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USING THE BARBIERI-REMIDDI LOWEST-ORDER EQUATION FOR CALCULATING THE DECAY RATE AND THE ENERGY SHIFT OF POSITRONIUM

By

Alireza Abbasabadi

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Chapter 1

Introduction

Positronium, which consists of one electron and one positron, is a pure quantum electrodynamic (QED) bound system. It is experimentally accessible and therefore its study is another sensitive test of QED. It also can be used to test any two-body relativistic equation for the case in which the interaction is purely electromagnetic. Since positronium consists of a particle and an antiparticle, which can annihilate and create photons, the lifetime of positronium is very short, 10^{-10} sec. If we want to get a gross structure of positronium, we may compare it with hydrogen atom. In this case, since electron and positron have equal masses, the reduced mass will be $\frac{1}{2} m_e$. Using this reduced mass, we find out that the Bohr radius of positronium atom is twice that of hydrogen atom, or the ionization energy is half of the hydrogen atom and etc.

The spin of positronium is sum of the spin of electron and spin of positron. Therefore it can have spin 0 (singlet state) or spin 1 (triplet state). Positronium with spin 0 is called parapositronium (p-Ps), and with spin 1 is called orthopositronium (O-Ps). We use the notation $^{2S+1}L_J$, where J, L and S are total angular momentum, orbital angular momentum and spin of the system respectively; and beside they are related by the relation $\vec{J} = \vec{L} + \vec{S}$. By using this notation, the states of positronium can be grouped as $^1S_0$, $^3S_1$; $^1P_1$, $^3P_0$, $^3P_1$, $^3P_2$ and so on.

We can show [1] the charge parity (charge conjugate) of positronium has the eigenvalues $(-)^{\ell+s}$, and for photon it has the eigenvalue (-). Therefore, for positronium decay into photons, we must have the following selection rule (since C, the charge conjugation, is a good quantum number)

$$(-)^{\ell+s} = (-)^n \tag{1.1}$$

where, n is the number of photons into which the positronium can decay. Therefore in ground-state ($\ell = 0$), parapositronium ($s = 0$) can decay into an even number of photons, and orthopositronium ($s = 1$) can decay into an odd number of photons. Since from conservation of energy-momentum, the decay of orthopositronium into a single photon is forbidden, the minimum number of photons that orthopositronium, in ground-state ($\ell = 0$), can decay into is three photons. For parapositronium, in ground-state ($\ell = 0$), the minimum number of photons that it can decay into is two photons.

The calculated decay rate, in lowest-order approximation, for p-Ps (parapositronium) is [2]

$$\Gamma_{th}(\text{p-Ps} \rightarrow 2\gamma) = \frac{\alpha^5}{2} \left(\frac{mc^2}{\hbar}\right) = 8.0325 \times 10^9 \text{ sec}^{-1} \quad , \tag{1.2}$$

while the measured value is [3]

$$\Gamma_{exp}(\text{p-Ps}) = (7.994 \pm .011) \times 10^9 \text{ sec}^{-1} \quad . \tag{1.3}$$

The calculated decay rate, in lowest-order approximation, for o-Ps (orthopositronium) is [4]

$$\Gamma_{th}(\text{o-Ps} \rightarrow 3\gamma) = \frac{2}{9\pi}(\pi^2 - 9)\,\alpha^6\,(\frac{mc^2}{\hbar})$$

$$= 7.2112 \times 10^6 \text{ sec}^{-1} \quad , \qquad (1.4)$$

while the measured value is [3]

$$\Gamma_{exp}(\text{o-Ps}) = (7.051 \pm .005) \times 10^6 \text{ sec}^{-1} \quad , \qquad (1.5)$$

(for a review of experimental advances in positronium see the review by Rich [5] and for new measurement of decay rates see [3], and for new development in QED see [6].)

The accuracy of the measurement of p-Ps, relation (1.3), is not high enough to test the radiative corrections to the decay rate (1.2). While for the o-Ps decay rate measurement, relation (1.5), its accuracy is high enough to test the radiative corrections to the decay rate (1.4). In fact the decay rates, including the radiative corrections, are

$$\Gamma_{th}(\text{p-Ps}) = \Gamma_o(\text{p-Ps})[1 - \frac{\alpha}{\pi}(5 - \frac{\pi^2}{4}) + \frac{2}{3}\alpha^2 \ln \alpha^{-1} + 0(\alpha^2)], \quad (1.6a)$$

$$\Gamma_{th}(\text{o-Ps}) = \Gamma_o(\text{0-Ps})[1 - \frac{\alpha}{\pi}(10.266 \pm .011) - \frac{1}{3}\alpha^2 \ln \alpha^{-1}$$

$$+ 0\{(\frac{\alpha}{\pi})^2\}] \quad , \qquad (1.7a)$$

where $\Gamma_o(\text{p-Ps})$, the lowest-order decay rate of p-Ps is given by (1.2), and $\Gamma_o(\text{o-Ps})$, the lowest-order decay rate of o-Ps is given by (1.4); the $o(\alpha)$ correction to p-Ps was done by Harris and Brown [7], and the $o(\alpha)$ correction to o-Ps (the value that we quoted in (1.7a)) and the order $\alpha^2 \ln \alpha^{-1}$ corrections, to both para-and orthopositronium, are given by Caswell and Lepage [8].

In relations (1.6a) and (1.7a), if we assume that the coefficients of α^2 and $(\frac{\alpha}{\pi})^2$ terms are unity, then we find

$$\Gamma_{th}(\text{p-Ps}) = (7.984 \pm .001) \times 10^9 \text{ sec}^{-1} \quad , \tag{1.6b}$$

$$\Gamma_{th}(\text{o-Ps}) = (7.0386 \pm .0002) \times 10^6 \text{ sec}^{-1} \quad . \tag{1.7b}$$

For the decay rate of p-Ps, comparison of (1.3) with (1.6b) shows that the theory and experiment are in agreement, but the accuracy of measurement is not high enough to check the coefficients of α and $\alpha^2 \ln \alpha^{-1}$ terms. In fact if we write equation (1.6a), up to the order α, in the following form

$$\Gamma_{th}(\text{p-Ps}) = \Gamma_o(\text{p-Ps})(1-k\alpha) \quad , \tag{1.8}$$

experimental result (1.6b) gives the following possible range of values for k

$$k \simeq .66 \pm .2 \quad . \tag{1.9}$$

We note that the calculated value of k

$$k = \frac{1}{\pi}\left(5 - \frac{\pi^2}{4}\right) \simeq .806 \tag{1.10}$$

is one of the possible values in (1.9). But since the range of the values of k in (1.9) is not narrow enough, we do not consider the agreement between the measured and calculated values of the decay rate of p-Ps as a sufficient criterion for the test of the radiative corrections.

For the decay rate of o-Ps, comparison of (1.5) with (1.7b) shows that the agreement between theory and experiment is not satisfactory, and in fact there is a discrepancy.

To resolve the unsatisfactory agreement between the Γ_{th}(o-Ps) and Γ_{exp}(o-Ps), Gidley et al. [3] initiated more measurements of the o-Ps decay rate.

Up to now all measured values of Γ(o-Ps) have been above the calculated one, therefore one may think that, maybe, there are other channels available for the o-Ps to decay to. It is suggested [9] that the following process could be responsible (at least partially) for the discrepancy between theory and experiment

$$\text{o-Ps} \rightarrow a^o + \gamma, \tag{1.11}$$

where, a^o is a neutral particle of mass $m_a < 2m_e$.

Amaldi et al. [10] measured the decay rate of the above process. For the mass m_a varying from 100 to 900 kev, they found the following range of upper limits for the ratio of decay rates of the above mode to the 3γ mode

$$R = \frac{\Gamma\ (\text{o-Ps} \rightarrow a^{o}\gamma)}{\Gamma\ (\text{o-Ps} \rightarrow 3\gamma)} \sim 5 \times 10^{-6} - 1 \times 10^{-6}. \qquad (1.12)$$

Therefore, their result shows that the process (1.11) can not resolve the discrepancy between Γ_{exp} (o-Ps) and Γ_{th} (o-Ps).

There is another quantity of interest, the ground-state hyperfine splitting of positronium, $\Delta E = E\ (s = 1) - E\ (s = 0)$, the energy difference between the $s = 0$ and $s = 1$ levels with $n = 1$. The measured value of this quantity is [11]

$$\Delta E_{exp}\ (\text{hfs}) = 203.3885 \pm .0010 \text{ GHZ} \quad , \qquad (1.13)$$

while the theoretical value is

$$\Delta E_{th}(\text{hfs}) = \frac{mc^2\alpha^4}{4} \left[\frac{7}{3} - \frac{\alpha}{\pi}\left(\frac{32}{9} + 2 \ln 2\right) + \frac{5}{6}\alpha^2 \ln \alpha^{-1} + 0\ (\alpha^2)\right]$$

$$\simeq 203.400 \text{ GHZ}. \qquad (1.14)$$

The first two terms were calculated by Karplus and Klein [12], and calculation of the $\alpha^2 \ln \alpha^{-1}$ term started by Fulton, Owen and Repko [13], and completed by Caswell and Lepage [14], and Bodwin and Yennie [15].

Contribution of α^2 term to the hyperfine splitting, if we assume its coefficient is unity, is of order .005 GHZ. Therefore in order to have a meaningful comparison of theoretical result, relation (1.14), with experimental result, relation (1.13), we need to calculate the o (α^2) correction to the hyperfine splitting.

As expressed by Buchmüller and Remiddi [16], most methods which proposed and used in the past - for calculating the radiative shifts of energy levels of positronium - work only up to the order $\alpha^2 \ln \alpha^{-1}$ correction and in practice they can not find α^2 correction to the ΔE, relation (1.14).

In Chapter 2, we discuss the method that Barbieri and Remiddi [17] introduced for solving the positronium problem, which in principle can be used for finding decay rate or energy splitting of positronium up to the any desired order in α.

Throughout our work, in subsequent Chapters, we use the Coulomb gauge. The reason, beside the others, is that in covariant gauge, for positronium, there are some spurious terms, such as o (α^3) and o $(\alpha^3 \ln \alpha^{-1})$ corrections to the energy levels, which appear in some Feynman diagrams (for a discussion of these problems and the cancellation of these contributions see [18]), while in Coulomb gauge we do not have such spurious terms.

In Chapter 3, after introducing a perturbative expansion for energy levels [16], we calculate the energy shift, δE, of the orthopositronium ($\delta E = E - 2\kappa$, where E is the ground-state energy of o-Ps and $\kappa = (1 - \frac{\alpha^2}{4})^{1/2}$). We consider those contributions which come from the one-photon-annihilation channel and contribute up to the first-order of perturbation theory (up to the o (α) correction).

The energy shift δE that we find is

$$\delta E = \frac{mc^2\alpha^4}{4} \left(1 - \frac{4\alpha}{\pi} - \frac{8\alpha}{9\pi}\right), \tag{1.15}$$

which agrees with the result of Karplus and Klein [12]. The interesting point is that we find the finite value for δE, without performing the wave-function and vertex renormalization subtractions (for the regularization and renormalization of QED in Coulomb gauge see [19] and [20]).

In Chapter 4, after writing down the perturbed wave-function for parapositronium, in terms of the zeroth-order wave-function, we calculate the decay rate of p-Ps. We consider those decays which contribute up to the first-order of perturbation theory (up to the o(α) correction). They are the decays of p-Ps into two photons.

The decay rate that we obtain is

$$\Gamma\,(\text{p-Ps} \rightarrow 2\,\gamma) = \frac{mc^2\alpha^5}{2\hbar}\left[1 - \frac{\alpha}{\pi}\left(5 - \frac{\pi^2}{4}\right)\right] \;, \tag{1.16}$$

which agrees with the result of Harris and Brown [7]. Here also we find the result (1.16), without performing the wave-function and vertex renormatization subtractions.

Throughout our work, we use the notation and conventions of BJorken and Drell [21]. We also use the natural units $\hbar = c = 1$, and take the electron mass $m_e = 1$. For regularization, we employ the method of dimensional regularization [22].

Chapter 2

Solving the Bethe-Salpeter Equation for Positronium [17], [23]

The formulation of the bound-state problem in QED, which is given by the relativistic Bethe-Salpeter (BS) equation [24], although from theoretical point of view is complete, it lacks a tractable and systematic computation procedure. In practice, in order to solve BS equation, which is an integral equation, one specifies a lowest-order equation, then by using this equation as a starting point, one does a perturbative calculation on it.

In general this lowest-order equation has a kinetic and an interaction term. The interaction term is responsible for the bound-states and supposedly is the largest part of the full BS kernel. In the limit of low-momenta these two terms should reduce to the kinetic energy and the Coulomb potential of the Schrödinger equation, and also it is advisable that in the limit of high-momenta the kinetic term describes the free propagation of the electron and positron.

In the following, we discuss a lowest-order equation which has the above properties and can be solved exactly.

The BS equation for the Green function G is [24]

$$G\,(P,\, p,\, q) = G_o\,(P,\, p,\, q) + \int \frac{d^4k}{(2\pi)^4}\, G_o\,(P,\, p,\, k)$$

$$\times \int \frac{d^4k'}{(2\pi)^4} K(P, k, k') \, G(P, k', q), \qquad (2.1)$$

and the lowest-order equation which is proposed by Barbieri and Remiddi [17] is

$$G_c(P, p, q) = G_o(P, p) \left[(2\pi)^4 \delta^4(p-q) \right.$$

$$\left. + \int \frac{d^4p'}{(2\pi)^4} K_c(P, p, p') \, G_c(P, p', q) \right]. \qquad (2.2)$$

First we specify G_o and K_c. In the C.M. frame where $P = (2w, \vec{o})$ ($2w$ is the total energy of positronium and $2w-2$ is the binding energy), G_o and K_c are chosen as

$$G_o(w, p) = \left[\frac{i}{\not{p} + w\gamma_o - 1 + i\varepsilon}\right]^{(1)} \left[\frac{i}{\not{p} - w\gamma_o - 1 + i\varepsilon}\right]^{(2)}_{T}, \quad (2.3)$$

and

$$K_c(w, \vec{p}, \vec{q}) = -i\lambda(\vec{p}, \vec{q}) \, R(w, \vec{p}) \, V_c(\vec{p} - \vec{q}) \, R(w, \vec{q}), \quad (2.4)$$

where

$$\lambda(\vec{p}, \vec{q}) = \frac{1}{16 \, E_p \, E_q \, N_p \, N_q} \left[(N_p + \vec{p} \cdot \vec{\gamma})(1 + \gamma_o)(N_q + \vec{q} \cdot \vec{\gamma}) \right]^{(1)}$$

$$\times \left[(N_q + \vec{q} \cdot \vec{\gamma})(1 - \gamma_o)(N_p + \vec{p} \cdot \vec{\gamma}) \right]^{(2)}_{T}, \qquad (2.5)$$

and

$$R\,(w,\,\vec{p}) = \frac{\sqrt{2}}{\sqrt{E_p + w}}\ ,\ E_p = \sqrt{\vec{p}^{\,2} + 1}\ ,\ N_p = E_p + 1 \quad , \qquad (2.6)$$

and V_c is the scalar Coulomb potential

$$V_c\,(\vec{k}) = \frac{-4\pi\alpha}{\vec{k}^{\,2}} \quad (\alpha = \frac{e^2}{4\pi}) \quad . \qquad (2.7)$$

In the above relations, (1) stands for electron line and (2) stands for positron line.

By specifying K_c and G_o, as we did, one finds the exact solution to the equation (2.2) for the Green function G_c. In C.M. frame this solution has the following form

$$G_c\,(w,\,p,\,q) = G_o\,(w,\,p)\,\Big[(2\pi)^4\,\delta^4\,(p - q)$$

$$+\, i\,R\,(w,\,\vec{p})\,H_c\,(w,\,\vec{p},\,\vec{q})\,R\,(w,\,\vec{q})\,\lambda\,(\vec{p},\,\vec{q})\,G_o\,(w,\,q)\Big], \quad (2.8)$$

where, the scalar function H_c is

$$H_c\,(w,\,\vec{p},\,\vec{q}) = \frac{4\pi\ \alpha}{|\vec{p} - \vec{q}|^2} + 4\pi\ \alpha\ \nu \int_0^1 d\,\rho\ \rho^{-\nu}$$

$$\times \Big\{|\vec{p} - \vec{q}|^2\,\rho - \big[1/(4w^2 - 4)\big]\,(E_p^2 - w^2)\,(E_q^2 - w^2)\,(1 - \rho)^2\Big\}^{-1}, \quad (2.9)$$

where

$$\nu = \frac{\gamma}{\sqrt{1 - w^2}} \, , \; \gamma = \frac{\alpha}{2} \, . \tag{2.10}$$

We note that H_c has pole at $w = \kappa$ (from now on we assume $w < \kappa$)

$$H_c (w, \vec{p}, \vec{q}) \Bigg|_{w \to \kappa} \to \frac{-32 \, \pi \, \gamma^2}{\kappa \, (\vec{p}^{\,2} + \gamma^2) \, (\vec{q}^{\,2} + \gamma^2)} \, \frac{1}{w - \kappa} \, , \tag{2.11}$$

where

$$\kappa = \sqrt{1 - \gamma^2} \, , \tag{2.12}$$

and it corresponds to the ground-state energy.

In equation (2.9), if we replace the real integral with contour integral [25]

$$\int_0^1 d \rho \; (. \; . \; .) \to \frac{i}{2 \sin \nu} \, e^{i \pi \nu} \int_c d \rho \; (. \; . \; .) \, , \tag{2.13}$$

where the path c begins at $\rho = 1 + oi$ and terminates at $\rho = 1 - oi$, after encircling the origin within the unite circle, then we find that H_c has poles at

$$w = \sqrt{1 - \frac{\gamma^2}{n^2}} \qquad n = 1, 2, 3, \ldots \quad , \tag{2.14}$$

which correspond to all excited states of energy (including the ground-state). But since we are interested in ground-state, the H_c which is given by equation (2.9) is the one that we will use in our calculations.

Now, since H_c has pole at $w = \kappa$, therefore by equation (2.8) G_c has pole at $w = \kappa$ too. At the pole G_c is

$$G_c\,(w, p, q)\Bigg|_{w \to \kappa} \to \frac{i}{2\,\kappa} \; \frac{\sum \psi_o\,(p)\,\bar{\psi}_o\,(p)}{w - \kappa} \quad , \tag{2.15}$$

where summation is over all degenerate states (for ground-state (ℓ = o) degeneracy is only due to spin). Using (2.15), one can find ψ_o's, the zeroth-order Barbieri-Remiddi wave-functions. For orthopositronium ground-state (s = 1, ℓ = 0)

$$\psi_o\,(p) = \frac{i\,\pi\,\alpha}{(\vec{p}^2 + \gamma^2)^2}\,\phi_o\,\frac{(\kappa - E_p)\sqrt{E_p + \kappa}}{E_p\,N_p}$$

$$\times \frac{(N_p - \vec{p}\cdot\vec{\gamma})\,(1 + \gamma_o)\,\hat{\xi}^{(m)}\cdot\vec{\gamma}\,(N_p - \vec{p}\cdot\vec{\gamma})}{(p_o + \kappa - E_p + i\varepsilon)\,(p_o - \kappa + E_p - i\varepsilon)} \quad , \tag{2.16}$$

and its "conjugate" transpose

$$\bar{\psi}_o^T(p) = \frac{-i\,\pi\,\alpha}{(\vec{p}^2 + \gamma^2)^2}\,\phi_o\,\frac{(\kappa - E_p)\sqrt{E_p + \kappa}}{E_p\,N_p}$$

$$\times \frac{(N_p - \vec{p}\cdot\vec{\gamma})\,(1-\gamma_o)\,\hat{\xi}^{(m)^*}\cdot\vec{\gamma}\,(N_p - \vec{p}\cdot\vec{\gamma})}{(p_o + \kappa - E_p + i\varepsilon)\,(p_o - \kappa + E_p - i\varepsilon)}\quad, \tag{2.17}$$

where ϕ_o and spin 1 polarization vectors $\xi^{(m)}$ are

$$\phi_o = \left(\frac{\alpha^3}{8\pi}\right)^{1/2},\ \xi^{(o)} = (0,\ 0,\ 1)\ ,\ \xi^{(\pm)} = \frac{1}{\sqrt{2}}\,(1,\ \pm\ i,\ o)\ . \tag{2.18}$$

For parapositronium ground-state ($s = o$, $\ell = o$)

$$\psi_o(p) = \frac{2\,i\,\pi\,\alpha}{(\vec{p}^2 + \gamma^2)^2}\,\phi_o\,\frac{(\kappa - E_p)\sqrt{E_p + \kappa}}{E_p}$$

$$\times \frac{(E_p + \gamma_o - \vec{p}\cdot\vec{\gamma}\,\gamma_o)\,\gamma_5}{(p_o + \kappa - E_p + i\varepsilon)\,(p_o - \kappa + E_p - i\varepsilon)}\quad, \tag{2.19}$$

and its "conjugate" transpose

$$\bar{\psi}_o^T = \frac{2\,i\,\pi\,\alpha}{(\vec{p}^2 + \gamma^2)^2}\,\phi_o\,\frac{(\kappa - E_p)\sqrt{E_p + \kappa}}{E_p}$$

$$\times \frac{(E_p - \gamma_o + \vec{p}\cdot\vec{\gamma}\,\gamma_o)\,\gamma_5}{(p_o + \kappa - E_p + i\varepsilon)\,(p_o - \kappa + E_p - i\varepsilon)}\quad. \tag{2.20}$$

In the above relations we defined $\bar{\psi}_o$ by the following relation

$$\bar{\psi}_o = \gamma_o \psi_o^* \gamma_o \quad . \tag{2.21}$$

One may use (2.15) for finding an integral equation for ψ_o. This equation can be found by taking the residuum of (2.2) at $w = \kappa$

$$\psi_o (p) = G_o (\kappa, p) \int \frac{d^4 p'}{(2\pi)^4} K_c (\kappa, p, p') \psi_o (p') \quad . \tag{2.22}$$

We can show that the following normalization condition for these wave-functions is satisfied

$$\int \frac{d^4 p}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \bar{\psi}_o (p) \frac{i}{\kappa} \frac{\partial}{\partial P_o} [G_o^{-1} (P, p, q)$$

$$- K_c (P, p, q)] \psi_o (q) = 1 \quad , \tag{2.23}$$

where

$$P = (2\kappa, \vec{o}), \ \kappa = (1 - \gamma^2)^{1/2} \ , \ \gamma = \frac{\alpha}{2} \quad . \tag{2.24}$$

In order to use the perturbation theory around $w = \kappa$, one needs the following quantity (in Chapters 3 and 4 we will use this quantity)

$$\hat{G}_c (\kappa) \equiv [G_c (w) - \frac{i}{2\kappa} \frac{\Sigma \psi_o \bar{\psi}_o}{w - \kappa}]_{w \to \kappa} \quad , \tag{2.25}$$

which is finite (pole contribution is subtracted), and can be found by using (2.8) and (2.9)

$$\hat{G}_c(\kappa, p, q) = (2\pi)^4 \delta^4 (p - q) G_o(\kappa, p) + iR(\kappa, \vec{p}) R(\kappa, \vec{q})$$

$$\times G_o(\kappa, p) \lambda(\vec{p}, \vec{q}) G_o(\kappa, q) \hat{H}_c(\kappa, p, q) \quad , \qquad (2.26)$$

where $\hat{H}_c$ is finite (pole contribution is subtracted), and is given in [23]. Up to the first order of perturbation theory, $\hat{H}_c$ can be written as

$$\hat{H}_c(\kappa, p, q) = 4\pi\alpha \left\{\frac{1}{A} + \frac{1}{B}\left[\frac{5}{2} + \frac{-2\gamma^2}{\vec{p}^2 + \gamma^2} + \frac{-2\gamma^2}{\vec{q}^2 + \gamma^2}\right.\right.$$

$$+ \frac{\gamma^2}{p_o + \kappa - E_p + i\varepsilon} + \frac{-\gamma^2}{p_o - \kappa + E_p - i\varepsilon} + \frac{\gamma^2}{q_o + \kappa - E_q + i\varepsilon}$$

$$\left.\left. + \frac{-\gamma^2}{q_o - \kappa + E_q - i\varepsilon} + \int_o^1 d\rho \, \frac{2B - A - B\rho}{A\rho + B(1 - \rho)^2}\right]\right\} \quad , \qquad (2.27)$$

where

$$A = |\vec{p} - \vec{q}|^2 \ , \ B = \frac{1}{4\gamma^2} (\vec{p}^2 + \gamma^2)(\vec{q}^2 + \gamma^2) \quad . \qquad (2.28)$$

In the course of calculations of positronium decay rate or energy splitting, in some Feynman diagrams we divide the contribution of each diagram into two parts. One which comes from the first term in $\hat{G}_c$, equation (2.26), and the other which comes from the second term in $\hat{G}_c$. Let's write (2.26) in the following form

$$\hat{G}_c(\kappa, p, q) = (2\pi)^4 \delta^4(p - q) G_o(\kappa, p) + \hat{R}(\kappa, p, q) \quad , \qquad (2.29)$$

where $\hat{R}$ is

$$\hat{R}(\kappa, p, q) = i\, R(\kappa, \vec{p})\, R(\kappa, \vec{q})\, G_o(\kappa, p)\, \lambda(\vec{p}, \vec{q})$$

$$\times\, G_o(\kappa, q)\, \hat{H}_c(\kappa, p, q) \quad . \qquad (2.30)$$

Beside $\hat{R}$, there is another quantity that we always refer to, it is $K_c(\kappa, \vec{p}, \vec{q})$, which is $K_c(w, \vec{p}, \vec{q})$ at $w = \kappa$, or from (2.4)

$$K_c(\kappa, \vec{p}, \vec{q}) = -i\, \lambda(\vec{p}, \vec{q})\, R(\kappa, \vec{p})\, V_c(\vec{p} - \vec{q})\, R(\kappa, \vec{q}) \quad . \qquad (2.31)$$

There are some remarks that we should mention. In 1952, Salpeter [26] derived an equation which is the approximate version of the BS equation and has the properties which we mentioned at the beginning of the present Chapter; but the exact analytic solution of this equation is not known. This equation and its improved version which was given by Cung et al. [27], before 1978 had been the starting point of nearly all perturbative calculations for the positronium and muonium. In 1978, the year that Barbieri and Remiddi [17] proposed their lowest-order equation, Caswell and Lepage [14] also proposed an equation which is essentially equivalent to the Barbieri-Remiddi equation.

It is also necessary to note that the Dirac equation, for a system which consists of an electron in an external Coulomb potential, can be solved exactly and it gives the correct energy levels up to the order α^4 (fine structure) [28]. One may think that the Dirac equation is a good lowest-order equation to start with; but this is not correct. The Dirac equation for an electron in an external Coulomb potential can be obtained from the BS equation, by considering the interaction kernel as a sum of the all irreducible crossed ladder graphs in the limit of infinite mass of positron [17]. In fact it is an equation for one fermion, not for two fermions.

In any case, we do not need to demand that the lowest-order equation, by itself, should give the correct order α^4 for energy levels of positronium. As long as a lowest-order equation, by using a perturbation expansion, gives the correct energy levels in terms of the successive powers of α, it will be a good lowest-order equation to start with (explicit calculation shows that beside powers of α, we have also $\ln \alpha$).

Chapter 3

Contribution of One-Photon-Annihilation Channel to the Energy-Shift of the Ground-State of Orthopositronium in the First-Order Perturbation Theory

The energy-shift of the orthopositronium ground-state, in terms of the perturbation kernel δK which is given by Barbieri and Remiddi [17], can be written [16], up to the first order in perturbation theory as

$$i\ \delta E = (\delta K) + (\delta K\ \hat{G}_c\ \delta K) + (\delta K)\ (\delta K') + O(\delta K^3), \qquad (3.1)$$

where $\delta K = K - K_c$, and is given graphically in Figure 1.

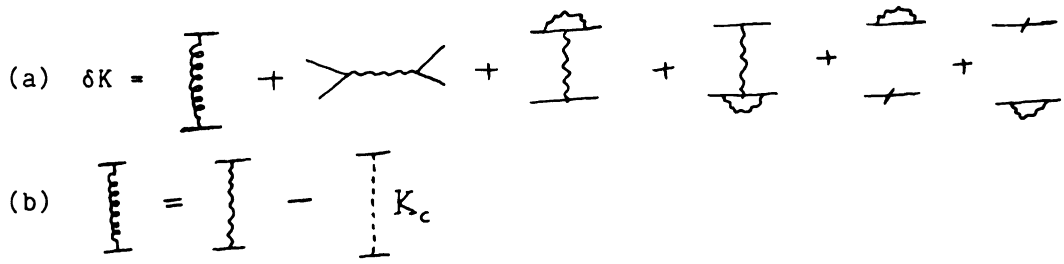


Figure 1. Kernel δK, where the Coulomb-like kernel K_c is given by (2.31). Lines with a dash indicate inverse propagators.

$\hat{G}_c$ and K_c are given by equations (2.29) and (2.31), respectively.

In (3.1), δE is defined as $\delta E = E - 2\kappa$ where E is the ground-state energy, and prime stands for $\frac{i}{2\kappa}\frac{\partial}{\partial\kappa}$, and $\langle\ldots\rangle$ means the expectation value with respect to ψ_o. For example,

$$\langle\delta K'\rangle \quad \int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\,\bar{\psi}_o(q)\left[\frac{i}{2\kappa}\frac{\partial}{\partial\kappa}\,\delta K(\kappa,\,q,\,p)\right]\psi_o(p), \qquad (3.2)$$

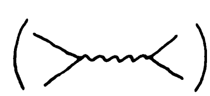
where ψ_o is the zeroth-order Barbieri-Remiddi wave-function [17], which for the orthopositronium ground-state is given by the equation (2.16).

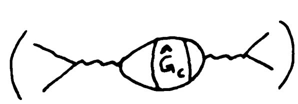
Now using (3.1) and the graphical representation of δK, Figure 1, one gets the energy-shift of the ground-state of orthopositronium, up to the first-order of perturbation theory.

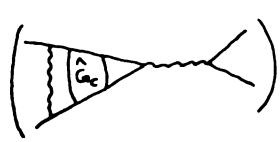
The contribution of one-photon-annihilation channel, up to the $o(\alpha)$ correction, can be written as

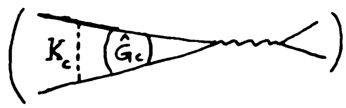
$$\delta E = \sum_{i=1}^{7}\delta E_i \quad , \qquad (3.3)$$

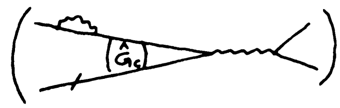
where, δE's are represented graphically in Figure 2.

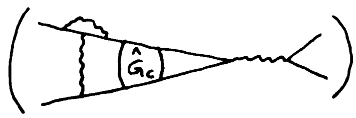
(a) $i\delta E_1 =$ 

(b) $i\delta E_2 =$ 

(c) $i\delta E_3 = 2$ 

(d) $i\delta E_4 = -2$ 

(e) $i\delta E_5 = 4$ 

(f) $i\delta E_6 = 4$ 

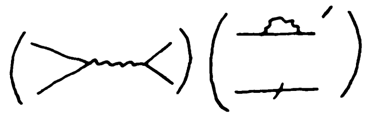
(g) $i\delta E_7 = 2$ 

Figure 2. Energy shifts δE's, where $\hat{G}_c$ and K_c are given by (2.29) and (2.31), respectively.

In the following sections we calculate the contributions δE's.

A. Contribution δE_1

Using Figure 2(a), contribution δE_1 can be written as

$$i\delta E_1 = \int \frac{d^4p}{(2\pi)^4} \langle -ie\gamma_\mu \, \psi_o \, (p) \rangle \frac{-ig^{\mu\nu}}{4\kappa^2}$$

$$\times \int \frac{d^4q}{(2\pi)^4} \langle \bar{\psi}_o^T \, (q) \, (-ie\gamma_\nu) \rangle \quad , \tag{3A.1}$$

where $\langle \ \rangle$ stands for trace. The double integral factorized, and it is the case for all other diagrams.

Using (2.16) for wave-function ψ_o and performing p_o integration (by virtue of Cauchy's theorem), and then performing the angular integrations, one gets

$$\int \frac{d^4p}{(2\pi)^4} <-ie\gamma_\mu \psi_o (p) >$$

$$= ie\phi_o \xi^\mu \frac{2\alpha}{3\pi} \int_o^\infty \frac{p^2 \, d\,p}{(\vec{p}^2 + \gamma^2)^2 \, E_p} \sqrt{E_p + \kappa}\,(2E_p + 1) \quad . \qquad (3A.2)$$

The integral in the right-hand side of (3A.2), up to the o(α) correction, is performed in Appendix B; using that result, equation (B.11), we find

$$\int \frac{d^4p}{(2\pi)^4} < - ie\gamma_\mu \psi_o (p) > = ie\phi_o \xi^\mu \sqrt{2} (1 + \frac{\alpha}{6}) \quad , \qquad (3A.3)$$

and since

$$\int \frac{d^4q}{(2\pi)^4} < \bar{\psi}_o^T (q) (-ie\gamma_\nu) > = [\int \frac{d^4p}{(2\pi)^4} < -ie\gamma_\nu \psi_o (p) >]^* \quad , (3A.4)$$

equation (3A.1) gives (up to the o(α) correction)

$$\delta E_1 = \frac{\alpha^4}{4} (1 + \frac{\alpha}{3}) \quad . \qquad (3A.5)$$

B. Contribution δE_2

For δE_2, Figure 2(b), let's first consider the quantity $A_{\mu\nu}$ which is represented graphically in Figure 3(a).

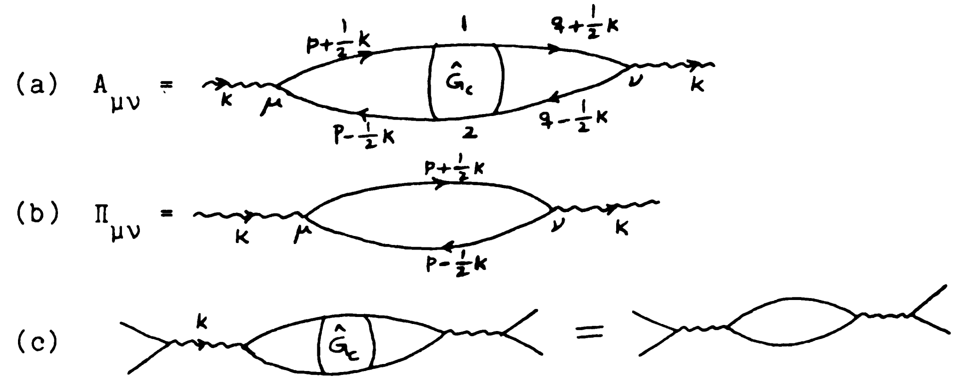


Figure 3. (a) Definition of quantity $A_{\mu\nu}$. (b) Graphical representation of $\Pi_{\mu\nu}$. (c) Equality which is correct up to the $o(\alpha)$ correction. $\hat{G}_c$ is given in (2.29) and $k = (2\kappa, \vec{o})$.

Using the Feynman rules and equation (2.29) for $\hat{G}_c$, the quantity $A_{\mu\nu}$ (μ, ν = 1, 2, 3) can be written as

$$A_{\mu\nu} = -\int \frac{d^n p}{(2\pi)^4} \frac{d^n q}{(2\pi)^4} \Big[\langle - ie\gamma_\mu \, iS^1(p + \tfrac{1}{2}k)(-ie\gamma_\nu) iS^2(p - \tfrac{1}{2}k)\rangle$$

$$\times (2\pi)^4 \, \delta^4(p - q) + iR(\kappa, \vec{p})R(\kappa, \vec{q})\hat{H}_c(\kappa, p, q)\langle - ie\gamma_\mu iS^1(p + \tfrac{1}{2}k)$$

$$\times \lambda^1(\vec{p}, \vec{q}) iS^1(q + \tfrac{1}{2}k)(-ie\gamma_\nu) iS^2(q - \tfrac{1}{2}k)\lambda^2(\vec{p}, \vec{q})$$

$$\times iS^2(p - \tfrac{1}{2}k)\rangle\Big] \quad , \qquad (3B.1)$$

where 1 stands for electron and 2 for positron. Let's write (3B.1) in the following form

$$A_{\mu\nu} = \Pi_{\mu\nu} + \Pi'_{\mu\nu} \quad , \qquad (3B.2)$$

where

$$\Pi_{\mu\nu} = -\int \frac{d^4p}{(2\pi)^4} \langle -ie\gamma_\mu iS(p + \tfrac{1}{2}k)(-ie\gamma_\nu)iS(p - \tfrac{1}{2}k)\rangle \quad , \qquad (3B.3)$$

and

$$\Pi'_{\mu\nu} = \frac{i\alpha\pi}{4} \int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} R(\kappa, \vec{p})R(\kappa, \vec{q})\hat{H}_c(\kappa, p, q)(E_pE_qN_pN_q)^{-1}$$
$$\times\{[(p + \tfrac{1}{2}k)^2 - 1][(p - \tfrac{1}{2}k)^2 - 1][(q + \tfrac{1}{2}k)^2 - 1][(q - \tfrac{1}{2}k)^2 - 1]\}^{-1}$$
$$\times\langle\gamma_\mu(\not{p} + \tfrac{1}{2}\not{k} + 1)(N_p + \vec{p}\cdot\vec{\gamma})(1 + \gamma_o)(N_q + \vec{q}\cdot\vec{\gamma})(\not{q} + \tfrac{1}{2}\not{k} + 1)\gamma_\nu$$
$$\times(\not{q} - \tfrac{1}{2}\not{k} + 1)(N_q + \vec{q}\cdot\vec{\gamma})(1 - \gamma_o)(N_p + \vec{p}\cdot\vec{\gamma})$$
$$\times(\not{p} - \tfrac{1}{2}\not{k} + 1)\rangle \quad , \qquad (3B.4)$$

which means that $\Pi_{\mu\nu}$ is the contribution of the first term of $\hat{G}_c$ and $\Pi'_{\mu\nu}$ is the contribution of the second term of $\hat{G}_c$ (see (2.29) for $\hat{G}_c$).

The quantity $\Pi_{\mu\nu}$ is represented graphically in Figure 3 (b), and obviously it is of order α. For $\Pi'_{\mu\nu}$ we want to show that its contribution is of order higher than α, therefore we can ignore it.

In equation (3B.4), we note that the most contribution comes from $\vec{p}$, $\vec{q} \sim \gamma$ and p_o, $q_o \sim \gamma^2$. In fact after performing p_o and q_o

integrations (poles at $p_o = \kappa - E_p$ and $q_o = \kappa - E_q$ give the most contribution) and keeping the lowest powers of $\vec{p}$ and $\vec{q}$, we get

$$\Pi'_{\mu\nu} \sim \alpha \int d^3p d^3q \frac{\bar{H}_c (\kappa, \vec{p}, \vec{q})}{(\vec{p}^2 + \gamma^2)(\vec{q}^2 + \gamma^2)} \quad , \tag{3B.5}$$

where $\bar{H}_c$ is

$$\bar{H}_c (\kappa, \vec{p}, \vec{q}) = 4\pi\alpha\{\frac{1}{A} + \frac{1}{B}\left[\frac{5}{2} - \frac{4\gamma^2}{\vec{p}^2 + \gamma^2} - \frac{4\gamma^2}{\vec{q}^2 + \gamma^2}\right.$$

$$\left. + \int_0^1 d\rho \frac{2B - A - B\rho}{A\rho + B(1 - \rho)^2}\right]\} \quad , \tag{3B.6}$$

and quantities A and B are given by (2.28).

By scaling $\vec{p} \to \gamma\vec{p}$ and $\vec{q} \to \gamma\vec{p}$, we note that $\bar{H}_c$ is of order $\frac{1}{\alpha}$ and therefore $\Pi'_{\mu\nu}$ is of order α^2; so we can ignore $\Pi'_{\mu\nu}$ and up to the $o(\alpha)$ correction (3B.2) gets the following form

$$A_{\mu\nu} = \pi_{\mu\nu} \quad , \tag{3B.7}$$

which is represented graphically in Figure 3(c).

Now, by virtue of Figures 2(b) and 3(c), contribution δE_2 is

$$i\delta E_2 = \int \frac{d^4p}{(2\pi)^4} < - ie\gamma_\mu\psi_o(p) >\left[\frac{-i}{k^2} \Pi^{\mu\nu}(k) \frac{-i}{k^2}\right]$$

integrations (poles at $p_o = \kappa - E_p$ and $q_o = \kappa - E_q$ give the most contribution) and keeping the lowest powers of $\vec{p}$ and $\vec{q}$, we get

$$\Pi'_{\mu\nu} \sim \alpha \int d^3p d^3q \, \frac{\bar{H}_c\,(\kappa,\, \vec{p},\, \vec{q})}{(\vec{p}^{\,2} + \gamma^2)(\vec{q}^{\,2} + \gamma^2)} \quad , \tag{3B.5}$$

where $\bar{H}_c$ is

$$\bar{H}_c\,(\kappa,\, \vec{p},\, \vec{q}) = 4\pi\alpha\Big\{\frac{1}{A} + \frac{1}{B}\,\Big[\frac{5}{2} - \frac{4\gamma^2}{\vec{p}^{\,2} + \gamma^2} - \frac{4\gamma^2}{\vec{q}^{\,2} + \gamma^2}$$

$$+ \int_0^1 d\rho \, \frac{2B - A - B\rho}{A\rho + B\,(1 - \rho)^2}\Big]\Big\} \quad , \tag{3B.6}$$

and quantities A and B are given by (2.28).

By scaling $\vec{p} \to \gamma\vec{p}$ and $\vec{q} \to \gamma\vec{p}$, we note that $\bar{H}_c$ is of order $\frac{1}{\alpha}$ and therefore $\Pi'_{\mu\nu}$ is of order α^2; so we can ignore $\Pi'_{\mu\nu}$ and up to the $o(\alpha)$ correction (3B.2) gets the following form

$$A_{\mu\nu} = \pi_{\mu\nu} \quad , \tag{3B.7}$$

which is represented graphically in Figure 3(c).

Now, by virtue of Figures 2(b) and 3(c), contribution δE_2 is

$$i\delta E_2 = \int \frac{d^4p}{(2\pi)^4} < -\, ie\gamma_\mu\psi_o(p) >\Big[\frac{-i}{k^2}\,\Pi^{\mu\nu}(k)\,\frac{-i}{k^2}\Big]$$

$$\times \int \frac{d^4q}{(2\pi)^4} < \bar{\psi}_o^T\ (q)(-ie\gamma_\nu) > \quad , \tag{3B.8}$$

where $\pi_{\mu\nu}$, the vacuum polarization, can be found from (3B.3), which up to the $o(\alpha)$ is

$$\Pi_{\mu\nu}\ (k) = \frac{i\alpha}{3\pi}\left[(1 + \frac{a}{2})\ \Gamma(-a) + \frac{8}{3}\right]\ (k_\mu\ k_\nu - k^2 g_{\mu\nu}) \quad ,$$

$$k = (2\kappa,\ \vec{o}),\quad a = \frac{n}{2} - 2 \quad (n \to 4),\quad \Gamma\ (-a) \to \frac{1}{-a},\quad a > o\ , \tag{3B.9}$$

(for finding $\pi_{\mu\nu}$, by using the cut-off method, see [29]).

Equation (3A.3) implies

$$\int \frac{d^4p}{(2\pi)^4} < -ie\gamma_\mu \psi_o(p) > \propto \xi_\mu \quad (\xi_o = o) \quad , \tag{3B.10}$$

therefore (3B.8) gets the following form

$$i\delta E_2 = \frac{-\alpha}{3\pi}\left[(1 + \frac{a}{2})\Gamma(-a) + \frac{8}{3}\right] \int \frac{d^4p}{(2\pi)^4} < -ie\gamma_\mu \psi_o(p) >$$

$$\times \frac{-ig^{\mu\nu}}{k^2} \int \frac{d^4q}{(2\pi)^4} < \bar{\psi}_o^T\ (q)(-ie\gamma_\nu) > \quad , \tag{3B.11}$$

which by comparing it with (3A.1), we find

$$i\delta E_2 = \frac{-\alpha}{3\pi}\left[(1 + \frac{a}{2})\Gamma(-a) + \frac{8}{3}\right](i\delta E_1) \quad . \tag{3B.12}$$

Charge renormalization takes care of the first term, therefore by virtue of (3A.5)

$$\delta E_2 = \frac{\alpha^4}{4} \left(\frac{-8\alpha}{9\pi}\right) \quad , \tag{3B.13}$$

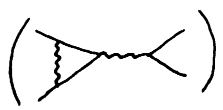
which is correct up to the $o(\alpha)$ correction.

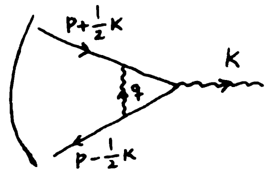
C. Contribution δE_3

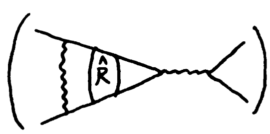
For δE_3, Figure 2(c), we decompose it as

$$\delta E_3 = \delta E_3{}^a + \delta E_3{}^b \quad , \tag{3C.1}$$

where, the first term comes from the first term of $\hat{G}_c$, equation (2.29), and the second term comes from the second term of $\hat{G}_c$; they are represented in Figure 4.

(a) $i\delta E_3{}^a = 2$ 

$= \frac{1}{2} e\phi_o \xi^{*\mu} \sqrt{2} \left(1 + \frac{\alpha}{6}\right)$ 

(b) $i\delta E_3{}^6 = 2$ 

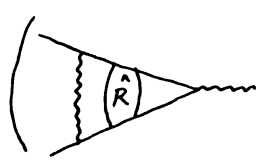
$= \frac{1}{2} e\phi_o \xi^{*\mu} \sqrt{2} \left(1 + \frac{\alpha}{6}\right)$ 

Figure 4. Contributions $\delta E_3{}^a$ and $\delta E_3{}^b$, where $\hat{R}$ is given by (2.30), and $k = (2\kappa, \vec{o})$.

We first calculate $\delta E_3{}^a$. From Figure 4(a)

$$\delta E_3{}^a = \frac{-i}{2}\, e\phi_o\, \xi^{*\mu} \sqrt{2}\, (1 + \frac{\alpha}{6})$$

$$\times \int \frac{d^4 p}{(2\pi)^4} < -ie\, \Lambda_\mu(p - \frac{1}{2}k,\ p + \frac{1}{2}k)\psi_o(p) > \quad , \qquad (3C.2)$$

where Λ_μ is due to the vertex correction and can be found by the following relation

$$\Lambda_\mu\ (p - \frac{1}{2}k,\ p + \frac{1}{2}k) = \frac{i\alpha}{4\pi^3} \int d^n q\ \frac{A_\mu + B_\mu}{D} \quad , \qquad (3C.3)$$

where

$$D = \vec{q}^{\,2}\, q^2\, [(q + p - \frac{1}{2}k)^2 - 1][(q + p + \frac{1}{2}k)^2 - 1] \quad , \qquad (3C.4)$$

$$A_\mu = [-\vec{q}^{\,2} g^{\rho\sigma} - q^\rho q^\sigma + q^o(q^\rho \delta^\sigma{}_o + q^\sigma \delta^\rho{}_o)]\gamma_\rho C_\mu \gamma_\sigma - B_\mu \quad , \qquad (3C.5)$$

$$B_\mu = [-\vec{q}^{\,2} g^{\rho\sigma} - q^\rho q^\sigma + q^o(q^\rho \delta^\sigma{}_o + q^\sigma \delta^\rho{}_o)]\gamma_\rho \not{q} \gamma_\mu \not{q} \gamma_\sigma \quad , \qquad (3C.6)$$

$$C_\mu = (\not{q} + \not{p} - \frac{1}{2}\not{k} + 1)\gamma_\mu(\not{q} + \not{p} + \frac{1}{2}\not{k} + 1) \quad . \qquad (3C.7)$$

Note that in the numerator of (3C.3), we added and subtracted the term B_μ, the term which gives ultraviolet divergence. So we find A_μ (and $\int d^n q\, \frac{A_\mu}{D}$) in 4 dimension (since it gives finite contribution) and B_μ in n dimension. Therefore

$$A_\mu = -\vec{q}^{\,2}\gamma^0 C_\mu \gamma_0 - \not{q} C_\mu \not{q} + q^0 \not{q} C_\mu \gamma_0 + q^0 \gamma_0 C_\mu \not{q} - 4\,\vec{q}^{\,2} q_\mu \not{q}$$

$$+ q^2\,\vec{q}^{\,2}\gamma_\mu - q^2\,q_0^2\gamma_\mu - 2\,q_0 q^2 q_\mu \gamma_0 \quad , \qquad (3C.8)$$

$$B_\mu = -2(2-n)\,\vec{q}^{\,2} q_\mu \not{q} + q^2[(3-n)\vec{q}^{\,2}\gamma_\mu + q_0^2\gamma_\mu + 2q_0 q_\mu \gamma_0] \quad . \qquad (3C.9)$$

From (3C.2) and equation (2.16) for ψ_0, we note that it is necessary to evaluate the quantity $\langle \Lambda_\mu F \rangle$, where

$$F = (N_p - \vec{p}\cdot\vec{\gamma})(1 + \gamma_0)\hat{\xi}\cdot\vec{\gamma}(N_p - \vec{p}\cdot\vec{\gamma}) \quad . \qquad (3C.10)$$

By virtue of (3C.3), we obtain

$$\langle \Lambda_\mu F \rangle = \frac{i\alpha}{4\pi^3}\int d^4q\,\frac{\langle A_\mu F \rangle}{D} + \frac{i\alpha}{4\pi^3}\int d^nq\,\frac{\langle B_\mu F \rangle}{D} \quad , \qquad (3C.11)$$

where

$$\langle A_\mu F \rangle = -32(E_p + 1)\,q^2\xi^\mu \quad , \qquad (3C.12)$$

$$\langle B_\mu F \rangle = [32\,(2-n)\,\vec{q}^{\,2} q_\mu^2 + 16\,(3-n)\,\vec{q}^{\,2} q_0^2 + 16\,(n-3)\,\vec{q}^{\,4}$$

$$+ 16\,q^2 q_0^2]\,\xi^\mu \quad . \qquad (3C.13)$$

In finding (3C.12) and (3C.13) we used these arguments: from the structure of integrals in (3C.11) we notice that for the first integral the small values of $q(q \sim \gamma)$ gives the $o(1)$ and $o(\alpha)$

contributions to $\delta E_3^{\,a}$, while the non-small values of q gives only the $o(\alpha)$ contribution. For the second integral only non-small values of q contribute and give the $o(\alpha)$ contribution. Therefore in (3C.12) we dropped terms such as $\gamma^2 q^2$, $\vec{p}\cdot\vec{q}\,q^2$, $(\vec{p}\cdot\vec{q})^2$ and etc. In (3C.13) we put $p = 0$, $\gamma = 0$. Beside these approximations we also used the relation $p_o = \kappa - E_p$ and $p \sim \gamma$, which follow from the fact that the main contribution, up to the $o(\alpha)$, comes from the pole in wave-function.

Using (3C.12) and (3C.13) in (3C.11), we notice that we need the following integrals, which can be found by the table of integrals which is provided in [19]

$$\int d^4q\,\frac{q^2}{D} = i\pi^2 \int_o^1 \frac{dx}{\sqrt{x}} \int_o^1 dy\,\frac{1}{k^2y^2 - k^2y + E_p^2 - \vec{p}^{\,2}x} \quad , \tag{3C.14}$$

$$\int d^nq\,\frac{\vec{q}^{\,2} q_\mu^{\,2}}{D} = \frac{-i\pi^2}{4}(1 - 3a)\Gamma(-a) \qquad (\mu = 1, 2, 3) \quad , \tag{3C.15}$$

$$\int d^nq\,\frac{q^2 q_o^{\,2}}{D} = -2i\pi^2 - i\pi^2\Gamma(-a) \quad , \tag{3C.16}$$

$$\int d^nq\,\frac{\vec{q}^{\,4}}{D} = \frac{-i\pi^2}{4}(3 - 7a)\Gamma(-a) \quad , \tag{3C.17}$$

$$\int d^nq\,\frac{\vec{q}^{\,2} q_o^{\,2}}{D} = \frac{-i\pi^2}{4}(1 + 3a)\Gamma(-a) \quad , \tag{3C.18}$$

In relations (3C.15) - (3C.18), we dropped $\vec{p}$ and p_o and γ, since we are interested in finding the values of these integrals up to the o(1), which corresponds to the o(α) correction to δE^a_3.

The integral in (3C.14) is performed in Appendix A and is given by equation (A.7), the result is

$$\int d^4q \, \frac{q^2}{D} = \frac{i\pi^3}{p} \sin^{-1}\left(\frac{p}{\sqrt{\vec{p}^2 + \gamma^2}}\right) - 2i\pi^2 \quad . \tag{3C.19}$$

Using these results in equation (3C.11), we find

$$\langle \Lambda_\mu F \rangle = \frac{4\alpha\xi^\mu}{\pi}\left[\frac{2\pi}{p}(E_p + 1)\sin^{-1}\left(\frac{p}{\sqrt{\vec{p}^2 + \gamma^2}}\right) - 8 + \Gamma(-a)\right] \quad , \tag{3C.20}$$

where the first term will give the o(1) and the rest the o(α) corrections to δE^a_3.

From equations (3C.2), (3C.10), (3C.20) and equation (2.16) for ψ_o, we find

$$\delta E^a_3 = \frac{\alpha^6}{16\pi^3}\sqrt{2}\left(1 + \frac{\alpha}{6}\right)\int d^3p \, \frac{\sqrt{E_p + \kappa}}{(\vec{p}^2 + \gamma^2)^2 E_p (E_p + 1)}$$

$$\times \left[\frac{2\pi(E_p + 1)}{p}\sin^{-1}\left(\frac{p}{\sqrt{\vec{p}^2 + \gamma^2}}\right) - 8 + \Gamma(-a)\right] \quad . \tag{3C.21}$$

Since in the square brackets the first term gives the o(1) and the second and third terms give the o(α) corrections to δE^a_3, for the second and third terms we use the approximation $E_p = 1$, $\kappa = 1$.

Therefore, using these approximate relations and then performing p integration on these two terms we find

$$\delta E_3^a = \frac{\alpha^5}{2\pi}\sqrt{2}\,(1 + \frac{\alpha}{6})\,[-\sqrt{2} + \frac{\sqrt{2}}{8}\,\Gamma(-a)$$

$$+ \alpha \int_0^\infty dp \frac{p\sqrt{E_p + \kappa}}{(\vec{p}^2 + \gamma^2)^2 E_p} \sin^{-1}\left(\frac{p}{\sqrt{\vec{p}^2 + \gamma^2}}\right)] \quad . \qquad (3C.22)$$

The integral in (3C.22) is performed in the Appendix A and is given by equation (A.20). Using that result we find, up to the o(α) correction

$$\delta E_3^a = \frac{\alpha^4}{4}\,[2 - \frac{4\alpha}{\pi} + \frac{\alpha}{3} + \frac{\alpha}{2\pi}\,\Gamma(-a)] \quad . \qquad (3C.23)$$

To find δE_3^b, we note that in the diagram of the Figure 5, up to the o(α) correction, only the instantaneous Coulomb interaction D_{oo} of the photon propagator $D_{\lambda\nu}$ contributes, where

$$D_{\lambda\nu}(p) = \frac{1}{p^2}\,[-g_{\lambda\nu} + \frac{-p_\lambda p_\nu + p_o(p_\lambda \delta_\nu{}^o + p_\nu \delta_\lambda{}^o)}{\vec{p}^2}] \quad . \qquad (3C.24)$$

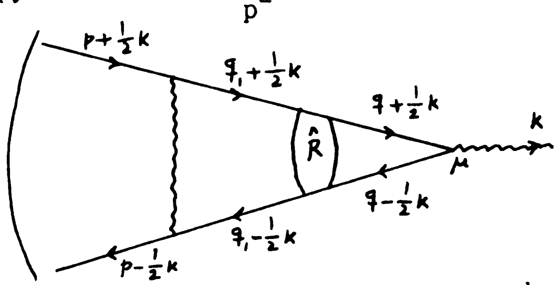


Figure 5. Diagram which is related to δE_3^b. $\hat{R}$ is defined in (2.30).

Dominant contribution comes from small p and q_1 (but it is not the case for q), more specifically, p and q_1 are of the order α, so we keep the lowest powers of p and q_1 and we set $E_p = E_{q_1} = 1$, where ever it is legitimate (for more details see Section D of Chapter 4 for calculation of M_4^b which is similar to the δE_3^b.)

After performing P_o, q_{10} and q_o integrations we find (poles are at $p_o = \kappa - E_p$, $q_{10} = \kappa - E_{q_1}$ and $q_o = \kappa - E_q$)

$$\delta E_3^b = \frac{\alpha^6}{384\pi^7} \sqrt{2}\,(1 + \tfrac{\alpha}{6}) \int d^3p d^3q d^3q_1\, (2E_q + 1) \sqrt{E_q + \kappa}$$

$$\times \bar{H}_c(\kappa, \vec{q}, \vec{q}_1)[(\vec{p}^2 + \gamma^2)^2(\vec{p} - \vec{q}_1)^2(\vec{q}_1^2 + \gamma^2)(\vec{q}^2 + \gamma^2)E_q]^{-1}, \quad (3C.25)$$

or after performing p integration

$$\delta E_3^b = \frac{\alpha^5}{192\pi^5} \sqrt{2}(1 + \tfrac{\alpha}{6})\int d^3q d^3q_1 \frac{(2E_q + 1)\sqrt{E_q + \kappa}\, \bar{H}_c(\kappa, \vec{q}, \vec{q}_1)}{(\vec{q}_1^2 + \gamma^2)^2(\vec{q}^2 + \gamma^2)\, E_q}, \quad (3C.26)$$

where $\bar{H}_c$ comes from $\hat{H}_c$, equation (2.27), after performing p_o, q_o and q_{10} integrations, or

$$\bar{H}_c(\kappa, \vec{q}, \vec{q}_1) = 4\pi\alpha \left\{\frac{1}{A} + \frac{1}{B}\left[\frac{5}{2} + \frac{-4\gamma^2}{\vec{q}_1^2 + \gamma^2}\right.\right.$$

$$\left.\left. + \frac{-\gamma^2 (2 + \kappa + E_q)}{\vec{q}^2 + \gamma^2} + \int_0^1 d\rho\, \frac{2B - A - B\rho}{A\rho + B(1 - \rho)^2}\right]\right\} \quad . \quad (3C.27)$$

The integral in (3C.27) does not contribute. It is so because by direct integration we can show (see Appendix C)

$$\int d^3q F(\vec{q}^2) \int d^3q_1 \left[\frac{1}{B}\frac{2B - A - B\rho}{A\rho + B(1-\rho)^2}\right] = 0 \quad , \qquad (3C.28)$$

where $F(\vec{q}^2)$ is any function of $\vec{q}^2$ which satisfies the condition (C.13) of the Appendix C

$$\int_0^\infty dq \frac{\vec{q}^2}{\vec{q}^2 + \gamma^2} F(\vec{q}^2) < \infty \quad , \qquad (3C.29)$$

which in our calculations it is the case.

Therefore, using (3C.27) in (3C.26) and performing q_1 integration

$$\delta E_3^b = \frac{\alpha^5}{24\pi^2}\sqrt{2}\,(1 + \frac{\alpha}{6}) \int d^3q \frac{(2E_q + 1)\sqrt{E_q + \kappa}}{(\vec{q}^2 + \gamma^2)^2 E_q}$$

$$\times \left[\frac{3}{2} - \frac{\gamma^2(2 + \kappa + E_q)}{\vec{q}^2 + \gamma^2}\right] \quad . \qquad (3C.30)$$

The second term in square brackets only for $q \sim \gamma$ contributes. For this term we use the approximations $E_q = 1$ and $\kappa = 1$. Therefore

$$\delta E_3^b = \frac{\alpha^5}{8\pi^2}\sqrt{2}\,(1 + \frac{\alpha}{6}) \int d^3q \frac{1}{(\vec{q}^2 + \gamma^2)^2}$$

$$\times \left[\frac{1}{2E_q}(2E_q + 1)\sqrt{E_q + \kappa} - \frac{4\sqrt{2}\,\gamma^2}{\vec{q}^{\,2} + \gamma^2}\right] \quad . \tag{3C.31}$$

The integration on the first term in square brackets is performed in Appendix B and is given by the equation (B.11). Using that result and performing the integration on the second term in the square brackets we obtain

$$\delta E_3^b = \frac{\alpha^4}{4}\left(1 + \frac{2\alpha}{3}\right) \quad . \tag{3C.32}$$

So, by virtue of equations (3C.1), (3C.23), and (3C.32), the contribution δE_3 is

$$\delta E_3 = \frac{\alpha^4}{4}\left[3 + \alpha - \frac{4\alpha}{\pi} + \frac{\alpha}{2\pi}\,\Gamma(-a)\right]. \tag{3C.33}$$

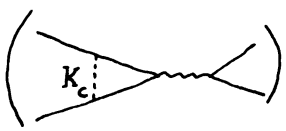
D. Contribution δE_4

For δE_4, Figure 2(d), we use the same decomposition that we had for δE_3, equation (3C.1), or

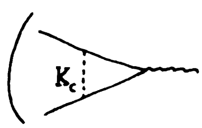
$$\delta E_4 = \delta E_4{}^a + \delta E_4{}^b \quad , \tag{3D.1}$$

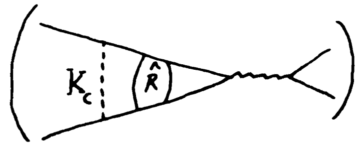
where $\delta E_4{}^a$ and $\delta E_4{}^b$ are represented in Figure 6.

(a) $i\delta E_4{}^a = -2$



$= -\frac{1}{2}\, e\phi_o\, \xi^{*\mu}\sqrt{2}\,(1 + \frac{\alpha}{6})$



(b) $i\delta E_4{}^b = -2$ 

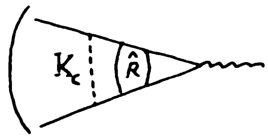
$= -\frac{1}{2}\, e\phi_o\, \xi^{*\mu}\sqrt{2}\,(1 + \frac{\alpha}{6})$ 

Figure 6. Contributions $\delta E_4{}^a$ and $\delta E_4{}^b$.

Using the following identity, which we gave it by the equation (2.22) (momentum integrations are implicit)

$$\psi_o = G_o K_c \psi_o \quad , \tag{3D.2}$$

we obtain the identity which is respresented in Figure 7,

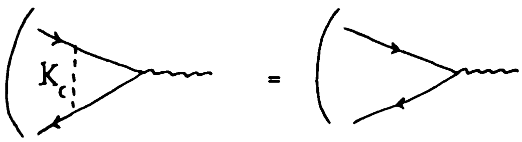


Figure 7. The identity which is the direct result of the equation (3D.2).

and from it we find the contribution $\delta E_4{}^a$

$$\delta E_4{}^a = -2\,\delta E_1 \quad , \tag{3D.3}$$

or by virtue of (3A.5)

$$\delta E_4{}^a = \frac{\alpha^4}{4}\,(-2 - \frac{2\alpha}{3}) \quad . \tag{3D.4}$$

Calculation of $\delta E_4{}^b$ is similar to $\delta E_3{}^b$. In fact, by applying the same approximation that we used for $\delta E_3{}^b$, at the very beginning we note the following relation which is correct up to the $o(\alpha)$ correction

$$\delta E_4{}^b = -\delta E_3{}^b \quad , \tag{3D.5}$$

(for the details of calculation see Section E of Chapter 4 for the calculation of $M_5{}^b$ which is similar to $\delta E_4{}^b$).

Equations (3D.5) and (3C.32) give

$$\delta E_4{}^b = \frac{\alpha^4}{4}\left(-1 - \frac{2\alpha}{3}\right) \quad , \tag{3D.6}$$

and equations (3D.1), (3D.4) and (3D.6) give

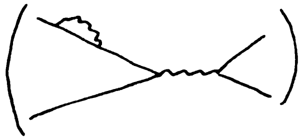
$$\delta E_4 = \frac{\alpha^4}{4}\left(-3 - \frac{4\alpha}{3}\right) \quad . \tag{3D.7}$$

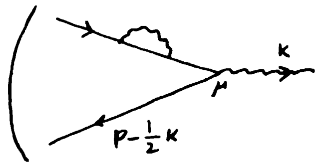
E. Contribution δE_5

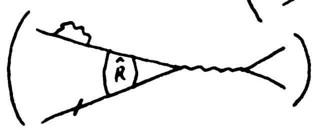
To find δE_5, Figure 2(e), we decompose its contribution into two parts, as we did in (3C.1) for δE_3

$$\delta E_5 = \delta E_5{}^a + \delta E_5{}^b \quad , \tag{3E.1}$$

where $\delta E_5{}^a$ and $\delta E_5{}^b$ are represented in Figure 8.

(a) $i\delta E_5^{\,a} = 4$ 

$= e\phi_o\xi^{*\mu}\sqrt{2}\,(1 + \frac{\alpha}{6})$ 

(b) $i\delta E_5^{\,b} = 4$ 

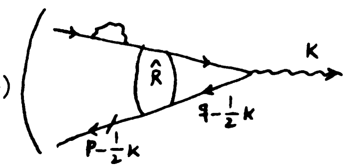
$= e\phi_o\xi^{*\mu}\sqrt{2}\,(1 + \frac{\alpha}{6})$ 

Figure 8. Contributions $\delta E_5^{\,a}$ and $\delta E_5^{\,b}$. Lines with a dash through them indicate inverse propagators.

For the electron self-energy we use the following form [19]

$$\Sigma(p) = -\frac{\alpha}{4\pi}\,\Gamma(-a)(\not p - 1) + \frac{\alpha}{4\pi}\Big\{\frac{19}{6}\,\vec{p}\cdot\vec{\gamma} - \frac{1}{2}p_o\gamma_o$$

$$-\int_0^1 \frac{dx}{\sqrt{x}}[(1 - x)\vec{p}\cdot\vec{\gamma} + 1]\ln X + 2\int_0^1 dx[(1 - x)\not p - 1]\ln Y$$

$$+ 2\,\vec{p}\cdot\vec{\gamma}\int_0^1 dx\,\sqrt{x}\int_0^1 du\,\ln Z\Big\} \quad , \qquad (3E.2)$$

where, the mass-renormalization is already performed and

$$X = 1 + \vec{p}^{\,2}(1 - x) \quad , \qquad Y = 1 - p^2(1 - x) - i\varepsilon \quad ,$$

$$Z = 1 - p_o^{\,2}(1 - u) + \vec{p}^{\,2}(1 - xu) - i\varepsilon \quad . \qquad (3E.3)$$

For the contribution $\delta E_5^{\,a}$, Figure 8(a), we write

$$\delta E_5{}^a = -4\pi\alpha\phi_o\xi^{*\mu}\sqrt{2}\,(1 + \frac{\alpha}{6})$$

$$\times \int \frac{d^4p}{(2\pi)^4} \langle\gamma_\mu S(p + \tfrac{1}{2}k)\Sigma(p + \tfrac{1}{2}k)\psi_o(p)\rangle \quad . \qquad (3E.4)$$

Using (3E.2) for $\Sigma(p + \frac{1}{2}k)$, by replacing p by $p + \frac{1}{2}k$ in (3E.2) and (3E.3), we find

$$\delta E_5{}^a = \alpha^2\phi_o\xi^{*\mu}\sqrt{2}\,\Gamma(-a)\int \frac{d^4p}{(2\pi)^4} \langle \gamma_\mu\psi_o(p)\rangle$$

$$-\alpha^2\phi_o\xi^{*\mu}\sqrt{2}\int \frac{d^4p}{(2\pi)^4} \langle \gamma_\mu\, S(p + \tfrac{1}{2}k)$$

$$\times \left[C_1\,\vec{p}\cdot\vec{\gamma} + C_2(p_o + \kappa)\gamma_o + C_3\right]\psi_o(p)\rangle \quad , \qquad (3E.5)$$

where

$$C_1 = \frac{19}{6} - \int_0^1 \frac{dx}{\sqrt{x}}(1 - x)\ln X - 2\int_0^1 dx(1 - x)\ln Y + 2\int_0^1 dx\int_0^1 du\ln Z \quad ,$$

$$C_2 = -\frac{1}{2} + 2\int_0^1 dx\;(1 - x)\ln Y \quad ,$$

$$C_3 = -\int_0^1 \frac{dx}{\sqrt{x}}\ln X - 2\int_0^1 dx\ln Y \quad . \qquad (3E.6)$$

The first integral in (3E.5) up to the order that we are interested is (see equation (3A.3))

$$\int \frac{d^4p}{(2\pi)^4} < \gamma_\mu \psi_o(p) > = -\sqrt{2}\, \xi^\mu \sqrt{2} \quad . \tag{3E.7}$$

For the second integral in (3E.5), we can show that it is of higher orders so we can ignore it. In fact, after performing the p_o integration (dominant contribution comes from the pole in the wave-function, or for $p_o = \kappa - E_p$; we close the contour in the upper half plane of p_o), then by scaling $\vec{p} \rightarrow \gamma\vec{p}$ we note that in order this integral contributes, the following trace should be, at most, of the order α^2

$$<\gamma_\mu(\not{p} + \kappa\,\gamma_o + 1)[C_1\, \vec{p}\cdot\vec{\gamma} + C_2\,(p_o + \kappa)\gamma_o + C_3]$$

$$\times\, (N_p - \vec{p}\cdot\vec{\gamma})(1 + \gamma_o)\hat{\xi}\cdot\vec{\gamma}(N_p - \vec{p}\cdot\vec{\gamma}) > \quad , \tag{3E.8}$$

where $p_o = \kappa - E_p$.

It is not difficult to see that we need to find C_1 up to the $O(1)$, but for C_2 and C_3 up to the order α^2. From the relations in (3E.3) and (3E.6), with the replacement p by $p + \frac{1}{2}k$, we find the following approximate relations

$$C_1 = \frac{10}{3} \quad ,$$

$$C_2 = -2 - 4(\vec{p}^{\,2} + \gamma^2)\ln[2(\vec{p}^{\,2} + \gamma^2)] - 2(\vec{p}^{\,2} + \gamma^2) \quad ,$$

$$C_3 = 2 + 4(\vec{p}^{\,2} + \gamma^2)\ln[2(\vec{p}^{\,2} + \gamma^2)] - \frac{4}{3}\,\vec{p}^{\,2} \quad , \tag{3E.9}$$

where we have used the following approximations

$$E_p = 1 + \frac{1}{2}\vec{p}^2 \quad , \kappa = 1 - \frac{1}{2}\gamma^2 \quad . \tag{3E.10}$$

After finding the trace in (3E.8) and using (3E.9), we note that it is of the order higher than α^2. Therefore the second integral in (3E.5) does not contribute.

Using (3E.7) in (3E.5) we obtain

$$\delta E_5{}^a = \frac{\alpha^4}{4}\left[-\frac{\alpha}{\pi}\Gamma(-a)\right] \quad . \tag{3E.11}$$

For the contribution $\delta E_5{}^b$, Figure 8(b), we write

$$\delta E_5{}^b = \frac{i\pi\alpha}{4}\phi_o\xi^{*\mu}\sqrt{2}(1 + \frac{\alpha}{6})\int\frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{R(\kappa,\vec{p})R(\kappa,\vec{q})}{E_pE_qN_pN_q}$$

$$\times \hat{H}_c(\kappa, q, p)\langle(N_p + \vec{p}\cdot\vec{\gamma})(1 - \gamma_o)(N_q + \vec{q}\cdot\vec{\gamma})$$

$$\times S(q - \tfrac{1}{2}k)\gamma_\mu S(q + \tfrac{1}{2}k)(N_q + \vec{q}\cdot\vec{\gamma})(1 + \gamma_o)$$

$$\times (N_p + \vec{p}\cdot\vec{\gamma})S(p + \tfrac{1}{2}k)\Sigma(p + \tfrac{1}{2}k)\psi_o(p)\rangle \quad , \tag{3E.12}$$

where we have used (2.30) for $\hat{R}$.

Using relations (4D.9) for simplifying the trace in (3E.12), and using (3E.2) for Σ, we note that (by scaling $\vec{p}\rightarrow\gamma\vec{p}$ and $\vec{q}\rightarrow\gamma\vec{q}$)

the most contribution comes from $\vec{p}$, $\vec{q} \sim \gamma$ and the poles at $p_o = \kappa - E_p$ and $q_o = \kappa - E_q$ (there is only one pole for q_o).

After performing the p_o and q_o integrations, we obtain

$$\delta E_5{}^b = \frac{\alpha^5}{16\pi^5}\left[\frac{-\alpha}{4\pi}\,\Gamma(-a)\right]\int d^3p d^3q\,\frac{\bar{H}_c(\kappa,\vec{q},\vec{p})}{(\vec{p}^{\,2}+\gamma^2)^2(\vec{q}^{\,2}+\gamma^2)}$$

$$+\ [\text{contribution from the second term of } \Sigma], \qquad (3E.13)$$

where $\bar{H}_c$ is given by (3B.6).

We can show that contribution of the second term of Σ is of higher orders, so we may neglect it. In fact, for the contribution of the second term of Σ we have a similar situation that we had for $\delta E_5{}^a$; here we need to find the following trace, at most, up to the order α^2

$$\langle (N_p + \vec{p}\cdot\vec{\gamma})(1-\gamma_o)(N_q - \vec{q}\cdot\vec{\gamma})\gamma_\mu(N_q - \vec{q}\cdot\vec{\gamma})(1+\gamma_o)(N_p - \vec{p}\cdot\vec{\gamma})$$

$$\times [C_1\,\vec{p}\cdot\vec{\gamma} + C_2(2\kappa - E_p)\gamma_o + C_3](N_p - \vec{p}\cdot\vec{\gamma})(1+\gamma_o)\hat{\xi}\cdot\vec{\gamma}$$

$$\times\ (N_p - \vec{p}\cdot\vec{\gamma})\rangle \quad . \qquad (3E.14)$$

It is not difficult to show that we need to know C_1 up to the $O(1)$, but C_2 and C_3 up to the order α^2. Therefore we may use (3E.9) for values of C_1, C_2 and C_3. After finding the trace in (3E.14) and using (3E.9), we note that it is of order higher than α^2. Therefore in (3E.13) only the first term contributes.

Using (3B.6) for $\bar{H}_c$ and then (3C.28), relation (3E.13) gets the following form

$$\delta E_5{}^b = \frac{-\alpha^7}{16\pi^5}\,\Gamma(-a)\int d^3p\,d^3q\,\frac{1}{(\vec{p}^{\,2}+\gamma^2)^2(\vec{q}^{\,2}+\gamma^2)}$$

$$\times\left[\frac{1}{A}+\frac{1}{B}\left(\frac{5}{2}+\frac{-4\gamma^2}{\vec{p}^{\,2}+\gamma^2}+\frac{-4\gamma^2}{\vec{q}^{\,2}+\gamma^2}\right)\right] \quad , \qquad (3E.15)$$

where A and B are given by (2.28).

Performing p and q integrations, we obtain

$$\delta E_5{}^b = \frac{\alpha^4}{4}\left[\frac{-\alpha}{2\pi}\,\Gamma(-a)\right] \quad , \qquad (3E.16)$$

and by virtue of (3E.1), (3E.11) and (3E.16) we find

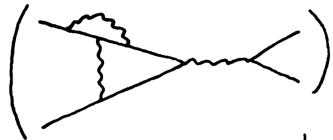
$$\delta E_5 = \frac{\alpha^4}{4}\left[\frac{-3\alpha}{2\pi}\,\Gamma(-a)\right] \quad . \qquad (3E.17)$$

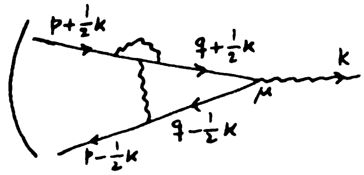
F. Contribution δE_6

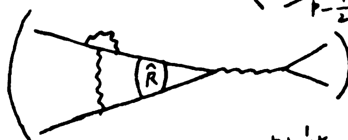
For δE_6, Figure 2(f), we decompose its contribution in the same way that we did in (3C.1) for δE_3

$$\delta E_6 = \delta E_6{}^a + \delta E_6{}^b \quad , \qquad (3F.1)$$

where $\delta E_6{}^a$ and $\delta E_6{}^b$ are represented in Figure 9.

(a) $i\delta E_6{}^a = 4$ 

$= e\phi_o \xi^{*\mu}\sqrt{2}\ (1 + \frac{\alpha}{6})$ 

(b) $i\delta E_6{}^b = 4$ 

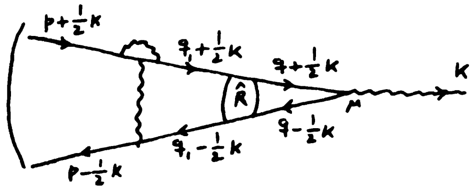
$= e\phi_o \xi^{*\mu}\sqrt{2}\ (1 + \frac{\alpha}{6})$ 

Figure 9. Contributions $\delta E_6{}^a$ and $\delta E_6{}^b$.

First we consider the vertex correction in these diagrams (see Figure 10.)

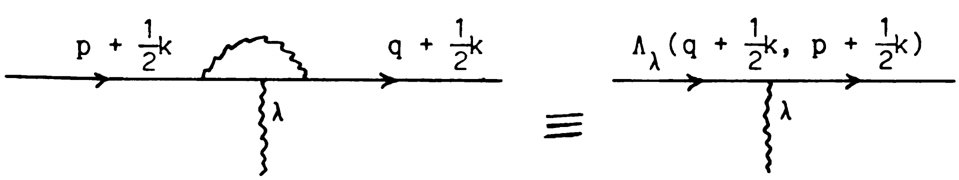


Figure 10. Vertex correction for $\delta E_6{}^a$ and $\delta E_6{}^b$.

Contributions $\delta E_6{}^a$ and $\delta E_6{}^b$ can be written as

$$\delta E_6{}^a = -16 i\pi^2\alpha^2\phi_o\xi^{*\mu}\sqrt{2}(1 + \frac{\alpha}{6})\int\frac{d^4q}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\langle D^{\lambda\nu}(p - q)$$

$$\times\ \gamma_\nu S(q - \tfrac{1}{2}k)\gamma_\mu S(q + \tfrac{1}{2}k)\Lambda_\lambda(q + \tfrac{1}{2}k,\ p + \tfrac{1}{2}k)\psi_o(p)\rangle \quad , \qquad (3F.2)$$

$$\delta E_6{}^b = -\pi^2\alpha^2\phi_o\xi^{*\mu}\sqrt{2}(1 + \frac{\alpha}{6})\int \frac{d^4q}{(2\pi)^4}\frac{d^4q_1}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}$$

$$\times \frac{R(\kappa,\ \vec{q})R(\kappa,\ \vec{q}_1)}{E_qE_{q_1}N_qN_{q_1}}\hat{H}_c(\kappa,\ q,\ q_1)\langle D^{\lambda\nu}(p - q_1)$$

$$\times\ \gamma_\nu S(q_1 - \frac{1}{2}k)(N_{q_1} + \vec{q}_1 \cdot \vec{\gamma})(1 - \gamma_o)(N_q + \vec{q} \cdot \vec{\gamma})$$

$$\times\ S(q - \frac{1}{2}k)\gamma_\mu S(q + \frac{1}{2}k)(N_q + \vec{q} \cdot \vec{\gamma})(1 + \gamma_o)(N_{q_1} + \vec{q}_1 \cdot \vec{\gamma})$$

$$\times\ S(q_1 + \frac{1}{2}k)\Lambda_\lambda(q_1 + \frac{1}{2}k,\ p + \frac{1}{2}k)\psi_o(p)\rangle \quad , \qquad (3F.3)$$

where $D^{\lambda\nu}$ is given by (3C.24).

Using relations (4D.9) for simplifying the trace in (3F.3) we note that (by scaling $\vec{p} \to \gamma\vec{p}$, $\vec{q} \to \gamma\vec{q}$ and $\vec{q}_1 \to \gamma\vec{q}_1$) the most contribution comes from $\vec{p}$, $\vec{q}$ and $\vec{q}_1 \sim \gamma$ and the poles at $p_o = \kappa - E_p$, $q_o = \kappa - E_q$ and $q_{10} = \kappa - E_{q_1}$ (for (3F.2) we have the same situation with the exception that there are only p and q variables.)

Performing the p_o, q_o and q_{10} integrations (for (3F.2) only p_o and q_o integrations) we see that up to the $o(\alpha)$ correction only $\nu = o$ contributes. This means that $\lambda = o$ (since $D^{oi} = o$, for $i = 1, 2, 3$). Therefore only D_{oo}, the instantaneous Coulomb interaction contributes. We summarize these results in Figure 11.

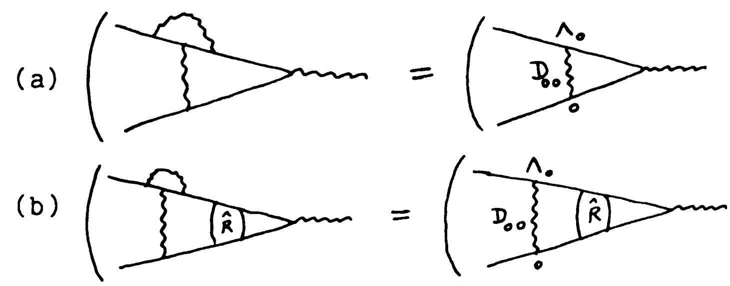


Figure 11. Equalities which hold up to the o(α) corrections, where D_{oo} is the instantaneous Coulomb interaction (see (3C.24).)

Therefore, contributions $\delta E_6{}^a$ and $\delta E_6{}^b$ get the following form

$$\delta E_6{}^a = \frac{\alpha^6}{32\pi^4}\,\xi^{*\mu}\int d^3p\,d^3q\left[(\vec{p}^{\,2} + \gamma^2)^2(\vec{q}^{\,2} + \gamma^2)(\vec{p} - \vec{q})^2\right]^{-1}$$

$$\times \langle\gamma_\mu(1 + \gamma_o)\Lambda_o(q + \tfrac{1}{2}k,\ p + \tfrac{1}{2}k)(1 + \gamma_o)\hat{\xi}\cdot\vec{\gamma}\rangle \qquad , (3F.4)$$

$$\delta E_6{}^b = \frac{\alpha^6}{256\pi^7}\,\xi^{*\mu}\int d^3p\,d^3q\,d^3q_1\ \bar{H}_c(\kappa,\ \vec{q}_1,\ \vec{q})$$

$$\times \left[(\vec{p}^{\,2} + \gamma^2)^2(\vec{q}_1^{\,2} + \gamma^2)(\vec{q}^{\,2} + \gamma^2)(\vec{p} - \vec{q}_1)^2\right]^{-1}$$

$$\times \langle\gamma_\mu(1 + \gamma_o)\Lambda_o(q_1 + \tfrac{1}{2}k,\ p + \tfrac{1}{2}k)(1 + \gamma_o)\hat{\xi}\cdot\vec{\gamma}\rangle \qquad , (3F.5)$$

where $\bar{H}_c$ is given by (3C.27).

Relations (3F.4) and (3F.5) show that we need to find the leading order of Λ_o. In the following we find Λ_o, but we keep in mind that Λ_o is sandwiched between two $(1 + \gamma_o)$'s.

Using Feynman rules the quantity Λ_o (p', p) (where p stands for $p + \frac{1}{2}k$ and p' stands for $q + \frac{1}{2}k$ or $q_1 + \frac{1}{2}k$) can be written as

$$\Lambda_o(p', p) = \frac{i\alpha}{4\pi^3} \int \frac{d^n q}{D} \left[F^{\mu\nu}\gamma_\nu(\not{q} + \not{p}' + 1)\gamma_o(\not{q} + \not{p} + 1)\gamma_\mu\right] \quad , \quad (3F.6)$$

where

$$D = q^2 \vec{q}^2[(q + p)^2 - 1][(q + p')^2 - 1] \quad ,$$

$$F^{\mu\nu} = -g^{\mu\nu} \vec{q}^2 - q^\mu q^\nu + q^o(q^\mu\delta^\nu_o + q^\nu\delta^\mu_o) \quad . \qquad (3F.7)$$

Since p_o and p_o' are of the o(1), and $\vec{p}$ and $\vec{p}'$ are of the o(α), (3F.6) gets the following form

$$\Lambda_o(p', p) = \frac{i\alpha}{4\pi^3} \int \frac{d^n q}{D} \left[F^{\mu\nu}\gamma_\nu(\not{q} + 1 + \gamma_o)\gamma_o(\not{q} + 1 + \gamma_o)\gamma_\mu\right] \quad , \quad (3F.8)$$

or if we use the fact that in (3F.4) and (3F.5), Λ_o is sandwiched between two $(1 + \gamma_o)$'s

$$\Lambda_o(p', p) = \frac{i\alpha}{4\pi^3} \int \frac{d^n q}{D} \left[q_o^2 q^2 + 2(n - 2)q_o^2 \vec{q}^2 + (3 - n)q^2 \vec{q}^2 \right.$$

$$\left. + 4 q_o q^2 + 4 q^2\right] \quad . \qquad (3F.9)$$

Using the table of integrals, which are provided in [19], we obtain the following relations

$$\int d^n q \, \frac{q_o^2 \, q^2}{D} = i\pi^2 \int_o^1 \frac{dx}{\sqrt{x}} \int_o^1 dy \, \{ \frac{1}{\Delta_1} [p_o' + y(p_o - p_o')]^2$$

$$-\frac{1}{2} x^a \Delta_1{}^a \Gamma(-a)\} \quad ,$$

$$\int d^n q \, \frac{q_o \, q^2}{D} = -i\pi^2 \int_o^1 \frac{dx}{\sqrt{x}} \int_o^1 dy \, \frac{1}{\Delta_1} [p_o' + y(p_o - p_o')] \quad ,$$

$$\int d^n q \, \frac{q^2}{D} = i\pi^2 \int_o^1 \frac{dx}{\sqrt{x}} \int_o^1 dy \, \frac{1}{\Delta_1} \quad ,$$

$$\int d^n q \, \frac{\vec{q}^2 q^2}{D} = i\pi^2 \, \Gamma(-a) \int_o^1 dx \, \Delta_2^a \quad , \qquad (3F.10)$$

and also the following relation

$$\int d^n q \, \frac{q_o^2 \, \vec{q}^2}{D} = -i\pi^2 \int_o^1 dx \int_o^1 dy \{ \frac{x^2}{\Delta_3} [p_o' + y(p_o - p_o')]^2$$

$$- \frac{1}{2} x^{a+1} \Delta_3^a \, \Gamma(-a)\} \quad , \qquad (3F.11)$$

where

$$\Delta_1 = [p_o' + y(p_o - p_o')]^2 - x[\vec{p}' + y(\vec{p} - \vec{p}')]^2 + 1 - p'^2 + y(p'^2 - p^2) \quad ,$$

$$\Delta_2 = [p' + x(p - p')]^2 + 1 - p'^2 + x(p'^2 - p^2) \quad ,$$

$$\Delta_3 = x[p' + y(p - p')]^2 + 1 - p'^2 + y(p'^2 - p^2) \quad . \qquad (3F.12)$$

It is not difficult to note that for the integrals in (3F.10), it is legitimate to use the approximate values of p and p', before performing these integrals.

The approximate relations that we use in these integrals are

$$p_o = P_o' = 1 \quad , \quad \vec{p} = \vec{p}\,' = o \quad , \quad \Delta_1 = \Delta_2 = 1 \quad . \qquad (3F.13)$$

Using (3F.13) in (3F.10) and then performing integrals we obtain

$$\int d^n q \, \frac{q_o^{\,2} q^2}{D} = -i\pi^2 \, \Gamma(-a) \quad ,$$

$$\int d^n q \, \frac{q_o q^2}{D} = -2i\pi^2 \quad ,$$

$$\int d^n q \, \frac{q^2}{D} = 2i\pi^2 \quad ,$$

$$\int d^n q \, \frac{\vec{q}^{\,2} q^2}{D} = i\pi^2 \, \Gamma(-a) \quad . \qquad (3F.14)$$

For the integral in (3F.11) it is not clear that we can use the approximate relations in (3F.13); so first we perform the x integration

$$\int d^n q \, \frac{q_o^{\,2} \vec{q}^{\,2}}{D} = -i\pi^2 \int_o^1 dy \, \frac{C}{A^2} [\tfrac{1}{2} A - B + \frac{B^2}{A} \ln (\frac{A + B}{B})$$

$$+ \frac{i\pi^2}{4}\int_o^1 dy\left[\Gamma(-a) + 1 - \frac{B}{A} - \frac{B^2}{A^2}\ln B + \left(\frac{B^2}{A^2} - 1\right)\ln(A + B)\right] \quad , \quad (3F.15)$$

where

$$A = \left[p' + y(p - p')\right]^2 \quad , \quad B = 1 - p'^2 + y(p'^2 - p^2) \quad ,$$

$$C = \left[p_o' + y(p_o - p_o')\right]^2 \quad . \qquad (3F.16)$$

Since the approximate values of A, B and C are (for all values of y, $o \le y \le 1$)

$$A = 1 \quad , \quad B = o \quad , \quad C = 1 \quad , \qquad (3F.17)$$

we can use these values in (3F.15) before performing the integration.

Using (3F.17) in (3F.15) and then performing the y integration we obtain

$$\int d^n q \, \frac{q_o^2 \vec{q}^2}{D} = \frac{i\pi^2}{4}\left[\Gamma(-a) - 1\right] \quad . \qquad (3F.18)$$

Now, by virtue of (3F.9), (3F.14) and (3F.18) we find

$$\Lambda_o(p', p) = \frac{\alpha}{4\pi}\Gamma(-a) \quad . \qquad (3F.19)$$

Relation (3F.19) is derived under the assumption that Λ_o is going to be sandwiched between two $(1 + \gamma_o)$'s. Therefore the proper way of writing (3F.19) is

$$(1 + \gamma_o)\Lambda_o(p', p)(1 + \gamma_o) = (1 + \gamma_o)\left[\frac{\alpha}{4\pi}\,\Gamma(-a)\right](1 + \gamma_o) \quad . \qquad (3F.20)$$

We note that, we have proved the legitimacy of using the following approximation in the integral in (3F.9)

$$p' = p = (1, \vec{o}) \quad . \qquad (3F.21)$$

However, if we use (3F.21) at the very beginning, i.e., using (3F.21) in (3F.6) where Λ_o is not yet sandwiched between two $(1 + \gamma_o)$'s, we find a term which is proportional to $(1 - \gamma_o)$ but undefined. Still there is a fast and simple way of finding the result (3f.20).

By considering the following approximation (instead of (3F.21))

$$p' = p = (2\kappa - E_p, \vec{p}) = (1 - \gamma^2 - \frac{1}{2}\vec{p}^2, \vec{p}^2) \quad , \qquad (3F.22)$$

we can write (at this point we only use the equality $p' = p$)

$$\Lambda_o\,(q_1 + \tfrac{1}{2}k,\ p + \tfrac{1}{2}k) = \Lambda_o(q + \tfrac{1}{2}k,\ p + \tfrac{1}{2}k)$$

$$= \Lambda_o(p + \tfrac{1}{2}k,\ p + \tfrac{1}{2}k) \quad . \qquad (3F.23)$$

Therefore, we need to find $\Lambda_o(p + \frac{1}{2}k,\ p + \frac{1}{2}k)$, which can be found by the Ward identity

$$\Lambda_o(p,\ p) = -\frac{\partial}{\partial p_o}\,\Sigma\,(p) \quad . \tag{3F.24}$$

Using equation (3E.2) for Σ, and approximate relations (3F.22), and keeping only the leading orders we obtain

$$\Lambda_o(p + \frac{1}{2}k,\ p + \frac{1}{2}k) = \frac{\alpha}{4\pi}\,\{\gamma_o\Gamma(-a) + 4(1 - \gamma_o)$$

$$\times\left[1 + \ln(2\vec{p}^{\,2} + 2\gamma^2)\right]\} \quad , \tag{3F.25}$$

where if we sandwich this Λ_o between two $(1 + \gamma_o)$'s we find the same form that we had in (3F.20).

Since the second term in (3F.25) does not contribute, we can represent the result in (3F.25) effectively by the Figure 12.

(a)

(b)

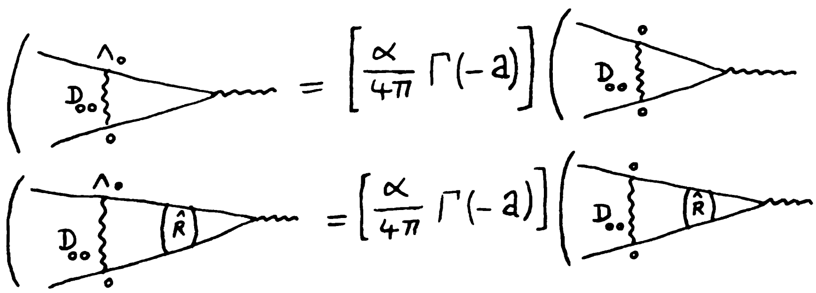


Figure 12. Equalities which are correct up to the $o(\alpha)$; they are the result of equation (3F.25).

Now, using the result (3F.20) or (3F.25) in the relations (3F.4) and (3F.5), we find

$$\delta E_6{}^a = \frac{\alpha^7}{16\pi^5}\,\Gamma(-a)\int d^3p\,d^3q\left[(\vec{p}^{\,2} + \gamma^2)^2(\vec{q}^{\,2} + \gamma^2)(\vec{p} - \vec{q})^2\right]^{-1} \quad , \qquad (3F.26)$$

$$\delta E_6{}^b = \frac{\alpha^7}{128\pi^8}\,\Gamma(-a)\int d^3p\,d^3q\,d^3q_1\ \bar{H}_c(\kappa,\ \vec{q}_1,\ \vec{q})$$

$$\times\ \left[(\vec{p}^{\,2} + \gamma^2)^2(\vec{q}_1^{\,2} + \gamma^2)(\vec{q}^{\,2} + \gamma^2)(\vec{p} - \vec{q}_1)^2\right]^{-1} \quad , \qquad (3F.27)$$

or by virtue of the equation (3C.27) for $\bar{H}_c$ and equation (3C.28) and then performing the integrals

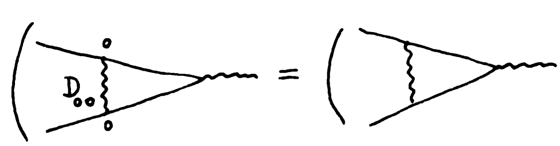
$$\delta E_6{}^a = \frac{\alpha^4}{4}\left[\frac{\alpha}{\pi}\,\Gamma(-a)\right] \quad , \qquad (3F.28)$$

$$\delta E_6{}^b = \frac{\alpha^4}{4}\left[\frac{\alpha}{2\pi}\,\Gamma(-a)\right] \quad . \qquad (3F.29)$$

Finally, relations (3F.1), (3F.28) and (3F.29) give

$$\delta E_6 = \frac{\alpha^4}{4}\left[\frac{3\alpha}{2\pi}\,\Gamma(-a)\right] \quad . \qquad (3F.30)$$

We note that, there is a close connection between δE_6, Figure 2(f), and δE_3, Figure 2(c). In fact, since we are interested in the leading order, we can derive the results (3F.28) and (3F.29) by observing that in the diagrams in the right-hand sides of the equalities in Figure 12, D_{oo} can be replaced by $D_{\mu\nu}$. We represent this approximation in Figure 13.

(a) 

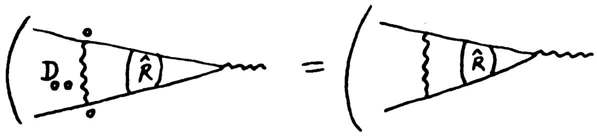
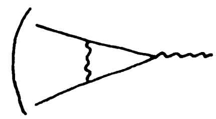
(b) 

Figure 13. Equalities which are correct up to leading order, i.e. o(1).

Comparing the results in the Figures 9, 11, 12 and 13 we obtain equalities which are represented in Figure 14.

(a) $i\delta E_6^{\,a} = e\phi_o \xi^{*\mu}\sqrt{2}\,\frac{\alpha}{4\pi}\,\Gamma(-a)$ 

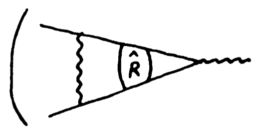
(b) $i\delta E_6^{\,b} = e\phi_o \xi^{*\mu}\sqrt{2}\,\frac{\alpha}{4\pi}\,\Gamma(-a)$ 

Figure 14. Equalities which are correct up to the o(α) correction.

Now, the comparison of the Figure 4 with Figure 14 gives the following relation which is correct up to the o(α) correction

$$\delta E_6 = \frac{\alpha}{2\pi}\,\Gamma(-a)\;\delta E_3 \quad , \qquad (3F.31)$$

and by virtue of the equation (3C.33) for δE_3, we recover the equation (3F.30) for δE_6.

There is also a relationship between δE_6 and δE_5, which can be found, simply, by comparing (3F.30) with (3f.17)

$$\delta E_6 = -\delta E_5 \quad . \qquad (3F.32)$$

G. Contribution δE_7

For calculating δE_7, Figure 2(g), by virtue of the Figure 15

$$i\delta E_7 = 2i\delta E_1$$

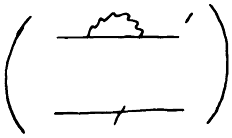


Figure 15. Equality which is the result of comparing Figure 2(a) with Figure 2(g).

we write

$$\delta E_7 = 2\delta E_1 \int \frac{d^4p}{(2\pi)^4} \frac{i}{\kappa} \frac{\partial}{\partial k_o} < -S^{-1}(p - \tfrac{1}{2}k)\bar{\psi}_o^T(p)$$

$$\times\ \Sigma(p + \tfrac{1}{2}k)\psi_o(p) > \Big|_{k_o = 2\kappa} \quad . \qquad (3G.1)$$

Using relations (2.16) and (2.17) for wave-functions and relation (3E.2) for Σ, we obtain

$$\delta E_7 = \frac{-i\alpha^6}{8\kappa\pi^4} \delta E_1 \Gamma(-a) \int d^4p (\kappa + E_p)(\kappa - E_p)^3$$

$$\times\ [(\vec{p}^2 + \gamma^2)^4 (p_o + \kappa - E_p + i\varepsilon)^2 (p_o - \kappa + E_p - i\varepsilon)^2]^{-1}$$

$$+ \frac{-i\alpha^6}{8\kappa\pi^4} \delta E_1 \int d^4p (\kappa + E_p)(\kappa - E_p)^2$$

$$\times\ [E_p (\vec{p}^2 + \gamma^2)^4 (p_o + \kappa - E_p + i\varepsilon)^2 (p_o - \kappa + E_p - i\varepsilon)^2]^{-1}$$

$$\times \frac{\partial}{\partial k_o} \left[(p_o - \tfrac{1}{2}k_o + E_p) F(p_o + \tfrac{1}{2}k_o, \vec{p}^2) \right] \Bigg|_{k_o = 2\kappa} \quad , \tag{3G.2}$$

where

$$F(p_o + \tfrac{1}{2}k_o, \vec{p}^2) = \frac{19}{6}\vec{p}^2 - \tfrac{1}{2}E_p(p_o + \tfrac{1}{2}k_o) - \int_o^1 \frac{dx}{\sqrt{x}} X \ln X$$

$$+ 2\int_o^1 dx \left\{ (1 - x)\left[E_p(p_o + \tfrac{1}{2}k_o) - \vec{p}^2 \right] - 1 \right\} \ln Y$$

$$+ 2\vec{p}^2 \int_o^1 dx \sqrt{x} \int_o^1 du \ln Z \quad , \tag{3G.3}$$

and X, Y and Z are given by (3E.3) (with the obvious replacement of p by $p + \frac{1}{2}k$).

After performing p_o integration (dominant contribution corresponds to the pole at $p_o = \kappa - E_p$) we find

$$\delta E_7 = \frac{-\alpha^6}{16\pi^3} \delta E_1 \ \Gamma(-a) \int d^3p \ \frac{\kappa + E_p}{(\vec{p}^2 + \gamma^2)^4}$$

$$- \frac{\alpha^6}{32\pi^3} \delta E_1 \int d^3p \ \frac{(\kappa + E_p)^2}{E_p(\vec{p}^2 + \gamma^2)^5}$$

$$\times \left[F + (\kappa - E_p)\left(2\frac{\partial F}{\partial k_o} - \frac{\partial F}{\partial p_o}\right) \right] \quad . \tag{3G.4}$$

The second term in square brackets of (3G.4), because of the following identity, is zero

$$\frac{\partial F}{\partial k_o} = \frac{1}{2}\frac{\partial F}{\partial p_o} \quad . \tag{3G.5}$$

Using relation (3G.3), we note that F is of order higher than α^2, i.e., the first term in square brackets also does not contribute. Therefore, only the first integral in (3G.4) survives. Using the approximation $E_p = \kappa = 1$ in this integral, we find

$$\delta E_7 = \frac{-\alpha}{2\pi}\,\Gamma(-a)\;\delta E_1 \quad , \tag{3G.6}$$

or by virtue of relation (3A.5) for δE_1

$$\delta E_7 = \frac{\alpha^4}{4}\left[\frac{-\alpha}{2\pi}\,\Gamma(-a)\right] \quad . \tag{3G.7}$$

H. Summary of the Calculation of the Total Energy-Shift of the Orthopositronium Ground-State due to the One-Photon-Annihilation Channel

Using relations (3.3), (3C.1), (3D.1), (3D.3), (3D.5) and (3F.32), we obtain the total energy-shift of the orthopositronium ground-state

$$\delta E = -\delta E_1 + \delta E_2 + \delta E_3^{\,a} + \delta E_7 \quad , \tag{3H.1}$$

which we represent it graphically in Figure 16.

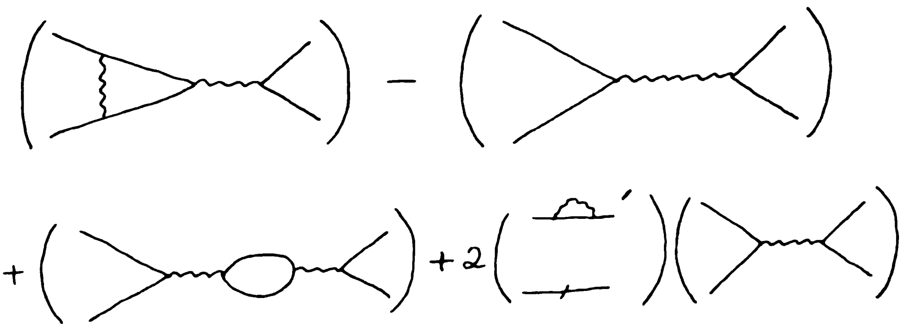
$i\delta E = 2$ 

Figure 16. Graphical representation of the total energy-shift, correct up to the order α correction.

Relation (3H.1), or its graphical representation Figure 16, shows that, up to the $o(\alpha)$ correction the second term of $\hat{G}_c$, $\hat{R}$, does not give any contribution to the energy-shift δE.

Using relation (3H.1), (3A.5), (3B.13), (3C.23) and (3G.7), we obtain

$$\delta E = \frac{\alpha^4}{4}\left(1 - \frac{4\alpha}{\pi} - \frac{8\alpha}{9\pi}\right) \quad , \qquad (3H.2)$$

whcih agrees with the result of Karplus and Klein [12].

We present all results of calculations of the present Chapter in the Table 1.

Table 1

Summary of the Calculation of the Energy-Shift

δE_1	$\frac{\alpha^4}{4}(1 + \frac{\alpha}{3})$
δE_2	$\frac{\alpha^4}{4}(\frac{-8\alpha}{9\pi})$
δE_3	$\frac{\alpha^4}{4}[3 + \alpha - \frac{4\alpha}{\pi} + \frac{\alpha}{2\pi}\Gamma(-a)]$
δE_4	$\frac{\alpha^4}{4}(-3 - \frac{4\alpha}{3})$
δE_5	$\frac{\alpha^4}{4}[\frac{-3\alpha}{2\pi}\Gamma(-a)]$
δE_6	$\frac{\alpha^4}{4}[\frac{3\alpha}{2\pi}\Gamma(-a)]$
δE_7	$\frac{\alpha^4}{4}[\frac{-\alpha}{2\pi}\Gamma(-a)]$
$\delta E = \sum_{i=1}^{7} \delta E_i$	$\frac{\alpha^4}{4}(1 - \frac{4\alpha}{\pi} - \frac{8\alpha}{9\pi})$

Chapter 4

Decay Rate of the Ground-State of Parapositronium in the First-Order Perturbation Theory

For the decay rate we use the perturbed wave-function ψ which can be written [30] as

$$C^{-1}\psi = \psi_o + (\delta K')\psi_o + \hat{G}_c\ \delta K\psi_o + o(\delta K^2) \quad , \qquad (4.1)$$

where $\hat{G}_c$ is given by (2.29) and C is the normalization constant and $\delta K = K - K_c$ is represented graphically in Figure 17.

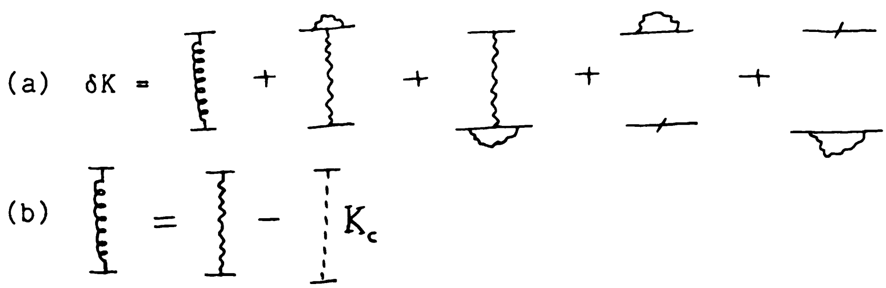


Figure 17. Kernel δK, where the Coulomb-like kernel K_c is given by (2.31). Lines with a dash indicate inverse propagators.

In relation (4.1), ψ_o is the zeroth-order Barbieri-Remiddi wave-function [17], which for the ground-state of parapositronium is given by (2.19), and Prime stands for $\frac{i}{2\kappa}\frac{\partial}{\partial\kappa}$, and $(\ldots)$ means the expectation value with respect to ψ_o (see (3.2)).

The normalization constant C, up to the order α, is [31]

$$C^{-2} = 1 + (\delta K') \quad . \tag{4.2}$$

The only $(\delta K')$ which contributes (up to $o(\alpha)$) is the one that we represent it in Figure 18. We can show that its

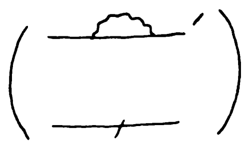
$(\delta K') = 2$ 

Figure 18. The only $(\delta K')$ which contributes, up to the $o(\alpha)$ correction.

value is the same as the $(\delta K')$ for orthopositronium. So by comparing relation (3G.6) with Figure 15, we obtain

$$(\delta K') = \frac{-\alpha}{2\pi}\, \Gamma(-a) \quad . \tag{4.3}$$

Therefore, from relations (4.1)-(4.3), the perturbed wavefunction, up to the $o(\alpha)$ correction, is

$$\psi = C[\psi_o + (\delta K')\psi_o + \hat{G}_c\, \delta K\, \psi_o] \quad , \tag{4.4}$$

where C, the normalization constant, is

$$C = 1 + \frac{\alpha}{4\pi}\, \Gamma(-a) \quad . \tag{4.5}$$

To find the decay rate up to the $o(\alpha)$ correction, we need to consider only the decays to two photons. It is so because for parapositronium, decays to odd number of photons are forbidden and decays to four photons are of the higher orders.

Let's express the decay amplitude as [31]

$$T(p\text{-}Ps \rightarrow 2\gamma) = \frac{1}{(2\omega\ 2\omega'\kappa)^{1/2}} M\ (p\text{-}Ps \rightarrow 2\gamma) \quad , \tag{4.6}$$

where, $\omega = |\vec{k}|$, $\omega' = |\vec{k}'|$ and $\vec{k}$ and $\vec{k}'$ are wave vectors of the two photons; M the invariant decay amplitude is

$$M(p\text{-}Ps \rightarrow 2\gamma) = \int \frac{d^4p}{(2\pi)^4} \langle \eta^T\ \psi(p) \rangle \quad . \tag{4.7}$$

The quantity η is represented graphically by irreducible graphs in the Figure 19, and we assume the mass and charge renormalizations are already performed.

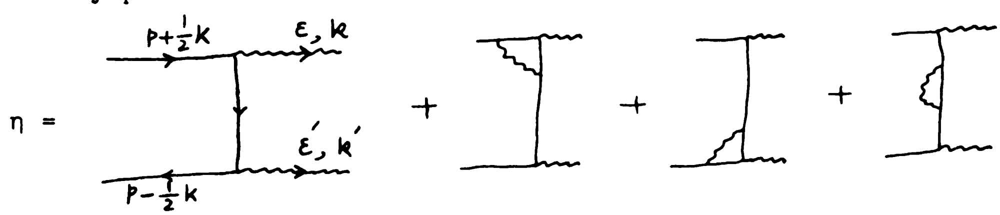


Figure 19. Quantity η, up to the $o(\alpha)$ correction.

In C. M. frame, M can be expressed as

$$M = f\ \hat{k} \cdot (\hat{\varepsilon} \times \hat{\varepsilon}') \quad , \tag{4.8}$$

where f is a constant and $\hat{k}$ is the direction of the decay line and $\hat{\varepsilon}$ and $\hat{\varepsilon}'$ are polarization vectors of the two back to back photons.

The decay rate is

$$\Gamma = \frac{1}{2!}\int \frac{d^3}{(2\pi)^3}\,\frac{d^3\,'}{(2\pi)^3}\,(2\pi)^4\,\delta^4(K - \mathbf{k} - \mathbf{k}')\,\sum_{\varepsilon,\varepsilon'}|T|^2 \quad , \quad (4.9)$$

where, $k = (2\kappa, \vec{o})$.

By virtue of (4.6), (4.8) and (4.9) we obtain

$$\Gamma = \frac{1}{8\pi\kappa}\,|f|^2 \quad , \qquad (4.10)$$

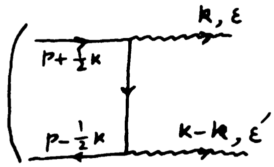
or, up to the order α correction

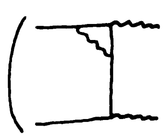
$$\Gamma = \frac{1}{8\pi}\,|f|^2 \quad . \qquad (4.11)$$

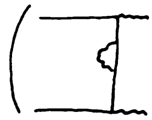
The invariant amplitude M, according to the Figures 17-19 and relations (4.5) and (4.7), can be expressed up to the o(α) correction as

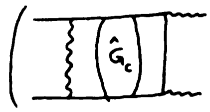
$$M = \sum_{i=1}^{8} M_i \quad , \qquad (4.12)$$

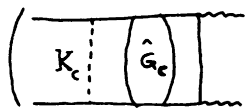
where M_i's are represented graphically in Figure 20.

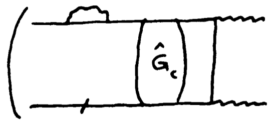
(a) M_1 = 2C 

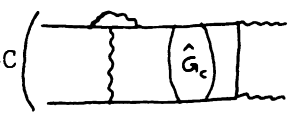
(b) M_2 = 4C 

(c) $M_3 = 2C$ 

(d) $M_4 = 2C$ 

(e) $M_5 = -2C$ 

(f) $M_6 = 4C$ 

(g) $M_7 = 4C$ 

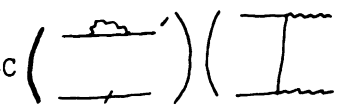
(h) $M_8 = 4C$ 

Figure 20. Contributions to the invariant decay amplitude, where $\hat{G}_c$, K_c and C are given by (2.29), (2.31) and (4.5) respectively. Lines with a dash indicate inverse propagators.

In the following Sections we calculate the contributions M_i's.

A. Contribution M_1

For M_1, Figure 20(a), we write

$$M_1 = 2C\int\frac{d^4p}{(2\pi)^4} < -ie\, \not{\epsilon}'\; i\; S(p + \frac{1}{2}k - \mathbf{k})(-ie\, \not{\epsilon})\psi_o(p) > \quad , \qquad (4A.1)$$

and using (2.19) for ψ_o, after performing p_o integration, we obtain

$$M_1 = \frac{2\alpha^2}{\pi} C\phi_o \hat{\mathbf{k}} \cdot \hat{\varepsilon} \times \hat{\varepsilon}' \int d^3p \frac{\sqrt{E_p + \kappa}}{E_p(\vec{p}^2 + \gamma^2)^2(E_p - \vec{p} \cdot \hat{\mathbf{k}})}$$

$$\times \left[1 + \frac{\kappa - E_p}{(E_p^2 + \kappa^2 - 2\vec{p} \cdot \vec{\mathbf{k}})^{1/2}}\right] \quad . \tag{4A.2}$$

In the square brackets of (4A.2), the first term is due to the pole of wave-function and it gives the $o(1)$ and also gives contribution to the $o(\alpha)$ correction; the second term is due to the pole in propagator and it gives contribution to the $o(\alpha)$ correction. Therefore, for the second term we may use the approximation $\kappa = 1$ (since for this term only non-small values of $\vec{p}$ contributes).

After performing angular integrations, (4A.2) gets the following form

$$M_1 = 8C\,\alpha^2\phi_o \hat{\mathbf{k}} \cdot \hat{\varepsilon} \times \hat{\varepsilon}' \left\{\int_0^\infty dp \frac{p\sqrt{E_p + \kappa}}{(\vec{p}^2 + \gamma^2)^2 E_p} \ln (E_p + p)\right.$$

$$\left. - \int_0^\infty dp \frac{\sqrt{E_p + 1}}{p^3 E_p} \ln \left[\frac{(E_p + p)(E_p - 1 + \sqrt{\vec{p}^2 + 2 - 2p})}{E_p - 1 + \sqrt{\vec{p}^2 + 2 + 2p}}\right]\right\} \quad . \tag{4A.3}$$

The first integral is performed in Appendix D. For the second integral, we found it by numerical integration. The results are

$$\text{first integral} = \sqrt{2}\left(\frac{\pi}{2\alpha} + G - 2\ln 2 - \frac{3\pi^2}{16}\right) \quad , \tag{4A.4}$$

$$\text{second integral} = \sqrt{2}\,\lambda \quad , \qquad (4A.5)$$

where, $G = .91596559\ldots$ is the Catalan Constant and $\lambda = .49172096\ldots$.

Therefore using (4.5) for C, contribution M_1 up to the $o(\alpha)$ correction is

$$M_1 = M_o\left\{1 + \frac{\alpha}{4\pi}\left[\Gamma(-a) + 8G - 8\lambda - 16\ln 2 - \frac{3\pi^2}{2}\right]\right\} \quad , \qquad (4A.6)$$

where

$$M_0 = 4\sqrt{2}\,\pi\,\alpha\,\phi_o\,\hat{\mathbf{k}}\cdot\hat{\varepsilon}\times\hat{\varepsilon}' \quad , \qquad (4A.7)$$

is the lowest order invariant amplitude.

B. Contribution M_2

For M_2, Figure 20(b), we write

$$M_2 = 4C\int\frac{d^4p}{(2\pi)^4} < -ie\,\boldsymbol{\varepsilon}'\; i\, S(p + \tfrac{1}{2}k - \mathbf{k})(-ie)$$

$$\times\, \Lambda_i(p + \tfrac{1}{2}k - \mathbf{k},\ p + \tfrac{1}{2}k)\varepsilon^i\psi_o(p)> \ (i = 1, 2, 3) \quad , \qquad (4B.1)$$

where momenta p, k and $\mathbf{k}$, and polarization vectors ε and ε' are shown in Figure 21.

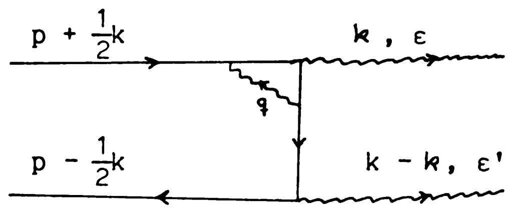


Figure 21. Diagram related to M_2, where $k = (2\kappa, \vec{o})$, $\boldsymbol{k} = (\kappa, \vec{\boldsymbol{k}})$ and $\boldsymbol{k}^2 = o$.

After performing p_o integration (dominant contribution comes from the pole in the wave-function, or for $p_o = \kappa - E_p$) and scaling $\vec{p} \to \gamma\vec{p}$ we note that the dominant contribution comes from $\vec{p}$ of order γ; therefore we neglect p wherever it is possible. We find up to the $o(\alpha)$ correction

$$M_2 = \frac{i}{\pi}\alpha^2\phi_o\sqrt{2}\int\frac{d^3p}{(\vec{p}^2 + \gamma^2)^2} < \not{\varepsilon}'(1 + \gamma_o - \not{\boldsymbol{k}})$$

$$\times\ \Lambda_i(p + \frac{1}{2}k - \boldsymbol{k},\ p + \frac{1}{2}k)\varepsilon_i(1 + \gamma_o)\gamma_5> \quad , \qquad (4B.2)$$

where we have used (4.5) for C.

Since we need only the leading order of Λ_i, we neglect p in Λ_i and perform the rest of the integration in (4B.2). We find

$$M_2 = i\pi\alpha\phi_o\sqrt{2}\ \varepsilon_i<\not{\varepsilon}'[(1 + \gamma_o) + \vec{\boldsymbol{k}}\cdot\vec{\gamma}(1 - \gamma_o)]$$

$$\times\ \Lambda_i\ (\frac{1}{2}k - \boldsymbol{k},\ \frac{1}{2}k)(1 + \gamma_o)\gamma_5> \quad , \qquad (4B.3)$$

where we introduced another factor of $(1 + \gamma_o)$ in order to simplify the calculation of the trace.

Λ_i can be found by the following relation

$$\Lambda_i = \frac{i\alpha}{4\pi^3} \int \frac{d^n q}{D} F^{\mu\nu} \gamma_\nu (\not{q} + \vec{k} \cdot \vec{\gamma} + 1) \gamma_i (\not{q} + \gamma_o + 1) \gamma_\mu \quad , \qquad (4B.4)$$

where

$$D = \vec{q}^{\,2} \, q^2 [(q + \tfrac{1}{2}k)^2 - 1][(q + \tfrac{1}{2}k - k)^2 - 1] \quad , \qquad (4B.5)$$

$$F^{\mu\nu} = -g^{\mu\nu} \vec{q}^{\,2} - q^\mu q^\nu + q^o (q^\mu \delta_o^\nu + q^\nu \delta_o^\mu) \quad . \qquad (4B.6)$$

Let's separate the term which gives the ultraviolet divergence from those which give finite contribution

$$\Lambda_i = \frac{i\alpha}{4\pi^3} \int \frac{d^n q}{D} F^{\mu\nu} \gamma_\nu \not{q} \gamma_i \not{q} \gamma_\mu$$

$$+ \frac{i\alpha}{4\pi^3} \int \frac{d^4 q}{D} [F^{\mu\nu} \gamma_\nu \not{q} \gamma_i (1 + \gamma_o) \gamma_\mu$$

$$+ F^{\mu\nu} \gamma_\nu (1 + \vec{k} \cdot \vec{\gamma}) \gamma_i (\not{q} + 1 + \gamma_o) \gamma_\mu] \quad , \qquad (4B.7)$$

where only the first integral gives ultraviolet divergence.

Using (4B.7) in (4B.3) we obtain

$$M_2 = -\frac{\sqrt{2}}{4\pi^2}\,\alpha^2\phi_o\varepsilon_i\left[\int\frac{d^nq}{D}\langle\rangle_n + \int\frac{d^4q}{D}\langle\rangle_4\right] \quad , \qquad (4B.8)$$

where, effectively

$$\langle\rangle_n \equiv \langle 2\vec{k}\cdot\vec{\gamma}[q_o^2q^2\gamma_i + (3-n)\vec{q}^2q^2\gamma_i$$

$$+ 2(2-n)q_i\,\vec{q}^2\,\vec{q}\cdot\vec{\gamma}]\gamma_o\gamma_5\not{\varepsilon}'\rangle \quad , \qquad (4B.9)$$

$$\langle\rangle_4 \equiv \langle 4\gamma_i(q_o\,\vec{q}^2\,\vec{k}\cdot\vec{\gamma} - q_o\,\vec{q}\cdot\vec{k}\,\vec{q}\cdot\vec{\gamma}$$

$$- q_o\,q^2\,\vec{k}\cdot\vec{\gamma})\gamma_o\gamma_5\not{\varepsilon}'\rangle \quad . \qquad (4B.10)$$

To find M_2, relation (4B.8), we need to perform the q integration. This can be done by using the table which is provided in [19]. We find

$$\int d^nq\,\frac{q_iq_j\vec{q}^2}{D} = i\pi^2 g_{ij}\left[\frac{1}{4}\,\Gamma(-a) + \frac{3}{4} - \frac{\pi^2}{16}\right]$$

+ term proportional to $k_i\ k_j$;
so gives zero contribution to
the trace of (4B.9) , (4B.11)

$$\int d^nq\,\frac{\vec{q}^2q^2}{D} = i\pi^2\,\Gamma(-a) \quad , \qquad (4B.12)$$

$$\int d^4q\,\frac{q_o\,\vec{q}^2}{D} = i\pi^2(1 - \ln 2) \quad . \qquad (4B.13)$$

The evaluation of these three integrals was easy, so we gave only the final results. However, the evaluation of the rest of the integrals that we need is not straight forward. In Appendix F, we have shown how to perform these integrals. The results are (see relations (F.11), (F.14) and (F.39))

$$\int d^n q \, \frac{q_o^2 \, q^2}{D} = i\pi^2[-\Gamma(-a) - 8 + \frac{\pi^2}{2} + 4\sqrt{2}\ \ln\ (1 + \sqrt{2})$$

$$-4\ \ln^2\ (1 + \sqrt{2})] \quad , \tag{4B.14}$$

$$\int d^4 q \, \frac{q_o \, q^2}{D} = i\pi^2[-\frac{\pi^2}{4} + 2\ \ln^2(1 + \sqrt{2})] \quad , \tag{4B.15}$$

$$\int d^4 q \, \frac{q_o q_j q_\ell}{D} = i\pi^2 \, k_j k_\ell [\frac{11}{2} - \frac{3\pi^2}{16} + 2\ \ln\ 2 - \frac{9\sqrt{2}}{2}\ \ln\ (1 + \sqrt{2})$$

$$+ \frac{3}{4}\ \ln^2(1 + \sqrt{2})]$$

$$+ i\pi^2 \delta_{j\ell}[-\frac{3}{2} + \frac{\pi^2}{16} - \ln\ 2 + \frac{3\sqrt{2}}{2}\ \ln\ (1 + \sqrt{2})$$

$$-\frac{1}{4}\ \ln^2(1 + \sqrt{2})] \quad . \tag{4B.16}$$

Now, by virtue of (4B.8) - (4B.16) we find

$$M_2 = M_o \frac{\alpha}{4\pi} [2\ \Gamma(-a) + \pi^2 - 4 - 8\ \ln\ 2 + 4\sqrt{2}\ \ln\ (1 + \sqrt{2})$$

$$-2\ln^2(1 + \sqrt{2})\Big] \quad , \qquad (4B.17)$$

where M_o is given by (4A.7).

C. Contribution M_3

For M_3, Figure 20(c), we write

$$M_3 = 2C\int\frac{d^4p}{(2\pi)^4} \langle -ie\,\not{\epsilon}'\; i\,S(p + \tfrac{1}{2}k - \mathbf{k})(-i)\Sigma(p + \tfrac{1}{2}k - \mathbf{k})$$

$$\times\; i\,S(p + \tfrac{1}{2}k - \mathbf{k})(-ie\,\not{\epsilon})\psi_o(p)\rangle \quad , \qquad (4C.1)$$

where the momenta p, k and $\mathbf{k}$, and polarization vectors ε and ε' are shown in Figure 22.

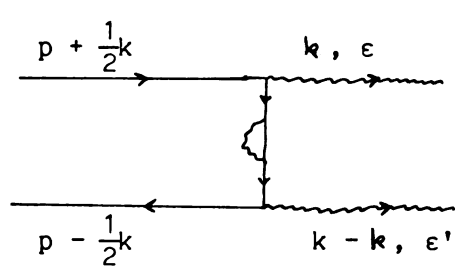


Figure 22. Diagram related to M_3, where $k = (2\kappa, \vec{o})$ and $\mathbf{k} = (\kappa, \vec{\mathbf{k}})$.

After performing p_o integration (the dominant contribution comes from the pole in the wave-function, or for $p_o = \kappa - E_p$) and scaling $\vec{p} \to \gamma\vec{p}$, we notice that the leading contribution to M_3 comes from $\vec{p}$ of the order γ. Therefore we neglect $\vec{p}$ and replace κ by 1 everywhere except for the term $\frac{1}{\vec{p}^2 + \gamma^2}$. We find

$$M_3 = \frac{i}{4\pi}\sqrt{2}\,\phi_o\,\alpha^2 < \not{\epsilon}'\,(1 + \vec{k}\cdot\vec{\gamma})\Sigma(\tfrac{1}{2}k - k)(1 + \vec{k}\cdot\vec{\gamma})$$

$$\times\ \not{\epsilon}(1 + \gamma_o)\gamma_5 > \int\frac{d^3p}{(\vec{p}^2 + \gamma^2)^2} \quad . \tag{4C.2}$$

By performing p integration and using relation (3E.2) for Σ and finding the trace, we obtain

$$M_3 = M_o\,\frac{\alpha}{4\pi}\left[-\Gamma(-a) + \int_0^1 \frac{dx}{\sqrt{x}}\ln X + 2\int_0^1 dx\,\ln Y\right] \quad , \tag{4C.3}$$

where M_o is given in (4A.7), X and Y are given in (3E.3), which in present case they are

$$X = Y = 2 - x \quad . \tag{4C.4}$$

Using (4C.4) in (4C.3), we obtain (up to the order α correction)

$$M_3 = M_o\,\frac{\alpha}{4\pi}\left[-\Gamma(-a) - 6 + 4\ln 2 + 4\sqrt{2}\ln(1 + \sqrt{2})\right] \quad . \tag{4C.5}$$

D. Contribution M_4

For M_4, Figure 20(d), we use the same decomposition that we used for δE_3, relation (3C.1), or

$$M_4 = M_4{}^a + M_4{}^b \quad , \qquad (4D.1)$$

where $M_4{}^a$ and $M_4{}^b$ are represented in Figure 23.

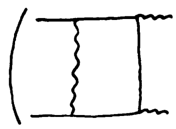
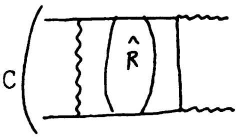
(a) $M_4{}^a = 2C$  , (b) $M_4{}^b = 2C$ 

Figure 23. Contributions $M_4{}^a$ and $M_4{}^b$.

Contribution $M_4{}^a$ can be written as

$$M_4{}^a = 32\pi^2\alpha^2 C\int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4} D^{\mu\nu}(q - p)$$

$$\times \langle \gamma_\nu S(q - \tfrac{1}{2}k)\not{\varepsilon}' S(q + \tfrac{1}{2}k - \boldsymbol{k})\not{\varepsilon} S(q + \tfrac{1}{2}k)\gamma_\mu \psi_o(p)\rangle \quad , \ (4D.2)$$

where momenta p, q, k and $\boldsymbol{k}$, and polarization vectors ε and ε' are shown in Figure 24, and ψ_o, $D^{\mu\nu}$ and C are given by the relations (2.19), (3C.24) and (4.5) respectively.

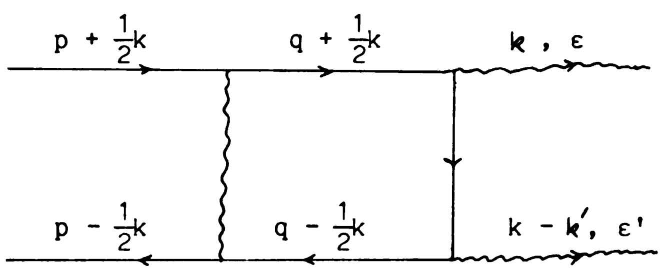


Figure 24. Diagram related to $M_4{}^a$.

After performing p_o integration (up to the $o(\alpha)$ correction, only the pole in wave-function contributes, or for $p_o = \kappa - E_p$) and scaling $\vec{p} \to \gamma\vec{p}$, we note that the o(1) and o(α) corrections both come from the small values of $\vec{p}$, or $\vec{p}$ of order γ.

By neglecting $\vec{p}^{\,2}$, γ^2 and higher orders in $\vec{p}$ and γ, wherever it is possible, we obtain (see Appendix E)

$$M_4{}^a = M_o\, C\, \frac{i\alpha^2}{4\pi^5} \int \frac{d^3p}{(\vec{p}^{\,2} + \gamma^2)^2} \int \frac{d^4q}{D} \left[\frac{4}{(\vec{q} - \vec{p})^2} + 1 - \frac{q_o{}^2}{\vec{q}^{\,2}} - \frac{4\, q_o{}^2}{q^2} - \frac{q_i}{\kappa_i} - \frac{q_o{}^2\, q_i}{\vec{q}^{\,2}\, \kappa_i}\right] \quad , \tag{4D.3}$$

where M_o is given in (4A.7) and i = 1, 2, 3 (no summation on i), and

$$D = [(q - \tfrac{1}{2}k)^2 - 1][(q + \tfrac{1}{2}k)^2 - 1][(q + \tfrac{1}{2}k - \kappa)^2 - 1] \quad . \tag{4D.4}$$

We should mention that in deriving (4D.3) we also neglected terms of the form $\vec{p} \cdot \vec{q}$. We note that in square brackets of (4D.3) the first term gives o(1) and o(α) corrections and the other terms give only o(α) correction which comes from non-small values of q; the o(1) comes from small q (of order γ). Therefore $\vec{p} \cdot \vec{q}$ in numerators of all terms, gives the o(α^2) correction which we are not interested, so we can neglect it.

For finding $M_4{}^a$ from (4D.3), we do not use any more approximation with respect to p and q. The only approximation that we use, wherever possible, is related to γ and κ.

Performing the p integration in (4D.3), we obtain

$$M_4^{\,a} = \frac{i\alpha}{2\pi^3}\, C\, M_o \int \frac{d^4q}{D}\Big[\frac{4}{\vec{q}^{\,2} + \gamma^2} + 1 - \frac{q_o^{\,2}}{\vec{q}^{\,2}} - \frac{4q_o^{\,2}}{q^2}$$

$$- \frac{q_i}{k_i} - \frac{q_o^{\,2}\, q_i}{\vec{q}^{\,2}\, k_i}\Big] \quad . \qquad (4D.5)$$

These integrals are performed in Appendix G. Using the results (G.2), (G.10), (G.26) and (G.38) of the Appendix G, we obtain

$$M_4^{\,a} = M_o\Big\{1 + \frac{\alpha}{4\pi}\Big[\Gamma(-a) + 4 \ln 2 - \frac{\pi^2}{2} + 2 \ln^2 (1 + \sqrt{2})$$

$$- 8\sqrt{2}\, \ln (1 + \sqrt{2})\Big]\Big\} \quad . \qquad (4D.6)$$

For $M_4^{\,b}$, Figure 23(b), we write

$$M_4^{\,b} = -2iC\pi^2\alpha^2 \int \frac{d^4p}{(2\pi)^4}\, \frac{d^4q_1}{(2\pi)^4}\, \frac{d^4q}{(2\pi)^4}\, R(\kappa, \vec{q})R(\kappa, \vec{q}_1)$$

$$\times \frac{\hat{H}_c\,(\kappa, q, q_1)}{E_q E_{q_1} N_q N_{q_1}}\, D^{\mu\nu}(p - q_1)\, \langle\rangle \quad , \qquad (4D.7)$$

where

$$\langle\rangle \equiv \langle\gamma_\nu S(q_1 - \tfrac{1}{2}k)(N_{q_1} + \vec{q}_1 \cdot \vec{\gamma})(1 - \gamma_o)(N_q + \vec{q} \cdot \vec{\gamma})$$

$$\times\ S(q - \tfrac{1}{2}k)\not{\epsilon}'\ S(q + \tfrac{1}{2}k - \boldsymbol{k})\not{\epsilon}\ S(q + \tfrac{1}{2}k)(N_q + \vec{q} \cdot \vec{\gamma})$$

$$\times\ (1 + \gamma_0)(N_{q_1} + \vec{q}_1 \cdot \vec{\gamma})S(q_1 + \tfrac{1}{2}k)\gamma_\mu \psi_0(p)> \quad . \qquad (4D.8)$$

In relation (4D.7), the momenta and polarization vectors are shown in Figure 24; $D^{\mu\nu}$, $R(\kappa, \vec{q})$, ψ_0 and $\hat{H}_c$ are given by the relations (3C.24), (2.6), (2.19) and (2.27) respectively.

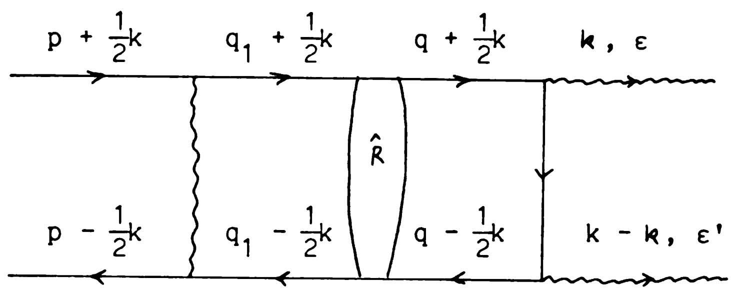


Figure 25. Diagram related to $M_4{}^b$.

Calculation of $M_4{}^b$ is similar to the $\delta E_3{}^b$ (Section C of Chapter 3). The only difference (apart from the wave-function and presence of polarization vectors) is the presence of the propagator $S(q + \frac{1}{2}k - \boldsymbol{k})$ which its pole at $q_0 = -(E_q{}^2 + \kappa^2 - 2\vec{q} \cdot \vec{\boldsymbol{k}})^{1/2}$ gives contribution of the $o(\alpha)$.

In fact in (4D.7) by using the following relations

$$(1 - \gamma_0)(N_q + \vec{q} \cdot \vec{\gamma})S(q - \tfrac{1}{2}k) = \frac{-(1 - \gamma_0)(N_q - \vec{q} \cdot \vec{\gamma})}{q_0 - \kappa + E_q - i\varepsilon} \quad , \qquad (4D.9a)$$

$$\times\ S(q - \tfrac{1}{2}k)\not{\varepsilon}'\ S(q + \tfrac{1}{2}k - \mathbf{k})\not{\varepsilon}\ S(q + \tfrac{1}{2}k)(N_q + \vec{q}\cdot\vec{\gamma})$$

$$\times\ (1 + \gamma_o)(N_{q_1} + \vec{q}_1\cdot\vec{\gamma})S(q_1 + \tfrac{1}{2}k)\gamma_\mu\psi_o(p)\rangle \quad . \qquad (4D.8)$$

In relation (4D.7), the momenta and polarization vectors are shown in Figure 24; $D^{\mu\nu}$, $R(\kappa, \vec{q})$, ψ_o and $\hat{H}_c$ are given by the relations (3C.24), (2.6), (2.19) and (2.27) respectively.

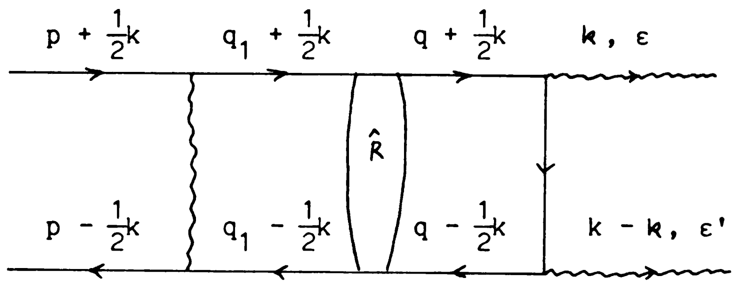


Figure 25. Diagram related to $M_4^{\ b}$.

Calculation of $M_4^{\ b}$ is similar to the $\delta E_3^{\ b}$ (Section C of Chapter 3). The only difference (apart from the wave-function and presence of polarization vectors) is the presence of the propagator $S(q + \frac{1}{2}k - \mathbf{k})$ which its pole at $q_o = -(E_q^{\ 2} + \kappa^2 - 2\vec{q}\cdot\vec{\mathbf{k}})^{1/2}$ gives contribution of the $o(\alpha)$.

In fact in (4D.7) by using the following relations

$$(1 - \gamma_o)(N_q + \vec{q}\cdot\vec{\gamma})S(q - \tfrac{1}{2}k) = \frac{-(1 - \gamma_o)(N_q - \vec{q}\cdot\vec{\gamma})}{q_o - \kappa + E_q - i\varepsilon} \quad , \qquad (4D.9a)$$

$$S(q + \frac{1}{2}k)(N_q + \vec{q} \cdot \vec{\gamma})(1 + \gamma_o) = \frac{(N_q - \vec{q} \cdot \vec{\gamma})(1 + \gamma_o)}{q_o + \kappa - E_q + i\varepsilon} \quad , \qquad (4D.9b)$$

$$(1 + \gamma_o)(N_q + \vec{q} \cdot \vec{\gamma})S(q + \frac{1}{2}k) = \frac{(1 + \gamma_o)(N_q - \vec{q} \cdot \vec{\gamma})}{q_o + \kappa - E_q + i\varepsilon} \quad , \qquad (4D.9c)$$

$$S(q - \frac{1}{2}k)(N_q + \vec{q} \cdot \vec{\gamma})(1 - \gamma_o) = \frac{-(N_q - \vec{q} \cdot \vec{\gamma})(1 - \gamma_o)}{q_o - \kappa + E_q - i\varepsilon} \quad , \qquad (4D.9d)$$

and after performing p_o, q_{10} and q_o integrations, we obtain ($\vec{p}$, $\vec{q}_1 \sim \gamma$)

$$M_4{}^b = \frac{\alpha^3}{8\pi^6} C\phi_o \hat{\mathbf{k}} \cdot \hat{\varepsilon} \times \hat{\varepsilon}' \int d^3p d^3q_1 d^3q \sqrt{E_q + \kappa}$$

$$\times \left[(\vec{p}^2 + \gamma^2)^2 (\vec{p} - \vec{q}_1)^2 (\vec{q}_1^2 + \gamma^2)(\vec{q}^2 + \gamma^2)(E_q - \vec{q} \cdot \hat{\mathbf{k}})E_q\right]^{-1}$$

$$\times \left[\bar{H}_c(\kappa, \vec{q}, \vec{q}_1) + \bar{H}_c'(\kappa, \vec{q}, \vec{q}_1) \frac{\kappa - E_q}{(E_q^2 + \kappa^2 - 2\vec{q} \cdot \vec{\mathbf{k}})^{1/2}}\right] \quad . \qquad (4D.10)$$

In deriving (4D.10), we used these observations: for p_o and q_{10} only poles at $p_o = \kappa - E_p$ and $q_{10} = \kappa - E_{q_1}$ contribute, which means that only D^{oo}, the instantaneous Coulomb interaction of the photon propagator $D^{\mu\nu}$ contributes; for q_o, both poles at $q_o = \kappa - E_q$ and $q_o = -(E_q{}^2 + \kappa^2 - 2\,\vec{q} \cdot \vec{\mathbf{k}})^{1/2}$ contribute.

In relation (4D.10), $\bar{H}_c'$ is

$$\bar{H}_c' = \bar{H}_c + \frac{\gamma^2(2 + \kappa + E_q)}{\vec{q}^2 + \gamma^2} \quad , \qquad (4D.11)$$

where $\bar{H}_c$ is given by (3C.27). $\bar{H}_c$ is related to the pole at $q_o = \kappa - E_q$, and $\bar{H}_c'$ is related to the pole at $q_o = -(E_q^2 + \kappa^2 - 2\vec{q} \cdot \vec{k})^{1/2}$.

In (4D.10), the first term in the second square brackets gives the o(1) and o(α) corrections. The o(1) comes from $\vec{q} \sim \gamma$ and o(α) comes from all values of $\vec{q}$. The second term gives only contributions to the o(α) correction which comes from non-small values of $\vec{q}$. Therefore in (4D.10) we may use the following approximate relation

$$\bar{H}_c'(\kappa, \vec{q}, \vec{q}_1) = \bar{H}_c(\kappa, \vec{q}, \vec{q}_1) \qquad (4D.12)$$

which is correct up to the o(α) correction.

Relation (4D.10), after performing p integration and using (4D.12), gets the following form

$$M_4{}^b = \frac{\alpha^2}{4\pi^4} C\phi_o \hat{k} \cdot \hat{\varepsilon} \times \hat{\varepsilon}' \int d^3q_1 d^3q \sqrt{E_q + \kappa}\, \bar{H}_c(\kappa, \vec{q}, \vec{q}_1)$$

$$\times \left[(\vec{q}_1^2 + \gamma^2)^2(\vec{q} + \gamma^2)(E_q - \vec{q} \cdot \hat{k})E_q\right]^{-1}$$

$$\times \left[1 + \frac{\kappa - E_q}{(E_q^2 + \kappa^2 - 2\vec{q} \cdot \vec{k})^{1/2}}\right] \quad . \qquad (4D.13)$$

By virtue of (3C.27) for $\bar{H}_c$, and using (3C.28) and then performing q_1 integration, we obtain

$$M_4{}^b = \frac{2\alpha^2}{\pi} C\phi_o \hat{\kappa} \cdot \hat{\varepsilon} \times \hat{\varepsilon}' \int d^3q \frac{1}{(\vec{q}^2 + \gamma^2)^2} \{\frac{3}{2} \frac{\sqrt{E_q + \kappa}}{E_q(E_q - \vec{q} \cdot \hat{\kappa})}$$

$$\times [1 + \frac{\kappa - E_q}{(E_q{}^2 + \kappa^2 - 2\vec{q} \cdot \vec{\kappa})^{1/2}}] - \frac{\alpha^2 \sqrt{2}}{\vec{q}^2 + \gamma^2}\} \quad . \qquad (4D.14)$$

Performing the q integration on the last term and comparing the integral of other terms with equation (4A.2) (therefore using the result (4A.6)), we find

$$M_4{}^b = M_o\{\frac{1}{2} + \frac{3\alpha}{8\pi} [\frac{1}{3} \Gamma(-a) + 8G - 8\lambda - 16 \ln 2 - \frac{3\pi^2}{2}]\} \quad . \qquad (4D.15)$$

Finally, (4D.1), (4D.6) and (4D.15) give

$$M_4 = M_o\{\frac{3}{2} + \frac{\alpha}{4\pi} [\frac{3}{2} \Gamma(-a) + 12G - 12\lambda - 20 \ln 2 - \frac{11\pi^2}{4}$$

$$-8\sqrt{2} \ln(1 + \sqrt{2}) + 2 \ln^2(1 + \sqrt{2})]\} \quad . \qquad (4D.16)$$

E. Contribution M_5

For M_5, Figure 20(e), we use the same decomposition that we used for δE_3, relation (3C.1), or

$$M_5 = M_5^{\,a} + M_5^{\,b} \quad , \tag{4E.1}$$

where $M_5^{\,a}$ and $M_5^{\,b}$ are represented in Figure 26.

(a) $M_5^{\,a} = -2C$

(b) $M_5^{\,b} = -2C$

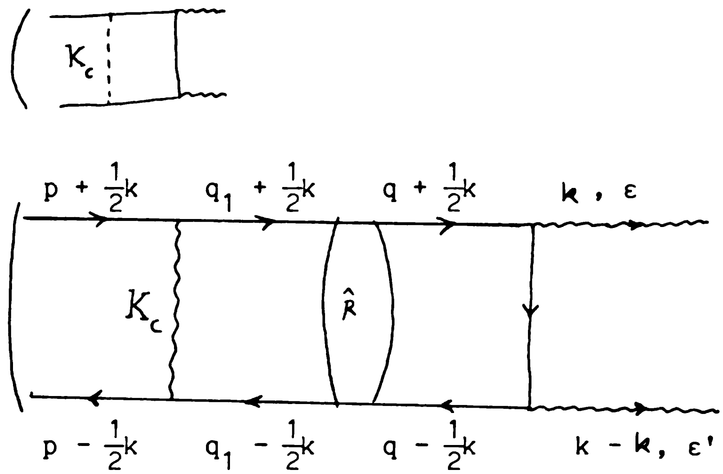


Figure 26. Contributions $M_5^{\,a}$ and $M_5^{\,b}$.

As for the case of $\delta E_4^{\,a}$, by virtue of the relation (3D.2) we find the identity which is given in Figure 27.

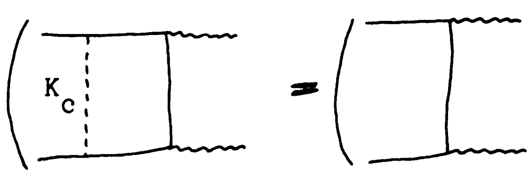


Figure 27. The identity which is the direct result of the relation (3D.2).

Therefore, we can write

$$M_5^{\,a} = -\,M_1 \quad , \tag{4E.2}$$

or from (4A.6) for M_1, we obtain

$$M_5{}^a = - M_o\{1 + \frac{\alpha}{4\pi}\left[\Gamma(-a) + 8G - 8\lambda - 16 \ln 2 - \frac{3\pi^2}{2}\right]\} \quad . \qquad (4E.3)$$

For $M_5{}^b$, Figure 26(b), calculation is very similar to the $M_4{}^b$. In fact, in $M_4{}^b$ by replacing $iD^{\mu\nu}(p - q_1)(-ie\gamma_\mu)^{(1)}(-ie\gamma_\nu)^{(2)}$ by $K_c(\kappa, \vec{q}_1, \vec{p})$, we obtain an expression for $(-)M_5{}^b$; where K_c is given by (2.31), and (1) stands for electron line and (2) for positron line. The (-) comes from the definations of $M_4{}^b$ and $M_5{}^b$ (compare Figure 23(b) with Figure 26(b).)

In the following, by using this replacement, we find $M_5{}^b$ from $M_4{}^b$.

As we noticed in the course of calculation of $M_4{}^b$, $\vec{p}$ and $\vec{q}_1$ were small ($\vec{p}$, $\vec{q}_1 \sim \gamma$) and only $\nu = \mu = o$ contributed. Therefore, in deriving (4D.10) from (4D.8) we used the following approximate relation

$$iD^{\mu\nu}(p - q_1)(-ie\gamma_\mu)^{(1)}(-ie\gamma_\nu)^{(2)} =$$

$$iD^{oo}(p - q_1)(-ie\gamma_o)^{(1)}(-ie\gamma_o)^{(2)} \quad , \qquad (4E.4)$$

which by virtue of (3C.24) for $D^{\mu\nu}$, is

$$iD^{\mu\nu}(p - q_1)(-ie\gamma_\mu)^{(1)}(-ie\gamma_\nu)^{(2)} = \frac{-4\pi i\alpha}{(\vec{p} - \vec{q}_1)^2}\gamma_o{}^{(1)}\gamma_o{}^{(2)} \quad . \qquad (4E.5)$$

Relation (2.31) for K_c, for small $\vec{p}$ and $\vec{q}_1$ gives

$$K_c(\kappa, \vec{q}_1, \vec{p}) = \frac{4\pi i\alpha}{(\vec{p} - \vec{q}_1)^2} \left(\frac{1 + \gamma_o}{2}\right)^{(1)} \left(\frac{1 - \gamma_o}{2}\right)^{(2)} \quad . \qquad (4E.6)$$

Relation (4D.8), for small $\vec{p}$ and $\vec{q}$ is (see relation (4D.7) for $M_4{}^b$)

$$\langle\rangle_{M_4}b = \langle \gamma_\nu{}^{(2)}(\ldots)\gamma_\mu{}^{(1)}(1 + \gamma_o)\gamma_5 \rangle \quad , \qquad (4E.7)$$

where $(1 + \gamma_o)\gamma_5$ comes from ψ_o, and we use subscript $M_4{}^b$ for $\langle\rangle$ to indicate that this trace is related to the $M_4{}^b$ (note that only $\nu = \mu = o$ contribute).

Let's write (4E.7) in the following form

$$\langle\rangle_{M_4}b = \langle(\ldots)\gamma_\mu{}^{(1)}(1 + \gamma_o)\gamma_5\gamma_\nu{}^{(2)} \rangle \quad . \qquad (4E.8)$$

To find $\langle\rangle_{M_5}b$, according to the results (4E.5) and (4E.6), in (4E.8) we replace $\gamma_o{}^{(1)}\gamma_o{}^{(2)}$ by $(-)\left(\frac{1 + \gamma_o}{2}\right)^{(1)} \left(\frac{1 - \gamma_o}{2}\right)^{(2)}$.

Now, using this replacement and following relations

$$\gamma_o{}^{(1)}(1 + \gamma_o)\gamma_5(\gamma_o)^{(2)} = \gamma_o(1 + \gamma_o)\gamma_5\gamma_o = -(1 + \gamma_o)\gamma_5 \quad , \qquad (4E.9)$$

$$(-)\left(\frac{1 + \gamma_o}{2}\right)^{(1)} (1 + \gamma_o)\gamma_5\left(\frac{1 - \gamma_o}{2}\right)^{(2)} = -\left(\frac{1 + \gamma_o}{2}\right)(1 + \gamma_o)\gamma_5\left(\frac{1 - \gamma_o}{2}\right)$$

$$= -(1 + \gamma_o)\gamma_5 \quad , \qquad (4E.10)$$

we infer the value of $\langle\rangle_{M_5}{}^b$

$$\langle\rangle_{M_5}{}^b = \langle\rangle_{M_4}{}^b \quad . \tag{4E.11}$$

Therefore, up to the o(α) correction

$$M_5{}^b = -M_4{}^b \quad , \tag{4E.12}$$

or by virtue of the relation (4D.15) for $M_4{}^b$

$$M_5{}^b = -M_o\Big\{\frac{1}{2} + \frac{\alpha}{4\pi}\Big[\frac{1}{2}\,\Gamma(-a) + 12G - 12\lambda - 24\ln 2 - \frac{9\pi^2}{4}\Big]\Big\} \quad . \tag{4E.13}$$

Finally, (4E.1), (4E.3) and (4E.13) give

$$M_5 = M_o\Big\{-\frac{3}{2} + \frac{\alpha}{4\pi}\Big[-\frac{3}{2}\,\Gamma(-a) - 20G + 20\lambda + 40\ln 2 + \frac{15\pi^2}{4}\Big]\Big\} \quad . \tag{4E.14}$$

F. Contribution M_6

For M_6, Figure 20(f), we use the following decomposition

$$M_6 = M_6{}^a + M_6{}^b \quad , \tag{4F.1}$$

where $M_6{}^a$ and $M_6{}^b$ are represented in Figure 28.

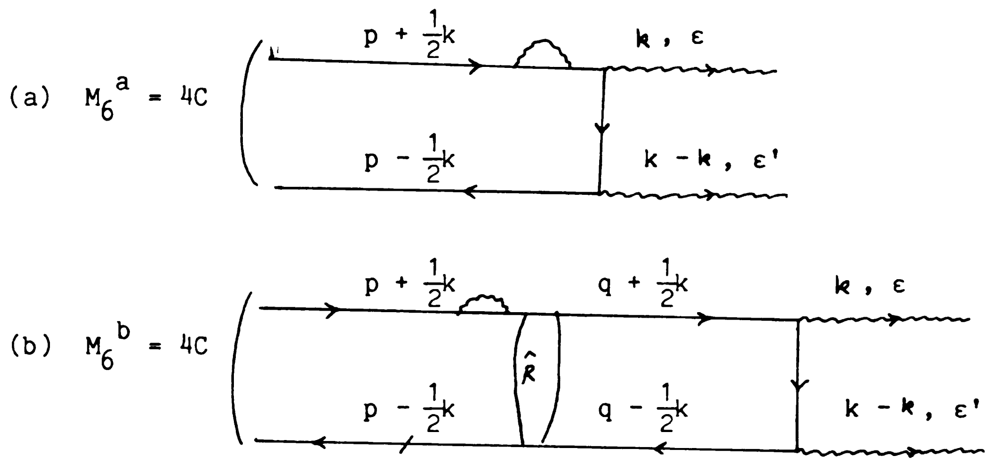


Figure 28. Contributions $M_6{}^a$ and $M_6{}^b$, where $\hat{R}$ and C are given by (2.30) and (4.5) respectively. Line with a dash indicates inverse propagator.

Calculations of these diagrams are similar to δE_5 (see Section E of Chapter 3).

For $M_6{}^a$, Figure 28(a), we write

$$M_6{}^a = -\,16\, i\pi\alpha C \int \frac{d^4p}{(2\pi)^4} < \not{\varepsilon}'\, S(p + \tfrac{1}{2}k - \mathbf{k})\not{\varepsilon}\, S(p + \tfrac{1}{2}k)$$

$$\times\ \Sigma(p + \tfrac{1}{2}k)\psi_o(p) > \quad , \qquad (4F.2)$$

and using (3E.2) for Σ, we find

$$M_6{}^a = 4\, i\alpha^2 C\ \Gamma(-a) \int \frac{d^4p}{(2\pi)^4} < \not{\varepsilon}' S(p + \tfrac{1}{2}k - \mathbf{k})\not{\varepsilon}\psi_o(p) >$$

$$- 4i\alpha^2 C \int \frac{d^4p}{(2\pi)^4} < \not{\epsilon}' \; S(p + \frac{1}{2}k - \mathbf{k})\not{\epsilon} S(p + \frac{1}{2}k)$$

$$\times [C_1\vec{p} \cdot \vec{\gamma} + C_2(p_o + \kappa)\gamma_o + C_3]\psi_o(p) > \quad , \qquad (4F.3)$$

where C_1, C_2 and C_3 are given in (3E.6).

The first integral in (4F.3) can be found, simply, by comparing (4A.1) with (4A.6). the integral, up to the order that we are interested, is

$$\int \frac{d^4p}{(2\pi)^4} < \not{\epsilon}' S(p + \frac{1}{2}k - \mathbf{k})\not{\epsilon} \; \psi_o(p) > = \frac{i}{8\pi\alpha} M_o \quad . \qquad (4F.4)$$

For the second integral of (4F.3), after performing the p_o integration (dominant contribution comes from the pole in wave-function, or for $p_o = \kappa - E_p$, and $\vec{p} \sim \gamma$; we close the countor in the upper half-plane of p_o) and then by scaling $\vec{p} \to \gamma\vec{p}$, we note that in order this integral contributes the following trace should be, at most, of the order α^2

$$<\not{\epsilon}'(\not{p} + \vec{\mathbf{k}} \cdot \vec{\gamma} + 1)\not{\epsilon}(\not{p} + \kappa\gamma_o + 1)[C_1\vec{p} \cdot \vec{\gamma} + C_2(p_o + \kappa)\gamma_o + C_3]$$

$$\times(E_p + \gamma_o - \vec{p} \cdot \vec{\gamma}\,\gamma_o)\gamma_5 > \quad , \qquad (4F.5)$$

where $p_o = \kappa - E_p$.

It is not difficult to show that in order to find this trace, we need to find C_1 up to the o(1) and C_2 and C_3 up to the $o(\alpha^2)$. These coefficients, up to these desired orders, are given by (3E.9).

After finding the trace in (4F.5) and using (3E.9), we note that this trace is of the order higher than α^2. Therefore the second integral in (4F.3) does not contribute and only the first integral gives contribution.

Using (4F.4), (4.5) and (4F.3), we obtain

$$M_6^{\,a} = M_o\left[\frac{-\alpha}{2\pi}\,\Gamma(-a)\right] \quad . \qquad (4F.6)$$

For $M_6^{\,b}$, Figure 28(b), we write

$$M_6^{\,b} = -\,\pi\alpha C\int\frac{d^4p}{(2\pi)^4}\,\frac{d^4q}{(2\pi)^4}\,\frac{R(\kappa,\,\vec{q})R(\kappa,\,\vec{p})}{E_qE_pN_qN_p}\,\hat{H}_c(\kappa,\,q,\,p)\langle\rangle \quad , \qquad (4F.7)$$

where

$$\begin{aligned}\langle\rangle \equiv\ &\langle(N_p + \vec{p}\cdot\vec{\gamma})(1 - \gamma_o)(N_q + \vec{q}\cdot\vec{\gamma})S(q - \tfrac{1}{2}k)\not{\epsilon}' \\ &\times S(p + \tfrac{1}{2}k - k)\not{\epsilon}S(q + \tfrac{1}{2}k)(N_q + \vec{q}\cdot\vec{\gamma})(1 + \gamma_o) \\ &\times(N_p + \vec{p}\cdot\vec{\gamma})S(p + \tfrac{1}{2}k)\Sigma(p + \tfrac{1}{2}k)\psi_o(p)\,\rangle \quad ,\end{aligned} \qquad (4F.8)$$

and we have used (2.30) for $\hat{R}$.

After using (3E.2) for Σ and (2.19) for ψ_o and also the relations (4D.9) for simplifying the trace in (4F.8), and then performing the p_o and q_o integrations (by scaling $\vec{p} \to \gamma\vec{p}$ and $\vec{q} \to \gamma\vec{q}$ we note that the dominant contribution comes from $\vec{p}$, $\vec{q} \sim \gamma$ and the poles at $p_o = \kappa - E_p$ and $q_o = \kappa - E_q$; for q_o there is only one pole), and then using the following approximate relations

$$E_p = 1 + \frac{1}{2}\vec{p}^{\,2} \quad , \quad E_q = 1 + \frac{1}{2}\vec{q}^{\,2} \quad , \quad \kappa = 1 - \frac{1}{2}\gamma^2 \quad , \qquad (4F.9)$$

we find

$$M_6{}^b = \frac{\alpha C}{8\pi^5} M_o\left[\frac{-\alpha}{4\pi}\Gamma(-a)\right]\int d^3p d^3q \frac{\bar{H}_c(\kappa, \vec{q}, \vec{p})}{(\vec{p}^{\,2} + \gamma^2)^2(\vec{q}^{\,2} + \gamma^2)}$$

$$+ \text{ contribution from the second term in } \Sigma, \qquad (4F.10)$$

where $\bar{H}_c$ is given by (3B.6).

We can show that contribution of the second term in Σ is of higher orders, so we can neglect it. In fact, for contribution of the second term in Σ we have a similar situation that we had for $M_6{}^a$; here we need to find the following trace, at most, up to the order α^2

$$<(N_p + \vec{p}\cdot\vec{\gamma})(1 - \gamma_o)(N_q - \vec{q}\cdot\vec{\gamma})\not{\epsilon}'(\not{p} + \vec{k}\cdot\vec{\gamma} + 1)\not{\epsilon}$$

$$\times(N_q - \vec{q}\cdot\vec{\gamma})(1 + \gamma_o)(N_p - \vec{p}\cdot\vec{\gamma})[C_1\,\vec{p}\cdot\vec{\gamma} + C_2(2\kappa - E_p)\gamma_o + C_3]$$

$$\times(E_p + \gamma_o - \vec{p} \cdot \vec{\gamma}\, \gamma_o)\gamma_5 > \quad . \tag{4F.11}$$

It is not difficult to show that we need to know C_1 up to the o(1) and C_2 and C_3 up to the $o(\alpha^2)$. Therefore we may use relations (3E.9) for these coefficients.

After finding the trace in (4F.11) and using (3E.9), we note that this trace is of the order higher than α^2; so we neglect contribution of the second term in Σ.

Therefore from (4F.10) we have

$$M_6{}^b = \frac{\alpha C}{8\pi^5} M_o[\frac{-\alpha}{4\pi} \Gamma(-a)] \int d^3p d^3q \, \frac{\bar{H}_c(\kappa, \vec{q}, \vec{p})}{(\vec{p}^2 + \gamma^2)^2(\vec{q}^2 + \gamma^2)} \quad . \tag{4F.12}$$

This integral is the one that we had in (3E.13). By comparing (3E.13) with (3E.16) we infer the value of this integral

$$\int d^3p d^3q \, \frac{\bar{H}_c(\kappa, \vec{q}, \vec{p})}{(\vec{p}^2 + \gamma^2)^2(\vec{q}^2 + \gamma^2)} = \frac{8\pi^5}{\alpha} \quad , \tag{4F.13}$$

and by using (4.5) for C, we find, up to the $o(\alpha)$ correction

$$M_6{}^b = M_o[\frac{-\alpha}{4\pi} \Gamma(-a)] \quad . \tag{4F.14}$$

Now, relations (4F.1), (4F.6) and (4F.16) give us the final result for contribution M_6

$$M_6 = M_o\left[\frac{-3\alpha}{4\pi}\,\Gamma(-a)\right] \quad . \tag{4F.15}$$

G. Contribution M_7

For M_7, Figure 20(g), we use the following decomposition

$$M_7 = M_7{}^a + M_7{}^b \quad , \tag{4G.1}$$

where, $M_7{}^a$ and $M_7{}^b$ are represented in Figure 29.

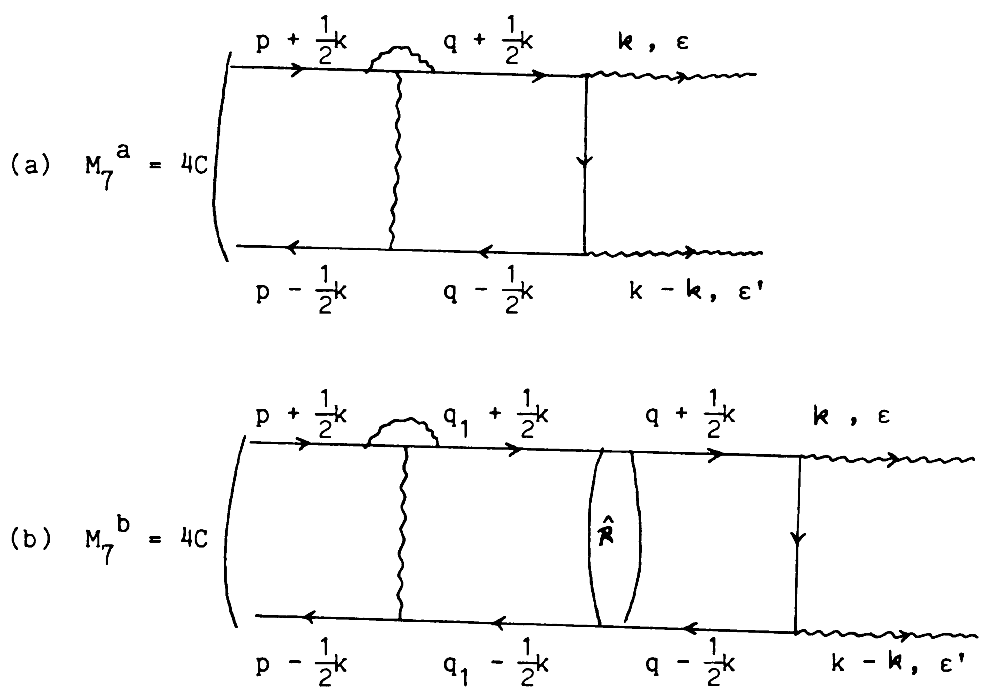


Figure 29. Contributions $M_7{}^a$ and $M_7{}^b$.

Calculation of these diagrams are very similar to δE_6 (see Section F of Chapter 3), so we try to give only the summary of the calculation.

For $M_7{}^a$ we write

$$M_7{}^a = 64\, C\pi^2\alpha^2 \int \frac{d^4q}{(2\pi)^4} \frac{d^4p}{(2\pi)^4} \langle\rangle_a \quad , \tag{4G.2a}$$

where

$$\langle\rangle_a \equiv \langle D^{\mu\nu}(p - q)\gamma_\nu S(q - \tfrac{1}{2}k)\not{\epsilon}' S(q + \tfrac{1}{2}k - \not{k})\not{\epsilon}$$

$$\times\, S(q + \tfrac{1}{2}k)\Lambda_\lambda(q + \tfrac{1}{2}k,\, p + \tfrac{1}{2}k)\psi_o(p) \rangle \quad . \tag{4G.2b}$$

For $M_7{}^b$, using (2.30) for $\hat{R}$, we write

$$M_7{}^b = -4\, iC\pi^2\alpha^2 \int \frac{d^4q}{(2\pi)^4} \frac{d^4q_1}{(2\pi)^4} \frac{d^4p}{(2\pi)^4} \frac{R(\kappa,\, \vec{q}_1)R(\kappa,\, \vec{q})}{E_q E_{q_1} N_q N_{q_1}}$$

$$\times\, \hat{H}_c(\kappa,\, q_1,\, q)\langle\rangle_b \quad , \tag{4G.3a}$$

where

$$\langle\rangle_b \equiv \langle D^{\mu\nu}(p - q_1)\gamma_\nu S(q_1 - \tfrac{1}{2}k)(N_{q_1} + \vec{q}_1 \cdot \vec{\gamma})(1 - \gamma_o)$$

$$\times (N_q + \vec{q} \cdot \vec{\gamma})S(q - \tfrac{1}{2}k)\not{\epsilon}' S(q + \tfrac{1}{2}k - \not{k})\not{\epsilon} S(q + \tfrac{1}{2}k)$$

$$\times (N_q + \vec{q} \cdot \vec{\gamma})(1 + \gamma_o)(N_{q_1} + \vec{q}_1 \cdot \vec{\gamma})S(q_1 + \tfrac{1}{2}k)$$

$$\times \Lambda_\lambda(q_1 + \tfrac{1}{2}k,\, p + \tfrac{1}{2}k)\psi_o(p) \rangle \quad . \tag{4G.3b}$$

In the above formula, Λ_λ is the vertex correction and $D^{\mu\nu}$ is given by (3C.24).

We note that relations (4G.2) and (4G.3), apart from a multiplicative factor and the wave-function, are the same as relations (3F.2) and (3F.3) respectively, if in the latter relations we use the following replacement

$$\gamma_\mu \to \not{\varepsilon}' S(q + \tfrac{1}{2}k - \mathbf{k})\not{\varepsilon} \quad . \qquad (4G.4)$$

Since for $M_7{}^a$ and $M_7{}^b$ we are interested in the leading order (which corresponds to the $o(\alpha)$ correction), (4G.4) can be written as

$$\gamma_\mu \to -\frac{1}{2}\not{\varepsilon}'(1 + \vec{\mathbf{k}} \cdot \vec{\gamma})\not{\varepsilon} \quad , \qquad (4G.5)$$

where, we used the same approximation that we applied for δE_6, namely $\vec{q} \sim \gamma$ and $q_o \sim \gamma^2$. In fact, (4G.5) shows that we can use all approximations that we employed for δE_6, namely, $\vec{p}$, $\vec{q}_1$, $\vec{q} \sim \gamma$, and $\lambda = \nu = o$, and that for p_o, q_{10} and q_o integrations the dominant poles correspond to $p_o = \kappa - E_p$, $q_{10} = \kappa - E_{q_1}$ and $q_o = \kappa - E_q$.

Therefore (4G.2a), after peforming p_o and q_o integrations, gets the following form which is very similar to (3F.4)

$$M_7{}^a = \frac{-i\alpha^3}{4\pi^3}\sqrt{2}\,\phi_o \int d^3p\, d^3q \left[(\vec{p}^2 + \gamma^2)^2(\vec{q}^2 + \gamma^2)(\vec{p} - \vec{q})^2\right]^{-1}$$

$$\times\langle \not{\epsilon}'(1 + \vec{k}\cdot\vec{\gamma})\not{\epsilon}(1 + \gamma_o)\Lambda_o(q + \tfrac{1}{2}k,\ p + \tfrac{1}{2}k)(1 + \gamma_o)\gamma_5 \rangle \quad , \quad (4G.6)$$

and (4G.3a) after performing p_o, q_{10} and q_o integrations, gets the following from which is very similar to (3F.5)

$$M_7{}^b = \frac{-i\alpha^3}{32\pi^6}\sqrt{2}\ \phi_o \int d^3p d^3q d^3q_1\ \bar{H}_c(\kappa,\ \vec{q}_1,\ \vec{q})$$

$$\times[(\vec{p}^{\,2} + \gamma^2)^2(\vec{q}_1^{\,2} + \gamma^2)(\vec{q}^{\,2} + \gamma^2)(\vec{p} - \vec{q}_1)^2]^{-1}$$

$$\times\langle \not{\epsilon}'(1 + \vec{k}\cdot\vec{\gamma})\not{\epsilon}(1 + \gamma_o)\Lambda_o(q_1 + \tfrac{1}{2}k,\ p + \tfrac{1}{2}k)(1 + \gamma_o)\gamma_5 \rangle \quad , \quad (4G.7)$$

where $\bar{H}_c$ is given by (3C.27), and we have used (4.5) for C.

Now by virtue of the relation (3F.20) (that we found in the Section F of Chapter 3), which in the present case can be written as

$$(1 + \gamma_o)\Lambda_o(q + \tfrac{1}{2}k,\ p + \tfrac{1}{2}k)(1 + \gamma_o) =$$

$$(1 + \gamma_o)[\tfrac{\alpha}{4\pi}\ \Gamma(-a)](1 + \gamma_o) \quad , \quad (4G.8a)$$

$$(1 + \gamma_o)\Lambda_o(q_1 + \tfrac{1}{2}k,\ p + \tfrac{1}{2}k)(1 + \gamma_o) =$$

$$(1 + \gamma_o)[\tfrac{\alpha}{4\pi}\ \Gamma(-a)](1 + \gamma_o) \quad , \quad (4G.8b)$$

relations (4G.6) and (4G.7) get the following forms

$$M_7{}^a = \frac{\alpha^3}{8\pi^5} M_o\Gamma(-a)\int d^3p d^3q[(\vec{p}^2 + \gamma^2)^2(\vec{q}^2 + \gamma^2)(\vec{p} - \vec{q})^2]^{-1} \quad , \qquad (4G.9)$$

$$M_7{}^b = \frac{\alpha^3}{64\pi^8} M_o\Gamma(-a)\int d^3p d^3q d^3q_1 \; \bar{H}_c(\kappa, \vec{q}_1, \vec{q})$$

$$\times[(\vec{p}^2 + \gamma^2)^2(\vec{q}_1^2 + \gamma^2)(\vec{q}^2 + \gamma^2)(\vec{p} - \vec{q}_1)^2]^{-1} \quad , \qquad (4G.10)$$

where M_o is given by (4A.6).

The integrals in (4G.9) and (4G.10) are those that we had in (3F.26) and (3F.27), respectively. Therefore

$$M_7{}^a = M_o[\frac{\alpha}{2\pi} \Gamma(-a)] \quad , \qquad (4G.11)$$

$$M_7{}^b = M_o[\frac{\alpha}{4\pi} \Gamma(-a)] \quad , \qquad (4G.12)$$

and by virtue of (4G.1)

$$M_7 = M_o[\frac{3\alpha}{4\pi} \Gamma(-a)] \quad . \qquad (4G.13)$$

By comparing (4G.11) with (4F.6), and (4G.12) with (4F.14) we obtain the following relations

$$M_7{}^a = -M_6{}^a \quad , \qquad (4G.14)$$

$$M_7{}^b = -M_6{}^b \quad , \qquad (4G.15)$$

and consequently

$$M_7 = -M_6 \quad . \tag{4G.16}$$

H. Contribution M_8

For M_8, Figure 20(h), by comparing Figure 20(h) with Figures 20(a) and 18, we find

$$M_8 = (\delta K') \, M_1 \quad , \tag{4H.1}$$

and by virtue of the relations (4.3) and (4A.6), we obtain

$$M_8 = M_o\left[\frac{-\alpha}{2\pi}\,\Gamma(-a)\right] \quad . \tag{4H.2}$$

I. Summary of the Calculations of the Total Invariant Amplitude and the Decay Rate of the Parapositronium Ground-State

By virtue of the relations (4.12), (4E.2), (4E.12) and (4G.16) we find the total invariant decay amplitude

$$M = M_2 + M_3 + M_4{}^a + M_8 \quad , \tag{4I.1}$$

which means that, up to the $o(\alpha)$ correction, the second term in $\hat{G}_c$, $\hat{R}$ (see relation (2.29)), does not contribute.

We represent M, relation (4I.1), graphically in Figure 30.

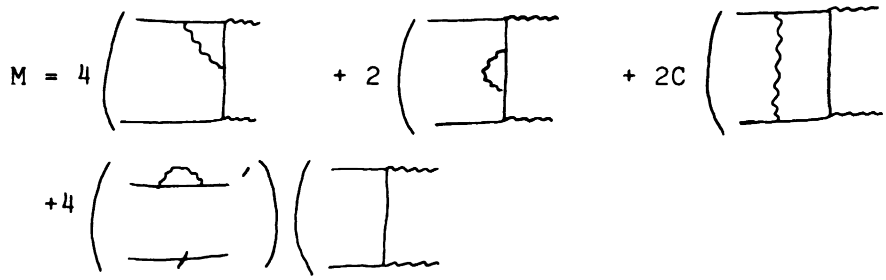


Figure 30. Total invariant decay amplitude, correct up to the $o(\alpha)$ correction. Line with a dash through it indicates inverse propagator, and the normalization constant C is given by (4.5).

In (4I.1), by virtue of (4B.17), (4C.5), (4D.6) and (4H.2) we obtain

$$M = M_o\left[1 + \frac{\alpha}{4\pi}\left(\frac{\pi^2}{2} - 10\right)\right] \quad , \qquad (4I.2)$$

where M_o, the lowest order invariant amplitude, is given by (4A.7).

Now that we know the total invariant decay amplitude M, by virtue of the relations (2.18), (4.8), (4.11), (4A.7) and (4I.2) we can find the decay rate (up to the order α correction)

$$\Gamma(p\text{-}Ps \rightarrow 2\gamma) = \frac{\alpha^5}{2}\left[1 - \frac{\alpha}{\pi}\left(5 - \frac{\pi^2}{4}\right)\right] \quad , \qquad (4I.3)$$

which agrees with the result of Harris and Brown [7].

We present all results of calculations of the present Chapter in the Table 2.

Table 2

Summary of the Calculations of the

Amplitudes and the Decay Rate

M_1	$M_o\{1 + \frac{\alpha}{4\pi}[\Gamma(-a) + 8G - 8\lambda - 16\ \ln 2 - \frac{3\pi^2}{2}]\}$
M_2	$M_o \frac{\alpha}{4\pi}[2\Gamma(-a) + \pi^2 - 4 - 8\ \ln 2 + 4\sqrt{2}\ \ln(1 + \sqrt{2}) - 2\ \ln^2(1 + \sqrt{2})]$
M_3	$M_o \frac{\alpha}{4\pi}[-\Gamma(-a) - 6 + 4\ \ln 2 + 4\sqrt{2}\ \ln(1 + \sqrt{2})]$
M_4	$M_o\{\frac{3}{2} + \frac{\alpha}{4\pi}[\frac{3}{2}\Gamma(-a) + 12G - 12\lambda - 20\ \ln 2 - \frac{11\pi^2}{4} - 8\sqrt{2}\ \ln(1 + \sqrt{2}) + 2\ \ln^2(1 + \sqrt{2})]\}$
M_5	$M_o\{-\frac{3}{2} + \frac{\alpha}{4\pi}[-\frac{3}{2}\Gamma(-a) - 20G + 20\lambda + 40\ \ln 2 + \frac{15\pi^2}{4}]\}$
M_6	$M_o[\frac{-3\alpha}{4\pi}\Gamma(-a)]$
M_7	$M_o[\frac{3\alpha}{4\pi}\Gamma(-a)]$
M_8	$M_o[\frac{-\alpha}{2\pi}\Gamma(-a)]$
$M = \sum_{i=1}^{8} M_i$	$M_o[1 + \frac{\alpha}{4\pi}(\frac{\pi^2}{2} - 10)]$
$\Gamma = \frac{\alpha^5}{2}(\frac{M}{M_o})^2 = \frac{\alpha^5}{2}[1 - \frac{\alpha}{\pi}(5 - \frac{\pi^2}{4})]$	

Chapter 5

Summary and Conclusion

Using the Barbieri-Remiddi lowest-order equation, equation (2.2) which we discussed it in Chapter 2, and working in Coulomb gauge we calculated the energy shift of orthopositronium and the decay rate of parapositronium.

For the contribution of one-photon-annihilation channel to the energy shift of the ground state of orthopositronium, using relation (3.1), up to the order α correction we found (see Chapter 3)

$$\delta E = \frac{\alpha^4}{4}\left(1 - \frac{4\alpha}{\pi} - \frac{8\alpha}{9\pi}\right) \quad , \qquad \text{(3H.2)}$$

which is in agreement with the result of Karplus and Klein [12].

The interesting point is that we derived relation (3H.2) without performing the wavefunction and the vertex renormalization subtractions. It means that relation (3.1) which gives the energy shift, is free of divergences and gives the finite result without needing to perform the wavefunction and the vertex renormalization subtractions (at least in the first-order perturbation theory.)

For the decay rate of the ground state of parapositronium, using relation (4.9), up to the order α correction we found (see Chapter 4)

$$\Gamma(\text{p-Ps} \to 2\gamma) = \frac{\alpha^5}{2}\left[1 - \frac{\alpha}{\pi}\left(5 - \frac{\pi^2}{4}\right)\right] \quad , \qquad (4I.3)$$

which is in agreement with the result of Harris and Brown [7].

Here also we derived the result (4I.3) without performing the wavefunction and the vertex renormalization subtractions. It implies that the relation (4.9) which gives the decay rate, is free of divergences and gives the finite result without needing to perform the wavefunction and the vertex renormalization subtractions (at least in the first-order perturbation theory).

In Chapters 3 and 4 we noticed that in derivations of the results (3H.2) and (4I.3) we encountered some diagrams which "rule of thumb" for finding their leading-order contributions would give higher orders in α than their actual orders. For instance, rule of thumb for the leading-order contributions of the diagrams of Figures 2(c) and 20(d) gives order α, and for the diagrams of Figures 2(f) and 20(g) gives order α^2. However, the explicit calculation showed that the leading-order contributions of the diagrams in Figures 2(c) and 20(d) were of the order 1 and for the diagrams in Figures 2(f) and 20(g) were of the order α.

The reason that the rule of thumb for these diagrams does not work is due to the kernel δK, which was introduced in Chapter 2; it can generate inverse powers of α.

We should also mention that in covariant gauge there are an infinite number of diagrams which contribute to a given order in α; whereas in Coulomb gauge we deal with a finite number of diagrams. This is due to the photon propagator $D_{\mu\nu}$. In the diagrams which one or more virtual photons are exchanged between electron and positron lines, the D_{oo} part of photon propagator, in covariant gauge, creates pole; whereas in Coulomb gauge D_{oo} does not give a pole (see relation (3C.24)).

In fact it was one of the reasons that we chose to work in Coulomb gauge; namely, dealing with a finite number of diagrams.

Our calculations showed that the method of Barbieri-Remiddi, for solving the positronium problem, is straight-forward and practical. Also in order to make our results useful for derivation of higher order corrections to the quantities that we calculated up to the order α correction, we presented all calculations of the related diagrams separately.

APPENDICES

Appendix A

Evaluation of an Integral related to the Contribution δE_3

Consider the following integral

$$I = \int_0^1 \frac{dx}{\sqrt{x}} \int_0^1 dy \frac{1}{k^2y^2 - k^2y + E^2 - \vec{p}^2x} \quad , \tag{A.1}$$

where

$$k^2 = 4(1 - \gamma^2) \quad , \quad E^2 = \vec{p}^2 + 1 \quad . \tag{A.2}$$

After performing the y integration, we find

$$I = \int_0^1 \frac{dx}{\sqrt{x}} \frac{2}{k[\vec{p}^2(1 - x) + \gamma^2]^{1/2}} \{\frac{\pi}{2}$$

$$-\tan^{-1}[\frac{2}{k} (\vec{p}^2(1 - x) + \gamma^2)^{1/2}]\} \quad . \tag{A.3}$$

We should mention that we are not interested in the quantity I by itself; it is the following quantity which we need to find up to the order $\frac{1}{\gamma}$

$$\int d^3p \frac{\sqrt{E + \kappa}}{(\vec{p}^2 + \gamma^2)^2 E} I \quad , \tag{A.4}$$

where $\kappa = (1 - \gamma^2)^{1/2}$.

By scaling $\vec{p} \to \gamma\vec{p}$, we note that in (A.4) the term involving $\tan^{-1}$ is of the order $\frac{1}{\gamma}$, therefore, for this term we may say $\vec{p}$ is of the order γ and use the following approximation

$$\tan^{-1}\left[\frac{2}{k}\,(\vec{p}^{\,2}(1 - x) + \gamma^2)^{1/2}\right] = \frac{2}{k}(\vec{p}^{\,2}(1 - x) + \gamma^2)^{1/2} \quad . \tag{A.5}$$

Using (A.5) in (A.3) we obtain

$$I = \frac{\pi}{k} \int_0^1 \frac{dx}{\sqrt{x}} \frac{1}{[\vec{p}^{\,2}(1 - x) + \gamma^2]^{1/2}} - 2 \quad , \tag{A.6}$$

which by change of variable $x \to x^2$ and performing the integral and using the approximate relation $k = 2$, we find

$$I = \frac{\pi}{p} \sin^{-1}\left(\frac{p}{\sqrt{\vec{p}^{\,2} + \gamma^2}}\right) -2 \quad . \tag{A.7}$$

As we mentioned before, we are interested in the quantity which is given in (A.4).

First we find the contribution of the first term of I in (A.7), which we define it as J

$$J = \int_0^\infty dp \, \frac{p\sqrt{E + \kappa}}{(\vec{p}^{\,2} + \gamma^2)^2 E} \sin^{-1}\left(\frac{p}{\sqrt{\vec{p}^{\,2} + \gamma^2}}\right) \quad . \tag{A.8}$$

Relation (A.8) can be written as

$$J = J_1 + J_2 \quad , \tag{A.9}$$

where

$$J_1 = \int_0^\infty dp \, \frac{p\sqrt{1+\kappa}}{(\vec{p}^2+\gamma^2)^2} \sin^{-1}\left(\frac{p}{\sqrt{\vec{p}^2+\gamma^2}}\right) \quad , \tag{A.10}$$

$$J_2 = -\int_0^\infty dp \, \frac{p}{(\vec{p}^2+\gamma^2)^2} \left(\sqrt{1+\kappa} - \frac{\sqrt{E+\kappa}}{E}\right)\sin^{-1}\left(\frac{p}{\sqrt{\vec{p}^2+\gamma^2}}\right) \quad . \tag{A.11}$$

Integration by parts gives the J_1, which up to the order $\frac{1}{\gamma}$ is

$$J_1 = \frac{\sqrt{2}\,\pi}{8\gamma^2} \quad . \tag{A.12}$$

For J_2, we show that it is of the order higher than $\frac{1}{\gamma}$, so it can be ignored.

First we note

$$\sqrt{1+\kappa} - \frac{\sqrt{E+\kappa}}{E} \geq 0 \quad , \tag{A.13}$$

therefore

$$|J_2| = \int_0^\infty dp \, \frac{p}{(\vec{p}^2+\gamma^2)^2} \left(\sqrt{1+\kappa} - \frac{\sqrt{E+\kappa}}{E}\right)\sin^{-1}\left(\frac{p}{\sqrt{\vec{p}^2+\gamma^2}}\right) \quad , \tag{A.14}$$

and since

$$\sin^{-1}\left(\frac{p}{\sqrt{\vec{p}^2 + \gamma^2}}\right) \leq \frac{\pi}{2}, \quad \sqrt{E + \kappa} \geq \sqrt{1 + \kappa}, \quad \sqrt{1 + \kappa} < \sqrt{2}, \qquad (A.15)$$

from (A.14) we have

$$|J_2| \leq \frac{\pi}{\sqrt{2}} \int_o^\infty dp \, \frac{p}{(\vec{p}^2 + \gamma^2)^2} \frac{E-1}{E} \quad . \qquad (A.16)$$

Using the following relations

$$E-1 = \frac{\vec{p}^2}{E + 1} \leq \frac{1}{2} \vec{p}^2 \quad , \qquad (A.17)$$

from (A.16) we obtain

$$|J_2| \leq \frac{\pi}{2\sqrt{2}} \int_o^\infty dp \, \frac{\vec{p}^2}{(\vec{p}^2 + \gamma^2)^2} \frac{p}{E} \quad , \qquad (A.18)$$

or

$$|J_2| < \frac{\pi}{2\sqrt{2}} \int_o^\infty dp \, \frac{p}{\vec{p}^2 + \gamma^2} \frac{1}{E}$$

$$= \frac{\pi}{2\sqrt{2}} \left(\int_o^1 dp \, \frac{p}{\vec{p}^2 + \gamma^2} \frac{1}{E} + \int_1^\infty dp \, \frac{p}{\vec{p}^2 + \gamma^2} \frac{1}{E}\right)$$

$$< \frac{\pi}{2\sqrt{2}} \left(\int_o^1 dp \, \frac{p}{\vec{p}^2 + \gamma^2} + \int_1^\infty dp \, \frac{1}{\vec{p}^2 + \gamma^2}\right)$$

$$< \frac{\pi}{2\sqrt{2}} \left[\frac{1}{2} \ln \left(\frac{1 + \gamma^2}{\gamma^2}\right) + \int_1^\infty dp \, \frac{1}{\vec{p}^2}\right]$$

$$= \frac{\pi}{2\sqrt{2}} \left[\frac{1}{2} \ln \left(\frac{1 + \gamma^2}{\gamma^2}\right) + 1\right] \quad , \tag{A.19}$$

which means that J_2 is not of order $\frac{1}{\gamma}$, so we neglect it.

Therefore, using (A.8), (A.9) and (A.12), we obtain the following relation which is correct up to the order $\frac{1}{\gamma}$

$$\int_0^\infty dp \frac{p\sqrt{E + \kappa}}{(\vec{p}^2 + \gamma^2)^2 E} \sin^{-1}\left(\frac{p}{\sqrt{\vec{p}^2 + \gamma^2}}\right) = \frac{\sqrt{2}\,\pi}{8\gamma^2} \quad . \tag{A.20}$$

Appendix B
Evaluation of an Integral
related to the Contribution δE_1

The following integral is the one that we need for finding δE_1 (its evaluation also serves as a sample for finding similar integrals)

$$I = \int_0^\infty \frac{\vec{p}^2 \, dp}{(\vec{p}^2 + \gamma^2)^2 E} \sqrt{E + \kappa} \, (2E + 1) \quad , \tag{B.1}$$

where

$$E = \sqrt{\vec{p}^2 + 1} \quad , \kappa = (1 - \gamma^2)^{1/2} \quad . \tag{B.2}$$

Scaling the momentum $\vec{p}$ by γ

$$\vec{p} \to \gamma\vec{p} \quad , \tag{B.3}$$

we find that the leading order of I is

$$I \sim o(\frac{1}{\gamma}) \quad . \tag{B.4}$$

We need to find the integral I up to the o(1). Consider the following relation

$$\sqrt{E + \kappa} = \sqrt{E + 1} \, (1 - \frac{1 - \kappa}{1 + E})^{1/2} \quad , \tag{B.5}$$

which by (B.2) it is

$$\sqrt{E + \kappa} = \sqrt{E + 1}\,\left[1 + o(\gamma^2)\right]^{1/2} \sim \sqrt{E + 1} \quad , \qquad (B.6)$$

where we ignored the second term in square brackets, since it gives contribution of order higher than o(1).

Therefore, (B.1) gives

$$I = \int_0^\infty \frac{\vec{p}^{\,2}\, dp}{(\vec{p}^{\,2} + \gamma^2)^2} \frac{\sqrt{E + 1}}{E} (2E + 1) \quad , \qquad (B.7)$$

or, using (B.2) for changing the variable p to E

$$I = \int_1^\infty \frac{dE}{(E^2 - \kappa^2)^2} (E + 1)(2E + 1) \sqrt{E - 1} \quad . \qquad (B.8)$$

By changing the integration variable

$$E = 1 + \frac{1}{2} x^2 \quad , \qquad (B.9)$$

we obtain

$$I = \frac{8}{\sqrt{2}} \int_0^\infty dx \frac{x^2(x^2 + 4)(x^2 + 3)}{(x^4 + 4x^2 + 4\gamma^2)^2} \quad , \qquad (B.10)$$

which by using table of integrals, up to the o(1) is

$$I = \frac{\sqrt{2}\pi}{4} \left(\frac{3}{\gamma} + 1\right) \quad . \tag{B.11}$$

Appendix C

Proof of an Identity

We want to show that the following integral is identically zero

$$I = \int d^3q\ F(\vec{q}^{\,2}) \int d^3p \frac{1}{(\vec{p}^{\,2} + \gamma^2)^2} \frac{1}{B} \frac{-A + 2B - B\rho}{A\rho + B(1 - \rho)^2} \quad , \qquad \text{(C.1)}$$

where

$$A = (\vec{p} - \vec{q})^2 \ , \quad B = \frac{1}{4\gamma^2} (\vec{p}^{\,2} + \gamma^2)(\vec{q}^{\,2} + \gamma^2) \quad , \quad o \leq \rho \leq 1 \quad , \qquad \text{(C.2)}$$

and $F(\vec{q}^{\,2})$ is any function of $\vec{q}^{\,2}$ which satisfies the condition (C.13).

We write (C.1) as

$$I = I_1 + I_2 \quad , \qquad \text{(C.3)}$$

where

$$I_1 = \frac{-1}{\rho} \int d^3q\ F(\vec{q}^{\,2}) \int d^3p \frac{1}{(\vec{p}^{\,2} + \gamma^2)^2} \frac{1}{B} \quad , \qquad \text{(C.4)}$$

$$I_2 = \frac{1}{\rho} \int d^3q\ F(\vec{q}^{\,2}) \int d^3p \frac{1}{(\vec{p}^{\,2} + \gamma^2)^2} \frac{1}{A\rho + B(1 - \rho)^2} \quad . \qquad \text{(C.5)}$$

After performing the p integration in (C.4), we obtain

$$I_1 = \frac{-4\pi^3}{\rho} \int_0^\infty dq \, \frac{\vec{q}^2}{\vec{q}^2 + 1} \, F(\gamma^2 \, \vec{q}^2) \quad . \tag{C.6}$$

For I_2, we write ($x = \cos \Theta$, where Θ is the angle between $\vec{p}$ and $\vec{q}$)

$$I_2 = \frac{2\pi}{\rho} \int \frac{d^3p}{(\vec{p}^2 + \gamma^2)^2} \int_0^\infty dq \, \vec{q}^2 \, F(\vec{q}^2)$$

$$\times \int_{-1}^{+1} dx \, \left[(\vec{p}^2 + \vec{q}^2 - 2pqx)\rho + B(1 - \rho)^2 \right]^{-1} \quad , \tag{C.7}$$

or after performing the x integration

$$I_2 = \frac{-4\pi^2}{\rho^2} \int_{-\infty}^\infty dq \, qF(\vec{q}^2) \int_0^\infty dp \, \frac{p}{\vec{p}^2 + \gamma^2}$$

$$\times \ln \left[(p - q)^2 \rho + B(1 - \rho)^2 \right] \quad . \tag{C.8}$$

By scaling $p \to \gamma p$, $q \to \gamma q$ and integration by parts, we obtain

$$I_2 = \frac{-4\pi^2}{\rho^2} \int_{-\infty}^\infty dq \, qF(\gamma^2 \, \vec{q}^2) \int_0^\infty \frac{dp}{\vec{p}^2 + 1}$$

$$\times \frac{p - q + \beta(\vec{q}^2 + 1)p}{(p - q)^2 + \beta(\vec{q}^2 + 1)(\vec{p}^2 + 1)} \quad , \tag{C.9}$$

where

$$\beta = \frac{(1 - \rho)^2}{4\rho} \quad . \tag{C.10}$$

In (C.9), the integration on variable p is not difficult to perform. After performing this integral and retaining only those terms which are an odd function of q, we obtain

$$I_2 = \frac{4\pi^3}{\rho} \int_0^\infty dq \, \frac{\vec{q}^{\,2}}{(\vec{q}^{\,2} + 1)} \, F(\gamma^2 \, \vec{q}^{\,2}) \quad . \tag{C.11}$$

Therefore, from (C.3), (C.6) and (C.11) we conclude

$$I = o \quad . \tag{C.12}$$

We note that this result is based on the following assumption

$$\int_0^\infty dq \, \frac{\vec{q}^{\,2}}{(\vec{q}^{\,2} + 1)} \, F(\gamma^2 \, \vec{q}^{\,2}) = \text{finite} \quad . \tag{C.13}$$

Appendix D

Evaluation of an Integral related to the Contribution M_1

Consider the following integral

$$I = \int_0^\infty dp \, \frac{p\sqrt{E + \kappa}}{(\vec{p}^2 + \gamma^2)^2 E} \ln(E + p) \quad , \tag{D.1}$$

where

$$E = (p^2 + 1)^{1/2} \quad , \quad \kappa = (1 - \gamma^2)^{1/2} \quad , \quad \gamma = \frac{\alpha}{2} \quad . \tag{D.2}$$

Scaling the momentum p by γ

$$p \to \gamma p \quad , \tag{D.3}$$

We notice the leading order of the integral in (D.1) is

$$I = o(\frac{1}{\gamma}) \quad . \tag{D.4}$$

We need to find the integral I up to the o(1). Using (B.6) in (D.1), we obtain

$$I = \int_0^\infty dp \, \frac{p\sqrt{E + 1}}{(p^2 + \gamma^2)^2 E} \ln(E + p) \quad . \tag{D.5}$$

Let's write (D.5) in the following form

$$I = I_1 + I_2 \quad , \tag{D.6}$$

where

$$I_1 = \sqrt{2} \int_o^\infty dp \frac{p}{(p^2 + \gamma^2)^2} \ln(E + p) \quad , \tag{D.7}$$

$$I_2 = \int_o^\infty dp \frac{p}{(p^2 + \gamma^2)^2 E} (\sqrt{E + 1} - E\sqrt{2}) \ln(E + p) \quad . \tag{D.8}$$

The integral I_1, using the integration by parts, up to the o(1) is

$$I_1 = \sqrt{2}(\frac{\pi}{4\gamma} - \frac{1}{2}) \quad . \tag{D.9}$$

For I_2, let's write it in the following form

$$I_2 = \int_o^\infty dp \frac{1}{p^3 E} (\sqrt{E + 1} - E\sqrt{2}) \ln(E + p)$$

$$-\int_o^\infty dp \frac{\gamma^2(2p^2 + \gamma^2)}{p^3(p^2 + \gamma^2)^2} (\sqrt{E + 1} - E\sqrt{2}) \ln(E + p) \quad . \tag{D.10}$$

Scaling p by γ shows that the leading order of the second integral is o(γ), therefore we may ignore it and write (D.10) as

$$I_2 = \int_o^\infty dp \frac{1}{p^3 E} (\sqrt{E + 1} - E\sqrt{2}) \ln(E + p) \quad . \tag{D.11}$$

Changing the variable p to E (see (D.2)), (D.11) gives

$$I_2 = \int_1^\infty dE \frac{1}{(E^2 - 1)^2} (\sqrt{E + 1} - E\sqrt{2}) \ln(E + \sqrt{E^2 - 1}) \quad , \qquad (D.12)$$

or by changing the variable $E = \frac{1 + x^2}{2x}$

$$I_2 = -4\sqrt{2} \int_1^\infty dx \frac{x}{(x^2 - 1)^3} (\sqrt{x} - 1)^2 (x + \sqrt{x} + 1) \ln x \quad , \qquad (D.13)$$

and if we change $x \to x^2$, we obtain

$$I_2 = -16\sqrt{2} \int_1^\infty dx \frac{x^3}{(x^4 - 1)^3} (x - 1)^2 (x^2 + x + 1) \ln x \quad . \qquad (D.14)$$

The method of partial fraction gives I_2 in a form of integrals which can be found in the standard tables of integrals. The result is

$$I_2 = \sqrt{2} \left(\frac{1}{2} + G - 2 \ln 2 - \frac{3\pi^2}{16}\right) \quad , \qquad (D.15)$$

where G = .915965594... is the Catalan's Constant.

Using (D.6), (D.9) and (D.15), we find up to the o(1)

$$I = \sqrt{2} \left(\frac{\pi}{2\alpha} + G - 2 \ln 2 - \frac{3\pi^2}{16}\right) \quad . \qquad (D.16)$$

Appendix E

Calculation of a Trace related to M_4

In this appendix we calculate a trace which is related to the decay amplitude $M_4{}^a$, equation (4D.2). This calculation serves as a sample for the method of approximation that we implemented for finding nearly all other traces that we had in our work. Traces were found, mostly, by the SCHOONSCHIP program.

The quantity that we want to evaluate is

$$T \equiv D^{\mu\nu}(p - q)\langle\gamma_\nu(\not{q} - \kappa\gamma_o + 1)\not{\epsilon}'(\not{q} + \vec{k} \cdot \vec{\gamma} + 1)\not{\epsilon}$$

$$\times(\not{q} + 1 + \kappa\gamma_o)\gamma_\mu(E_p + \gamma_o - \vec{p} \cdot \vec{\gamma}\,\gamma_o)\gamma_5\rangle \quad , \qquad \text{(E.1)}$$

where $iD^{\mu\nu}$, the photon propagator in Coulomb gauge is given by (3C.24), and $\langle\rangle$ stands for trace.

As it can be seen from (4D.2), we are interested in the following quantity (up to the order α)

$$I \equiv \alpha^2 \int \frac{d^3p}{(\vec{p}^2 + \gamma^2)^2} \int \frac{d^4q}{D}\, T \quad , \qquad \text{(E.2)}$$

which is related to the $M_4{}^a$, relation (4D.2) after performing the p_o integration (we note that, up to the order α correction, only the pole at $p_o = \kappa - E_p$ contributes $(\vec{p} \sim \gamma)$). The quantity D in (E.2) is given by (4D.4).

Relation (E.2) with scaling $\vec{p} \to \gamma\vec{p}$ shows that I is of the order

$$I \sim \alpha \int \frac{d^4q}{D} T \quad . \tag{E.3}$$

The most contribution to I is of the o(1), and it comes from the pole at $q_o = \kappa - E_q \sim \gamma^2$ (if $\vec{q} \sim \gamma$ and if we assume <> ~ 1 in (E.1))

$$\int \frac{d^4q}{D} \sim \int \frac{d^3q}{\alpha^2} \sim \alpha \quad , \quad T \sim \frac{1}{\alpha^2} \quad , \quad \vec{q} \sim \gamma \quad , \tag{E.4}$$

which means

$$I \sim 1 \quad (\text{only for } \vec{q} \sim \gamma) \quad . \tag{E.5}$$

If we assume $\vec{q}$ is not small (not of the order γ), and <> ~ 1, then

$$\int \frac{d^4q}{D} \sim \int \frac{d^4q}{1} \sim 1 \quad \text{and } T \sim 1 \quad (\text{for non-small } q) \quad , \tag{E.6}$$

which means

$$I \sim o(\alpha) \quad , \quad (\text{for non-small } q) \quad . \qquad ((E.7)$$

In short, the o(1) contribution to I comes from small values of q, while the o(α) contribution to I comes from all values of q (we always keep in mind that $\vec{p}$ is of the order γ).

The trace <> in (E.1) has the following orders

$$<> \sim 1 + o(\alpha) \qquad \text{for small } q \ (\vec{q} \sim \gamma) \quad , \tag{E.8}$$

$$<> \sim 1 \qquad \text{for non-small } q \quad , \tag{E.9}$$

and-using the above considerations for I-all higher orders of <> give rise to the $o(\alpha^2)$ and higher order contributions to I (relation (E.2)).

Therefore, for finding T, relation (E.1), we will use relations (E.8) and (E.9) in order to simplify the calculation. One of the implications of (E.8) and (E.9) is that we may use the following approximations

$$E \sim 1 + \frac{1}{2}\vec{p}^2 \sim 1 \quad , \quad \kappa \sim 1 - \frac{1}{2}\gamma^2 \sim 1 \quad , \tag{E.10}$$

$$\vec{p}\cdot\vec{q} \sim o \quad , \quad q^2\,\vec{q}\cdot\vec{p} \sim o \quad , \quad \vec{q}\cdot\vec{k}\,\vec{p}\cdot\vec{q} \sim o \quad ,\ldots \tag{E.11}$$

(for any term which contains p).

Using (E.10) and (E.11) in relation (E.1), we obtain

$$T = \left[\frac{q^2 - 2q_o^2 + 4}{(\vec{q} - \vec{p})^2} + \frac{2q^2 - 4q_o^2}{(q - p)^2}\right] < \not{\epsilon}'\not{\epsilon}\,\vec{k}\cdot\vec{\gamma}\,\gamma_o\,\gamma_5 >$$

$$+ \left[\frac{-q^2}{(\vec{q} - \vec{p})^2} + \frac{-2q^2}{(q - p)^2}\right] < \not{\epsilon}'\not{\epsilon}\,\vec{q}\cdot\vec{\gamma}\,\gamma_o\,\gamma_5 > \quad , \tag{E.12}$$

or since in numerators we can add any term of the forms $\vec{p}^2$, $\vec{p} \cdot \vec{q}$, p^2, etc. (any term which contains p gives higher orders so its introduction does not change the result), we may write relation (E.12) in the following form

$$T = [\frac{4}{(\vec{q} - \vec{p})^2} + 1 - \frac{q_o^2}{(\vec{q} - \vec{p})^2} - \frac{4q_o^2}{(q - p)^2}] \langle \not{e}'\not{e}\ \vec{k} \cdot \vec{\gamma}\ \gamma_o\ \gamma_5 \rangle$$

$$+ [- 1 - \frac{q_o^2}{(\vec{q} - \vec{p})^2}] \langle \not{e}'\not{e}\ \vec{q} \cdot \vec{\gamma}\ \gamma_o\ \gamma_5 \rangle \quad . \tag{E.13}$$

The first term in (E.13), $\frac{4}{(\vec{q} - \vec{p})^2}$, gives the o(1) and o($\alpha$) contributions to I. They come from all values of $\vec{q}$; small $\vec{q}$ gives both o(1) and o(α), non-small $\vec{q}$ gives only o(α). We leave this term in its present form.

For all other terms in (E.13), their contributions are of the o(α) and they come from non-small values of $\vec{q}$ and q_o (for small $\vec{q}$, with either the pole at $q_o \sim 1$ or $q_o \sim \gamma^2$, we get contributions of orders higher then α for I).

Therefore we may use the following approximate relations

$$\frac{q_o^2}{(\vec{q} - \vec{p})^2} \approx \frac{q_o^2}{\vec{q}^2} \quad , \quad \frac{q_o^2}{(q - p)^2} \approx \frac{q_o^2}{q^2} \quad , \tag{E.14}$$

and write relation (E.13) in the form

$$T = \left[\frac{4}{(\vec{q} - \vec{p})^2} + 1 - \frac{q_o^{\ 2}}{\vec{q}^2} - \frac{4q_o^{\ 2}}{q^2}\right]\langle \not{\epsilon}'\not{\epsilon}\ \vec{k} \cdot \vec{\gamma}\ \gamma_o\gamma_5\rangle$$

$$+ \left(- 1 - \frac{q_o^{\ 2}}{\vec{q}^2}\right)\langle \not{\epsilon}'\not{\epsilon}\ \vec{q} \cdot \vec{\gamma}\ \gamma_o\gamma_5\rangle \quad , \qquad \text{(E.15)}$$

or since integration on q, relation (E.2), for the $\vec{q} \cdot \vec{\gamma}$ term gives a quantity proportional to $\vec{k} \cdot \vec{\gamma}$, (E.15) may be written as

$$T = \left[\frac{4}{(\vec{q} - \vec{p})^2} + 1 - \frac{q_o^{\ 2}}{\vec{q}^2} - \frac{4q_o^{\ 2}}{q^2} - \frac{q_i}{k_i} - \frac{q_o^{\ 2}\ q_i}{\vec{q}^2\ k_i}\right]$$

$$\times\langle \not{\epsilon}'\not{\epsilon}\ \vec{k} \cdot \vec{\gamma}\ \gamma_o\ \gamma_5\rangle \quad , \qquad \text{(E.16)}$$

where i = 1, 2, 3 (no summation on i).

For performing the integrations in (E.2), the easiest way is first to perform the p integration, which transforms the first term of (E.16), $\frac{4}{(\vec{q} - \vec{p})^2}$, to the form $\frac{4}{\vec{q}^2 + \gamma^2}$; then we proceed to perform the q integration, by the Feynman parameterization method. For doing this, for the first term, $\frac{4}{\vec{q}^2 + \gamma^2}$, we use the exact form of D, relation (4D.4); but for all other terms we use the approximation $\kappa = 1$.

This method of approximation, in practice, facilitates calculations; it allows us, before performing the integrations, to get rid of unnecessary terms (terms which give those higher order corrections that we are not interested).

Appendix F

Evaluation of some Integrals related to the Contribution M_2

In this Appendix we evaluate three integrals that we need in evaluation of M_2. In these integrals D is defined by (4B.5) and q integrations are performed by using the Tables provided in [19].

One of the integrals that we need is

$$\int d^n q \, \frac{q_o^2 q^2}{D} = i\pi^2[I_1 + I_2 - \Gamma(-a)] \quad , \tag{F.1}$$

where

$$I_1 = \int_o^1 \frac{dx}{\sqrt{x}} \int_o^1 dy \, \frac{(1-y)^2}{1 + (1-x)y^2} \quad , \tag{F.2}$$

$$I_2 = \frac{1}{2} \int_o^1 \frac{dx}{\sqrt{x}} \int_o^1 dy \, \ln[x + x(1-x)y^2] \quad . \tag{F.3}$$

For I_1, after changing x to x^2 and performing the x integration, we obtain

$$I_1 = 2\int_o^1 dy \, \frac{(1-y)^2}{y\sqrt{1+y^2}} \ln(y + \sqrt{1+y^2}) \quad , \tag{F.4}$$

and by changing the variable $y + \sqrt{1 + y^2} = t$

$$I_1 = -2 + 2\sqrt{2}\,\ln(1+\sqrt{2}) - 2\ln^2(1+\sqrt{2}) + 4\int_1^{1+\sqrt{2}} dt\,\frac{\ln t}{t^2-1}\quad , \qquad (F.5)$$

or

$$I_1 = -2 + \frac{\pi^2}{4} + 2\sqrt{2}\,\ln(1+\sqrt{2}) - 3\ln^2(1+\sqrt{2}) \quad . \qquad (F.6)$$

For I_2, after performing y integration and changing x to x^2, we find

$$I_2 = \int_0^1 dx\left[2\ln x + \ln(2-x^2) - 2 + \frac{2}{\sqrt{1-x^2}}\tan^{-1}\sqrt{1-x^2}\right], \qquad (F.7)$$

or

$$I_2 = -6 + 2\sqrt{2}\,\ln(1+\sqrt{2}) + 2\int_0^1 dx\,\frac{1}{\sqrt{1-x^2}}\tan^{-1}\sqrt{1-x^2}\quad , \qquad (F.8)$$

and by changing $\sqrt{1-x^2} = t$

$$I_2 = -6 + 2\sqrt{2}\,\ln(1+\sqrt{2}) + 2\int_0^1 dt\,\frac{1}{\sqrt{1-t^2}}\tan^{-1}t \quad , \qquad (F.9)$$

or [31]

$$I_2 = -6 + \frac{\pi^2}{4} + 2\sqrt{2}\,\ln(1+\sqrt{2}) - \ln^2(1+\sqrt{2}) \quad . \qquad (F.10)$$

Therefore from (F.1) by virtue of (F.6) and (F.10) we obtain

$$\int d^n q \, \frac{q_o^2 \, q^2}{D} = i\pi^2\Big[-\Gamma(-a) - 8 + \frac{\pi^2}{2} + 4\sqrt{2}\,\ln(1 + \sqrt{2})$$

$$- 4\,\ln^2(1 + \sqrt{2})\Big] \quad . \qquad \text{(F.11)}$$

The next integral is

$$\int d^4 q \, \frac{q_o \, q^2}{D} = i\pi^2 \int_o^1 \frac{dx}{\sqrt{x}} \int_o^1 dy \, \frac{-y}{1 + (1 - y)^2 - x(1 - y)^2} \quad . \qquad \text{(F.12)}$$

Changing $x \to x^2$, $y \to 1 - y$ and performing the x integration we find

$$\int d^4 q \, \frac{q_o \, q^2}{D} = i\pi^2 \int_o^1 dy \, \frac{2(y - 1)}{y\sqrt{1 + y^2}} \ln(y + \sqrt{1 + y^2}) \quad , \qquad \text{(F.13)}$$

and by change of variable $y + \sqrt{1 + y^2} = t$ and then performing the t integration we obtain

$$\int d^4 q \, \frac{q_o \, q^2}{D} = i\pi^2 \left[- \frac{\pi^2}{4} + 2\,\ln^2(1 + \sqrt{2})\right] \quad . \qquad \text{(F.14)}$$

The last integral is

$$\int d^4 q \, \frac{q_o \, q_j \, q_\ell}{D} = i\pi^2(f \, \mathbf{k}_j \, \mathbf{k}_\ell + g \, \delta_{j\ell}) \;\; (j, \ell = 1, 2, 3) \quad , \qquad \text{(F.15)}$$

where

$$f = \int_0^1 \frac{dx}{\sqrt{x}} \int_0^x dy \int_0^y dz \, \frac{z(y - z)^2}{x\Delta^2} \quad , \tag{F.16}$$

$$g = \frac{1}{2} \int_0^1 \frac{dx}{\sqrt{x}} \int_0^x dy \int_0^y dz \, \frac{z}{x\Delta} \quad , \tag{F.17}$$

$$\Delta = \frac{z^2}{x} - (y - z)^2 + 2(y - z) \quad . \tag{F.18}$$

From (F.15) we infer the following relation (since we are interested in the leading order: $\vec{\kappa}^2 = \kappa^2 = 1$)

$$\int d^4q \, \frac{q_o \, \vec{q}^{\,2}}{D} = i\pi^2(f + 3g) \quad . \tag{F.19}$$

Comparing (4B.13) with (F.19) we find

$$f + 3g = 1 - \ln 2 \quad , \tag{F.20}$$

therefore we need to calculate f or g, but not both of them. We choose to calculate g.

From (F.17) and (F.18) we have

$$g = \frac{1}{2} \int_0^1 \frac{dx}{\sqrt{x}} \int_0^x dy \int_0^y dz \, \frac{z}{z^2 - x(y - z - 1)^2 + x} \quad , \tag{F.21}$$

or by using the following change of variables

$$z = \alpha \quad , \quad y - z - 1 = \beta \quad , \quad x = x \quad , \tag{F.22}$$

$$g = \frac{1}{2} \int_0^1 \frac{dx}{\sqrt{x}} \int_{-1}^{x-1} d\beta \int_0^{x-1-\beta} d\alpha \, \frac{\alpha}{\alpha^2 - x\beta^2 + x} \quad . \tag{F.23}$$

Performing the α integration, we find

$$g = \frac{1}{4} \int_0^1 \frac{dx}{\sqrt{x}} \left\{ \int_0^x du \ln[(1 - x)u^2 + x^2] - \int_{-1}^{x-1} d\beta \ln (x - x\beta^2) \right\} \quad , \tag{F.24}$$

where in the first integral we changed β to $u-1$. By performing the β and u integrations we obtain

$$g = \frac{1}{2} \int_0^1 \frac{dx}{\sqrt{x}} \left[\ln(2 - x) - \ln 2 + \frac{x}{\sqrt{1 - x}} \tan^{-1} \sqrt{1 - x} \right] \quad , \tag{F.25}$$

or by changing x to x^2

$$g = -2 - \ln 2 + 2\sqrt{2} \ln(1 + \sqrt{2}) + \int_0^1 dx \, \frac{x^2}{\sqrt{1-x^2}} \tan^{-1} \sqrt{1-x^2} \quad . \tag{F.26}$$

Let's represent the remaining integral by J

$$J = \int_0^1 dx \, \frac{x^2}{\sqrt{1 - x^2}} \tan^{-1} \sqrt{1 - x^2} \quad , \tag{F.27}$$

where by changing $1 - x^2 = y^2$ gets the following form

$$J = \int_0^1 dy \sqrt{1 - y^2} \tan^{-1} y \quad . \tag{F.28}$$

We write J in the following form

$$J = J_1 + J_2 \quad , \tag{F.29}$$

where

$$J_1 = \int_0^1 dy \frac{1}{\sqrt{1 - y^2}} \tan^{-1} y \quad , \tag{F.30}$$

and it is the same integral that we had in (F.9), so

$$J_1 = \frac{\pi^2}{8} - \frac{1}{2} \ln^2(1 + \sqrt{2}) \quad , \tag{F.31}$$

and J_2 is

$$J_2 = -\int_0^1 dy \frac{y}{\sqrt{1 - y^2}} (y \tan^{-1} y) \quad . \tag{F.32}$$

Let's try to find J_2 by integration by parts

$$J_2 = -\int_0^1 dy \sqrt{1 - y^2} \left(\tan^{-1} y + \frac{y}{1 + y^2}\right) \quad , \tag{F.33}$$

which by (F.28) we can write it as

$$J_2 = -J - \int_o^1 dy \sqrt{1 - y^2}\, \frac{y}{1 + y^2} \quad , \tag{F.34}$$

and by changing $1 - y^2 = x^2$ and then performing x integration we get

$$J_2 = -J + 1 - \sqrt{2}\,\ln(1 + \sqrt{2}) \quad . \tag{F.35}$$

Using (F.29), (F.31) and (F.35) we find

$$J = \frac{1}{2} + \frac{\pi^2}{16} - \frac{\sqrt{2}}{2}\ln(1 + \sqrt{2}) - \frac{1}{4}\ln^2(1 + \sqrt{2}) \quad . \tag{F.36}$$

Therefore (F.25), (F.28) and (F.36) give

$$g = -\frac{3}{2} + \frac{\pi^2}{16} - \ln 2 + \frac{3\sqrt{2}}{2}\ln(1 + \sqrt{2}) - \frac{1}{4}\ln^2(1 + \sqrt{2}) \quad , \tag{F.37}$$

and by virtue of (F.20), f is

$$f = \frac{11}{2} - \frac{3\pi^2}{16} + 2\ln 2 - \frac{9\sqrt{2}}{2}\ln(1 + \sqrt{2}) - \frac{3}{4}\ln^2\,(1 + \sqrt{2}) \quad . \tag{F.38}$$

Finally, the integral in (F.15) by virtue of (F.37) and (F.38) is

$$\int d^4q\, \frac{q_o\, q_j\, q_\ell}{D} = i\pi^2 \mathbf{k}_j \mathbf{k}_\ell \Big[\frac{11}{2} - \frac{3\pi^2}{16} + 2\ln 2 - \frac{9\sqrt{2}}{2}\ln(1 + \sqrt{2})$$

$$+ \frac{3}{4}\ln^2(1 + \sqrt{2})\Big]$$

$$+ i\pi^2 \delta_{j\ell}\left[- \frac{3}{2} + \frac{\pi^2}{16} - \ln 2 + \frac{3\sqrt{2}}{2} \ln(1 + \sqrt{2})\right.$$

$$\left.- \frac{1}{4} \ln^2(1 + \sqrt{2})\right] \quad (i,j = 1, 2, 3) \quad . \tag{F.39}$$

Appendix G
Evaluation of some Integrals related to the Contribution M_4

In this Appendix we evaluate four integrals that we need in evaluation of M_4. In these integrals D is defined by (4D.4), and q integrations (except for the integral in (G.3)) are performed by using the Tables given in [19].

The first integral is (in all integrals, except (G.11), we put $\gamma = o$)

$$\int \frac{d^4q}{D} \left(1 - \frac{q_i}{k_i}\right) = -i\pi^2 \int_o^1 dx \int_o^x dy \frac{x}{4y^2 - 4xy + 1}$$

$$= -i\pi^2 \int_o^1 dx \frac{x}{\sqrt{1 - x^2}} \tan^{-1}\left(\frac{x}{\sqrt{1 - x^2}}\right) \quad , \qquad \text{(G.1)}$$

and by integration by parts we obtain

$$\int \frac{d^4q}{D} \left(1 - \frac{q_i}{k_i}\right) = -i\pi^2 \quad . \qquad \text{(G.2)}$$

The next integral is

$$I_1 = \int \frac{d^4q}{D} \frac{-4q_o^2}{q^2} \quad , \qquad \text{(G.3)}$$

which using the Feynman parametrization method, we find

$$I_1 = -2i\pi^2 \int_0^1 dx \int_0^x dy \int_0^y dz \left[\frac{1}{\Delta} + \frac{2(x^2 - 1)}{\Delta^2}\right] \quad , \tag{G.4}$$

where

$$\Delta = (2y - z - x)^2 + 1 - x^2 \quad . \tag{G.5}$$

By changing the variables

$$2y - z - x = v \quad , \quad z = u \quad , \quad x = x \quad , \tag{G.6}$$

the integral gets the following form

$$I_1 = -2i\pi^2 \int_0^1 dx \int_0^x du \int_0^{x-u} dv \left[\frac{1}{v^2 + 1 - x^2} + \frac{2(x^2 - 1)}{(v^2 + 1 - x^2)^2}\right] \quad , \tag{G.7}$$

or after v integration

$$I_1 = 2i\pi^2 \int_0^1 dx \int_0^x du \, \frac{x - u}{(x - u)^2 + 1 - x^2} \quad . \tag{G.8}$$

Changing the variables $x - u = t$, $x = x$ and performing the t integration, we find

$$I_1 = -i\pi^2 \int_0^1 dx \, \ln(1 - x^2) \quad , \tag{G.9}$$

which after x integration and using (G.3), we obtain

$$\int \frac{d^4q}{D} \frac{-4q_o^{\,2}}{q^2} = -i\pi^2(2 \ln 2-2) \quad . \tag{G.10}$$

Now we consider the following integral (wherever possible we put $\gamma = 0$)

$$I_2 = \int \frac{d^4q}{D} \frac{4}{\vec{q}^{\,2} + \gamma^2} \quad , \tag{G.11}$$

which by using Tables of [19] it gives

$$I_2 = -4i\pi^2 \int_o^1 \frac{dx}{\sqrt{x}} \int_o^x dy \int_o^y dz \frac{1}{\Delta^2} \quad , \tag{G.12}$$

where approximately (we ignore higher orders, since they give $o(\alpha^2)$ corrections)

$$\Delta = \frac{1}{x}(2z - y)^2 + (x - y)(2 + y - x) + \gamma^2 \quad . \tag{G.13}$$

By following transformations

$$2z - y = u \,, \quad x - y = v \,, \quad x = x \quad , \tag{G.14}$$

I_2 gets the following form

$$I_2 = -4i\pi^2 \int_o^1 \frac{dx}{\sqrt{x}} \int_o^x dv \int_o^{x-v} du \frac{1}{\Delta^2} \tag{G.15}$$

where

$$\Delta = \frac{u^2}{x} + v(2 - v) + \gamma^2 \quad . \tag{G.16}$$

After performing u integration, we find

$$I_2 = -2i\pi^2 \int_0^1 dx\sqrt{x} \int_0^x dv \frac{x - v}{a[(x - v)^2 + ax]}$$

$$- 2i\pi^2 \int_0^1 dv \frac{1}{a\sqrt{a}} \int_v^1 dx \tan^{-1}(\frac{x - v}{\sqrt{ax}}) \quad , \tag{G.17}$$

where

$$a = v(2 - v) + \gamma^2 \quad . \tag{G.18}$$

In (G.17), after using integration by parts on the x variable of the second integral, we obtain

$$I_2 = -2i\pi^2 \int_0^1 dv \frac{1}{a\sqrt{a}} \tan^{-1} (\frac{1 - v}{\sqrt{a}})$$

$$-i\pi^2 \int_0^1 dx\sqrt{x} \int_0^x dv \frac{1}{a} \frac{x - 3v}{ax + (x - v)^2} \quad , \tag{G.19}$$

and by changing the variable $\frac{1-v}{\sqrt{a}} = t$ in the first integral, and performing v integration in the second integral, we find (up to the $o(\alpha)$ correction)

$$I_2 = -2i\pi^2 \int_0^{\frac{1}{\gamma}} dt \tan^{-1} t$$

$$-i\pi^2\int_0^1 \frac{dx}{\sqrt{x}}\,[\frac{1}{2}\ln x - \ln \gamma - \frac{x^2 - 5x + 2}{2(2 - x)^2}\ln(\frac{2 - x}{4})]$$

$$-i\pi^2\int_0^1 \frac{dx}{\sqrt{x}}\,\frac{(x - 6)\sqrt{1 - x}}{(x - 2)^2}\tan^{-1}\sqrt{1 - x} \quad . \qquad \text{(G.20)}$$

After performing the first and second integrals, and changing the variable $\sqrt{1 - x} = t$ in the third integral, we obtain

$$I_2 = -i\pi^2[\frac{\pi}{\gamma} - 1 - \frac{5}{2}\sqrt{2}\ln(1 + \sqrt{2})]$$

$$+ 2i\pi^2\int_0^1 dt\,\frac{1}{\sqrt{1 - t^2}}\tan^{-1}t$$

$$- 2i\pi^2\int_0^1 dt\,\frac{1}{\sqrt{1 - t^2}}\,\frac{1 - 3t^2}{(1 + t^2)^2}\tan^{-1}t \quad . \qquad \text{(G.21)}$$

Relation (F.31) gives the value of the first integral, and for the second integral we use the following relation

$$\tan^{-1}t = \frac{1}{2}\cos^{-1}\frac{1 - t^2}{1 + t^2} \quad , \qquad \text{(G.22)}$$

and the change of variable $\frac{1 - t^2}{1 + t^2} = s$, or

$$\int_0^1 dt\,\frac{1}{\sqrt{1 - t^2}}\,\frac{1 - 3t^2}{(1 + t^2)^2}\tan^{-1}t = \frac{\sqrt{2}}{8}\int_0^1 ds\,\frac{2s - 1}{\sqrt{s - s^2}}\cos^{-1}s \qquad \text{(G.23)}$$

$$= -\frac{\sqrt{2}}{4}\int_0^1 ds \, \frac{s}{\sqrt{s + s^2}} \tag{G.24}$$

$$= -\frac{1}{2} + \frac{1}{4}\sqrt{2}\ln(1 + \sqrt{2}) \quad , \tag{G.25}$$

where, using integration by parts in (G.23) we found (G.24).

Therefore, using (G.11), (G.21), (F.31) and (G.25) we obtain (up to the $o(\alpha)$ correction)

$$\int \frac{d^4q}{D} \frac{4}{\vec{q}^2 + \gamma^2} = -i\pi^2\Big[\frac{\pi}{\gamma} - 2 - 2\sqrt{2}\ln(1 + \sqrt{2})$$

$$- \frac{\pi^2}{4} + \ln^2(1 + \sqrt{2})\Big] \quad . \tag{G.26}$$

Next, we consider the following integral

$$I_3 = \int \frac{d^4q}{D}\left(-\frac{q_o^2}{\vec{q}^2} - \frac{q_o^2 \, q_i}{\vec{q}^2 \, k_i}\right) \quad (i = 1, 2, 3) \quad , \tag{G.27}$$

which using Tables of [19], gives

$$I_3 = i\pi^2\int_0^1 \frac{dx}{x\sqrt{x}} \int_0^x dy \int_0^y dz \, (x - y + 1)\left[\frac{(2z - y)^2}{x\Delta^2} - \frac{1}{2\Delta}\right] \quad , \tag{G.28}$$

where

$$\Delta = \frac{1}{x}(2z - y)^2 + (x - y)(2 + y - x) \quad . \tag{G.29}$$

Using following transformation

$$2z - y = u , \quad x - y = v , \quad x = x \quad , \tag{G.30}$$

I_3 gets the following form

$$I_3 = i\pi^2 \int_0^1 \frac{dx}{\sqrt{x}} \int_0^x (v + 1) dv \int_0^{x-v} du \{ \frac{u^2}{[u^2 + v(2 - v)x]^2}$$

$$- \frac{1}{2} \frac{1}{u^2 + v(2 - v)x} \} \quad , \tag{G.31}$$

and after performing u integration

$$I_3 = \frac{i\pi^2}{2} \int_0^1 \frac{dx}{\sqrt{x}} \int_0^x dv \, \frac{(v + 1)(v - x)}{v^2(1 - x) + x^2} \quad . \tag{G.32}$$

Performing v integration in (G.32) gives

$$I_3 = \frac{i\pi^2}{2} \int_0^1 \frac{dx}{\sqrt{x}} \, [\tfrac{1}{2}\ln(2-x) + \frac{x}{1 - x} - \frac{1}{(1 - x)^{3/2}} \tan^{-1}\sqrt{1 - x}] \; , \tag{G.33}$$

and after performing the x integration on the first term, and changing the variable $x = 1 - t^2$ in other terms, we obtain

$$I_3 = i\pi^2[-1 + \sqrt{2}\,\ln(1 + \sqrt{2})]$$

$$+ i\pi^2 \int_0^1 dt [\frac{\sqrt{1 - t^2}}{t} - \frac{1}{\sqrt{1 - t^2}} \tan^{-1} t - \frac{\sqrt{1 - t^2}}{t^2} \tan^{-1} t] \quad . \tag{G.34}$$

Using the following relation (it can be found by integration by parts)

$$\int_o^1 dt \frac{1}{t^2} (\sqrt{1 - t^2} \tan^{-1} t) = 1 + \int_o^1 dt \left(\frac{-1}{\sqrt{1 - t^2}} \tan^{-1} t + \frac{\sqrt{1 - t^2}}{t} \right.$$

$$\left. - \frac{t\sqrt{1 - t^2}}{1 + t^2} \right) \quad , \qquad (G.35)$$

(G.34) gets the following form

$$I_3 = i\pi^2[-2 + \sqrt{2} \ln(1 + \sqrt{2})] + i\pi^2 \int_o^1 dt \frac{t\sqrt{1 - t^2}}{1 + t^2} \quad , \qquad (G.36)$$

or

$$I_3 = -i\pi^2[3 - 2\sqrt{2} \ln(1 + \sqrt{2})] \quad , \qquad (G.37)$$

and finally by (G.27)

$$\int \frac{d^4q}{D} \left(- \frac{q_o^{\ 2}}{\vec{q}^{\,2}} - \frac{q_o^{\ 2}}{\vec{q}^{\,2}} \frac{q_i}{k_i} \right) = -i\pi^2[3 - 2\sqrt{2} \ln(1 + \sqrt{2})] \quad . \qquad (G.38)$$

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