$\mathbb{Z}_{2p}$ - ACTIONS ON THE 2-DIMENSIONAL AND THE SOLID KLEIN BOTTLES

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ABSTRACT

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AND THE SOLID KLEIN BOTTLES

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In this thesis PL homeomorphisms of periods p and $2p$ are classified on both the 2-dimensional Klein bottle K^2 and the solid Klein bottle K, where p is an odd prime number.

It is shown that up to weak equivalence there is only one class of homeomorphisms of period p on K^2 and only three equivalence classes of homeomorphisms of period $2p$ on K^2, distinguished by the fixed point sets of their p^{th} powers.

Also, free cyclic actions of odd period are classified on K as well as cyclic actions of period $2p$. In the first case it is shown that, up to weak equivalence, only one such action exists, while in the second case there are three such homeomorphisms, distinguished by the fixed point sets of their p^{th} power.

Finally, semi-free action on K are classified for any finite period n. It is shown that these exist only for n equals two and for all odd values of n such that $F(h^n) = \emptyset$.

TO MY PARENTS

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INTRODUCTION

Let X and X' be topological spaces. Two homeomorphisms h and g on X and X' respectively are said to be weakly equivalent (written $h \sim_{\omega} g$) if there exists a homeomorphism $t : X \to X'$ such that $t^{-1}gt = h^i$ for some positive integer $i \neq 1$.

All maps and spaces considered in this thesis are in the piecewise linear category.

In this thesis we classify piecewise linear homeomorphisms of periods p and $2p$, p an odd prime, on both the 2-dim and the solid Klein bottles.

Chapter 1 deals with the homeomorphisms of periods p and $2p$ on the 2-dim Klein bottle K^2, p odd prime. Proposition 1 of section 1.1 asserts that up to weak equivalence there is a unique homeomorphism of period p on K^2. Theorem 1 of section 1.2 gives all the homeomorphisms of period $2p$ on K^2 up to weak equivalence, p odd prime. In fact there are three such homeomorphisms h_i distinguished by the fixed point sets of $h_i{}^p$, $i = 1, 2, 3$.

Chapter 2 is divided into three sections. Section 2.1 deals with the free homeomorphisms of odd period on the solid Klein bottle K. Proposition 3 of this section states that there is only one such homeomorphism up to weak equivalence. Section 2.2 provides a complete classification of homeomorphisms of period $2p$ on K, p odd prime. The main theorem of this section is the classification Theorem 1 which asserts that up to weak equivalence there are only three such homeomorphisms h_i, distinguished by $F(h_i^p)$, $i = 1, 2, 3$. Finally section 2.3 deals with the semi-free periodic actions on K. These actions are given in Theorem 1 of that section.

CHAPTER 0

PRELIMINARIES AND DEFINITIONS

Throughout this thesis, we work in the PL (piecewise linear) category and all spaces and maps will be piecewise linear.

A homeomorphism $h : X \to X$ is <u>periodic</u> if $h^n =$ identity for $n > 1$ in $\mathbb{Z}$. If $n = 2$, h is said to be an involution.

Let h be a periodic map of a space X. The cyclic group generated by h will be denoted by $\langle h \rangle$. If h is periodic on X, then the <u>orbit space</u> of h is the quotient space obtained by identifying x with $h^i(x)$ for all i and all x in X. The orbit space of h will be denoted by $X/\langle h \rangle$. The identification map $p_h : X \to X/\langle h \rangle$ is called the <u>orbit map</u>. When there is no confusion p_h will be simply written as p.

Two actions of $\langle h \rangle$ and $\langle f \rangle$ on X are said to be <u>weakly equivalent</u> if there is a homeomorphism t of X such that $\langle tht^{-1} \rangle = \langle f \rangle$ and $tht^{-1} = f^i$ for some i. Equivalently h and f are weakly equivalent if there are homeomorphisms t and $\bar{t}$ that make the diagram commutative

$$\begin{tikzcd} X \arrow[r, "\bar{t}"] \arrow[d, "p_h"'] & X \arrow[d, "p_f"] \\ X/\langle h\rangle \arrow[r, "t"] & X/\langle f\rangle \end{tikzcd}$$

i·e $p_f\bar{t} = tp_h$. With a special consideration of the fixed point sets of h and f this implies that $\bar{t}^{-1} f \bar{t}$ is a linear transformation (automorphism) of X with respect to h i·e it belongs to the group of deck transformations $A(X, h)$ with the connection with the proposition on page 5, this implies weak equivalence. If $tht^{-1} = f$, then h and f are said to be equivalent.

The set $\{x \in X | h(x) = x\}$ of the fixed points of h will be denoted by $F(h)$ or $Fix(h)$.

Concerning the action we assume that (1) for every $h^i \in \langle h\rangle$, $F(h^i)$ is a subcomplex of X; (2) the natural cell structure of the orbit space $X/\langle h\rangle$ and the orbit map $p : X \to X/\langle h\rangle$ are simplicial and (3) p maps each simplex homeomorphically.

According to [16] these conditions are not restrictive, from the PL point of view.

The following proposition concerning the free action of periodic homeomorphisms proves to be very useful in proving weak equivalence, where $h : X \to X$ <u>acts freely</u> or is <u>free</u> if $F(h) = \emptyset = F(h^i)$ for all i for which $h^i \neq$ identity.

<u>Proposition</u>.

Let $(X, \langle h \rangle)$ be a free periodic action in which h is of finite period, and X is a connected manifold. Let $\overline{X} = X/\langle h \rangle$ be the orbit space and $p : X \to \overline{X}$ be the orbit map. Let $x \in X$, $\overline{x} = p(x)$. Then $\overline{X}$ is a connected manifold, (X, p) a regular cover, and $\langle h \rangle \cong \pi_1(\overline{X}, \overline{x}) / p_*(\pi_1(X, x))$.

For a proof see proposition 8.2, Chapter 5 of [11]. Also see [11] for terminology of covering spaces.

A periodic homeomorphism $h : X \to X$ acts <u>semi-freely</u> on X, if $F(h) = F(h^i)$ for all $1 < i < n$, where n is the period of h. In other words any point $x \in X$ is either fixed by all h^i or is moved by all h^i, $1 < i < n$.

A compact not necessarily connected 2-manifold F is said to be <u>2-sided</u> in X if there exists a neighborhood of F of the form $F \times [-1, 1]$ with $F = F \times 0$ and $(F \times [-1, 1]) \cap \partial X = \partial F \times [-1, 1]$.

A simple closed curve J embedded in a closed surface S is <u>2-sided</u> if there is a neighborhood of J in S of the form $J \times [-1, 1]$ with $J \approx J \times \{0\}$. A simple closed curve J embedded in a closed surface S is <u>one-sided</u>

if it doesn't separate any connected neighborhood of J in S.

A 3-manifold X in which every embedded sphere bounds a 3-cell is called <u>irreducible</u>. If in addition X doesn't contain a 2-sided projective plane P^2, X is called <u>P^2-irreducible</u>.

A surface F is <u>properly embedded</u> in X if $F \cap \partial X = \partial F$.

A properly embedded disk E in X with ∂E does not bound a disk in ∂X is called a <u>meridional disk</u>.

Let F be two-sided surface in X. The manifold X' obtained by <u>splitting (cutting)</u> X at (along) F is the manifold whose boundary contains two copies, F^+ and F^-, of F such that the identification of F^+ and F^- defines a natural projection $f : (X', F^+ \cup F^-) \to (X, F)$ with $f|(X' - (F^+ \cup F^-))$ a homeomorphism onto $X - F$. Note that X' is homeomorphic to $X - (F \times (-1, 1))$.

The <u>2-dimentional Klein bottle</u> may be regarded as the identification space obtained as follows:
$\{[r z] : \frac{1}{2} \le r \le 2, \ | z | = 1, \ \frac{1}{2} z \sim -\frac{1}{2} z, \ 2 z \sim -2z\}$.
Also it can be considered as the space obtained by gluing together two Mobius bands along their boundaries, where a <u>Mobius band</u> is the space that is formed by identifying

(s, t) with $(-s, t+1)$ in $[-1, 1] \times \mathbb{R}$.

The <u>3-dim Klein bottle</u> is the space that is obtained from $D^2 \times \mathbb{R}$ by identifying (z, t) with $(\bar{z}, t+1)$. An element of this space sith representative (z, t) will be denoted by $[z, t]$. $D^2 = \{z \in C \mid |z| \leq 1\}$, $\mathbb{R}$ the field of real numbers. C = field of complex numbers. $S^1 = \{z \in C \mid |z| = 1\}$.

CHAPTER ONE

$\mathbb{Z}_{2p}$ - ACTIONS ON THE 2-DIMENSIONAL KLEIN BOTTLE

In this chapter we classify all the PL homeomorphisms of periods p and $2p$, where p is an odd prime on the two-dimensional Klein bottle.

<u>Notations:</u>

In this chapter we denote the 2-dim Klein bottle by K while in Chapter 2 we will use K^2 for the 2-dim Klein bottle and K for the solid Klein bottle.

<u>Section 1.1.</u> $\mathbb{Z}_p$ - ACTIONS ON THE TWO-DIMENSIONAL KLEIN BOTTLE K.

We consider only the case where p is an odd prime number. For $p = 2$ see [12] (also see [8]). Our main result is the following:

<u>Proposition 1.</u>

If $h : K \to K$ is a pl homeomorphism of period p on the 2-dim Klein bottle K, where p is an odd prime, then $F(h) = \emptyset$ and h is weakly equivalent to $h_1 : K \xrightarrow{\approx} K$, where h_1 is defined by $h_1([r\, z]) = [w\, r\, z]$, $w = e^{2\pi i/p}$.

First we prove the following.

Lemma 2.

If h is as in proposition 1, then $F(h) = \emptyset$ and the orbit space $K/\langle h\rangle \approx K$.

Proof

Let $B = K/\langle h\rangle$. By [4] we have $\chi(k) + (p-1)\chi(F(h)) = p\chi(B)$, where χ indicates the Euler characteristic. Since $\chi(K) = 0$, $(p-1)\chi(F(h)) = p\chi(B)$. Since p is odd, dim $(F(h)) \neq 1$. So $F(h)$ is either 0-dim or $\emptyset$. Assume that dim $(F(h)) = 0$ and that $F(h)$ consists of k points. Assume $k > 0$. Then $(p-1)k = p\chi(B)$ and this implies that $k = \frac{p}{p-1}\chi(B) \cdot k > 0$ yields that $(p-1) | \chi(B)$ and since B is a surface, $\chi(B) \leq 2$, so $p \leq 3$. But p is an odd prime so $p = 3$. Hence $\chi(B) = 2$ and then $B \approx S^2$, the 2-sphere. But this is impossible since K is nonorientable. So k must be 0 contradicting our assumption $k > 0$. So $F(h) = \emptyset$. Moreover, $\chi(B) = 0$. Since $F(h) = \emptyset$, $\partial B = \emptyset$ and since B is nonorientable $B \approx K$.

Proof of Proposition 1.

It is clear that $F(h_1) = \emptyset$ and $K/\langle h_1\rangle \approx K$ (lemma 2). Let $h_2 = h$. By lemma 2 above $K/\langle h_2\rangle \approx K$. $\langle h_1\rangle$ and $\langle h\rangle$ act freely on K. Let $q_i : K \to K/\langle h_i\rangle$, $i = 1, 2,$ be the orbit maps. Let t be any homeomorphism: $K/\langle h_1\rangle \to K/\langle h_2\rangle$. Since $F(h_i) = \emptyset$, q_1 and q_2 are p-covering projections of K by K. But $\pi_1(K)$ has a unique normal subgroup of index p, so t can be lifted to a homeomorphism $\bar{t} : K \to K$ such that the diagram below is commutative

$$\begin{tikzcd} K \arrow[r, "\bar{t}"] \arrow[d, "q_1"'] & K \arrow[d, "q_2"] \\ K/\langle h_1 \rangle \arrow[r, "t"] & K/\langle h_2 \rangle \end{tikzcd}$$

i.e $q_2\bar{t} = tq_1$. By the commutativity of the diagram

$$\begin{tikzcd} K \arrow[r, "\bar{t}"] \arrow[d, "q_1"'] & K \arrow[r, "h_2"] \arrow[dr, "q_2"'] & K \arrow[d, "q_2"] & K \arrow[l, "\bar{t}"'] \arrow[d, "q_1"] \\ K \arrow[rr, "t"] & & K & K \arrow[l, "t"'] \end{tikzcd}$$

We obtain $q_1 = q_1\bar{t}^{-1}h_2\bar{t}$. That is $\bar{t}^{-1}h_2\bar{t}$ is a non-trivial covering transformation on K with respect to q_1. But the group of covering transformations $A(K,q_1) \cong \langle h_1 \rangle$ (see Chapter 0). So $\bar{t}^{-1}h_2\bar{t} = h_1^{\,i}$ for some $1 \leq i < p$. This shows that $h \sim_w h_1$ and completes the proof of proposition 1.

Section 1.2. $\mathbb{Z}_{2p}$-ACTIONS ON THE 2-DIMENSIONAL KLEIN BOTTLE.

Theorem 1.

If $h : K \to K$ is a PL homeomorphism of period $2p$, where p is an odd prime, then h is weakly equivalent to one of the following $2p$- periodic homeomorphisms depending on whether $F(h^p)$ is $\emptyset$, $\approx S^1$ or $\approx S^1 \dot{\cup} S^1$ respectively

$$h_1([r\ z]) = [\frac{w}{r}\ z]$$

$$h_2([r\ z]) = [-\frac{w}{r}\ z]$$

$$h_3([r\ z]) = [w\ r\ z]$$

where $w = e^{\pi i/p}$.

The proof of theorem 1 occupies the rest of this section.

Since h is of period $2p$, h^2 is of period p on K, so by lemma 2 of Section 1.1, $F(h^2) = \emptyset$. The map h induces an involution h^- on $K/\langle h^2\rangle$, uniquely determined by h, such that $h^-q = qh$ where $q : K \to K/\langle h^2\rangle$ is the orbit map.

Now h^p is an involution on K. By [12] (also see [8]) $F(h^p)$ is one of the following sets: $\emptyset$, two points, one 2-sided nonseparating simple closed curve and two points, one 2-sided separating simple closed curve and finally two one-sided simple closed curves. In other words $F(h^p)$ is homeomorphic to one of the following sets: $\emptyset$, S^0, $S^1 \dot{\cup} S^0$, S^1, $S^1 \dot{\cup} S^1$. So we consider the following two cases

Case 1: $F(h^p) = \emptyset$

Case 2: $F(h^p) \neq \emptyset$

Case 1.

$F(h^p) = \emptyset$. In this case h acts freely on K. We prove the following:

Proposition 2.

If $h : K \to K$ is a homeomorphism of period $2p$ on K, p odd prime, and if h acts freely, then h is weakly equivalent to h_1, where h_1 is as in theorem 1.

Proof

Let $h = h_2$. Let $M_i = K/\langle h_i{}^2\rangle$ and $q_i : K \to M_i$, $q'_i : M_i \to M_i/\langle \bar{h}_i\rangle$, $i = 1, 2$, be the orbit maps, where $\bar{h}_1$ and $\bar{h}_2$ are the induced involutions by h_1 and h_2 on M_1 and M_2 respectively. Note that $F(\bar{h}_i) = q_i(F(h_i{}^p))$, hence $F(\bar{h}_i) = \emptyset$, $i = 1, 2$. As in lemma 2 Section 1.1, $K/\langle \bar{h}_i\rangle \approx K$. By lemma 2, $M_i \approx K$, so $M_i/\langle \bar{h}_i\rangle \approx K/\langle \bar{h}_i\rangle \approx K$. Note also $K/\langle \bar{h}_i\rangle \approx K/\langle h_i\rangle$. Let $g_i = q'_i q_i : K \to M_i/\langle \bar{h}_i\rangle$. Note that q_i is a p- covering projection for $i = 1, 2$ and q'_i is a 2- covering projection for $i = 1, 2$. Since $F(h_i{}^p) = \emptyset = F(h^2) = F(h)$, $\langle h_i\rangle$ acts freely on K, so g_i is a $2p$- covering projection for $i = 1, 2$. Let $t : K/\langle h_1\rangle \to K/\langle h_2\rangle$ be any homeomorphism. By [14] up to equivalence there is only one subgroup of indes 2 of $\pi_1(K)$ corresponding to the double cover of K by K. So M_1 and M_2 are equivalent, hence $\exists\ \bar{t} : M_1 \overset{\approx}{\to} M_2$ such that $tq'_1 = q'_2\bar{t}$

$$\begin{tikzcd}
K \arrow[r, "\bar{\bar{t}}"] \arrow[d, "q_1"'] & K \arrow[d, "q_2"] \\
M_1 \arrow[r, "\bar{t}"] \arrow[d, "q'_1"'] & M_2 \arrow[d, "q'_2"] \\
K/\langle h_1\rangle \arrow[r, "t"] & K/\langle h_2\rangle
\end{tikzcd}$$

Now since q_1 and q_2 are p- covering projections and $\pi_1(K)$ has a unique normal subgroup of index p, t can be lifted to $\bar{\bar{t}} : K \overset{\approx}{\to} K$ such that $\bar{t}q_1 = q_2\bar{\bar{t}}$. Hence $tq'_1q_1 = q'_2q_2\bar{\bar{t}}$ i.e $tg_1 = g_2\bar{\bar{t}}$. As in the proof of proposition 1, section 1.1, $\bar{\bar{t}}\, h_2\, \bar{\bar{t}}^{\,-1}$ is a covering transformation on K. Note that g_1 and g_2 are $2p$- covering transformations and the group of covering transformations of K with respect to g_1, $A(K, g_1) \cong \langle h_1\rangle$ (see Chapter 0). So we obtain that $\bar{\bar{t}}\, h_2\, \bar{\bar{t}}^{\,-1} = h_1^{\,i}$ for some $1 \le i < 2p$.

Hence $h_2 = h \sim_w h_1$. This finishes the proof of proposition 2 and case 1.

Case 2.

$F(h^p) \neq \emptyset$. By [12] (or [8]) $F(h^p)$ is homeomorphic to one of the following sets: S^0, $S^0 \dot{\cup} S^1$, S^1, $S^1 \dot{\cup} S^1$. If $F(h^p) \approx S^0 = \{x, y\}$, then since $h(F(h^p)) = F(h^p)$ and $F(h) = \emptyset$, $h(x) = y$ and $h(y) = x$. But this implies that $h^2 x = x$ and $h^2 y = y$ contradicting the fact that $F(h^2) = \emptyset$. So $F(h^p)$ cannot be $\approx S^0$.

If $F(h^p) \approx S^1 \dot{\cup} S^0$, then since $F(h^p)$ is invariant under h, we get $h(S^0) = S^0$ and $h(S^1) \approx S^1$. As above this contradicts $F(h^2) = \emptyset$. Hence $F(h^p)$ cannot be $\approx S^0 \dot{\cup} S^1$.

Now we are left with the two possibilities: $F(h^p) \approx S^1$ or $S^1 \dot{\cup} S^1$.

Lemma 3.

If $F(h^p) \approx S^1$ or $S^1 \dot{\cup} S^1$, then $K/\langle h^p \rangle$ is homeomorphic to Mobius band or an annulus respectively.

Proof

Let $B = K/\langle h^p \rangle$. Since h^p is an involution on K, we have by [4] that $\chi(K) + \chi(F(h^p)) = 2\chi(B)$. But $F(h^p)$ is one-dim, so $\chi(F(h^p)) = 0$. And since $\chi(K) = 0$, $\chi(B) = 0$. So B is homeomorphic to one of the spaces: K, $T^2 =$ the 2-dim torus, Mobius band M, an annulus A.

Now we claim that $\partial(B) = q(F)$, where $q : K \to B$ is the orbit map and $F = F(h^p)$. If $x \in K - F$, then x has a neighborhood $U_x \approx \mathbb{R}^2$ and $q(U_x) \approx U_x$. $q(U_x)$ is a neighborhood of $q(x) \in B$ and since $q(U_x) \approx U_x \approx \mathbb{R}^2$, $q(x) \in \mathrm{Int}(B)$. If $x \in F$, then x has a neighborhood $U_x \approx \mathbb{R}^2$. $F \cap U_x$ separates U_x into two components $\approx \mathbb{R}^2_+$ and $q(U_x) \approx \mathbb{R}^2_+$ and is a neighborhood of $q(x) \in B$, i.e $q(x) \in \partial B$. This proves the claim. So $\partial B \neq \emptyset$, hence B cannot be $\approx K$ or T^2, for these have no boundary.

If $F \approx S^1$, then B has one boundary component and $B \approx M$.

If $F \approx S^1 \cup S^1$, then ∂B has two boundary components and $B \approx A$. □

Now, let h_2 and h_3 be as in theorem 1. We prove the following lemma.

<u>Lemma 4.</u>

Let $h : K \to K$ be a periodic homeomorphism of period $2p$, p odd prime, and assume that $F(h^p) \approx S^1$. Then $h \sim_w h_2$.

<u>Proof</u>

h induces $h^- : K/\langle h^p\rangle \to K/\langle h^p\rangle$ of period p and h^- acts freely on $K/\langle h^p\rangle$. Similarly h_2 induces $h^-_2 :$ $K/\langle h_2{}^p\rangle \to K/\langle h_2{}^p\rangle$ of period p and h^-_2 acts freely on $K/\langle h_2{}^p\rangle$. By lemma 3 above $K/\langle h^p\rangle \approx M$, a Mobius band and $k/\langle h_2{}^p\rangle \approx M$. Let $F = F(h^p)$ and $F_2 = F(h_2{}^p)$. Let $K' = K/\langle h^p\rangle$, $K_2 = K/\langle h_2{}^p\rangle$ and $q : K \to K'$, $q_1 : K \to K_2$ be the orbit spaces and orbit maps respectively. Also let $q'_1 : K_2 \to K_2/\langle \bar{h}_2\rangle \approx K/\langle h_2\rangle$, $q' : K \to K'/\langle \bar{h}\rangle \approx K/\langle h\rangle$ be the orbit maps of K_2 and K'. $q_1(F_2) = J_2 \approx S^1$ and $q(F) = J \approx S^1$ are the boundaries of K_2 and K' respectively. $q'_1(J_2)$ and $q'(J)$ are the boundaries of $K/\langle h_2\rangle$ and $K/\langle h\rangle$ respectively. Since $\bar{h}$ and $\bar{h}_2$ are free of period p on M, an arguement like the one used in section 1.1 using the Euler characteristic yields $K_2/\langle \bar{h}_2\rangle \approx M \approx K'/\langle \bar{h}\rangle$. Let $t : q'(J) \to q'_1(J_2)$ be a homeomorphism. Extend t to a homeomorphism on all of $K/\langle h\rangle$ still call it t.

$$\begin{array}{ccc}
K & \xrightarrow{\;\bar{\bar{t}}\;} & K \\
{\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q_1} \\
K' & \xrightarrow{\;\bar{t}\;} & K_2 \\
{\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle q'_1} \\
K/\langle h\rangle & \xrightarrow{\;t\;} & K/\langle h_2\rangle
\end{array}$$

Since q' and q'_1 are p- covering projections and $\pi_1(M) \cong \mathbb{Z}$ has an unique normal subgroup of index p, we can lift t to a homeomorphism $\bar{t} : K' \to K_2$ such that $q'_1 \bar{t} = tq'$ and $\bar{t}(J) = J_2$. Now $q_1 : K - F_2 \to K_2 - J_2$ and $q : K - F \to K' - J$ are double covering projections and $\pi_1(K_2 - J_2) \cong \mathbb{Z} \cong \pi_1(K' - J)$ we can lift $\bar{t}$ to a homeomorphism $\bar{\bar{t}} : K - F \to K - F_2$ such that $q_1 \bar{\bar{t}} = \bar{t} q$. Since $\bar{\bar{t}}(F) = F_2$ and $\bar{\bar{t}}$ is continuous this is true on all of K.

Now, q' is a covering projection and the diagram below commutes

$$\begin{tikzcd}
K' \arrow[r, "\bar{t}"] \arrow[d, "q'"'] & K_2 \arrow[rr, "\bar{h}_2"] \arrow[dr, "q'_1"'] & & K_2 \arrow[dl, "q'_1"] & K' \arrow[l, "\bar{t}"'] \arrow[d, "q'"] \\
K/\langle h\rangle \arrow[rr, "t"] & & K/\langle h_2\rangle & & K/\langle h\rangle \arrow[ll, "t"']
\end{tikzcd}$$

hence for some i, $\bar{t}^{-1} \bar{h}_2 \bar{t} = h^{-i}$. Also, since $qh = \bar{h} q$ and $q_1 h_2 = \bar{h}_2 q_1$, it follows that: $qh^i = h^{-i} q$. Hence $qh^i = h^{-i} q = \bar{t}^{-1} \bar{h}_2 \bar{t} q = \bar{t}^{-1} \bar{h}_2 q_1 \bar{\bar{t}} = \bar{t}^{-1} q_1 h_2 \bar{\bar{t}} = q \bar{\bar{t}}^{-1} h_2 \bar{\bar{t}}$. Thus, if $x \in F$, then $h^{i+p}(x) = h^i(x)$, note that here $q|F$ is a homeomorphism. If $x \notin F$, then $h^{i+p}(x) = \bar{\bar{t}}^{-1} h_2 \bar{\bar{t}}(x)$ or $h^i(x) = \bar{\bar{t}}^{-1} h_2 \bar{\bar{t}}(x)$.

Therefore $h \sim_w h_2$.

Finally in order for the proof of theorem 1 to be complete we prove the following.

<u>Lemma 5.</u>

Let $h : K \to K$ be a periodic homeomorphism of period $2p$, p odd prime, with $F(h^p) \approx S^1 \cup S^1$. Then $h \sim_w h_3$.

<u>Proof</u>

Let $K_3 = K/\langle h_3{}^p \rangle$. By lemma 3, $K_3 \approx A$. As in lemma 4 we obtain that $h \sim_w h_3$. □

This finishes case 2 and completes the proof of theorem 1.

CHAPTER TWO

$\mathbb{Z}_{2p}$ - ACTIONS ON THE SOLID KLEIN BOTTLE K

Section 2.1. FREE $\mathbb{Z}_{2k+1}$ - ACTIONS ON K.

Lemma 1.

If $h : K \to K$ is a (PL) homeomorphism of period $2k+1$, $k \geq 1$, on K, then $\operatorname{Fix} h$ is either $\emptyset$ or $\approx S^1$.

Proof

Let $n = 2k + 1$. n can be written as $n = p_1^{t_1} p_2^{t_2} \ldots p_m^{t_m}$ where $p_1, \ldots, p_m$ are distinct odd primes and $t_1, \ldots, t_m$ are positive integers. If $m = 1$, then h is of period $p_1^{t_1}$ on K which is a homology 1- sphere. Hence by [4] $\operatorname{Fix} h$ is either $\emptyset$ or a homology 1-sphere. $F(h)$ cannot be 2-dimensional because $p_1^{t_1} \neq 2$. So $F(h)$ is either $\emptyset$ or $\approx S^1$. If $m = 2$, then $n = p_1^{t_1} p_2^{t_2}$ and $h^{p_1^{t_1}}$ is of period $p_2^{t_2}$. As above $F(h^{p_1^{t_1}})$ is either $\emptyset$ or $\approx S^1$. If $F(h^{p_1^{t_1}}) = \emptyset$, $F(h) = \emptyset$. Since $F(h^{p_1^{t_1}})$ is invariant under h, then if $F(h^{p_1^{t_1}}) \approx S^1$ and $F(h) \neq \emptyset$, h is of period $p_1^{t_1}$ on $F(h^{p_1^{t_1}}) \approx S^1$ a homology 1- sphere and again by [4] $F(h) \approx S^1$. Hence if $m = 2$, $F(h) = \emptyset$ or $\approx S^1$.

Now assume that the result is proven for $m = i$. Let $c = p_1^{t_1} \ldots p_i^{t_i}$. Let the period of h be $c\, p_{i+1}^{t_{i+1}}$. Then $h^{p_{i+1}^{t_{i+1}}}$ is of period c on K so by the induction hypothesis $F(h^{p_{i+1}^{t_{i+1}}})$ is either $\emptyset$ or $\approx S^1$. If $F(h^{p_{i+1}^{t_{i+1}}}) = \emptyset$, $F(h) = \emptyset$. If $F(h^{p_{i+1}^{t_{i+1}}}) \approx S^1$, then by [4] $F(h) \approx S^1$ or $\emptyset$.

Hence $F(h) = \emptyset$ or $\approx S^1$. $\square$

Remark. The proof above shows that if $F(h) \approx S^1$, then $F(h^i) = F(h) \approx S^1$ for all $1 < i < 2k + 1$.

In the rest of this section we consider the free $\mathbb{Z}_{2K+1}$ - actions on K, i.e $F(h^i) = \emptyset$ for all $1 < i < 2k + 1$.

Lemma 2.

Let $h : K \to K$ be a homeomorphism of period $2k + 1$, $k \geq 1$. If $\langle h \rangle$ acts freely on K, then $K/\langle h \rangle \approx K$.

Proof

Let $B = K/\langle h \rangle$ and let $p : K \to B$ be the orbit map. Since $\langle h \rangle$ acts freely on K, K is a regular $2k + 1$ covering of B [11, theorem 8.2, Chapter 5]. Hence $p_{\#}(\pi_1(K)) \cong \mathbb{Z}$ is a normal subgroup of index $2k + 1$ of $\pi_1(B)$. So we have a short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\alpha} \pi_1(B) \xrightarrow{\beta} \mathbb{Z}_{2k+1} \to 0$$

Since B is covered by a contractible space and no nontrivial finite group can act freely on a finite dimensional, contractible space [6; page 287], $\pi_1(B)$ has no torsion subgroup. Let $a = \alpha$(generator of $\mathbb{Z}$) and b be such that $\beta(b)$ is a generator of $\mathbb{Z}_{2k+1}$. Since $p_{\#}(\pi_1(K))$ is normal in $\pi_1(B)$, $b\,a\,b^{-1} \in \langle a \rangle$. So $b\,a\,b^{-1} = a$ or a^{-1}. If $b\,a\,b^{-1} = a^{-1}$, then $\pi_1(B)/[\pi_1(B), \pi_1(B)] \cong H_1(B)$ is finite (for the coset $\overline{b} = b + [\pi_1(B), \pi_1(B)]$ is of order $2k + 1$). Hence

$$\chi(B) = \sum_{i=0}^{3} (-1)^i \rho_i = 1 + \rho_2 \quad \text{because} \quad \rho_1 = 0 = \rho_3, \quad \rho_3 = 0$$

because B is nonorientable. But this implies that $\chi(B) \geq 1$ contradicting the fact that $\chi(B) = 0$. Hence $b\,a\,b^{-1} \neq a^{-1}$. So we must have $b\,a\,b^{-1} = a$ and so $\pi_1(B)$ is abelian. From the principal theorem for abelian groups $\pi_1(B) = \mathbb{Z} + \mathrm{Tor}(\pi_1(B)) = \mathbb{Z} + 0 = \mathbb{Z}$. Hence $\pi_1(B) = \mathbb{Z}$.

Note that B is compact, nonorientable, irreducible with K^2 (2-dim Klein bottle) as its boundary. Moreover, B contains no 2-sided projective planes P^2, because if B contains such P^2, then $p^{-1}(P^2)$ will be a 2-sphere S, and since K is irreducible S bounds a a 3-cell C. Then $p(C)$ is a 3-manifold bounded by P^2. Now by [5] theorem 11.7 $B \approx K$. $\square$

<u>Proposition 3.</u>

Up to weak equivalence there is exactly one free $\mathbb{Z}_{2k+1}$ $(k \geq 1)$ action on K.

<u>Proof</u>.

Let $h_1 : K \to K$ be defined by $h_1([z, t]) = [z, t + \frac{2k}{2k+1}]$. h_1 is a homeomorphism of period $2k+1$ and $F(h^i) = \emptyset$ for all $1 < i \leq 2k$. So by lemma 2 above $k/\langle h_1 \rangle \approx K$. Now let $h : K \to K$ be any homeomorphism of period $2K + 1$ such that $F(h^i) = \emptyset$, $1 < i < 2k + 1$. Lemma 2 implies that $k/\langle h \rangle \approx K$. Let $p_1 : K \to K/\langle h_1 \rangle$ and $p : K \to K/\langle h \rangle$ be the orbit maps. p_1 and p are $(2k + 1)$ - coverings of $K/\langle h_1 \rangle$ and $K/\langle h \rangle$ respectively. Let $t : K/\langle h_1 \rangle \to K/\langle h \rangle$ be a homeomorphism. Since tp_1 and p are $(2k + 1)$ - covering projections of K and since $\pi_1(K/\langle h \rangle) \cong \mathbb{Z}$ has a unique normal subgroup of index $2k + 1$, $\exists$ a homeomorphism $\bar{t} : K \to K$ making the diagram

$$\begin{tikzcd} K \arrow[r, "\bar{t}"] \arrow[d, "p_1"'] & K \arrow[d, "p"] \\ K/\langle h_1 \rangle \arrow[r, "t"] & K/\langle h \rangle \end{tikzcd}$$

commutative, i.e such that $tp_1 = p\bar{t}$. Now as in proposition 1, section 1.1, $h \sim h_1^i$ for some $1 \le i \le 2k$. Hence any two free $\mathbb{Z}_{2k+1}$ - actions on K are weakly equivalent. □

Section 2.2. $\mathbb{Z}_{2p}$ - ACTIONS ON K

In this section we classify all $\mathbb{Z}_{2p}$ - actions on the solid Klein bottle K, up to weak equivalence, where p is a prime number. The case p = 2 is studied in [13], so we study here the case where p is an odd prime.

Our main result is the following

<u>Theorem 1.</u>

If $h : K \to K$ is a homeomorphism of period 2p on K, p an odd prime, then $F(h) = \emptyset$ and $F(h^2) = \emptyset$ and h is weakly equivalent to one of the following period 2p homeomorphisms on K depending on whether $F(h^p) \approx M$, A or S^1 respectively:

$$h_1([z, t]) = [-z, t + \frac{p-2}{p}], \quad F(h_1^p) \approx \text{Mobius band } M$$

$$h_2([z, t]) = [\bar{z}, t + \frac{p-1}{p}], \quad F(h_2^p) \approx \text{Annulus } A$$

$$h_3([z, t]) = [-z, t + \frac{p-1}{p}], \quad F(h_3^p) \approx S^1 .$$

The rest of this section is devoted to the proof of theorem 1.

Let $h : K \to K$ be of period $2p$. Since $(h^2)^p = 1d$, h^2 is of period p on K. Since K is a homology 1-sphere, then [4] gives that $F(h^2)$ is a homology r-sphere, where $r \leq 1$ and $1 - r$ is even. Hence r is either -1 or 1. $p \neq 2$, so $F(h^2)$ is either $\emptyset$ or $\approx S^1$. Note that $F(h) \subseteq F(h^2)$. If $F(h^2) = \emptyset$, then $F(h) = \emptyset$. If $F(h^2) \approx S^1$, then since h is an involution on $F(h^2) \approx S^1$, it follows from Smith [15] that $F(h)$ is a homology r-sphere, $r \leq 1$. Hence $r = -1, 0, 1$ i.e $F(h)$ is one of the sets: $\emptyset$, S^0, S^1. So we consider the following cases:

<u>Case 1:</u> $F(h^2) = \emptyset$, hence $F(h) = \emptyset$.

<u>Case 2:</u> $F(h^2) \approx S^1$ and $F(h)$ is $\emptyset$, S^0 or $\approx S^1$.

<u>Case 1:</u> $F(h^2) = \emptyset$.

In this case h is determined up to weak equivalence by the $F(h^p)$. We prove the following

<u>Proposition 2.</u>

Let $h : K \to K$ be of period $2p$ on K, p odd prime, with $F(h^2) = \emptyset$. Then h is weakly equivalent to one of the homeomorphisms h_1, h_2, h_3 depending on whether $F(h^p) \approx M$, A or $\approx S^1$ respectively. (The maps h_1, h_2, h_3 are the ones in theorem 1).

Proof of Proposition 2.

h^p is an involution on K, hence by [13] $F(h^p)$ is homeomorphic to one of the sets: $I \dot{\cup} pt$, $D^2 \dot{\cup} I$, M, A, S^1. Since $F(h) = \emptyset$ and $F(h^p)$ is invariant under h, $F(h^p)$ cannot be $I \dot{\cup} pt$ or $D^2 \dot{\cup} I$, otherwise in the first case $h(pt) = pt$ and $h(I) = I$ so h would have at least 2 fixed points. In the other case $h(D^2) = D^2$ and $h(I) = I$ and Brouwer fixed point theorem implies that h has at least 2 fixed points. Hence $F(h^p)$ is homeomorphic to one of the sets M, A or S^1.

The map h induces a homeomorphism h^- of period p on $K/\langle h^p \rangle$ defined by $h^- q = qh$ where $q : K \to K/\langle h^p \rangle$ is the orbit map. Note that $\langle h^- \rangle$ acts freely on $K/\langle h^p \rangle$. From [13] if $F(h^p) \approx S^1$ or M, then $K/\langle h^p \rangle \approx K$ and if $F(h^p) \approx A$, then $K/\langle h^p \rangle \approx D^2 \times S^1$, the solid torus.

Now, we show that any homeomorphism h of period $2p$ with $F(h^2) = \emptyset$ is weakly equivalent to one of the homeomorphism h_1, h_2, h_3.

(i) Assume $F(h^p) \approx M$. We show that h is weakly equivalent to h_1. Note that $F(h_1) = F(h_1^2) = \emptyset$ and $K/\langle {h_1}^p \rangle \approx K$, also since $F(h^p) \approx M$, $K/\langle h^p \rangle \approx K$. The induced maps h_1^- and h^- are free on $K/\langle {h_1}^p \rangle$ and $K/\langle h^p \rangle$ respectively. Since $K/\langle {h_1}^p \rangle \approx K \approx K/\langle h^p \rangle$ and h_1^- and h^- are free of odd period p it follows from lemma 2, Section 2.1 that $(K/\langle h_1^p\rangle)/\langle \bar{h}_1\rangle \approx K$ and that $(K/\langle h^p\rangle)/\langle \bar{h}\rangle \approx K$. Let $K_1 = K/\langle h_1^p\rangle$ and $K_2 = K/\langle h^p\rangle$. Let $q_1 : K \to K_1$, $q_2 : K_1 \to K_1/\langle \bar{h}_1\rangle \approx K/\langle h_1\rangle$, $q : K \to K_2$ and $q' : K_2 \to K_2/\langle \bar{h}\rangle \approx K/\langle h\rangle$ be the orbit maps. Since h_1^p and h^p are involutions, $q_1(F(h_1^p)) = M_1$ is a Mobius band in $\partial(K_1)$ and $q(F(h^p)) = M_2$ is a Mobius band in $\partial(K_2)$. $F(h_1^p)$ is invariant under h_1 and $F(h^p)$ is invariant under h, hence M_1 and M_2 are invariant under $\bar{h}_1$ and $\bar{h}$ respectively. Since $\partial(K/\langle h_1\rangle)$ and $\partial(K/\langle h\rangle)$ are 2-dim Klein bottles, $q_2(M_1)$ and $q'_2(M_2)$ are Mobius bands in $\partial(K/\langle h_1\rangle)$ and $\partial(K/\langle h\rangle)$ respectively.

h_1^p is an involution on K so by [9] and [10] $\exists$ a meridional disk D in K which is invariant under h_1^p. Cut K along D we obtain $K' \approx D^2 \times I$ and by [9] $h_1^p/K' \sim f$, $f(z, t) = (\bar{z}, 1 - t)$. Let $\bar{D}$ be an h_1^p invariant disk in K', then $q_1(\bar{D}) = E_1$ is a meridional disk in K_1. $\bar{h}$ acts freely on K_1 so $E_1, \bar{h}_1(E_1), \ldots, h_1^{p-1}(E_1)$ are mutually disjoint. Hence $q_2(E_1)$ is a disk in $K/\langle h_1\rangle$. Similarly for h^p we obtain E and $q'(E)$ a meridional disk in $K/\langle h\rangle$. Let $c_1 = \partial(q_2(E_1))$, $c = \partial(q'(E))$ and $e_1 = \partial(q_2(M_1))$, $e = \partial(q'(M_2))$. e_1 and e separate $\partial(K/\langle h_1\rangle)$ and $\partial(K/\langle h\rangle)$ respectively each into two Mobius bands. Also c_1 and e_1 meet in 2 points so do c and e.

Note that $\partial(K/\langle h_1\rangle) - c_1 - e_1$, and $\partial(K/\langle h\rangle) - c - e$ each consists of two open rectangles. Now let $t_1 : c_1 \cup e_1 \overset{\approx}{\to} c \cup e$ and extend t_1 to $t_2 : \partial(K/\langle h_1\rangle) \to \partial(K/\langle h\rangle)$ such that $t_2 q_2(M_1) = q'(M_2)$. Then extend to t_3 on $q_2(E_1)$, finally accross the open 3-cell $K/\langle h_1\rangle - \partial(K/\langle h_1\rangle) - q_2(E_1)$. The final map is a homeomorphism $t : K/\langle h_1\rangle \to K/\langle h\rangle$ with $t(q_2(M_1)) = q'(M_2)$.

$$\begin{tikzcd}
K \arrow[r, "t"] \arrow[d, "q_1"'] & K \arrow[d, "q"] \\
K_1 \arrow[r, "\bar{t}"] \arrow[d, "q_2"'] & K_2 \arrow[d, "q'"] \\
K/\langle h_1\rangle \arrow[r, "t"] & K/\langle h\rangle
\end{tikzcd}$$

$\langle \bar{h}_1\rangle$ and $\langle \bar{h}\rangle$ are free so q_2 and q' are p-covering projections and since $\pi_1(K) \cong \mathbb{Z}$ has a unique normal subgroup of index p, t can be lifted to a homeomorphism $\bar{t} : K_1 \to K_2$ such that $tq_2 = q'\bar{t}$ and $\bar{t}(M_1) = M_2$. Now q_1 is a double cover : $K - F(h_1{}^p) \to K_1 - M_1$ and q a double covering projection: $K - F(h^p) \to K_2 - M_2$. But $\pi_1(K_1 - M_1) \cong \mathbb{Z}$, $\bar{t}$ can be lifted to $\bar{\bar{t}} : K - F(h_1{}^p) \to K - F(h^p)$.

By continuity $\bar{\bar{t}}$ can be extended to all of K. So $q\bar{\bar{t}} = \bar{t}q_1$. Now as in the proof of lemma 4, section 1.2 we see that $h \sim_w h_1$.

(ii) Assume $F(h^p) \approx A$, an annulus. We show that $h \sim_w h_2$. Note that $F(h_2) = F(h_2^{2}) = \emptyset$, $K/\langle h_2^p\rangle$ $D^2 \times S^1$, a solid torus. Similarly $K/\langle h^p\rangle \approx D^2 \times S^1$. Since h_2^- and h^- act freely on $K/\langle h_2^p\rangle$ and $K/\langle h^p\rangle$ respectively and p is odd, then by [7] $K_1/\langle h_2^-\rangle \approx D^2 \times S^1$ and $K_2/\langle h^-\rangle \approx D^2 \times S^1$, notations as in (i) above. h_2^p and h^p are involutions imply that $q_1(F(h_2^p)) = A_1$ and $q(F(h^p)) = A_2$ are annuli in ∂K_1 and ∂K_2 respectively. As in (i) $q_2(A_1)$ and $q'(A_2)$ are annuli in $\partial(K/\langle h_2\rangle)$ and $\partial(K/\langle h\rangle)$ the latters are homeomorphic to $S^1 \times S^1$. h_2^- and h^- free imply that q_2 and q' are p-covering projections. Note that $K_1/\langle h_2^-\rangle \approx K/\langle h_2\rangle$ and $K_2/\langle h^-\rangle \approx K/\langle h\rangle$. Hence $K/\langle h_2\rangle \approx D^2 \times S^1 \approx K/\langle h\rangle$. As in (i) there are meridional disks E_1 and E in K_1 and K_2 respectively such that $q_2(E_1)$ and $q'(E)$ are meridional disks in $K/\langle h_1\rangle$ and $K/\langle h\rangle$ respectively.

Let $c_1 = \partial(q_2E_1) \cap q_2A_1$, c_1' the complement of c_1 in $\partial(q_2E_1)$, $c_2 = \partial(q'E) \cap q'A_2$, c_2' its complement in $\partial(q'E)$. Let $t : c_1 \to c_2$ be a homeomorphism. Extend t to a homeomorphism : $q_2(A_1) \to q'(A_2)$ then on c_1' on to c_2', then on q_2E_1 on to $q'E$, next on the open rectangle $\partial(K/\langle h_1\rangle) - q_2A_1$ on to $\partial(K/\langle h\rangle)$ -

$q'(A_2)$, finally across the remaining 3-cell in $K/\langle h_1 \rangle$. Still call the new homeomorphism t. Note that $t(q_2A_1) = q'A_2$. As in (i) lift t to $\bar{t} : K_1 \to K_2$ such that $q'\bar{t} = tq_2$. Since $\pi_1(K_1 - A_1) \cong \pi_1(K_2 - A_2) \cong \mathbb{Z}$ lift $\bar{t}$ to $\bar{\bar{t}} : K - F(h_2^p) \to K - F(h^p)$, by continuity extend $\bar{\bar{t}}$ on all of K. So $q\bar{\bar{t}} = \bar{t}q_1$ on K. Hence as in (i) $h \sim_w h_2$.

(iii) Assume $F(h^p) \approx S^1$. We show that $h \sim_w h_3$. We use the same notations of (i). By [13] $K_1 \approx K \approx K_2$, where $K_1 = K/\langle h_3^p \rangle$, $K_2 = K/\langle h^p \rangle$. $q_1(F(h_3^p)) = c_1$ and $q(F(h^p)) = c_2$ are simple closed curves in the interiors of K_1 and K_2 respectively. By lemma 2 (section 2.1) $K_1/\langle h_3^- \rangle \approx K \approx K_2/\langle h^- \rangle$. q_2c_1 and $q'c_2$ are s.c.c in the interiors of $K/\langle h_3 \rangle$ and $K/\langle h \rangle$ respectively. Let $t : K/\langle h_3 \rangle \to K/\langle h \rangle$ be a homeomorphism mapping $q_2(c_1)$ on to $q'(c_2)$. As in (1) lift t to $\bar{t} : K_1 \to K_2$ mapping c_1 on to c_2 and $q'\bar{t} = tq_2$. By [13] $F(h_3^p)$ and $F(h^p)$ are the "cores" of K, so c_1 and c_2 are the cores of K_1 and K_2 respectively. Hence $\pi_1(K_1 - c_1) \cong \pi_1(K^2)$, K^2 the 2-dim Klein bottle. $\pi_1(K^2)$ has a unique normal subgroup of index 2 corresponding to K^2 [14]. So we can lift $\bar{t}$ to $\bar{\bar{t}}$: $K - F(h_3^p) \to K - F(h^p)$. As in (i) we conclude that h is weakly equivalent to h_3.

This completes the proof of proposition 2. □

This takes care of case 1.

Case 2. $F(h^2) \approx S^1$ and $F(h) = \emptyset$ or S^0 or $\approx S^1$.

We consider the following 3 subcases:

Subcase 2.1. $F(h^2) \approx S^1$ and $F(h) \approx S^0$.

We show that this case cannot happen.

Proposition 3.

There is no homeomorphism h of period $2p$, p odd prime, on K with $F(h^2) \approx S^1$ and $F(h) \approx S^0$.

Proof

h^p is an involution on K. Hence by [13] $F(h^p)$ is homeomorphic to one of the sents: $I \dot{\cup} pt$, $D^2 \dot{\cup} I$, M, A, S^1. Note that $F(h) \subset F(h^p)$. If $F(h^p) \approx M$, A or S^1, then $F(h^p)$ is a homology 1-sphere. Since $F(h^p)$ is invariant under h, h is a period p homeomorphism on $F(h^p)$. Hence by [4] $F(h)$ is a homoloyg r-sphere with $r \leq 1$ and $1 - r$ is even. So $r = -1$ or 1, i.e $F(h)$ is either $\emptyset$ or $\approx S^1$ contradicting our assumption that $F(h) \approx S^0$. So $F(h^p)$ cannot be M, A or S^1. If $F(h^p) \approx D^2 \dot{\cup} I$, then since $h(F(h^p)) = F(h^p)$, $h(D^2) = D^2$ and $h(I) = I$. So h has one fixed point in D^2 and one fixed point x in I. So h must interchange the sides of $I - x$ and since p is odd h^p interchanges the sides of $I - x$ contradicting the fact that $I \subseteq F(h^p)$. Hence $F(h^p) \not\approx D^2 \dot{\cup} I$. Finally if $F(h^p) \approx I \dot{\cup} pt$, then since $F(h^p)$ is invariant under h, we have $h(I) = I$ so h must have a fixed point in I. We get a contradiction as above. $\square$

<u>Subcase 2.2.</u> $F(h^2) \approx S^1$ and $F(h) \approx S^1$.

Note that $F(h) = F(h^2) \cap F(h^p)$ and h^p is an involution on K. Since $F(h^p)$ is invariant under h and $F(h) \approx S^1$, $F(h^p)$ cannot be 2-dimensional, otherwise h must interchange the two components of $U - F(h)$, where $U \subset F(h^p)$ is a small neighborhood of x in $F(h)$ and since p is odd this implies that h^p interchanges the two components of $U - F(h)$ contradicting the fact that $U \subset F(h^p)$. So $F(h^p)$ cannot be $D^2 \dot{\cup} I$, M or A. Since $F(h) \subset F(h^p)$ and $F(h) \approx S^1$, $F(h^p)$ cannot be $\approx I \dot{\cup}$ pt. Hence the only possibility is the $F(h^p) \approx S^1$. So now we have $F(h) = F(h^i) \approx S^1$ for all $1 < i < 2p$. We shall show that this cannot happen.

First we show the following:

<u>Lemma 4.</u>

If there exists a homeomorphism $h : K \to K$ of period $2p$, p odd prime, with $F(h) \approx S^1$, then $K/\langle h\rangle \approx K$.

<u>Proof</u>

h induces an orientation preserving homeomorphism $h^\sim$ on the orientable double cover $D^2 \times S^1$ of K with period $2p$ and $qh^\sim = hq$ where $q : D^2 \times S^1 \to K$ is the covering projection. By definition of $h^\sim$, $F(h^\sim) \neq \emptyset$ and $F(h^\sim) = q^{-1}(F(h))$. By [7] $F(h^\sim) \approx S^1$ and $F(h^\sim) = F(h^{\sim i})$, $1 < i < 2p$. Hence $h^{\sim p}$ is an involution on

$D^2 \times S^1$ with fixed point set a simple closed curve. Tollefson [17] shows that if an orientation preserving homeomorphism f on $D^2 \times S^1$ has a simple closed curve as its fixed point set, then f is equivalent to a rotation about the core $0 \times S^1$ of $D^2 \times S^1$. In particular $F(f) = 0 \times S^1$ i.e it is unknotted. So $Fix(h^p) = 0 \times S^1$ and since $F(\tilde{h}^p) = F(\tilde{h})$, $F(\tilde{h}) \approx 0 \times S^1$. Hence by [7] $D^2 \times S^1/\langle \tilde{h} \rangle \approx D^2 \times S^1$.

Now consider the diagram

$$\begin{tikzcd}
K \arrow[d, "q"'] & D^2 \times S^1 \arrow[l, "p"'] \arrow[d, "\tilde{q}"] \\
B & D^2 \times S^1/\langle \tilde{h} \rangle \approx D^2 \times S^1 \arrow[l, "p'"']
\end{tikzcd}$$

where $B = K/\langle h \rangle$, $q : K \to B$ the orbit map, $\tilde{q} : D^2 \times S^1 \to D^2 \times S^1$ the orbit map of $D^2 \times S^1$ and $p : D^2 \times S^1 \to K$ the covering projection of the orientable double cover $D^2 \times S^1$ of K. $p' : D^2 \times S^1 \to B$ is defined by $p'\tilde{q} = qp$. Since $q|(K - F(h))$ is a $2p$ - sheeted cover of $B - q(F(h))$ and $\tilde{q}|(D^2 \times S^1 - F(\tilde{h}))$ is a $2p$ - cover of $D^2 \times S^1 - \tilde{q}(F(\tilde{h}))$ and since p is a covering map, then given $b \in B-q(F(h))$ we can find an open neighborhood V_b such that $p'^{-1}(V_b) = U_1 \dot{\cup} U_2$ with

U_i open and $p'(U_i) \approx V_b$, $i = i, 2$. Note also that $F(h^\sim)$ is a double cover of $F(h)$ and $q(F(h)) \approx F(h)$ and $\tilde{q}(F(h^\sim)) \approx F(h^\sim)$. Hence $D^2 \times S^1$ double covers B. Since B is non orientable and $\partial B = K^2$ and is double covered by $D^2 \times S^1$, $B \approx K$. □

The proof of lemma 4 can be used to prove the following:

<u>Corollary 5.</u>

If $h : K \to K$ is a homeomorphism of period n, where n is an odd integer, with $F(h)$ a simple closed curve $\approx$ core of K, then $K/\langle h\rangle \approx K$.

<u>Proof</u>

Since n is odd and $F(h) \approx S^1$, then by the remark following lemma 1, section 2.1, $F(h) = F(h^i)$ for all $1 < i < n$. h induces $h^\sim$ on $D^2 \times S^1$ with $F(h^\sim) \approx S^1 \approx 0 \times S^1$, the core of $D^2 \times S^1$. Now proceed as in the proof of lemma 4. □

Since $F(h) = F(h^i) \approx S^1$, $F(h^p) \approx S^1$ and since h^p is an involution on K, $F(h^p) \approx$ core of K. So we can assume that $F(h) =$ core of K, $q(F(h)) =$ core of $K/\langle h\rangle \approx K$.

Now h^2 is of period p and $F(h^2) \approx S^1$ and by the above remark $F(h^2) =$ core of K. Hence to be done with this subcase we prove the following

Proposition 6.

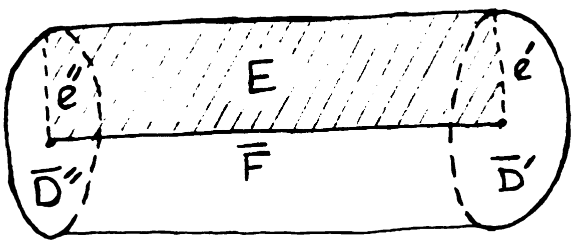
There is no homeomorphism $h : K \to K$ of period p, p an odd prime, with $F(h)$ = core of $K \approx S^1$.

Proof

Let $q : K \to K_1 = K/\langle h\rangle$ be the orbit map. By corollary 4, $K_1 \approx K$, $q(F(h))$ = Core of K_1.

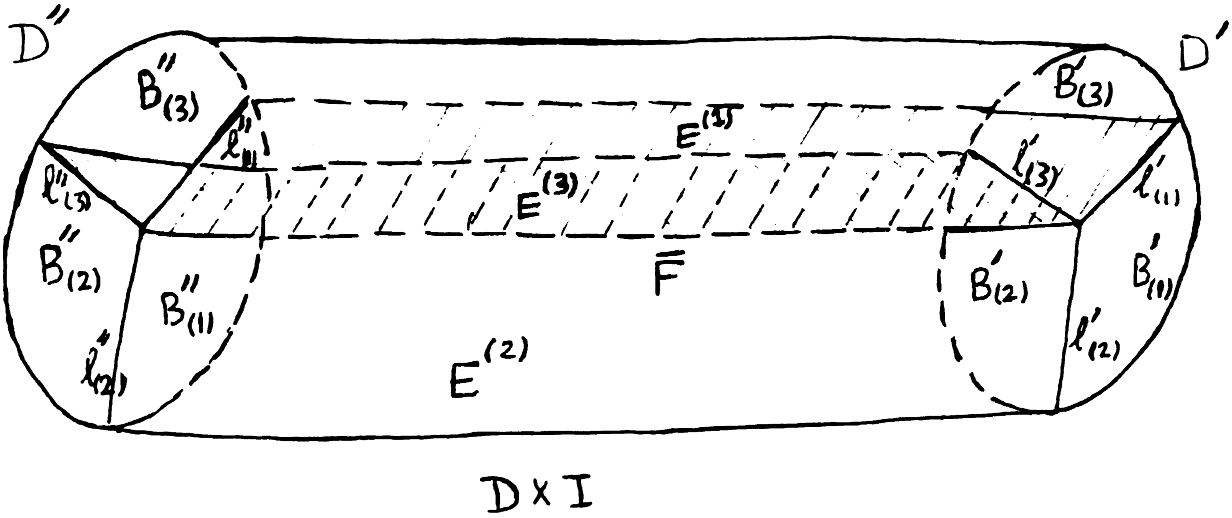
Let $\bar{D}$ be a meridional disk in $K/\langle h\rangle$ such that $\bar{D} \cap q(F)$ is a point (Note that $q(F)$ is the core of K_1). $q^{-1}(\bar{D})$ is either one disk or n disks meeting at a common interior point. Note that $\partial K = \partial K_1 = K^2$ the 2-dim Klein bottle. $q|K^2$ is a p- sheeted cover of $\partial K_1 = K^2$. $\partial\bar{D}$ is a simple loop $\alpha : I \to K^2$. Let $e_0 = \alpha(0) = \alpha(1)$ and let $\tilde{e}_0 \in q^{-1}(e_0)$. Let $a = [\alpha]$, and let β be an orientation reversing loop which meets α transversaly once at e_0 and let $b = [\beta]$. So $\pi_1(\partial K_1, e_0) = \langle a, b | bab^{-1} = a^{-1}\rangle$. $\pi_1(K^2)$ has a unique normal subgroup of index $p \cong \pi_1(K^2)$, hence $(q|\partial K)_\#$ $\pi_1(\partial K, \tilde{e}_0) = \langle a^p, b | bab^{-1} = a^{-1}\rangle$. Let $\eta : I \to K$ be a loop which wraps once around the component of $q^{-1}(\partial\bar{D}) = q^{-1}(\alpha)$ containing $\tilde{e}_o$, with $\eta(0) = \eta(1) = \tilde{e}_o$. Then $(q|\partial K)_\#([\eta]) = a^p$ and so $q|\eta(I)$ covers $\alpha(I) = \partial\bar{D}$ in a p to 1 fashion. Hence $q^{-1}(\bar{D})$ is a meridional disk D in K. Since $hq^{-1}(\bar{D}) = q^{-1}(\bar{D})$, $h(D) = D$ i.e D is invariant under h.

Now cut K along D and K_1 along $\bar{D}$ to obtain $D \times I$ and $D_1 \times I$ respectively, where $D \approx D_1 \approx D^2$, the standard disk. $\partial(D \times I)$ contains two copies D' and D'' of D and $\partial(D_1 \times I)$ contains two copies $\bar{D}'$ and $\bar{D}''$ of $\bar{D}$. Let e be an arc $\approx I$ in $\bar{D}$ joining the center of $\bar{D}(= \bar{D} \cap q(F(h)))$ and $\partial\bar{D}$. So $\bar{D}'$ contains a copy e' of e and $\bar{D}''$ contains another copy e'' of e. Let $F \subset D \times I$ be $F(h)$ after cutting K and $\bar{F} \subset D_1 \times I$ be $q(h(F))$ after cutting K_1. Let E be a disk in $D_1 \times I$ bounded by e', e'', $\bar{F}$ and an arc in $(\partial D_1) \times I$ joining the point $e' \cap \bar{D}'$ and the point $e'' \cap \bar{D}''$. See the figure below



$D_1 \times I$

$q^{-1}(E)$ consists of p disks $E^{(i)}$, $i = 1, \ldots, p$ having F as a common edge. Let $\ell'_{(i)} = E^{(i)} \cap D'$ and $\ell''_{(i)} = E^{(i)} \cap D''$ in $D \times I$. $\ell'_{(i)}$ and $\ell''_{(i)}$ are pairwise disjoint arcs for all i joining F to $\partial D'$ and $\partial D''$ respectively. Let $B'_{(i)}$ be the part of D' that lies between $\ell'_{(i)}$ and $\ell'_{(i+1)}$ and $\partial D'$. Define $B''_{(i)}$ similarly. See the figure below $(p = 3)$.



The case $p = 3$

Let ψ be the identification map of $D \times I$ and ψ' the identification map of $D_1 \times I$. ψ' identifies e' with e'' and $\bar{D}'$ with $\bar{D}''$ so that $D_1 \times I/\psi' = K_1$. So ψ identifies $\ell'_{(i)}$ with $\ell''_{(i)}$, $i = 1, \ldots, p$. So it must identify $B'_{(i)}$ with $B''_{(i)}$ for $i = 1, \ldots, p$. Hence ψ identifies D' with D'' in an orientation preserving manner. Hence $D \times I/\psi = D^2 \times S^1$. So K cannot be restored from $D \times I$. But $D \times I/\psi$ must yield K. □

<u>Remark 7.</u>

In the above proposition p may be replaced by any odd positive integer.

This follows from corollary 5, section 2.2 and lemma 1 section 2.1 and the proof of proposition 6. This completes subcase 2.2.

Now we handle the last subcase

<u>Subcase 2.3.</u> $F(h^2) \approx S^1$ and $F(h) = \emptyset$.

Since h^p is an involution on K by [13] $F(h^p)$ is $\approx I\dot{\cup}pt$, $D^2 \dot{\cup} I$, M, A, S^1. As before, since $F(h) = \emptyset$, $F(h^p)$ cannot be $I \dot{\cup} pt$ or $D^2 \dot{\cup} I$. So we are left with the 3 cases $F(h^p) \approx M$, A, S^1. We shall show that none of these cases is possible.

First we prove the following:

<u>Lemma 8.</u>

Let $F = F(h^2)$ and let $x_0 \in F$. Let $i : F \to K$ be the inclusion map. Then $\text{image}(\pi_1(F, x_0) \xrightarrow{i_*} \pi_1(K, x_0)) = \pi_1(K, x_0)$ i.e F generates $\pi_1(K, x_0)$.

<u>Proof</u>

We follow here a method of proof used by Conner and Raymond [2]. Let $f = h^2$. Let (K^*, q) be the universal cover of K and let $\tilde{x}_0 \in q^{-1}(x_0)$. Note that $h^2 = f$ is of period p on K. Lift the $\mathbb{Z}_p$ action of f on K to a $\mathbb{Z}_p$ action on K^*. From covering space theory f induces $\tilde{f}$ on K^* with period p and $q\tilde{f} = fq$. Let $E = F(\tilde{f})$. From Smith theory E is connected and acyclic and since E is connected, $q(E) = F$. Let $\alpha \in \pi_1(K, x_0)$

and let $\lambda(t)$ be a loop in K based at x_0 representing α. Lift $\lambda(t)$ into a covering path $\hat{\lambda}(t)$ with $\hat{\lambda}(0) = \tilde{x}_0$ and $\hat{\lambda}(1) = \alpha(\tilde{x}_0) \in E$ (for $q\alpha = q$ implies $q(\alpha(\tilde{x}_0)) = q(\tilde{x}_0) = x_0$, $\tilde{f}\alpha = \alpha\tilde{f}$ implies $\tilde{f}\alpha(\tilde{x}_0) = \alpha\tilde{f}(\tilde{x}_0) = \alpha(\tilde{x}_0)$, hence $\alpha(\tilde{x}_0) \in E$). $\exists$ a path $\hat{\lambda}'(t)$ in E with $\hat{\lambda}'(0) = \tilde{x}_0$ and $\hat{\lambda}'(1) = \hat{\lambda}(1) = \alpha(\tilde{x}_0)$.

Then

$$\psi(t) = \begin{cases} \hat{\lambda}(2t) & 0 \le t \le \frac{1}{2} \\ \hat{\lambda}'(2-2t) & \frac{1}{2} \le t \le 1 \end{cases}$$

is a loop based at $\tilde{x}_0$ representing $[\psi] \in \pi_1(K^*, \tilde{x}_0) = 0$. But $q(\hat{\lambda}'(t))$ represents $\beta \in \pi_1(F, x_0)$ (for $q(\hat{\lambda}'(0)) = q(\tilde{x}_0) = x_0$, $q(\hat{\lambda}'(1)) = q(\hat{\lambda}(1)) = q(\alpha(\tilde{x}_0)) = q(\tilde{x}_0) = x_0$, $\hat{\lambda}' \subset E$ implies $q(\hat{\lambda}'(t)) \subset F$) and $[\psi] = 0$ implies $\beta = \alpha$ in $\pi_1(K, x_0)$, for $\hat{\lambda} \sim \hat{\lambda}'$ implies $q(\hat{\lambda}) = q(\hat{\lambda}')$ i.e $\alpha = \beta$. $\square$

Now we are ready to handle the remaining cases for $F(h^p)$.

(i) $F(h^p) \approx S^1$. From Chapter 1, section 1.2, we wee that $F(h^p)$ and $F(h^2)$ are in the interior of K which has $\mathbb{R}^3$ as a universal cover. h^p induces an involution α on $\mathbb{R}^3$ with $q\alpha = h^p q$, where $q : \mathbb{R}^3 \to \mathrm{Int}(K)$ is the covering projection. Since

$\pi_1(F(h^p), x) \cong \pi_1(K, x)$ (lemma 8), $F(\alpha) \approx \mathbb{R}$. Let $\overline{F} = q^{-1}(F(h^2))$. By lemma 8 we obtain the fact that $\overline{F} \approx \mathbb{R}$. Since $F(h^2) \cap F(h^p) = F(h) = \emptyset$, $F(\alpha) \cap \overline{F} = \emptyset$, otherwise $\tilde{x} \in F(\alpha) \cap \overline{F}$ would imply $p(\tilde{x}) \in F(h^p) \cap F(h^2)$. It is easy to see that $\alpha(\overline{F}) = \overline{F}$. Hence α is an involution on $\mathbb{R}^3 - \overline{F} \approx \mathbb{R}^3 - \mathbb{R}$. By Alexander's duality theorem [3] we have $\tilde{H}_i(\mathbb{R}^3 - \mathbb{R}) = H_i(S^3 - S^1)$. But $H_r(S^3 - S^1) = 0$ for $r > 1$ and $\mathbb{Z}$ for $r = 0, 1$. In other words $\mathbb{R}^3 - \mathbb{R} \approx \mathbb{R}^3 - \overline{F}$ is a homology 1-sphere. Now $F(\alpha) \subseteq \mathbb{R}^3 - \overline{F}$ is a homology r-sphere with $r = -1, 0, 1$ by [15]. That is $F(\alpha)$ is a homology 1-sphere, S^0, $\emptyset$. But $F(\alpha) \approx \mathbb{R}$ has the homology of a point. This contradiction shows that $F(h^p)$ cannot be $\approx S^1$.

(ii) $F(h^p) \approx A$, an annulus. By [9] and [10] $\exists$ an h^p-invariant meridional disk D embedded in K and D in general position with respect to $F(h^p)$. So $F(h^p) \cap D$ is a properly embedded arc I in D. Cut K along D to obtain a component $U \approx D^2 \times I$ and U contains two copies of D, say D' and D'' each contains a copy of I, say $I' \subset D'$ and $I'' \subset D''$. Let F be $F(h^p)$ after cutting K. $F \cap D' = I'$ and $F \cap D'' = I''$. By Brown [1] F is an unknotted disk in U. Let $\overline{F}$ be the image of $F(h^2)$ in U. Since $F(h^2) \cap F(h^p) = \emptyset$, $\overline{F} \cap F = \emptyset$. K is obtained from U by identifying $(x,0)$ in D' with $(\emptyset(x), 1)$ in D''

where $\emptyset$ is an orientation reversing homeomorphism $D' \to D''$. But $\bar{F} \cap F = \emptyset$ so in order to obtain $F(h^2)$ we have to let $\bar{F}$ travel twice around K, otherwise $F(h^2) \cap F(h^p) \neq \emptyset$. But $\bar{F}$ under the identification must be $F(h^2)$, but it is not, it is $2F(h^2)$. Since $F(h^2)$ generates $\pi_1(K, x)$, lemma 8, this cannot happen. Hence $F(h^p)$ cannot be A.

(iii) $F(h^p) \simeq M$, Mobius band. As in (ii) this cannot occur.

This completes case 2 and finishes the proof of Theorem 1.

Section 2.3. SEMI-FREE ACTIONS ON K

In this section we study the semi-free actions on the solid Klein bottle K. We state our results in the following.

<u>Theorem 1.</u>

Let $h : K \to K$ be a periodic homeomorphism of period n acting semi-freely on K.

Then

(1) There is no such h if n is even and $n > 2$.

(2) There are 5 such h up to equivalence if $n = 2$.

(3) There is no such h if n is odd and $F(h) =$ "core" of K.

(4) There is a unique such h, up to weak equivalence if n is odd and $F(h) = \emptyset$.

Proof

Case 1. n is odd, of course $n > 1$.

By lemma 1, section 2.1, $F(h)$ is either $\emptyset$ or a simple closed curve $\approx S^1$. If $F(h) = \emptyset$, then by definition of h, $F(h^i) = \emptyset$ for all $1 \leq i < n$. Hence $\langle h \rangle$ acts freely on K. By proposition 3, section 2.1, h is weakly equivalent to h_1, where $h_1 : K \to K$ is defined by $h_1([z, t]) = [z, t + \frac{n-1}{n}]$. This proves (4).

If $F(h) \approx S^1$, then $F(h^i) \approx S^1$, $1 \leq i < n$. If $F(h) =$ core of K, i.e $F(h^i)$ is unknotted then by remark 7, section 2.2, such an h doesn't exist. This proves (3).

Case 2. n is even.

If $n = 2$, then h is an involution on K. All involutions on K are classified up to equivalence in [13]. In fact, there are 5 such involutions, up to equivalence, distinguished by their fixed point sets.

If n is even > 2, then n has one of the two forms : $n = 2^\alpha$, $\alpha > 1$ or $n = 2^\alpha m$, m is an odd positive integer.

If $n = 2^\alpha$, α positive integer > 1, then by [13] such an h doesn't exist.

Finally, $n = 2^{\alpha}m$, m positive integer > 1. If $F(h) = \emptyset$, then this cannot happen, otherwise since h acts semi-freely, $F(h^{\frac{n}{2}}) = \emptyset$. But $h^{\frac{n}{2}}$ is an involution on K and by [13] there is no involution on K with empty fixed point set.

Now, assume $F(h) \neq \emptyset$. Since $h^{2^{\alpha}}$ has an odd period m, lemma 1, section 2.1 gives $F(h^{2^{\alpha}}) \approx S^1$ and since h acts semi-freely on K, $F(h) \approx S^1$. Hence $F(h^i) = F(h) \approx S^1$ for all $1 < i < n$. Since n is even $F(h) =$ core of K. Now we have $h^{2^{\alpha}}$ is a homeomorphism of odd period with fixed point set $\approx S^1 =$ core of K. But this cannot occur by remark 7, section 2.2. This yields (1) and finishes the proof. $\square$

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