pence and David H.Y. Yen.

While preparing this dissertation, I held teaching assistantships awarded by the Department of Metallurgy, Mechanics and Materials Science and research assistantships supported jointly by the National Science Foundation under Grant MSM 85-12825 (C.O. Horgan), Department of Metallurgy, Mechanics and Materials Science and the Composite Materials and Structures Center of Michigan State University under a REED Program Grant. The support of these institutions is gratefully acknowledged.

A final note of appreciation must go to my husband, Hsien-Ming Chou, for his support and encouragement.

TABLE OF CONTENTS

LIST OF FIGURES

1. INTRODUCTION

Void nucleation and growth in solids have been of concern for a long time because of the fundamental role such phenomena play in fracture and other failure mechanisms. (See e.g. Goods and Brown [1] for a discussion of cavity nucleation in metals). The phenomenon of sudden void formation ("cavitation") has also been observed experimentally in vulcanized rubber by Gent and Lindley [2]. Nonlinear theories of solid mechanics have been used recently to account for such phenomena. The impetus for much of the recent theoretical developments has been supplied by the work of Ball [3]. In [3], Ball has made an extensive study of a class of <u>bifurcation problems</u> for the equations of nonlinear elasticity which model the appearance of a cavity in the interior of an apparently solid homogeneous isotropic elastic body once a critical load has been attained. An alternative interpretation for such problems in terms of the growth of a <u>pre</u>-<u>existing</u> micro-void is given in [4]. Further investigations of such bifurcation problems have been carried out in [5-12]. It is worth noting that cavitation can be shown to occur only when <u>finite</u> strain measures are taken into account (see e.g. [4], [9]). The corresponding problems in linearized elasticity or in the infinitesimal strain theory of plasticity do <u>not</u> exhibit such bifurcations.

The purpose of the present study is to further investigate this bifurcation approach to void nucleation in two specific contexts. First of all, we carry out an investigation of the problem of static tensile dead-loading of a solid sphere composed of a particular class of homogeneous isotropic incompressible nonlinearly elastic materials, namely those of power-law type. Secondly, we study the elastodynamic version of this problem for the special case of a neo-Hookean material.

In Section 2, we formulate the basic boundary-value problem that arises when a solid sphere, composed of an incompressible isotropic elastic material, is subjected to a prescribed uniform radial tensile dead-load p_0 on its boundary. One solution to this problem, for all values p_0, corresponds to a trivial homogeneous state in which the sphere remains undeformed while stressed. However, for sufficiently large values of p_0, one has in addition other possible radially symmetric configurations involving an internal traction-free spherical cavity. Such solutions have been shown by Ball [3] to bifurcate from the homogeneous solution at a critical value of p_0, say p_{cr}, at which the homogeneous solution becomes unstable. The possibility for these bifurcated solutions to exist depends on the constitutive law for the material under consideration.

In Section 3, attention is confined to a particular class of homogeneous isotropic incompressible elastic materials, namely those of power-law type. Such nonlinearly elastic materials were first introduced by Ogden [13] and have been employed in a wide variety of problems since then (see e.g. [14], [15]). An extensive discussion of the properties of this class of materials has been provided recently by Zee and Sternberg [16]. Our interest here is in examining the dependence of the critical loads at which cavitation occurs on the constitutive parameter n appearing in the definition of this class of materials (see equation (3.1)). In Section 3, we first examine the behavior of these materials under certain homogeneous deformations namely uniaxial stress, equibiaxial stress and pure shear. These results are shown in Figures 1-3. The explicit relationship between the applied load p_0 and the deformed cavity radius is examined and plotted in Figure 4. It is found that as the hardening parameter n increases, the critical load p_{cr} at which bifurcation takes place also increases. For the special case of a neo-Hookean material, for which $n = 1$, we recover results due to Ball [3]. The stress distribution in the sphere is also described. An interesting feature concerning the principal stresses immediately after cavitation is the presence of a boundary layer near the cavity wall. To see this, we have plotted the stresses in Figures 5-10 for applied dead loads p_0 slightly larger than p_{cr}.

In Section 4, we consider the radially symmetric motion of an isotropic incompressible elastic solid sphere composed of a neo-Hookean material which is set into motion at time $t = 0$ by a suddenly applied uniform radial tensile dead-load p_0. If the material were compressible, the medium would respond to such a loading by propagating a dilatation wave inward from the boundary. In the incompressible case, the effect of the tensile load is felt immediately throughout the medium and the response takes the form of a nonlinear oscillation. Such oscillation problems were first investigated by Knowles [17], [18] for hollow circular cylinders and have received considerable attention since then (see e.g. [19-22] and the references cited therein). In Section 4, we show that one solution to the dynamic problem described above, for all values of p_0, corresponds to a trivial homogeneous <u>static</u> state in which the sphere remains undeformed while stressed. However, for sufficiently large values of p_0, one has in addition another possible radially symmetric motion involving an internal traction-free cavity. A relationship between the applied load p_0 and cavity radius c(t) at time t is obtained in the form of a second-order nonlinear ordinary differential equation (see equation (4.19)). By adapting the techniques of Knowles [17], [18], we show that periodic oscillations can occur if and only if the applied tensile dead-load p_0 is such that

$$p_0 \geq 5\mu/2, \tag{1.1}$$

where μ denotes the shear modulus for infinitesimal deformations of the neo-Hookean material. As $p_0 \to 5\mu/2+$, the deformed cavity radius $c(t) \to 0+$. It is shown that the value of the "critical load" at which an internal void may be initiated in the dynamic problem <u>coincides</u> with that for the static problem. For values of $p_0 > 5\mu/2$, following the application of such a load at time $t = 0$, the cavity would expand until its radius reaches a maximum value given by equation (4.27), then would contract to zero and repeat the cycle.

2. BIFURCATION PROBLEM FOR A SPHERE: FORMULATION AND SOLUTION

2.1 Formulation:

We are concerned here with a sphere composed of a homogeneous incompressible isotropic elastic material. Let the undeformed solid sphere be denoted by
$D_0 = \{(r,\theta,\phi) \mid 0 \le r < b,\ 0 < \theta \le 2\pi,\ 0 \le \phi \le \pi\}$. The sphere is subjected to a prescribed uniform radial tensile dead-load of magnitude p_0 on its boundary $r = b$. The resulting deformation is a one-to-one mapping which takes the point with spherical polar coordinates (r,θ,ϕ) in the undeformed region D_0 to the point (R,Θ,Φ) in the deformed region D. We assume that the deformation is radially symmetric so that

$$R = R(r) > 0,\ 0 < r < b;\ R(0+) \ge 0,\ \Theta = \theta,\ \Phi = \phi,\ \text{on } D_0, \tag{2.1}$$

where R(r) is to be determined.

The principal stretches associated with the radially symmetric deformation (2.1) are

$$\lambda_r = \dot{R}(r),\ \lambda_\theta = \lambda_\phi = \frac{R(r)}{r}\ , \tag{2.2}$$

where the dot denotes differentiation with respect to the argument. The spherical polar components of the deformation gradient tensor $\underset{\sim}{F}$ associated with (2.1) are given by

$$\left.\begin{aligned} &F_{rr} = \dot{R}(r),\ F_{\theta\theta} = F_{\phi\phi} = \frac{R}{r}, \\ &F_{r\theta} = F_{\theta r} = F_{\theta\phi} = F_{\phi\theta} = F_{r\phi} = F_{\phi r} = 0. \end{aligned}\right\} \tag{2.3}$$

Incompressibility then requires that the Jacobian determinant $J = \operatorname{Det} \underset{\sim}{F} = 1$, which upon integration yields

$$R(r) = (r^3 + c^3)^{1/3}, \tag{2.4}$$

where $c \geq 0$ is a constant to be determined. If it is found that $c = 0$, (2.4) implies that the body remains a solid sphere in the current configuration. On the other hand, if c is found to be greater than zero, then $R(0+) = c > 0$ and so there is a cavity of radius c centered at the origin in the current configuration. In this event, the cavity surface is assumed to be traction-free.

The strain-energy density per unit undeformed volume for a homogeneous isotropic incompressible elastic material is denoted by

$$W = W(\lambda_1, \lambda_2, \lambda_3), \tag{2.5}$$

where λ_i $(i = 1,2,3)$ are the principal stretches. The function W is invariant with respect to interchange of the λ_i and is taken to satisfy the normalization condition $W(1,1,1) = 0$. In the sequel, we proceed formally and assume that W possesses sufficient regularity properties to permit the subsequent analysis.

The principal components of the Cauchy stress tensor $\underset{\sim}{\tau}$ are given by

$$\tau_{ii} = \lambda_i \frac{\partial W}{\partial \lambda_i} - p, \text{ (no sum on i)}, \tag{2.6}$$

where p is the hydrostatic pressure associated with the incompressibility constraint $\lambda_1\lambda_2\lambda_3 = 1$. For the radially symmetric deformation with principal stretches given by (2.2), the principal stress components are

$$\left.\begin{aligned} \tau_{RR}(r) &= v^{-2}\, W_1(v^{-2}, v, v) - p(r), \\ \tau_{\Theta\Theta}(r) &= \tau_{\Phi\Phi}(r) = v\, W_2(v^{-2}, v, v) - p(r), \end{aligned}\right\} \tag{2.7}$$

where, following Ball [3], we have introduced the notation

$$v = v(r) = \frac{R}{r} = \left(1 + \frac{c^3}{r^3}\right)^{1/3} . \tag{2.8}$$

Notice that in (2.7) we consider $\tau(r)$ rather than the more conventional $\tau(R)$. The subscript notation on W in (2.7) denotes differentiation with respect to the appropriate argument. In (2.7), we have also used $W_2(v^{-2}, v, v) = W_3(v^{-2}, v, v)$, which follows from the invariance of W with respect to interchange of its three arguments.

The dead-load boundary condition now requires that

$$\tau_{RR}(b) = p_0\left[\frac{b}{R(b)}\right]^2 = p_0\ [v(b)]^{-2}, \tag{2.9}$$

where the constant $p_0 > 0$ is prescribed. We note that the boundary conditions of vanishing shear tractions are satisfied identically. In addition if $c > 0$, then the condition for a traction-free cavity surface

$$\tau_{RR}\ (0) = 0, \tag{2.10}$$

must also hold.

In the absence of body forces, the sphere will be in equilibrium provided that div $\underset{\sim}{\tau} = 0$, which will hold provided that

$$\frac{\partial \tau_{RR}}{\partial r} + 2\, \frac{\dot{R}}{R}\ [\tau_{RR} - \tau_{\Theta\Theta}] = 0, \tag{2.11}$$

holds throughout the sphere.

Thus, the problem to be solved is the following: _For a prescribed value of the dead-load traction_ $p_0 > 0$, _we seek a pressure field_ $p(r)$ _and a constant_ $c \geq 0$ _such that_ (2.11) _and_ (2.9) _are satisfied where_ τ_{RR}, $\tau_{\Theta\Theta}$, $\tau_{\Phi\Phi}$ _are given by_ (2.7) _and_ (2.8). _In addition if_ $c > 0$, _then_ (2.10) _must also be satisfied_.

2.2 _Solutions_:

It may be readily shown that one solution to the foregoing problem, for all values of p_0, is

$$p(r) = W_1\ (1,\ 1,\ 1) - p_0,\ c = 0. \tag{2.12}$$

This corresponds to the trivial homogeneous state of deformation

$$R(r) = r, \tag{2.13}$$

with corresponding stresses $\tau_{RR} = \tau_{\Theta\Theta} = \tau_{\Phi\Phi} = p_0$.

Next we describe solutions for which $c > 0$, corresponding to the presence of a traction-free cavity at the origin. For this purpose, we adopt an approach developed by Horgan and Pence [10] and rewrite the differential equation (2.11) in the form

$$\frac{d}{dr}[v^{-2}\ W_1(v^{-2},v,v) - p(r)] + \frac{2v^{-4}}{r}[v^{-1}\ W_1(v^{-2},v,v) - v^2 W_2(v^{-2},v,v)] = 0,$$

$$\text{on } 0 < r < b, \tag{2.14}$$

where we have used (2.7),(2.8). On integration of (2.14), we have

$$p(r) - p(0) = v^{-2}(r)\ W_1(v^{-2},\ v,\ v) + 2\ J(r),\ 0 < r < b, \tag{2.15}$$

where

$$J(r) = \int_0^r \left\{ v^{-5}(s)W_1[v^{-2}(s),v(s),v(s)] - v^{-2}(s)W_2[v^{-2}(s),v(s),v(s)]\right\} \frac{ds}{s} ,$$

$$0 < r < b. \tag{2.16}$$

On substitution into (2.7) we obtain

$$\tau_{RR}(r) = -p(0) - 2J(r),\ 0 < r < b. \tag{2.17}$$

The traction-free cavity surface condition (2.10), together with (2.17) and $J(0) = 0$, now yields

$$p(0) = 0. \tag{2.18}$$

Finally the boundary condition (2.9) at $r = b$ is satisfied if

$$-2J(b) = p_0\ [v(b)]^{-2} . \tag{2.19}$$

The condition (2.19) may be written in a compact fashion on utilizing the change of variables $s \to v$ in the integral (2.16). From (2.8) it is seen that this change of variable is one-to-one and invertible if and only if $c > 0$. Introducing the function

$$\hat{W}(x) = W(x^{-2},\ x,\ x), \tag{2.20}$$

and adopting the notation

$$\hat{W}_1(x) = \frac{d}{dx}\hat{W}(x), \tag{2.21}$$

(2.19) may be written as

$$p_0 = (1 + \frac{c^3}{b^3})^{2/3} \int_{(1 + \frac{c^3}{b^3})^{1/3}}^{\infty} \frac{\hat{W}_1(v)}{(v^3 - 1)}\, dv,\ c > 0. \tag{2.22}$$

Equation (2.22) was first established by Ball [3] for the n-dimensional version of the problem described here (see equation (5.18) of [3]). Thus, for a given dead-load p_0, solutions involving a traction-free internal cavity of radius c exist provided that c is a positive root of (2.22). The associated pressure field is given by

$$p(r) = v^{-2}(r)W_1(v^{-2}, v, v) + 2J(r), \quad 0 < r < b. \tag{2.23}$$

The <u>critical</u> <u>load</u> p_{cr} at which an internal cavity may be initiated is found by formally letting $c \to 0+$ in (2.22), and so

$$p_{cr} = \int_1^{\infty} \frac{\hat{W}_1(v)}{(v^3 - 1)} dv. \tag{2.24}$$

This result was first established by Ball [3] in n-dimensions (see equation (5.22) of [3]).

In summary then, we have seen that for all values of the applied dead-load traction p_0, one obtains the trivial solution (2.12) corresponding to the homogeneous state of deformation (2.13). Moreover, if positive roots c of (2.22) exist, then one obtains the <u>additional</u> solutions involving a traction-free internal cavity described above. Such solutions have been shown by Ball [3] to bifurcate from the trivial solution at the critical value p_{cr} at which the trivial solution becomes unstable.

2.3 <u>The critical load</u>:

Since the integral in (2.24) is improper, p_{cr} may or may not be finite, and so cavitation may or may not take place. As regards the lower limit in (2.24), it is shown in Appendix A that

$$\frac{d\hat{W}(1)}{dv} = 0, \qquad \frac{d^2\hat{W}(1)}{dv^2} = 12\mu, \tag{2.25}$$

where μ denotes the shear modulus for infinitesimal deformations of the material. Thus by l'Hôpital's rule, the limit of the integrand in (2.24) is finite as $v \to 1$. An analogous issue was discussed by Horgan and Pence [11] in the context of a composite sphere under tensile dead-loading on its boundary (see equation (17) and the Appendix of [11]). Consequently the question of whether or not p_{cr} is finite depends on the behavior of $\hat{W}(v)$ for large values of stretch v. Sufficient conditions to guarantee that p_{cr} be finite were given by Ball [3] for both incompressible and compressible materials. Here we provide an ad hoc treatment of this issue. Suppose, for example, that the strain-energy density per unit undeformed volume for a homogeneous incompressible isotropic elastic material can be written in the polynomial form

$$\hat{W}(v) = a_0 + a_1 v + a_2 v^2 + \cdots\cdots + a_n v^n \text{ , } (n > 1), \tag{2.26}$$

so that

$$\hat{W}_1(v) = a_1 + 2a_2 v + \cdots\cdots + na_n v^{n-1}. \tag{2.27}$$

From (2.24), (2.27) we see that p_{cr} will be finite if

$$v^{n-4} < v^{-1} \qquad \text{for large } v. \tag{2.28}$$

Thus if

$$n < 3, \tag{2.29}$$

the value of p_{cr} given by (2.24) will be finite.

We now consider some specific examples:

Example 1. The neo-Hookean material:

The strain-energy density function for this material is given by

$$W(\lambda_1, \lambda_2, \lambda_3) = \frac{\mu}{2} (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3), \ \lambda_1\lambda_2\lambda_3 = 1, \tag{2.30}$$

where λ_1, λ_2, λ_3, are the principal stretches, and $\mu > 0$ is the shear modulus for infinitesimal deformations. By virtue of (2.2), (2.8) and (2.20) we thus have

$$\hat{W}(v) = \frac{\mu}{2} (v^{-4} + 2v^2 - 3). \tag{2.31}$$

Therefore

$$\hat{W}(v) \to \mu v^2 \qquad \text{for large } v. \tag{2.32}$$

Thus comparing with (2.26), we get $n = 2$ and so by (2.29), the critical load p_{cr} is finite. In fact Ball [3] has shown that

$$p_{cr} = 5\mu/2. \tag{2.33}$$

(See also Section 3 of the present work.)

Example 2. <u>The Mooney-Rivlin material</u>:

The strain-energy density function for this material is

$$W(\lambda_1, \lambda_2, \lambda_3) = \frac{\mu_1}{2} (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) + \frac{\mu_2}{2} (\lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 - 3),$$

$$\lambda_1\lambda_2\lambda_3 = 1, \tag{2.34}$$

where λ_1, λ_2 and λ_3 are the principal stretches, and μ_1, μ_2 are positive constants. By virtue of (2.2), (2.8), and (2.20) we thus have

$$\hat{W}(v) = \frac{\mu_1}{2} (v^{-4} + 2v^2 - 3) + \frac{\mu_2}{2} (v^4 + 2v^{-2} - 3) . \tag{2.35}$$

Therefore

$$\hat{W}(v) \to \frac{\mu_2 v^4}{2} \qquad \text{for large } v. \tag{2.36}$$

Thus comparing with (2.26), we see that $n = 4$, and so by (2.29), the critical load p_{cr} is not finite. Of course it is well known that the Mooney-Rivlin model is not a very accurate constitutive model for large stretches (see, for example, Ogden [15] pp. 492-493 for a discussion of biaxial deformation of a rectangular sheet).

Example 3. The Rivlin-Saunders material:

Experimental work of Rivlin and Saunders [23] suggests consideration of a strain-energy density function of the form

$$W(\lambda_1, \lambda_2, \lambda_3) = \frac{\mu_1}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) + f(\lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 - 3),$$
$$\lambda_1\lambda_2\lambda_3 = 1, \tag{2.37}$$

where f is, as yet, an unspecified function, with $f(0) = 0$ and μ_1 is a positive constant. By virtue of (2.2), (2.8) and (2.20) we thus have

$$\hat{W}(v) = \frac{\mu_1}{2}(v^{-4} + 2v^2 - 3) + f(v^4 + 2v^{-2} - 3). \tag{2.38}$$

In what follows, we discuss two special forms of (2.38):

$$\text{(i)} \quad \hat{W}(v) = \frac{\mu_1}{2}(v^{-4} + 2v^2 - 3) + \frac{\mu_2}{2}(v^4 + 2v^{-2} - 3)^{\alpha}, \ \alpha > 0, \ \mu_2 > 0. \tag{2.39}$$

Clearly the special case $\alpha = 1$ corresponds to the Mooney-Rivlin material (2.34) considered in Example 2 above. We see that if $4\alpha > 2$, $(\alpha > 1/2)$,

$$\hat{W}(v) \to \frac{\mu_2 v^{4\alpha}}{2} \qquad \text{for large } v. \tag{2.40}$$

On comparing with (2.26), we have $n = 4\alpha$, and so by (2.29) to ensure that p_{cr} is finite, we require that $4\alpha < 3$, i.e., $\alpha < 3/4$, and so p_{cr} is finite for the material (2.39) if

$$\frac{1}{2} < \alpha < \frac{3}{4} . \tag{2.41}$$

For $\alpha \le 1/2$,

$$\hat{W}(v) \rightarrow \mu_1 v^2 \qquad \text{for large } v, \tag{2.42}$$

and so p_{cr} is again finite on comparing with (2.26) and (2.29). In summary then, for the material (2.39), p_{cr} is finite if

$$0 < \alpha < \frac{3}{4} . \tag{2.43}$$

We remark that the commonly used version of (2.39) with $\alpha = 2$ does not yield a finite value of p_{cr}. It is of interest to observe here that Simmonds [25] has recently shown that a circular rubber-like plate composed of the material (2.39) suffers a finite deflection under a concentrated vertical load at its center only if $\alpha > 1$. For a membrane [26], the corresponding result holds only if $\alpha > 2$. See also the discussion on pp. 281-282 of the book by Libai and Simmonds [27].

(ii) Another special form of (2.37) has been considered by Gent and Thomas (1958) [24], in which f is taken to be the logarithm function. Thus we have

$$\hat{W}(v) = \frac{\mu_1}{2}(v^{-4} + 2v^2 - 3) + \frac{\mu_2}{2}\ln(v^4 + 2v^{-2} - 2), \; \mu_2 > 0, \tag{2.44}$$

so that

$$\hat{W}_1(v) = \frac{\mu_1}{2}(-4v^{-5} + 4v) + \frac{\mu_2}{2}\frac{4v^3 - 4v^{-3}}{v^4 + 2v^{-2} - 2} . \tag{2.45}$$

Thus

$$\hat{W}_1(v) \rightarrow 2\mu_1 v \qquad \text{for large } v, \tag{2.46}$$

and so from (2.24) we see that the critical load p_{cr} is finite for the material (2.44).

3. SOLUTIONS FOR A CLASS OF INCOMPRESSIBLE ELASTIC MATERIALS

3.1 A class of incompressible elastic materials:

We now consider a particular constitutive law, namely that of power-law type, and provide an explicit solution for the bifurcation problem discussed generally in Section 2. Thus consider

$$W(\lambda_1, \lambda_2, \lambda_3) = \frac{\mu}{2n}(\lambda_1^{2n} + \lambda_2^{2n} + \lambda_3^{2n} - 3), \ \lambda_3 = (\lambda_1\lambda_2)^{-1}, \ \mu > 0, \ n > 0, \tag{3.1}$$

where λ_1, λ_2, λ_3 are the principal stretches, and the constants μ, n are constitutive parameters. Constitutive models of the form (3.1) were first introduced by Ogden [13] and have been widely investigated since then (see e.g. [14], [15]). The constant μ in (3.1) is the shear modulus for infinitesimal deformations and n is the hardening exponent. The special case when $n = 1$ in (3.1) corresponds to the neo-Hookean material.

We recall from Section 2 that the critical load p_{cr} is given by (2.24), i.e.,

$$p_{cr} = \int_1^\infty \frac{\hat{W}_1(v)}{(v^3 - 1)}\, dv, \tag{3.2}$$

where the notation (2.20) is used. Expressed in the notation of (2.20), the strain-energy density (3.1) can be written in polynomial form as

$$\hat{W}(v) = \frac{\mu}{2n}\,(v^{-4n} + 2v^{2n} - 3),\ \mu > 0,\ n > 0. \tag{3.3}$$

To ensure the existence of p_{cr}, we recall from (2.29) that 2n should be less than 3, i.e.,

$$n < \frac{3}{2}\ . \tag{3.4}$$

It is of interest to observe that a restriction similar to (3.4) also arises in the work of Carroll [28] concerned with the problem of inflation of a hollow sphere composed of the material (3.1).

The response of the material described by (3.1) to certain basic pure homogeneous deformations will now be discussed. A recent investigation of these issues was carried out by Zee and Sternberg [16], and we now summarize those results in [16] which are relevant to our problem here. The pure homogeneous deformations considered are as follows:

$$\left.\begin{array}{l}
\text{(i)}\ \ \underline{\text{Uniaxial}}\ \underline{\text{stress}}: \\
\qquad \tau_{11} = \tau_{22} = 0,\ \tau_{33}(\lambda) = \mu\,(\lambda^{2n} - \lambda^{-n}),\ \lambda_3 = \lambda,\ \lambda_1 = \lambda_2 = \lambda^{-1/2}, \\
\text{(ii)}\ \ \underline{\text{Equibiaxial}}\ \underline{\text{stress}}: \\
\qquad \tau_{33} = 0,\ \tau_{22}(\lambda) = \mu\,(\lambda^{2n} - \lambda^{-4n}),\ \lambda_1 = \lambda_2 = \lambda,\ \lambda_3 = \lambda^{-2}, \\
\text{(iii)}\ \underline{\text{Pure}}\ \underline{\text{shear}}: \\
\qquad \tau_{22} = 0,\ \tau_{11}(\lambda) = \mu\,(\lambda^{2n} - \lambda^{-2n}),\ \lambda_1 = \lambda_2^{-1} = \lambda,\ \lambda_3 = 1.
\end{array}\right\} \tag{3.5}$$

The normal stresses $\tau_{11}(\lambda)$, $\tau_{22}(\lambda)$, as well as $\tau_{33}(\lambda)$, for each of the pure homogeneous deformations (3.5), are monotonic increasing functions of λ for $0 < \lambda < \infty$. The stress-stretch relation (3.5) appropriate to (i) (uniaxial stress) is plotted in Figure 1 for the values of the exponent n given by n = 5/4, 1, 3/4, 1/2, and 1/4.

(cf. Figure 3 of [16]). Note that the material hardens as n increases. The graphs of $\tau_{22}(\lambda)$ and $\tau_{11}(\lambda)$, corresponding to the cases (ii) and (iii), are qualitatively similar to Figure 1 and are plotted in Figures 2, 3 respectively.

It is of interest to remark on the character of the system of governing partial differential equations, namely the displacement equations of equilibrium [16]

$$C_{ijkl}(\underset{\sim}{F})u_{k,lj} - p_{,j}F^{-1}_{ji} = 0; \; J = \det \underset{\sim}{F} = 1, \tag{3.6}$$

where $C_{ijkl}(\underset{\sim}{F})$ are the components of the fourth-order tensor defined by

$$C_{ijkl}(\underset{\sim}{F}) = C_{klij}(\underset{\sim}{F}) = \frac{\partial^2 W(\underset{\sim}{F})}{\partial F_{ij}\partial F_{kl}} \; . \tag{3.7}$$

Necessary and sufficient conditions for <u>ellipticity</u> of the system of equations (3.6), (3.7) have been obtained by Zee and Sternberg in [16]. For the special case of the material (3.1), these conditions are particularly simple. Thus from the results of [16], p.85, ellipticity holds for the material (3.1) <u>at all deformations</u> if

$$n \geq \frac{1}{2} \; . \tag{3.8}$$

In what follows, we assume that (3.8) holds, and so recalling (3.4), we thus have

$$\frac{1}{2} \leq n < \frac{3}{2} \; . \tag{3.9}$$

3.2 Cavitation solutions:

Consider a quasi-static loading process in which the solid sphere is subjected to a dead-load p_0 that increases slowly from zero. Cavity formation and growth is described by the relationship $p_0 = p_0(c)$ given in (2.22).

For the material described by (3.1) (recalling the notation (2.2), (2.8)) we have $\lambda_1 = v^{-2}$, $\lambda_2 = \lambda_3 = v$ and so the first derivative with respect to λ_2 is 0, i.e., $W_2(v^{-2}, v, v) = 0$, and the first derivative with respect to λ_1 is given by

$$W_1\,(v^{-2}, v, v) = \mu(v^{-4n+2} - v^{2n+2}),\ \frac{1}{2} \le n < \frac{3}{2}\ . \tag{3.10}$$

On using the notation (2.20), we thus obtain

$$\hat{W}_1(v) = 2\mu(v^{2n-1} - v^{-4n-1}),\ \frac{1}{2} \le n < \frac{3}{2}\ , \tag{3.11}$$

and so

$$\frac{\hat{W}_1(v)}{v^3 - 1} = 2\mu\,\frac{v^{2n-1} - v^{-4n-1}}{v^3 - 1}\ ,\ \frac{1}{2} \le n < \frac{3}{2}\ . \tag{3.12}$$

When the relationship (2.22) between the applied pressure p_0 and deformed cavity radius c is specialized to the particular strain-energy function (3.1) (and (3.12) is used), one obtains

$$p_0 = p_0(c) = 2\mu\,(1 + \frac{c^3}{b^3})^{2/3}\int_{(1 + \frac{c^3}{b^3})^{1/3}}^{\infty} \frac{v^{2n-1} - v^{-4n-1}}{v^3 - 1}\,dv,\ \frac{1}{2} \le n < \frac{3}{2}. \tag{3.13}$$

Before proceeding with an analysis of the relationship (3.13), it is convenient to record here corresponding expressions for the stresses subsequent to cavitation given by (2.7). On using (2.17), (2.18), (2.20), (2.21) we find

$$\tau_{RR}(r) = \int_{(1 + \frac{c^3}{r^3})^{1/3}}^{\infty} \frac{\hat{W}_1(v)}{v^3 - 1} \, dv, \tag{3.14}$$

while from (2.7) we obtain

$$\tau_{\Theta\Theta} = \tau_{\Phi\Phi} = vW_2(v^{-2}, v, v) - v^{-2}W_1(v^{-2}, v, v) + \tau_{RR}(r). \tag{3.15}$$

On using $W_2(v^{-2}, v, v) = 0$, (3.10) and (3.12) we obtain

$$\tau_{RR}(r) = 2\mu \int_{(1 + \frac{c^3}{r^3})^{1/3}}^{\infty} \frac{v^{2n-1} - v^{-4n-1}}{v^3 - 1} \, dv, \quad \frac{1}{2} \le n < \frac{3}{2}, \tag{3.16}$$

and

$$\tau_{\Theta\Theta} = \tau_{\Phi\Phi} = \tau_{RR}(r) - \mu \, [v^{-4n}(r) - v^{2n}(r)], \tag{3.17}$$

where we recall from (2.8) that $v(r) = (1 + \frac{c^3}{r^3})^{1/3}$.

We confine attention to the range of values of n in (3.9), namely $1/2 \le n < 3/2$. For specific values of n in this range, namely $n = 1/2, 3/4, 1, 5/4$, the integrals in (3.13) and (3.16) may be evaluated explicitly. The relevant integrals can be evaluated by using results of Ryshik and Gradstein [29]. We assemble these integrals in Appendix B. The corresponding expressions occurring in (3.13) then become

$$n = \frac{1}{2}: \; p_0 = \mu, \tag{3.18}$$

$$n = \frac{3}{4}:\ p_0 = 2\mu\left(1 + \frac{c^3}{b^3}\right)^{2/3}\left[\frac{1}{3}\ln\frac{1 + (1 + c^3/b^3)^{1/2}}{1 + c^3/b^3} + \frac{1}{3}\left(1 + \frac{c^3}{b^3}\right)^{-1} + \frac{1}{3}\ln\frac{c^3/b^3}{(1 + c^3/b^3)^{1/2} - 1}\right], \tag{3.19}$$

$n = 1$:

$$p_0 = 2\mu\left(1 + \frac{c^3}{b^3}\right)^{-2/3}\left[\frac{5 + 4\,c^3/b^3}{4}\right], \tag{3.20}$$

$n = \frac{5}{4}$:

$$\begin{aligned} p_0 = 2\mu\left(1 + \frac{c^3}{b^3}\right)^{2/3}\Bigg\{ & \frac{4}{3\sqrt{3}}\pi - \frac{1}{3}\ln\left[\frac{(1 + c^3/b^3)^{1/6} - 1}{(1 + c^3/b^3)^{1/3} - 1}\right] \\ & + \frac{1}{3}\ln\left[1 + \left(1 + \frac{c^3}{b^3}\right)^{1/6}\right] - \frac{1}{6}\ln\left[1 - \left(1 + \frac{c^3}{b^3}\right)^{1/6} + \left(1 + \frac{c^3}{b^3}\right)^{1/3}\right] \\ & + \frac{1}{6}\ln\left[1 + \left(1 + \frac{c^3}{b^3}\right)^{1/6} + \left(1 + \frac{c^3}{b^3}\right)^{1/3}\right] - \frac{1}{\sqrt{3}}\arctan\frac{2(1+c^3/b^3)^{1/6} - 1}{\sqrt{3}} \\ & - \frac{1}{6}\ln\left[1 + \left(1 + \frac{c^3}{b^3}\right)^{1/3} + \left(1 + \frac{c^3}{b^3}\right)^{2/3}\right] - \frac{1}{\sqrt{3}}\arctan\frac{2(1+c^3/b^3)^{1/6} + 1}{\sqrt{3}} \\ & + \frac{1}{5}\left(1 + \frac{c^3}{b^3}\right)^{-5/3} + \frac{1}{2}\left(1 + \frac{c^3}{b^3}\right)^{-2/3} \\ & - \frac{1}{\sqrt{3}}\arctan\frac{\sqrt{3}(1 + c^3/b^3)^{1/3}}{2 + (1 + c^3/b^3)^{1/3}}\Bigg\}. \end{aligned} \tag{3.21}$$

Equations (3.18) - (3.21) provide a relationship $p_0 = p_0(c)$ between the dimensionless applied dead-load p_0/μ and the dimensionless cavity radius c/b. The critical load p_{cr} is the value at which the curve $p_0 = p_0(c)$ bifurcates from the straight line $c = 0$ corresponding to the trivial homogeneous solution. On letting $c \to 0+$ in (3.18) - (3.21) and applying l'Hôpital's rule where appropriate, the critical load p_{cr} is tabulated below.

n	1/2	3/4	1	5/4
p_{cr}	μ	1.5909μ	2.5μ	4.7426μ

As one might expect, the values of p_{cr} increase as the hardening parameter n increases. The graphs of $p_0(c)$ according to (3.18)-(3.21) are shown in Figure 4. From Figure 4 (and (3.18)), it is clear that the case $n = 1/2$ is special. We recall from (3.8) that this is the limiting value of n for which ellipticity holds.

The corresponding principal stresses, given by (3.16),(3.17), are:

$n = \frac{1}{2}$:

$$\tau_{RR}(r) = \mu \left(1 + \frac{c^3}{r^3}\right)^{-2/3}, \tag{3.22}$$

$$\tau_{\Theta\Theta}(r) = \tau_{\Phi\Phi}(r) = \mu \left(1 + \frac{c^3}{r^3}\right)^{1/3}. \tag{3.23}$$

$n = \frac{3}{4}$:

$$\tau_{RR}(r) = 2\mu \left[\frac{1}{3} \ln \frac{1 + (1 + c^3/r^3)^{1/2}}{1 + c^3/r^3} + \frac{1}{3}\left(1 + \frac{c^3}{r^3}\right)^{-1} + \frac{1}{3} \ln \frac{c^3/r^3}{(1 + c^3/r^3)^{1/2} - 1} \right], \tag{3.24}$$

$$\tau_{\Theta\Theta}(r) = \tau_{\Phi\Phi}(r) = \mu \left[\frac{2}{3} \ln \frac{1 + (1 + c^3/r^3)^{1/2}}{1 + c^3/r^3} - \frac{1}{3} (1 + \frac{c^3}{r^3})^{-1} + \frac{2}{3} \ln \frac{c^3/r^3}{(1 + c^3/r^3)^{1/2} - 1} + (1 + \frac{c^3}{r^3})^{1/2} \right] . \tag{3.25}$$

$n = 1$:

$$\tau_{RR}(r) = 2\mu \left[(1 + \frac{c^3}{r^3})^{-1/3} + \frac{1}{4}(1 + \frac{c^3}{r^3})^{-4/3} \right] , \tag{3.26}$$

$$\tau_{\Theta\Theta}(r) = \tau_{\Phi\Phi}(r) = \tau_{RR}(r) - \mu \left[(1 + \frac{c^3}{r^3})^{-4/3} - (1 + \frac{c^3}{r^3})^{2/3} \right] . \tag{3.27}$$

$n = \frac{5}{4}$:

$$\begin{aligned} \tau_{RR}(r) = 2\mu \Bigg\{ & \frac{4}{3\sqrt{3}} \pi - \frac{1}{3} \ln \left[\frac{(1 + c^3/r^3)^{1/6} - 1}{(1 + c^3/r^3)^{1/3} - 1} \right] \\ & + \frac{1}{3} \ln \left[1 + (1 + \frac{c^3}{r^3})^{1/6} \right] - \frac{1}{6} \ln \left[1 - (1 + \frac{c^3}{r^3})^{1/6} + (1 + \frac{c^3}{r^3})^{1/3} \right] \\ & + \frac{1}{6} \ln \left[1 + (1 + \frac{c^3}{r^3})^{1/6} + (1 + \frac{c^3}{r^3})^{1/3} \right] - \frac{1}{\sqrt{3}} \arctan \frac{2(1+c^3/r^3)^{1/6} - 1}{\sqrt{3}} \\ & - \frac{1}{6} \ln \left[1 + (1 + \frac{c^3}{r^3})^{1/3} + (1 + \frac{c^3}{r^3})^{2/3} \right] - \frac{1}{\sqrt{3}} \arctan \frac{2(1+c^3/r^3)^{1/6} + 1}{\sqrt{3}} \\ & + \frac{1}{5} (1 + \frac{c^3}{r^3})^{-5/3} + \frac{1}{2} (1 + \frac{c^3}{r^3})^{-2/3} \\ & - \frac{1}{\sqrt{3}} \arctan \frac{\sqrt{3}(1 + c^3/r^3)^{1/3}}{2 + (1 + c^3/r^3)^{1/3}} \Bigg\} , \end{aligned} \tag{3.28}$$

$$\tau_{\Theta\Theta}(r) = \tau_{\Phi\Phi}(r) = \tau_{RR}(r) - \mu \left[(1 + \frac{c^3}{r^3})^{-5/3} - (1 + \frac{c^3}{r^3})^{5/6} \right] . \tag{3.29}$$

The graphs of $\tau_{RR}(r)$, $\tau_{\Theta\Theta}(r)$ and $\tau_{\Phi\Phi}(r)$ corresponding to (3.24)-(3.29), i.e. for values of n = 3/4, 1, 5/4, are shown in Figures 5-10. An interesting feature concerning these stresses immediately after cavitation is the presence of a boundary layer near the cavity wall. To see this, we have plotted the stresses in Figures 5-10 for applied dead loads p_0 slightly larger than p_{cr}. A similar boundary-layer phenomenon was observed in [11] for the problem of tensile dead-loading of a composite sphere composed of two neo-Hookean materials.

4. ELASTODYNAMIC PROBLEM FOR A NEO-HOOKEAN SOLID SPHERE SUBJECTED TO A SUDDENLY APPLIED DEAD-LOAD

4.1 Formulation:

In this Section, we consider the radially symmetric motion of an isotropic incompressible elastic solid sphere composed of a neo-Hookean material. The undeformed sphere has radius b, and it is set into motion at time $t = 0$ by a suddenly applied uniform radial tensile dead-load p_0. In this incompressible case, the effect of the tensile load is felt immediately throughout the medium, and the response takes the form of a nonlinear oscillation.

Large amplitude oscillations of hollow incompressible elastic cylinders were first considered by Knowles [17,18]. Methods similar to those used in [17] and [18] have been applied to the case of symmetric motions of a hollow thick-walled incompressible elastic sphere in [19], and an unbounded incompressible elastic medium containing a spherical cavity has been treated in [20]. See [21] for a review of some of this work. See also the recent paper [22] for a treatment, using phase-plane arguments, of radial motion of thick spherical shells composed of incompressible materials.

The emphasis in [17-20] is on the characteristics of the motion, such as the period and amplitude, and on conditions which will ensure the existence of periodic motions. In this Section we use the techniques developed in [17-20] to investigate the dynamic analog of the bifurcation problem described in Sections 2 and 3. For simplicity of presentation we restrict our attention to the case of a neo-Hookean material. We use similar notation to that introduced in Section 2. Thus a point which at time t has spherical coordinates (R,Θ,Φ) is assumed to have been at the point (r,θ,ϕ) in the undeformed state. The motion is thus described by

$$R = R(r,t) > 0, \ 0 < r < b; \ R(0+,t) \geq 0; \ \Theta = \theta, \ \Phi = \phi, \tag{4.1}$$

where $R(r,t)$ is to be determined. Since the material is assumed to be incompressible, the deformation gradient $\underset{\sim}{F}$ obeys $\det \underset{\sim}{F} = 1$, $t \geq 0$. For the motion (4.1), this implies $R^2 \partial R/\partial r = r^2$, which when integrated gives

$$R = R(r,t) = [r^3 + c^3(t)]^{1/3}, \ c(t) \geq 0, \ t \geq 0, \tag{4.2}$$

where $c(t)$ is to be determined. The motion is completely determined once $c(t)$ is known. If it is found that $c(t) = 0$ for $t \geq 0$, (4.2) implies that the body remains a solid sphere in the current configuration. On the other hand if $c(t) > 0$ (i.e. $R(0+, t) > 0$), there is a cavity of radius $c(t)$ centered at the origin in the current configuration. In this event, the cavity surface is assumed to be traction-free.

For the neo-Hookean material, the strain energy density per unit undeformed volume is given by

$$W(\lambda_1,\lambda_2,\lambda_3) = \frac{\mu}{2} (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3), \ \lambda_1\lambda_2\lambda_3 = 1, \tag{4.3}$$

where λ_i $(i = 1,2,3)$ are the principal stretches, and $\mu > 0$ is the shear modulus for infinitesimal deformations.

For the radially symmetric motion (4.1), the principal stretches are given by $\lambda_r = \partial R(r,t)/\partial r$, $\lambda_\theta = \lambda_\phi = R(r,t)/r$. The principal components of the Cauchy stress tensor $\underset{\sim}{\tau}$ are again given by (2.6) which for the material (4.3) and the motion (4.1) can be written as

$$\tau_{RR}(R,t) = \mu \frac{(R^3 - c^3)^{4/3}}{R^4} - P(R,t),$$

$$\tau_{\Theta\Theta}(R,t) = \tau_{\Phi\Phi}(R,t) = \mu \frac{R^2}{(R^3 - c^3)^{2/3}} - P(R,t), \quad t \geq 0, \qquad (4.4)$$

where $P(R,t)$ represents the arbitrary hydrostatic pressure.

It is assumed that the sphere is in an undeformed state and at rest at time $t = 0$, so that $R(r,0) = r$, $\dot{R}(r,0) = 0$, and so from (4.2) we deduce that the current cavity radius $c(t)$ must satisfy the initial conditions

$$c(0) = 0, \quad \dot{c}(0) = 0, \qquad (4.5)$$

where the dot denotes differentiation with respect to time.

A dead-load p_0 is suddenly applied and maintained at the surface of the sphere so that the boundary conditions are

$$\left.\begin{aligned} \tau_{RR}(A,t) &= 0, \\ \tau_{RR}(A,t) &= p_0\left(\frac{b}{A}\right)^2, \end{aligned}\right\} \begin{aligned} t &< 0, \\ t &\geq 0, \end{aligned} \qquad (4.6)$$

where p_0 is a positive constant and $A = R(b,t) = \{b^3 + c(t)^3\}^{1/3}$ is the deformed outer radius. In addition if $c(t) > 0$, then the condition for a traction-free cavity surface

$$\tau_{RR}(c,t) = 0, \quad t \geq 0, \qquad (4.7)$$

must also hold.

The equations of motion, in the absence of body force, governing the radially symmetric motion of the sphere reduce to the single equation

$$\frac{\partial \tau_{RR}}{\partial R} + \frac{1}{R}(2\tau_{RR} - \tau_{\Theta\Theta} - \tau_{\Phi\Phi}) = \rho\ddot{R}, \ t \geq 0, \tag{4.8}$$

where ρ is the constant mass density of the material. Thus the problem to be solved is the following: <u>For a prescribed value of the dead-load traction</u> $p_0 > 0$, <u>we seek a pressure field</u> $P(R,t)$, <u>and a time dependent function</u> $c(t) \geq 0$, <u>such that</u> (4.2), (4.5), (4.8) <u>and</u> (4.6) <u>are satisfied where</u> τ_{RR}, $\tau_{\Theta\Theta}$, $\tau_{\Phi\Phi}$ <u>are given by</u> (4.4). <u>In addition if</u> $c(t) > 0$, <u>then</u> (4.7) <u>must also be satisfied</u>.

4.2 Solutions:

It is readily shown that one solution to the foregoing problem, for all values of p_0, is

$$P(R,t) = \mu - p_0 \ , \ c(t) = 0, \ t \geq 0. \tag{4.9}$$

This corresponds to the trivial homogeneous (static) state of deformation

$$R(r,t) = r, \ t \geq 0, \tag{4.10}$$

with corresponding stresses $\tau_{RR} = \tau_{\Theta\Theta} = \tau_{\Phi\Phi} = p_0$.

Next we describe solutions for which $c(t) > 0$, corresponding to the presence of a traction-free cavity at the origin. On substitution from (4.4) into (4.8) we obtain

$$\frac{\partial}{\partial R}\left[\mu \frac{(R^3 - c^3)^{4/3}}{R^4} - P(R,t)\right] + \frac{2\mu}{R}\left[\frac{(R^3 - c^3)^{4/3}}{R^4} - \frac{R^2}{(R^3 - c^3)^{2/3}}\right] = \rho\ddot{R}. \tag{4.11}$$

The incompressibility condition (4.2) is now used to compute the acceleration d^2R/dt^2 in terms of the acceleration $d^2c(t)/dt^2$ of particles on the cavity surface, so that we have

$$\frac{d^2R}{dt^2} = 2cR^{-5}(R^3 - c^3)\left(\frac{dc}{dt}\right)^2 + c^2 R^{-2} \frac{d^2c}{dt^2}. \tag{4.12}$$

Equation (4.12) is now introduced into the right hand side of (4.11) to yield

$$\frac{\partial}{\partial R}\left[\mu \frac{(R^3 - c^3)^{4/3}}{R^4} - P(R,t)\right] + \frac{2\mu}{R}\left[\frac{(R^3 - c^3)^{4/3}}{R^4} - \frac{R^2}{(R^3 - c^3)^{2/3}}\right]$$

$$= 2\rho cR^{-5}(R^3 - c^3)\left(\frac{dc}{dt}\right)^2 + \rho c^2 R^{-2} \frac{d^2c}{dt^2}. \tag{4.13}$$

Equation (4.13) is now integrated with respect to R, to yield

$$\mu \frac{(R^3 - c^3)^{4/3}}{R^4} - P(R,t) + P(c,t) + 2\mu\int_c^R \left[\frac{(\xi^3 - c^3)^{4/3}}{\xi^5} - \frac{\xi}{(\xi^3 - c^3)^{2/3}}\right] d\xi$$

$$= 2\rho c\left(\frac{dc}{dt}\right)^2 \int_c^R \frac{(\xi^3 - c^3)}{\xi^5}\, d\xi + \rho c^2 \frac{d^2c}{dt^2}\int_c^R \frac{d\xi}{\xi^2}\,. \tag{4.14}$$

The integral on the left hand side of (4.14) may be simplified, on integration by parts, to yield

$$2\mu\int_c^R \left[\frac{(\xi^3 - c^3)^{4/3}}{\xi^5} - \frac{\xi}{(\xi^3 - c^3)^{2/3}} \right] d\xi \tag{4.15}$$

$$= -\frac{\mu(R^3 - c^3)^{4/3}}{2R^4} + 2\mu\int_c^R \frac{(\xi^3 - c^3)^{1/3}}{\xi^2}\, d\xi - 2\mu\int_c^R \xi(\xi^3 - c^3)^{2/3} d\xi .$$

The first integral on the right hand side of (4.15) is also simplified on integrating by parts to yield

$$2\mu\int_c^R \frac{(\xi^3 - c^3)^{1/3}}{\xi^2}\, d\xi = -2\mu\frac{(R^3 - c^3)^{1/3}}{R} + 2\mu\int_c^R \frac{\xi}{(\xi^3 - c^3)^{2/3}}\, d\xi . \tag{4.16}$$

Thus on combining (4.15), (4.16) and evaluating the integrals on the right hand side of (4.14) directly, we rewrite (4.14) as follows:

$$\mu\frac{(R^3 - c^3)^{4/3}}{R^4} - P(R,t)$$

$$= -P(c,t) + 2\mu\left[\frac{(R^3 - c^3)^{4/3}}{4R^4} + \frac{(R^3 - c^3)^{1/3}}{R} \right]$$

$$+ 2\rho c\left(\frac{dc}{dt}\right)^2 \left[\frac{c^3}{4R^4} - \frac{1}{R} + \frac{3}{4c} \right] + \rho c^2 \frac{d^2c}{dt^2}\left[\frac{1}{c} - \frac{1}{R} \right] . \tag{4.17}$$

Equation (4.17) is now introduced into the right hand side of the first of (4.4), and then the traction-free cavity surface condition (4.7) is imposed. This leads to $P(c,t) = 0$, $t \geq 0$, and so we obtain

$$\tau_{RR}(R,t) = 2\mu\left[\frac{(R^3 - c^3)^{4/3}}{4R^4} + \frac{(R^3 - c^3)^{1/3}}{R} \right]$$

$$+ 2\rho c\left(\frac{dc}{dt}\right)^2 \left[\frac{c^3}{4R^4} - \frac{1}{R} + \frac{3}{4c} \right] + \rho c^2 \frac{d^2c}{dt^2}\left[\frac{1}{c} - \frac{1}{R} \right] . \tag{4.18}$$

Finally the boundary condition (4.6) at $R = A = (b^3 + c(t)^3)$ is satisfied if

$$
p_0 \left[\frac{b}{(b^3 + c^3)^{1/3}} \right]^2 = 2\mu \left[\frac{b^4}{4(b^3 + c^3)^{4/3}} + \frac{b}{(b^3 + c^3)^{1/3}} \right]
$$

$$
+ 2\rho c \left(\frac{dc}{dt}\right)^2 \left[\frac{c^3}{4(b^3 + c^3)^{4/3}} - \frac{1}{(b^3 + c^3)^{1/3}} + \frac{3}{4c} \right]
$$

$$
+ \rho c^2 \frac{d^2 c}{dt^2} \left[\frac{1}{c} - \frac{1}{(b^3 + c^3)^{1/3}} \right], \quad t \geq 0. \tag{4.19}
$$

The relationship (4.19) between the applied load p_0 and cavity radius $c(t)$ is the <u>dynamic</u> counterpart of (2.22), for the neo-Hookean material. In fact, on formally replacing $c(t)$ in (4.19) by the constant c, it is readily verified that one recovers (3.20).

4.3 <u>The basic differential equation</u>:

To treat the differential equation (4.19), we adopt the techniques of Knowles [17] and consider the quantity

$$
x(t) = \frac{c(t)}{b} > 0, \tag{4.20}
$$

where b is the original undeformed radius of the solid sphere. In this notation (4.19) becomes a nonlinear second-order ordinary differential equation for the dimensionless cavity radius $x(t)$. From (4.20) we have

$$
c(t) = bx(t), \quad \frac{dc}{dt} = b\,\frac{dx}{dt}, \quad \frac{d^2c}{dt^2} = b\,\frac{d^2x}{dt^2} \,. \tag{4.21}
$$

On introducing the notation

$$f(x) = \frac{2\mu}{\rho b^2}\left[\frac{1}{4(1+x^3)^{4/3}} + \frac{1}{(1+x^3)^{1/3}}\right], \qquad (4.22)$$

and using (4.21), we rewrite (4.19) as

$$\frac{p_0}{\rho b^2(1+x^3)^{2/3}} = x^2\frac{d^2x}{dt^2}\left[\frac{1}{x} - \frac{1}{(1+x^3)^{1/3}}\right]$$

$$+ 2x\left(\frac{dx}{dt}\right)^2\left[\frac{x^3}{4(1+x^3)^{4/3}} - \frac{1}{(1+x^3)^{1/3}} + \frac{3}{4x}\right] + f(x),\ t \geq 0. \qquad (4.23)$$

Since the motion starts when the sphere is undeformed and at rest (see equation (4.5)), we deduce from (4.5), (4.20), (4.21) that the initial conditions

$$x(0) = 0,\ \frac{dx(0)}{dt} = 0, \qquad (4.24)$$

must also hold.

4.4 <u>Oscillations</u>:

With the notation $v = dx/dt$, $d^2x/dt^2 = v\,dv/dx$, it is possible to write the differential equation (4.23) in the form

$$\frac{d}{dx}\left\{x^4\left[\frac{1}{x} - \frac{1}{(1+x^3)^{1/3}}\right]v^2\right\} + 2x^2 f(x) = \frac{2x^2 p_0}{\rho b^2(1+x^3)^{2/3}}. \qquad (4.25)$$

Using (4.22), we find that (4.25) may be integrated with respect to x over the interval from zero to x to yield

$$x^4\left[\frac{1}{x} - \frac{1}{(1+x^3)^{1/3}}\right]v^2 + \frac{4\mu}{\rho b^2}\left[\frac{(1+x^3)^{2/3}}{2} - \frac{1}{4(1+x^3)^{1/3}} - \frac{1}{4}\right]$$

$$= \frac{2p_0}{\rho b^2}\left[(1+x^3)^{1/3} - 1\right],\ t \geq 0. \qquad (4.26)$$

It is well known from the theory of vibrations that the motion $x(t)$ is periodic if and only if the 'energy curves' (4.26) are closed curves in the x-v plane with a finite period $\oint dx/v$. The energy curve in the x-v plane is symmetric about the x-axis. This curve, given by (4.26), starts at the initial point $x = 0$, $v = 0$ at time $t = 0$. If p_0 is sufficiently large to produce an internal cavity, x and v then move into the region $x > 0$, $v > 0$ as t increases from zero. If v passes through a maximum and returns to zero as x increases from zero, the curve will be closed. According to (4.26), this will happen for a given p_0 if there is a root $x > 0$ of (4.26) when $v = 0$. Setting $v = 0$ in (4.26) we obtain

$$\frac{p_0}{\mu} = [(1 + x^3)^{1/3} - 1]^{-1} \left[(1 + x^3)^{2/3} - \frac{1}{2(1 + x^3)^{1/3}} - \frac{1}{2} \right]. \qquad (4.27)$$

The right hand side of (4.27) is a <u>monotone</u> increasing function of x for $x > 0$. As $x \to 0+$ in (4.27), we find, using l'Hôpital's rule, that

$$\frac{p_0}{\mu} \to \frac{5}{2}+ \; . \qquad (4.28)$$

For a given $p_0 > 5\mu/2$, we denote by x_m the non-zero root of (4.27) (there is <u>only one</u> since the right hand side of (4.27) is monotonic increasing). The quantity x_m is the maximum cavity radius in the oscillation process. If $p_0 < 5\mu/2$, no positive root of (4.27) exists, and hence periodic motions do not occur for this range of applied tensile loads. <u>Thus we have shown that the value of the "critical load" at which an internal cavity may be initiated in the dynamic problem coincides with that for the static problem</u>. (Recall equation

(2.33)). Thus following application of a pressure $p_0 > 5\mu/2$, an internal cavity would form and expand until it would reach the value x_m, given by the root of (4.27), then would contract to zero and repeat the cycle.

It is of interest to note that Knowles and Jakub [20] found that no periodic motions exist for values of pressure <u>above</u> $5\mu/2$ for the problem of an unbounded solid, composed of a neo-Hookean material, containing a spherical cavity which is set into motion by the sudden application of a spatially uniform radial pressure to the cavity wall. In fact, for this problem, the deformed cavity radius tends to infinity as the applied pressure tends to the value $5\mu/2$. A related observation was made by Gent and Lindley [2] and by Ball [3] for the corresponding static problems.

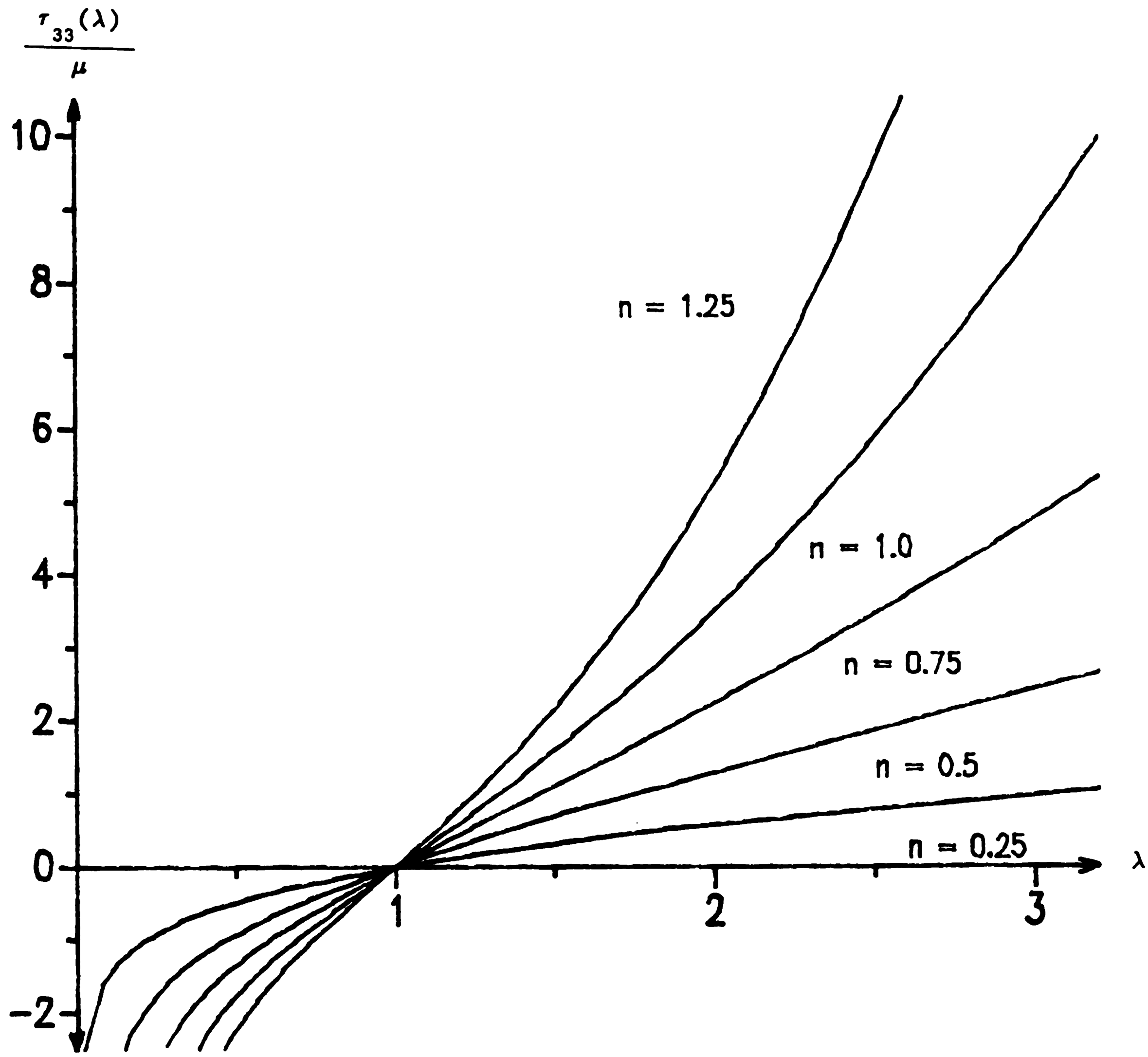

Figure 1. Behavior of the power law material under uniaxial stress.

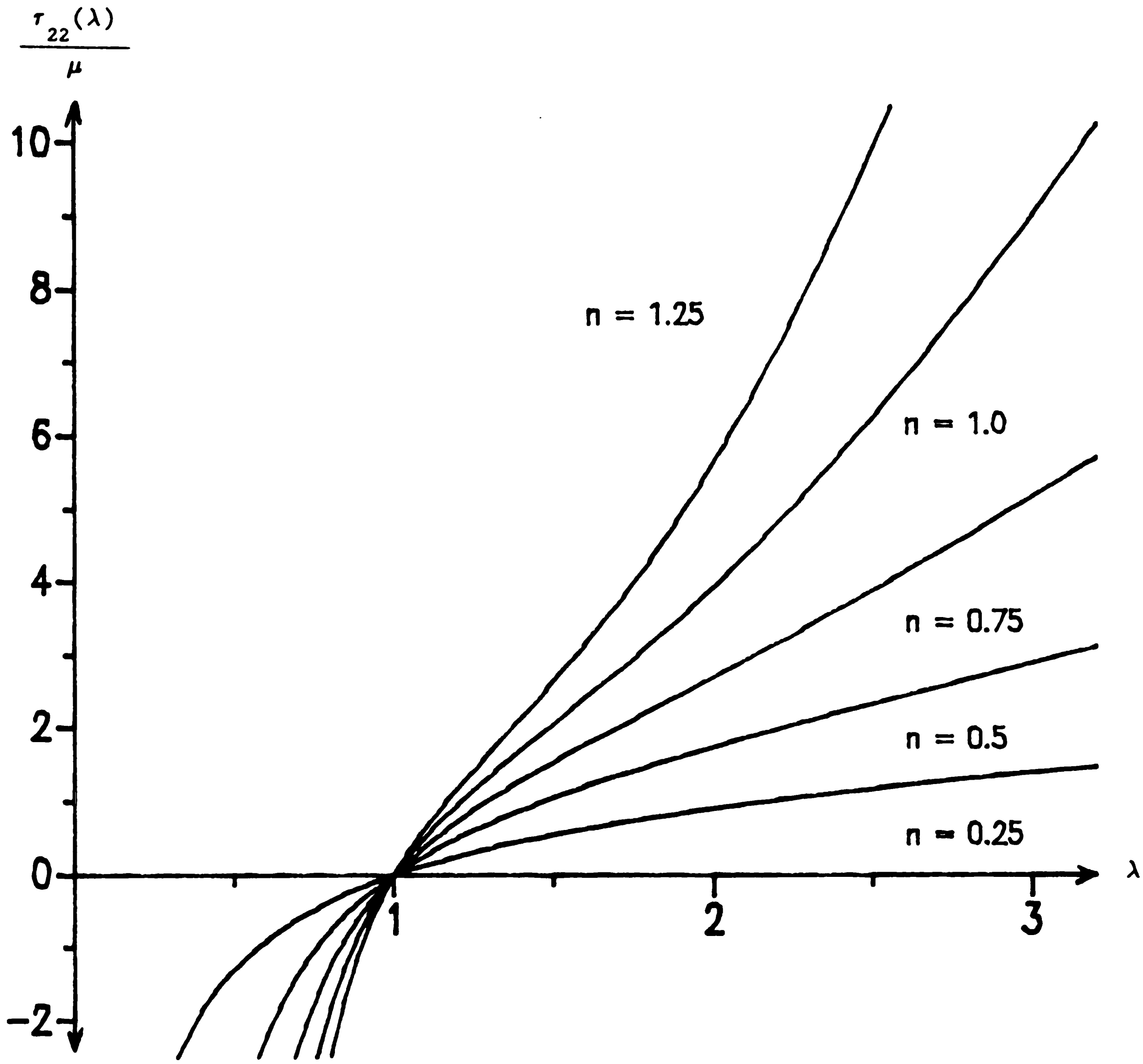

Figure 2. Behavior of the power law material under equibiaxial stress.

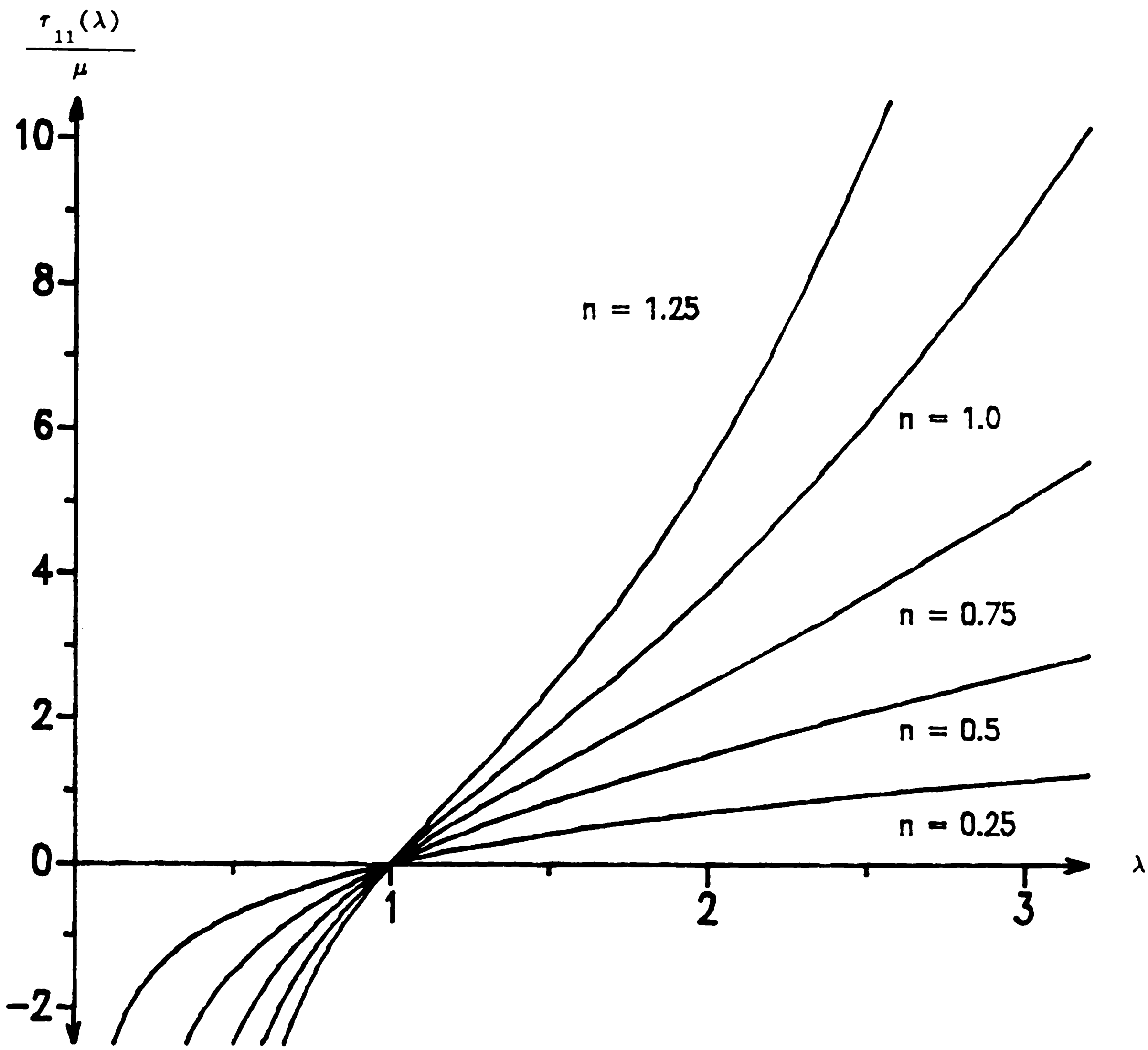

Figure 3. Behavior of the power law material under pure shear.

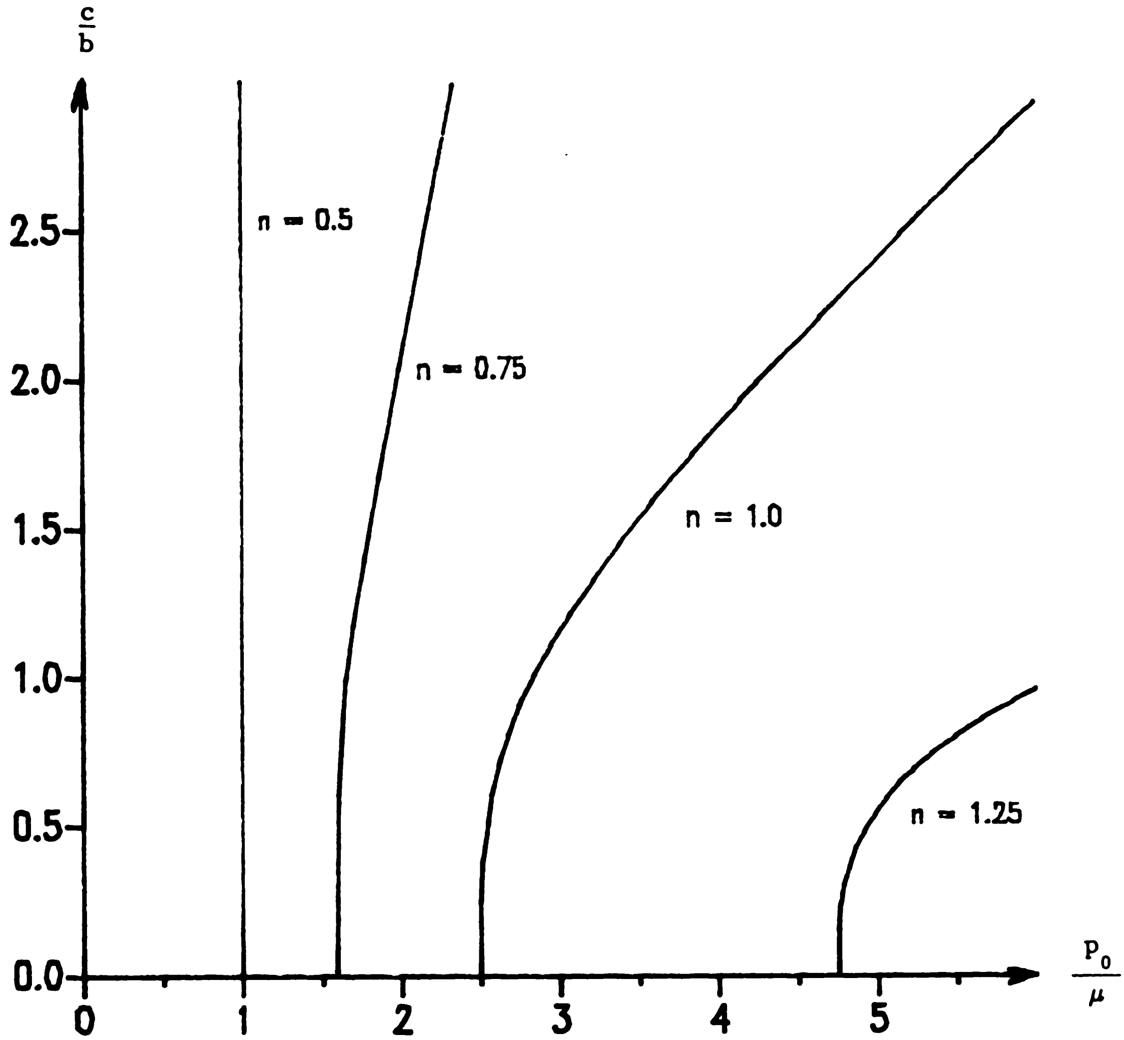

Figure 4. Variation of the deformed cavity radius c with applied dead load p_0 for a power law material with strain energy density given by (3.1).

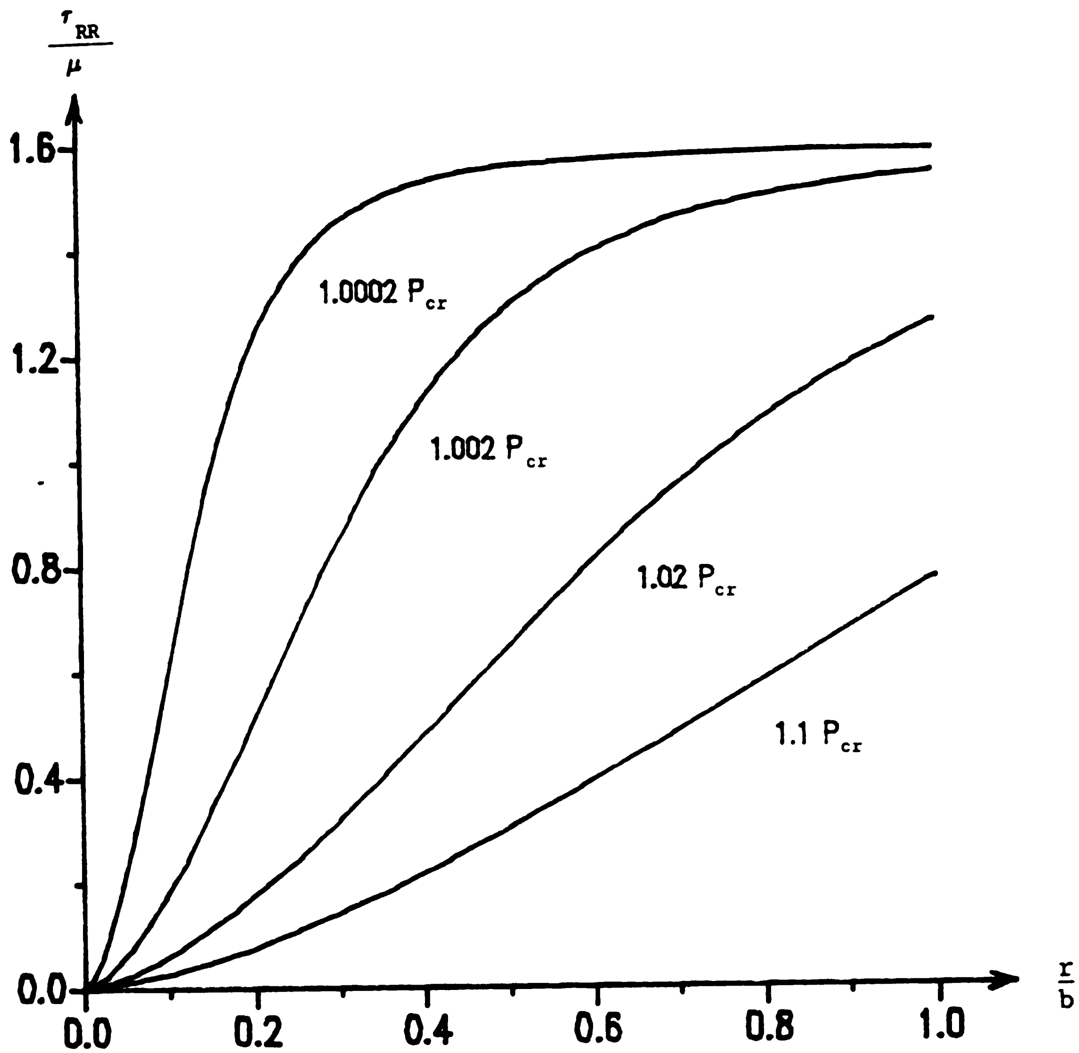

Figure 5. Variation of the radial stress $\tau_{RR}(r)$ with undeformed radius r subsequent to cavitation for a power law material (3.1) with $n = 3/4$.

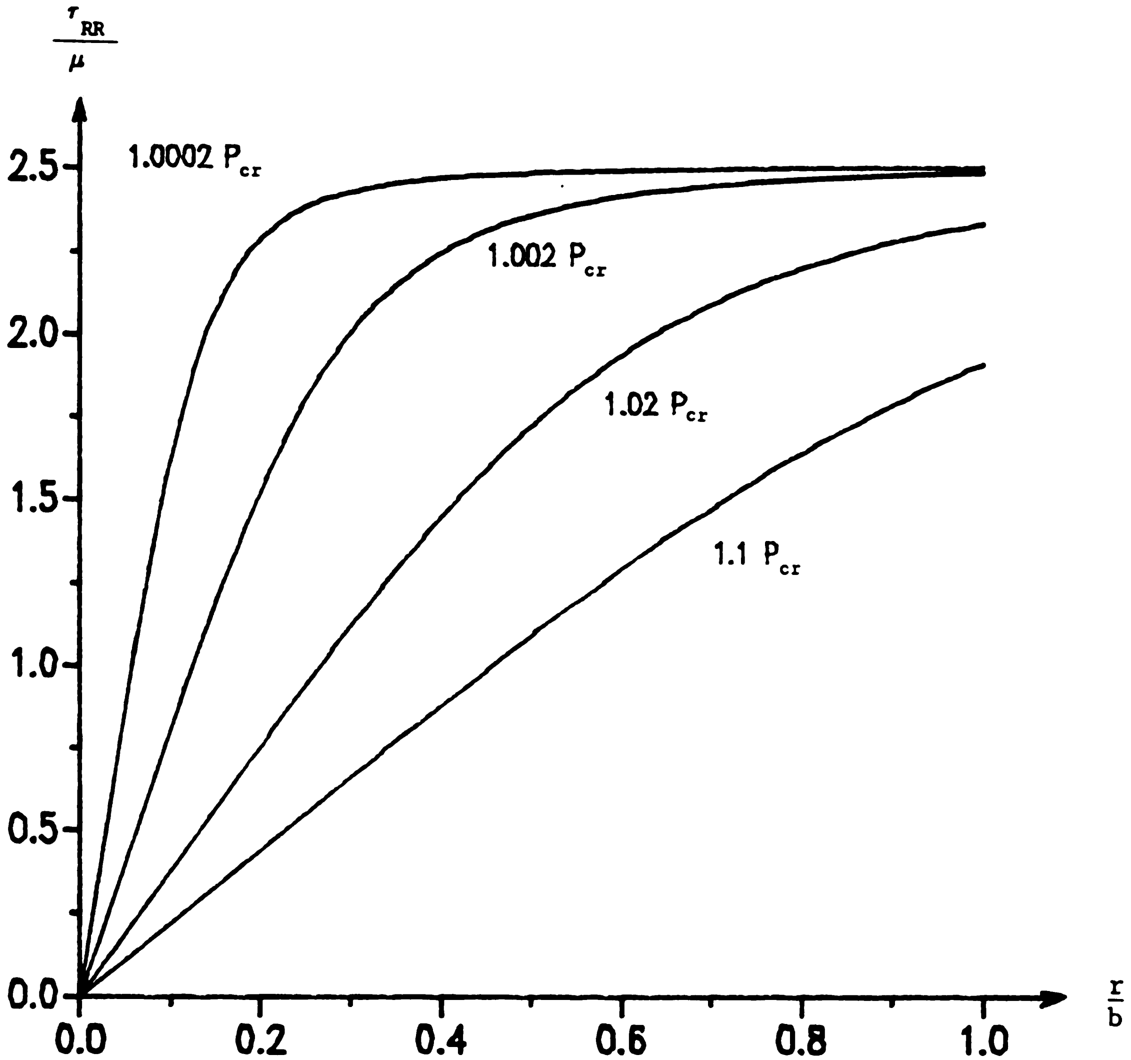

Figure 6. Variation of the radial stress $\tau_{RR}(r)$ with undeformed radius r subsequent to cavitation for a power law material (3.1) with n = 1.

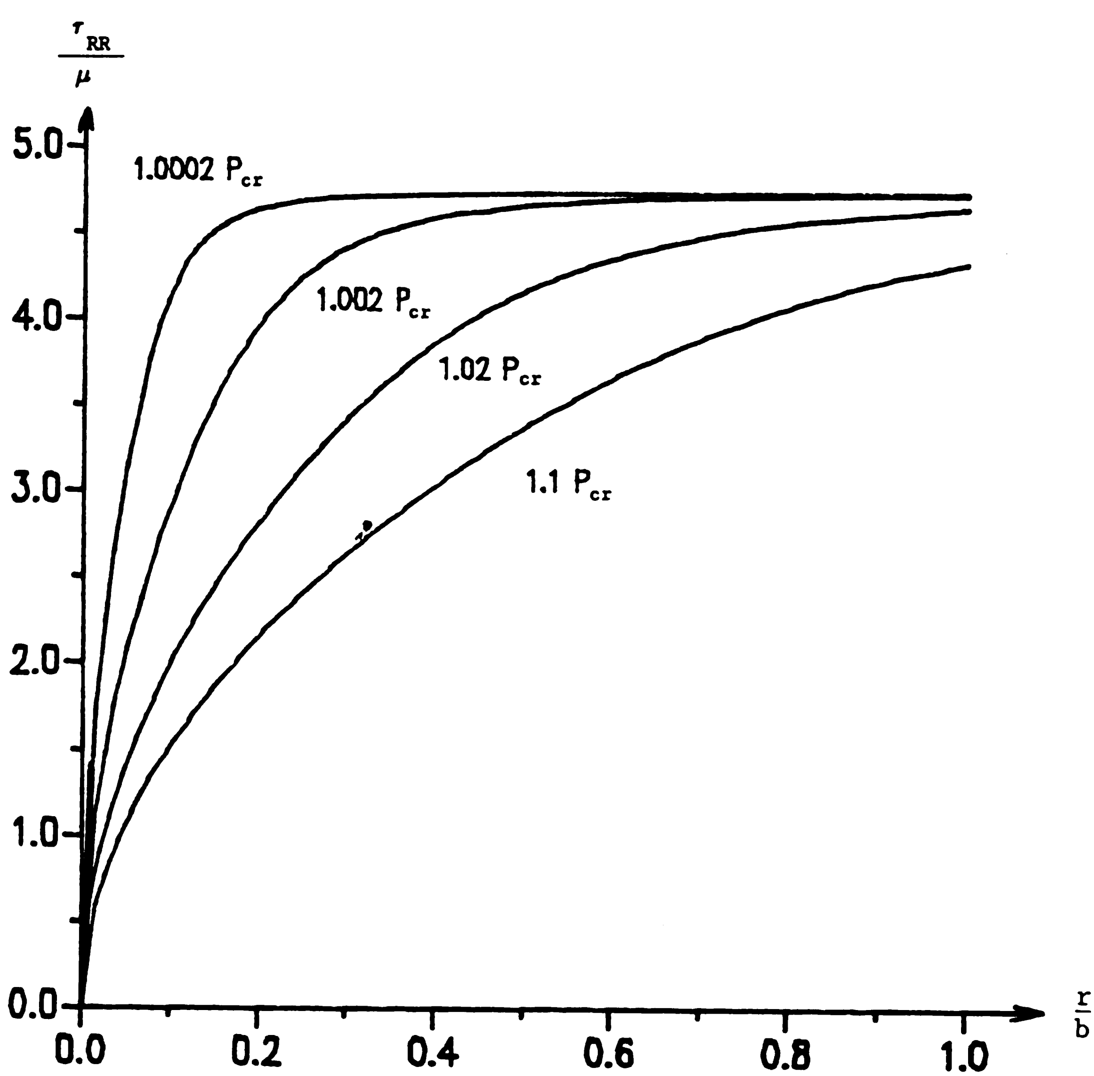

Figure 7. Variation of the radial stress $\tau_{RR}(r)$ with undeformed radius r subsequent to cavitation for a power law material (3.1) with n = 5/4.

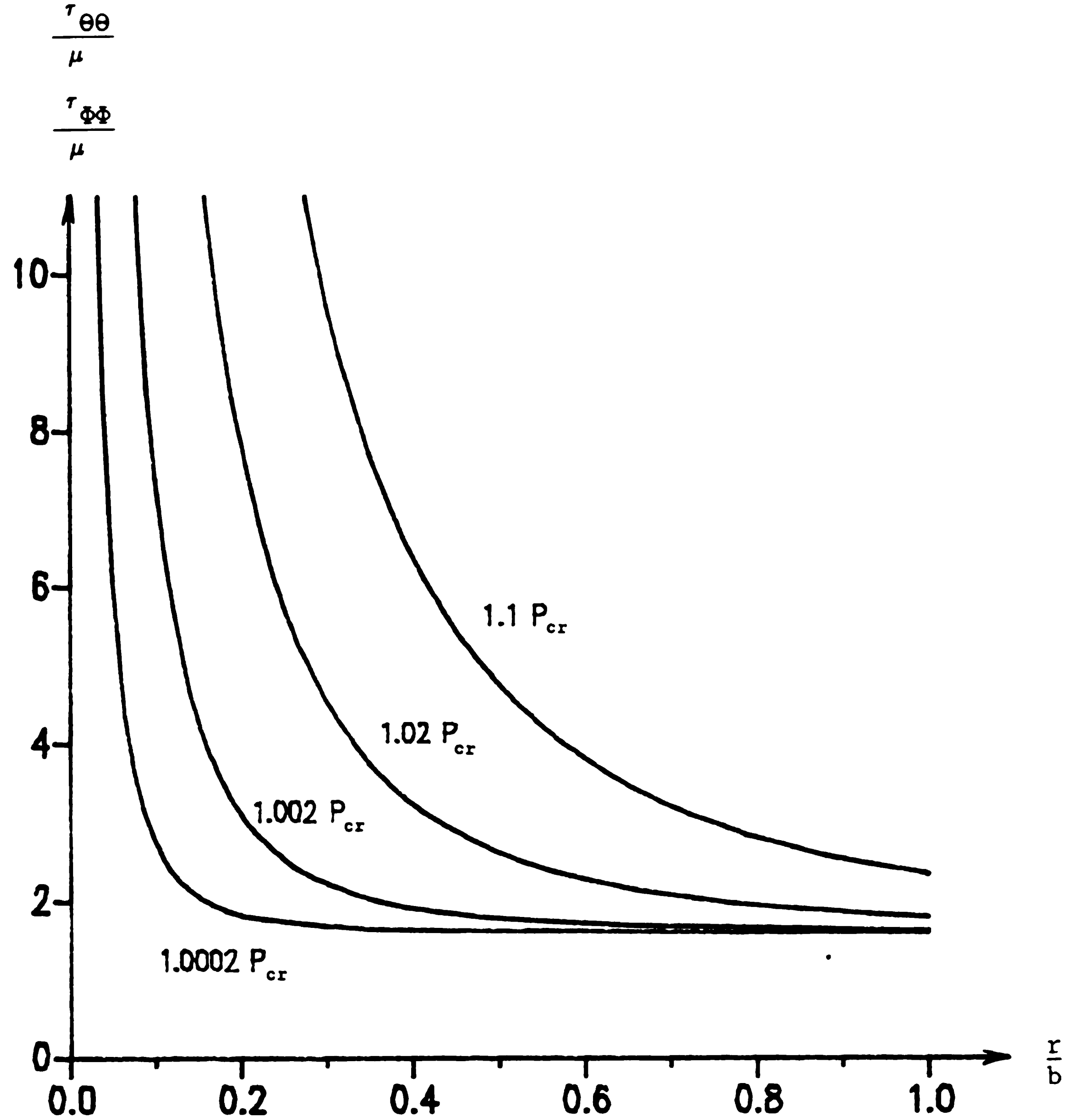

Figure 8. Variation of the stresses $\tau_{\theta\theta}(r)$, $\tau_{\Phi\Phi}(r)$ with undeformed radius r subsequent to cavitation for a power law material (3.1) with n = 3/4.

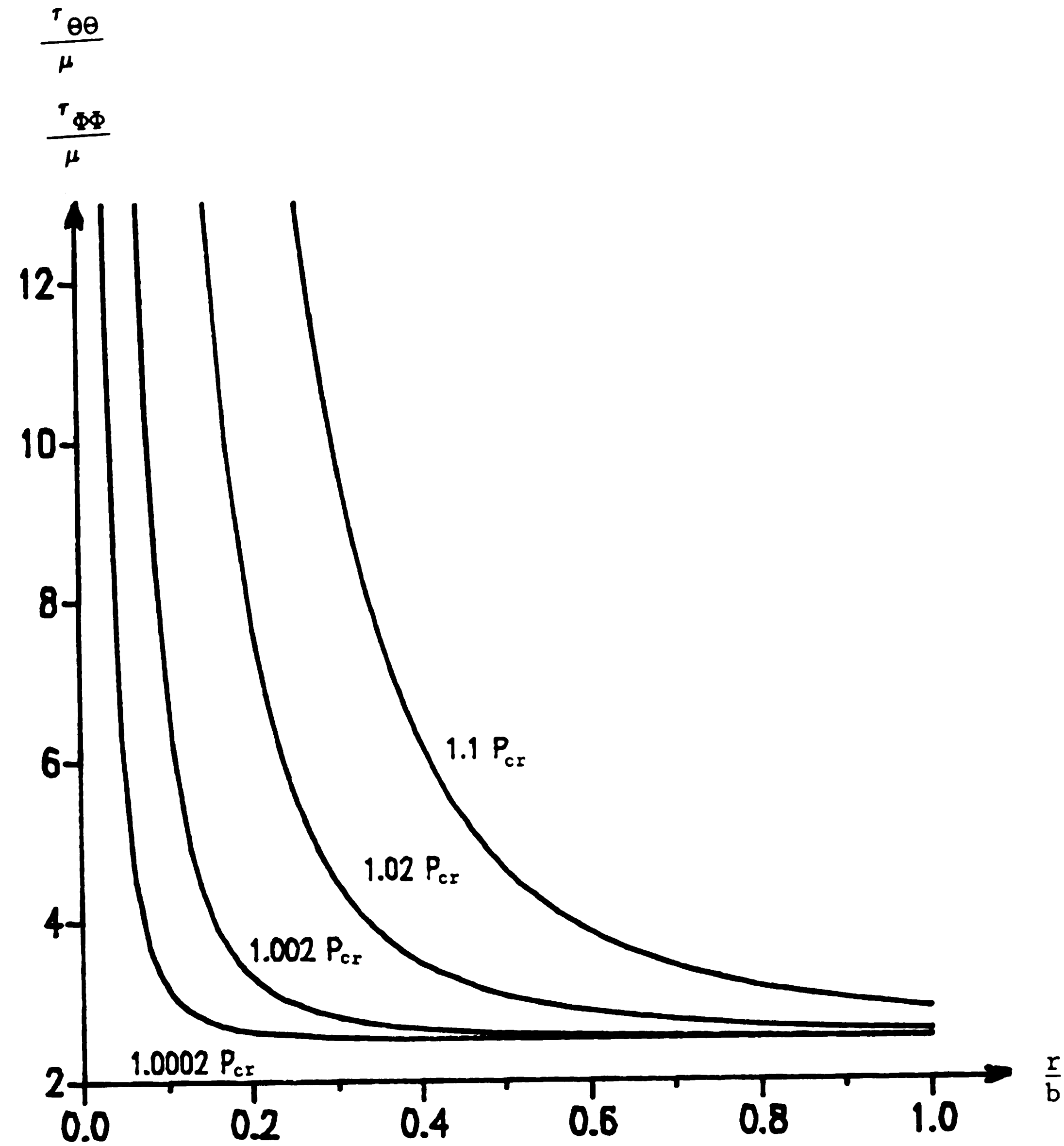

Figure 9. Variation of the stresses $\tau_{\theta\theta}(r)$, $\tau_{\Phi\Phi}(r)$ with undeformed radius r subsequent to cavitation for a power law material (3.1) with n = 1.

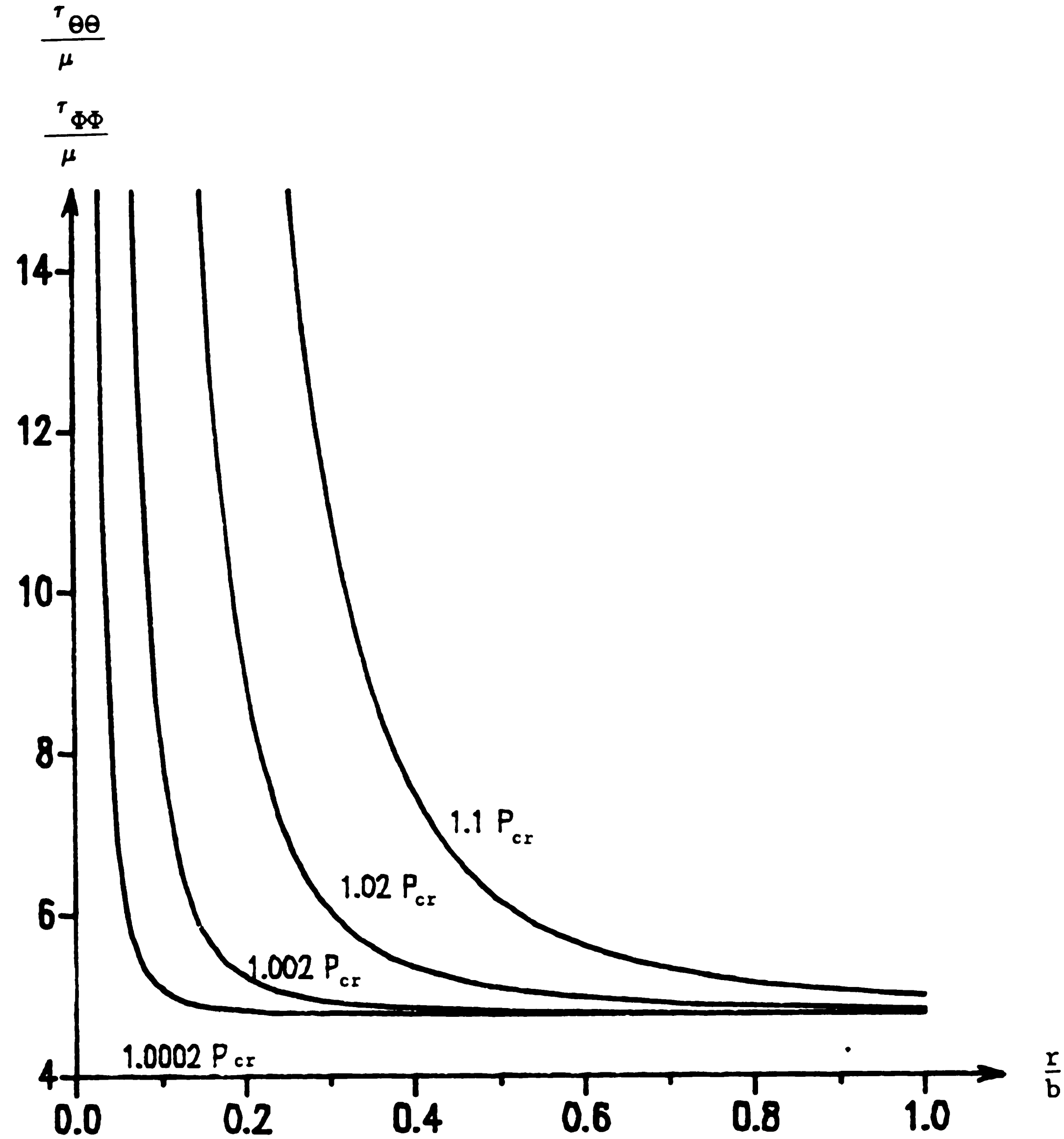

Figure 10. Variation of the stresses $\tau_{\Theta\Theta}(r)$, $\tau_{\Phi\Phi}(r)$ with undeformed radius r subsequent to cavitation for a power law material (3.1) with n = 5/4.

Appendices

Appendix A: Verification of (2.25)

Equation (2.25) has been established recently by Horgan and Pence [11]. For completeness here, we provide a brief review of their argument. First we recall from (2.20) that

$$\hat{W}(v) = W(v^{-2}, v, v), \tag{A.1}$$

and so

$$\frac{d\hat{W}(v)}{dv} = -2v^{-3}W_1(v^{-2}, v, v) + 2W_2(v^{-2}, v, v), \tag{A.2}$$

on using the chain rule and the fact that $W_2(v^{-2},v,v) = W_3(v^{-2},v,v)$. Thus

$$\frac{d\hat{W}(1)}{dv} = 2[W_2(1, 1, 1) - W_1(1, 1, 1)] = 0, \tag{A.3}$$

which establishes $(2.25)_1$ as desired.

To verify $(2.25)_2$, we recall from finite elasticity theory (see e.g. Ogden [15]) that the shear modulus for infinitesimal deformations of an incompressible homogeneous isotropic material with strain-energy density $\bar{W}(I_1, I_2)$ is given by

$$\mu = 2\left[\frac{\partial \bar{W}}{\partial I_1} + \frac{\partial \bar{W}}{\partial I_2}\right]_{I_1 = I_2 = 3}. \tag{A.4}$$

Here $I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$, $I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2$, are the usual first and second invariants. Thus from (A.1) we have

$$\hat{W}(v) = \bar{W}(\tilde{I}_1(v), \tilde{I}_2(v)), \tag{A.5}$$

where

$$\tilde{I}_1(v) = v^{-4} + 2v^2, \quad \tilde{I}_2(v) = 2v^{-2} + v^4. \tag{A.6}$$

Using the chain-rule, and observing that

$$\frac{d\tilde{I}_1}{dv} = \frac{d\tilde{I}_2}{dv} = 0, \text{ when } v = 1, \tag{A.7}$$

it is readily verified that

$$\frac{d^2\hat{W}(1)}{dv^2} = \left[\frac{\partial\bar{W}}{\partial I_1}\frac{d^2\tilde{I}_1}{dv^2} + \frac{\partial\bar{W}}{\partial I_2}\frac{d^2\tilde{I}_1}{dv^2}\right]\Bigg|_{\substack{I_1 = I_2 = 3,\\ v = 1,}} \tag{A.8}$$

and so it follows from (A.4), on using (A.6), that

$$\frac{d^2\hat{W}(1)}{dv^2} = 12\,\mu, \tag{A.9}$$

which establishes $(2.25)_2$ as desired.

Appendix B. Verification of (3.18) - (3.21) and (3.22), (3.24), (3.26), (3.28)

Here we present the details of the derivation of equations (3.18) - (3.21) and (3.22), (3.24), (3.26), (3.28). We first treat the indefinite integral which is needed to evaluate both (3.13) and (3.16): (Constants of integration will not be written down.)

$$I = \int \frac{v^{2n-1} - v^{-4n-1}}{v^3 - 1}\, dv, \quad \frac{1}{2} \le n < \frac{3}{2}. \tag{B.1}$$

It is convenient to record here the values of $2n - 1$ and $-4n - 1$ corresponding to $n = 1/2, 3/4, 1, 5/4$, respectively. The integral (B.1) will be decomposed into the two parts involving these exponents.

n	$1/2$	$3/4$	1	$5/4$
$2n - 1$	0	$1/2$	1	$3/2$
$-4n - 1$	-3	-4	-5	-6

(i) Evaluation of I for n = 1/2.

When $n = 1/2$,

$$I = \int \frac{1 - v^{-3}}{v^3 - 1}\, dv = \int \frac{v^{-3}(v^3 - 1)}{v^3 - 1}\, dv = \int v^{-3}\, dv = -\frac{1}{2v^2}\ . \tag{B.2}$$

immediately evaluated to yield the desired expressions (3.18), (3.22).

(ii) Evaluation of I for n = 3/4.

First, we record here the indefinite integrals (2.128) of Ryshik and Gradstein [29],

$$\int \frac{dv}{v^k z_3^\ell} = -\frac{1}{(k-1)av^{k-1}z_3^{\ell-1}} - \frac{b(3\ell + k - 4)}{a(k-1)} \int \frac{dv}{v^{k-3}z_3^\ell}, \quad k \neq 1, \tag{B.3}$$

where $z_3 = a + bv^3$, $a \neq 0$, b and $\ell > 0$ are constants.

When n = 3/4, from (B.1) we see that

$$I = \int \frac{\sqrt{v}}{(v^3 - 1)}\, dv - \int \frac{dv}{v^4(v^3 - 1)} = I_1 - I_2. \tag{B.4}$$

To evaluate I_2, we use (B.3) with $k = 4$, $a = -1$, $b = 1$, $\ell = 1$, and get

$$I_2 = \int \frac{dv}{v^4(v^3 - 1)} = \frac{1}{3v^3} - \int \frac{dv}{v(v^3 - 1)}. \tag{B.5}$$

The second integral of (B.5) is evaluated as follows:

$$\int \frac{dv}{v(v^3 - 1)} = \int \frac{v^2\, dv}{v^3(v^3 - 1)} = \int -\left(\frac{v^2}{v^3}\right) dv + \int \frac{v^2}{(v^3 - 1)}\, dv$$

$$= -\frac{1}{3}\ln v^3 + \frac{1}{3}\ln (v^3 - 1) = -\frac{1}{3}\ln \frac{v^3}{v^3 - 1}, \tag{B.6}$$

and so, from (B.5), we have

$$I_2 = \frac{1}{3v^3} + \frac{1}{3}\ln \frac{v^3}{v^3 - 1}. \tag{B.7}$$

In order to evaluate I_1 in (B.4) we use a change of variables, i.e., $r = v^{3/2}$, and so

$$I_1 = \int \frac{\sqrt{v}\, dv}{v^3 - 1} = \int \frac{2/3\, dr}{r^2 - 1} = \frac{1}{3} \ln \frac{r - 1}{r + 1} = \frac{1}{3} \ln \frac{v^{3/2} - 1}{v^{3/2} + 1}. \tag{B.8}$$

Thus on combining (B.7) and (B.8) in (B.4) we obtain an expression for I. The definite integrals in (3.13) and (3.16) are then immediately evaluated to yield the desired expressions (3.19) and (3.24).

(iii) Evaluation of I for n = 1.

When n = 1,

$$I = \int \frac{v - v^{-5}}{v^3 - 1}\, dv = \int \frac{v^{-5}\,(v^6 - 1)}{v^3 - 1}\, dv = \int v^{-5}\,(v^3 + 1)\, dv$$

$$= \int (v^{-2} + v^{-5})\, dv = -\frac{1}{v} - \frac{1}{4v^4}. \tag{B.9}$$

On using (B.9), the definite integrals in (3.13), (3.16) are immediately evaluated to yield the desired expressions (3.20), (3.26).

(iv) Evaluation of I for n = 5/4.

When n = 5/4,

$$I = \int \frac{v\sqrt{v}\, dv}{v^3 - 1} - \int \frac{dv}{v^6(v^3 - 1)} = I_3 - I_4. \tag{B.10}$$

To evaluate I_4, we use (B.3) with $k = 6$, $a = -1$, $b = 1$, $\ell = 1$, to get

$$I_4 = \frac{1}{5v^5} + \int \frac{dv}{v^3(v^3 - 1)} . \tag{B.11}$$

The integral in (B.11) is evaluated by using (B.3) with $k = 3$, $a = -1$, $b = 1$, $\ell = 1$, to get

$$\int \frac{dv}{v^3(v^3 - 1)} = \frac{1}{2v^2} + \int \frac{dv}{v^3 - 1} . \tag{B.12}$$

The last integral in (B.12) can be evaluated using standard integral tables. For example, (2.143) of Ryshik and Gradstein [29] gives

$$\int \frac{dv}{v^3 - 1} = -\frac{1}{3} \ln \frac{(1 + v + v^2)^{1/2}}{v - 1} - \frac{1}{\sqrt{3}} \arctan \frac{\sqrt{3}v}{2 + v} . \tag{B.13}$$

Thus, on using (B.13), (B.12), (B.11), we obtain

$$I_4 = \frac{1}{5v^5} + \frac{1}{2v^2} - \frac{1}{3} \ln \frac{(1 + v + v^2)^{1/2}}{v - 1} - \frac{1}{\sqrt{3}} \arctan \frac{\sqrt{3}v}{2 + v} . \tag{B.14}$$

In order to evaluate I_3, we use a change of variable, i.e., $r = v^{1/2}$, to get

$$I_3 = \int \frac{v\sqrt{v}\, dv}{v^3 - 1} = \int \frac{r^4 dr}{r^6 - 1} = \int \frac{r dr}{r^3 + 1} + \int \frac{r dr}{r^3 - 1} . \tag{B.15}$$

By using (2.145.3) and (2.145.7) of [29], we have

$$I_3 = -\frac{1}{6} \ln \frac{(1 + \sqrt{v})^2}{1 - \sqrt{v} + v} + \frac{1}{\sqrt{3}} \arctan \frac{2\sqrt{v} - 1}{\sqrt{3}}$$

$$+ \frac{1}{6} \ln \frac{(\sqrt{v} - 1)^2}{1 + \sqrt{v} + v} + \frac{1}{\sqrt{3}} \arctan \frac{2\sqrt{v} + 1}{\sqrt{3}}. \tag{B.16}$$

Thus on combining (B.14) and (B.16), we obtain an expression for I from (B.10). The definite integrals in (3.13) and (3.16) are then readily evaluated to obtain the desired expressions (3.21) and (3.28).

List of References

References:

1. S.H. Goods and L.M. Brown, The nucleation of cavities by plastic deformation. Acta Met., 27 (1979), 1-15.

2. A.N. Gent and P.B. Lindley, Internal rupture of bonded rubber cylinders in tension. Proc. R. Soc. Lond. A., 249 (1958) 195-205.

3. J.M. Ball, Discontinuous equilibrium solutions and cavitation in nonlinear elasticity. Phil. Trans. R. Soc. Lond. A., 306 (1982) 557-610.

4. C.O. Horgan and R. Abeyaratne, A bifurcation problem for a compressible nonlinearly elastic medium: growth of a micro-void. J. Elasticity, 16 (1986) 189-200.

5. C.A. Stuart, Radially symmetric cavitation for hyperelastic materials. Ann. Inst. Henri Poincaré-Analyse non linéaire, 2 (1985) 33-66.

6. P. Podio-Guidugli, G. Vergara Caffarelli and E.G. Virga, Discontinuous energy minimizers in nonlinear elastostatics: an example of J. Ball revisited. J. Elasticity, 16 (1986) 75-96.

7. J. Sivaloganathan, Uniqueness of regular and singular equilibria for spherically symmetric problems of nonlinear elasticity. Arch. Ration. Mech. Analysis, 96 (1986) 97-136.

8. J. Sivaloganathan, A field theory approach to stability of radial equilibria in nonlinear elasticity. Math. Proc. Camb. Phil. Soc., 99 (1986) 589-604.

9. D.-T. Chung, C.O. Horgan and R. Abeyaratne, A note on a bifurcation problem in finite plasticity related to void nucleation. Int. J. Solids and Structures, 23 (1987) 983-988.

10. C.O. Horgan and T.J. Pence, Void nucleation in tensile dead-loading of a composite incompressible nonlinearly elastic sphere. J. Elasticity, 20 (1988) (in press).

11. C.O. Horgan and T.J. Pence, Cavity formation at the center of a composite incompressible nonlinearly elastic sphere. (to be published).

12. S.S. Antman and P.V. Negrón-Marrero, The remarkable nature of radially symmetric equilibrium states of aeolotropic nonlinearly elastic bodies. J. Elasticity, 18 (1987), 131-164.

13. R.W. Ogden, Large deformation isotropic elasticity I: on the correlation of theory and experiment for incompressible rubberlike solids. Proc. R. Soc; Lond: A., 326 (1972) 565-584.

14. R.W. Ogden, Elastic deformations of rubberlike solids, in Mechanics of Solids - the Rodney-Hill 60th Anniversary Volume, pp. 499-537 (H.G. Hopkins and M.J. Sewell eds.). Pergamon Press (1982).

15. R.W. Ogden, Nonlinear Elastic Deformations. Halsted Press (1984).

16. L. Zee and E. Sternberg, Ordinary and strong ellipticity in the equilibrium theory of incompressible hyperelastic solids. Arch. Ration. Mech. Analysis, 83 (1983) 53-90.

17. J.K. Knowles, Large amplitude oscillations of a tube of incompressible elastic material. Q. Appl. Math., 18 (1960) 71-77.

18. J.K. Knowles, On a class of oscillations in the finite deformation theory of elasticity. J. Appl. Mech., 29 (1962) 283-286.

19. Z.H. Guo and R. Solecki, Free and forced finite amplitude oscillations of an elastic thick-walled hollow sphere made of incompressible material. Arch. Mech. Stos., 15 (1963) 427-433.

20. J.K. Knowles and M.T. Jakub, Finite dynamic deformations of an incompressible elastic medium containing a spherical cavity. Arch. Rational Mech. Anal., 18 (1964) 367-378.

21. A.C. Eringen and E.S. Suhubi, Elastodynamics, Vol. 1, Academic Press (1975).

22. C. Calderer, The dynamical behavior of nonlinear elastic spherical shells. J. Elasticity, 13 (1983) 17-47.

23. R.S. Rivlin and D.W. Saunders, Large elastic deformations of isotropic materials-VII. Experiments on the deformation of rubber. Phil. Trans. R. Soc. Lond. A., 243 (1951) 251-288.

24. A.N. Gent and A.G. Thomas, Forms for the stored (strain) energy function for vulcanized rubber. J. Polym. Sci., 28 (1958) 625-628.

25. J.G. Simmonds, A necessary condition on the strain-energy density for a circular rubber-like plate to have a finite deflection under a concentrated load, J. Applied Mechanics, (in press).

26. J.P. Fulton and J.G. Simmonds, Large deformations under vertical edge loads of annular membranes with various strain energy densities. Int. J. of Nonlinear Mechanics, 21 (1986) 257-267.

27. A. Libai and J.G. Simmonds, The Nonlinear Theory of Elastic Shells; One Spatial Dimension. Academic Press (1988).

28. M.M. Carroll, Pressure maximum behavior in inflation of incompressible elastic hollow spheres and cylinders. Q. Appl. Math., 45 (1987) 141-154.

29. M. Ryshik and I.S. Gradstein, Tables of Series, Products, and Integrals. Veb. Deutscher Verlag Der Wissenschaften Berlin (1963).