ABSTRACT

THE STRUCTURE OF THE GENERALIZED CENTER AND HYPERCENTER OF A FINITE GROUP

By

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Let $G$ be a finite group. Following Ore, a subgroup $K$ of $G$ is called quasinormal in $G$ if $KH = HK$ for all subgroups $H$ of $G$. As a generalization of quasinormality, we say that a subgroup $K$ of $G$ is $\pi$-quasinormal in $G$ if $KP = PK$ for all Sylow subgroups $P$ of $G$. Kegel introduced this concept. He proved that a maximal $\pi$-quasinormal subgroup is normal and that a $\pi$-quasinormal subgroup is subnormal. Later, Deskins studied $\pi$-quasinormality and proved that if $K$ is a $\pi$-quasinormal subgroup of $G$, then $K/K_G$ is nilpotent, where $K_G$ is the largest normal subgroup of $G$ contained in $K$.

The aim here is to study $\pi$-quasinormality further and to obtain generalizations of some of the results on normality and quasinormality by imposing the weaker requirement of $\pi$-quasinormality in place of normality and quasinormality. Throughout, $G$ denotes a finite group.

Gaschütz has studied the finite groups in which every subnormal subgroup is normal while Zacher has studied the groups whose subnormal subgroups are quasinormal. They have characterized, respectively, such finite solvable groups. Extending their results, we characterize finite solvable groups in which every subnormal subgroup is $\pi$-quasinormal. The main results we obtain in this direction

are:

(1) Let $G$ be solvable and $D(G)$ be its hypercommutator subgroup (the smallest normal subgroup of $G$ such that $G/D(G)$ is nilpotent). If all subnormal subgroups of $G$ are $\pi$-quasinormal in $G$, then $D(G)$ is a Hall subgroup of odd order and every subgroup of $D(G)$ is normal in $G$. In particular, $D(G)$ is abelian.

Note that, since $G/D(G)$ is nilpotent, every subgroup of $G/D(G)$ is $\pi$-quasinormal in $G/D(G)$.

(2) If $G$ (not necessarily solvable) has a normal Hall subgroup $N$ such that all subnormal subgroups of $N$ are normal in $G$ and all subnormal subgroups of $G/N$ are $\pi$-quasinormal in $G/N$, then the subnormal subgroups of $G$ are $\pi$-quasinormal in $G$.

This result implies that the conclusions of (1) are not only necessary but are also sufficient.

Generalizing the notion of the center of a group $G$, Ore defined the quasicenter $Q(G)$ of $G$ to be the subgroup generated by all elements $g$ of $G$ such that $\langle g\rangle$ is quasinormal in $G$. Mukherjee has studied the quasicenter in detail and proved that $Q(G)$ is nilpotent. Like the hypercenter, he also defined and investigated the hyperquasicenter $Q^*(G)$ and proved that $Q^*(G)$ is supersolvable.

In an obvious manner, we generalize the above concepts and the work of Mukherjee. We define the generalized center $Z_{Gn}(G)$ of $G$ to be the characteristic subgroup generated by all elements $g$ of $G$ such that $\langle g\rangle$ is $\pi$-quasinormal in $G$. This leads us to the definition of the generalized hypercenter. Let $(Z_{Gn}(G))_0 = 1$ and $(Z_{Gn}(G))_{i+1}/(Z_{Gn}(G))_i$ be the generalized center of $G/(Z_{Gn}(G))_i$. We get an ascending chain of characteristic subgroups:

$1 = (Z_{Gn}(G))_0 < Z_{Gn}(G) = (Z_{Gn}(G))_1 < (Z_{Gn}(G))_2 < \ldots < (Z_{Gn}(G))_m = Z^*_{Gn}(G)$. The terminal member $Z^*_{Gn}(G)$ of this chain is called the generalized hypercenter of G.

Some of the results we prove here are: (1) Every Sylow subgroup of $Z_{Gn}(G)$ is generated by the elements g of G such that $\langle g\rangle$ is $\pi$-quasinormal in G; (2) $Z_{Gn}(G)$ is nilpotent; (3) $Z^*_{Gn}(G) = \cap\{N \mid N \triangleleft G$ and $Z_{Gn}(G/N) = \bar{1}\}$; (4) $Z^*_{Gn}(G)$ is supersolvable; (5) G has the Sylow tower property of supersolvable groups if and only if $G/Z^*_{Gn}(G)$ has this property; (6) G is supersolvable if and only if $G/Z^*_{Gn}(G)$ is supersolvable; and (7) $Z^*_{Gn}(G)$ is the product of all generalized hypercentral subgroups of G, i.e., $Z^*_{Gn}(G) = \langle H \mid H \triangleleft G$ and for all $M \triangleleft G$ with $M \lneq H$, $H/M \cap Z_{Gn}(G/M) \neq \bar{1}\rangle$.

Huppert has studied the structure of a group when its i-th maximal subgroups (i = 2,3) are normal while Janko has studied the structure of a solvable group whose 4-th maximal subgroups are normal. Mann has improved their results by requiring that i-th maximal subgroups be quasinormal instead of normal. We further improve these results and close our present investigation on $\pi$-quasinormality. We prove the results of Huppert, Janko and Mann under the weaker requirement of $\pi$-quasinormality. Our results are:

(1) If every second maximal subgroup of G is $\pi$-quasinormal in G, then G is supersolvable. Furthermore, if $|G|$ is divisible by at least three different primes, then G is nilpotent.

(2) If every third maximal subgroup of G is $\pi$-quasinormal in G, then: (i) if $|G|$ is divisible by three or more different primes, then G is supersolvable; (ii) the commutator subgroup G' of G is nilpotent; and (iii) the rank of $G = r(G) \leq 2$.

(3) Let $G$ be solvable. If every fourth maximal subgroup of $G$ is $\pi$-quasinormal in $G$, then: (i) if $|G|$ is divisible by four or more different primes, then $G$ is supersolvable; (ii) $r(G) \leq 3$.

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IN MEMORY OF MY MOTHER

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## INTRODUCTION

Let G be a finite group. A subgroup K of G is called quasinormal in G if K permutes with every subgroup of G, that is, if $KH = HK$ for all subgroups H of G. This concept was introduced by Ore [16]. As a generalization of this, we say that a subgroup of G is $\pi$-quasinormal in G if it permutes with every Sylow subgroup of G. The $\pi$-quasinormal subgroups were first defined and studied by Kegel [13]. Later, they were also studied by Deskins [5].

It is obvious that a normal or quasinormal subgroup is always $\pi$-quasinormal, but the converse is not true in general. However, Kegel has shown that a $\pi$-quasinormal subgroup is necessarily a subnormal subgroup and that a maximal $\pi$-quasinormal subgroup is normal. Deskins has proved that if K is a $\pi$-quasinormal subgroup of G, then $K/K_G$ is nilpotent, where $K_G$ is the largest normal subgroup of G contained in K.

In the present thesis, we further investigate $\pi$-quasinormality and generalize a number of results on normality and quasinormality. Throughout, the groups are finite.

A group G is called a (q)-group ((t)-group) if every subnormal subgroup of G is quasinormal (normal) in G. The (q)-groups and the (t)-groups have been studied by Zacher [18] and Gaschütz [7] respectively, who have characterized such finite solvable groups.

A solvable (t)- or (q)-group is always supersolvable. Gaschütz has shown that if $G$ is a finite solvable (t)-group and $G/L$ is the maximal nilpotent factor group of $G$, then $G/L$ is Hamiltonian and $L$ is an abelian Hall subgroup of odd order whose subgroups are normal in $G$. A similar result was proved by Zacher for solvable (q)-groups.

As a generalization of (q)-groups and of (t)-groups, we say that a group $G$ is a ($\pi$-q)-group if every subnormal subgroup of $G$ is $\pi$-quasinormal in $G$. In Chapter II, we study finite ($\pi$-q)-groups and characterize such solvable groups. We extend the above results, among others, to finite solvable ($\pi$-q)-groups and give the conditions under which a ($\pi$-q)-group is either a (q)-group or a (t)-group. We also give examples of ($\pi$-q)-groups that are not (q)-groups.

Generalizing the notion of the center $Z(G)$ of a group $G$, Ore [16] defined the quasicenter $Q(G)$ of $G$ to be the subgroup generated by all elements $g$ of $G$ such that $\langle g \rangle$ is quasinormal in $G$. The structure of the quasicenter and its properties have been studied in detail by Mukherjee [15]. He proved that $Q(G)$ is nilpotent and that the Sylow subgroups of $Q(G)$ are generated by the elements $g$ such that $\langle g \rangle$ is quasinormal in $G$. Mukherjee also extended, in a natural way, the concept of the hypercenter $Z^*(G)$ of $G$. He defined and studied the hyperquasicenter $Q^*(G)$ of $G$, and proved that 1) $Q^*(G)$ is supersolvable; 2) $Q^*(G) = \cap\{N \mid N \triangleleft G \text{ and } Q(G/N) = \bar{1}\}$ and 3) $G$ is supersolvable if and only if $G/Q^*(G)$ is supersolvable.

In Chapter III, we generalize the above concepts and the work of Mukherjee. We define the generalized center $Z_{Gn}(G)$ of a group $G$ to be the subgroup generated by all elements $g$ of $G$ such that

$\langle g \rangle$ is $\pi$-quasinormal in $G$. Such elements are called the generalized central elements of $G$. It is obvious that $Z_{Gn}(G)$ is a characteristic subgroup of $G$ and that if $G$ is nilpotent, then $Z_{Gn}(G) = G$.

The definition of the generalized center leads to the definition of the generalized hypercenter. It is defined in the same way as the hypercenter and hyperquasicenter. Let $(Z_{Gn}(G))_0 = 1$ and $(Z_{Gn}(G))_{i+1}/(Z_{Gn}(G))_i$ be the generalized center of $G/(Z_{Gn}(G))_i$. This yields an ascending chain of characteristic subgroups: $1 = (Z_{Gn}(G))_0 < Z_{Gn}(G) = (Z_{Gn}(G))_1 < (Z_{Gn}(G))_2 < \cdots < (Z_{Gn}(G))_m = Z^*_{Gn}(G)$. The terminal member $Z^*_{Gn}(G)$ of this chain is called the generalized hypercenter of $G$.

Some of the results we prove here are: 1) If $\langle g \rangle$ is $\pi$-quasinormal in $G$, then $\langle g^r \rangle$ is also $\pi$-quasinormal in $G$ for all integers $r$; 2) Every Sylow subgroup of $Z_{Gn}(G)$ is generated by the generalized central elements of $G$; 3) $Z_{Gn}(G)$ is nilpotent; 4) $Z^*_{Gn}(G) = \bigcap\{N \mid N \lhd G \text{ and } Z_{Gn}(G/N) = \bar{1}\}$; 5) $G$ is supersolvable if and only if $G/Z^*_{Gn}(G)$ is supersolvable; 6) $G$ has the Sylow tower property of supersolvable groups if and only if $G/Z^*_{Gn}(G)$ has the same property; and 7) $Z^*_{Gn}(G)$ is supersolvable.

A number of mathematicians have studied the structure of the group when its i-th maximal subgroups satisfy some imbedding property. In this direction, Huppert [9], Janko [12] and Mann [14] have proved the following theorems for a finite group $G$.

(Huppert). If each second maximal subgroup of $G$ is normal in $G$, then $G$ is supersolvable. If the order of $G$ is divisible by at least three different primes, then $G$ is nilpotent.

(Huppert). Let each third maximal subgroup of G be normal in G. Then: (i) the commutator subgroup $G'$ is nilpotent; (ii) the rank of $G = r(G) \leq 2$; (iii) if $|G|$ is divisible by at least three different primes, then G is supersolvable.

(Janko). Let G be solvable. If each fourth maximal subgroup of G is normal in G, then: (i) $r(G) \leq 3$; (ii) if $|G|$ is divisible by at least four different primes, then G is supersolvable.

(Mann). Let G be solvable and each n-th maximal subgroup of G be quasinormal in G. Then: (i) $r(G) \leq n-1$; (ii) if $|G|$ is divisible by at least $n-k+1$ distinct primes, then $r(G) \leq k$, where $k \geq 1$.

We conclude our investigation by improving the above results. In Chapter IV, we prove these results under the weaker assumption of $\pi$-quasinormality instead of normality or quasinormality.

For the sake of completeness, we have collected some basic definitions and known results in Chapter I.

## CHAPTER I

## DEFINITIONS AND KNOWN RESULTS

This chapter contains, for easy reference, some basic definitions and results which are frequently used throughout the present work. Some proofs are also included. The results without references can be found in Huppert's book [8]. <u>All groups considered here are finite</u>.

### 1.1 <u>Maximal subgroups, subnormal subgroups, solvable groups and supersolvable groups</u>.

<u>Definition 1.1.1</u>: A subgroup $H$ of a group $G$ is <u>maximal</u> in $G$ if there is no proper subgroup $K$ of $G$ such that $H \lneq K$. A chain of subgroups $H_n < H_{n-1} < \ldots < H_1 < H_0 = G$ of $G$ is a maximal chain if each $H_i$ is maximal in $H_{i-1}$ for $i = 1,2,\ldots,n$. A subgroup $H$ of $G$ is an <u>n-th maximal subgroup</u> of $G$ if there is at least one maximal chain of subgroups $H_n < H_{n-1} < \ldots < H_1 < H_0 = G$ with $H_n = H$.

It should be noted that an n-th maximal subgroup of $G$ may also be an m-th maximal subgroup of $G$ for $m \neq n$. For example, consider $S_4$, the symmetric group on four letters, and if $H$ is a Sylow 2-subgroup of $S_3$ in $S_4$, then $H$ is a second as well as a third maximal subgroup of $S_4$.

<u>Theorem 1.1.2</u>: If $H$ is a maximal subgroup of a solvable group $G$, then $[G:H]$ is a power of a prime.

Definition 1.1.3: A subgroup H of a group G is subnormal in G if there is a chain of subgroups: $G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots \trianglerighteq G_n = H$ such that $G_i$ is normal in $G_{i-1}$ for $i = 1,2,\ldots,n$.

It is well-known that if H and K are two subnormal subgroups of G, then $\langle H,K\rangle$ and $H \cap K$ are again subnormal subgroups of G. It follows from the definition that if H is subnormal in G and K is a subgroup of G, then $H \cap K$ is subnormal in K and if $H \le K$, then H is subnormal in K.

Definition 1.1.4: For a subgroup H of a group G, the core of H in G, denoted by $H_G$, is the largest normal subgroup of G contained in H, and the subnormal core of H in G, denoted by $H_{SG}$, is the largest subnormal subgroup of G contained in H.

The subnormal core of a subgroup need not be the core of the subgroup. For example, let $G = A_4$, the alternating group of degree 4, and H be a subgroup of order 2. Since H is subnormal in G, $H_{SG} = H$. But H is not normal in G and so $H_G = 1 \ne H_{SG}$. However, for maximal subgroups, equality does hold.

Lemma 1.1.5: If M is a maximal subgroup of G, then $M_{SG} = M_G$.

Proof: Clearly, $M_G \le M_{SG}$. To prove $M_{SG} \le M_G$, it is enough to show that $M_{SG}$ is normal in G.

If M is normal in G, then $M_{SG} = M_G$ and hence $M_{SG} \le M_G$. On the other hand, if M is not normal in G, then we consider the subnormal chain: $M_{SG} = N_0 \ntrianglelefteq N_1 \ntrianglelefteq N_2 \ntrianglelefteq \cdots \ntrianglelefteq N_k = G$. Since $N_1$ is subnormal in G and $M_{SG} \ntrianglelefteq N_1$, it follows that $N_1 \not\le M$. Hence $G = \langle N_1, M\rangle$. Note that $N_1$ normalizes $M_{SG}$. We will show that M also normalizes $M_{SG}$. Let $x \in M$. Then $x^{-1}M_{SG}x$ is subnormal in

$G$. Since $x^{-1}M_{SG}x \leq x^{-1}Mx = M$, it follows that $x^{-1}M_{SG}x \leq M_{SG}$. But $|x^{1}M_{SG}x| = |M_{SG}|$ and so $x^{-1}M_{SG}x = M_{SG}$. Hence $M$ normalizes $M_{SG}$. This means that $M_{SG}$ is normal in $G$. Therefore, $M_{SG} \leq M_G$.

We now turn our attention to the results on the Sylow subgroups and Hall subgroups. We begin with a lemma known as the Frattini lemma.

Lemma 1.1.6: Let $H$ be a normal subgroup of $G$. If $P$ is a Sylow subgroup of $H$, then $G = HN_G(P)$, where $N_G(P)$ is the normalizer of $P$ in $G$.

Using the fact that the Sylow subgroups, for a given prime, of a group $G$ are conjugate in $G$, we prove the following:

Proposition 1.1.7: A subnormal Sylow subgroup of $G$ is characteristic in $G$.

Proof: It suffices to show that a subnormal Sylow subgroup $P$ of $G$ is normal in $G$. For this, we may assume without loss of generality that $P \ntrianglelefteq K \ntrianglelefteq G$. Clearly, $P$ is a Sylow subgroup of $K$. Let $g \in G$. Then $g^{-1}Kg = K$ and so $g^{-1}Pg$ is a Sylow subgroup of $K$. But $P$ is normal in $K$. Hence $g^{-1}Pg = P$, which implies that $P$ is normal in $G$.

The next result, due to P. Hall, is a generalization of the Sylow theorems that holds in finite solvable groups.

Theorem 1.1.8: Let $G$ be a solvable group of order $ab$, where $(a,b) = 1$. Then $G$ contains at least one subgroup of order $a$, and any two such subgroups are conjugate.

Definition 1.1.9: Let $|G| = p^n a$, where $(p^n,a) = 1$. A subgroup of $G$ of order $a$ is called a p-complement of $G$.

Hall's theorem tells us that every finite solvable group has a p-complement for every prime $p$.

Definition 1.1.10: A subgroup $H$ of a group $G$ is called a Hall subgroup of $G$ if $(|H|, [G:H]) = 1$.

The following result which holds for Hall subgroups of a solvable group is an extension of Proposition 1.1.7.

Proposition 1.1.11: A subnormal Hall subgroup of a solvable group is characteristic.

Definition 1.1.12: A group $G$ is called supersolvable if every homomorphic image of $G$ contains a cyclic normal subgroup.

The subgroups and the factor groups of supersolvable groups are again supersolvable. An extension of a supersolvable group by a supersolvable group is not necessarily supersolvable. However, an extension of a cyclic group by a supersolvable group is supersolvable.

A remarkable characterization of supersolvable groups was obtained by Huppert [9].

Theorem 1.1.13: A group $G$ is supersolvable if and only if every maximal subgroup of $G$ is of prime index.

Definition 1.1.14: The commutator subgroup $G'$ of a group $G$ is the subgroup generated by all commutators $x^{-1}y^{-1}xy$ for all $x, y$ in $G$.

Theorem 1.1.15: The commutator subgroup of a supersolvable group is nilpotent.

Let $H$ and $K$ be two subgroups of $G$. Then $[H,K]$ denotes the subgroup $\langle h^{-1}k^{-1}hk \mid h \in H, k \in K \rangle$. Let $\gamma_1(G) = G$ and

$\gamma_{i+1}(G) = [\gamma_i(G), G]$. This yields a chain of normal subgroups of G.

Definition 1.1.16: The lower central series of G is the normal series

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \dots$$

The terminal member of this series is called the hypercommutator subgroup D(G) of G.

The above definition of the hypercommutator is equivalent to the following:

Definition 1.1.17: The hypercommutator subgroup D(G) of a group G is the intersection of all normal subgroups N of G such that G/N is nilpotent. That is, D(G) is the smallest normal subgroup of G such that G/D(G) is nilpotent.

## 1.2 Generalized Normality

Definition 1.2.1: Two subgroups H and K of a group G permute if $\langle H,K\rangle = HK = KH$. A subgroup H of G is quasinormal in G if H permutes with every subgroup of G.

The concept of a quasinormal subgroup is a generalization of the notion of a normal subgroup, which was introduced by Ore [16] and later studied by Iwasawa [11], Itô and Szép [10] and Deskins [5].

Clearly, a normal subgroup is always a quasinormal subgroup. But the converse is not, in general, true as shown below by an example. However, Ore proved that a quasinormal subgroup is necessarily a subnormal subgroup.

Example: Let $G = \langle x,y \mid x^{p^2} = y^p = 1,\ y^{-1}xy = x^{p+1}\rangle$, where p is an odd prime. Then $\langle y\rangle$ is a quasinormal subgroup of G which is not normal in G.

As a generalization of quasinormality, we have

Definition 1.2.2: A subgroup of a group $G$ is called $\pi$-quasinormal in $G$ if it permutes with every Sylow subgroup of $G$.

The above concept was first introduced and studied by Kegel [13]. It was later studied by Deskins [5]. Note that a quasinormal subgroup is always a $\pi$-quasinormal subgroup, but the converse is not necessarily true.

Let us point out here that every subgroup of a nilpotent group is $\pi$-quasinormal in the group. This is so because all the Sylow subgroups of a nilpotent group are normal in the group. It is easy to see that there are nilpotent groups in which every subgroup is not quasinormal. This confirms that a $\pi$-quasinormal subgroup is not always a quasinormal subgroup.

The following results describe some of the basic properties of $\pi$-quasinormal subgroups.

Lemma 1.2.3: If $H$ is $\pi$-quasinormal in $G$ and $\theta$ is a homomorphism of $G$, then $H^\theta$ is $\pi$-quasinormal in $G^\theta$. In particular, $H/N$ is $\pi$-quasinormal in $G/N$ for all normal subgroups $N$ of $G$ contained in $H$.

Proof: The lemma follows from the fact that the Sylow subgroups of $G^\theta$ are the images of the Sylow subgroups of $G$ under $\theta$ and the fact that the property of permutability of subgroups is preserved by $\theta$.

Lemma 1.2.4: Let $N \leq H \leq G$ and $N$ be normal in $G$. If $H/N$ is $\pi$-quasinormal in $G/N$, then $H$ is $\pi$-quasinormal in $G$.

Proof: Let $P$ be a Sylow subgroup of $G$. Since $PN/N$ is a Sylow subgroup of $G/N$ and $H/N$ is $\pi$-quasinormal in $G/N$, we have $(H/N)(PN/N) = (PN/N)(H/N)$. Therefore, for $h$ in $H$ and $k$ in $P$, there exist some $h_0$ in $H$ and $k_0$ in $P$ such that $(hN)(kN) = (k_0N)(h_0N)$. Hence $hkN = k_0h_0N$, which implies that $h_0^{-1}k_0^{-1}hk$ belongs to $N$. This means that $h_0^{-1}k_0^{-1}hk$ belongs to $H$ since $N$ is contained in $H$. Therefore, there is some $h_1$ in $H$ such that $h_0^{-1}k_0^{-1}hk = h_1$ and so $hk = k_0(h_0h_1)$. From this, it follows that $HP = PH$. This proves that $H$ is $\pi$-quasinormal in $G$.

The results in the next theorem are due to Kegel [13].

Theorem 1.2.5: For a group $G$, we have:

(i) If $H \leq K \leq G$ and $H$ is $\pi$-quasinormal in $G$, then $H$ is also $\pi$-quasinormal in $K$.

(ii) If $H$ is a maximal $\pi$-quasinormal subgroup of $G$, then $H$ is normal in $G$.

(iii) If $H$ is a $\pi$-quasinormal subgroup of $G$, then $H$ is subnormal in $G$.

Theorem 1.2.6 (Deskins [5]). If $K$ is a $\pi$-quasinormal subgroup of $G$ and $K_G$ is the core of $K$ in $G$, then $K/K_G$ is nilpotent.

## CHAPTER II

## FINITE GROUPS WHOSE SUBNORMAL SUBGROUPS ARE $\pi$-QUASINORMAL

We recall from Chapter I the fact that a $\pi$-quasinormal subgroup of a group is always subnormal in the group. Since a subnormal subgroup of a group is not necessarily $\pi$-quasinormal in the group, it seems natural to ask: "What can be said about the structure of a group if all of its subnormal subgroups are $\pi$-quasinormal in the group?" We call such a group a $(\pi$-q)-group and, in this chapter, we study finite $(\pi$-q)-groups. Especially we study and characterize, in Section 2, finite solvable $(\pi$-q)-groups. Theorems 2.2.3, 2.2.4, and 2.2.5 establish the characterization.

A group G is called a (q)-group ((t)-group) if every subnormal subgroup of G is quasinormal (normal) in G. The (q)-groups and the (t)-groups have been studied by Zacher [18] and Gaschütz [7] respectively, and they have characterized such finite solvable groups.

A solvable (t)- or (q)-group is always supersolvable. The main result of Gaschütz states that if G is a finite solvable (t)-group and if D(G) is its hypercommutator subgroup, then G/D(G) is Hamiltonian (a group in which every subgroup is normal) and D(G) is an abelian Hall subgroup of odd order whose subgroups are normal in G. A similar result was proved by Zacher for the solvable (q)-groups.

We extend the above results, among others, to finite solvable ($\pi$-q)-groups and give the conditions under which a ($\pi$-q)-group is either a (q)-group or a (t)-group. We also give examples of ($\pi$-q)-groups that are not (q)-groups.

## 2.1 Preliminary Results and Examples

Definition 2.1.1: A group is a ($\pi$-q)-group if all of its subnormal subgroups are $\pi$-quasinormal in the group.

Remark: In view of Theorem 1.2.5(i), the above definition is equivalent to the following definition:

A ($\pi$-q)-group is a group in which $\pi$-quasinormality is a transitive relation.

The next two inheritance properties of ($\pi$-q)-groups are immediate consequences of Theorem 1.2.5(i) and Lemma 1.2.3.

(2.1.2) A subnormal subgroup of a ($\pi$-q)-group is again a ($\pi$-q)-group.

Note that a non-subnormal subgroup of a ($\pi$-q)-group is not necessarily a ($\pi$-q)-group, as confirmed by the following example: Let $G = A_5$, the alternating group of degree 5, and let $H$ denote $A_4$ in $A_5$. Since $A_5$ is simple, it follows that $A_5$ is a ($\pi$-q)-group. But $H$ is not a ($\pi$-q)-group because its subnormal subgroups of order 2 do not permute with the Sylow 3-subgroups of $H$.

We will show later that if $G$ is a solvable ($\pi$-q)-group, then all subgroups of $G$ are ($\pi$-q)-groups.

(2.1.3) The factor groups of a ($\pi$-q)-group are again ($\pi$-q)-groups.

Proof: Let $G$ be a $(\pi\text{-}q)$-group and $N$ be a normal subgroup of $G$. Suppose $H/N$ is a subnormal subgroup of $G/N$. Then $H$ is subnormal in $G$ and so $H$ is $\pi$-quasinormal in $G$. Hence by Lemma 1.2.3, $H/N$ is $\pi$-quasinormal in $G/N$, which implies that $G/N$ is a $(\pi\text{-}q)$-group.

Proposition 2.1.4: If $G_1$ and $G_2$ are two $(\pi\text{-}q)$-groups and $(|G_1|, |G_2|) = 1$, then $G = G_1 \times G_2$ is a $(\pi\text{-}q)$-group.

Proof: Let $H$ be a subnormal subgroup of $G = G_1 \times G_2$ and $P$ be a Sylow $p$-subgroup of $G$. To prove that $G$ is a $(\pi\text{-}q)$-group, we must show that $H$ and $P$ permute. Since $H$ is subnormal in $G$, it follows that for every Sylow subgroup $S$ of $G$, $H \cap S$ is a Sylow subgroup of $H$. This, together with $(|G_1|, |G_2|) = 1$, implies that $H = (H \cap G_1) \times (H \cap G_2)$. Clearly, we may assume without loss of generality that $P \leq G_1$. Since $H \cap G_1$ is subnormal in $G_1$ and $G_1$ is a $(\pi\text{-}q)$-group, $H \cap G_1$ is $\pi$-quasinormal in $G_1$. Hence $H \cap G_1$ permutes with $P$. Moreover, $H \cap G_2$ centralizes (a fortiori, permutes with) $P$ and, therefore, $(H \cap G_1) \times (H \cap G_2) = H$ permutes with $P$. This proves the proposition.

Remark: In the above proposition, the condition that $(|G_1|, |G_2|) = 1$ cannot be omitted. The following example shows this.

Let $G_1 = S_3 = \langle x,y \mid x^3 = y^2 = 1,\ yx = x^2y \rangle$ and $G_2 = \langle z \mid z^3 = 1 \rangle$. Then $\langle xz \rangle$ is subnormal in $G_1 \times G_2$. But $\langle xz \rangle$ is not $\pi$-quasinormal in $G_1 \times G_2$ since it does not permute with the Sylow 2-subgroup $\langle y \rangle$ of $G_1 \times G_2$.

As noted in Chapter I, every subgroup of a nilpotent group is $\pi$-quasinormal but not necessarily quasinormal. Since all subgroups of a nilpotent group are subnormal, it follows that a nilpotent group is always a $(\pi\text{-}q)$-group but not always a (q)-group. The following examples show that there also exist non-nilpotent groups which are $(\pi\text{-}q)$-groups but not (q)-groups.

Example 1. Let $p$ be a prime greater than 3. Denote by $G_1$ the direct product of the groups $\langle a,b \mid a^{p^2} = b^p = 1,\ b^{-1}ab = a^{1+p}\rangle$ and $\langle c \mid c^{p^2} = 1\rangle$. Let $G = G_1 \times S_3$, where $S_3$ is the symmetric group on 3 letters. Then $G$ is a $(\pi\text{-}q)$-group by Proposition 2.1.4. But $G$ is not a (q)-group since its subnormal subgroup $\langle b\rangle$ does not permute with $\langle ac\rangle$. Note that $G$ is solvable but not nilpotent.

Example 2. Let $H$ be a non-abelian simple group and $K$ be a nilpotent group that is not a (q)-group. Let $(|H|, |K|) = 1$. Such $K$ certainly exists (for instance, $G_1$ of last example when $p > |H|$). Now $H \times K$ is a non-solvable $(\pi\text{-}q)$-group which is not a (q)-group.

Remark: Note that a (q)-group is always a $(\pi\text{-}q)$-group. Hence it follows that the class of $(\pi\text{-}q)$-groups is larger than the class of (q)-groups defined by Zacher in [18].

## 2.2 Solvable $(\pi\ q)$-groups

Here we characterize the solvable $(\pi\text{-}q)$-groups. We begin with the following observation.

Lemma 2.2.1: Let $G$ be a $(\pi\text{-}q)$-group. If $N$ is a solvable minimal normal subgroup of $G$, then the order of $N$ is a prime.

Proof: Since a characteristic subgroup of a normal subgroup is normal in the group and since $N$ is a minimal normal subgroup, it follows that $N$ does not contain any proper characteristic subgroups. By the solvability of $N$, the commutator subgroup $N'$ of $N$ is a proper characteristic subgroup of $N$. Hence $N' = 1$ and so $N$ is abelian. Now, if $p$ is a prime divisor of $|N|$, then the Sylow p-subgroup $N_p$ of $N$ is characteristic in $N$. Since $N_p \neq 1$, it follows that $N = N_p$. Hence $N$ is a p-subgroup. This means that every subgroup of $N$ (being subnormal in $G$) is $\pi$-quasinormal in $G$.

Let $P$ be any Sylow p-subgroup of $G$. Then $N$ is a normal subgroup of $P$ and $N \cap Z(P) \neq 1$, where $Z(P)$ is the center of $P$. Let $g$ be a non-identity element of $N \cap Z(P)$. Since $\langle g \rangle$ is $\pi$-quasinormal in $G$, it follows that $\langle g \rangle$ is subnormal in $\langle g \rangle Q = Q\langle g \rangle$ for all Sylow q-subgroups $Q$ of $G$ for primes $q \neq p$. From this and the fact that $\langle g \rangle$ is a Sylow p-subgroup of $\langle g \rangle Q$, we obtain that $\langle g \rangle$ is normal in $\langle g \rangle Q$ for all $Q$, $q \neq p$. But $\langle g \rangle$ is normal in $P$ and hence $\langle g \rangle$ is normal in $G$. Since $N$ is a minimal normal subgroup of $G$, we have $N = \langle g \rangle$, which implies that $|N| = p$, as desired.

As mentioned earlier, Zacher [18] has shown that a solvable (q)-group is supersolvable. Using the above lemma, we prove the same for a solvable ($\pi$-q)-group.

<u>Theorem 2.2.2</u>: A solvable ($\pi$-q)-group $G$ is supersolvable.

<u>Proof</u>: We use induction on the order of $G$. Let $N$ be a minimal normal subgroup of $G$. Then $N$ is solvable and hence $|N| = p$, a prime. By (2.1.3), $G/N$ is a solvable ($\pi$-q)-group. Therefore, $G/N$ is supersolvable by induction. Since $N$ is cyclic, it follows that $G$ itself is supersolvable.

Our next theorem generalizes the main result, stated in the introduction, of Gaschütz [7] and the corresponding result of Zacher [18].

Theorem 2.2.3: Let $G$ be a solvable $(\pi\text{-}q)$-group and $D(G)$ be the hypercommutator subgroup of $G$. Then

(i) $D(G)$ is a Hall subgroup of $G$ of odd order, and

(ii) every subgroup of $D(G)$ is normal in $G$.

In particular, $D(G)$ is an abelian subgroup of $G$.

Remark: Note that if $G$ is a (t)-group, then $G/D(G)$ is a nilpotent (t)-group and hence every subgroup of $G/D(G)$ is normal in $G/D(G)$. In our case, the subgroups of $G/D(G)$ are $\pi$-quasinormal in $G/D(G)$ and that is what we should expect since we have replaced normality by $\pi$-quasinormality. Also note that every complement of $D(G)$ in $G$ is nilpotent.

Proof of Theorem 2.2.3: We proceed by induction on the order of $G$. Let $p$ be the largest prime divisor of $|G|$ and $P$ be a Sylow p-subgroup of $G$. Since $G$ is supersolvable by Theorem 2.2.2, $P$ is normal in $G$. Also $D(G)$ is contained in the q-complement of $G$ for the smallest prime divisor $q$ of $|G|$ because $G$ is q-nilpotent. Hence the order of $D(G)$ is odd. We now have two cases according to whether or not $p$ divides the order of $D(G)$.

Case 1. $p$ does not divide $|D(G)|$. Then $P \cap D(G) = 1$ and so $P$ centralizes $D(G)$. By induction, $D(G/P) = D(G)P/P$ is a Hall subgroup of $G/P$ and every subgroup of $D(G)P/P$ is normal in $G/P$. This means that $D(G)$, since $(|P|, |D(G)|) = 1$, is a Hall subgroup of $G$ and, for $H \leq D(G)$, $HP$ is normal in $G$. Since $H$ is centralized by $P$, $H$ is a normal Hall (hence characteristic) subgroup of $HP$, which implies that $H$ is normal in $G$.

Case 2. p divides $|D(G)|$. We will show that $P \leq D(G)$. Let $a$ be an element of $P$. Since $\langle a\rangle$ is subnormal in $G$, $\langle a\rangle$ is $\pi$-quasinormal in $G$. Let $K$ be a p-complement of $G$. Then $\langle a\rangle K = K\langle a\rangle$ is a subgroup. Since $\langle a\rangle$ is a subnormal Sylow p-subgroup of $\langle a\rangle K$, $\langle a\rangle$ is normal in $\langle a\rangle K$. But $a$ is an arbitrary element of $P$ and so every subgroup of $P$ is normalized by $K$. Hence every element of $K$ induces a power automorphism in the elementary abelian group $P/\Phi(P)$ of order prime to $p$, where $\Phi(P)$ is the Frattini subgroup of $P$. Thus, for every $k$ in $K$, there exists a positive integer $m(k)$ such that $a^k \equiv a^{m(k)} \bmod \Phi(P)$ for all $a$ in $P$. Let $\gamma_n(G)$ be the terminal member of the lower central series of $G$. Then $\gamma_n(G) = D(G)$. Since $p$ divides $|D(G)|$, $K$ does not centralize $P$. For, otherwise $K$ would be normal in $G$ and so $G/K$ would be nilpotent, which would imply that $D(G) \leq K$, an impossibility. Hence there is some $y$ in $K$ which does not centralize $P$. Now it follows from Theorem 11.7 in [17] that $m(y) \not\equiv 1 \bmod p$ and so, for all $x$ in $P - \Phi(P)$, the commutator $((\ldots((x,y_1),y_2),\ldots),y_n) = x^{(m(y)-1)^n} \not\equiv 1 \bmod \Phi(P)$, where $y_1 = y_2 = \ldots = y_n = y$. This says that if $A/\Phi(P) \leq P/\Phi(P)$ and $|A/\Phi(P)| = p$, then $\gamma_n(G)$ contains an element $g$ such that $g \in P - \Phi(P)$ and $g\Phi(P)$ generates $A/\Phi(P)$. The Burnside Basis Theorem yields that $P \leq \gamma_n(P) = D(G)$. By induction, $D(G/P) = D(G)P/P = D(G)/P$ is a Hall subgroup of $G/P$, which implies that $D(G)$ is a Hall subgroup of $G$.

Next we prove that every subgroup of $D(G)$ is normal in $G$. Clearly, we need only show that every cyclic subgroup of $D(G)$ is normal in $G$. Let $\langle c\rangle$ be any cyclic subgroup of $D(G)$. Then

$c = uv = vu$ for some p-element $u$ and p'-element $v$ and $\langle c\rangle = \langle u\rangle \times \langle v\rangle$. Since $G$ is supersolvable, its commutator subgroup $G'$ is nilpotent. But $D(G) \leq G'$ and so $D(G)$ is nilpotent. Hence all subgroups of $D(G)$, in particular $\langle u\rangle$ and $\langle v\rangle$, are $\pi$-quasinormal in $G$. It now follows that $\langle u\rangle$ is a subnormal Sylow p-subgroup of $\langle u\rangle Q = Q\langle u\rangle$ for all Sylow q-subgroups $Q$ of $G$ and primes $q \neq p$. Therefore, $\langle u\rangle$ is normal in $\langle u\rangle Q$ for all $Q$ and primes $q \neq p$ and so $\langle u\rangle$ is normal in $G[p]$, where $G[p]$ is the normal subgroup of $G$ generated by all p'-elements of $G$. Since $G/G[p]$ is nilpotent, $D(G) \leq G[p]$. This, together with $P \leq D(G)$, implies that $G[p] = G$ and so $\langle u\rangle$ is normal in $G$. Since $\langle v\rangle P/P$ is a subgroup of $D(G)/P = D(G/P)$, we have by induction that $\langle v\rangle P/P$ is normal in $G/P$. Thus $\langle v\rangle P$ is normal in $G$. But $\langle v\rangle P$ is nilpotent. Hence $\langle v\rangle$ is characteristic in $\langle v\rangle P$ and so $\langle v\rangle$ is normal in $G$. Therefore, $\langle c\rangle$ is normal in $G$ and this takes care of case 2.

Finally, note that $D(G)$ is Hamiltonian of odd order. It is well-known that such a group is always abelian. Hence $D(G)$ is abelian and the proof of the theorem is complete.

The next theorem gives sufficient conditions for a group $G$ to be a $(\pi\text{-}q)$-group. Here we do not require that $G$ be solvable.

<u>Theorem 2.2.4</u>: Let the group $G$ have a normal Hall subgroup $N$ such that

(i) $G/N$ is a $(\pi\text{-}q)$-group, and

(ii) every subnormal subgroup of $N$ is normal in $G$.

Then $G$ is a $(\pi\text{-}q)$-group.

Proof: Let $H$ be a subnormal subgroup of $G$. Then we must show that $H$ is $\pi$-quasinormal in $G$. Let $N \cap H \neq 1$. Since $N \cap H$ is subnormal in $N$, $N \cap H$ is normal in $G$ by (ii). Now consider the factor group $G/N \cap H$. By induction, $H/N \cap H$ is $\pi$-quasinormal in $G/N \cap H$. It follows from Lemma 1.2.4 that $H$ is $\pi$-quasinormal in $G$.

Next suppose that $N \cap H = 1$. By the improved version of Schur and Zassenhaus theorem (after the well-known theorem of Feit and Thompson), $G$ splits over $N$ and all complements of $N$ in $G$ are conjugate. Let $M$ be any complement of $N$ in $G$. Then $M$, being isomorphic to $G/N$, is a $(\pi\text{-}q)$-group. Since $H$ is subnormal in $G$ and $(|N|, |M|) = 1$, it follows, as in Proposition 2.1.4, that $H = (H \cap M)(H \cap N)$. But $H \cap N = 1$ and so $H = H \cap M$. This means that $H \leq M$. Hence every complement of $N$ is a $(\pi\text{-}q)$-group and contains $H$. Note that $(|H|, |N|) = 1$.

Now consider the subgroup $HN$. Then $H$ is a subnormal Hall subgroup of $HN$. We may assume without loss of generality that $H \lhd K \lhd HN$. Since $H$ is a normal Hall subgroup of $K$, $H$ is the unique subgroup of $K$ of order $|H|$, which implies that $H$ is normal in $HN$. This means that $H$ permutes with every Sylow subgroup of $N$. Let $p$ be a prime divisor of the order of $G$ and $G_p$ be a Sylow $p$-subgroup of $G$. If $p$ divides the order of $N$, then $G_p \leq N$ and, therefore, $HG_p = G_pH$. On the other hand, if $p$ does not divide the order of $N$, then there is a complement $L$ of $N$ in $G$ such that $G_p \leq L$. Since $H$ is a subnormal subgroup of $L$ and $L$ is a $(\pi\text{-}q)$-group, the subgroups $H$ and $G_p$ must permute. Hence $H$ is $\pi$-quasinormal in $G$. This proves the theorem.

From this we obtain the following result and see that the conditions (i) and (ii) of Theorem 2.2.3 are not only necessary but are also sufficient.

Theorem 2.2.5: Let $G$ have a normal Hall subgroup $N$ such that

(i) $G/N$ is a solvable $(\pi\text{-}q)$-group, and

(ii) $N$ is solvable and all of its subnormal subgroups are normal in $G$.

Then $G$ is a solvable $(\pi\text{-}q)$-group.

Proof: Since $G/N$ and $N$ are both solvable, it follows that $G$ is solvable. The rest is obvious from Theorem 2.2.4.

Remark: The condition (i) of the above theorem is automatically satisfied if the factor group $G/N$ is nilpotent.

Corollary 2.2.6: Let $G$ be a solvable $(\pi\text{-}q)$-group. Then all subgroups of $G$ are again solvable $(\pi\text{-}q)$-groups.

Proof: Let $K$ be any subgroup of $G$ and consider $K \cap D(G)$. It follows from Theorem 2.2.3 that $K \cap D(G)$ is a normal Hall subgroup of $K$ and its subnormal subgroups are normal in $K$. But $D(G)K/D(G) \cong K/K \cap D(G)$ and hence $K/K \cap D(G)$ is nilpotent. Now $K$ is a solvable $(\pi\text{-}q)$-group by Theorem 2.2.5.

## 2.3 $(\pi\text{-}q)$-groups With Special Sylow Subgroups

In this section, we obtain conditions for a $(\pi\text{-}q)$-group, not necessarily solvable, to be a $(q)$-group or a $(t)$-group. We need the following definition.

Definition 2.3.1: A group $G$ is called quasi-Hamiltonian if all of its subgroups are quasinormal in $G$.

Iwasawa [11] has shown the existence of quasi-Hamiltonian p-groups that are not Hamiltonian. This suggests the next theorem.

Theorem 2.3.2: Let $G$ be a ($\pi$-q)-group. If all of its Sylow subgroups are quasi-Hamiltonian, then $G$ is a (q)-group.

Proof: Let $K$ be a subnormal subgroup of $G$. Then we must show that $K$ permutes with every subgroup of $G$. Since the factor groups of $G$ satisfy the conditions of the theorem, it is sufficient to consider the case when $K_G$, the core of $K$ in $G$, is $\langle 1\rangle$. Since $K$ is $\pi$-quasinormal in $G$, it follows from Theorem 1.2.6 that $K$ is nilpotent. Hence every subgroup of $K$, which is subnormal in $G$, is $\pi$-quasinormal in $G$. Let $p$ be a prime divisor of the order of $K$. Then, since the Sylow p-subgroup $K_p$ of $K$ is $\pi$-quasinormal in $G$, $K_p$ is a subnormal and hence normal Sylow p-subgroup of the subgroup $K_pG_q = G_qK_p$ for all Sylow q-subgroups $G_q$ and primes $q \neq p$. Therefore, $K_p$ is normalized by every p'-element of $G$. Since $K_p$ permutes with every Sylow p-subgroup of $G$, it follows that $K_p$ is contained in every Sylow p-subgroup of $G$. But the Sylow subgroups of $G$ are quasi-Hamiltonian. Hence $K_p$ permutes with every p-subgroup of $G$. Now let $g$ be any element of $G$. Then $\langle g\rangle = \langle u\rangle \times \langle v\rangle$ for some p-element $u$ and p'-element $v$, and so $\langle g\rangle$ and $K_p$ permute. Since $K$ is the direct product of its Sylow subgroups, it follows that $K$ and $\langle g\rangle$ permute, which implies that $K$ permutes with every subgroup of $G$. Hence $K$ is quasinormal in $G$. This completes the proof.

Theorem 2.3.3: Let $G$ be a ($\pi$-q)-group. If all of its Sylow subgroups are Hamiltonian, then $G$ is a (t)-group.

Proof: Let $K$ be a subnormal subgroup of $G$. Duplicating the above argument, we see that $K_p$ is normalized by every p'-element of $G$ and, since the Sylow subgroups of $G$ are Hamiltonian, $K_p$ is a normal subgroup of every Sylow p-subgroup of $G$. Hence $K_p$ is normal in $G$, which implies that $K$ is normal in $G$.

## CHAPTER III

## GENERALIZED CENTER AND HYPERCENTER OF A FINITE GROUP

Generalizing the notion of the center $Z(G)$ of a group $G$, Ore [16] defined the quasicenter $Q(G)$ of $G$ to be the subgroup generated by all elements $g$ of $G$ for which $\langle g \rangle$ is quasinormal in $G$. The structure of the quasicenter and its properties have been studied in detail by Mukherjee [15]. He proved that $Q(G)$ is nilpotent and that the Sylow subgroups of $Q(G)$ are generated by the elements $g$ such that $\langle g \rangle$ is quasinormal in $G$. Mukherjee [15] also defined and studied the hyperquasicenter $Q^*(G)$ of $G$ which is a generalization of the concept of the hypercenter $Z^*(G)$ of $G$. Some of the results he proved are: (1) $Q^*(G)$ is supersolvable; (2) $Q^*(G) = \cap\{N \mid N \lhd G$ and $Q(G/N) = \bar{1}\}$; (3) If $T \lhd G$ and $T \leq Q^*(G)$, then $Q^*(G/T) = Q^*(G)/T$; and (4) $G$ is supersolvable if and only if $G/Q^*(G)$ is supersolvable.

In this chapter, we generalize the above concepts. We define and investigate another center and hypercenter of a finite group, and extend Mukherjee's results.

We define the generalized center $Z_{Gn}(G)$ of a group $G$ to be the subgroup generated by all elements $g$ of $G$ for which $\langle g \rangle$ is $\pi$-quasinormal in $G$. In a natural way, the definition of the generalized center leads to the definition of the generalized hypercenter $Z^*_{Gn}(G)$. Let $(Z_{Gn}(G))_0 = 1$ and $(Z_{Gn}(G))_{i+1}/(Z_{Gn}(G))_i$ be the generalized center of $G/(Z_{Gn}(G))_i$. From this we get an ascending chain of characteristic subgroups: $1 = (Z_{Gn}(G))_0 < Z_{Gn}(G) = (Z_{Gn}(G))_1 < (Z_{Gn}(G))_2 < \cdots < (Z_{Gn}(G))_m = Z^*_{Gn}(G)$. The terminal member $Z^*_{Gn}(G)$ of this chain is called the generalized hypercenter of $G$.

The main results proved here are: (1) If $\langle g\rangle$ is $\pi$-quasinormal in $G$, then every subgroup of $\langle g\rangle$ is again $\pi$-quasinormal in $G$; (2) Every Sylow subgroup of $Z_{Gn}(G)$ is generated by the elements $g$ of $G$ such that $\langle g\rangle$ is $\pi$-quasinormal in $G$; (3) $Z_{Gn}(G)$ is nilpotent; (4) If $G = H \times K$, then $Z_{Gn}(G) = Z_{Gn}(H) \times Z_{Gn}(K)$ and $Z^*_{Gn}(G) = Z^*_{Gn}(H) \times Z^*_{Gn}(K)$; (5) $Z^*_{Gn}(G) = \cap\{N \mid N \lhd G$ and $Z_{Gn}(G/N) = \bar{1}\}$; (6) If $T \lhd G$ and $T \leq Z^*_{Gn}(G)$, then $Z^*_{Gn}(G/T) = Z^*_{Gn}(G)/T$; (7) $Z^*_{Gn}(G)$ is supersolvable; (8) $G$ is supersolvable if and only if $G/Z^*_{Gn}(G)$ is supersolvable; (9) $G$ has the Sylow tower property of supersolvable groups if and only if $G/Z^*_{Gn}(G)$ has the same property; (10) $Z^*_{Gn}(G)$ is contained in the intersection of the maximal supersolvable subgroups of $G$; and (11) $Z^*_{Gn}(G)$ is the product of all generalized hypercentral subgroups of $G$ (these subgroups are defined in section 3.3).

## 3.1 Generalized Center

The aim here is to generalize the notions of the center and of the quasicenter of a group $G$ to that of the generalized center of $G$ and to obtain some generalizations of Mukherjee's results. We begin with the following simple observation.

Lemma 3.1.1: Let $p$ be a prime divisor of the order of $G$ and $H$ be a $p$-subgroup of $G$. If $H$ is $\pi$-quasinormal in $G$, then

$H$ is normal in $HG_q$ for every Sylow q-subgroup $G_q$ of $G$ and primes $q \neq p$.

Proof: By Theorem 1.2.5(iii), $H$ is subnormal in $G$. Since $H$ permutes with $G_q$, $HG_q$ is a subgroup of $G$. Hence $H$ is subnormal in $HG_q$. But $H$ is a p-subgroup and so $H$ is a subnormal Sylow p-subgroup of $HG_q$. By Proposition 1.1.7, $H$ is normal in $HG_q$.

Definition 3.1.2: The generalized center $Z_{Gn}(G)$ of a group $G$ is the subgroup generated by all elements $g$ of $G$ such that $\langle g \rangle$ is $\pi$-quasinormal in $G$. Such elements shall be called the generalized central elements of $G$.

Clearly, $Z_{Gn}(G)$ is a characteristic subgroup of $G$ which contains the quasicenter $Q(G)$ and the center $Z(G)$ of $G$. We shall show later, by way of an example, that $Q(G)$ can be a proper subgroup of $Z_{Gn}(G)$.

Remark: Note that if $G$ is nilpotent, then $Z_{Gn}(G) = G$. This is so because every subgroup of a nilpotent group is $\pi$-quasinormal.

It is a simple fact that the subgroups of a cyclic normal subgroup of a group are again normal in the group. In the following theorem, we prove a similar result for the cyclic $\pi$-quasinormal subgroups.

Theorem 3.1.3: If $\langle g \rangle$ is $\pi$-quasinormal in $G$, then every subgroup of $\langle g \rangle$ is also $\pi$-quasinormal in $G$.

Proof: Let $\langle g^n \rangle$ be a subgroup of $\langle g \rangle$, where $n$ is an integer. To prove the theorem, we must show that $\langle g^n \rangle G_p = G_p \langle g^n \rangle$ for an arbitrary but fixed Sylow p-subgroup $G_p$ of $G$. By Theorem 1.2.5(iii), $\langle g \rangle$ is subnormal in $G$. Hence $\langle g^n \rangle$ is subnormal in $G$. Let $P$ be the Sylow p-subgroup of $\langle g^n \rangle$ and $K$ be the p-complement of $<g^n>$. Since $P$ is a subnormal p-subgroup of $G$, it follows that $P$ is contained in every Sylow p-subgroup of $G$. Hence $P \leq G_p$ and so $PG_p = G_pP = G_p$. Let $Q$ be the Sylow q-subgroup of $<g>$ for prime $q \neq p$. Since $<g>$ is subnormal in $G$, $Q$ is subnormal in $G$. This means that $Q$ is a subnormal Sylow q-subgroup of $<g>G_p = G_p<g>$, which implies that $Q$ is normal in $<g>G_p$. From this, we conclude that the p-complement $H$ of $<g>$ is normal in $<g>G_p$. Since $K \leq H$ and $H$ is cyclic, it now follows that $K$ is normal in $<g>G_p$ and so $KG_p = G_pK$. This, together with $PG_p = G_pP = G_p$, implies that $P \times K = <g^n>$ permutes with $G_p$, and the proof is complete.

Remark: In view of Lemma 1.2.3, it is obvious that if $G \cong \bar{G}$ under the isomorphism $\theta$, then $(Z_{Gn}(G))^{\theta} = Z_{Gn}(\bar{G})$.

The following example shows that every subgroup of $Z_{Gn}(G)$ is not necessarily $\pi$-quasinormal in $G$. In particular, it shows that if $<a>$ and $<b>$ are $\pi$-quasinormal in $G$, then $<ab>$ is not necessarily $\pi$-quasinormal in $G$.

Example: Let $G = <a,b><x>$, where $a^3 = b^3 = 1$, $ab = ba$, $a^x = a^2$, $b^x = b$ and $x^2 = 1$. Clearly, $Z_{Gn}(G) = <a,b>$. Hence $ab \in Z_{Gn}(G)$. But $<ab>$ is not $\pi$-quasinormal in $G$ since $<ab>$ does not permute with $<x>$ which is a Sylow 2-subgroup of $G$. For, otherwise $<ab>$ would be normalized by $x$, an impossibility.

Let $H$ be a subgroup of a group $G$. In general, there seems to be no relationship between $Z_{Gn}(G)$ and $Z_{Gn}(H)$, even if $H$ is normal in $G$. For example, if $G = A_4$ and $H$ is its Sylow 2-subgroup, then $Z_{Gn}(G) = 1 \lneq Z_{Gn}(H) = H$. On the other hand, if $G$ is a nilpotent group and $H$ is a proper (normal or non-normal) subgroup of $G$, then $Z_{Gn}(H) = H \lneq Z_{Gn}(G) = G$. This leads us to investigate the relationship between the generalized center of a group and the generalized centers of its direct factors. We prove

<u>Proposition 3.1.4</u>: If $G = H \times K$, then $Z_{Gn}(G) = Z_{Gn}(H) \times Z_{Gn}(K)$.

<u>Proof</u>: First we show that $Z_{Gn}(H) \times Z_{Gn}(K) \leq Z_{Gn}(G)$. Let $h$ be an element of $H$ such that $\langle h\rangle$ is $\pi$-quasinormal in $H$ and $G_p$ be a Sylow p-subgroup of $G$. Since $G_p = H_p \times K_p$ for some $H_p$ and $K_p$ and since $\langle h\rangle$ is centralized by $K$, it follows that $\langle h\rangle G_p = G_p\langle h\rangle$. This means that $\langle h\rangle$ is $\pi$-quasinormal in $G$ and hence $h \in Z_{Gn}(G)$. Thus $Z_{Gn}(H) \leq Z_{Gn}(G)$. Similarly, $Z_{Gn}(K) \leq Z_{Gn}(G)$ and so $Z_{Gn}(H) \times Z_{Gn}(K)$ is contained in $Z_{Gn}(G)$.

Next to show that $Z_{Gn}(G) \leq Z_{Gn}(H) \times Z_{Gn}(K)$, let $g$ be an element of $G$ such that $\langle g\rangle$ is $\pi$-quasinormal in $G$. Let $P$ be the Sylow p-subgroup of $\langle g\rangle$. Clearly, $P = \langle hk\rangle$ for some p-elements $h \in H$ and $k \in K$. We will show that $\langle h\rangle$ is $\pi$-quasinormal in $H$. For this, let $H_q$ be any Sylow q-subgroup of $H$, $q \neq p$ and $x$ be an element of $H_q$. By Theorem 3.1.3, $P$ is $\pi$-quasinormal in $G$. Hence it follows from Lemma 3.1.1 that $P$ is normal in $PG_q$ for every Sylow q-subgroup $G_q$ of $G$, $q \neq p$. In particular, $x^{-1}(hk)x = (hk)^n = h^nk^n$ for some integer $n$. This implies that $h^{-n}(x^{-1}hx) = k^n(x^{-1}kx)^{-1} \in H \cap K = 1$. Hence $x^{-1}hx = h^n$ and so $\langle h\rangle$ permutes with every $H_q$ for primes $q \neq p$. Now let $H_p$ be any Sylow p-subgroup of $H$. Then there exists a Sylow p-subgroup $G_p$ of $G$ such that $H_p \leq G_p$. But $G = H \times K$ and therefore $G_p = H_p \times (K \cap G_p)$. Since $P = \langle hk\rangle$ is $\pi$-quasinormal in $G$, it follows that $P \leq G_p$. Hence $hk \in H_p \times (K \cap G_p)$, which implies that $h \in H_p$. Consequently,

$\langle h\rangle H_p = H_p\langle h\rangle = H_p$. This means that $\langle h\rangle$ is $\pi$-quasinormal in $H$ and so $h \in Z_{Gn}(H)$. Similarly, $k \in Z_{Gn}(K)$. Thus $P \leq Z_{Gn}(H) \times Z_{Gn}(K)$. Since $\langle g\rangle$ is the direct product of its Sylow subgroups, it follows that $\langle g\rangle \leq Z_{Gn}(H) \times Z_{Gn}(K)$. Hence $Z_{Gn}(G) \leq Z_{Gn}(H) \times Z_{Gn}(K)$. This completes the proof.

Remark: Mukherjee [15] has proved the above result for the quasicenters under the added hypothesis of $(|H|, |K|) = 1$.

If $G$ is nilpotent, then $Z_{Gn}(G) = G$ but the quasicenter $Q(G)$ is not necessarily $G$ itself. For example, let $G = \langle a,b,c\rangle\langle x\rangle$, where $a^2 = b^2 = c^2 = x^2 = 1$, $ab = ba$, $bc = cb$, $ac = ca$, $a^x = ab$, $b^x = b$, $c^x = c$. It is shown by Mukherjee [15] that $|Q(G)| = 8$. But $|G| = 16$ and so $Q(G) \lneq G = Z_{Gn}(G)$. Using this example, we can construct a non-nilpotent group $G_1$ such that $Q(G_1) \lneq Z_{Gn}(G_1)$.

Let $G_1 = G \times S_3$, where $G$ is the group defined above and $S_3$ is the symmetric group on 3 letters. Then $Z_{Gn}(G_1) = Z_{Gn}(G) \times Z_{Gn}(S_3) = G \times A_3$. Since $Q(G_1) \leq Q(G) \times Q(S_3) = Q(G) \times A_3$, it follows that $Q(G_1) \lneq Z_{Gn}(G_1)$.

Next we determine the structure of the generalized center of a group. For this, we need the following lemma.

Lemma 3.1.5: Every Sylow subgroup of $Z_{Gn}(G)$ of a group $G$ is generated by generalized central elements of $G$.

Proof: Let $p_1, p_2, \ldots,$ and $p_n$ be the prime divisors of the order of $Z_{Gn}(G)$. For $1 \leq i \leq n$, denote by $S_i$ the subgroup generated by all $p_i$-elements of $G$ that are the generalized central elements of $G$. Clearly, each $S_i$ is $\pi$-quasinormal in $G$ and hence is subnormal in $G$. Furthermore, since a generalized central $p_i$-element of $G$ belongs to every Sylow $p_i$-subgroup $P_i$ of $G$, it follows that $S_i \leq \cap P_i$. Therefore, $S_i$ is a $p_i$-subgroup of $G$.

Let $p_j$ and $p_k$ be any two different prime divisors of the order of $Z_{Gn}(G)$. Then, by the preceding paragraph and Lemma 3.1.1, it follows that $S_j$ is normal in $S_jP_k$. Hence $S_j$ and $S_k$ permute and so $S_jS_k$ is a subgroup for all primes $p_j$ and $p_k$ with $p_j \neq p_k$ and $1 \leq j,k \leq n$. Therefore, $S_1S_2 \ldots S_n$ is a subgroup which is contained in $Z_{Gn}(G)$. Now, let $g$ be any generalized central element of $G$. Since $\langle g \rangle$ is the direct product of its Sylow subgroups each of which is cyclic and $\pi$-quasinormal in $G$, it follows that every Sylow subgroup of $\langle g \rangle$ is contained in some $S_i$. Hence $\langle g \rangle \leq S_1S_2 \ldots S_n$, which implies that $Z_{Gn}(G) \leq S_1S_2 \ldots S_n$. Thus $Z_{Gn}(G) = S_1S_2 \ldots S_n$. From this, the assertion in the lemma follows immediately.

<u>Theorem 3.1.6</u>: $Z_{Gn}(G)$ of a group $G$ is nilpotent.

<u>Proof</u>: The proof of this theorem is essentially embodied in that of the preceding lemma. For, if $P$ is any Sylow subgroup of $Z_{Gn}(G)$, then $P$ is subnormal (in fact $\pi$-quasinormal) in $G$ and so $P$ is a subnormal Sylow subgroup of $Z_{Gn}(G)$. Hence $P$ is normal in $Z_{Gn}(G)$, which implies that $Z_{Gn}(G)$ is nilpotent.

<u>Remark</u>: Note that every Sylow subgroup $P$ of $Z_{Gn}(G)$ is characteristic in $G$. This follows from the fact that $P$ is characteristic in $Z_{Gn}(G)$ and $Z_{Gn}(G)$ is characteristic in $G$. Also note that $G$ is nilpotent if and only if $G = Z_{Gn}(G)$.

In the next few results, we prove some additional properties of the generalized center.

<u>Proposition 3.1.7</u>: If $N$ is a minimal normal subgroup of $G$, then $N \leq C_G(Z_{Gn}(G))$, the centralizer of $Z_{Gn}(G)$ in $G$.

<u>Proof</u>: If $N \cap Z_{Gn}(G) = 1$, then clearly $N \leq C_G(Z_{Gn}(G))$. On the other hand, if $N \cap Z_{Gn}(G) \neq 1$, then $N = N \cap Z_{Gn}(G)$ by

the minimality of N. This means that $N \leq Z_{Gn}(G)$ and so N is nilpotent. Hence every Sylow subgroup of N is normal in G. But N is a minimal normal subgroup of G and so N must be a p-subgroup for some prime p. Let P be the Sylow p-subgroup of $Z_{Gn}(G)$. Then N is a normal subgroup of P. Hence $N \cap Z(P) \neq 1$, where Z(P) is the center of P. Since P is normal (in fact characteristic) in G, $Z(P) \lhd G$. Therefore, $N \cap Z(P) \lhd G$ and so $N \cap Z(P) = N$. Hence $N \leq Z(P)$. The desired result is now obvious since $Z_{Gn}(G)$ is nilpotent.

Theorem 3.1.8: If the smallest prime divisor of $|G|$ divides $|Z_{Gn}(G)|$, then $Z(G) \neq 1$.

Proof: Let $p_1, p_2, \ldots,$ and $p_n$ be the prime divisors of the order of G, where $p_1 < p_2 < \ldots < p_n$. By hypotheiss, $p_1$ divides $|Z_{Gn}(G)|$. Denote by $P_i$ a Sylow $p_i$-subgroup of G for $i = 1,2,\ldots,n$ and by $P_1^*$ the Sylow $p_1$-subgroup of $Z_{Gn}(G)$. Then $G = P_1P_2\ldots P_n$ and $P_1^* \neq 1$. Since $P_1^*$ is characteristic in G, it follows that $P_1^*$ is a normal subgroup of $P_1$. Hence $P_1^* \cap Z(P_1) \neq 1$.

Let y be a non-identity element of $P_1^* \cap Z(P_1)$. Then y centralizes $P_1$ and we shall show that every $P_i$ for $i > 1$ is also centralized by y. Let g be any generalized central element of G such that $g \in P_1^*$ and z be an element of $P_i$ for $i > 1$. Lemma 3.1.1 yields that $\langle g \rangle$ is normalized by every $p_1'$-element of G. In particular, $\langle g \rangle\langle z \rangle = \langle z \rangle\langle g \rangle$ is a subgroup and $\langle g \rangle \lhd \langle g \rangle\langle z \rangle$. But $\langle g \rangle\langle z \rangle$ is supersolvable and $p_i > p_1$ and so $\langle z \rangle \lhd \langle g \rangle\langle z \rangle$. Thus $zg = gz$. From this we see that y centralizes z since y belongs to $P_1^*$ which is generated by generalized central elements of G. Hence $y \in Z(G)$ and the theorem is proved.

Corollary 3.1.9: Let $p$ be the smallest prime divisor of $|Z_{Gn}(G)|$. If every prime divisor $q$ of $|G|$ with $q < p$ does not divide $p-1$, then $Z(G) \neq 1$.

Proof: Let $P$ be a Sylow $p$-subgroup of $G$ and $P^*$ be the Sylow $p$-subgroup of $Z_{Gn}(G)$. As in the theorem, it follows that there is an element $y$ in $P^* \cap Z(P)$ such that $y \neq 1$ and $y$ centralizes $P$ and all Sylow $r$-subgroups of $G$ for primes $r > p$.

Let $z$ be a $q$-element of $G$ with prime $q < p$ and $g$ be a generalized central element of $G$ such that $g \in P^*$. By Lemma 3.1.1, $\langle g \rangle$ is normalized by $z$ and so $\langle g \rangle \langle z \rangle$ is a subgroup. Since $\langle z \rangle$ is a Sylow $q$-subgroup of $\langle g \rangle \langle z \rangle$ and $q$ does not divide $p-1$, it follows from the Sylow theorems that $\langle z \rangle$ is normal in $\langle g \rangle \langle z \rangle$. Hence $gz = zg$, which implies, as in the theorem, that $y$ centralizes $z$ and so $y \in Z(G)$.

Theorem 3.1.10: If $p$ is a prime divisor of $|Z_{Gn}(G)|$ and $g$ is an element of $Z_{Gn}(G)$ such that all the prime divisors of $|g|$ are less than $p$, then $g$ centralizes every $p$-element of $G$.

Proof: We may assume that $g$ is a $q$-element and prime $q < p$. We may also assume that $g$ is a generalized central element of $G$ since the Sylow $q$-subgroup of $Z_{Gn}(G)$ is generated by such elements. Let $x$ be a $p$-element of $G$. By Lemma 3.1.1, $\langle g \rangle \langle x \rangle$ is a subgroup and $\langle g \rangle \lhd \langle g \rangle \langle x \rangle$. Since $\langle g \rangle \langle x \rangle$ is supersolvable and $p > q$, $\langle x \rangle \lhd \langle g \rangle \langle x \rangle$. Thus $gx = xg$.

## 3.2 Generalized Hypercenter

In this section we generalize the concept of the hypercenter and hyperquasicenter to that of the generalized hypercenter and study the structure and some properties of this new hypercenter.

<u>Definition 3.2.1</u>: For a group $G$, let $(Z_{Gn}(G))_0 = 1$ and $(Z_{Gn}(G))_{i+1}/(Z_{Gn}(G))_i$ be the generalized center of $G/(Z_{Gn}(G))_i$. Then we define the <u>generalized hypercenter</u> $Z^*_{Gn}(G)$ of $G$ to be the terminal member of the ascending chain of characteristic subgroups: $1 = (Z_{Gn}(G))_0 < Z_{Gn}(G) = (Z_{Gn}(G))_1 < (Z_{Gn}(G))_2 <\dots< (Z_{Gn}(G))_m = Z^*_{Gn}(G)$.

It is obvious that $Z^*_{Gn}(G)$ contains the hypercenter and hyperquasicenter of $G$. The hypercenter $Z^*(G)$ is always nilpotent but $Z^*_{Gn}(G)$ is not necessarily nilpotent. The generalized hypercenter $Z^*_{Gn}(S_3)$ of the symmetric group $S_3$ on three letters is $S_3$ itself which is not nilpotent. However, we shall show that $Z^*_{Gn}(G)$ is supersolvable.

The following example shows that the hyperquasicenter of a group can be trivial even though the generalized hypercenter is nontrivial.

<u>Example</u>: Let $G$ be the symmetric group $S_3$ wreathed by the cyclic group $C_3$ of order 3, i.e., $G$ is the direct product of three copies of $S_3$ extended by an automorphism of order 3 which permutes the copies in a cycle. For this group, $Q(G) = Q^*(G) = 1$ but $Z_{Gn}(G) = Z^*_{Gn}(G)$ has order 27.

The hypercenter $Z^*(G)$ of a group $G$ is known to be the intersection of all normal subgroups $N$ of $G$ such that $Z(G/N) = \bar{1}$. A similar result is true for the generalized hypercenter.

<u>Theorem 3.2.2</u>: For a group $G$, $Z^*_{Gn}(G) = \bigcap\{N \mid N \lhd G$ and $Z_{Gn}(G/N) = \bar{1}\}$.

<u>Proof</u>: Consider the chain: $1 = (Z_{Gn}(G))_0 < Z_{Gn}(G) = (Z_{Gn}(G))_1 < (Z_{Gn}(G))_2 <\dots< (Z_{Gn}(G))_m = Z^*_{Gn}(G)$ and let

$T = \bigcap\{N \mid N \lhd G$ and $Z_{Gn}(G/N) = \bar{1}\}$. Then $T \le Z^*_{Gn}(G)$ since $Z_{Gn}(G/Z^*_{Gn}(G)) = \bar{1}$. To prove $Z^*_{Gn}(G) \le T$, let $K$ be a normal subgroup of $G$ such that $Z_{Gn}(G/K) = \bar{1}$ and $g$ be a generalized central element of $G$. From Lemma 1.2.3, it follows that $gK$ is a generalized central element of $G/K$. Hence $gK$ is an element of $Z_{Gn}(G/K) = \bar{1}$ and so $g$ belongs to $K$. This implies that $Z_{Gn}(G) = (Z_{Gn}(G))_1 \le K$.

Next we assume that $(Z_{Gn}(G))_i \le K$, $1 \le i < m$, and wish to show that $(Z_{Gn}(G))_{i+1} \le K$. Let $x(Z_{Gn}(G))_i$ be a generalized central element of $G/(Z_{Gn}(G))_i$. Since $G/K$ is a homomorphic image of $G/(Z_{Gn}(G))_i$, it follows that $xK$ is a generalized central element of $G/K$. But $Z_{Gn}(G/K) = \bar{1}$ and so $x$ belongs to $K$. This means that $Z_{Gn}(G/(Z_{Gn}(G))_i) = (Z_{Gn}(G))_{i+1}/(Z_{Gn}(G))_i \le K/(Z_{Gn}(G))_i)$. Hence $(Z_{Gn}(G))_{i+1} \le K$, which implies that $Z^*_{Gn}(G) \le K$. From this, it follows that $Z^*_{Gn}(G) \le T$ and the proof is complete.

The next proposition follows from the above theorem.

<u>Proposition 3.2.3</u>: Let $N$ be a normal subgroup of $G$. Then, $Z^*_{Gn}(G)N/N \le Z^*_{Gn}(G/N)$.

<u>Proof</u>: Let $K/N$ be a normal subgroup of $G/N$ such that $Z_{Gn}(G/N/K/N) = \bar{1}$. Since $G/N/K/N \cong G/K$, it follows that $Z_{Gn}(G/K) = \bar{1}$. Hence $Z^*_{Gn}(G) \le K$ and so $Z^*_{Gn}(G)N/N \le KN/N = K/N$. This means that $Z^*_{Gn}(G)N/N \le \bigcap\{K/N \mid K/N \lhd G/N$ and $Z_{Gn}(G/N/K/N) = \bar{1}\} = Z^*_{Gn}(G/N)$.

<u>Remark</u>: The inclusion in the above proposition could be proper, too. For example, consider $G = A_4$, the alternating group of degree 4, and let $N$ be the Sylow 2-subgroup of $A_4$. Then $Z^*_{Gn}(G)N/N = \bar{1} \lneq Z^*_{Gn}(G/N) = G/N$.

The following result which we shall need later is a special case of the preceding proposition.

Proposition 3.2.4: Let $T$ be a normal subgroup of $G$ such that $T \leq Z^*_{Gn}(G)$. Then $Z^*_{Gn}(G/T) = Z^*_{Gn}(G)/T$.

Proof: Consider the chain: $\bar{1} = T/T < W_1/T < W_2/T < \ldots < W_m/T = Z^*_{Gn}(G/T)$, where $W_1/T = Z_{Gn}(G/T)$ and $W_i/T/W_{i-1}/T = Z_{Gn}(G/T/W_{i-1}/T)$ for $i = 2,3,\ldots,m$. Since $T \leq Z^*_{Gn}(G)$, it follows from Proposition 3.2.3 that $Z^*_{Gn}(G) \leq W_m$. To prove that $W_m \leq Z^*_{Gn}(G)$, we first show that $W_1 \leq Z^*_{Gn}(G)$. For this, let $gT$ be a generalized central element of $G/T$. Then $gZ^*_{Gn}(G)$ is a generalized central element of $G/Z^*_{Gn}(G)$ since $G/Z^*_{Gn}(G)$ is a homomorphic image of $G/T$. Hence $g$ belongs to $Z^*_{Gn}(G)$, which implies that $W_1 \leq Z^*_{Gn}(G)$.

We now assume that $W_i \leq Z^*_{Gn}(G)$, $2 \leq i < m$, and will show that $W_{i+1} \leq Z^*_{Gn}(G)$. Let $\theta$ be the natural isomorphism from $G/T/W_i/T$ onto $G/W_i$. It follows that $Z_{Gn}(G/W_i) = (Z_{Gn}(G/T/W_i/T))^{\theta} = W_{i+1}/W_i$. Now replacing $T$ by $W_i$ in the argument used above, one can show that $W_{i+1} \leq Z^*_{Gn}(G)$. Hence $W_m \leq Z^*_{Gn}(G)$ and so $W_m = Z^*_{Gn}(G)$, which means that $Z^*_{Gn}(G)/T = Z^*_{Gn}(G/T)$.

In general, there is no relationship between the generalized hypercenter of the group and the generalized hypercenter of a subgroup of the group. This is confirmed by the following examples.

(1) Let $H$ be a proper subgroup of the symmetric group $S_3$ on three letters. Since $Z^*_{Gn}(S_3) = S_3$, we have $Z^*_{Gn}(H) \lneq Z^*_{Gn}(S_3)$.

(2) Let $G = A_4$. Since $Z^*_{Gn}(G) = 1$, $Z^*_{Gn}(G) \lneq Z^*_{Gn}(H)$ for every proper subgroup $H$ of $G$.

The last example suggests the next result.

Proposition 3.2.5: Let H be a subgroup of a group G such that $Z^*_{Gn}(G) \leq H$. Then $Z^*_{Gn}(G) \leq Z^*_{Gn}(H)$. In particular, $Z^*_{Gn}(Z^*_{Gn}(G)) = Z^*_{Gn}(G)$.

Proof: We use induction on the order of G. Since $Z^*_{Gn}(G) \leq H$, $Z_{Gn}(G) \leq H$. This means that all generalized central elements of G are in H. Now it follows from Theorem 1.2.5(i) that all generalized central elements of G are also the generalized central elements of H. Hence $Z_{Gn}(G) \leq Z_{Gn}(H) \leq Z^*_{Gn}(H)$ and so, by Proposition 3.2.4, $Z^*_{Gn}(H/Z_{Gn}(G)) = Z^*_{Gn}(H)/Z_{Gn}(G)$. Since $Z^*_{Gn}(G/Z_{Gn}(G)) = Z^*_{Gn}(G)/Z_{Gn}(G) \leq H/Z_{Gn}(G)$, it follows by induction that $Z^*_{Gn}(G)/Z_{Gn}(G) \leq Z^*_{Gn}(H/Z_{Gn}(G)) = Z^*_{Gn}(H)/Z_{Gn}(G)$, which implies that $Z^*_{Gn}(G) \leq Z^*_{Gn}(H)$.

To prove the particular case, set $H = Z^*_{Gn}(G)$. This yields that $Z^*_{Gn}(G) \leq Z^*_{Gn}(Z^*_{Gn}(G)) \leq Z^*_{Gn}(G)$.

It is easily proved that if G and $\bar{G}$ are isomorphic under the map $\theta$, then $Z^*_{Gn}(\bar{G})$ is the image of $Z^*_{Gn}(G)$ under $\theta$. Using this and Proposition 3.1.4, we prove the following result.

Proposition 3.2.6: If $G = H \times K$, then $Z^*_{Gn}(G) = Z^*_{Gn}(H) \times Z^*_{Gn}(K)$.

Proof: We use induction on the order of G. Clearly, we may assume that $Z_{Gn}(G) = Z_{Gn}(H) \times Z_{Gn}(K) \neq 1$. By induction and Proposition 3.2.4, $Z^*_{Gn}(H/Z_{Gn}(H) \times K/Z_{Gn}(K)) = Z^*_{Gn}(H/Z_{Gn}(H)) \times Z^*_{Gn}(K/Z_{Gn}(K)) = Z^*_{Gn}(H)/Z_{Gn}(H) \times Z^*_{Gn}(K)/Z_{Gn}(K)$. Let $\theta$ be the natural isomorphism from $G/Z_{Gn}(G)$ onto $H/Z_{Gn}(H) \times K/Z_{Gn}(K)$. Then, $Z^*_{Gn}(H)/Z_{Gn}(H) \times Z^*_{Gn}(K)/Z_{Gn}(K) = (Z^*_{Gn}(G/Z_{Gn}(G)))^{\theta} = (Z^*_{Gn}(G)/Z_{Gn}(G))^{\theta}$. Now apply $\theta^{-1}$ to obtain $Z^*_{Gn}(G)/Z_{Gn}(G) = (Z^*_{Gn}(H) \times Z^*_{Gn}(K))/Z_{Gn}(G)$, which yields the desired result.

Remark: If $G$ is an extension of $H$ by $K$, then it is not true in general that $Z^*_{Gn}(G) = Z^*_{Gn}(H) \cdot Z^*_{Gn}(K)$, as confirmed by $A_4$.

We now investigate the structure of the generalized hypercenter. Our aim here is to prove that it is supersolvable. For this, we need the following theorem. However, we first recall a definition from [8].

Definition 3.2.7: Let $p_1 > p_2 > \ldots > p_n$ be the natural ordering of the prime divisors of the order of a group $G$ and $P_i$ be a Sylow $p_i$-subgroup of $G$ for $i = 1,2,\ldots,n$. Then $G$ has the Sylow tower property of supersolvable groups if for each $k$, $1 \le k \le n$, $P_1P_2\ldots P_k$ is a normal subgroup of $G$.

Let us point out that a group having this property is necessarily solvable.

Theorem 3.2.8: $Z^*_{Gn}(G)$ of a group $G$ has the Sylow tower property of supersolvable groups.

Proof: We proceed by induction on the order of $G$. Thus, in view of Proposition 3.2.5, we may assume that $Z^*_{Gn}(G) = G$. Let $p$ be the largest prime divisor of the order of $G$ and $G_p$ be a Sylow $p$-subgroup of $G$. We shall show that $G_p$ is normal in $G$.

First suppose that $p$ does not divide the order of $G/Z_{Gn}(G)$. Then, $G_p \le Z_{Gn}(G)$. Since $Z_{Gn}(G)$ is nilpotent, it follows that $G_p$ is normal in $G$. Next suppose that $p$ divides the order of $G/Z_{Gn}(G)$. Since $p$ is the largest prime divisor of the order of $G/Z_{Gn}(G)$ and $Z^*_{Gn}(G/Z_{Gn}(G)) = Z^*_{Gn}(G)/Z_{Gn}(G) = G/Z_{Gn}(G)$, by induction $(G/Z_{Gn}(G))_p = G_pZ_{Gn}(G)/Z_{Gn}(G)$ is normal in $G/Z_{Gn}(G)$. This means that $G_pZ_{Gn}(G) \lhd G$. But the Sylow subgroups of $Z_{Gn}(G)$ are normal in $G$. Hence $G_pZ_{Gn}(G) = G_pL \lhd G$, where $L$ is the $p$-complement of $Z_{Gn}(G)$. To prove that $G_p \lhd G$, it suffices to show that $L \leq N_G(G_p)$ since $G = G_p L N_G(G_p) = L N_G(G_p)$ by Frattini Lemma. For this, let $x$ be any q-element of $G$ for a prime $q \neq p$ such that $\langle x \rangle$ is $\pi$-quasinormal in $G$. By Lemma 3.1.1, $\langle x \rangle \lhd \langle x \rangle G_p$. Let $g$ be any element of $G_p$. Then $\langle x \rangle \langle g \rangle$ is a supersolvable subgroup of $G$ and $\langle x \rangle \lhd \langle x \rangle \langle g \rangle$. Since $p > q$, $\langle g \rangle$ is also normal in $\langle x \rangle \langle g \rangle$. Hence $g$ and $x$ centralize each other and so $x \in N_G(G_p)$. But, by Lemma 3.1.5, all such $x$ generate $L$. Hence $L \leq N_G(G_p)$, which implies that $G_p$ is normal in $G$.

To complete the argument, consider $G/G_p$. By induction, $Z^*_{Gn}(G/G_p) = G/G_p$ has the Sylow tower property of supersolvable groups, which clearly implies that $G = Z^*_{Gn}(G)$ has the same property. This proves the theorem.

We are now ready to prove the supersolvability of $Z^*_{Gn}(G)$.

<u>Theorem 3.2.9</u>: $Z^*_{Gn}(G)$ of a group $G$ is supersolvable.

<u>Proof</u>: We proceed by induction on the order of $G$. If $Z^*_{Gn}(G) \lneq G$, then $Z^*_{Gn}(Z^*_{Gn}(G)) = Z^*_{Gn}(G)$ is supersolvable by induction. Hence we may assume that $Z^*_{Gn}(G) = G$. This means that every factor group $G/K$ of $G$ for $K \neq 1$ is supersolvable by induction since $Z^*_{Gn}(G/K) = Z^*_{Gn}(G)/K = G/K$.

If the Frattini subgroup $\Phi(G)$ of $G$ is not identity, then $G/\Phi(G)$ is supersolvable. Now a theorem of Huppert [9] yields that $G$ is supersolvable. So we assume that $\Phi(G) = 1$. Let $p$ be the largest prime divisor of $|G|$. By Theorem 3.2.8, the Sylow p-subgroup $P$ of $G$ is normal in $G$. Hence $\Phi(P) \leq \Phi(G)$, which implies that $\Phi(P) = 1$. This means that $P$ is elementary abelian. Let $M$ be a maximal subgroup of $G$. We will show that $[G:M]$ is a prime.

It follows from Theorem 3.2.8 that $G$ is solvable. Hence $[G:M]$ is a power of a prime. If $[G:M]$ is not a power of prime $p$, then $P \leq M$ and, since $G/P$ is supersolvable, $[G/P:M/P] = [G:M]$ is a prime. On the other hand, if $[G:M]$ is a power of $p$, we consider $Z_{Gn}(G)$ and proceed as follows: It is obvious that $Z_{Gn}(G) \neq 1$. If a prime $q \neq p$ divides the order of $Z_{Gn}(G)$, then the Sylow q-subgroup $\bar{Q}$ of $Z_{Gn}(G)$ is normal in $G$ and therefore $\bar{Q} \leq M$. But $G/\bar{Q}$ is supersolvable, which means that $[G/\bar{Q}:M/\bar{Q}] = [G:M]$ is a prime. Hence we may assume that $Z_{Gn}(G)$ is a p-subgroup. Since $Z_{Gn}(G)$ is generated by the generalized central elements of $G$ and the powers of a generalized central element of $G$ are again the generalized central elements of $G$, it follows that $G$ contains generalized central elements of order $p$. Let $N$ be the subgroup generated by all these elements of order $p$. Since the conjugates of a $\pi$-quasinormal subgroup are $\pi$-quasinormal in the group, it follows that $N$ is normal in $G$. If $N \leq M$, then, as before, $[G:M]$ is a prime. On the other hand, if $N \not\leq M$, then there is an element $y$ of order $p$ such that $\langle y \rangle$ is $\pi$-quasinormal in $G$ and $y \notin M$. Since $P$ is abelian, $\langle y \rangle$ is a normal subgroup of $P$. It follows from Lemma 3.1.1 that $\langle y \rangle$ is normalized by p'-elements of $G$. Hence $\langle y \rangle$ is normal in $G$. This means that $M\langle y \rangle$ is a subgroup and so $M\langle y \rangle = G$, which implies that $[G:M] = |\langle y \rangle| = p$, a prime. Now by a theorem of Huppert (Theorem 1.1.13), $G = Z^*_{Gn}(G)$ is supersolvable and the proof is complete.

The above result leads to the following simple observation.

<u>Theorem 3.2.10</u>: A group $G$ is supersolvable if and only if $G = Z^*_{Gn}(G)$.

Proof: If $G = Z^*_{Gn}(G)$, then $G$ is supersolvable by the preceding theorem. On the other hand, if $G$ is supersolvable, then $G$ has a cyclic normal subgroup $\langle g\rangle \neq 1$. Since a normal subgroup is always $\pi$-quasinormal, it follows that $\langle g\rangle \leq Z_{Gn}(G) \leq Z^*_{Gn}(G)$. Hence $Z^*_{Gn}(G/\langle g\rangle) = Z^*_{Gn}(G)/\langle g\rangle$. But by induction, $Z^*_{Gn}(G/\langle g\rangle) = G/\langle g\rangle$ and so $Z^*_{Gn}(G) = G$.

In the next result, we obtain a condition for a group to be supersolvable.

Theorem 3.2.11: A group $G$ is supersolvable if and only if $G/Z^*_{Gn}(G)$ is supersolvable.

Proof: If $G$ is supersolvable, then its factor group $G/Z^*_{Gn}(G)$ is supersolvable, too. Conversely, if $G/Z^*_{Gn}(G)$ is supersolvable, then by Theorem 3.2.10, $Z^*_{Gn}(G/Z^*_{Gn}(G)) = G/Z^*_{Gn}(G)$. But $Z^*_{Gn}(G/Z^*_{Gn}(G))$ is identity and therefore $G = Z^*_{Gn}(G)$. Since $Z^*_{Gn}(G)$ is supersolvable, it follows that $G$ is supersolvable.

It is known that the product of two supersolvable subgroups is not necessarily supersolvable. However, if one of the subgroups is the generalized hypercenter, then the following is true.

Theorem 3.2.12: Let $S$ be a supersolvable subgroup of a group $G$. Then $SZ^*_{Gn}(G)$ is supersolvable.

Proof: We use induction on $|G|$. If $SZ^*_{Gn}(G) \lneq G$, then $SZ^*_{Gn}(SZ^*_{Gn}(G))$ is supersolvable by induction. Proposition 3.2.5 yields that $Z^*_{Gn}(G) \leq Z^*_{Gn}(SZ^*_{Gn}(G))$, which implies that $SZ^*_{Gn}(G)$ is supersolvable. So assume that $SZ^*_{Gn}(G) = G$. Since $G/Z^*_{Gn}(G) = SZ^*_{Gn}(G)/Z^*_{Gn}(G) \cong S/S \cap Z^*_{Gn}(G)$ and $Z^*_{Gn}(G/Z^*_{Gn}(G))$ is identity, it follows that $Z^*_{Gn}(S/S \cap Z^*_{Gn}(G))$ is identity. But $S/S \cap Z^*_{Gn}(G)$ is supersolvable and so $Z^*_{Gn}(S/S \cap Z^*_{Gn}(G)) = S/S \cap Z^*_{Gn}(G)$. This means that $S/S \cap Z^*_{Gn}(G)$ is identity, which implies that $S \leq Z^*_{Gn}(G)$. Hence $SZ^*_{Gn}(G) = Z^*_{Gn}(G)$. But $Z^*_{Gn}(G)$ is supersolvable and therefore $SZ^*_{Gn}(G)$ is supersolvable.

Baer proved in [4] that the hypercenter of a group $G$ is the intersection of the maximal nilpotent subgroups of $G$. Since the generalized hypercenter is supersolvable, it seems reasonable to conjecture that the generalized hypercenter of $G$ is the intersection of the maximal supersolvable subgroups of $G$. We have not yet been able to prove this conjecture completely. However, we prove part of the conjecture in the following theorem.

<u>Theorem 3.2.13</u>: $Z^*_{Gn}(G)$ of a group $G$ is contained in the intersection of the maximal supersolvable subgroups of $G$.

<u>Proof</u>: We must show that $Z^*_{Gn}(G)$ is contained in every maximal supersolvable subgroup of $G$. For this, let $M$ be any maximal supersolvable subgroup of $G$. By Theorem 3.2.12, $MZ^*_{Gn}(G)$ is supersolvable. Hence either $MZ^*_{Gn}(G) = G$ or $MZ^*_{Gn}(G) = M$. If $MZ^*_{Gn}(G) = G$, then $G$ is supersolvable. This implies that $M = G$ and therefore $Z^*_{Gn}(G) \leq M$. On the other hand, if $MZ^*_{Gn}(G) = M$, then it follows immediately that $Z^*_{Gn}(G) \leq M$.

<u>Remark</u>: Note that many of Mukherjee's results (see [15] and the introduction of this chapter) on $Q(G)$ and $Q^*(G)$ are immediate consequences of our results on $Z_{Gn}(G)$ and $Z^*_{Gn}(G)$, since $Q(G) \leq Z_{Gn}(G)$ and $Q^*(G) \leq Z^*_{Gn}(G)$ and since the conclusions have already been proved for $Z_{Gn}(G)$ and $Z^*_{Gn}(G)$.

We conclude this section with a result which may be of some independent interest. It gives a condition for a group to have the Sylow tower property of supersolvable groups.

Theorem 3.2.14: A group $G$ has the Sylow tower property of supersolvable groups if and only if $G/Z^*_{Gn}(G)$ has the same property.

Proof: We need only prove that if $G/Z^*_{Gn}(G)$ has the Sylow tower property of supersolvable groups, then $G$ has the same property. For this, we first show that if for any group $H$, $H/Z_{Gn}(H)$ has the Sylow tower property of supersolvable groups, then $H$ has this property, too. We proceed by induction on the order of $H$.

Let $N$ be a normal subgroup of $H$. We shall show that $H/N/Z_{Gn}(H/N)$ has the Sylow tower property of supersolvable groups. Since $H/N/Z_{Gn}(H)N/N \cong H/Z_{Gn}(H)N$ and $H/Z_{Gn}(H)N$ is a homomorphic image of $H/Z_{Gn}(H)$, it follows that $H/N/Z_{Gn}(H)N/N$ has the Sylow tower property of supersolvable groups. Hence $H/N/Z_{Gn}(H/N)$ has the same property because $H/N/Z_{Gn}(H/N)$ is a homomorphic image of $H/N/Z_{Gn}(H)N/N$ by Proposition 3.2.3.

Let $p$ be the largest prime divisor of the order of $H$ and $H_p$ be a Sylow p-subgroup of $H$. One can easily verify by essentially duplicating the argument used in Theorem 3.2.8 that $H_p$ is a normal subgroup of $H$. Now it follows from the observation made in the preceding paragraph that $H/H_p/Z_{Gn}(H/H_p)$ has the Sylow tower property of supersolvable groups. Hence by induction, $H/H_p$ has this property which, in fact, implies that $H$ has this property.

To establish the theorem, consider the chain: $1 = (Z_{Gn}(G))_0 < Z_{Gn}(G) = (Z_{Gn}(G))_1 < (Z_{Gn}(G))_2 < \ldots < (Z_{Gn}(G))_m = Z^*_{Gn}(G)$, where $(Z_{Gn}(G))_i/(Z_{Gn}(G))_{i-1} = Z_{Gn}(G/(Z_{Gn}(G))_{i-1})$, $1 \leq i \leq m$. Since $G/Z^*_{Gn}(G) = G/(Z_{Gn}(G))_m \cong G/(Z_{Gn}(G))_{m-1}/(Z_{Gn}(G))_m/(Z_{Gn}(G))_{m-1} = G/(Z_{Gn}(G))_{m-1}/Z_{Gn}(G/(Z_{Gn}(G))_{m-1})$ and $G/Z^*_{Gn}(G)$ has the Sylow tower property of supersolvable groups, it follows by what we have shown above that $G/(Z_{Gn}(G))_{m-1}$ has the Sylow tower property of supersolvable groups. Now repeated use of this argument shows that $G$ has the same property, the desired conclusion.

### 3.3 Generalized Hypercentral Subgroups

In this section, the generalized hypercenter is shown to be the product of all generalized hypercentral subgroups. The notion of the generalized hypercentral subgroups is an extension of the concept of the hypercentral subgroups introduced by Baer. He proved that the hypercenter is the product of all normal hypercentral subgroups. In our case the normality is included in the definition.

Definition 3.3.1: We shall call a normal subgroup $H$ of a group $G$ a generalized hypercentral (GH-central) subgroup of $G$ if for all $M \lneq H$ and $M$ normal in $G$, $H/M \cap Z_{Gn}(G/M) \neq \bar{1}$.

Proposition 3.3.2: If $H$ is a GH-central subgroup of $G$, then for each $N \triangleleft G$ and $N \lneq H$, $H/N$ is a GH-central subgroup of $G/N$.

Proof: Let $K/N \lneq H/N$ and $K/N \triangleleft G/N$. Then $K \lneq H$ and $K \triangleleft G$. Hence $H/K \cap Z_{Gn}(G/K) \neq \bar{1}$. Consequently, there is an element $hK$ of $H/K$ such that $hK \in Z_{Gn}(G/K)$ and $h \notin K$. Suppose $G/K \cong G/N/K/N$ under the map $\theta$. Since $Z_{Gn}(G/N/K/N) = (Z_{Gn}(G/K))^{\theta}$, it follows that $(hK)^{\theta}$ is a non-identity element of $H/N/K/N \cap Z_{Gn}(G/N/K/N)$, which implies that $H/N$ is a GH-central subgroup of $G/N$.

Lemma 3.3.3: If $H \leq Z_{Gn}(G)$ and $H \triangleleft G$, then $H$ is a GH-central subgroup of $G$. In particular, $Z_{Gn}(G)$ is a GH-central subgroup of $G$.

Proof: Suppose $K \lhd G$ and $K \lneq H$. Since $G/K$ is a homomorphic image of $G$, it follows from Lemma 1.2.3 that $Z_{Gn}(G)K/K = Z_{Gn}(G)/K \leq Z_{Gn}(G/K)$. But $H/K \leq Z_{Gn}(G)/K$. Hence $H/K \cap Z_{Gn}(G/K) = H/K \neq \bar{1}$.

Theorem 3.3.4: For $1 \leq i \leq m$, every member $(Z_{Gn}(G))_i$ of the chain: $1 = (Z_{Gn}(G))_0 < Z_{Gn}(G) = (Z_{Gn}(G))_1 < (Z_{Gn}(G))_2 <\ldots< (Z_{Gn}(G))_m = Z^*_{Gn}(G)$ is a GH-central subgroup of $G$.

Proof: We use induction on $|G|$. In view of the preceding Lemma, we prove the theorem for $i = 2,3,\ldots,m$. Consider the group $G/Z_{Gn}(G)$ and form the chain: $\bar{1} = (Z_{Gn}(G))_1/Z_{Gn}(G) < (Z_{Gn}(G))_2/Z_{Gn}(G) < \ldots < (Z_{Gn}(G))_m/Z_{Gn}(G) = Z^*_{Gn}(G)/Z_{Gn}(G)$, where $Z_{Gn}(G/Z_{Gn}(G)/(Z_{Gn}(G))_{i-1}/Z_{Gn}(G)) = (Z_{Gn}(G))_i/Z_{Gn}(G)/(Z_{Gn}(G))_{i-1}/Z_{Gn}(G)$ for $i = 2,3,\ldots,m$. Now suppose $M \lneq (Z_{Gn}(G))_i$ and $M \lhd G$, $i > 1$. If $Z_{Gn}(G) \leq M$, then $(Z_{Gn}(G))_i/Z_{Gn}(G)/M/Z_{Gn}(G) \cap Z_{Gn}(G/Z_{Gn}(G)/M/Z_{Gn}(G)) \neq \bar{1}$ since by induction, $(Z_{Gn}(G))_i/Z_{Gn}(G)$ is a GH-central subgroup of $G/Z_{Gn}(G)$. From this we see that $(Z_{Gn}(G))_i/M \cap Z_{Gn}(G/M) \neq \bar{1}$. On the other hand, if $Z_{Gn}(G) \nleq M$, then there is a generalized central element $g$ of $G$ such that $g \notin M$. By Lemma 1.2.3, $gM \in Z_{Gn}(G/M)$ and so $gM \in (Z_{Gn}(G))_i/M \cap Z_{Gn}(G/M)$. Hence $(Z_{Gn}(G))_i$ is a GH-central subgroup of $G$.

Lemma 3.3.5: If $N_1$ and $N_2$ are GH-central subgroups of a group $G$, then the product $N_1N_2$ is also a GH-central subgroup of $G$.

Proof: Let $M \lhd G$ with $M \lneq N_1N_2$. If $M \lneq N_1$ or $M \lneq N_2$, then $N_1N_2/M \cap Z_{Gn}(G/M) \neq \bar{1}$ since $N_i/M \cap Z_{Gn}(G/M) \neq \bar{1}$ for $i = 1$ or 2. If $M \cap N_1 = N_1$ and $M \cap N_2 = N_2$, then $N_1N_2 \leq M$, which is impossible. Thus we may assume without loss of generality that $M \cap N_1 = W \lneq N_1$. Since $W$ is also normal in $G$, $N_1/W \cap Z_{Gn}(G/W) \neq \bar{1}$.

Let $xW$ be a non-identity element of $N_1/W \cap Z_{Gn}(G/W)$. Since $G/M$ is a homomorphic image of $G/W$ and $x \notin M$, $xM$ is a non-identity element of $N_1N_2/M \cap Z_{Gn}(G/M)$. Hence $N_1N_2$ is a GH-central subgroup of $G$.

From the above results we now obtain the following characterization of the generalized hypercenter.

<u>Theorem 3.3.6</u>: $Z^*_{Gn}(G)$ of a group $G$ is the product of all GH-central subgroups of $G$.

<u>Proof</u>: It follows from Lemma 3.3.5 that the product $P$ of all GH-central subgroups of $G$ is GH-central. Hence $Z^*_{Gn}(G) \leq P$ by Theorem 3.3.4. If $Z^*_{Gn}(G) \lneq P$, then $P/Z^*_{Gn}(G) \cap Z_{Gn}(G/Z^*_{Gn}(G)) \neq \bar{1}$ which means that $Z_{Gn}(G/Z^*_{Gn}(G)) \neq \bar{1}$, an impossibility. Thus $Z^*_{Gn}(G) = P$.

## CHAPTER IV

## FINITE GROUPS WHOSE i-th MAXIMAL SUBGROUPS ARE π-QUASINORMAL

A number of mathematicians have studied the structure of the group when its i-th maximal subgroups satisfy some imbedding property. The most natural imbedding property is perhaps normality. The other examples of this property are quasinormality, π-quasinormality and subnormality. Huppert [9] and Janko [12] have described the structure of the group when its i-th maximal subgroups are normal and Mann [14] has studied the structure of the group when its i-th maximal subgroups are subnormal or quasinormal. In this chapter, we investigate the structure of the group when its i-th maximal subgroups are π-quasinormal. Our primary concern is to improve the following theorems.

(Huppert [9]). If each second maximal subgroup of G is normal in G, then G is supersolvable. If the order of G is divisible by at least three different primes, then G is nilpotent.

(Huppert [9]). Let each third maximal subgroup of G be normal in G. Then: (i) G' is nilpotent; (ii) the rank of G = $r(G) \leq 2$; (iii) if $|G|$ is divisible by at least three different primes, then G is supersolvable.

(Janko [12]). Let G be solvable. If each fourth maximal subgroup of G is normal in G, then: (i) $r(G) \leq 3$; (ii) if $|G|$ is divisible by at least four distinct primes, then G is supersolvable.

(Mann [14]). Let $G$ be solvable, and each n-th maximal subgroup of $G$ be quasinormal in $G$. Then: (i) $r(G) \leq n-1$; (ii) if $|G|$ is divisible by at least $n-k+1$ distinct primes, then $r(G) \leq k$, where $k \geq 1$.

In Section 2, we prove the above results under the weaker assumption that each i-th maximal subgroup ($i = 2,3,4$) be $\pi$-quasinormal instead of normal or quasinormal.

## 4.1 Definitions and Assumed Results

Definition 4.1.1: For a group $G$, a series of subgroups

$$G = G_0 > G_1 > G_2 > \ldots > G_k = 1$$

is called a chief series if every $G_i$ is a maximal normal subgroup of $G$ contained in $G_{i-1}$ for $i = 1,2,\ldots,k$. The factor groups $G_i/G_{i+1}$ are called the chief factors of $G$.

It is known that if $G$ is solvable, then every chief factor of $G$ is an elementary abelian p-group for some prime $p$.

Definition 4.1.2: Let $G$ be a solvable group. Then the rank of $G$, denoted by $r(G)$, is the maximal integer $n$ such that $G$ has a chief factor of order $p^n$ for some prime $p$.

We list for an easy reference two known results of which we shall make rather frequent use.

(4.1.3) Huppert [8]. If all proper subgroups of the non-nilpotent group $G$ are nilpotent, then $G$ is solvable; $|G| = p^a q^b$ for distinct primes $p$ and $q$; the Sylow p-subgroup $G_p$ is normal and each Sylow q-subgroup $G_q$ is cyclic.

(4.1.4) Doerk [6]. If each maximal subgroup of G is supersolvable, then: (i) G is solvable; (ii) G has a Sylow tower for the natural (descending) ordering of prime divisors of $|G|$, or G satisfies the hypotheses of (4.1.3); (iii) if G itself is not supersolvable, then G has exactly one normal Sylow subgroup.

## 4.2 Generalized Results

For a group G, we prove the following theorems:

Theorem 4.2.1: If each second maximal subgroup of G is $\pi$-quasinormal in G, then G is supersolvable. Furthermore, if $|G|$ is divisible by at least three different primes, then G is nilpotent.

Theorem 4.2.2: If each third maximal subgroup of G is $\pi$-quasinormal in G, then:

(i) if $|G|$ is divisible by three or more different primes, then G is supersolvable;

(ii) the commutator subgroup G' of G is nilpotent;

(iii) the rank of $G = r(G) \leq 2$.

Theorem 4.2.3: Let G be solvable. If every fourth maximal subgroup of G is $\pi$-quasinormal in G, then:

(i) if $|G|$ is divisible by four or more different primes, then G is supersolvable;

(ii) $r(G) \leq 3$.

Proof of theorem 4.2.1: Let M be a maximal subgroup of G. Then every maximal subgroup of M is $\pi$-quasinormal in G. This means that all maximal subgroups of M are $\pi$-quasinormal in M by Theorem 1.2.5(i) and therefore they are normal in M by Theorem 1.2.5(ii).

Hence $M$ is nilpotent and so all proper subgroups of $G$ are nilpotent. Now by (4.1.3), $G$ is solvable. In addition, if $|G|$ is divisible by three or more different primes, then $G$ is nilpotent and we have disposed of this case.

Next we consider the case where $|G|$ is divisible by, at most, two distinct primes. To prove that $G$ is supersolvable, we must show that $[G:M]$, the index of $M$ in $G$, is a prime for an arbitrary but fixed maximal subgroup $M$ of $G$ since a theorem of Huppert states that a group is supersolvable if and only if its maximal subgroups have prime index. If $M_G \neq 1$, then, since by Lemma 1.2.3 $G/M_G$ satisfies the hypothesis of the theorem, $G/M_G$ is supersolvable by induction. From this it follows that $[G/M_G:M/M_G] = [G:M]$ is a prime. Therefore, we may assume that $M_G = 1$, and form the maximal chain: $M_1 < M < G$, where $M_1$ is maximal in $M$. Since $M_1$ is $\pi$-quasinormal in $G$, by Theorem 1.2.5 (iii) $M_1$ is subnormal in $G$. Hence $M_1 \leq M_{SG} = M_G = 1$, which implies that $M_1 = 1$. But $M$ is nilpotent and, therefore, $|M| = p$, a prime. Now consider $[G:M]$ which is a power of a prime since $G$ is solvable. If $[G:M]$ is a power of $p$, then $G$ is a p-group and we are finished. On the other hand, if $[G:M] = q^m$, $q \neq p$, then $|G| = pq^m$. Let $G_q$ be a Sylow q-subgroup of $G$ and $L$ be a maximal subgroup of $G_q$. Then $G_q$ is maximal in $G$, and $L$ is $\pi$-quasinormal in $G$. Since $M$ is a Sylow p-subgroup of $G$, $LM = ML$ is a subgroup of $G$. But $M$ is maximal in $G$ and $LM \neq G$. Therefore, $LM = M$. This implies that $L \leq M$ and so $L = 1$. Hence $|G_q| = q$ showing that $[G:M] = q$, a prime. This completes the proof.

Remark: If we simply require that every second maximal subgroup of $G$ be subnormal in $G$, then $G$ is not necessarily supersolvable, as confirmed by $A_4$, the alternating group of degree 4.

Proof of Theorem 4.2.2: (i) From Theorem 1.2.5(i) and Theorem 4.2.1, it follows that every maximal subgroup of $G$ is supersolvable. Hence $G$ is solvable by (4.1.4). Moreover, if the order of $G$ is divisible by at least four different primes, then $G$ is supersolvable. Thus we need only consider the case in which $|G|$ is divisible by three different primes. Before proceeding, it should be noted that every second maximal subgroup of $G$ is nilpotent by Theorem 1.2.5(i) and (ii) and, therefore, every third maximal subgroup of $G$ is also nilpotent.

Let $|G| = p^{\alpha}q^{\beta}r^{\gamma}$, where $p > q > r$ and $\alpha,\beta,\gamma > 0$. Suppose that $G$ is not supersolvable. Then, since (4.1.3) does not hold, it follows from (4.1.4) that the Sylow p-subgroup $G_p$ is normal in $G$ and no other Sylow subgroup of $G$ is normal in $G$. Since $G$ is solvable, there exist Sylow subgroups $G_q$ and $G_r$ such that $G_qG_r$ is a subgroup. Let $H = G_qG_r$. If $H$ is not maximal in $G$, then $G_q$ is contained in a third maximal subgroup of $G$. Since each third maximal subgroup is nilpotent and subnormal (being $\pi$-quasinormal; Theorem 1.2.5(iii)), it follows that $G_q$ is subnormal in $G$. But a subnormal Sylow subgroup is always normal and so $G_q$ is normal in $G$, a contradiction. Hence $H$ is maximal in $G$.

Now suppose that $\beta \geq 2$. Since every maximal subgroup of $G$ is supersolvable, $H$ is supersolvable, too. Therefore, $G_r$ is properly contained in a maximal subgroup of $H$. This means that $G_r$ is contained in a third maximal subgroup of $G$. Hence, as before,

$G_r$ is normal in $G$, again a contradiction. Thus $\beta = 1$. By a similar argument, $\gamma = 1$ and so $|G| = p^{\alpha}qr$. Next suppose that $L$ is a maximal subgroup of $G_p$ and consider the following maximal chain:

$$L < G_p < G_pG_q < G \quad .$$

From this we see that $L$ is $\pi$ quasinormal in $G$. Hence $L$ permutes with $H$ and therefore $LH$ is a subgroup. Since $H$ is maximal in $G$ and $LH \neq G$, $LH = H$. Thus $L \leq H$ and so $L = 1$, which means that $\alpha = 1$. Hence $G$ is supersolvable, a contradiction to our assumption that $G$ is not supersolvable. Therefore, we have the desired result.

(ii) In view of part (i) and the fact that the commutator subgroup of a supersolvable group is always nilpotent, we may assume that $G$ is not supersolvable and $|G|$ is divisible by two different primes $p$ and $q$. We may further assume without loss of generality (see (4.1.4)) that $G_p$ is normal in $G$. Then $G_q$ is not normal in $G$ and we will show that $G_q$ is either abelian or cyclic.

Suppose $(G_q)_G \neq 1$. Since $|G/(G_q)_G|$ is divisible by both primes $p$ and $q$, $(G/(G_q)_G)'$ is nilpotent by induction. Clearly, $(G/G_p)'$ is nilpotent. Since $(G/G_p)' \cong G'/G' \cap G_p$ and $(G/(G_q)_G)' \cong G'/G' \cap (G_q)_G$, it follows that $G'/(G' \cap G_p) \cap (G' \cap (G_q)_G) \cong G'$ is nilpotent. Next suppose that $(G_q)_G = 1$. If $G_q$ is maximal in $G$, then every second maximal subgroup of $G_q$ is $\pi$-quasinormal (hence subnormal) in $G$. Since $(G_q)_G = (G_q)_{SG} = 1$, all second maximal subgroups of $G_q$ are 1. Therefore, $|G_q| \leq q^2$ which implies that $G_q$ is abelian. On the other hand, if $G_q$ is not maximal in $G$, then there exists a maximal subgroup $M$ of $G$ such that $G_q < M < G$.

Now if $G_q$ is not maximal in $M$, then, as in part (i), $G_q$ is normal in $G$, a contradiction. Therefore, $G_q < M < G$ is a maximal chain. Hence every maximal subgroup of $G_q$ is subnormal (being $\pi$-quasinormal) in $G$. Since $G_q$ is not subnormal in $G$, $G_q$ must have a unique maximal subgroup and so $G_q$ is cyclic. Now to show that $G'$ is nilpotent we need only note that $G' \leq G_p$ since $G/G_p$ ($\cong G_q$) is abelian. This proves part (ii).

(iii) Again, the only case that requires a proof is the one in which $|G|$ is divisible by two distinct primes $p$ and $q$. We further assume that $G$ is not supersolvable, otherwise $r(G) = 1$. As in part (ii), we suppose that $G_p$ is the only Sylow subgroup of $G$ which is normal in $G$.

Let $N_i$ be normal in $G$ and $N_i \neq 1$ for $i = 1$ and $2$. If $p$ and $q$ both divide $|G/N_i|$, then by induction $r(G/N_i) \leq 2$, and if $G/N_i$ is a $p$ or $q$-group, then $r(G/N_i) = 1 \leq 2$. Hence if $N_1 \cap N_2 = 1$, then $r(G/N_1 \cap N_2) = r(G) = \max\{r(G/N_i)\} \leq 2$ and we are done. Thus we may assume that $G$ has a unique minimal normal subgroup $N$. Since $G_p$ is normal in $G$, $N$ is a $p$-subgroup. It now suffices to show that $|N| \leq p^2$ because we already have $r(G/N) \leq 2$ by induction.

Let $G_q$ be a Sylow $q$-subgroup of $G$. If $N \neq G_p$, then $NG_q \neq G$. Hence $NG_q$ is supersolvable. Since $G_p$ is normal in $G$, it follows that its center $Z(G_p)$ is normal in $G$. Thus $N \leq Z(G_p)$ and so every subgroup of $N$ is normal in $G_p$. Let $T$ ($\neq 1$) be a normal subgroup of $NG_q$ such that $T \leq N$. Then $T$ is normal in $G_p$. Hence $T$ is normal in $G_pG_q = G$ and so $T = N$ by the uniqueness of $N$. Therefore, $N$ is also a minimal normal subgroup of $NG_q$, which implies that $|N| = p < p^2$. On the other hand, if $N = G_p$, then $G_p$ is abelian and $G_q$ is maximal in G. Since G has a unique minimal normal subgroup, it follows that $(G_q)_G = (G_q)_{SG} = 1$. Hence every second maximal subgroup of $G_q$ is 1 and so $|G_q| \leq q^2$. First, suppose that $|G_q| = q^2$ and consider the following maximal chain:

$$L < G_p < G_pK < G ,$$

where L is maximal in $G_p$ and K is maximal in $G_q$. Now L is $\pi$-quasinormal in G, and so $LG_q$ is a subgroup of G. But $LG_q \neq G$, and therefore $LG_q = G_q$. From this we conclude that $L = 1$, which shows that $|N| = |G_p| = p \leq p^2$. Next, suppose that $|G_q| = q$. Then, in the same manner, it follows that every second maximal subgroup of $G_p$ is 1, which in turn proves that $|N| = |G_p| \leq p^2$. This completes the proof of part (iii) and of the theorem.

Remark: The group $A_4$ shows that if the order of a group is divisible by two different primes and if its third maximal subgroups are $\pi$-quasinormal, then the group need not be supersolvable in general.

Proof of Theorem 4.2.3: (i) Let M be an arbitrary but fixed maximal subgroup of G. By Theorem 1.2.5(i), third maximal subgroups of M are $\pi$-quasinormal in M. Since $|G|$ is divisible by at least four different primes and G is solvable, $|M|$ is divisible by at least three different primes. Hence by part (i) of Theorem 4.2.2, M is supersolvable. Now G is supersolvable by a theorem of Huppert [9].

(ii) We use induction on the order of $G$. In view of part (i), the only cases that need proof are the ones in which $|G|$ is divisible by three and two different primes, respectively. We treat these cases separately. Before proceeding, we should observe that each second maximal subgroup of $G$ is supersolvable by Theorem 1.2.5(i) and Theorem 4.2.1. Also note that every third maximal subgroup of $G$ is nilpotent. This follows from Theorem 1.2.5(i) and (ii) and the fact that a group is nilpotent if and only if its maximal subgroups are normal.

Case 1. $|G|$ is divisible by two primes, $p$ and $q$. Then, as in part (iii) of Theorem 4.2.2, we can assume that $G$ has a unique minimal normal subgroup $N$ and by induction $r(G/N) \leq 3$. Without loss of generality, let $|N| = p^n$. It is sufficient to show that $n \leq 3$. For this, let $G_p$ be a Sylow p-subgroup and $G_q$ be a Sylow q-subgroup of $G$. Since $N$ is a normal p-subgroup, $N \leq G_p$.

First, suppose that $N \neq G_p$, and consider $NG_q$. If $NG_q$ is not maximal in $G$, then $NG_q$ is supersolvable. Now if $G_q$ is not maximal in $NG_q$, then $G_q$ is, or is contained in, a fourth maximal subgroup of $G$ which is subnormal and nilpotent. This implies that $G_q$ is subnormal (hence normal) in $G$, a contradiction. Thus $G_q$ is maximal in $NG_q$ and so $|N| = p < p^3$. On the other hand, if $NG_q$ is maximal in $G$, then we claim that $|G_q| \leq q^2$. To show this, let $|G_q| \geq q^3$, and consider the chain:

$$L_2 < NL_2 < NL_1 < NG_q < G ,$$

where $L_2$ is maximal in $L_1$, $L_1$ is maximal in $G_q$ and $L_2 \neq 1$. This implies that $L_2$ is contained in a fourth maximal subgroup of

$G$. But fourth maximal subgroups are nilpotent and subnormal, and so $L_2$ is subnormal in $G$. Thus $L_2$ is contained in every Sylow $q$-subgroup of $G$, which means that there is a non-trivial normal $q$-subgroup of $G$, a contradiction. Hence $|G_q| \leq q^2$. We now have two possibilities: (a) Suppose $|G_q| = q^2$. Let $L$ be a maximal subgroup of $G_q$ and $H$ be a maximal subgroup of $N$. If $H = 1$, then $|N| = p$ and if $H \neq 1$, then we form the following maximal chain:

$$H < N < NL < NG_q < G \quad .$$

From this we see that $H$ is $\pi$-quasinormal in $G$. Therefore $HG_q$ is a subgroup and is clearly maximal in $NG_q$. Consider the chain:

$$G_q < HG_q < NG_q < G \quad .$$

If $G_q$ is not maximal in $HG_q$, then $G_q$ is subnormal in $G$, a contradiction. Hence $G_q$ must be maximal in $HG_q$. Since $HG_q$ is supersolvable, we have $|H| = p$, which shows that $|N| = p^2 < p^3$, the desired conclusion. (b) Now suppose that $|G_q| = q$ and form the maximal chain:

$$A < B < N < NG_q < G \quad .$$

If $A = 1$ or $B = 1$, then $|N| \leq p^2$ and if $A \neq 1$, then, as before, the chain:

$$G_q < AG_q < NG_q < G$$

implies that $|A| = p$, which means that $|N| = p^3$ and we are finished.

Next suppose that $N = G_p$. Then $G_q$ is maximal in G and, since N is the unique minimal normal subgroup of G, $(G_q)_G = (G_q)_{SG} = 1$ and so every third maximal subgroup of $G_q$ is 1. Hence $|G_q| \leq q^3$. First, suppose that $|G_q| = q^3$ and let L be a maximal subgroup of $G_q$ and K be a maximal subgroup of L. Then

$$N < NK < NL < G = NG_q$$

is a maximal chain. Let M be a maximal subgroup of N. Since M is $\pi$-quasinormal, $MG_q$ is a subgroup. But $MG_q \neq G$ and hence $G_q = MG_q$ by the maximality of $G_q$. Thus $M = 1$ and so $|N| = |G_p| = p$. Likewise, it can be shown that if $|G_q| = q^2$, then $|N| \leq p^2$ and if $|G_q| = q$, then $|N| \leq p^3$. This completes the proof of case 1.

<u>Case 2</u>. $|G|$ is divisible by three distinct primes p, q, and r. Let G/K be any proper factor group of G. If $|G/K|$ is a prime-power, then $r(G/K) = 1 < 3$; if $|G/K|$ is divisible by two distinct primes, then by case 1, $r(G/K) \leq 3$; and if $|G/K|$ is divisible by all primes p, q, and r, then by induction $r(G/K) \leq 3$. So, as before, we may assume that N is the unique minimal normal subgroup of G. Without loss of generality, let $|N| = p^n$. We must show that $n \leq 3$. Since G is solvable, there exists a subgroup M such that $M = G_q G_r$ for some Sylow subgroups $G_q$ and $G_r$. Now if $N \neq G_p$, then $NM \neq G$. Hence from the chain:

$$G_q < NG_q < NM < G \quad ,$$

it follows that $G_q$ is maximal in $NG_q$, for otherwise $G_q$ would be subnormal in G, which is impossible. But $NG_q$ is supersolvable.

Hence $[NG_q : G_q] = |N| = p$. On the other hand, if $N = G_p$, then M is maximal in G. Since $N \cap M_G = 1$, it follows from the uniqueness of N that $M_{SG} = M_G = 1$. Thus every third maximal subgroup of M is 1. This implies that N is either a third or a second maximal subgroup of G. Note that N cannot be a first maximal subgroup of G. First, suppose that N is a third maximal subgroup of G and let H be a maximal subgroup of N. Then H is π-quasinormal in G and, since $M = G_q G_r$, $HM = MH$ is a subgroup. But M is maximal and $HM \neq G$. Hence $HM = M$, which means that $H = 1$. Thus $|N| = p$ and we are done. Similarly, one can verify that $|N| \leq p^2$ when N is second maximal in G. This proves case 2 and the theorem.

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