MONOTONE UNION PROPERTIES IN TOPOLOGICAL SPACES

Thesis for the Degree of Ph. D.
MICHIGAN STATE UNIVERSITY
TINUOYE MICHAEL ADENIRAN
1969

This is to certify that the

thesis entitled

Monotone Union Properties
in Topological Spaces

presented by

Tinuoye Michael Adeniran

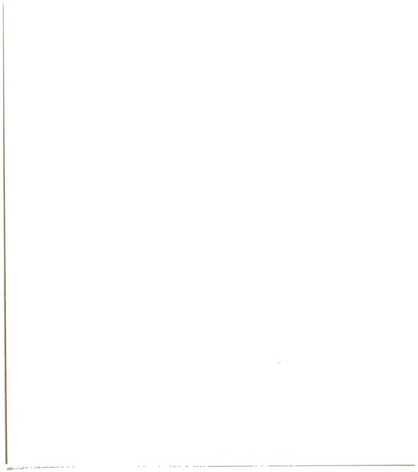
has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Mathematics

Major professor

Date 8-29-69

O-169



ABSTRACT

MONOTONE UNION PROPERTIES IN TOPOLOGICAL SPACES

By

Tinuoye Michael Adeniran

A topological space X has the absolute monotone union property if whenever

(i) $A = \bigcup_{i=1}^{\infty} A_i$

(ii) $A_i \overset{T}{\equiv} A_{i+1}$ (topological equivalence), for each i,

where $\{A_i\}$ is a monotone increasing sequence indexed by the positive integers, then A is necessarily topologically equivalent to X. If for each i, A_i is open, A has the open monotone union property.

The thesis investigates topological spaces having some of these properties, our attention being drawn mainly to one- and two-dimensional spaces. Given a sequence $\{A_i | A_i \subset A_{i+1}\}$ and a property P such that each A_i has the given property, we investigate whether property P is absolute; that is whether the monotone union $\underset{i=1}{\overset{\infty}{\bigcup_m}} A_i$ has the given property. Finally some results are obtained when we look at open monotone union property in invertible locally connected plane continua.

MONOTONE UNION PROPERTIES
IN TOPOLOGICAL SPACES

By

Tinuoye Michael Adeniran

A THESIS

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

1969

G61594
4-17-70

Dedicated to

TINUOLA

OYELADUN

and ROSEMARIE

ACKNOWLEDGMENTS

My indebtness to Professor Patrick H. Doyle goes far beyond the courtesy usually accorded advisers in this kind of work. It can be said without any exaggeration that without his singular efforts which include excellent supervision and guidance, various suggestions, continuous encouragements, a very firm stand against some opposing forces, and a great deal of understanding, this work could not have been completed in its final form. My gratitude to him will probably be made more manifest by following his various suggestions and fine examples in the future.

I would also like to thank Dr. H. Davis for his help in reading the manuscript and in giving very valuable suggestions.

It is also appropriate for me to thank all those (too numerous to be listed) who have encouraged me, positively or otherwise, to determinedly go through the many pitfalls that befell me in the past four years and in the preparation of this thesis.

TABLE OF CONTENTS

TABLE OF CONTENTS

LIST OF FIGURES

SECTION 1

INTRODUCTION

A topological space X is said to have the <u>absolute monotone union property</u> if whenever

$$A = \bigcup_{i=1}^{\infty} A_i$$

where $\{A_i | A_i \subset A_{i+1}\}$ is a monotone increasing sequence indexed by the positive integers with each A_i topologically equivalent to X and if $A_i \overset{T}{=} A_{i+1}$, then A is necessarily topologically equivalent to X.

Kwun [11] shows that there are many manifolds with related property. In particular he proves that if X is a closed PL-manifold of dimension n, $n \neq 4$, and $p \in X$, then X - p has the open monotone union property (defined below). Without explicit definition, Brown [2] proves that the monotone union of open n-cells is an open cell. Kapoor [10] investigates the monotone union property in complexes. More extensive work has been done by Doyle [3,4] however; it is in [4] that the concept of absolute monotone union property (as defined above) is introduced and, compared with earlier works, extensively used. Our approach, in this work, is akin to the treatment of monotone union properties in the last work cited.

Section 2 defines $A(X, \mathcal{C})$ and investigates some topological spaces that have or do not have the absolute monotone union property relative to some classes $\mathcal{C}$ of spaces. A characterization of the rationals is also obtained (Theorem 2.8). Given a topological space X, what subsets of X have the absolute monotone union property in X? Section 3 exhibits some spaces all of whose subsets have this property and further investigates the nature of such spaces.

We next relax the definition of absolute monotone union property be requiring that only the condition $A_i \subset A_{i+1}$ needs hold and not necessarily $A_i \overset{T}{=} A_{i+1}$. Then given a property P, and if each A_i has property P, does the monotone union $\overset{\infty}{\underset{i=1}{(m)}} A_i$ have property P? Some topological properties are looked at in this perspective in Section 4; and in Section 5 we use weak topology to get some more properties, specifically the separation axioms.

If each A_i is open, we call the resulting property an <u>open monotone union property</u>. This property is applied in Section 6 to invertible plane continua that are locally connected.

SECTION 2

ABSOLUTE MONOTONE UNION PROPERTY
RELATIVE TO A CLASS $\mathcal{C}$

In this section we present examples of topological spaces that possess the absolute monotone union property with respect to the class $\mathcal{C}$ of topological spaces to which they belong. We also give examples of spaces that do not have this property.

Definition 2.1: Let $\mathcal{C}$ be a class of topological spaces, and let X be a member of $\mathcal{C}$. X is said to have the absolute monotone union property with respect to $\mathcal{C}$, denoted $A(X, \mathcal{C})$, if whenever there exists a monotone increasing sequence of copies of $X:\{M_i | M_i \in \mathcal{C},\ M_i \subset M_{i+1}\}$ such that $M_i \subset Y$, $Y \in \mathcal{C}$, $M_i \overset{T}{=} X$, then

$$X \overset{T}{=} \underset{i=1}{\overset{\infty}{\textcircled{M}}} M_i$$

(where the symbol $\underset{i=1}{\overset{\infty}{\textcircled{M}}}$ indicates a monotone union over sets indexed by the integers). If M_i, X and Y are in a topological space Z, X is said to have the absolute monotone union property in Z.

If X is a finite space, $A(X, \mathcal{T})$ holds, where $\mathcal{T}$ is the class of all topological spaces.

Definition 2.2: A space X has dimension 0 at a point p if p has arbitrarily small neighbourhoods whose boundaries are empty. A nonempty space X has dimension 0, dim X = 0, if X has dimension 0 at each of its points. We say then that X is a 0-dimensional space.

Lemma 2.1: A countable metric space is 0-dimensional.

Proof: If U_p is a neighbourhood of any point $p \in X$ of a countable space X, let $\delta > 0$ be a real number such that $S_\delta(p)$, the spherical neighbourhood about p with radius δ, is contained in U_p. Let $\{p_1, p_2, \ldots\}$ be an enumeration of X and $d(x,y)$ be the metric. Then there exists a real number $0 < \delta' < \delta$ such that $\delta' \neq d(p_i, p)$ for all i, and such that $S_{\delta'}(p) \subset U$. Then $Bd(S_{\delta'}(p))$, the boundary of $S_{\delta'}(p)$, is empty. Since p is arbitrary, X is 0-dimensional.

Theorem 2.2: Let Q be the space of rationals, M the class of separable metric spaces, then $A(Q,M)$ holds.

Proof: Let $\{M_i | M_i \subset M_{i+1}\}$ be a monotone increasing sequence of rationals, $M_i \subset Y$. Since each M_i is countable, each M_i is 0-dimensional by Lemma 2.1. Furthermore a countable union of countable spaces is countable; hence $\bigcup_{i=1}^{\infty} M_i$ is countable and therefore is 0-dimensional.

Claim: $M_i \overset{T}{=} Q$ for each i, and $\bigcup_{i=1}^{\infty} M_i \overset{T}{=} Q$. To prove these, we use the General Imbedding Theorem of Hurewicz and Walman [8]:

"Suppose X is an arbitrary space and dim $X \leq n < \infty$, then X is homeomorphic to a subset of I^{2n+1}."

That is the 0-dimensional M_i (and $\overset{\infty}{\underset{i=1}{\mathbb{M}}} M_i$) is homeomorphic to a subset of I, the closed unit interval; hence each M_i (and $\overset{\infty}{\underset{i=1}{\mathbb{M}}} M_i$) can be mapped homeomorphically onto Q, the rationals. We therefore have

$$Q \overset{T}{=} M_i, \; M_i \subset M_{i+1} \text{ and } Q \overset{T}{=} \overset{\infty}{\underset{i=1}{\mathbb{M}}} M_i$$

($\overset{\infty}{\underset{1}{\mathbb{M}}} M_i$ is countable and each point is a limit point)

and the definition of $A(Q,\mathcal{M})$ is satisfied.

The characterization of Q is Theorem 2.8 below.

<u>Theorem 2.3</u>: If P is the space of irrationals, and $\mathcal{M}$ is as in Theorem 2.2, then P does not have the absolute monotone union property relative to $\mathcal{M}$.

<u>Proof</u>: Let Q be the rationals and therefore $P = E^1 - Q$ where E^1 is the real line. Let $\{r_1, r_2, ..\}$ be an enumeration of the rationals Q. Then $Q = \bigcup_{i=1}^{\infty} r_i$, r_i is a rational number. Since $P \cup \{r_i\}$ is an irrational space for each i, we can successively adjoin the rationals to the space P to get a monotone increasing sequence of irrational spaces:

$$P \subset P \cup \{r_1\} \subset P \cup \{r_1, r_2\} \subset \ldots\ldots \subset P \cup \{r_1, r_2 \ldots r_n\} \subset \ldots$$

For convenience, denote $P \cup \{r_1, r_2, \ldots r_j\} = P \cup (\bigcup_{i=1}^{j} r_j)$ by P_j, each $r_i \in Q$. Then for each j, $P \overset{T}{=} Y_j$. This follows from the fact that each P_j is 0-dimensional and we can therefore use the General Imbedding Theorem quoted in 2.2. However

$$\underset{j=1}{\overset{\infty}{\bigcup\!\!\!\!M}} P_j = P \cup (\bigcup_{i=1}^{\infty} r_i) = (E^1 - Q) \cup Q = E^1$$

But since P is the irrational space $P \overset{T}{\neq} E^1$ and therefore $A(P,M)$ does not hold.

It is shown above that if a subset A of X has the absolute monotone union property relative to a class C, X-A does not necessarily have the property. The foregoing example shows that $A(X,C)$ is not hereditary; that is if X possesses the absolute monotone union property with respect to C, it does not follow that every subspace of X has that property:

<u>Theorem 2.4</u>: Let X be the joined curve, that is

$X = \{(0,y) \mid -1 \leq y \leq 1\} \cup \{(x, \sin \pi/x) \mid 0 < x \leq \frac{1}{2}\} \cup$

a simple arc joining the points $P_1(0,-1)$ and $P_2(1/2,0)$ (Figure 1). Since X is unique in the class of all topological spaces in X (for if there were to be any other space Y homeomorphic to X, then X would be properly imbeddable in one of its subsets), X has the absolute monotone union property relative to T. We now consider a

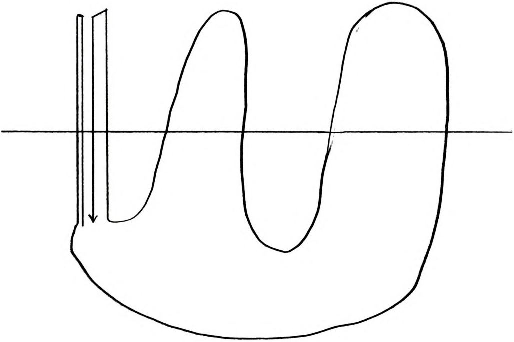


Figure 1

connected subspace Y_0, a subarc homeomorphic to $[0,\infty)$. In E^1, let

$$Y_1 = [-1,\infty)$$
$$Y_2 = [-2,\infty)$$
$$Y_n = [-n,\infty)$$

Then $Y_i \subset Y_{i+1}$, $Y_i \overset{T}{=} Y_0$ for each i. But

$\overset{\infty}{\underset{i=1}{\mathcal{M}}} Y_i = (-\infty,\infty) \overset{T}{\neq} Y_0$ as is required for $A(Y_0,\mathcal{T})$

to hold.

Since the irrationals P can be embedded in the space X of Example 2.4 above, and we have proved that $A(P,\mathcal{M})$ does not hold, this is another example showing that $A(X,\mathcal{C})$ does not hold.

<u>Theorem 2.5</u>: The property of having absolute monotone union property is a topological invariant; that is if $A(X,\mathcal{C})$ holds for a topological space X in a class $\mathcal{C}$, and if $X \overset{T}{\equiv} Y$, then $A(Y,\mathcal{C})$ holds.

<u>Proof</u>: Let $A(X,\mathcal{C})$ hold, and let $\{X_i | X_i \subset C,\ c \in \mathcal{C}\}$ satisfy the definition of $A(X,\mathcal{C})$. In addition suppose $f : X \longrightarrow Y$ is a homeomorphism of X onto Y. Then let $g_i : X \longrightarrow X_i$ and $g : X \longrightarrow \underset{i=1}{\overset{\infty}{\textcircled{M}}} X_i$ be the homeomorphisms of X onto X_i and X onto $\underset{i=1}{\overset{\infty}{\textcircled{M}}} X_i$ respectively. Consider the following diagram:

$$\begin{tikzcd} & & X_i \\ Y \arrow[r, "f^{-1}"] & X \arrow[ur, "g_i"] \arrow[dr, "g"'] & \\ & & \underset{i=1}{\textcircled{M}} X_i \end{tikzcd}$$

The homeomorphism of $f : x \longrightarrow Y$ gives the obvious homeomorphism $f^{-1} : Y \longrightarrow X$. And the facts that a composition of two (or more) homeomorphisms is a homeomorphism yields

$$g_i \cdot f^{-1} : Y \longrightarrow X_i \quad \text{and}$$

$$g \cdot f^{-1} : Y \longrightarrow \underset{i=1}{\overset{\infty}{\textcircled{M}}} X$$

as the required two homeomorphisms of Y onto X_i and Y onto $\underset{i=1}{\overset{\infty}{\textcircled{M}}} X_i$ respectively. Thus $Y \overset{T}{\equiv} X_i$, $X_i \subset X_{i+1}$, and $Y \overset{T}{=} \underset{i=1}{\overset{\infty}{\textcircled{M}}} X_i$ and therefore Y has the absolute

monotone union property relative to $\mathcal{C}$.

Theorem 2.6: The Cantor set K does not have the absolute monotone union property relative to $\mathcal{M}$, the class of separable metric spaces.

Proof: Let K_1 be the Cantor set in the closed interval $[0,1]$. One of the characterizations of the Cantor set is that it is a totally disconnected, compact, perfect, metric space [6]. Thus $K \overset{T}{=} K_1$. Let K_2' be the Cantor set in the closed interval $[0,2]$. By characterizing the Cantor set as the set of all points in a closed interval having no units in their ternary expansion, $K_1 \overset{T}{=} K_2 \overset{T}{=} K$ and $K_1 \subset K_2 = K_2' \cup K_1$.

We inductively construct K_n so that K_n' is the Cantor set in the closed interval $[0,n]$ with $K \overset{T}{=} K_n$, and $K_{n-1} \subset K_n = K_n' \cup K_{n-1}$. The first given characterization of the Cantor set implies that for each positive integer k, $k \neq \infty$, K is topologically equivalent to $\underset{i=1}{\overset{k}{\textcircled{M}}} K_i$, the non-compactness of the monotone union $\underset{i=1}{\overset{\infty}{\textcircled{M}}} K_i$ implies that K and $\underset{i=1}{\overset{\infty}{\textcircled{M}}} K_i$ are not topologically the same, and therefore K fails to have the desired property.

In the proof of Theorem 2.2 and the ensuing discussions one is tempted to ask whether some of the discussed spaces, and the space Q of rationals in particular, have the absolute monotone union property with respect to any other class $\mathcal{C}$ of topological spaces besides $\mathcal{M}$. The

following example of a non-metric space shows that going outside M does not yield the desired property:

Let $\{r_1, r_2, \ldots\}$ be an enumeration of the rationals, Q. Construct the rational comb space R as follows: (Figure 2)

$$R = \{(0, r_i)\} \cup \{(\tfrac{1}{n}, r_i) \mid n = 1,2,3,\ldots\},\ r_i \in Q.$$

This gives a sequence $\{Q_{1/n}\}$ of 1/n - rationals converging to $Q_0 = \{(0, r_i)\}$ as limit. We next define a topology on R as follows:

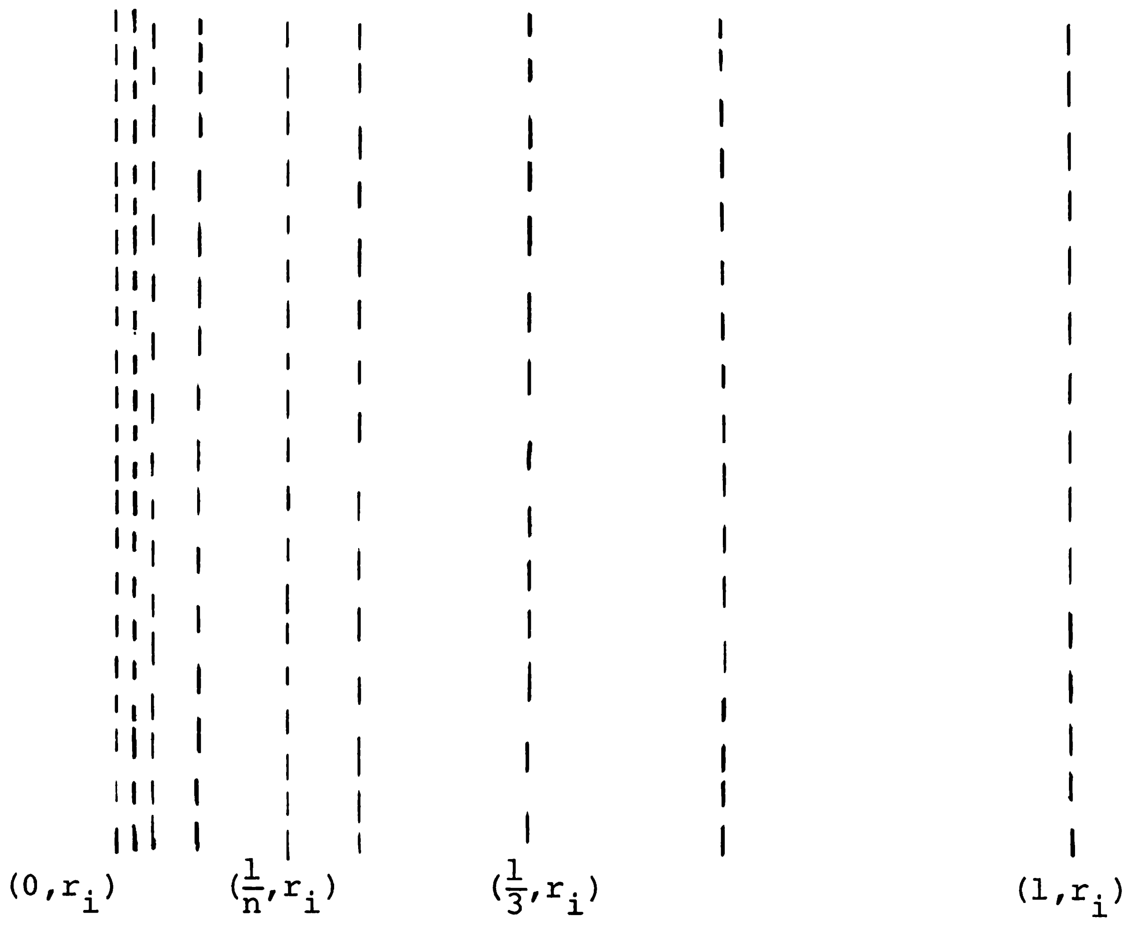


Figure 2

On R - $\{(0,r_i)\}$, use the usual rational topology; for a neighbourhood N_0 of $(0,r_j)$ take a neighbourhood in $\{(0,r_i)\}$ and all but a finite number of the vertical $\{(1/n,r_i|)\}$'s.

Claim: R, together with the topology described above, is not a metric space. For let S_1 and S_2 be two points on $\{(0,r_i)\}$, the rationals of convergence. Since any open subset of R containing S_1 meets any containing S_2, we cannot get two disjoint open sets containing S_1 and S_2 respectively; thus R is not T_2. However a metric space is T_2, hence the conclusion that R is not metric.

So let R_0 be the rationals of convergence $\{(0,r_i)\}$. For R_1 take $\{(0,r_i)\} \cup \{(1,r_i)\} = R_1$. For R_2, let $\{(0,r_i)\} \cup \{(1,r_i)\} \cup \{(1/2,r_i)\} = R_2$.

.
.
.
.

For R_n, let $\{(0,r_i)\} \cup \cup \{(1/k,r_i)|k = 1,2,\ldots,n\}$ be R_n. This construction gives the monotone increasing sequence:

$$R_0 \subset R_1 \subset R_2 \subset \ldots. \subset R_n \subset \ldots.$$

Because each R_i is the rationals, for i = 0,1,2,..., each $R_i \overset{T}{\equiv} Q$, the rationals space. But $\biguplus_{i=1}^{\infty} R_i = R$ which is not metric. Hence the non-metric R is not topologically equivalent to the metric space Q thus completing the proof.

<u>Remark</u>: Let X be a space such that X cannot be imbedded in itself. X has the absolute monotone union property relative to the class of all topological spaces. The proof that the Cantor set does not have the property of absolute monotone union suggests that many spaces imbeddable in themselves may not have this property.

<u>Definition 2.3a</u>: A <u>series, $(K,\leq)$</u>, is a nonempty nondegenerate simply ordered set.

<u>Definition 2.3b</u>: A <u>continuous series, $(K,\leq)$</u>, is a series with the following properties:

(i) If K_1 and K_2 are any two nonempty subsets of K such that every element of K belongs to either K_1 or K_2, and every element of K_1 precedes every element of K_2, then there is at least one element x in K such that

a. any element that precedes x belongs to K_1 and

b. any element following x belongs to K_2

(Dedekind's Postulate)

(ii) If a and b are elements of the set K and if $a < b$, then there is at least one element x in K such that $a < x < b$. (Postulate of Density)

<u>Definition 2.3</u>: <u>A linear continuous series</u>, $(K,\leq)$, is a continuous series which satisfies the following property:

The set $(K,\leq)$ contains a countable subset Q in such a way that between any two elements

of the set K there exists an element of Q. Using the elementary properties of the real line E^1 and the definitions 2.3a, b and 2.3, we state the following theorem:

Theorem 2.7: E^1, together with the usual ordering of the reals, is a linear continuous series.

Theorem 2.8: (A characterization of the Rationals in E^1): Let X be a countably infinite space in E^1 all of whose points are limit points. Then X is the Rationals.

Proof: Since the rationals are countably infinite and dense in E^1, it will suffice to show that any two countable dense series having neither a first nor a last element are ordinally similar, thus we would have characterized the rationals. To this end, Huntington [7] has used the method of George Cantor to prove precisely that.

In the proof that the rationals have the absolute monotone union property with respect to $\mathcal{M}$ the properties used are, in fact, those that characterize the rationals; this allows us to generalize Theorem 2.2 to the following.

Corollary 2.9: If M is a countably infinite space all of whose points are limit points, then M has the absolute monotone union property relative to the class of all separable metric spaces.

Theorem 2.10: Let X be a subset of E^1 such that X has the absolute monotone union property in E^1. Then X is

either an open interval, or a totally disconnected set having at most a finite number of isolated points and at least a limit point.

Proof: (The proof depends upon the continuum hypothesis). There are two cases.

Case I. X contains an open nonempty subset S. Then there exists an increasing sequence $\{U_i | U_i \subset U_{i+1}\}$ of proper open sets in E^1 such that $S \stackrel{T}{\equiv} U_i$ and $E^1 \stackrel{T}{\equiv} \bigcup_{i=1}^{\infty}(M) U_i$. Thus $S = E^1$ has the absolute monotone union property relative to E^1.

Before considering the second case, we need the following:

Lemma 2.11: If S is a totally disconnected countably infinite subset of E^1 with infinitely many isolated points, then $A(S,M)$ does not hold.

Proof: Enumerate the isolated points of S as $\{p_1, p_2, \ldots.\}$ and call the set P. For each p_i, let $U^0_{p_i}$ be a finite open neighbourhood of p_i such that $U^0_{p_i} \cap S = \{p_i\}$ and such that $U^0_{p_i} \cap U^0_{p_j} = \emptyset$ for $i \neq j$. Next let $U^1_{p_i}$ be the finite open neighbourhood $U^0_{p_i}$ to which has been added an extra point p_{i1}, for each i, since $U^1_{p_i}$ is an open interval containing p_i and p_{i1}, it is clear that $U^0_{p_i}$ is topologically equivalent to $U^1_{p_i}$.

In a similar fashion, let $U^2_{p_i}$ be the finite open neighbourhood $U^0_{p_i}$ to which have been added p_{i1} and p_{i2} for each i. By using induction, define $U^k_{p_i}$ as the finite open neighbourhood $U^0_{p_i}$ to which has been added the set $\{p_{i_1}, p_{i_2}, \ldots, p_{i_k}\}$ for each i, where $\{p_{i_k}\}$ is the rationals in $U^0_{p_i}$. (Figure 3)

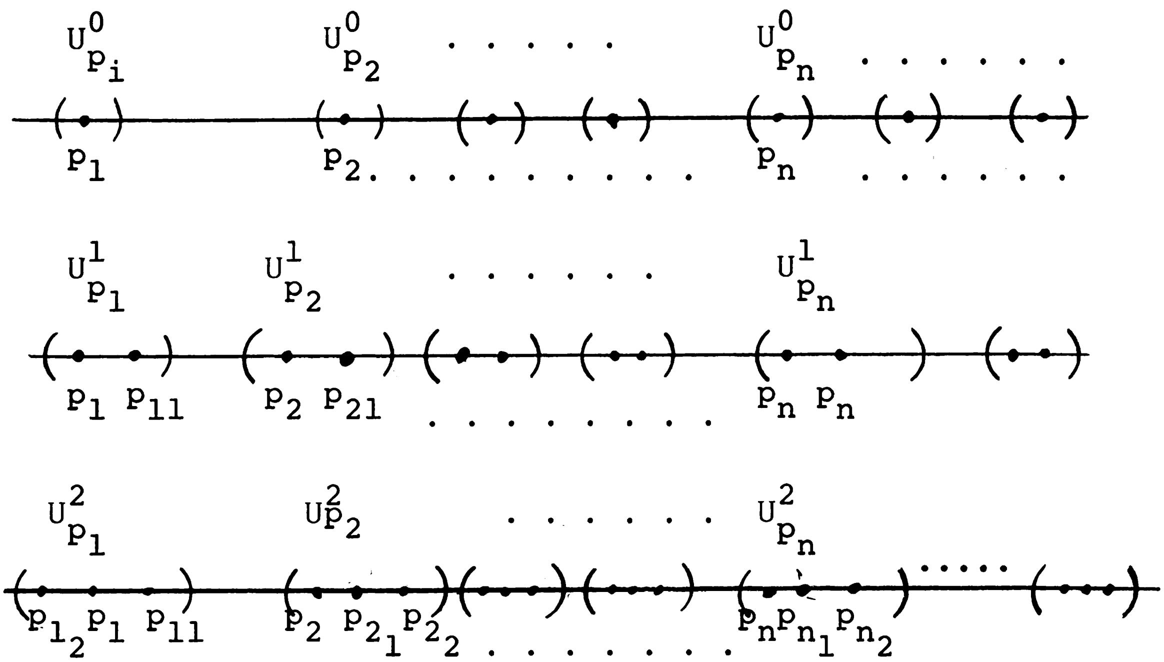


Figure 3

Denote the p_{i_1}'s and the p_i's in $\bigcup_{i=1}^{\infty} {}^{1}_{p_i}$ by P^1; i.e., $P^1 = P \cup \{p_{i_1} | i = 1,2,\ldots\}$ in such a way that each point is a limit point of the union. Also let

$$P^2 = P^1 \cup \{P_{i_2} | i = 1,2,\ldots\}.$$

Inductively after defining p^{k-1}, let

$$P^k = P^{k-1} \cup \{P_{ik} | i = 1,2,\ldots\}.$$

Then we have $P =^T P^1 =^T P^2 =^T \ldots\ldots\ =^T P^k =^T \ldots\ldots\ldots$ since for each j, P_j is an enumeration of isolated points

just as P is. Furthermore, $P \subset P^1 \subset P^2 \subset \ldots \subset P^k \subset \ldots$.

Thus we have

$S = S \cup P =^T (S-P) \cup P^1 =^T (S-P) \cup P^2 =^T \ldots$

$=^T (S \cup P) \cup P^k =^T \ldots$

In addition,

$S \cup P \subset (S-P) \cup P^1 \subset (S-P) \cup P^2 \subset \ldots \subset (S-P) \cup P^k \subset \ldots$

thereby getting an absolute monotone union of $(S-P) \cup P^j$

$$\overset{\infty}{\underset{j=1}{\textcircled{M}}} (S-P) \cup P^j.$$

However, successive addition of points p_{ik} to $U^0_{p_i}$ for each i yields a dense set of points in $U^0_{p_i}$. Therefore the monotone union $\overset{\infty}{\underset{j=1}{\textcircled{M}}} (S - P) \cup P^j$ is a dense set in $\bigcup_{i=1}^{\infty} U^0_{p_i}$. But P, and therefore S, is not dense in $\bigcup_{i=1}^{\infty} U^0_{p_i}$. And so $S \neq^T \overset{\infty}{\underset{j=1}{\textcircled{M}}} (S - P) \cup P^j$, thus showing that S does not have the absolute monotone union property. We now consider the second case.

<u>Case II</u>: X is totally disconnected.

A. If X is finite, we have seen that X has the absolute monotone union property with respect to any class of topological spaces, and therefore $A(X,M)$; hence $A(X,E^1)$.

B. If X is infinite, what is the topological structure of X? Firstly, X must have a limit point for if otherwise X is topologically Z, the integers. But Z consists only of infinitely many isolated points and by

Lemma 2.11, Z does not have the absolute monotone union property in E^1. Secondly, X cannot have infinitely many isolated points. For if X has infinitely many isolated points, again Lemma 2.11 shows that the desired property fails for X.

SECTION 3

SPACES WITH HEREDITARY ABSOLUTE MONOTONE UNION PROPERTY

Section 2 discussed some topological spaces having the absolute monotone union properties relative to a class $\mathcal{C}$; also discussed were some spaces whose subsets have or do not have these properties. An example was given to show that given a space X and A(X,$\mathcal{C}$), not all subsets of X necessarily have the absolute monotone union properties in that space. This section deals with those spaces all of whose subsets have the absolute monotone union properties in them. To be more precise, we being with a definition.

<u>Definition 3.1</u>: A topological space X is said to have the hereditary absolute monotone union property, denoted HA(S,X), if every proper subset S of X has the absolute monotone union property in X.

<u>Remark</u>: Henceforth in this section all spaces shall be considered infinite unless otherwise stated.

<u>Theorem 3.1</u>: Let Z be the integers. The Z has the HA(S,Z) property.

Proof: Let S be an infinite subspace of the integers. S is homeomorphic to Z; hence there exists an infinite subset Y_0 in Z such that $Z - Y_0$ is infinite and $S \overset{T}{=} Y_0$. Let $Y_1 = Y_0 \cup \{Z_1\}$ where $Z_1 \notin Y_0$. It is clear that Y_1 is homeomorphic to S and $Y_0 \subset Y_1$. Inductively form the monotone sequence $\{Y_i | Y_{i+1}, Y_i \overset{T}{=} S\}$ after Y_{k-1} has been got by letting Y_k to be $Y_{k-1} \cup \{Z_k\}$, $Z_k \notin Y_0$. Since $Y_0 \bigcup_{i=1}^{\infty} Z_i \overset{T}{=} S$ we have $S \overset{T}{=} \boxed{m}_{i=1}^{\infty} Y_i$, thus satisfying the absolute monotone union property in Z.

Definition 3.2: Let $X = \{a_i | i \in Z^+\}$ be a set of points indexed by the positive integers. Let $U_1 = \{a_1\}$.

$$U_2 = \{a_1, a_2\}$$

$$U_3 = \{a_1, a_2, a_3\}$$

$$\vdots$$

$$U_n = \{a_1, a_2, \ldots a_{n-1}, a_n\} \ldots$$

The set $T = \{U_i | i \in Z^+\}$ along with $\emptyset$ and X is called the tower topology on X.

Theorem 3.2: Let $X = \{a_i | i \in Z^+\}$ with the tower topology. Then X has the HA(S,X) property.

Proof: Let S be a subset of X. If S is finite, we are done. So assume that S in an infinite subset of X. We first claim that S has the same topology as X: For if $S = \{b_1, b_2, \ldots | b_i = a_j \text{ for some } a_j \in X\}$, suppose the first element b_1 of S corresponds to a_k in X for some $k \in Z^+$.

Then $U_k = \{a_1, a_2, \ldots a_k\} = \{a_1, a_2, \ldots b_1\}$ is open in X; hence $U_k \cap S = \{b_1\}$ is open in S. If b_2 corresponds to say a_j in X, then U_j is open in X and $U_j \cap S = \{b_1, b_2\}$ is open in S. Thus for each $j \in Z^+$, $\{b_1, b_2, \ldots b_j\}$ is open in S; and therefore S has the same topology as X. It is now easy to see that any infinite subset S has the absolute monotone union property in X.

Theorem 3.3: A space X with the discrete topology has the HA(S,X) property.

Proof: Every infinite subset S of X is open and is topologically equivalent to any other infinite subspace of X with the same cardinality. There exists, therefore, an infinite subspace S' of X such that S - S' is infinite, card S = card (X - S') and $S \overset{T}{=} X - S'$. Let $S_1 = X - S'$. For some $x_1 \in X - S'$, let $S_2 = S_1 \cup \{x_1\}$. For some $x_2 \in X - S'$, let $S_3 = S_2 \cup \{x_2\}$. Assuming that S_{k-1} has been thus obtained, let $S_k = S_{k-1} \cup \{x_{k-1}\}$ where $x_{k-1} \in X - S'$. This yields a monotone sequence $\{S_i | S_i \subset S_{i+1}\}$.

Claim: (i) $S_i \overset{T}{=} S$, (ii) $S \overset{T}{=} \underset{i=1}{\overset{\infty}{Ⓜ}} S_i$: For each S_i is open and thus $\underset{i=1}{\overset{\infty}{ⓜ}} S_i$ is open and infinite. There is therefore a bijection f which carries any open set in S to an open set in $\underset{i=1}{\overset{\infty}{ⓜ}} S_i$. And for any open set in $\underset{i=1}{\overset{\infty}{ⓜ}} S_i$, its inverse image is open in S by the discrete topology, f is thus a bicontinuous bijection and the topological equivalences claimed follow.

Theorem 3.4: The space Q of the rationals does not have the HA(S,Q) property.

Proof: Take as a subset of Q the integers Z. Let $\{r_1, r_2, \ldots\} = Q'$ be an enumeration of the proper fractions (rationals) in Q. Form a monotone increasing sequence $\{R_i\}$ as follows: Take as R_1 the set r_1, Z, and observe that $R_1 =^T Z$. Let $\{r_1, r_2\}Z = R_2$; that is R_2 is the set of all points of the form $\{r_1, z, r_2 z \mid r_1, r_2 \in Q', z \in Z\}$. Similarly $R_2 =^T Z$ and $R_1 \subset R_2$. Inductively let $R_k = \{r_1, r_2, \ldots r_k\}Z$, $r_i \in Q'$ for $i = 1, 2, \ldots, k$. Then $R_k =^T Z$, $R_k \subset R_{k+1}$. However, $\underset{i=1}{\overset{\infty}{Ⓜ}} R_i = \{r_1, r_2, \ldots,\}Z = Q'Z =^T Q$, hence Z does not have the absolute monotone union property in Q since $\underset{i=1}{\overset{\infty}{Ⓜ}} R_i \neq^T Z$.

Lemma 3.5: Let M be a metric space having the HA(S,M) property. Then M has at most one limit point.

Proof: Suppose M has at least two limit points, p_1 and p_2. Then there exist in M two convergent sequences $\{a_i\}$ and $\{b_i\}$ converging to p_1 and p_2 respectively. Let $S = \{a_i\} \cup p_1 \cup p_2$. S is a convergent sequence in M having p_1 as its limit point. Let $S_1 = \{a_i\} \cup b_1 \cup p_1 \cup p_2$. The only limit point of S_1 is p_1, and $S \overset{T}{=} S_1$. Let $S_2 = \{a_i\} \cup \{b_1, b_2\} \cup p_1 \cup p_2$; similarly, the only limit point of S_2 is p_1 and as before $S \overset{T}{=} S_2$ and $S_1 \subset S_2$. After S_{k-1} has been obtained, let

$$S_k = \{a_i\} \cup \{b_1, b_2, \ldots b_{k-1}, b_k\} \cup p_1 \cup p_2.$$

S_k has p_1 as its only limit point, $S \overset{T}{=} S_k$, and $S_{k-1} \subset S_k$. Next consider $\underset{i=1}{\overset{\infty}{ⓜ}} S_i$. This is the set: $\{a_i\} \cup \{b_i\} \cup p_i \cup p_2$. Because $a_i \longrightarrow p_1$ and $b_i \longrightarrow p_2$, $\underset{i=1}{\overset{\infty}{ⓜ}} S_i$ has p_1, p_2 as two limit points whereas S has only one, namely p_1. Hence $S \overset{T}{\neq} \underset{i=1}{\overset{\infty}{ⓜ}} S_i$.

<u>Theorem 3.6</u>: Let M be a metric space having the HA(S,M) property. Then M is a compact metric space with exactly one limit point, or M is a discrete space with no limit points.

<u>Proof</u>: If M in Lemma 3.5 has a limit point, M has to be compact. For suppose M is not compact and has a limit point. In particular let M be the union of the nonnegative integers and $\{1/n\}$. The limit point of M is 0 and there is no other. Let $S_1 = \{1/n\} \cup \{0,-1\}$, S_1 is compact and has 0 as its limit point. Let $S_2 = \{1/n\} \cup \{0,-1,-2\}$. $S_1 \overset{T}{=} S_2$, $S_1 \subset S_2$, and S_2 is compact with the limit point 0. Thus for each k, let $S_k = \{1/n\} \cup \{0,-1,2,\ldots -k\}$. $S_1 \overset{T}{=} S_2 \overset{T}{=} \ldots \overset{T}{=} S_k$, S_{k-1}, $\subset S_k$ and each S_k is compact.

But although $\underset{i=1}{\overset{\infty}{ⓜ}} S_i = \{1/n\} \cup \{0,-1,-2,\ldots\}$ has the limit point 0, this last set is not compact. There does not exist, therefore, a subset S of M such that $S \overset{T}{=} S_i$ and $S \overset{T}{=} \underset{i=1}{\overset{\infty}{ⓜ}} S_i$. Hence M has to be compact if it has a limit and the HA(S,X) property holds.

If M has no limit point, then M is topologically equivalent to Z which is discrete, and we have seen that Z has the HA(S,X) property.

<u>Corollary</u> 3.7: The Cantor set K does not have the HA(S,K) property

<u>Proof</u>: K has more than one limit point.

SECTION 4

ABSOLUTE PROPERTIES UNDER MONOTONE UNIONS

Definition 4.1: A property P is absolute under monotone unions (aumu) in a class $\mathcal{C}$ of topological spaces if for each $Y \in \mathcal{C}$, $X_i \subset Y$, $X_i \subset X_{i+1}$ and for each i, each X_i has property P implies that $\bigcup_{i=1}^{\infty} X_i$ has property P. We here investigate some topological properties that are or are not aumu with respect to a class $\mathcal{C}$ of topological spaces.

Theorem 4.1: Connectedness is an absolute property under monotone unions in the class $\mathcal{C}$ of all topological spaces.

Proof: Let Y be a topological space having a sequence $\{X_i | X_i \subset X_{i+1}\} X_1 \neq \emptyset$ of subsets with the property that for each i, X_i is connected. Then $\bigcap_{i=1}^{\infty} X_i \subset X_1$; hence there exists at least one point in common with the family $\{X_i\}$ of connected subsets of Y. The union $\bigcup_{i=1}^{\infty} X_i$ of this family is therefore connected and the assertion is proved.

Theorem 4.2: Arcwise (path) connectedness is an absolute property under monotone unions in the class $\mathcal{C}$ of all topological spaces.

Proof: Let Y, $\{X_i\}$ be as in 4.1 and X_i is arcwise (path) connected. For any two points $x,y \in \bigcup_{i=1}^{\infty} X_i$, there exists

some $j \in Z^+$ such that $x,y \in X_j$. Since X_j is arcwise (path) connected, so is $\underset{i=1}{\overset{\infty}{ⓜ}} X_i$.

Theorem 4.3: The property of being locally connected is not aumu in C.

Proof: Modify the space X in 2.4 by letting the joined sine curve be open at the point (0,1). Taking as X_1 an open interval beginning at the point (0,1), as X_2 an open interval containing X_1 as X_n an open interval containing X_{n-1}, a monotone increasing sequence $\{X_i | X_i \subset X_{i+1}\}$ of open intervals each beginning at the point (0,1) is thus obtained. Each X_i is locally connected. But it is easy to see that each point x in $\{(x,y) | x = 0, -1 < y < 1\} \subset X$ has a neighbourhood not containing any connected neighbourhood of x.

Theorem 4.4: Disconnectedness is not absolute under monotone unions in C.

Proof: If P is the set of irrationals in E^1, form a monotone increasing sequence $\{P_i | P_i \subset P_{i+1}\}$ as in Theorem 2.3: Each P_i is disconnected but $\underset{i=1}{\overset{\infty}{ⓜ}} P_i = E^1$ is not.

Corollary 4.5: The property of being 0-dimensional is not aumu in the class C of topological spaces.

Proof: Each P_i in 4.4 is 0-dimensional but E^1 is not.

Lemma 4.6: A 0-dimensional space Y is disconnected.

Proof: Let Y be 0-dimensional. In particular for each

$p \varepsilon Y$ there exists arbitrarily small neighbourhoods of y which are both open and closed because the boundaries of such neighbourhoods are empty. Since these neighborhoods are neither Y nor empty, Y cannot be connected for the only subsets of a connected space both open and closed are the empty set and the space itself.

Theorem 4.7: Let C_0 be the class of all countable metric spaces. Then disconnectedness is an absolute property under monotone unions in C_0.

Proof: Let $Y \varepsilon C_0$, $\{X_i \subset Y | X_i \subset X_{i+1}\}$, a sequence of disconnected subsets of Y. Every subset of a countable (metric) space Y is countable, hence for each i, X_i is countable (and therefore 0-dimensional). We need to show $\underset{i=1}{\overset{\infty}{\textcircled{m}}} X_i$ is disconnected. To do this, observe that a countable union of countable subsets is a countable subset. Therefore $\underset{i=1}{\overset{\infty}{\textcircled{m}}} X_i$ is countable and is therefore 0-dimensional (by Lemma 2.1). Apply Lemma 4.6 and we have proved that $\underset{i=1}{\overset{\infty}{\textcircled{m}}} X_i$ is disconnected.

Corollary 4.8: The property of being countable is absolute under monotone unions in any class C.

Remark: For any cardinality c greater or equal to the cardinality of the rationals, the property of having cardinality c is absolute under monotone unions in any class C, for if for each i card $X_i = c$ and $X_i \subset X_{i+1}$, then card $\left(\underset{i=1}{\overset{\infty}{\textcircled{m}}} X_i\right) = c$.

Definition 4.2: A set F is called an F_σ - set (or an F_σ) if F is the union of at most countably many closed sets. A set G is called a G_δ - set (or a G_δ) if G is the intersection of at most countably many open sets.

Theorem 4.9: The property of being F_σ is aumu in any class $\mathcal{C}$ of topological spaces.

Proof: Let $Y \varepsilon \mathcal{C}$ such that $\{X_i | X_i \subset X_{i+1}\}$ is a sequence of subsets of Y with the property that each X_i is an F_σ. Then for each i,

$$X_i = \bigcup_{i=1}^{\infty} C_{ij}$$

where each C_{ij} is a closed set. Then $\bigcup_{i=1}^{\infty} X_i = \bigcup_{i=1}^{\infty} [\bigcup_{j=1}^{\infty} C_{ij}]$. The right side is a countable union of countably many closed sets; and as such it is the union of countably many closed sets. Therefore $\bigcup_{i=1}^{\infty} X_i$ is an F_σ - set.

Lemma 4.10: Let x be a rational point in E^n. Then $\{x\}$ is a G_δ.

Proof: Let V_x^1 be a spherical neighbourhood of radius 1 with center at x; let V_x^2 be a spherical neighbourhood of radius 1/2 with center at x. In general, let V_x^n be a spherical neighbourhood of radius 1/n and center at x. Then $\bigcap_{j=1}^{\infty} V_x^j$ is the countable intersection of open sets V_x^j and this intersection therefore is G_δ. But $\bigcap_{j=1}^{\infty} V_x^j = \{x\}$, and therefore $\{x\}$ is G_δ.

Lemma 4.11: Let $R = \{r_1, r_2, \ldots r_k\}$ be a finite set of rationals in E^1. Then R is a G_δ - set.

Proof: For each $r_j \in R$ let V^1_{rj} be a spherical neighbourhood of radius 1 and center at r_j. In general V^n_{rj} be the n^{th} spherical neighbourhood of r_j of radius $1/n$ and center at r_j; for each $r_j \in R$. Then

$$\bigcup_{q=1}^{\infty} \bigcup_{j=1}^{\infty} V^q_{rj}$$

is an open set containing R, since a (finite) union of open sets is open. Let

$$Y = \bigcup_{j=1}^{k} \bigcap_{q=1}^{\infty} V^q_{rj}$$

By Lemma 4.10, for each $j = 1,2,\ldots,k$, $\bigcap_{q=1} V^q_{rj} = \{r_j\}$. Hence $Y = \bigcup_{j=1}^{k} r_j = R$, that is Y is a finite union of G_δ - sets and is therefore a G_δ - set.

Lemma 4.12: The set Q of the rationals in E^1 is not a G_δ - set.

Proof: Suppose $Q = \bigcap_{i=1}^{\infty} V^i$ where each V^i is open in E^1. Since Q is dense, each V^i is also dense. Let Y be the family

$$\{V^i\} \cup \{E - r \mid r \in Q\}$$

Y is a family of open sets since $E - r$ is open for each r, Y is countable and dense in E^1, and E^1 is a locally

compact space. By a theorem of Baire [5] the intersection of any countable family of open dense sets in a locally compact space is dense. Thus by this, $\bigcap_{A \varepsilon Y} A$ has to be dense. However

$$\bigcap_{A \varepsilon Y} A = \bigcap_{i=1}^{\infty} V^i \cap \{E - r \mid r \varepsilon Q\}$$

is empty and therefore is not the intersection of countably many open sets in E^1. So Q is not a G_δ - set.

<u>Theorem 4.13</u>: The property of being a G_δ - set is not absolute under monotone union in any class C.

<u>Proof</u>: Let $C = \{E^1\}$, and let $Q = \{r_1, r_2, \ldots\}$ be an enumeration of the rationals in E^1. Form a monotone sequence X_i as follows:

$$\begin{aligned} \text{Let } X_1 &= \{r_1\} \\ X_2 &= \{r_1, r_2\} \\ &\vdots \\ X_k &= \{r_1, r_2, \ldots, r_k\} \\ &\vdots \end{aligned}$$

By Lemma 4.10 and 4.11 X_1 is a G_δ and for each $j \varepsilon Z^+$, X_j is G_δ. $X_i \subset E^1$ and for each i $X_i \subset X_{i+1}$, however

$$\textcircled{m}_{i=1}^{\infty} X_i = Q$$

is the set of all rationals in E^1 and it has been seen in Lemma 4.12 that Q is not a G_δ.

<u>Definition 4.3</u>: A space X is said to be Lindelöf if every open covering of X has a countable open subcovering.

<u>Theorem 4.14</u>: The property of being Lindelöf is aumu in any class $\mathcal{C}$.

<u>Proof</u>: Let Y be an element of $\mathcal{C}$, $\{X_i \mid X_i \subset X_{i+1}\}$ a sequence of subsets of Y such that for each i, X_i is Lindelöf. Let $X = \underset{i=1}{\overset{\infty}{\text{Ⓜ}}} X_i$, and let $\{U_\alpha \mid \alpha \in A\}$ be an arbitrary open covering of X. Then $X = \underset{i=1}{\overset{\infty}{\text{Ⓜ}}} X_i = \bigcup_{\alpha \in A} U_\alpha$. Therefore for each $\alpha \in A$ there exists some $j \in Z^+$ such that $U_\alpha \cap X_j$ is not empty.

<u>Claim</u>: $\{U_\alpha \cap X_j \mid j \in Z^+ \text{ and } \alpha \in A\}$ is a countable open subcovering for X. For, for each $\alpha \in A$, $U_\alpha \cap X_j \subset X_j$, thus getting a countable subfamily $\{U_\alpha \cap X_j\}$ of $\{U_\alpha\}$. Furthermore

$$\bigcup_{\alpha \in A} \{U_\alpha \cap X_j\} = \underset{i=1}{\overset{\infty}{\text{Ⓜ}}} X_i = X$$

So X is Lindelöf.

<u>Definition 4.4</u>: A space X is <u>first countable</u> (or, satisfies the first axiom of countability) if X has a countable basis at each point p of X.

<u>Theorem 4.15</u>: The property of being first countable is aumu in any class $\mathcal{C}$.

<u>Proof</u>: Let $Y \in \mathcal{C}$, $X_i \subset Y$, $X_i \subset X_{i+1}$ for all i, and X_i is first countable. Let $X = \underset{i=1}{\overset{\infty}{\text{Ⓜ}}} X_i$ and suppose $p \in X$. Then

for some $j \in Z^+$, $p \in X_j \subset X$. By the first countability of X_j, there exists at most a countably infinite family $\{U'_k(p) \mid k \in Z^+\}$ of neighbourhoods of p having the following property: For each open G containing the point p there is some $U'_k(p) \subset G$ in X_j.

For each k, $U'_k(p)$ is open in X_j implies that there exists an open set U_k in X such that $U_k \cap X_j = U'_k(p)$. Since $p \in U'_k(p)$, $p \in U_k$ as well. So let

$$\{U_k(p) \subset X \mid U_k(p) \cap X_j = U'_k(p),\ k \in Z^+\}$$

be the countably infinite family of neighbourhoods of p in X. Thus for any G open in X and containing p, it is easy to see that there exists some $U_k(p)$ such that $U_k(p) \subset G$. This proves that X is first countable.

<u>Definition 4.5</u>: A space is <u>second countable</u> (or, satisfies the second axiom or countability) if it has a countable basis.

<u>Theorem 4.16</u>: Second countability is an absolute property under monotone unions in any class C of topological spaces.

<u>Proof</u>: Let $Y \in C$ with $X_i \subset Y$, $X_i \subset X_{i+1}$, and for each i, X_i is second countable. Let $X = \underset{i=1}{\overset{\infty}{m}} X_i$, for each i, let $\{V^i_j \mid j \in Z^+\}$ be a countable basis for X_i. We claim that the set

$$\mathcal{U} = \left\{ \bigcup_{i=1}^{\infty} V^i_j \mid \{V^i_j\} \text{ is a basis for } X_i,\ j \in Z^+ \right\}$$

is a basis for X. For let U be open in X. Then for each $i \in Z^+$, $U \cap X_i$ is either empty or open in X_i. If $U \cap X_i$ is empty for some i, there is nothing to prove. So assume that $U \cap X_i$ is not empty and therefore open. By the second countability of X_i, the open subset (in X_i) $U \cap X_i = \bigcup_j V_j^i$ for some $j \in Z^+$. Furthermore since $U \subset \underset{i=1}{\overset{\infty}{\textcircled{m}}} X_i$, we have

$$U = \bigcup_i \{U \cap X_i \mid U \cap X_i \neq \emptyset\}$$

Thus since each $U \cap X_i$ is a countable Union of the V_j^i's, U is a countable union of V_j^i, for some $i, j \in Z^+$, and therefore every open set in X is the union of members of the countable family $\mathcal{U}$. Therefore $\underset{i=j}{\overset{\infty}{\textcircled{m}}} X_i$ is second countable.

<u>Theorem 4.17</u>: Compactness is not an absolute property under monotone unions in the class $\mathcal{C}$ of all topological spaces.

<u>Proof</u>: Let $Y = \{1/n\} \cup \{0,1,2,\ldots\} \subset E^1$, $n = 1,2,\ldots$ Construct a monotone sequence of compact subsets $\{X_i\}$ as follows:

$$X_1 = \{1/n\} \cup \{0,1\}$$

$$X_2 = \{1/n\} \cup \{0,1,2\}$$

$$\vdots$$

$$X_k = \{1/n\} \cup \{0,1,2,\ldots k\} \quad n = 1,2,3,\ldots$$

$$\vdots$$

It is obvious to see that each X_i is compact (since it is closed and bounded). $X_i \subset X_{i+1}$; it is equally obvious to see that

$$\textcircled{m}_{i=1}^{\infty} X_i = Y = \{1/n\} \cup \{0,1,2,\ldots\}$$

is not compact for it is not a bounded set.

Theorem 4.18: The property of being a T_1-space is aumu in any class $\mathcal{C}$ of topological spaces.

Proof: Let Y be a topological space with the sequence $\{X_i \mid X_i \subset X_{i+1}\}$ of subsets of Y such that each X_i is T_1. Let $X = \textcircled{m}_{i=1}^{\infty} X_i$, and let x,y be two arbitrary distinct points in X. Then there exists, for some $j \in Z^+$, $X_j \subset X$ such that $x, y \in X_j$. X_j being T_1, there exist open sets U', V' in X_j such that $x \in U'$, $x \notin V'$, $y \in V'$, $y \notin U'$. Now there exist open sets U,V in X such that

$$U \cap X_j = U'$$

$$V \cap X_j = V'$$

Since $x \in U'$, $x \in U$; similarly $y \in V$. $x \in X_j$ and $x \notin V'$ imply that $x \notin V$. Similarly, $y \notin U$. We thus find sets U, V open in X with $x \in U$, $y \in V$, $x \notin V$ and $y \notin U$. By definition X is T_1 and the theorem is proved.

Theorem 4.19: Being T_2 is not an absolute property under monotone unions in some classes $\mathcal{C}$ of topological spaces.

Proof: Let $Y = E^2$, I^1 the open unit interval (0,1).

Let

$X_1 = (0 \times I^1) \cup (1 \times I^1)$

$X_2 = (0 \times I^1) \cup (1 \times I^1) \cup (1/2 \times I^1)$

$X_3 = (0 \times I^1) \cup (1 \times I^1) \cup (1/2 \times I^1) \cup (1/3 \times I^1)$

$\vdots$

$X_k = (0 \times I^1) \cup (1 \times I^1) \cup \ldots \cup (1/k \times I^1)$

$\vdots$

Each X_i is a finite union of open intervals in E^2 and since each I^1 is T_2, each X_i is T_2. Let $X = \bigcup_{i=1}^{\infty} X_i$ (Figure 4).

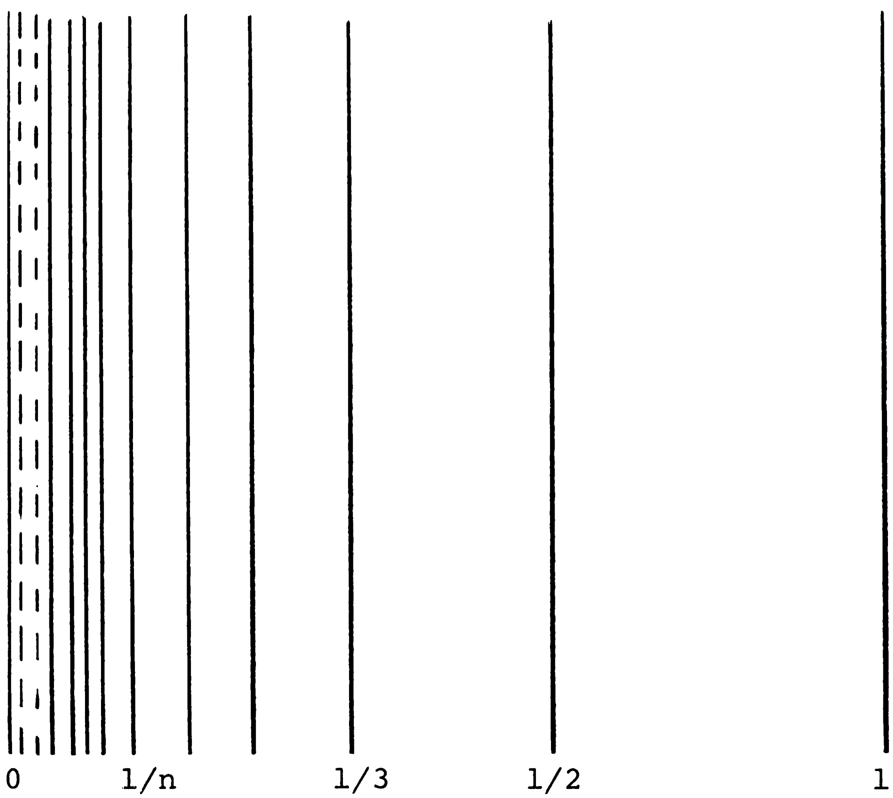


Figure 4

Put the following topology on X: On $X - (0 \times I^1)$ use the usual topology on E^1. For a neighbourhood of a point x in $0 \times I^1$, take an open interval in I^1 of $0 \times I^1$ and all but a finite number of the $(1/k \times I^1)$'s, $k = 1,2,\ldots$ Using the argument similar to the construction of a non-metric space in Section 2, we conclude that for any two distinct points x, y in $0 \times I^1$, there do not exist open sets U_x, U_y with $U_x \cap U_y = \emptyset$. Hence $X = \underset{i=1}{\overset{\infty}{\textcircled{m}}} X_i$ is not T_2.

Corollary 4.20: Any separation axiom beyond T_2 is not an absolute property under monotone unions in some classes $\mathcal{C}$ of topological spaces.

Proof: Any space satisfying any separation axiom beyond T_2 satisfies T_2, hence the corollary (using the definition of separation axioms in [5]).

Corollary 4.21: Let $\{X_i | X_i \subset X_{i+1}\}$ be a sequence of spaces such that each X_i is T_2 (regular, Tychonoff, normal). Then $\underset{i=1}{\overset{\infty}{\textcircled{m}}} X_i$ is at least T_1.

Proof: A simple application of Theorems 4.18, 4.19 and Corollary 4.20.

Corollary 4.22: Metrizability is not an absolute property under monotone unions in the class of all topological spaces.

Proof: Each X_i in theorem 4.19 is a metric space. But X is not normal and therefore not metric.

Another proof of Corollary 4.22 is afforded by the following: Let $Q = \{r_1, r_2, \ldots\}$ be an enumeration of the rationals in the plane on or above the x-axis.

$$
\begin{aligned}
\text{Let } X_1 &= \{r_1\} \\
X_2 &= \{r_1, r_2\} \\
&\vdots \\
X_k &= \{r_1, r_2, \ldots r_k\} \\
&\vdots
\end{aligned}
$$

Each X_i, being finite, is metrizable. So let $X = \underset{i=1}{\overset{\infty}{(\mathrm{m})}} X_i$ with the following topology: if (x,y) is a point of X and $\varepsilon > 0$, let

$$(x,y) + \{(r,0) \mid \text{either} \left| r - \left(a + \frac{b}{\sqrt{3}}\right) \right| < \varepsilon \text{ or}$$

$$\left| r - \left(a - \frac{b}{\sqrt{3}}\right) \right| < \varepsilon\}$$

be a neighbourhood of (x,y)

Geometrically, such a neighbourhood with center at (x,y) can be obtained by constructing an equilateral triangle (Figure 5) with base on the x-axis and apex at (x,y). If $y = 0$, let the point (x,y) be the desired triangle. Then (x,y) + all rationals on the x-axis whose distances from a base vertex of the triangle are less than ε is an ε-neighbourhood with center at (x,y). R. H. Bing has shown [1] that although this space X has a countable basis, it is not regular and therefore not metric.

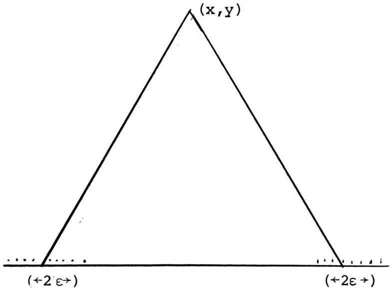


Figure 5

Definition 4.6: A space X is said to be a nontrivial product if X = Y x Z and neither X nor Y reduces to a single point.

Theorem 4.23: Being a nontrivial product is not an absolute property under monotone unions in $\mathcal{C}$.

Proof: Let $\mathcal{C} = E^3$. Let T_1 and T_2 be two solid tori as shown below (Figure 6). There exists a homeomorphism $h: E^3 \xrightarrow{\text{onto}} E^3$ such that $h: T^2 \longrightarrow T^2$ is an onto homeomorphism, and such that h is the identity exterior to some sphere [12]. Then $h(T^1) = h^2(T^2)$ is a torus such that $T^1 \subset$ interior $h^2(T^2)$. Then

$$M^3 = \bigcup_{n=1}^{\infty} h^n(T^2)$$

is a 3-mainfold which is a monotone union of $h^n(T^2)$. But each $h^j(T^2)$ is a copy of $E^2 x\ S^1$; hence M^3 is an open-monotone union of copies of $E^2 x\ S^1$ [4].

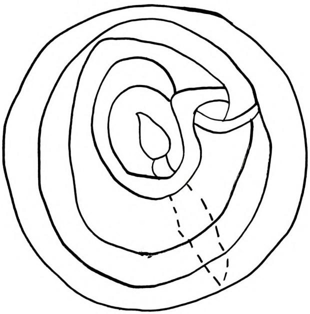


Figure 6

<u>Claim</u>: M^3 is not a nontrivial product. Suppose the contrary. Then M^3 is either (i) E^1 x C_1 or (ii) S^1 x C_2 where C_1, C_2 are subsets of M^3. Consider the homotopy groups of M^3 and of E^1 x C_1 and S^1 x C_2. For each k, $\pi_k(m^3) = 0$. Thus for π_1, we have (i) $\pi_1\ (E^1$ x $C_1) = 0$ or (ii) $\pi_1\ (S^1$ x $C_2) = 0$. For (i) $\pi_1(E^1$ x $C_1) = 0$ we have $\pi_1(E^1$ x $C_1) = \pi_1(E^1) * \pi_1(C_1)$. Since $\pi_1(E^1) = 0$, $\pi_1(C_1)$ has to be 0; but $\pi_1(C_1) = 0$ implies $C_1 = E^2$. And therefore for (i) $M^3 = E^1$ x $E^2 = E^3$ which is a contradiction since $M^3 \neq E^3$. For (ii) $\pi_1(S^1$ x $C_2) =$

$\pi_1 S^1) * \pi_1(C_2) = 0$ is impossible since $\pi_1(S^1) = Z \neq 0$.

Since the only two possibilities of M^3 have been proved not to apply, $M^3 = \underset{n=1}{\overset{\infty}{(m)}} h^n(T^2)$ has to be a trivial product even though each $h^n(T^2)$ is a nontrivial product.

Theorem 4.24: Let $C' = \{E^2\}$, the property of being a connected open nontrivial product is absolute under monotone unions in C'.

Proof: Let $X_i \subset E^2$, $X_i \subset X_{i+1}$ and for each i, X_i is a connected open nontrivial product. So let $X_i = Y \times Z$ where neither Y nor Z reduces to single points. Then Y and Z are locally compact connected sets, and $Y \times Z$ can be imbedded in a 2-mainfold. Hence by a theorem of Jones and Young [9], Y and Z can be either an arc, a simple closed curve (and therefore homeomorphic to S^1), a ray, or an open curve (homeomorphic to an open interval). But the homogeneity of Y and Z reduces these possibilities of Y and Z to either S^1 or I, where I is an open interval, and therefore to E^1. So we have

either (i) $X_i = E^1 \times E^1$

or (ii) $X_i = E^1 \times I$

or (iii) $X_i = E^1 \times S^1$, $X_i \subset E^2$.

(i) is not possible, since $X_i \neq E^2 = E^1 \times E^1$. So we consider (ii). If each $X_i = E^1 \times I$, then $\underset{i=1}{\overset{\infty}{(M)}} X_i = E^2$ which is a nontrivial product. (iii). If each $X_i = E^1 \times S^1$, then X_i is an annulus A. And the monotone union $\underset{i=1}{\overset{\infty}{(M)}} X_i$

gives two possibilities.

$$\underset{i=1}{\overset{\infty}{\textcircled{M}}} X_i = \begin{cases} \text{a) } E^2 \\ \text{b) an annulus } A \end{cases}$$

Since an annulus A is a nontrivial product ($A = E^2 \times S^1$) and E^2 is also nontrivial, we have thus proved the theorem.

SECTION 5

ABSOLUTE PROPERTIES IN WEAK TOPOLOGY

It has been shown above that there are some topological properties that are not absolute under monotone unions. In particular any separation axiom beyond T_1 is not absolute under monotone unions in some classes $\mathcal{C}$ of spaces. Here the concept of weak topology is used in order that these separation axioms be absolute.

<u>Definition 5.1</u>: Let X be a set, and let $\mathcal{S} = \{A_\alpha | \alpha \in \mathcal{A}\}$ be a sequence of subsets of X such that each A_α has a topology. Assume that for any $\alpha, \beta \in \mathcal{A}$ the following two properties hold:

(i) The topologies of A_α and A_β agree on $A_\alpha \cap A_\beta$

(ii) Either (a) each $A_\alpha \cap A_\beta$ is open in A_α and in A_β or (b) each $A_\alpha \cap A_\beta$ is closed in A_α and in A_β.

The <u>weak topology</u> in X induced (or determined) by $\mathcal{S}$ is

$$\mathcal{T}_{\mathcal{S}} = \{U \subset X | \text{for all } \alpha,\ U \cap A_\alpha \text{ is open in } A_\alpha\}$$

In literature X is also said to have a topology <u>coherent</u> with $\mathcal{S}$, if X has weak topology $\mathcal{T}_{\mathcal{S}}$ induced by $\mathcal{S}$. Denote such a topological space by $(X, \mathcal{T}_{\mathcal{S}})$ to distinguish it from other topologies.

<u>Lemma 5.1</u>: Let $(X, \mathcal{T}_s)$ be a topological space having the weak topology induced by $S = \{X_i | X_i \subset X_{i+1}\}$. Then any subset U open in X_j is open in $(X, \mathcal{T}_s)$.

<u>Proof</u>: Let $U \subset X_j$ be an open subset in X_j. Then for any k, $U \cap X_k$ is open in $X_j \cap X_k$ by definition 5.1 (i). Noting that A open (closed) in Y and Y open (closed) in Z implies that A is open (closed) in Z, and using Definition 5.1 (ii) (a) and (b), observe that $U \cap X_k$ is open in $(X, \mathcal{T}_s)$.

<u>Corollary 5.2</u>: Any subset V closed in X is closed in $(X, \mathcal{T}_s)$.

<u>Corollary 5.3</u>: A subset $B \subset X$ is open (closed) in X if $B \cap X_i$ is open (closed) in X_i for each X_i in S.

<u>Notation</u>: For the remainder of this section, $\mathcal{W}$ shall denote the class of topological spaces that are coherent with some family S of subspaces.

<u>Theorem 5.4</u>: T_2 is an absolute property under monotone unions in $\mathcal{W}$.

<u>Proof</u>: Let $\{X_i | X_i \subset X_{i+1}\}$ be a sequence of T_2-spaces and such that $X = \underset{i=1}{\overset{\infty}{(m)}} X_i$ has a topology coherent with $\{X_i\}$. Let x,y be two distinct points of X. Then for some $j \in Z^+$, $x,y \in X_j$. Since X_j is T_2, there exist open neighbourhoods U_x, U_y of x and y respectively such that $U_x \cap U_y = \emptyset$ in X_j. But by Lemma 5.1 U_x and U_y are open in $(X, \mathcal{T}_s)$.

Definition 5.2: A space Y is a Urysohn space if for every distinct points $x, y \in Y$ there exist open neighbourhoods U_x, U_y of x and y respectively such that $\overline{U}_x \cap \overline{U}_y = \emptyset$.

Corollary 5.5: The property of being a Urysohn space is aumu in class $\mathcal{W}$.

Definition 5.3: A Hausdorff space Y is regular if each $y \in Y$ and any closed set A not containing y have disjoint neighbourhoods.

Lemma 5.6: Y is regular if and only if for each $y \in Y$ and closed A not containing y, there is a neighbourhood V of y such that $\overline{V} \cap A = \emptyset$.

Theorem 5.7: Regularity is an absolute property under monotone unions in $\mathcal{W}$.

Proof: Let $\{X_i | X_i \subset X_{i+1}\}$ be a sequence of regular spaces and let X be its monotone union with the weak topology induced by $\{X_i\}$. Given $y \in X$ and a closed subset $A \subset X$ not contained y we want open neighbourhoods U_y of y and V containing A with $U_y \cap V = \emptyset$.

Case I: A is contained in a finite number of elements of $\{X_i\}$. Then for some k large enough, $y \in X_k$ and $A \subset X_k$, $y \notin A$. By the regularity of X_k, there exist U_y and V open in X_k, and therefore open in X, such that $y \in U_y$, $A \subset V$, and $U_y \cap V = \emptyset$.

Case II: A is contained in infinitely many X_i's. Again for some $j \in Z^+$, $y \in X_j$.

(i) If for this j, $A \cap X_j = \emptyset$. Then X_j being T_2 (for it is regular) there exist $x \in X_j$, $x \neq y$, and an open neighbourhood U_y of y with $x \notin \bar{U}_y$. Then let $V = X - \bar{U}_y$. V is an open subset of X containing A and $V \cap U_y \neq \emptyset$.

(ii) If $A \cap X_j \neq \emptyset$. A closed in X implies that $A \cap X_j$ is closed in X_j and therefore closed in X (Corollaries 5.2 and 5.3). Note that $y \notin A \cap X_j$. By the regularity of X_j then there exists an open neighbourhood U_y of y such that $\bar{U}_y \cap (A \cap X_j) = \emptyset$ in X_j. (Lemma 5.6)

Since $\bar{U}_y \cap X_j = \bar{U}_y$, we have $\bar{U}_y \cap A = \emptyset$. So let $V = X - \bar{U}_y$. V is open in X, contains A and has an empty intersection with U_y. Thus X is regular.

Definition 5.3: A Hausdorff space is normal if each pair of disjoint closed sets have disjoint neighbourhoods.

Lemma 5.8: Y is normal if and only if for each closed subset A and an open subset U containing A there is an open subset V such that

$$A \subset V \subset \bar{V} \subset U.$$

Theorem 5.9: Normality is an absolute property under monotone unions in $\mathcal{W}$.

Proof: Let $\{X_i | X_i \subset X_{i+1}\}$ be a sequence of normal spaces with $X = \textcircled{m}_{i=1}^{\infty} X_i$ having a topology coherent with $\{X_i\}$. Let C,D be two disjoint closed sets in X.

Case I: C,D are each contained in a finite number of elements of $\{X_i\}$. Then there exists a positive integer k large enough so that $C,D \subset X_k$. By the normality of X_k, there exist open disjoint subsets U,V in X_k such that $C \subset U$, $D \subset V$, $U \cap V = \emptyset$. But U and V are also open in X; hence the assertion.

Case II: One of the closed sets is contained in infinitely many X_i's. Without loss of generality, let D be contained in infinitely many X_i's. Thus for some $j \in Z^+$, $C \subset X_j$.

(i) If for the j, $D \cap X_j = \emptyset$, then the normality of X_j implies that for the closed set $C \subset X_j$ and open subset U' of X_j containing C, there exists an open set V with $C \subset V \subset \overline{V} \subset U'$ in X_j, by Lemma 5.8. V open in X_j implies V open in X. So here let $U = X - \overline{V}$. Then $C \subset V$, $D \subset U$, and $U \cap V = (X - V) \cap \overline{V} = \emptyset$.

(ii) If $X_j \cap D \neq \emptyset$. Since $C \cap D = \emptyset$, $C \cap X_j \cap D = \emptyset$. But D closed in X implies $D \cap X_j$ is closed in X_j. We therefore have two disjoint closed sets C and $X_j \cap D$ in X_j. By the normality of X_j there exists an open set U in X_j with $C \subset U$ and

$$\overline{U} \cap (X_j \cap D) = \emptyset; \text{ that is } \overline{U} \cap D = \emptyset$$

Hence X is normal.

Case III: C,D are each contained in infinitely many of the elements X_i. Let $\{X_j'\}$ be the infinite subsequence of $\{X_i\}$ such that either $X_j' \cap C \neq \emptyset$ or $X_j' \cap D \neq \emptyset$. Thus

$$C = \overset{\infty}{\underset{i}{\bigcup}} (C \cap X_i') \text{ and } D = \overset{\infty}{\underset{i}{\bigcup}} (D \cap X_i')$$

Each $C \cap X_j'$ and $D \cap X_k'$ is closed in X_j' and in X_k' respectively. Let X_k' be an arbitrary element in $\{X_j'\}$. By its normality, there exist U_k, V_k open in X_k such that $C \cap X_k' \subset U_k$ and $D \cap X_k' \subset V_k$, with $U_k \cap V_k = \emptyset$. Similarly for the integer $k + 1$.

Set

$$U = \bigcup_{i=1}^{\infty} U_i, \quad V = \bigcup_{i=1}^{\infty} V_i$$

where U_i, V_i are the open sets satisfying the normality conditions: $C \cap X_i' \subset U_i$, $D \cap X_i' \subset V_i$, and $U_i \cap V_i = \emptyset$ for each X_i' in $\{X_j'\}$.

<u>Claim:</u> $U \cap V = \emptyset$. For if not, let $x \in U \cap V$. This implies that for some $i, j \in Z^+$, $x \in U_i$ and $x \in V_j$, i.e., $U_i \cap V_j \neq \emptyset$. Let $i \leq j$. Then U_i, $V_j \subset X_j'$. We thus have $C \cap X_i' \subset U_i \subset X_j'$ which implies that there exist $U_j \subset X_j'$ such that $C \cap X_i' \subset C \cap X_j' \subset U_j$, with $U_j \cap V_j = \emptyset$. Therefore x cannot be in U_i and in U_j: that is $U_i \cap V_j = \emptyset$. Since x is arbitrary $U \cap V = \emptyset$, thus X is normal.

SECTION 6

OPEN MONOTONE UNIONS AND INVERTIBLE PLANE CONTINUA

In this section we define open monotone union property and then apply it to some spaces. Although this property is used in spaces other than the 2-euclidean spaces, our attention is drawn mostly to the application of the property to locally connected invertible plane continua.

<u>Definition 6.1</u>: Let A be a topological space. A has the <u>open monotone union property</u> if whenever a space $X = \underset{i=1}{\overset{\infty}{Ⓜ}} A_i$ of open sets A_i exists such that

$$\text{(i) } A_i \subset A_{i+1}$$

$$\text{(ii) } A_i \overset{T}{=} A,$$

$$\text{then } X \overset{T}{=} A.$$

If A, A_i and X are in a topological space Y, we say A has the open monotone union (omu) property in Y.

<u>Two Examples</u>

(i) Let $A = E^1$ and let $A_n = (-n,n)$, an open interval from $-n$ to n, for each $n \in Z^+$. Then $A_i \subset A_{i+1}$ and $A_i \overset{T}{=} A = E^1$. It is easy to see that $\underset{i=1}{\overset{\infty}{Ⓜ}} A_i \overset{T}{=} E^1$.

(ii) Again by modifying the space X in $2 \cdot 4$ so that it opens at the point (0,1), let this be A with monotone increasing open intervals starting from the point (0,1).

Definition 6.2: A topological space X is said to be *invertible* if, for each non-empty open set U of X, there exists a homeomorphism h of X onto itself such that $h(X-U) \subset U$. The map h is called an *inverting homeomorphism* for U.

Definition 6.3: A continuum is a compact connected set having at least two points.

Theorem 6.1: Let C be a locally connected invertible continuum in E^2 such that C contains an open set U, $\overline{U} \neq C$. If U has the open monotone union property in C, then there exists a point $x \in C$ such that $C - \{x\} \overset{T}{=} U$.

Proof: Since $\overline{U} \neq C$, let V_1 be closed in $C - \overline{U}$ such that $V_1 \overset{T}{=} C - U = V$, $C - V_1 \overset{T}{=} U$. Let V_2 be an open set of diameter 1/2 with $V_2 \overset{T}{=} V$, $\overline{V}_2 \subset V_1$. Let V_n be the n^{th} open set of diameter $1/n$ and such that $V_n \overset{T}{=} V$, $C - V \overset{T}{=} U$ as before. We thus get a sequence $\{V_i\}$ of open sets $\overline{V}_{i+1} \subset V_i$ and $\bigcap_{i=1}^{\infty} V_i = x$, say. Then

$$\underset{i=1}{\overset{\infty}{\mathrm{M}}} (C - V_i) = C - \{x\},$$

hence the theorem.

The next result is a slight generalization of Theorem 6.1. It is found in Doyle [4], but it is given here for the continuity of our presentation.

Theorem 6.2: Let C be an open connected set in E^1. If C has the open monotone union property, then $C \overset{T}{=} E^n$.

Proof: Since C is embeddable as an open set in S^n, there is a topological copy C' of C in E^n and $\overline{C}'$ is compact. Then in E^n, C' lies interior to a sphere of radius r. Assume that the origin is in C' or else there exists a transformation which can perform the shifting. Then the sphere of radius r lies interior to a copy C" of C'. By continuing this construction and making spheres at least one unit larger in radius, we get a sequence $\{C^{(n)}\}$ of copies of C and $\mathop{M}\limits_{n=1}^{\infty} C^{(n)} = E^n$. ($C' = C^{(1)}$ and $C'' = C^{(2)}$ here). So if C has the open monotone union property, it is E^n topologically.

Theorem 6.3: Let $U \subsetneqq M$ have the omu property in M, $\overline{U} \neq M$ where M is an invertible plane continuum. Then there exist $U_1, U_2 \subset M$ such that $U_1 \subset U \subset U_2$ with U_1, U_2 having the omu property in M and $U_1 \overset{T}{=} U \overset{T}{=} U_2$.

Proof: Let $\{M_j\}$ be a sequence of open sets in M with $M_j \subset M_{j+1}$, $M_j \overset{T}{=} U$ and

$$U \overset{T}{=} \mathop{M}\limits_{j=1}^{\infty} M_j \subset M.$$

Let $h_j : M_j \longrightarrow U$ be a homeomorphism of M_j onto U (the existence of such an h_j being asserted by the fact that each $M_j \overset{T}{=} U$). Then $h_j^{-1}(U) \subset M_j \subset M_{j+1} \overset{T}{=} U$. Let $i : M_j \longrightarrow M_{j+1}$ be the inclusion map and consider the following diagram.

$$\begin{tikzcd} M_j \arrow[r, "i"] \arrow[d, "h_j"] & M_{j+1} \arrow[d, "h_{j+1}"] \\ U \arrow[r, "\text{identity}"', "\text{map}"] & U \end{tikzcd}$$

Thus it is clear that for each j, the equations

$$\text{(i) } h_{j+1} \cdot i \cdot h_j^{-1}(U) \subset U$$

and

$$\text{(ii) } h_{j+1} \cdot i \cdot h_j^{-1}(U) \neq U$$

hold since it is assumed that $M_j \neq M_{j+1}$. So let $U_1 = h_{j+1} \cdot i \cdot h_j^{-1}(U)$.

<u>Claim</u>: (i) U_1 has the open monotone union property in M: Since U has the omu property, $M_j = h_j^{-1}(U)$ has the omu property. But the inclusion map equally preserves this property; hence $i \cdot h_j^{-1}(U) \subset M_{j+1}$ has the said property in M. Similarly since h_{j+1} is a homeomorphism, $h_{j+1} | [i \cdot h_j^{-1}(U)]$ is a homeomorphic image of $i \cdot h_j^{-1}(U)$ and therefore has the omu property; that is $h_{j+1} \cdot i \cdot h_j^{-1}(U) = U_1 \subset U$ has the open monotone union property.

Net let $\{M_j\}$ be the aforesaid sequence. Then for each k, $U \overset{T}{=} M_k$. So let $h_k : M \longrightarrow M$ be a homeomorphism of M onto itself such that $h_k(M_k)$ is topologically equivalent to U. Since $M_k \subset M_{k+1}$, we have $U = h_k(M_k) \subset h_K(M_{k+1})$ (proper inclusion).

So for the desired U_2, pick any k such that $M_k \overset{T}{=} U$, select a homeomorphism h_k of M onto itself with $h_k(M_k) = U$, and let $U_2 = h_k(M_{k+1})$ thus obtaining $U \subset U_2 = h_k(M_{k+1})$.

<u>Claim</u>: (ii) U_2 has the omu property: For since h_k is a homeomorphism, we can choose $h_k | M_k$ as an inclusion of M_k in M_{k+1}, hence a homeomorphism. Thus if U has the omu property, so does $h_k(M_{k+1})$.

Next since $U_1 \overset{T}{=} U$ by the homeomorphism of the composite function $h_{j+1} \cdot i \cdot h_j^{-1}$ and $U \overset{T}{=} U_2$ by a similar process as described above, and since U_1 and U_2 have the omu property as claimed, we can repeat the process by substituting U_1 for U and U_2 for U respectively to obtain the desired result.

<u>Theorem 6.4</u>: If $U \subset M$ is a nondense open set having the open monotone union property in an invertible plane continuum M, then U is connected.

<u>Proof</u>: From Theorem 6.1, there exists an x in M such that $M-\{x\} \overset{T}{=} U$. Since U has the omu property, there exists a strictly increasing sequence $\{U_i\}$ of copies of U such that

$U = \text{(M)}_{i=1}^{\infty} U_i$ so that we now have

$$U \overset{T}{=} \text{(M)}_{i=1}^{\infty} U_i \overset{T}{=} M - \{x\}.$$

From Theorem 6.3, there exist sets 0_1, $0_2 \subset M$ such that $0_1 \subset U \subset 0_2$, 0_1 and 0_2 have the omu property in M and $0_1 \overset{T}{=} U \overset{T}{=} 0_2$. By the same theorem there exists $0_1'$ such that $0_1' \subset 0_1$ with the properties just mentioned above. So suppose P_1 is a component of U. Then there exists a copy P_2 of P_1 such that $U_1 \subset P_2$. But $P_2 \subset U_3$ since $P_2 \subset U \overset{T}{=} U_2 \subset U_3$. We thus have inductively

$$U_{2i-1} \subset P_{2i} \subset U_{2i+1} \subset \cdots\cdots\cdots$$

For each i, $U_{2i-1} \cup P_{2i}$ is an open (for each of U_{j-1} and P_j is open), connected (since $U_{j-1} \subset P_j$, $j = 2k$, $k \in Z^+$) set; therefore

$$\text{(M)}_{i=1}^{\infty} (U_{2i-1} \cup P_{2i})$$

is open, connected and is a strictly increasing union of copies of U, thus U is connected.

<u>Definition 6.4</u>: A point p of X is said to be a <u>cut point</u> of X if $X-\{p\} = Y \cup Z$ where Y and Z are separated; otherwise p is a <u>non-cut point</u> of X.

<u>Corollary 6.5</u>: Let C, U, and x be as in Theorem 6.1. Then x is not a cut point of X.

<u>Proof</u>: If x were a cut point and $C - x \overset{T}{=} U = Y \cup Z$, where Y and Z are separated, then U is not connected thereby contradicting Theorem 6.4.

<u>Theorem 6.6</u>: Let S^1 be the 1-sphere and suppose $U \subset S^1$ is a connected nondense locally connected open set in S^1. Then U has the omu property in S^1.

<u>Proof</u>: Let p be any point of S^1. For some $\varepsilon > 0$ let $(p-\varepsilon, p+\varepsilon)$ be an ε-neighbourhood of p and call it V_1. Let $V_2 = (p-2\varepsilon, p+2\varepsilon)$ and so $V_1 \subset V_2$. In general, let $V_n = (p-n\varepsilon, p+n\varepsilon)$. Then $V_i \subset V_{i+1}$ and $V_i \overset{T}{=} U$. Since $U \overset{T}{=} \underset{i=1}{\overset{\infty}{Ⓜ}} V_i$, U has the desired property in S^1.

<u>Remark 1</u>: The same proof goes for any open connected non-dense locally connected set U in S^n, for $n > 1$.

<u>Remark 2</u>: The property of connectedness in 6.6 cannot be dispensed with since by 6.4 it has been shown that if U is not connected, U may fail to have the omu property in $S^1(S^n)$.

<u>Definition 6.5</u>: Let X have the open monotone union property, and suppose $\{A_i | A_i \subset A_{i+1}\}$ is the sequence for which $A_i \overset{T}{=} X$ and $\underset{i=1}{\overset{\infty}{Ⓜ}} A_i \overset{T}{=} X$. Then for each i, A_i is called the <u>monotone subspace</u> of X.

<u>Theorem 6.7</u>: Let X have the open monotone union property in an invertible plane continuum M, and let $D \subset M$ be a

compact subset of M, $D \neq M$. Then D can be imbedded in one of the monotone subspaces of X.

Proof: Let $D \subset M$ be compact, $D \neq M$, and let h be an inverting homeomorphism of M. D being compact, M-D is open. We can as well assume that the homeomorphism h is such that $h(D) \subset M-D$. It is also clear that $h(D) \neq M-D$. By Theorem 6.1, there exists a point x in M such that $M-\{x\} \overset{T}{=} X$. Since $h(D) \subset M-D \subset M-\{x\}$ for such a point x, we have $h(D) \subset M-\{x\} \overset{T}{=} X = \underset{i=1}{\overset{\infty}{Ⓜ}} X_i$ where $\{X_i | X_i \subset X_{i+1}\}$ satisfies the definition of X having the omu property. But now h(D) is compact, hence h(D) lies in a connected subset of M-D. By the monotonicity of the sequence $\{X_i\}$ we see that h(D) lies in one of the X_i's which is a monotone subspace of X; hence the theorem holds.

BIBLIOGRAPHY

1. Bing, R. H., "A connected countable Hausdorff space," Proc. Amer. Math. Soc. 4(1953) 474.

2. Brown, M., "The monotone union of open n-cells is an open n-cell," Proc. Amer. Math. Soc. 12(1961) 812-814.

3. Doyle, P. H., "On monotone unions of manifolds," Proc. Amer. Math. Soc. 19(1968) 854-856.

4. Doyle, P. H., "Topics in topological manifolds" (Notes).

5. Dugundji, J., Topology, Allyn and Bacon (1966).

6. Hocking, J. G. and Young, G. S., Topology, Addison-Wesley Pub. Co. (1961).

7. Huntington, E. V., The Continuum, and other types of serial orders, Harvard Univ. Press (1917).

8. Hurewicz, W. and Wallman, H., Dimension Theory, Princeton Univ. Press (1941).

9. Jones, F. B., and Young, G. S., "Product spaces in n-manifords," Proc. Amer. Math. Soc. 10(1959) 307-308.

10. Kapoor, S. F., "Inverting and monotone properties of complexes," Thesis, Mich. State Univ. (1967).

11. Kwun, K. W., "Open manifolds with monotone union property," Proc. Amer. Math. Soc. 17(1966) 1091-1093.

12. Whitehead, J.H.C., "A certain open manifold whose group is unity," Quart., J. of Math (Oxford), 6(1935) 268-279.