TOPOLOGICAL PROPERTIES OF COMPACTIFICATIONS OF A HALF-OPEN INTERVAL

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ABSTRACT

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by David Parham Bellamy

We make the following definitions: Let $A = [1,\infty)$. A Hausdorff compactification (S,h) of A is a pseudocone, where $h: A \to S$ is the natural embedding. $S-h(A)$ is the base of S; $h(1)$ is the vertex of S; and if the base of S is homemorphic to a topological space X, S is a pseudocone over X. Unless explicitly stated otherwise, both A and the base of a pseudocone S will be considered identified with their images in S. When it is convenient to specify that X is the base of a pseudocone, the pseudocone will be denoted by $P(X)$. This is not meant to imply any sort of functorial relationship.

The broad questions considered are: "Under what conditions does there exist a pseudocone over X?" and "What is the relationship between the properties of $P(X)$ and those of X?"

An outline of principal results follows. The numbering of results below does not correspond to the numbering in the thesis.

Chapter I. Existence and general properties

Theorem 1: Let $P(X)$ be a pseudocone. Then $P(X)$ is a continuum irreducible between its vertex and any point on X.

Theorem 2: The base of a psuedocone is a compact Hausdorff

continuum.

Theorem 3: Let X be a compact Hausdorff continuum which is irreducible about some separable subset of itself and such that there exists a connected, locally path connected, locally compact Hausdorff space Y and an embedding $f: X \to Y$ such that $f(X)$ is a G_δ subset of Y. Then there exists a pseudocone over X.

Corollary 1: If X is a compact metric continuum, there exists a pseudocone over X.

Corollary 2: If X is a compact Hausdorff continuum with a separable dense path component, there exists a pseudocone over X.

Chapter II. Retracts of pseudocones

Theorem: Let X be a compact Hausdorff space. The following are equivalent:

1. X has a separable dense path component.

2. There exists a pseudocone $P(X)$ and an embedding $f: P(X) \to X \times I$ such that $f(X) = X \times \{0\}$.

3. There exists a pseudocone $P(X)$ such that X is a retract of $P(X)$.

Chapter III. Pseudocones over metric spaces

Proposition: Every pseudocone over a metric space is metrizable.

Theorem: A Peano continuum X is a retract of every pseudocone $P(X)$.

Chapter IV. A generalization of Peano continua

Peano continua are characterized in terms of pseudocones as are compact metric continua X with the property that $X = \bigcup_{i=1}^{\infty} W_i$ where each W_i is a Peano continuum and $W_{i+1} \supseteq W_i$.

Chapter V. The Stone-Čech compactification of A

The base of $\beta(A)$ is called A^*.

Proposition 1: Every metric continuum and every pseudocone is a continuous image of every nondegenerate subcontinuum of A^*.

Corollary: Every nondegenerate subcontinuum of A^* has cardinal number 2^c.

Proposition 2: A^* is an indecomposable continuum.

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INTRODUCTION

In this thesis, A will always denote the half-open interval $[1,\infty)$. Let (S,h) be a Hausdorff compactification of A where $h: A \to S$ is the natural embedding and let $X = S-h(A)$. It is the purpose of this thesis to give partial answers to the following questions: What properties must X possess? What relationships exist between the topological properties of X and those of S? This point of view motivates the following definitions:

A compactification (S,h) of A is a <u>pseudocone</u>. $S-h(A)$ is the <u>base</u> of S; $h(1)$ is the <u>vertex</u> of S; and if the base of a pseudocone S is homeomorphic to a topological space X, S is a <u>pseudocone over X</u>. The notation $P(X)$ will be used to denote a pseudocone over X whenever it is useful. This is not meant to imply any sort of functorial relationship. Unless explicitly stated otherwise, both A and X will be considered identified with their images in a pseudocone $P(X)$.

The first chapter is devoted to general properties of pseudocones and proofs of the existence of pseudocones over certain classes of compact Hausdorff continua. In the second chapter the question of when X is a retract of $P(X)$ is investigated. The third chapter discusses pseudocones over metric spaces; it is shown that every pseudocone over a metric space is metrizable, and machinery is developed to prove a stronger result

about retractions in this case. In chapter four, a characterization of Peano continua in terms of pseudocones is given, and a class of metric continua more general than Peano continua is investigated in some detail. The fifth and final chapter is devoted to an examination of some of the continua-theoretic properties of the Stone-Čech compactification of A.

CHAPTER I

EXISTENCE AND GENERAL PROPERTIES

The following two lemmas give the most important properties shared by all pseudocones.

Lemma 1: *A pseudocone $P(X)$ is a continuum irreducible between two points a and b, where a is its vertex and b is any element of X.*

Proof: Let U be any open subset of $P(X)$ missing a and b. A is dense in $P(X)$, so $U \cap A \neq \emptyset$. Let $y \in U \cap A$. Since A is locally compact, A is open in $P(X)$. Therefore, $[1,y)$ is open in $P(X)$. Also, $[1,y]$ is closed in $P(X)$. Let

$$B = [1,y) \cap (S-U).$$

Since $y \in U$,

$$B = [1,y] \cap (S-U)$$

$B \neq \emptyset$ since $1 = a \in B$. $B \neq S-U$ since $b \notin B$. Thus B is a nonvoid proper closed and open subset of $S-U$. Hence $S-U$ is not connected and the proof is complete.

Lemma 2: *The base of a pseudocone is a compact Hausdorff continuum.*

Proof: Let $P(X)$ be a pseudocone. For each positive integer n, let $A_n = [n,\infty)$ and set

$$P_n(X) = A_n \cup X.$$

Then

$$X = \bigcap_{n=1}^{\infty} P_n(X)$$

But $P_n(X) = \overline{A}_n$, so that each $P_n(X)$ is a compact Hausdorff continuum. The intersection is monotone, so X is also a compact Hausdorff continuum.

The principal results concerning the existence of pseudocones follow.[1]

Theorem 1: Let X be a compact Hausdorff continuum which is irreducible about some separable subset B of itself and such that there exists a locally path connected locally compact Hausdorff space Y and an embedding $f: X \to Y$ such that $f(X)$ is a G_δ subset of Y. Then there exists a pseudocone over X which embeds in $Y \times I$ with base $f(X) \times \{0\}$.

Proof: Let X be identified with its image in Y and let $\langle U_i \rangle_{i=1}^{\infty}$ be a sequence of open sets in Y such that:

1) $\overline{U}_1$ is compact
2) for each i, $\overline{U}_{i+1} \subseteq U_i$
3) $\bigcap_{i=1}^{\infty} U_i = X$
4) each U_i is path connected.

Let D be a countable dense subset of B and let $\langle a_i \rangle_{i=1}^{\infty}$ be a sequence from D such that for each $x \in D$, the set $N_x = \{i : a_i = x\}$ is nonvoid and N_x is infinite if x is an isolated point of D. Note that every neighborhood of any point in B contains elements from $\langle a_i \rangle$ of arbitrarily large index.

For each i, let $\omega_i\colon [i,i+1] \to U_i$ be a path from a_i to a_{i+1}. Set $\omega = \bigcup_{i=1}^{\infty} \omega_i$. Then ω is a continuous map from A into $\overline{U}_1$. Define $\omega^*\colon A \to \overline{U}_1 \times I$ (where $I = [0,1]$) by:

$$\omega^*(t) = (\omega(t), \tfrac{1}{t})$$

Then ω^* is an embedding of A into $\overline{U}_1 \times I$. Identify $\overline{U}_1$ with $\overline{U}_1 \times \{0\}$. Since $\overline{U}_1 \times I$ is a compact Hausdorff space, $\overline{\omega^*(A)}$ is a pseudocone. It remains to be shown that X is the base of $\overline{\omega^*(A)}$. Let X_0 be the base of $\overline{\omega^*(A)}$. Let p be any element of $(\overline{U}_1 \times I) - (X \cup \omega^*(A))$. If $p \notin \overline{U}_1$, $p = (x,t)$ and

$$(\overline{U}_1 \times (\tfrac{t}{2},1]) - \omega^*([1,\tfrac{2}{t}])$$

is an open neighborhood of p in $\overline{U}_1 \times I$ which misses $\omega^*(A)$. If $p \in \overline{U}_1$, let V be an open neighborhood of p in $\overline{U}_1$ such that $\overline{V} \cap X = \emptyset$. Then $\{(\overline{U}_1 - \overline{U}_i)\}_{i=2}^{\infty}$ is an open covering of $\overline{V}$ and by compactness there is an n such that $\overline{V} \subseteq \overline{U}_1 - \overline{U}_n$. Thus, $\overline{V} \cap U_n = \emptyset$. $V \times [0,\frac{1}{n})$ is an open neighborhood of p in $\overline{U}_1 \times I$. Suppose

$$x \in \omega^*(A) \cap (V \times [0,\tfrac{1}{n}))$$

Then $x = (\omega(t),\frac{1}{t})$. Since $\frac{1}{t} < \frac{1}{n}$, $t > n$. Therefore $\omega(t) \in U_n$ and $\omega(t) \notin V$, a contradiction. Thus, $\omega^*(A) \cap (V \times [0,\frac{1}{n})) = \emptyset$ and $X_0 \subseteq X$.

Now suppose $p \in B$. Let V be any open neighborhood of

p in $\overline{U}_1 \times I$. V contains a neighborhood of p of the form $W \times [0,\varepsilon)$ where W is open in $\overline{U}_1$ and $\varepsilon > 0$. By the choice of the sequence $\langle a_i \rangle$, there exists an $n > \frac{1}{\varepsilon}$ such that $a_n \in W$. Then $\omega^*(n) = (a_n, \frac{1}{n})$ and $\omega^*(n) \in W \times [0,\varepsilon) \subseteq V$. Therefore, $B \subseteq X_0$. By Lemma 2, X_0 is a continuum, and since X is irreducible about B, $X_0 = X$. This completes the proof.

Corollary 1: *If* M *is a compact metric continuum, there exists a pseudocone over* M.

Proof: M is itself separable, and embeds in the Hilbert Cube as a G_δ set.

Corollary 2: *If* X *is a compact Hausdorff continuum which contains a separable dense path component* D, *there exists a pseudocone over* X.

Proof: Let $\{a_i\}_{i=1}^{\infty}$ be a countable dense subset of D and let $\omega: A \to X$ be a continuous map such that for each positive integer i, $\omega(i) = a_i$. This is possible since D is path connected.

Define $\omega^*: A \to X \times I$ by

$$\omega^*(t) = (\omega(t), \frac{1}{t})$$

Then $\omega^*(A) \cup X \times \{0\}$ is the desired pseudocone.

Since a pseudocone is separable and completely regular, it embeds in I^c. This implies that the base of a pseudocone has cardinality at most 2^c. Corollary 2 implies that there exists pseudocone over I^c, showing that this cardinality can be realized.

CHAPTER II

RETRACTS OF PSEUDOCONES

The question of what properties a continuum must possess to be a retract of every pseudocone over itself appears to be difficult. A simple result is given here and a more general result is established for the metric case in Chapter III.

<u>Proposition 1</u>: <u>An absolute neighborhood retract</u> X <u>is a retract of every pseudocone</u> $P(X)$.

<u>Proof</u>: Let $P(X)$ be a pseudocone. Since X is an absolute neighborhood retract, there is an open neighborhood U of X in $P(X)$ such that X is a retract of U. $P(X) - U$ is a compact subset of A and hence there is a real number x such that $P(X) - U \subseteq [1,x)$. Let $r\colon U \to X$ be a retraction and define $g\colon P(X) \to U$ by:

$$
\begin{aligned}
g(t) &= x \quad \text{if} \quad t \in [1,x] \\
&= t \quad \text{if} \quad t \notin [1,x)
\end{aligned}
$$

Then $r \circ g\colon P(X) \to X$ is the desired retraction.

The question of whether, given X, there exists a pseudocone $P(X)$ such that X is a retract of $P(X)$; and whether, given $P(X)$, X is a retract of $P(X)$ are simpler, and characterizations are given for these cases.

<u>Lemma 1</u>: <u>Given a pseudocone</u> $P(X)$, X <u>is a retract of</u> $P(X)$ <u>if and only if there is an embedding</u> $f\colon P(X) \to X \times I$ <u>such that</u>

$f(X) = X \times \{0\}$.

<u>Proof</u>: If $P(X)$ can be embedded in $X \times I$ in the prescribed fashion, then $f^{-1} \circ p \circ f$ is the desired retraction, where $p: X \times I \to X \times \{0\}$ is the projection.

Conversely suppose X is a retract of $P(X)$. Let $r: P(X) \to X$ be a retraction and define $f_0: P(X) \to I$ by:

$$f_0(x) = \frac{1}{x} \quad \text{if} \quad x \in A$$
$$= 0 \quad \text{if} \quad x \in X$$

Let $f: P(X) \to X \times I$ be defined by:

$$f(x) = (r(x), f_0(x))$$

Since r and f_0 are both continuous, so is f. To show f is one to one, suppose $f(x) = f(y)$. Then $f_0(x) = f_0(y)$ so that $x = y$ if either $x \in A$ or $y \in A$. Thus suppose $x,y \in X$. Then $r(x) = r(y)$ and $x = y$ since r is the identity on X. The compactness of $P(X)$ and the Hausdorff property of $X \times I$ make f an embedding, and the proof is complete.

<u>Corollary 1</u>: <u>Let</u> $P(X)$ <u>be a pseudocone</u>. <u>There exists an embedding</u> $f: P(X) \to X \times I$ <u>such that</u> $f(X) = X \times \{0\}$ <u>if and only if there exists a topological space</u> Y, <u>a point</u> $y \in Y$, <u>and an embedding</u> $g: P(X) \to X \times Y$ <u>such that</u> $g(X) = X \times \{y\}$.

<u>Proof</u>: Let $r: X \times Y \to X \times \{y\}$ be the projection. Then $g^{-1} \circ r \circ g$ is a retraction from $P(X)$ to X. Thus, by Lemma 1,

there is an embedding $f\colon P(X) \to X \times I$ with the desired property. The converse is clear.

Lemma 2: _Let_ X _be a topological space with a dense path component_ D _and_ $f\colon X \to Y$ _a continuous surjection. Then_ Y _has a dense path component_ C, _and_ C _is separable if_ D _is_.

Proof: It is clear that the path component of Y containing $f(D)$ is dense and is separable if D is.

Theorem 1: _Let_ X _be a compact Hausdorff space. Then the following three conditions are equivalent_.

1) X _has a separable dense path component_.

2) _There exists a pseudocone_ $P(X)$ _and an embedding_ $f\colon P(X) \to X \times I$ _such that_ $f(X) = X \times \{0\}$.

3) _There exists a pseudocone_ $P(X)$ _such that_ X _is a retract of_ $P(X)$.

Proof: 1) implies 2) by the construction used in proving _Corollary 2_, Chapter I.

2) implies 3) by _Lemma 1_.

3) implies 1) by _Lemma 2_ and the separability of A.

Corollary 2: _Let_ X _be a compact metric space. Then the following three conditions are equivalent_.

1) X _has a dense path component_.

2) _There exists a pseudocone_ $P(X)$ _and an embedding_ $f\colon P(X) \to X \times I$ _such that_ $f(X) = X \times \{0\}$.

3) <u>There exists a pseudocone</u> $P(X)$ <u>such that</u> X <u>is a retract of</u> $P(X)$.

It seems natural to ask whether the hypothesis $f(X) = X \times \{0\}$ is necessary to prove that X is a retract of $P(X)$ in <u>Lemma 1</u>. Can this hypothesis be eliminated or perhaps weakened to $f(X) \subseteq X \times \{0\}$. The following example shows that this is not possible.

<u>Example 1</u>. In E^3, Euclidean 3-space set:

$$X = \{(x,y,0) : x^2 + y^2 \leq 1\}$$

$$Y = \{(1,y,0) : -1 \leq y \leq 1\}$$

$$Z = \{(x,y,0) : 1 < x \leq 1 + \frac{1}{\pi},\ y = \sin \frac{1}{x-1}\}$$

and let $W = X \cup Y \cup Z$.

By <u>Theorem 1</u>, Chapter I, there exists a pseudocone S with base W in the set

$$R = \{(x,y,z) \in E^3 : x^2 + y^2 \leq 4,\ 0 \leq z \leq 1\}$$

Define $f_0 : R \to W \times I$ by:

$$f_0(x,y,z) = (\frac{x}{2}, \frac{y}{2}, z),$$

and let f be the restriction of f_0 to S. Then f is an embedding and $f(W) \subseteq W$, but by <u>Corollary 2</u>, W is not a retract of S since W has no dense path component.

In view of the above result concerning dense path components of continua, it may be reasonable to ask whether path connectedness of X is sufficient to imply that every $P(X)$ retracts onto X. This is not the case, for it is fairly easy to construct a pseudocone over a Warsaw circle which cannot retract onto its base as follows:

Example 2: In E^3 let:

$$B = \{(x,y,0): y = \sin \frac{1}{x} \text{ and } 0 < x \le \frac{1}{\pi}\}$$

$$C = \{(0,y,0): -2 \le y \le 1\}$$

$$D = \{(x,-2,0): 0 \le x \le \frac{1}{\pi}\}$$

$$E = \{(\frac{1}{\pi},y,0): -2 \le y \le 0\}$$

Set $X = B \cup C \cup D \cup E$. Then X is a Warsaw circle. Construct a pseudocone over X as follows:

For each positive integer i, let:

D_i be the segment from $(\frac{1}{\pi},-2,\frac{1}{i})$ to $(0,-2,\frac{1}{i+1})$

$$C_i = \{(0,y,\frac{1}{i+1}): -2 \le y \le 0\}$$

$$N_i = \{(x,0,\frac{1}{i+1}): 0 \le x \le \frac{1}{i\pi}\}$$

$$R_i = \{(x,y,\frac{1}{i+1}): (x,y,0) \in B \text{ and } x \ge \frac{1}{i\pi}\}$$

$$E_i = \{(\frac{1}{\pi},y,\frac{1}{i+1}): -2 \le y \le 0\}.$$

Set $B_i = R_i \cup N_i$ for each i. Then $P(X) = X \cup [\bigcup_{i=1}^{\infty} (B_i \cup C_i \cup D_i \cup E_i)]$ is a pseudocone over X, and, roughly speaking, each D_i is over D running from the $z = \frac{1}{i}$ level to the $z = \frac{1}{i+1}$ level, each B_i, C_i, and E_i is over, B, C, and E, respectively, at the $z = \frac{1}{i+1}$ level.

To show that X is not a retract of P(X), let:

$$U = \{(x,y,0) \in X: \ y > -2\}$$

$$V = \{(0,y,0) \in X: \ -\frac{5}{3} < y < -\frac{4}{3}\}$$

$$W = \{(x,y,0) \in X: \ x > \frac{1}{2\pi} \text{ and } y > -2\}$$

For each positive integer n, let $p_n = (\frac{1}{\pi}, 0, \frac{1}{n+1})$ and let $q_n = (0, -\frac{3}{2}, \frac{1}{n+1})$. Let $I_n \subseteq C_n \cup B_n$ be the arc from q_n to p_n and let

$$p = (\frac{1}{\pi}, 0, 0) = \lim_{n\to\infty} p_n$$

and

$$q = (0, -\frac{3}{2}, 0) = \lim_{n\to\infty} q_n$$

Now suppose $r: P(X) \to X$ is a retraction. Then:

$$\lim_{n\to\infty} r(p_n) = p \quad \text{and} \quad \lim_{n\to\infty} r(q_n) = q.$$

Thus for some sufficiently large integer k, $r(p_k) \in W$; $r(q_k) \in V$; and $I_k \subseteq r^{-1}(U)$. Thus $r|I_k$ is a path in U from $r(q_k)$ to $r(p_k)$, which is impossible since $r(q_k)$ and $r(p_k)$ lie in different path components of U.

CHAPTER III

PSEUDOCONES OVER METRIC SPACES

In this chapter, certain relationships between pseudocones and compactifications of closed subsets of A are studied. Many of the techniques used rely upon the metric structure, but some of the questions involved are applicable to the non-metric case. These are treated briefly in Chapter V.

Definition 1: Let B be a closed noncompact subset of A and let B' be a compactification of B. Identify B with its image in B'. Let $P(X)$ be a pseudocone. B' *extends to* $P(X)$ if and only if there exists an embedding $f: B' \to P(X)$ such that $f(B' - B) = X$ and $f|B: B \to A$ is the inclusion map. If such a pseudocone exists but is not specified, B' will be said merely *to extend to a pseudocone*. If B' extends to $P(X)$, then $P(X)$ *naturally contains* B'.

The following result will be of use in some of the development.

Lemma 1: *Let* B *be a closed unbounded subset of* A *and let* $B' = B \cup Y$ *where both* B' *and* Y *are compact and* $B \cap Y = \emptyset$. *Then* B' *is metrizable if and only if* Y *is*.

Proof: If B' is metrizable, Y is since $Y \subseteq B'$.

If Y is metrizable, Y is separable since it is compact. Thus there exists an embedding f_0 of Y into the Hilbert Cube

C. Since C is an absolute retract, f_0 extends to a map $f_1\colon B' \to C$. Define $f_2\colon B' \to I$ by:

$$\begin{aligned} f_2(x) &= \frac{1}{x} \quad \text{if} \quad x \in B \\ &= 0 \quad \text{if} \quad x \in Y \end{aligned}$$

Let $f\colon B' \to C \times I$ be defined by

$$f(x) = (f_1(x),\ f_2(x)).$$

Then f is an embedding and B' is metrizable since $C \times I$ is.

Of particular interest is the case $B = Z^+$, the positive integers. The principal questions investigated are:

1) Given a compactification X of Z^+, when does X extend to a pseudocone?

2) Given a pseudocone S, when does S naturally contain a compactification of a copy of Z^+?

The first question is answered as follows:

Theorem 1: Let X be a metric compactification of Z^+ and let $i\colon Z^+ \to X$ be the natural embedding. A necessary and sufficient condition that X extend to a pseudocone is that $X - i(Z^+)$ be connected.

Proof: The necessity is clear from Lemma 2, Chapter I.

To prove the sufficiency, identify Z^+ with $i(Z^+)$, let $Y = X - Z^+$, and suppose Y is connected. Let $f\colon X \to C$ be an embedding where C denotes the Hilbert Cube. Let d be the usual metric on C as a subset of Hilbert space, and let $a_n = d(f(n), f(Y))$. Then $\lim_{n\to\infty} a_n = 0$. Set

$$U_n = \{p \in C \colon d(p, f(Y)) < 2 \max(a_n, a_{n+1})\}$$

For each $n \in Z^+$, let $\omega_n \colon [n, n+1] \to U_n$ be a map such that $\omega_n(n) = f(n)$ and $\omega_n(n+1) = f(n+1)$. This is possible since the convexity of C and the connectedness of Y imply that each U_n is path connected. Let

$$\omega = \bigcup_{i=1}^{\infty} \omega_i$$

Then $\omega \colon A \to C$ is a continuous map. Define $g \colon A \cup Y \to C \times I$ by:

$$g(x) = (\omega(x), \tfrac{1}{x}) \quad \text{if} \quad x \in A$$
$$= (f(x), 0) \quad \text{if} \quad x \in Y$$

Then $g|X$ is an embedding since ω and f agree on Z^+. It is also clear that $g|A$ is an embedding and that the complement of $g(A)$ in the closure of $g(A)$ lies in $C \times \{0\}$. But if $p \in C \times \{0\}$ and $p \notin f(Y) \times \{0\}$ there is an n such that $p \notin \overline{U}_k \times \{0\}$ for every $k \geq n$. Let $j \geq n$ be an integer such that $a_j \geq a_k$ for every $k \geq n$. Such a j exists since the a_i's tend to zero. Then $p \notin \overline{U}_j$ and

$$(C - \overline{U}_j) \times [0, \tfrac{1}{n})$$

is a neighborhood of p missing $g(A)$. But Y is contained in the closure of $g(Z^+)$ and hence in that of $g(A)$. Therefore $g(A \cup Y)$ is the desired pseudocone.

The central role of connectedness in the preceeding theorem makes the following result of some interest.

Proposition 1: *Let $\langle M,d\rangle$ be a compact metric space and let $\langle a_i\rangle_{i=1}^{\infty}$ be a sequence of points in M. If $\lim_{i\to\infty} d(a_i,a_{i+1}) = 0$, then the set F of cluster points of $\langle a_i\rangle$ is connected.*

Proof: Suppose not. Let $F = K \cup L$ sep. Since F is closed, K and L are compact and $d(K,L) > 0$. Let $d(K,L) = 3\varepsilon$. Set

$$U = \{x \in M: d(x,K) < \varepsilon\}$$

$$V = \{x \in M: d(x,L) < \varepsilon\}$$

Then $U \cup V$ contains all but finitely many of the a_i and each of U and V contain infinitely many of the a_i. Choose ℓ such that, for each integer $r \geq \ell$, $d(a_r,a_{r+1}) < \varepsilon$ and $a_r \in U \cup V$. Without loss of generality assume $a_\ell \in U$. Then there exists a smallest integer n greater than ℓ such that $a_n \in V$. Thus, $d(a_{n-1},a_n) > \varepsilon$, contradicting the choice of ℓ. Hence F is connected and the proof is complete.

Corollary 1: *Let $Z^+ \cup Y$ be a metric compactification of Z^+ with metric d, where $Z^+ \cap Y = \emptyset$. Suppose $d(n,n+1)$ tends to zero as n tends to infinity. Then Y is a compact metric continuum.*

Proposition 1 also has the following peripheral corollary:

Corollary 2: Let $\langle M,d \rangle$ be a totally disconnected compact metric space and let $\langle a_i \rangle_{i=1}^{\infty}$ be a sequence from M. Then $\langle a_i \rangle$ converges if and only if $\lim_{i\to\infty} d(a_i, a_{i+1}) = 0$.

The second question if also readily answered.

Theorem 2: Let X be a compact metric continuum and P(X) a pseudocone over X. Then P(X) naturally contains a compactification of a copy of Z^+.

Proof: Let $\{a_i\}_{i=1}^{\infty}$ be a countable dense subset of X. For each $i \in Z^+$ let $\langle b_k^i \rangle_{k=1}^{\infty}$ be a sequence of points from A such that:

1) $\lim_{k\to\infty} b_k^i = a_i$ in P(X)

2) For each i, $b_1^i > i$

3) For each i the sequence $\langle b_k^i \rangle$ is monotone increasing.

Then $\{1\} \cup \{b_k^i: k,i \in Z^+\}$ is a copy of Z^+ in A with the desired properties.

The metric structure of the pseudocones in question allows an even stronger condition to be placed on the copy of Z^+ considered.

Lemma 2: Let Y be a compact metric continuum and let P(Y) be a pseudocone over Y. Let d be a metric on P(Y). Then P(Y) naturally contains a compactification of a copy $\{a_i\}_{i=1}^{\infty}$

<u>of</u> Z^+ <u>with the property that</u>

$$\lim_{n\to\infty} \text{dia}\,([a_n,a_{n+1}]) = 0,$$

<u>where</u>, dia $([a_n,a_{n+1}])$ <u>denotes the diameter of the interval</u> $[a_n,a_{n+1}]$ <u>with respect to the metric</u> d.

<u>Proof</u>: By <u>Theorem 2</u>, there exists a copy

$$C = \{c_i\}_{i=1}^{\infty}$$

of Z^+ in A such that

1) $c_1 = 1$
2) $c_i > c_j$ if and only if $i > j$
3) $\overline{C} = C \cup Y$, where $\overline{C}$ denotes the closure of

C in $P(Y)$.

Parameterize A so that $C_n = n$ for each $n \in Z^+$.

The set $\{a_i\}_{i=1}^{\infty}$ is constructed as follows:

Define $a_1 = 1$.

Suppose a_{n-1} has been defined. Define k_n to be the largest element of the set

$$\{\ell \in Z^+ : \ell \le a_{n-1}\}$$

Set

$$D_n = \{y \in A : d(y,a_{n-1}) \le \frac{1}{k_n}\}$$

$$p_n = \sup\{t : [a_{n-1},t] \subseteq D_n\}$$

Now define $a_n = \min(p_n, k_n + 1)$.

Then $\langle a_i \rangle_{i=1}^{\infty}$ is an increasing sequence of elements of A. To show that $\langle a_i \rangle$ is unbounded, suppose not. Let r be the smallest integer greater than every a_i. Let $a = \sup\{a_i\}$. Then

$$a \in (r-1, r] \quad \text{and} \quad a = \lim_{i\to\infty} a_i$$

By the definition of convergence and the Cauchy criterion, there exists an integer n_0 such that for every $n > n_0$, both $a_n > r-1$ and $d(a_n, a_{n+1}) < \frac{1}{r-1}$. But $a_{n+1} = p_{n+1}$, and $k_{n+1} = r-1$, and if $d(p_{n+1}, a_n) < \frac{1}{r-1}$, there is a closed neighborhood

$$S = [p_{n+1} - \varepsilon,\ p_{n+1} + \varepsilon]$$

such that $S \subseteq D_{n+1}$. Then

$$[a_n, p_{n+1} + \varepsilon] \subseteq D_{n+1},$$

contradicting the choice of p_{n+1}.

Therefore $\langle a_i \rangle$ is an unbounded sequence. Let $B = \{a_i\}_{i=1}^{\infty}$. Since B contains Z^+, $\overline{B}$ contains Y. Since B is closed in A, $\overline{B} = B \cup Y$. It is clear from the construction that $\lim_{n\to\infty} \text{dia}\,([a_n, a_{n+1}]) = 0$, so the proof is complete.

The preceeding results will now be applied to the proof of the following Theorem.

<u>Theorem 3</u>: <u>Let</u> X <u>be a Peano continuum</u>, P(Y) <u>a pseudocone, and</u> f: $Y \to X$ <u>a continuous map</u>. <u>Then</u> X <u>is a retract of</u> $P(Y) \cup_f X$.

Proof: Let $P(Y) \cup_f X = R$. Then R is metrizable by Lemma 1. Let d be a metric on R. The closure of A in R is a pseudocone, and by Lemma 2, A contains a set $\{a_i\}_{i=1}^{\infty}$ such that:

1) $a_1 = 1$
2) $a_i > a_j$ if and only if $i > j$
3) $\{a_i\}$ is unbounded
4) $\lim_{n\to\infty} \text{dia}\,([a_n, a_{n+1}]) = 0.$

Parameterize A so that $a_n = n$ for each $n \in Z^+$.

Now let $\{b_n\}_{n=1}^{\infty}$ be a set of points in X such that for each $n \in Z^+$,

$$d(n, b_n) = d(n, X).$$

This is possible since X is compact.

Note that:

$$\begin{aligned} 0 &\le \lim_{n\to\infty} d(b_n, b_{n+1}) \\ &\le \lim_{n\to\infty} d(b_n, n) + \lim_{n\to\infty} d(n, n+1) + \lim_{n\to\infty} d(b_{n+1}, n+1) \\ &= 0 \end{aligned}$$

And thus $\lim_{n\to\infty} d(b_n, b_{n+1}) = 0$.

Now X is uniformly locally arcwise connected, and thus, given $\varepsilon > 0$, there exists a $\delta > 0$ such that if $x, y \in X$ and $d(x,y) < \delta$, then there exists a path $\omega\colon I \to X$ from x to y

such that $\text{dia}(\omega(I)) < \varepsilon$. (For brevity this will henceforth be written "a path of diameter $< \varepsilon$".)

For each $n \in Z^+$ let

$$R_n = \{t > 0\text{: whenever } x,y \in X \text{ and } d(x,y) \le d(b_n,b_{n+1}), \text{ there exists a path from } x \text{ to } y \text{ of diameter less than } t\},$$

and define $\varepsilon_n = 2 \inf R_n$. Then from uniform local arcwise connectedness and the fact that $d(b_n,b_{n+1})$ tends to zero it follows that

$$\lim_{n\to\infty} \varepsilon_n = 0$$

Now, for each $n \in Z^+$, let $r_n\colon [n,n+1] \to X$ be a path of diameter less than ε_n such that $r_n(n) = b_n$ and $r_n(n+1) = b_{n+1}$.

Let $r_0 = \bigcup_{i=1}^{\infty} r_i$. Then r_0 is a continuous map from A to X.

Define $r\colon R \to X$ by

$$\begin{aligned} r(x) &= r_0(x) \quad \text{if } x \in A \\ &= x \quad\quad\ \text{if } x \in X \end{aligned}$$

Since $r|A$ and $r|X$ are continuous, to prove the continuity of r it suffices to prove that if $\langle p_i\rangle_{i=1}^{\infty}$ is a sequence of points from A converging to $p \in X$, then $\lim_{i\to\infty} r(p_i) = p$ also. To show this, for each positive integer n let q_n be the largest integer less than or equal to p_n. Then

$$\lim_{i\to\infty} d(q_i,p_i) = 0$$

and

$$\lim_{i\to\infty} d(q_i,b_{q_i}) = 0$$

But $p_i \in [q_i,q_i+1]$ and $\lim_{i\to\infty} \text{dia}\,(r[q_i,q_i+1]) = 0$. Therefore $\lim_{i\to\infty} d(r(p_i),r(q_i)) = \lim_{i\to\infty} d(r(p_i,b_{q_i}) = 0$ since $r(q_i) = b_{q_i}$.
Thus

$$\begin{aligned} 0 &\le \lim_{i\to\infty} d(p_i,r(p_i)) \\ &\le \lim_{i\to\infty} d(p_i,q_i) + \lim_{i\to\infty} d(q_i,b_{q_i}) + \lim_{i\to\infty} d(b_{q_i},r(p_i)) \\ &= 0 \end{aligned}$$

or

$$\lim_{i\to\infty} d(p_i,r(p_i)) = 0$$

and thus

$$\lim_{i\to\infty} r(p_i) = \lim_{i\to\infty} p_i = p$$

and the proof is complete.

Corollary 3: Let X be a Peano continuum and $P(X)$ a pseudocone over X. Then X is a retract of $P(X)$.

Proof: Let f be the identity map on X.

Corollary 4: Let X be a Peano continuum and $P(X)$ a pseudocone over X. Then there exists an embedding $f: P(X) \to X \times I$ such that $f(X) = X \times \{0\}$.

Proof: Clear from Lemma 1, Chapter II.

CHAPTER IV

A GENERALIZATION OF PEANO CONTINUA

In view of the above results on dense path components of bases of pseudocones, it is natural to ask what other properties of continua related to path connectedness can be examined in terms of pseudocones.

Definition 1: A pseudocone $P(X)$ is said to be even if and only if there exists a retraction $r: P(X) \to X$ and a continuous map $h: A \to A$ such that, for each $t \in A$, $h(t) > t$ and $r(t) = r \circ h(t)$.

Lemma 1: *Let* $h: A \to A$ *be a continuous map such that, for each* $t \in A$, $h(t) > t$. *Then* $h(A) = [p,\infty)$ *for some* $p \in A$.

Proof: It is clear that $h(A)$ is an unbounded interval. Let $p = \inf h(A)$ and let $q > p$. Then if $t > q$, $h(t) > q$, so that $p = \inf h[1,q]$. Since $[1,q]$ is compact, $p \in h[1,q] \subseteq h(A)$, and the proof is complete.

The Hahn-Mazurkiewicz Theorem now yields a characterization of Peano continua in terms of pseudocones.

Theorem 1: *Let* X *be a topological space.* X *is a Peano continuum if and only if there exists an even pseudocone over* X.

Proof: Suppose X is a Peano continuum. Let $f_0: I \to X$ be a continuous surjection such that $f_0(0) = f_0(1)$. Such a map exists by the Hahn-Mazurkiewicz Theorem. Define $f: A \to X \times I$ by

$$f(t) = (f_0(t-[t]), \tfrac{1}{t})$$

where $[t]$ denotes the greatest integer function of t. Set $P(X) = X \times \{0\} \cup f(A)$. To see that $P(X)$ is an even pseudocone, let $r: P(X) \to X$ be the restriction to $P(X)$ of the projection $p: X \times I \to X \times \{0\}$ and define $h: A \to A$ by $h(t) = 1+t$.

Conversely, suppose $P(X)$ is an even pseudocone. Let $r: P(X) \to X$ and $h: A \to A$ be the given maps. $r(A)$ is dense in X since A is dense in $P(X)$. Let $h(A) = [p,\infty)$. Suppose $x \in r(A)$ and let $y = \inf\, (r|A)^{-1}(x)$. Since $(r|A)^{-1}(x)$ is closed in A, $y \in (r|A)^{-1}(x)$. It follows that $y \in [1,p]$, for if not, there exists $t \in A$ such that $h(t) = y$. Then

$$r(t) = r \circ h(t) = r(y).$$

This is a contradiction to the choice of y, since $t < y$. Thus,

$$x \in r([1,p]),$$

and $r([1,p]) = r(A)$. Therefore $r([1,p])$ is dense in X. But $r([1,p])$ is compact and hence closed. Thus $r([1,p]) = X$, and X is a Peano continuum by the Hahn-Mazurkiewicz Theorem.

The preceeding proof rests heavily upon the fact that a Hausdorff space is a Peano continuum if and only if it is a continuous image of a closed interval. A generalization of this notion is the following:

Definition 2: A compact metric space X is an almost-Peano continuum if there is a continuous surjection $f: A \to X$.

This is a proper generalization of Peano continua since the Warsaw circle is a continuous image of A.

A concept related to this is that of almost local connectedness.[2] A topological space is almost locally connected if and only if every open set contains a connected open set. This property is investigated below in some detail and leads to a characterization of almost-Peano continua.

Lemma 2: *A pseudocone is almost locally connected.*

Proof: Every open set in a pseudocone meets A and hence contains an open interval.

If X is an almost-Peano continuum and $f: X \to Y$ is a continuous surjection where Y is a compact Hausdorff space, then clearly Y is an almost-Peano continuum. However, the property of being almost locally connected is not preserved by continuous maps in general.

Example 1: Let X be the cone over the Cantor set. By Corollary 2, Chapter II, there exists a pseudocone S over X such that X is a retract of S. Let $r: S \to X$ be a retraction. Then r is a continuous surjection, and S is almost locally connected by Lemma 2. However, X is not almost locally connected since X with its vertex deleted contains no connected open set.

The following two results establish some conditions under which almost local connectedness is preserved.

Proposition 1: Let X and Y be topological spaces and let $f: X \to Y$ be a continuous open surjection. If X is almost locally connected, so is Y.

Proof: Let U be an open subset of Y. $f^{-1}(U)$ is open in X and hence contains a connected open set V. Then $f(V)$ is an open connected subset of U in Y.

Proposition 2: Let S be a locally compact Hausdorff space and suppose S is a countable union of closed, almost locally connected subsets. Then S is almost locally connected.

Proof: Suppose $S = \bigcup_{i=1}^{\infty} W_i$ where W_i are closed and almost locally connected. Let $U \subseteq S$ be open. Since S is locally compact, U contains an open set V with compact closure.

Then

$$\overline{V} = \bigcup_{i=1}^{\infty} (\overline{V} \cap W_i)$$

and thus there exists some j such that the interior with respect to $\overline{V}$ of $\overline{V} \cap W_j$ is nonvoid. Let this interior be denoted by Y. Then Y is open in $\overline{V}$ and hence there exists a set Y_0 open in S such that $Y = \overline{V} \cap Y_0$. Set $Y_1 = Y_0 \cap V$. Y_1 is open in S and $Y_1 \subseteq \overline{V} \cap W_j$. Thus Y_1 is open in W_j, and since W_j is almost locally connected, Y_1 contains a connected set Y_2 which is open in Y_1 and hence in S. Then $Y_2 \subseteq U$ and the proof is complete.

Proposition 2 suggests the following concept.

Definition 3: A locally compact Hausdorff space X is *completely almost locally connected*, or *completely alc*, if and only if X is a countable union of compact locally connected subsets.

Lemma 2: *A locally compact Hausdorff space X is completely alc if and only if X is a countable union of compact connected locally connected subsets.*

Proof: Let $X = \bigcup_{i=1}^{\infty} W_i$. Since each W_i is locally connected, its components are both open and closed in W_i. Since the set of components of W_i cover W_i and no proper subcollection does, W_i has at most finitely many components. Since the components of W_i are closed they are compact and since they are open they are locally connected. Then the set of components of the W_i is the desired countable collection of continua. The converse is clear.

In particular, then, a locally compact metric space is completely alc if and only if it is a countable union of Peano continua. This fact leads to the desired characterization of almost-Peano continua.

Lemma 3: *Let X and Y be locally compact Hausdorff spaces and suppose $f\colon X \to Y$ is a continuous surjection. Then if X is completely alc, so is Y.*

Proof: If $X = \bigcup_{i=1}^{\infty} W_i$, where each W_i is a compact locally connected continuum, then $Y = \bigcup_{i=1}^{\infty} f(W_i)$ and each $f(W_i)$ is a compact locally connected continuum.

<u>Theorem 2</u>: <u>Let</u> X <u>be a compact metric space</u>. <u>Then the following four conditions are equivalent</u>.

1) X <u>is an almost-Peano continuum</u>.

2) X <u>is completely alc and path connected</u>.

3) $X = \bigcup_{i=1}^{\infty} W_i$ <u>where each</u> W_i <u>is a Peano continuum and</u> $W_i \subseteq W_{i+1}$ <u>for each</u> i.

4) <u>There exists a pseudocone</u> $P(X)$ <u>and a retraction</u> $r: P(X) \to X$ <u>such that</u> $r|A: A \to X$ <u>is a surjection</u>.

<u>Proof</u>: 1) implies 2) Since path connectedness is preserved by continuous maps and $A = \bigcup_{i=1}^{\infty} [1,i]$, X is completely alc and path connected by <u>Lemma 3</u>.

2) implies 3) Let $X = \bigcup_{i=1}^{\infty} M_i$ where each M_i is a Peano continuum. For each i, let $a_i \in M_i$ and let $f_i: I \to X$ be a path from a_i to a_{i+1}. Set

$$W_i = \bigcup_{j=1}^{i} (M_j \cup f_j(I)).$$

Then each W_i is a Peano continuum, $X = \bigcup_{i=1}^{\infty} W_i$, and $W_i \subseteq W_{i+1}$ for each i.

3) implies 4) Let $X = \bigcup_{i=1}^{\infty} W_i$ where for each i, W_i is a Peano continuum and $W_i \subseteq W_{i+1}$. Let $p \in W_1$. For each i, let $f_i: [i,i+1] \to W_i$ be a continuous surjection such that $f_i(i) = f_i(i+1) = p$. Set

$$f_0 = \bigcup_{i=1}^{\infty} f_i$$

Then f_0 maps A onto X. Define $f\colon A \to X \times I$ by:

$$f(t) = (f_0(t), \tfrac{1}{t}).$$

Identify X with $X \times \{0\}$ and A with $f(A)$ and set $P(X) = X \cup A$. Let $p\colon X \times I \to X$ be the projection and define $r\colon P(X) \to X$ to be $p|P(X)$. Then $r|A = f_0$ so that $r|A$ is a surjection as required.

4) implies 1) This is clear since $r|A\colon A \to X$ is the required map.

CHAPTER V

THE STONE-ČECH COMPACTIFICATION OF A

Let X be a completely regular topological space. The following notation will be used: $\beta(X)$ is the Stone-Čech compactification of X and X is considered to be a subset of $\beta(X)$. If Y is a compact Hausdorff space and $f: X \to Y$ is a continuous map, f^* is the unique extension of f to $\beta(X)$. $\beta(X) - X$ is denoted X^*.

The following lemmas will be useful in some of the development.

Lemma 1: Let $P(X)$ be a pseudocone and let U be an open set which meets X. Then $U \cap A$ is unbounded.

Proof: Suppose not. Then $U \cap A \subseteq [1,x]$ for some $x \in A$. Thus,

$$U \cap (P(X) - [1,x])$$

is a nonempty open subset of $P(X)$ which misses A. This is impossible since A is dense in $P(X)$.

Lemma 2: Let $P(X)$ be a pseudocone and let $i: A \to P(X)$ be the inclusion map. If $f: A \to A$ is a continuous map such that $f(t) \to \infty$, as $t \to \infty$, then $i \circ f^*(A^*) = X$.

Proof: $(i \circ f)^*(\beta(A))$ is a closed subset of $P(X)$ containing $(i \circ f)(A)$. Since

$$(i\circ f)(A) = i[p,\infty)$$

for some $p \in A$ and X is contained in the closure of $i[p,\infty)$, it follows that

$$X \subseteq (i\circ f)^*(\beta(A)).$$

But

$$X \cap i[p,\infty) = \phi$$

and

$$\begin{aligned}(i\circ f)^*(\beta(A)) &= (i\circ f)(A) \cup (i\circ f)^*(A^*)\\ &= i[p,\infty) \cup (i\circ f)^*(A^*).\end{aligned}$$

Thus

$$X \subseteq (i\circ f)^*(A^*).$$

To prove $(i\circ f)^*(A^*) \subseteq X$, suppose $x \in A^*$ and $(i\circ f)^*(x) \in i(A)$. Let U be an open set in $\beta(A)$ such that:

$$x \in U \subseteq \overline{U} \subseteq (i\circ f)^{*^{-1}}(i(A))$$

Then $i^{-1}((i\circ f)^*(\overline{U}))$ is a compact subset of A and hence bounded. Therefore there exists a $y \in A$ such that for every $t > y$,

$$f(t) > \sup i^{-1}((i\circ f)^*(U)).$$

But for $t \in A$,

$$\begin{aligned}f(t) &= i^{-1}((i\circ f)(t))\\ &\ i^{-1}((i\circ f)^*(t))\end{aligned}$$

and thus for $t > y$,

$$i^{-1}((i \circ f)^*(t)) > \sup i^{-1}((i \circ f)^*(U))$$

Therefore $U \cap (y,\infty) = \emptyset$, and by Lemma 1, $U \cap A^* = \emptyset$. But $x \in U \cap A^*$, a contradiction, and the proof is complete.

The importance of the Stone-Čech compactification of A to the study of pseudocones stems from the following result.

Proposition 1: *Let X be a Hausdorff space. Then there exists a pseudocone over X if and only if X is a continuous image of A^*.*

Proof: Suppose $P(X)$ is a pseudocone over X. Let $i: A \to P(X)$ be the inclusion map. Then $(i \circ 1_A)^*|A^*$ maps A^* onto X by Lemma 2.

Conversely, if $f: A^* \to X$ is a continuous surjection, then

$$\beta(A) \cup_f X$$

is the desired pseudocone.

An examination of continuous images of other spaces associated with $\beta(A)$ may also be of interest. In what follows, a simple characterization of continuous images of $\beta(A)$ is given, and some information is obtained concerning continuous images of nondegenerate subcontinua of A^*. A complete characterization in the latter case appears to be difficult.

Proposition 2: Let X be a compact Hausdorff space. Then the following three conditions are equivalent:

1) X is a continuous image of $\beta(A)$.

2) There exists a pseudocone S of which X is a continuous image.

3) X has a separable dense path component.

Proof: 1) implies 2) This is clear since $\beta(A)$ is a pseudocone.

2) implies 3) S has a separable dense path component. Thus by Lemma 2, Chapter II, so does X.

3) implies 1) Let D be a separable dense path component of X and let $\{a_i\}_{i=1}^{\infty}$ be a countable dense subset of D. Since D is path connected, there exists a map $f: A \to D$ such that for each positive integer n, $f(n) = a_n$. Then $f(A)$ is dense in D and hence in X. Thus $f^*(\beta(A))$ is dense in X, and since $f^*(\beta(A))$ is compact, $f^*(\beta(A)) = X$, and the proof is complete.

Lemma 3: Suppose X is a separable compact Hausdorff continuum. Then there exists a subcontinuum M of $X \times I$ such that M is irreducible between two points and $X \times \{0\} \subseteq M$.

Proof: Let $\langle a_i \rangle_{i=1}^{\infty}$ be a sequence of points in X such that the range of $\langle a_i \rangle$ is dense in X. Let $\{L_i\}_{i=1}^{\infty}$ be a collection of subcontinua of X such that for each i, L_i is irreducible between a_i and a_{i+1}. In $X \times I$ define $\{W_i\}_{i=1}^{\infty}$

and $\{b_i\}_{i=1}^{\infty}$ as follows:

$$W_{2k-1} = L_k \times \{\tfrac{1}{k}\}$$

$$W_{2k} \quad = \{a_{k+1}\} \times [\tfrac{1}{k+1}, \tfrac{1}{k}]$$

$$b_{2k-1} = (a_k, \tfrac{1}{k})$$

$$b_{2k} \quad = (a_{k+1}, \tfrac{1}{k})$$

for each positive integer k. Then for each i, W_i is irreducible between b_i and b_{i+1}, and $W_i \cap W_{i+1} = \{b_{i+1}\}$. Set

$$M = \bigcup_{i=1}^{\infty} W_i \cup (X \times \{0\})$$

Then M is the closure in $X \times I$ of $\bigcup_{i=1}^{\infty} W_i$ and hence a continuum.

To prove that M is irreducible between two points, let $x \in X \times \{0\}$ and suppose F is a proper closed subset of M containing both b_1 and x. It remains to be shown that F is not connected. There are two cases to be considered:

1) If for some $i > 1$, $b_1 \notin F$, then

$$\bigcup_{j=1}^{i-1} (W_j \cap F)$$

is a nonvoid proper closed and open subset of F.

2) If each $b_i \in F$, then since $\bigcup_{i=1}^{\infty} W_i$ is dense in M, there exists some W_i which meets $M - F$. Then $F \cap W_i$ can be expressed as the disjoint union of two closed sets, B and C, such that $b_i \in B$ and $b_{i+1} \in C$. Then

$$\bigcup_{j=1}^{i-1} (W_j \cap F) \cup B$$

is a proper nonvoid closed and open subset of F.

Thus F is not connected and the proof is complete.

Corollary 1: Suppose X is a separable compact Hausdorff continuum. There exists a compact Hausdorff continuum M containing X such that X is a retract of M and M is irreducible between two points.

Proof: Identify X with $X \times \{0\}$ in $X \times I$, and let M be as in Lemma 3. A retraction is obtained by restriction to M of the projection $p: X \times I \to X$.

Corollary 2: Let X be a compact metric continuum. Then there exists a compact metric continuum M containing X such that X is a retract of M and M is irreducible between two points.

Proof: Let M be as in Corollary 1. M is metric since $X \times I$ is.

Lemma 4: Let U and V be unbounded open subsets of A such that $\overline{U} \cap \overline{V} = \emptyset$ and $\inf U < \inf V$. Then there exist sequences $\langle p_n \rangle_{n=1}^{\infty}$, $\langle q_n \rangle_{n=1}^{\infty}$, $\langle r_n \rangle_{n=1}^{\infty}$ and $\langle s_n \rangle_{n=1}^{\infty}$ such that:

1) For each positive integer n, $p_n < q_n < r_n < s_n < p_{n+1}$.
2) $U \subseteq \bigcup_{n=1}^{\infty} [p_n, q_n]$ and $V \subseteq \bigcup_{n=1}^{\infty} [r_n, s_n]$.

Indication of proof: Define

$$p_1 = \inf U$$

$$q_1 = \sup \{t \in U: [p_1,t] \cap V = \emptyset\}$$

$$r_1 = \inf V$$

$$s_1 = \sup \{t \in V: [r_1,t] \cap U = \emptyset\}$$

Then p_n, q_n, r_n, and s_n are defined inductively, replacing U and V with $U \cap [s_{n-1},\infty)$ and $V \cap [s_{n-1},\infty)$ respectively, in the definition of p_n and r_n and replacing p_1 and r_1 by p_n and r_n respectively in the definition of q_n and s_n. Verification of the properties 1) and 2) is then an elementary exercise in real analysis.

Lemma 5: *Suppose M is a metric continuum irreducible between two points, a and b, and suppose W is a subcontinuum of A^* and x and y are distinct elements of W. Then there exists a continuous surjection $g: W \to M$ such that $g(x) = a$ and* $g(y) = b$.

Proof: By Corollary 1, Chapter I, there exists a pseudo-cone $P(M)$. Let $i: A \to P(M)$ denote the inclusion map. By Lemma 1, Chapter III, $P(M)$ is metrizable. Thus there exist sequences $\langle a_n \rangle_{n=1}^{\infty}$ and $\langle b_n \rangle_{n=1}^{\infty}$ of elements of A such that:

1) For each positive integer n, $a_n < b_n < a_{n+1}$.
2) $\lim_{n\to\infty} i(a_n) = a$ and $\lim_{n\to\infty} i(b_n) = b$.

Now let U and V be open sets in $\beta(A)$ such that

1) $x \in U$ and $y \in V$

2) $\overline{U} \cap \overline{V} = \emptyset$.

3) $\inf (U \cap A) < \inf (V \cap A)$.

By Lemma 1, $U \cap A$ and $V \cap A$ are unbounded, and by Lemma 4 there exists sequences $\langle p_n \rangle_{n=1}^{\infty}$, $\langle q_n \rangle_{n=1}^{\infty}$, $\langle r_n \rangle_{n=1}^{\infty}$, and $\langle s_n \rangle_{n=1}^{\infty}$ such that:

1) For each positive integer n, $p_n < q_n < r_n < s_n < p_{n+1}$.

2) $U \cap A \subseteq \bigcup_{i=1}^{\infty} [p_i, q_i]$ and $V \cap A \subseteq \bigcup_{i=1}^{\infty} [r_i, s_i]$.

Define $f\colon A \to A$ as follows:

Set
$$f(1) = 1$$
$$f(t) = a_n \quad \text{for} \quad t \in [p_n, q_n]$$
$$f(t) = b_n \quad \text{for} \quad t \in [r_n, s_n]$$

and extend f linearly to each of the intervals $[1, p_1]$, $[q_n, r_n]$, and $[s_n, p_{n+1}]$. Then since the sequences $\langle p_n \rangle$, $\langle q_n \rangle$, $\langle r_n \rangle$, and $\langle s_n \rangle$ are unbounded, $f(t)$ is defined for each $t \in A$ and $f(t) \to \infty$ as $t \to \infty$. Therefore, by <u>Lemma 2</u>, $(i \circ f)^*(A^*) = M$, and $(i \circ f)^*(W)$ is a subcontinuum of M.

Now $(i \circ f)^{*-1}(\{i(a_j)\}_{j=1}^{\infty} \cup \{a\})$ is a closed set containing $U \cap A$, and hence containing the closure in $\beta(A)$ of $U \cap A$. Let K be any open set in $\beta(A)$ containing x. Then $K \cap U$ is an open set containing x, and since A is dense in $\beta(A)$, $K \cap A \cap U = \emptyset$. Therefore, x belongs to the closure of $U \cap A$ in $\beta(A)$, and hence

$$(i \circ f)^*(x) \in \{i(a_j)\}_{j=1}^{\infty} \cup \{a\}$$

But $(i \circ f)^*(x) \in M$ and $\{i(a_j)\}_{j=1}^{\infty} \cap M = \emptyset$. Thus $(i \circ f)^*(x) = a$. By a parallel argument, $(i \circ f)^*(y) = b$. Then $(i \circ f)^*(W)$ is a subcontinuum of M containing both a and b, and since M is irreducible between a and b, $(i \circ f)^*(W) = M$. Setting $g = (i \circ f)^* | W$ completes the proof.

Lemma 6: Suppose S is a pseudocone and W is a nondegenerate subcontinuum of A^*. Then there exists a continuous surjection $h: W \to S$.

Proof: Let x and y be distinct elements of W and let U and V be open sets in $\beta(A)$ such that:

1) $x \in U$ and $y \in V$.
2) $\overline{U} \cap \overline{V} = \emptyset$.
3) $\inf (U \cap A) < \inf V \cap A$.

Then let $\langle p_k \rangle_{k=1}^{\infty}$, $\langle q_k \rangle_{k=1}^{\infty}$, $\langle r_k \rangle_{k=1}^{\infty}$, and $\langle s_k \rangle_{k=1}^{\infty}$ be sequences from A such that for each positive integer k, $p_k < q_k < r_k < s_k < p_{k+1}$, and

$$U \cap A \subseteq \bigcup_{k=1}^{\infty} [p_k, q_k]$$

$$V \cap A \subseteq \bigcup_{k=1}^{\infty} [r_k, s_k]$$

Define $f: A = A$ as follows:

set $f(t) = 1$ for $t \leq p_1$

$f(t) = 1$ for $t \in [p_k, q_k]$

$f(t) = k$ for $t \in [r_k, s_k]$

and extend f linearly to each of the intervals $[q_n, r_n]$ and $[s_n, p_{n+1}]$.

Now let $j: A \to S$ be the inclusion map, and for simplicity of notation, set $g = (j \circ f)^*$. Then $g^{-1}(j(1))$ is a closed set containing $U \cap A$ and hence $g(x) = j(1)$. Let B denote the closure of $j(Z^+)$ in S. Then $g^{-1}(B)$ is a closed set containing $V \cap A$ and hence $g(y) \in B$. But if $g(y) = j(n)$ for some $n \in Z^+$, $g^{-1}(j[1,n+1)) \cap V$ is an open subset of $\beta(A)$ containing y, and by Lemma 1, $g^{-1}(j[1,n+1)) \cap V \cap A$ is unbounded, which is impossible since

$$g^{-1}(j[1,n+1)) \cap V \cap A \subseteq [1,s_n].$$

Therefore $g(y)$ belongs to the base of S, and by Lemma 1, Chapter I, S is irreducible between $g(x)$ and $g(y)$. Since $g(W)$ is a subcontinuum of S containing both $g(x)$ and $g(y)$, $g(W) = S$ and $g|W$ is the desired map.

Theorem 1: *Let X be a compact Hausdorff continuum and let W be a nondegenerate subcontinuum of A^*. If X is metrizable or if X has a separable dense path component, then X is a continuous image of W.*

Proof: If X is metrizable, by Corollary 2, there exists a metric continuum M irreducible between two points and a continuous surjection $h: M \to X$. By Lemma 5 there exists a continuous surjection $g: W \to M$. Thus $h \circ g$ is a continuous map of W onto X.

If X has a separable dense path component, by Proposition 2 there exists a pseudocone S and a continuous surjection $h: S \to X$. By Lemma 6, there exists a continuous surjection $g: W \to S$. Then $h \circ g$ is the desired map.

Corollary 3: *If W is a nondegenerate subcontinuum of A^*, the cardinal number of W is 2^c.*

Proof: Since $\beta(A)$ embeds in I^c, the cardinal number of W is at most 2^c. Since I^c is separable and path connected, by Theorem 1 there exists a continuous surjection $f: W \to I^c$. Thus the cardinal number of W is at least 2^c.

Corollary 4: *Let X be a compact metric continuum and $f: X \to A^*$ a continuous map. Then f is a constant map.*

Proof: $f(X)$ is a subcontinuum of A^* and has cardinal number at most c. By Corollary 3, such a continuum is a single point.

The remainder of this chapter is devoted to two miscellaneous topics related to $\beta(A)$. The notion of natural containment was introduced in Chapter III and discussed there for metric pseudocones. In the following two results, $\beta(A)$ is contrasted with certain pseudocones over continua with separable dense path components with respect to this property.

Proposition 3: *Let B be a closed subset of A. Then $\beta(A)$ naturally contains $\overline{B}$ if and only if $A - B$ is bounded.*

Proof: Suppose $A - B$ is bounded. Then let $t \in A$ such that:

$$[t,\infty) \subseteq B.$$

Now A^* lies in the closure of $[t,\infty)$ and hence in $\overline{B}$. But since B is closed in A, $\overline{B} - B \subseteq A^*$. Hence $\overline{B} - B = A^*$.

Conversely, suppose $A - B$ is unbounded. Let $\langle a_n \rangle_{n=1}^{\infty}$ be a sequence of points in $A - B$ such that for each positive integer n, $a_n > n$. Then $\{a_n\}_{n=1}^{\infty}$ is closed in A. Hence by normality there exists a function

$$f: A \to I$$

such that

$$\begin{aligned} f(x) &= 1 \quad \text{for} \quad x \in B \\ &= 0 \quad \text{for} \quad x \in \{a_n\}_{n=1}^{\infty} \end{aligned}$$

Then $f^{*-1}([0,1))$ is an open set in $\beta(A)$ missing B but meeting A^*. Therefore A^* is not contained in $\overline{B}$, and the proof is complete.

Proposition 4: *Let X be a compact Hausdorff space with a separable dense path component. Then there exists a pseudocone $P(X)$ which naturally contains a compactification of Z^+.*

Proof: Let D be a dense path component of X with a countable dense subset $\{a_i\}_{i=1}^{\infty}$. Let $f: A \to X$ be a continuous function such that $f(n) = a_n$ for each positive integer n.

Define $g: A \to X \times I$ by

$$g(t) = (f(t), \frac{1}{t})$$

and set $P(X) = g(A) \cup (X \times \{0\})$.

Now let $(x,0) \in X \times \{0\}$, and let U be any open set in $X \times I$ containing $(x,0)$. Then U contains an open set of the form $V \times [0,\varepsilon)$ for some $\varepsilon > 0$ and some open set V containing x. Then

$$V - \{a_n : n < \frac{1}{\varepsilon}\} = W$$

is an open set and hence meets $\{a_n\}_{n=1}^{\infty}$. Suppose $a_i \in W$. Then $i > \frac{1}{\varepsilon}$ and hence $\frac{1}{i} < \varepsilon$ and $g(i) = (a_i, \frac{1}{i})$ belongs to $W \times [0,\varepsilon)$ and hence to U. Thus X is contained in the closure of $g(Z^+)$ and the proof is complete.

There appear to be no known examples of non-metric indecomposable continua. The following result provides such an example.

<u>Theorem 2</u>: A^* <u>is an indecomposable continuum</u>.

<u>Proof</u>: Suppose $A^* = X \cup Y$ where X and Y are proper closed subsets of A^*. It will be shown that X is not connected. Let $x \in X - Y$ and $y \in Y - X$ and let U and V be open sets in $\beta(A)$ such that:

1) $x \in U$ and $y \in V$
2) $\overline{U} \cap \overline{V} = \overline{U} \cap Y = \overline{V} \cap X = \emptyset$

Choose sequences $\langle p_i\rangle_{i=1}^{\infty}$, $\langle q_i\rangle_{i=1}^{\infty}$, and $\langle r_i\rangle_{i=1}^{\infty}$ from A as follows:

Let $p_i \in U \cap A$. Then choose $q_1 > p_1$ such that $q_1 \in V$. This is possible since $V \cap A$ is unbounded. Then choose $r_1 > q_1$ such that $(q_1, r_1) \subseteq V$. This is possible since V is open and hence q_1 lies in some open interval in V.

Proceeding inductively, suppose p_k, q_k, and r_k have been chosen for $k < n$ such that for each k:

1) $p_k \in U$.

2) The interval (q_k, r_k) is contained in V.

3) $p_k < q_k < r_k$, and if $k < n-1$, then $r_k < p_{k+1}$.

Then since $U \cap A$ is unbounded, it is possible to choose $p_n > r_{n-1}$ such that $p_n \in U$. Since $V \cap A$ is unbounded, there exists a $q_n > p_n$ such that $q_n \in V$. Since V is open, r_n may be chosen greater than q_n such that $(q_n, r_n) \subseteq V$.

Then the sequences $\langle p_n\rangle$, $\langle q_n\rangle$, and $\langle r_n\rangle$ are unbounded, for if not they have a common supremum t, and it follows $t \in \overline{U} \cap \overline{V}$, a contradiction. Define $f\colon A \to I$ as follows:

First set

$$
\begin{aligned}
f(t) &= 0 && \text{if } t \le p_1 \\
f(p_i) &= 0 && \text{if } i \text{ is odd} \\
&= 1 && \text{if } i \text{ is even} \\
f(q_i) &= 1/3 && \text{if } i \text{ is odd} \\
&= 2/3 && \text{if } i \text{ is even} \\
f(r_i) &= 1/3 && \text{if } i \text{ is even} \\
&= 2/3 && \text{if } i \text{ is odd}
\end{aligned}
$$

Then extend f linearly to each of the intervals $[p_i, q_i]$, $[q_i, r_i]$, and $[r_i, p_{i+1}]$.

Then $f^{*-1}(0)$ is a closed subset of $\beta(A)$ containing $\{p_{2k+1}\}_{k=1}^{\infty}$. Hence $f^{*-1}(0)$ meets A^*. But any limit point of $\{p_{2k+1}\}_{k=1}^{\infty}$ lies in $\overline{U}$ and hence not in Y. Thus $f^{*-1}(0)$ meets X, or $0 \in f^*(X)$. Similarly, $1 \in f^*(X)$.

But let $a \in f^{*-1}(1/3, 2/3)$. Then a is a limit point of $f^{-1}(1/3, 2/3)$, and

$$f^{-1}(1/3,2/3) = \bigcup_{k=1}^{\infty} (q_k, r_k) \subseteq V$$

Thus $a \in \overline{V}$ and hence $a \notin X$. Therefore, $f^*(X) \cap (1/3, 2/3) = \emptyset$, and hence X is not connected. This completes the proof.

Observe, however, that X^* is a decomposable continuum if X is a half Euclidean space of dimension greater than one. Let

$$X = \{(x_1, \ldots, x_n) \in E^n : x_1 \geq 0\}$$

$$X_1 = \{(x_1, \ldots, x_n) \in X : x_n \geq 0\}$$

$$X_2 = \{(x_1, \ldots, x_n) \in X : x_n \leq 0\}$$

For each positive integer k, let R_k be that set of points in X with norm greater than or equal to k. Then $X^* = \bigcap_{k=1}^{\infty} \overline{R}_k$, where the closure is taken in $\beta(X)$. Thus X^* is a continuum. Similar arguments show that $\overline{X}_1 - X_1$ and $\overline{X}_2 - X_2$ are continua, and the reader can easily show that these are proper subcontinua of X^*. But $X^* = (\overline{X}_1 - X_1) \cup (\overline{X}_2 - X_2)$, completing the argument.

APPENDIX

UNSOLVED PROBLEMS SUGGESTED BY THIS THESIS

This investigation raises several questions which are as yet unanswered. Following is a brief discussion of some of these which the author considers most important to the further development of the theory.

I. Existence of pseudocones

Except in the metric case and the case of spaces with separable dense path components the results on existence of pseudocones are not particularly useful. A characterization of those continua over which there exist pseudocones in terms of intrinsic topological properties without the hypothesis of the existence of an embedding space would be most enlightening. Lacking this, more readily verified sufficient conditions would be useful. For example, does separability of X imply the existence of a pseudocone over X?

Another type of problem in this connection is the following: If a collection $\{X_\alpha\}_{\alpha \in J}$ of continua is specified such that for each α there exists a pseudocone $P(X_\alpha)$, under what conditions is there a pseudocone over $\prod_{\alpha \in J} (X_\alpha)$? A similar question can be asked about inverse limits of continua.

II. Retractions

Even for the metric case, the problem of when a continuum

X is a retract of every pseudocone over itself is unsolved except for Peano continua. It is also not clear whether this property is preserved by continuous surjections or even by continuous monotone surjections.

III. Metric pseudocones

The following problem originally motivated this study: Given a totally bounded metric ρ on A, let S denote the completion by Cauchy sequences of the metric space $\langle A,\rho\rangle$. What is the relationship between the properties of ρ as a real-valued function and the topological properties of the base of S? A satisfactory answer to this question still seems difficult. In view of the concept of natural containment, the same problem may be of interest for metrics on positive integers.

IV. Extension of maps on pseudocones

The map $h: A \to A$, in the definition of an even pseudocone, has an extension to the entire pseudocone, and its restriction to the base must be the identity function. In a more general setting it may be asked under what conditions a map from A to A extends to a map from an entire pseudocone P(X) to itself, and under what conditions such a map induces a self homeomorphism of X. This homeomorphism question may be of special interest in case the pseudocone under consideration is $\beta(A)$.

V. The Stone-Čech compactification of A

One question of interest here is whether $\beta(A)$ is the only pseudocone over A^* up to homeomorphism, and if so whether $\beta(A)$ and the one-point compactification of A are the only pseudocones which are topologically determined by their bases.

The proper subcontinua of A^* seem to share many of the properties of A^*. Is A^* hereditarily indecomposable, or possibly even homeomorphic to each nondegenerate subcontinuum of itself? If not, do there exist pseudocones over all, or any, of the nondegenerate proper subcontinua of A^*? Other questions about A^*, such as whether it has the fixed point property and whether its Čech homology groups are trivial may also be of interest.

Aside from the study of pseudocones, A^* also raises the question of whether a non-metric indecomposable continuum must have uncountably many composants, or even more than one composant, since the proof for the metric case depends upon the second axiom of countability. Also, if X is a locally compact metric space such that X^* is connected, what are necessary and sufficient conditions that X^* be an indecomposable continuum?

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