

A NEW CHARACTERIZATION OF
CESARO-PERRON INTEGRALS USING
PEANO DERIVATES

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ABSTRACT

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By

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Using the methods of O. Perron the Z_n -integral is defined. A majorant of a function f on an interval $[a,b]$ is, by definition, the n -th exact Peano derivative of a function P which satisfies $-\infty < \delta_{n+1}P(x) \geq f(x) \quad (x \in [a,b])$, where $\delta_{n+1}P$ is the $(n+1)$ th lower generalized derivate of P . Using a modification of a theorem of James it is shown that such majorants lead to a reasonable definition of integration. Some of the useful properties of this integral follow.

1. Every n -th exact Peano derivative is Z_{n-1} -integrable.
2. The Z_n -integral is a positive linear functional defined on certain of the Lebesgue classes.
3. The indefinite Z_n -integral is itself an n -th exact Peano derivative.
4. The Z_0 -integral is equivalent to the Perron integral and the Z_{n+1} -integral properly generalizes the Z_n -integral.

5. A non-negative function is Z_n -integrable if and only if it is Lebesgue integrable.
6. The Z_n -integral of a product may be computed using an integration by parts formula if certain natural restrictions are fulfilled.
7. The Z_n -integral is exactly equivalent to the C_nP -integral of Burkill.

The need to correct and simplify Burkill's work motivated this study. This approach brings to light some interesting relationships between Cesàro derivatives and Peano derivatives. Namely:

8. A function is C_n -continuous on $[a,b]$ if and only if it is an exact Peano derivative.
9. The $(n+1)$ th Peano derivative of f is the C_n -derivative of the n -th Peano derivative of f .

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INTRODUCTION

In a paper published in 1914, O. Perron, [11], attempted to give a new characterization of the Lebesgue integral but in the process defined a new and more general integral. He apparently was only trying to simplify Lebesgue's definition, and indeed it was easily seen that for the class of bounded measurable functions the ideas of Perron and Lebesgue were the same. Perron realized however that the integral did not depend upon the boundedness of the integrand. A year later H. Bauer, [6], developed some of the properties of the integral of Perron and in so doing established that the new integral is more general than Lebesgue's. In fact, he showed that Perron's integral is capable of reconstructing a function from its finite derivative, while the Lebesgue integral is not capable of this without using extra assumptions.

Perron's idea was very similar to what we know today as the theorem of Vitali-Caratheódory (not completed until 1918). That is, if f is Lebesgue

integrable on $[a,b]$ and $\epsilon > 0$ then there is a function u which is lower semi-continuous and bounded below and a function v , upper-semicontinuous and bounded above such that $v \leq f \leq u$ on $[a,b]$ and $\int_a^b (u-v) < \epsilon$. If we define $U(x) = \int_a^x u$ and $V(x) = \int_a^x v$ then U is an example of what Perron called an Oberfunktion (majorant or superfunction) for f and V is an example of an Unterfunktion (minorant or subfunction) for f on $[a,b]$. That is, the upper derivate of V is $\leq f$ on $[a,b]$ and the lower derivate of U is $\geq f$ on $[a,b]$ (also, the upper derivate of V is bounded above and the lower derivate of U is bounded below). The idea of using these functions to define the integral comes from the formulas

$$U(b) \leq \int_a^b f \leq V(b) \quad \text{and} \quad U(b) - V(b) < \epsilon.$$

For full details the reader should consult the excellent account given in [4;II, p.157f].

More recently the idea of using majorants and minorants to define integrals has been adapted to derivatives other than the ordinary derivatives. In 1931, J.C. Burkill, [8], applied the method to approximate derivatives and thus defined the AP integral, and in 1932-33 ([1] and [2]) he defined a new derivative, the

Cesàro derivative and then developed the corresponding integral. Unfortunately there were serious errors. Still more recently R.D. James [3] and L. Gordon [9] have used the method in dealing with still other types of derivatives.

Our purpose here is to define yet another integral, one which is naturally associated, according to Perron's method, with the Peano derivatives. We show that the integral has several desirable properties, among which is a certain "Fundamental Theorem," a form of "continuity" for the integral and an integration by parts theorem. We shall show that the integral we give is identical to that of Burkill (thus correcting his errors) and also give some interesting relationships between the Peano derivatives and the Cesàro derivatives.

CHAPTER I

BURKILL'S $C_n P$ - INTEGRAL

Burkill ([1] and [2]) attempted to define a notion of differentiation, continuity and integration inductively. We present his definitions.

Let the $C_0 P$ - integral denote the usual Perron integral (see [4, II p.157f] or [5, p.201f]). Then C_0 - continuity is the ordinary continuity and the C_0 derivative is the ordinary first derivative. Suppose that the notions of C_{n-1} - continuity, differentiation and the $C_{n-1} P$ - integral are defined. If f is $C_{n-1} P$ - integrable on $[a, b]$ then the number $(C_{n-1} P) \int_a^b f(x) dx$ denotes the value of this integral.

Definition 1.1. For a positive integer n , the n -th Cesàro mean of a $C_{n-1} P$ - integrable function f in $(x, x+h)$ is defined to be

$$C_n(f, x, h) = \frac{n}{h^n} (C_{n-1} P) \int_x^{x+h} (x+h-\xi)^{n-1} f(\xi) d\xi.$$

The number h may be positive or negative. Note that the existence of the integral is being assumed.

Definition 1.2. A function f is called C_n -continuous at x_0 provided

$$\lim_{h \rightarrow 0} C_n(f, x_0, h) = f(x_0).$$

Thus in particular f is C_{n-1}^P -integrable in some neighborhood of x_0 .

Definition 1.3. Define the (two-sided) upper n -th Cesàro derivate of f at x_0 by

$$C_n D^* f(x_0) = \limsup_{h \rightarrow 0} \frac{C_n(f, x_0, h) - f(x_0)}{h/(n+1)}.$$

The lower derivate, $C_n D_* f(x_0)$, is defined using the lower limit of the same ratio. If the upper and lower derivatives agree we call the common value the n -th Cesàro derivative and write it as $C_n Df(x_0)$. Right and left derivatives may be defined with the usual conventions. Burkill uses this notion of continuity and differentiation to define an integral according to the method of Perron. For this we need

Definition 1.4. Let f, M be defined on $[a, b]$. Then M is called a C_n -majorant of f on $[a, b]$ provided

- a) M is C_n -continuous on $[a, b]$,
- b) $C_n D_* M(x) \geq f(x)$ for each $x \in [a, b]$,
- c) $C_n D_* M(x) > -\infty$ for each $x \in [a, b]$,
- d) $M(a) = 0$.

Then C_n -minorants are defined by replacing b) and c) by

$$b') \quad C_n D^* M(x) \leq f(x) \quad \text{for each } x \in [a, b],$$

$$c') \quad C_n D^* M(x) < +\infty \quad \text{for each } x \in [a, b].$$

Definition 1.5. The upper $C_n P$ -integral of f is

$$(C_n P^*) \int_a^b f = \inf \{M(b) : M \text{ is a } C_n\text{-majorant of } f \text{ on } [a, b]\}.$$

The lower $C_n P$ -integral of f is

$$(C_n P_*) \int_a^b f = \sup \{m(b) : m \text{ is a } C_n\text{-minorant of } f \text{ on } [a, b]\}.$$

Suppose now that we have a proof of the following theorem.

Theorem. Let M be defined on $[a, b]$. Let $C_n D_* M \geq 0$ on $[a, b]$. Then M is non-decreasing on $[a, b]$.

Then if f is any function and M is a C_n -majorant and m is a C_n -minorant of f on $[a, b]$, we see that $C_n D_*(M-m) \geq C_n D_* M - C_n D^* m \geq 0$ and so $M-m$ is non-decreasing. But $M(a) = m(a) = 0$, so that $M(b) \geq m(b)$. Thus it follows that $(C_n P_*) \int_a^b f \leq (C_n P^*) \int_a^b f$. When these are equal and finite we say f is $C_n P$ -integrable and put $(C_n P) \int_a^b f$ for the common value. Moreover, if f is $C_n P$ -integrable then for each $\epsilon > 0$ there is a majorant M and a minorant m such that

$M(b) - m(b) < \epsilon$. And so $0 \leq M(x) - m(x) < \epsilon$ for each $x \in [a, b]$. From this it easily follows that if f is $C_n P$ -integrable on $[a, b]$ then it is also $C_n P$ -integrable on any sub-interval, and in particular on $[a, x]$, $a \leq x \leq b$. (We make the usual convention that $\int_a^a f = 0$ always). If we define $F(x) = (C_n P) \int_a^x f$, then we see easily that F is the uniform limit of a sequence of majorants of f . For these reasons and others the theorem stated above is central to the development of these integrals and will be called the "Validity Theorem."

The first and most fundamental problem with Burkill's paper is that his proof of the validity theorem is in error. Moreover, because of the way in which the n -th Cesaro mean is defined an integration by parts theorem is essential to the development. The induction by which Burkill defined the $C_n P$ -integral depends essentially on integration by parts to transform a $C_n P$ -integral to one of lower order. However Burkill gives only the briefest sketch of a proof of this theorem and although we believe that the details can be supplied the complete proof is very long.

We will take a different approach. We first define an integral, the Z_n -integral, which at first does not seem to be related to the $C_n P$ -integral. The definition does not use an induction and so is technically much simpler. The notion of derivation on which it is based is also much neater than the notion of C_n -derivatives. On the other hand we shall show that the Z_n -integral is equivalent to the $C_n P$ -integral.

CHAPTER II

PEANO DERIVATIVES

Definition 2.1. Let $F(x)$ be defined on an interval $[a, b]$. Let $x_0 \in (a, b)$. Let n be a natural number. If there are constants $\alpha_1, \dots, \alpha_n$ depending on x_0 but not on h such that

$$(2.1) \quad F(x_0+h) - F(x_0) - \sum_{k=1}^n \alpha_k \frac{h^k}{k!} = o(h^n) \quad \text{as } h \rightarrow 0$$

then α_n is called the generalized derivative or Peano derivative of F at x_0 . This is denoted by $F_{(n)}(x_0)$. It is easily seen that if $F_{(n)}(x_0)$ exists then so do $F_{(k)}(x_0)$ ($1 \leq k \leq n$) and then

$$F(x_0+h) - F(x_0) - \sum_{k=1}^n \frac{h^k}{k!} F_{(k)}(x_0) = o(h^n) \quad \text{as } h \rightarrow 0.$$

In particular $F_{(1)}(x_0) = F^{(1)}(x_0)$, the ordinary derivative. We also say that $F_{(0)}(x_0) = F(x_0)$ when F is continuous. By restricting h , say $h > 0$, we may also define one-sided generalized derivatives, denoted $F_{(n)+}(x_0)$ etc.

A function f defined and finite on an interval I will be called an n -th exact Peano derivative (e. P. d.) on I provided that there is a (continuous) function F on I such that $F_{(n)}(x_0) = f(x_0)$ for each point x_0 in the interior

of I and, in case I contains its end points, that the corresponding one-sided n -th derivatives of F at these points equal f there. Similarly if we say that F has an n -th Peano derivative in I we mean that $F_{(n)}$ exists in the interior and the one-sided derivatives exist at the end points when these are in I .

Definition 2.2. Let n be a natural number. Let F be defined in the interval $[a, b]$. Let $x_0 \in [a, b]$. If $n = 0$ we assume F is continuous at x_0 . If $n > 0$ we assume $F_{(n-1)}(x_0)$ exists. Define $\theta_n(F, x_0, h)$ for h such that $x_0 + h \in (a, b)$ by

$$(2.2) \quad \frac{h^n}{n!} \theta_n(F, x_0, h) = F(x_0 + h) - \sum_{k=0}^{n-1} \frac{h^k}{k!} F_{(k)}(x_0).$$

Note that in case x_0 is a or b we agree that all these generalized derivatives are "one-sided." Define

$$(2.3) \quad \begin{aligned} \Delta_n F(x_0) &= \bar{F}_{(n)}(x_0) = \limsup_{h \rightarrow 0} \theta_n(F, x_0, h) \\ \delta_n F(x_0) &= \underline{F}_{(n)}(x_0) = \liminf_{h \rightarrow 0} \theta_n(F, x_0, h). \end{aligned}$$

Then $\Delta_n F(x_0)$ is called the n -th upper generalized derivate of f at x_0 and $\delta_n F(x_0)$ is called the n -th lower generalized derivate of f at x_0 . It is clear that $F_{(n)}(x_0)$ exists if and only if $\Delta_n F(x_0) = \delta_n F(x_0)$ and both are finite. In this case $F_{(n)}(x_0)$ is the common value. But as $\Delta_n F$, $\delta_n F$ need not be finite we can say that $F_{(n)}(x_0)$

exists (in the finite or infinite sense) whenever $\Delta_n F(x_0) = \delta_n F(x_0)$. Again the "one-sided" derivatives are easy modifications of (2.3) and, as above, when we speak about $\Delta_n F(x_0)$, etc., on a closed interval we shall mean it in the one-sided sense at end points.

These derivatives have been extensively studied. (For example see [10]). We would like to note that these derivatives are more general than the ordinary derivatives. For example if $f(x)$ is the characteristic function of the rationals then $x^3 f(x)$ has a second generalized derivative at 0 but no second ordinary derivative there. The same is true of the function $g(x) = x^3 \sin x^{-2}$, $x \neq 0$, $g(0) = 0$.

Definition 2.3. Let F be defined in the interval $[a,b]$. Then F will be called 1-convex in $[a,b]$, if it is non-decreasing there. It will be called 2-convex if it is convex and continuous in $[a,b]$. If $n > 2$ then F will be called n -convex if $F^{(n-2)}$ exists in $[a,b]$ and is 2-convex there. We also define n -convexity in an open interval (a,b) by simply dropping the restrictions at a and at b .

Proposition 2.1. If $f^{(n)}(x)$ exists then $f_{(n)}(x)$ does as well, and they are equal.

Proof. Apply Taylor's theorem. $\square$

Proposition 2.2 Assume $f_{(n)}(x)$ ($n \geq 1$) exists on $[a, b]$ and $F(t) = \int_c^t f$ where $c \in (a, b)$. Then

$$F_{(k)}(x) = f_{(k-1)}(x) \quad (k = 1, \dots, n+1; x \in (a, b)).$$

$$\begin{aligned} \text{Proof. } F(x+h) - F(x) &= \int_x^{x+h} f(t) dt = \\ &= \int_0^h f(x+s) ds = \\ &= \int_0^h \left[f(x) + \sum_{k=1}^n \frac{s^k}{k!} f_{(k)}(x) + o(s^n) \right] ds \\ &= hf(x) + \sum_{k=1}^n \frac{h^{k+1}}{(k+1)!} f_{(k)}(x) + o(h^{n+1}). \quad \square \end{aligned}$$

Notation. Let $\delta_{n,+}f(x)$ signify

$$\liminf_{h \rightarrow 0+} \theta_n(f, x, h),$$

the lower right hand generalized derivate of f at x . We similarly define the symbols $\delta_{n,-}f(x)$, $\Delta_{n,+}f(x)$, and $\Delta_{n,-}f(x)$.

Corollary 2.3. Under the conditions of proposition 2.2

$$\delta_{n+2}F(x) \geq \delta_{n+1}f(x).$$

Proof. Let $M < \delta_{n+1}f(x)$. Then for all sufficiently small positive h ,

$$\int_0^h [f(x+s) - f(x) - \sum_{k=1}^n \frac{s^k}{k!} f_{(k)}(x)] ds > \int_0^h \frac{s^{n+1}}{(n+1)!} M ds.$$

That is,

$$F(x+h) - F(x) - \sum_{k=1}^{n+1} \frac{h^k}{k!} F_{(k)}(x) > \frac{h^{n+2}}{(n+2)!} M,$$

thus $\delta_{n+2,+} F(x) > M$, and so

$$\delta_{n+2,+} F(x) \geq \delta_{n+1} f(x).$$

If $h < 0$ and n is even then

$$\int_0^h [f(x+s) - f(x) - \sum_{k=1}^n \frac{s^k}{k!} f_{(k)}(x)] ds > \int_0^h \frac{s^{n+1}}{(n+1)!} M ds,$$

while if n is odd

$$\int_0^h [f(x+s) - f(x) - \sum_{k=1}^n \frac{s^k}{k!} f_{(k)}(x)] ds < \int_0^h \frac{s^{n+1}}{(n+1)!} M ds.$$

Thus in either case we have

$$\frac{(n+2)!}{h^{n+2}} \{F(x+h) - F(x) - \sum_{k=1}^n \frac{h^k}{k!} F_{(k)}(x)\} > M,$$

for h sufficiently small and negative. Thus

$$\delta_{n+2,-} F(x) > M \text{ whence}$$

$$\delta_{n+2,-} F(x) \geq \delta_{n+1} f(x).$$

The rest is obvious. $\square$

Corollary 2.4. If f is defined in $[a, b]$

and if $f_{(m)}$ exists there and if

$$F(x) = \int_a^x d\xi_1 \int_a^{\xi_1} d\xi_2 \dots \int_a^{\xi_{k-1}} f(\xi_k) d\xi_k$$

for $x \in [a, b]$, then

$$f_j(x) = F_{k+j}(x) \quad (0 \leq j \leq m; x \in [a, b])$$

and

$$\delta_{m+k+1}F \geq \delta_{m+1}f.$$

Proof. This follows by iteration from proposition 2.2 and corollary 2.3. $\square$

We need the next three propositions in the next chapter.

Lemma 2.5. Let f' be finite in (a, b) and let $f'(a+)$ exist (finite or infinite). Then $f(a+)$ exists.

Proof. If $f'(a+)$ is finite then f' is bounded in some interval (a, x_0) so that f is uniformly continuous there and so $f(a+)$ exists.

If $f'(a+) = +\infty$ then f' is positive in some interval (a, x_0) and so, by the mean value theorem, f is monotone in (a, x_0) . Thus $f(a+)$ exists. The case $f'(a+) = -\infty$ is similar. $\square$

Lemma 2.6. Let f be continuous in $[a, b]$ and let f have a finite derivative on (a, b) . Let $f'(a+)$ exist. Then $f'^+(a)$ exists and equals $f'(a+)$.

Proof. Apply L'Hôpital's rule to the limit

$$\lim_{x \rightarrow 0} \frac{f(a+x) - f(a)}{x} . \quad \square$$

Proposition 2.7. Let n be a natural number. Let f be defined in $[a, b]$ so that $f^{(n)}$ is finite in (a, b) , $f^{(n)}(a+)$ exists (finite or infinite) and $f_{(n)+}(a)$ is finite. Then

$$f_{(n)+}(a) = f^{(n)+}(a) = f^{(n)}(a+).$$

Proof. As $f_{(n)+}(a)$ is finite, f is continuous at a and hence, in some interval $[a, b_1]$. By lemma 2.5, $f^{(n-1)}(a+), \dots, f'(a+)$ exist. By lemma 2.6, $f'^+(a) = f'(a+)$ and since $f'^+(a) = f_{(1)+}(a)$ which is finite, we see that f' is continuous at a . Then lemma 2.6 applies also to f' . Thus

$$f^{(2)}(a+) = f^{(2)+}(a) = f_{(2)+}(a)$$

which is finite so that $f^{(2)}$ is continuous at a .

Finitely many such steps finish the proof. $\square$

CHAPTER III

THE THEOREM OF R.D. JAMES

In this section we prove a theorem that is very important to the development of what follows. It is merely a modification of a theorem of James [3].

Theorem 3.1. If f is defined in $[a,b]$ and if $\delta_n F \geq 0$ in $[a,b]$ then F is n -convex in (a,b) .

The proof of this theorem requires some additional definitions and lemmas.

Let n be a natural number. Let f be defined on an interval J and let $x_0, x_1, \dots, x_n$ be distinct points of J . Then there is a unique polynomial P of degree $\leq n$ such that $P(x_j) = f(x_j)$ ($j = 0, \dots, n$). We write $P(x) = P(x_0, x_1, \dots, x_n; x)$ to express the dependence on $x_0, \dots, x_n$. Since $P^{(n)}$ is a constant, we define

$$V_n(f, x_k) = V_n(f; x_0, x_1, \dots, x_n) = P^{(n)}/n! .$$

It is easily seen that

$$P(x) = \sum_{k=0}^n f(x_k) \frac{(x-x_0) \dots (x-x_{k-1})(x-x_{k+1}) \dots (x-x_n)}{(x_k-x_0) \dots (x_k-x_{k-1})(x_k-x_{k+1}) \dots (x_k-x_n)},$$

since this is clearly a polynomial of degree $\leq n$ having the required properties. If we expand this expression we see that

$$V_n(f; x_0, \dots, x_n) = \sum_{k=0}^n \frac{f(x_k)}{w'(x_k)}, \quad \text{where}$$

$$w(x) = \prod_{k=0}^n (x-x_k).$$

It is also easily verified that

$$P(x_0, \dots, x_n; x) = \frac{(x-x_0)P(x_1, \dots, x_n; x) - (x-x_n)P(x_0, \dots, x_{n-1}; x)}{x_n - x_0}$$

since this polynomial also has the required values at the points $x_0, \dots, x_n$. From this we see that

$$V_n(f; x_0, \dots, x_n) = \frac{V_{n-1}(f; x_1, \dots, x_n) - V_{n-1}(f; x_0, \dots, x_{n-1})}{x_n - x_0}.$$

Another property easily seen from the definition is the fact that a permutation of the points $x_0, \dots, x_n$ leaves P and $V_n(f; x_0, \dots, x_n)$ unchanged.

Definition 3.1. The function f is called n -convex (d) in J provided $V_n(f; x_0, \dots, x_n) \geq 0$ for all choices of $n+1$ distinct points $x_0, \dots, x_n$ from J .

We note that "2-convex (d)" means "convex" and "1-convex (d)" means "non-decreasing."

Proposition 3.2. Let f be n -convex (d) on (a,b) . Let $x_0, \dots, x_n$ be distinct points of (a,b) and $x_{n-1} < x_n$. Then

$$V_{n-1}(f; x_0, \dots, x_{n-2}, x_{n-1}) \leq V_{n-1}(f; x_0, \dots, x_{n-2}, x_n).$$

Proof. We see that

$$\begin{aligned} & V_{n-1}(f; x_0, \dots, x_{n-2}, x_n) - V_{n-1}(f; x_0, \dots, x_{n-2}, x_{n-1}) = \\ &= V_{n-1}(f; x_0, \dots, x_{n-2}, x_n) - V_{n-1}(f; x_{n-1}, x_0, \dots, x_{n-2}) = \\ &= (x_n - x_{n-1}) V_n(f; x_{n-1}, x_0, \dots, x_{n-2}, x_n) \geq 0. \quad \square \end{aligned}$$

Remark. We see from proposition 3.2 (and the remarks above it) that $V_{n-1}(f; x_1, \dots, x_n)$ is a monotone function in each of the variables $x_1, \dots, x_n$ when f is n -convex (d).

Proposition 3.3. Let $n \geq 2$. Let f be n -convex (d) on (a,b) . Then f'^+ and f'^- exist and are finite in (a,b) .

Proof. Let $x \in (a,b)$, $x^* \in (a,x)$. If $n > 2$ choose $x_1, \dots, x_{n-2}$ such that

$$a < x_1 < x_2 < \dots < x_{n-2} < x^*.$$

Set

$$F(t) = V_{n-1}(f; x_1, \dots, x_{n-2}, x, t) \quad (x < t < b).$$

By proposition 3.2, F is non-decreasing on (x, b) and

$$F(t) \geq V_{n-1}(f; x_1, \dots, x_{n-2}, x^*, t) \geq V_{n-1}(f; x_1, \dots, x_{n-2}, x^*, x)$$

here. Thus $F(x+)$ exists and is finite. Let

$$g(t) = \prod_{k=1}^{n-2} (t - x_k), \quad h(t) = f(t)/g(t), \quad \text{for } t \in (x^*, b).$$

Let

$$F_1(t) = \sum_{j=1}^{n-2} \frac{f(x_j)}{g'(x_j)(x_j - x)(x_j - t)}, \quad \text{for}$$

$$x^* < t < b. \quad (\text{Thus } g \equiv 1 \text{ and } F_1 \equiv 0 \text{ if } n = 2).$$

Then

$$\begin{aligned} F(t) &= F_1(t) + \frac{f(x)}{g(x)(x-t)} + \frac{f(t)}{g(t)(t-x)} = \\ &= F_1(t) + \frac{h(t) - h(x)}{t-x} \quad (x < t < b). \end{aligned}$$

It follows that $h'^+(x) = F(x+) - F_1(x)$. Since $f = hg$

and g is a polynomial we see that $f'^+(x)$ exists

and is finite. To finish we treat f'^- similarly. $\square$

Corollary 3.4. If $n \geq 2$ and f is n -convex (d) then f is continuous.

Proof. It is evidently both right and left continuous by the above. $\square$

Proposition 3.5. Let $n \geq 2$ and f be n -convex (d) in (a,b) . Then f'^{+} and f'^{-} are $(n-1)$ -convex (d) in (a,b) .

Proof. Let $a < x_1 < x_2 < \dots < x_n < b$. Let $h \in (0, b-x_n)$. Let $y_j = x_j + h$ ($j = 1, \dots, n$). Let $w_x(t) = \Pi(t-x_j)$, $w_y(t) = \Pi(t-y_j)$. Then it is easily seen that $w'_x(x_j) = w'_y(y_j)$. Then from proposition 3.2,

$$V_{n-1}(f; x_1, \dots, x_n) \leq V_{n-1}(f; y_1, \dots, y_n)$$

so that

$$\sum_{j=1}^n \frac{f(y_j) - f(x_j)}{w'_x(x_j)} \geq 0.$$

Thus

$$V_{n-1}(f'^{+}; x_1, \dots, x_n) = \sum_{j=1}^n \frac{f'^{+}(x_j)}{w'_x(x_j)} \geq 0. \quad \square$$

Proposition 3.6. Let f be continuous in (a,b) and $f'^{+} = 0$ in (a,b) . Then f is constant.

Proof. We first show that if g is continuous in (a,b) and $g'^{+} > 0$ there, then g is non-decreasing. To see this let α, β be in (a,b) and $\alpha < \beta$. Let

$$x_0 = \sup\{t \in [\alpha, \beta] : g(t) \geq g(\alpha)\}.$$

It follows easily from the continuity of g that $g(x_0) \geq g(\alpha)$. Moreover, we claim that $x_0 = \beta$. If this is not the case then, since $g'^{+}(x_0) > 0$, there is a point t in $(x_0, \beta]$ such that $g(t) \geq g(x_0) \geq g(\alpha)$,

and this contradicts the definition of x_0 . Thus $x_0 = \beta$ so that $g(\beta) \geq g(\alpha)$.

From this one sees easily that if g is continuous and $g'^+ \geq 0$ in (a,b) then g is non-decreasing. The proposition at hand is immediate from this and the corresponding statement about a function g such that $g'^+ \leq 0$. $\square$

Proposition 3.7. Let f be continuous on (a,b) such that f'^+ is also continuous. Then f is differentiable.

Proof. Let g be such that $g' = f'^+$. Then $(f-g)'^+ = 0$ so $f-g$ is constant. $\square$

Proposition 3.8. Let $n > 2$. Let f be n -convex (d) in (a,b) . Then f' is $(n-1)$ -convex (d) there.

Proof. As f'^+ is $(n-1)$ -convex (d) (proposition 3.5), and continuous (corollary 3.4), we see by proposition 3.7 that f' exists, and so $f' = f'^+$ is $(n-1)$ -convex (d). $\square$

Corollary 3.9. If f is n -convex (d) in (a,b) then f is n -convex in (a,b) . That is: for $n > 2$,

if f is n -convex (d) in (a,b) then $f^{(n-2)}$ exists and is convex.

Proof. This follows trivially from proposition 3.8 by iteration. $\square$

In fact n -convexity (d) is equivalent to n -convexity, but we do not need this result and so omit the proof. (A proof may be found in [7], however this paper is quite complicated. We have included the proof of corollary 3.9 for completeness and because of its simplicity. The proof is due to J. Marik.)

Definition 3.2. The generalized symmetric derivative, $D^{2n}F(x_0)$, of order $2n$ of a function F at x_0 is defined by the relation

$$\frac{1}{2}\{F(x_0+h)+F(x_0-h)\} - \sum_{k=0}^n \frac{h^{2k}}{(2k)!} D^{2k}F(x_0) = o(h^{2n})$$

as $h \rightarrow 0$, in a manner analogous to the definition of the Peano derivatives. The generalized symmetric derivative, $D^{2n+1}F(x_0)$, of order $2n+1$ is defined by

$$\frac{1}{2}\{F(x_0+h)-F(x_0-h)\} - \sum_{k=0}^n \frac{h^{2k+1}}{(2k+1)!} D^{2k+1}F(x_0) = o(h^{2n+1})$$

as $h \rightarrow 0$. These derivatives are generalizations of the Peano derivatives in the sense that if $F_k(x_0)$ exists

then so does $D^k F(x_0)$ but not necessarily conversely.

We can proceed as in Chapter II to define upper and lower symmetric derivatives.

Thus we define $\Psi_n(f, x_0, h)$ for $n = 2m$ by

$$\frac{h^{2m}}{(2m)!} \Psi_{2m}(f, x_0, h) = \frac{1}{2} \{ f(x_0+h) + f(x_0-h) \} - \sum_{k=0}^{m-1} \frac{h^{2k}}{(2k)!} D^{2k} f(x_0)$$

(provided $D^{2k} f(x_0)$ ($k = 0, \dots, m-1$) exist) and, for $n = 2m+1$, by

$$\begin{aligned} \frac{h^{2m+1}}{(2m+1)!} \Psi_{2m+1}(f, x_0, h) &= \frac{1}{2} (f(x_0+h) - f(x_0-h)) \\ &\quad - \sum_{k=0}^{m-1} \frac{h^{2k+1}}{(2k+1)!} D^{2k+1} f(x_0) \end{aligned}$$

(provided $D^{2k-1} f(x_0)$ ($k = 1, \dots, m$) exist). The k -th generalized upper symmetric derivative of f at x_0 is

$$\gamma_k f(x_0) = \limsup_{h \rightarrow 0} \Psi_k(f, x_0, h).$$

The function F is said to satisfy conditions A_{2m} in (a, b) if it is continuous in $[a, b]$ and if, for $1 \leq k \leq m-1$, each $D^{2k} F(x)$ exists and is finite in (a, b) , and if

$$\lim_{h \rightarrow 0} h \Psi_{2m}(F, x_0, h) = 0$$

for all x_0 in $(a, b) \setminus E$, where E is a countable set.

The conditions A_{2m+1} are defined analogously. This condition can be thought of as a sort of "n-th order smoothness."

We say that the finite function f has a discontinuity of the first kind at $c \in (a,b)$ or an ordinary discontinuity, provided that $f(c+)$, $f(c-)$ both exist in the finite or infinite sense but that at least one of these is different from $f(c)$.

The function F is said to satisfy conditions B_{2m-2} in (a,b) if it is continuous in (a,b) , if, for $1 \leq k \leq m-1$, each $D^{2k}F(x)$ exists and is finite in $[a,b]$ and if no $D^{2k}F(x)$ has an ordinary discontinuity in (a,b) . A similar definition is given for conditions B_{2m-1} .

Proposition 3.10. If F satisfies conditions A_n and B_{n-2} in (a,b) and if $\gamma_n F(x) \geq 0$ in (a,b) then $F(x)$ is n -convex (d) in (a,b) .

Proof. This is theorem 4.2 of James [3]. $\square$

We shall see that theorem 3.1 follows from proposition 3.10. In the first place, it is easily seen that $\delta_k F(x) \leq \Delta_k F(x) \leq \gamma_k F(x)$ whenever the left hand side has a meaning, so that if $\delta_k F(x) \geq 0$ on $[a,b]$ then also $\gamma_k F(x) \geq 0$ on $[a,b]$. To finish we need

only note that conditions A_n and B_{n-2} are automatically satisfied in case $\delta_n F(x) \geq 0$. The proof of this fact is in Section 8 of James' paper. Thus theorem 3.1 is proved.

Proposition 3.11. Let F be defined in $[a,b]$ and let $\delta_n F(x) \geq 0$ in $[a,b]$. Then F is n -convex in $[a,b]$.

Proof. By theorem 3.1 we need only show that $F^{(n-2)+}$ exists at a and that $F^{(n-2)}$ is continuous at a . (Similarly for b .) But as $F^{(n-2)}$ is convex in (a,b) we know that $F^{(n-2)}(a+)$ exists. Thus by proposition 2.7, $F^{(n-2)+}(a) = F_{(n-2)+}(a) = F^{(n-2)}(a+)$. Thus $F^{(n-2)}$ exists and is continuous in $[a,b]$ so that F is n -convex there. $\square$

CHAPTER IV

THE CENTRAL THEOREM

In this section we shall state and prove the validity theorem for the Z_n -integral which we define in the next section.

Theorem 4.1. Let f be defined on $[a,b]$ and suppose $\delta_n f(x) \geq 0$ on $[a,b]$. Then $f^{(n-1)}$ exists and is non-decreasing on $[a,b]$.

Proof. We know $f_{(n-1)}$ must exist at each point of $[a,b]$. By theorem 3.11 $f^{(n-2)}$ exists and is 2-convex in $[a,b]$. Moreover $f^{(n-2)'} +$ exists in $[a,b)$ and $f^{(n-2)'} -$ exists in $(a,b]$. Applying a one-sided version of Taylor's theorem we see that $f^{(n-2)'} + = f_{(n-1)} +$ and $f^{(n-2)'} - = f_{(n-1)} -$. Thus $f^{(n-1)}$ exists in $[a,b]$ and is the derivative of a convex function and hence is non-decreasing (and in fact continuous). $\square$

We shall need the following simple consequence of theorem 4.1.

Theorem 4.2. Let f and g be defined on an interval I such that $f_{(n)} = g_{(n)}$ on I . Then f and g differ by a polynomial of degree no more than $n-1$.

Proof. As $(f-g)_{(n)} = 0$ we have $(f-g)^{(n-1)}$ is a constant by theorem 4.1. The rest is easy. $\square$

CHAPTER V

THE Z_n - INTEGRAL

Definition 5.1. Let n be a natural number. Let M, f be defined in $[a, b]$. Then M is called an n -majorant of f in $[a, b]$ if there is a function P on $[a, b]$ such that

- 1) $M = P_{(n)}$ on $[a, b]$,
- 2) $\delta_{n+1} P(x) \geq f(x)$ for each $x \in [a, b]$,
- 3) $\delta_{n+1} P(x) > -\infty$ for each $x \in [a, b]$.

The function P will be called a pre-majorant. Then n -minorants are defined similarly, replacing 2), 3) by

- 2') $\Delta_{n+1} P(x) \leq f(x)$ for each $x \in [a, b]$,
- 3') $\Delta_{n+1} P(x) < +\infty$ for each $x \in [a, b]$,

and then P is called a pre-minorant.

Remark. For $n=0$ we have exactly the definition of majorant for the Perron integral.

Definition 5.2. Let f be defined on $[a, b]$. The upper Z_n -integral of f on $[a, b]$ is

$$(Z_n^*) \int_a^b f = \inf \{ M(b) - M(a) : M \text{ is an } n\text{-majorant of } f \text{ on } [a, b] \}.$$

The lower Z_n -integral of f on $[a,b]$ is

$$(Z_{n*}) \int_a^b f = \sup\{m(b) - m(a) : m \text{ is a } n\text{-minorant of } f \text{ on } [a,b]\}.$$

(The infimum of an empty set is of course $+\infty$, etc.)

If the upper and lower Z_n -integrals are finite and equal then we write $(Z_n) \int_a^b f$ for the common value and say that f is Z_n -integrable on $[a,b]$.

The Z_0 -integral is then exactly the Perron integral.

Remark. If M is an n -majorant of f and m is an n -minorant and if P and p are an associated pre-majorant and pre-minorant respectively, then

$\delta_{n+1}(P-p) \geq 0$ and so $(P-p)_{(n)} = M-m$ is non-decreasing by theorem 4.1. Thus $M(b) - M(a) \geq m(b) - m(a)$. From this it follows that $(Z_n^*) \int_a^b f \geq (Z_{n*}) \int_a^b f$ for every function f .

We can now prove a simple theorem which can be taken as the motivation for this integral.

Theorem 5.1. If $F_{(n+1)}$ exists and is finite on $[a,b]$ then $F_{(n+1)}$ is Z_n -integrable there and

$$(Z_n) \int_a^b F_{(n+1)} = [F_{(n)}]_a^b.$$

Proof. $F_{(n)}$ is at the same time an n -majorant and an n -minorant of $F_{(n+1)}$. Thus

$$(Z_n^*) \int_a^b F_{(n+1)} \leq [F_{(n)}]_a^b \quad \text{and}$$

$$(Z_{n*}) \int_a^b F_{(n+1)} \geq [F_{(n)}]_a^b.$$

But as we have remarked above,

$$(Z_n^*) \int_a^b F_{(n+1)} \geq (Z_{n*}) \int_a^b F_{(n+1)} \quad \text{and so}$$

$$(Z_n) \int_a^b F_{(n+1)} = [F_{(n)}]_a^b. \quad \square$$

We will frequently make use of the following simple result in the later proofs.

Proposition 5.2. A function f defined on $[a, b]$ is Z_n -integrable there if and only if for each $\epsilon > 0$ there is an n -majorant M and an n -minorant m such that

$$M(b) - M(a) - (m(b) - m(a)) < \epsilon.$$

Proposition 5.3. Let M be an n -majorant of f on $[a, b]$. Let c be any constant. Then $M(x) + c$ is an n -majorant.

Proof. Let $P_{(n)} = M$ on $[a, b]$. Let

$$Q(x) = P(x) + cx^n/n!.$$

Then $Q_{(n)}(x) = M(x) + c$ and $\delta_{n+1}Q(x) = \delta_{n+1}P(x)$. $\square$



Proposition 5.4. If $f \leq g$ on $[a,b]$ then

$$(Z_n^*) \int_a^b f \leq (Z_n^*) \int_a^b g.$$

Proof. Clearly any n -majorant of g is also an n -majorant of f . The rest is clear. $\square$

Occasionally we shall use the more classical notation $\int_a^b f(t)dt$ or $(Z_n) \int_a^b f(t)dt$ instead of $(Z_n) \int_a^b f$.

CHAPTER VI

PROPERTIES OF THE Z_n -INTEGRAL

Proposition 6.1. A Z_n -integrable function f on $[a,b]$ is finite almost everywhere on $[a,b]$.

Proof. Let M be an n -majorant and m an n -minorant of f on $[a,b]$. We may assume $M(a) = m(a) = 0$. Define $R(x) = M(x) - m(x)$. Then R is non-decreasing by theorem 4.1. Let P, p be defined on $[a,b]$ so that $P_{(n)} = M$, $p_{(n)} = m$. Then $P-p$ is $(n+1)$ -convex by theorem 3.11 and so $(P-p)^{(n)} = (P-p)_{(n)} = R$ (proposition 2.1). Suppose $f(x) = +\infty$. Then $\delta_{n+1}P(x) = +\infty$ and, as $\Delta_{n+1}p(x) < +\infty$, we have $(P-p)_{(n+1)}(x) = +\infty$. Similarly, if $f(x) = -\infty$, then $\Delta_{n+1}p(x) = -\infty$ and since $\delta_{n+1}P(x) > -\infty$ we have $(P-p)_{(n+1)}(x) = +\infty$. But $R' = (P-p)^{(n)'} = (P-p)^{(n+1)}$ exists and is finite a.e. in $[a,b]$ and so f is finite a.e. in $[a,b]$. $\square$

Proposition 6.2. If f is Z_n -integrable on $[a,b]$ and if $c \in (a,b)$ then f is Z_n -integrable on each of $[a,c]$ and $[c,b]$. Moreover

$$(Z_n) \int_a^b f = (Z_n) \int_a^c f + (Z_n) \int_c^b f.$$



Proof. Let $\epsilon > 0$ be given. Let M, m be respectively an n -majorant and an n -minorant for f on $[a, b]$ such that $[M-m]_a^b < \epsilon$. It follows immediately from the definitions that M is an n -majorant on $[a, c]$ and on $[c, b]$ and that m is an n -minorant on each of these. Then, by the remark following definition 5.2 we have

$$[M-m]_a^c \geq 0 \quad \text{and} \quad [M-m]_c^b \geq 0.$$

But then $0 \leq [M-m]_a^c + [M-m]_c^b = [M-m]_a^b < \epsilon$. Thus $0 \leq [M-m]_a^c \leq \epsilon$ and $0 \leq [M-m]_c^b < \epsilon$ so that f is Z_n -integrable on each of $[a, c]$ and $[c, b]$. Moreover

$$\begin{aligned} [m]_a^b &= [m]_a^c + [m]_c^b \leq (Z_n) \int_a^c f + (Z_n) \int_c^b f \leq \\ &\leq [M]_a^c + [M]_c^b = [M]_a^b. \end{aligned}$$

Since the first and last terms differ by less than ϵ the formula is proved as well. $\square$

Proposition 6.3. If $f(x)$ is Z_n -integrable on $[a, c]$ and on $[c, b]$, $a < c < b$, then it is Z_n -integrable on $[a, b]$ and the usual formula holds.

Proof. Let M_1, M_2 be n -majorants of f on $[a, c]$ and $[c, b]$ respectively. By proposition 5.3 we may assume that $M_1(a) = 0$ and $M_2(c) = M_1(c)$. Let M be defined by



$$M(x) = \begin{cases} M_1(x) & x \in [a, c] \\ M_2(x) & x \in [c, b]. \end{cases}$$

Then M is an n -majorant of f on $[a, b]$. To see this let $P_{(n)} = M_1$ and $Q_{(n)} = M_2$. Then

$$R(x) = \begin{cases} P(x) & x \in [a, c] \\ Q(x) - Q(c) + P(c) & x \in [c, b], \end{cases}$$

is a pre-majorant on $[a, b]$ since

$$R_{(n)} = M \text{ and } \delta_{n+1, +} R(c) = \delta_{n+1, +} P(c) \geq f(c)$$

and also $> -\infty$, and $\delta_{n+1, -} R(c) = \delta_{n+1, -} Q(c) \geq f(c)$ and also $> -\infty$.

If m_1 and m_2 are n -minorants on $[a, c]$ and on $[c, b]$ then we construct an n -minorant m on $[a, b]$ in an analogous way. Moreover, if $[M_1 - m_1]_a^c < \frac{\epsilon}{2}$ and $[M_2 - m_2]_c^b < \frac{\epsilon}{2}$ then

$$[M - m]_a^b = [M_1 - m_1]_a^c + [M_2 - m_2]_c^b < \epsilon.$$

Furthermore,

$$[m]_a^b \leq (Z_n) \int_a^c f + (Z_n) \int_c^b f \leq [M]_a^b.$$

Thus $(Z_n) \int_a^b f$ exists and equals $(Z_n) \int_a^c f + (Z_n) \int_c^b f$. $\square$

Definition 6.1. Let f be Z_n -integrable in $[a, b]$. Then, if $x \in (a, b]$, we see from proposition 6.2 that f is Z_n -integrable over $[a, x]$. We put



$(Z_n) \int_a^a f = 0$ for any function. Then $(Z_n) \int_a^x f$ is defined for each $x \in [a, b]$. An indefinite Z_n -integral of f is any function of the form $F(x) = c + (Z_n) \int_a^x f$ where c is constant.

Proposition 6.4. Let f be Z_n -integrable on $[a, b]$. Let F be any indefinite Z_n -integral of f on $[a, b]$. Then F is the uniform limit of a sequence of n -majorants on $[a, b]$ (similarly for n -minorants).

Proof. Let $\epsilon > 0$. Let M, m be respectively an n -majorant and an n -minorant of f on $[a, b]$ such that $M(a) = m(a) = F(a)$ and $M(b) - m(b) < \epsilon$. As $M - m$ is non-decreasing on $[a, b]$ (by the remark after definition 5.2), we have $0 \leq M(x) - m(x) < \epsilon$ for each $x \in [a, b]$. Since M, m are respectively an n -majorant and an n -minorant on every sub-interval of $[a, b]$ we have

$$m(x) - m(a) \leq (Z_n) \int_a^x f \leq M(x) - M(a).$$

Thus

$$0 \leq M(x) - M(a) - (Z_n) \int_a^x f = M(x) - F(x) < \epsilon.$$

The rest is obvious. $\square$

Proposition 6.5. Let F be an indefinite Z_n -integral of f on $[a, b]$. Let M be an n -majorant

of f on $[a,b]$ such that $F(a) = M(a)$. Then $M-F$ is non-decreasing.

Proof. Let $a \leq x_1 < x_2 \leq b$. Then M is an n -majorant of f on $[x_1, x_2]$ and so

$$F(x_2) - F(x_1) = (Z_n) \int_{x_1}^{x_2} f \leq M(x_2) - M(x_1).$$

Thus $M(x_1) - F(x_1) \leq M(x_2) - F(x_2)$. $\square$

Remark. Similarly, if m is an n -majorant than $F-m$ is non-decreasing.

Proposition 6.6. Let F be an indefinite Z_n -integral of a function f on $[a,b]$. Let M be an n -majorant of f on $[a,b]$. Then $M-F$ is continuous.

Proof. Let m be an n -minorant of f on $[a,b]$. Then $M-m$ is an n -th e.p.d. of some function G on $[a,b]$. As $\delta_{n+1}G \geq 0$ on $[a,b]$ we see that $G^{(n-1)}$ is convex and continuous in $[a,b]$ by theorem 3.11. Moreover $G^{(n)-}$ exists in $(a,b]$ and $G^{(n)+}$ exists in $[a,b)$. But then it follows from Taylor's theorem that $G^{(n)+} = G_{(n)+}$ and $G^{(n)-} = G_{(n)-}$. Since $G_{(n)}$ exists we see that $G^{(n)} = G_{(n)}$, so that $G_{(n)}$ is an ordinary derivative. Thus, since $G_{(n)} = G^{(n)}$ is non-decreasing, we see by the intermediate value property for derivatives that it is continuous. But $M-F$ is the uniform limit of



functions of the form $M-m$ so that $M-F$ is also continuous. $\square$

Theorem 6.7. Let F be an indefinite Z_n -integral of a function f on $[a,b]$. Then F is an n -th e.P.d. on $[a,b]$.

Proof. Let M be an n -majorant of f on $[a,b]$ and let $P_{(n)} = M$ there. Then as $M-F$ is continuous we see that there is a function G on $[a,b]$ such that $G^{(n)} = M-F$ there. But then

$$(P-G)_{(n)} = P_{(n)} - G^{(n)} = F. \quad \square$$

Corollary 6.8. Let F be an indefinite Z_n -integral of f on $[a,b]$. Then F is Z_{n-1} -integrable.

Proof. This is immediate from theorem 6.7 and theorem 5.1. $\square$

According to corollary 6.8, we can form the iterated integral

$$(6.1) \quad \int_a^x d\xi_0 \int_a^{\xi_0} d\xi_1 \dots \int_a^{\xi_{n-1}} f(\xi_n) d\xi_n$$

whenever f is Z_n -integrable, where the innermost integral is a Z_n -integral, the next is a Z_{n-1} -integral, etc., and the outermost is a Z_0 -integral. We shall use the symbol



$$\int_a^x f(\xi) d(\xi, n)$$

for the $(n+1)$ -fold iterated integral in (6.1). Note also that $\int_a^x f(\xi) d(\xi, n)$ is a continuous function of x .

Theorem 6.9. Let $F(x)$ be an indefinite Z_n -integral of a function f on $[a, b]$. Let $G_{(n)} = F$ on $[a, b]$. Then $G_{(n+1)}(x) = f(x)$ for almost every $x \in [a, b]$.

Proof. Let $\epsilon > 0$. Let $m(\cdot)$ denote Lebesgue measure and let $m^*(\cdot)$ denote Lebesgue outer measure. Let M be an n -majorant of f on $[a, b]$ such that $M(a) = F(a)$ and $M(b) - F(b) < \epsilon^2$. Let $P_{(n)} = M$. Let $R(x) = M(x) - F(x)$. Then R is continuous (proposition 6.6) and non-decreasing (proposition 6.5) and so

$$(L) \int_a^b R' \leq R(b) - R(a) < \epsilon^2.$$

Moreover $(P-G)^{(n)} = R$ on $[a, b]$.

Let

$$A(\epsilon) = \{x: \delta_{n+1} G(x) < f(x) - \epsilon\}.$$

Let

$$B = \{x: R' = (P-G)^{(n+1)} \text{ exists and is finite}\}.$$

If $x \in A(\epsilon)$ then $\delta_{n+1} G(x) < \delta_{n+1} P(x) - \epsilon$ and so

$\delta_{n+1} P(x) - \delta_{n+1} G(x) > \epsilon$. (Note that $\delta_{n+1} P(x) > -\infty$ and

$\delta_{n+1}G(x) < +\infty$. If $x \in B \cap A(\epsilon)$ then $\delta_{n+1}P(x) - \delta_{n+1}G(x) = R'(x)$. To see this, write $P = (P-G) + G$ so $\delta_{n+1}P \geq R' + \delta_{n+1}G$, while $G = P - (P-G)$ and so $\delta_{n+1}G \geq \delta_{n+1}P - R'$. Thus $B \cap A(\epsilon)$ is a subset of the set

$$B(\epsilon) = \{x: R'(x) > \epsilon\}.$$

We now see, $\epsilon m(B(\epsilon)) \leq (L) \int_{B(\epsilon)} R' \leq (L) \int_a^b R' < \epsilon^2$.

It follows that $m(B(\epsilon)) < \epsilon$ and so $m^*(A(\epsilon)) < \epsilon$.

So in particular, for each positive r , $m^*(A(r)) < r$.

So letting $r = \frac{\epsilon}{2^n}$ we see $m^*(A(\frac{\epsilon}{2^n})) < \frac{\epsilon}{2^n}$. From this

we see that $m^*(A(0)) < \epsilon$ and as ϵ is arbitrary this

implies that $\delta_{n+1}G \geq f$ a.e. in $[a, b]$. In the same

way we can show that $\Delta_{n+1}G \leq f$ a.e. and so

$G_{(n+1)} = f$ a.e. on $[a, b]$. $\square$

Proposition 6.10. Every Z_n -integrable function is measurable.

Proof. Let f be Z_n -integrable. Applying theorem 6.9 let $G_{(n+1)} = f$ a.e. Then

$$f(x) = \lim_{k \rightarrow \infty} (k^{n+1} (n+1)! \{G(x + \frac{1}{k}) - \sum_{m=0}^n G_{(m)}(x) / k^m\})$$

for almost every x . Thus f is the limit a.e. of a sequence of continuous functions (as G is continuous) and so is measurable. $\square$



Theorem 6.11. The Z_0 -integral is identical to the Perron integral. If f is Z_n -integrable on $[a,b]$ then f is also Z_{n+1} -integrable there and

$$(Z_n) \int_a^b f = (Z_{n+1}) \int_a^b f.$$

Proof. The first statement is obvious. Let f be Z_n -integrable on $[a,b]$. Let $\epsilon > 0$. Let M, m be respectively an n -majorant and an n -minorant of f on $[a,b]$ such that $[M-m]_a^b < \epsilon$.

Let $P_{(n)} = M, p_{(n)} = m$ on $[a,b]$.

Let $Q(x) = \int_a^x P, q(x) = \int_a^x p$.

Then by proposition 2.2, $Q_{(n+1)} = M, q_{(n+1)} = m$ and by corollary 2.3 and its analogue for upper derivatives,

$$\delta_{n+2}Q(x) \geq \delta_{n+1}P(x) \geq f(x) \quad \text{on } [a,b]$$

and

$$\delta_{n+2}Q(x) > -\infty \quad \text{on } [a,b].$$

Also

$$\Delta_{n+2}q(x) \leq \Delta_{n+1}p(x) \leq f(x)$$

and

$$\Delta_{n+2}q(x) < +\infty \quad \text{on } [a,b].$$

Thus M is an $(n+1)$ -majorant and m is an $(n+1)$ -minorant and so f is Z_{n+1} -integrable and $0 \leq [M]_a^x - (Z_{n+1}) \int_a^x f \leq \epsilon$.

Hence

$$(Z_n) \int_a^x f = (Z_{n+1}) \int_a^x f. \quad \square$$

Recall also that the Perron integral is an extension of the finite Lebesgue integral.

Proposition 6.12. Let f be a non-negative measurable function on $[a,b]$. Then

$$(Z_n^*) \int_a^b f = (Z_{n*}) \int_a^b f = (L) \int_a^b f.$$

(The last integral is the finite or infinite Lebesgue integral). In particular if $f \geq 0$ and if f is Z_n -integrable then f is Lebesgue integrable and

$$(Z_n) \int_a^b f = (L) \int_a^b f.$$

Proof. If $(L) \int_a^b f$ is finite then all follows immediately from theorem 6.11. If $(L) \int_a^b f = +\infty$ let $f_k = \min\{f, k\}$. Then f_k is Lebesgue integrable and it is easy to see that

$$(Z_{n*}) \int_a^b f \geq (Z_{n*}) \int_a^b f_k = (Z_n) \int_a^b f_k = (L) \int_a^b f_k$$

$$(k = 1, 2, \dots).$$

Then since $\lim (L) \int_a^b f_k = +\infty$, we also have $(Z_{n*}) \int_a^b f = +\infty$.

The rest follows at once. $\square$

Proposition 6.13. Let f, g, h be functions on $[a,b]$. Suppose that $(Z_n^*) \int_a^b f + (Z_n^*) \int_a^b g$ has a meaning, and that $f(x) + g(x) \geq h(x)$ whenever the left hand side has meaning. Then



$$(6.2) \quad (Z_n^*) \int_a^b h \leq (Z_n^*) \int_a^b f + (Z_n^*) \int_a^b g.$$

Proof. Let M_1 be an n -majorant of f and M_2 an n -majorant of g . (If one or the other of these does not exist then the right hand side of formula (6.2) is $+\infty$ in which case the result is obvious). Then $M_1 + M_2$ is an n -majorant for h . To see this let $P_{(n)} = M_1$, $Q_{(n)} = M_2$. Then $(P+Q)_{(n)} = M_1 + M_2$. Also

$$\delta_{n+1}(P+Q) \geq \delta_{n+1}P + \delta_{n+1}Q > -\infty$$

and

$$\delta_{n+1}(P+Q) \geq f + g \geq h$$

whenever the middle term has meaning. If $f + g$ has no meaning then we see that $\delta_{n+1}(P+Q) = +\infty$. Namely, one of $f(x)$, $g(x)$ is $+\infty$ so that one of $\delta_{n+1}P(x)$, $\delta_{n+1}Q(x)$ is $+\infty$ and the other is not $-\infty$ whence

$$\delta_{n+1}(P+Q)(x) = +\infty.$$

Thus $\delta_{n+1}(P+Q) \geq h$ on $[a, b]$ and $M_1 + M_2$ is an n -majorant. The result follows. $\square$

Remark. We of course have a similar theorem for lower integrals.

Corollary 6.14. Let f and g be Z_n -integrable in $[a, b]$. Let $h(x) = f(x) + g(x)$ whenever the right side has meaning. Then h is Z_n -integrable in $[a, b]$ and

$$(Z_n) \int_a^b h = (Z_n) \int_a^b f + (Z_n) \int_a^b g.$$



Proof.

$$\begin{aligned} (Z_{n*}) \int_a^b f + (Z_{n*}) \int_a^b g &\leq (Z_{n*}) \int_a^b h \leq \\ &\leq (Z_n^*) \int_a^b h \leq (Z_n^*) \int_a^b f + (Z_n^*) \int_a^b g. \end{aligned}$$

But the terms on the extremities are equal. $\square$

Proposition 6.15. If f is Z_n -integrable in $[a, b]$ and if $f = g$ a.e. in $[a, b]$ then g is also Z_n -integrable there and $(Z_n) \int_a^b f = (Z_n) \int_a^b g$.

Proof. Let $h(x) = 0$ whenever $f(x) = g(x)$ and $h(x) = +\infty$ otherwise. Then

$$0 = (L) \int_a^b h = (Z_n) \int_a^b h,$$

and by corollary 6.14

$$(Z_n) \int_a^b f = (Z_n) \int_a^b (h+f).$$

Moreover $g \leq h+f$ whenever the right side has a meaning and so

$$(Z_n^*) \int_a^b g \leq (Z_n) \int_a^b f.$$

Similarly considering the function defined by

$k(x) = -h(x)$ we see that

$$(Z_{n*}) \int_a^b g \geq (Z_n) \int_a^b f. \quad \square$$

Remark. According to proposition 6.15 the Z_n -integral may be defined naturally on certain of the



Lebesgue classes (those which contain a Z_n -integrable function). Combining propositions 6.15 and 6.14 we see that the Z_n -integral, considered as a functional on these classes, is additive. Again proposition 5.4 and 6.12 can be used to show that the Z_n -integral is a positive functional.

Theorem 6.16. Let f and P be defined in $[a,b]$ so that $P_{(n)}$ exists and so that

- a) $\delta_{n+1}P(x) \geq f(x)$ a.e. in $[a,b]$,
- b) $\delta_{n+1}P(x) > -\infty$ in $[a,b]$.

Then

$$(Z_n^*) \int_a^b f \leq [P_{(n)}]_a^b.$$

Proof. Let $h(x) = f(x)$ when $\delta_{n+1}P(x) \geq f(x)$, $h(x) = -\infty$ otherwise. Then $P_{(n)}$ is an n -majorant for h and so

$$(Z_n^*) \int_a^b h \leq [P_{(n)}]_a^b.$$

Define $k(x) = 0$ when $\delta_{n+1}P(x) \geq f(x)$, $k(x) = +\infty$ otherwise. Then $f(x) \leq h(x) + k(x)$ whenever the right side has meaning and so

$$(Z_n^*) \int_a^b f \leq (Z_n^*) \int_a^b h + 0 \leq [P_{(n)}]_a^b. \quad \square$$

From this theorem we see that we may enlarge the class of majorants to include those functions which

satisfy property 2) of definition 5.1, only almost everywhere. A similar result holds of course for minorants.

Proposition 6.17. Let f be Z_n -integrable on $[a,b]$. Let c be any finite number. Then cf is Z_n -integrable on $[a,b]$ and

$$(Z_n) \int_a^b (cf) = c(Z_n) \int_a^b f.$$

The proof of this proposition is easy when the cases $c \geq 0$ and $c < 0$ are considered separately. We omit the details.

Combining this result with our previous results we see that the Z_n -integral is a positive linear functional on the set of those Lebesgue classes for which it makes sense.

Proposition 6.18. Suppose g and h are Z_n -integrable functions and f is any measurable function on $[a,b]$. Suppose $g \leq f \leq h$ on $[a,b]$. Then f is Z_n -integrable as well.

Proof. Since we have, for almost every x , that $0 \leq f(x) - g(x) \leq h(x) - g(x)$ and since $h-g$ is Lebesgue integrable we see that $f-g$ is Lebesgue and hence Z_n -integrable. We now simply apply linearity. $\square$



Remark. In the same way we can show that if f is Z_n -integrable and bounded below (or above) by a Lebesgue integrable function, then f is already Lebesgue integrable.

Proposition 6.19, Dominated convergence theorem. Suppose g and h are Z_n -integrable functions on $[a,b]$. Let $\{f_k\}$ be any sequence of measurable functions on $[a,b]$ such that $g \leq f_k \leq h$ on $[a,b]$. Suppose $\lim f_k = f$ a.e. in $[a,b]$. Then f and each f_k is Z_n -integrable and

$$\lim_{k \rightarrow \infty} (Z_n) \int_a^b f_k = (Z_n) \int_a^b f.$$

Proof. The integrability is obvious from proposition 6.18. To see the last formula simply apply the usual Lebesgue theorem to the sequence $h-f_k$ and the limit $h-f$. Then apply linearity. $\square$

Proposition 6.20. Let $\{f_k\}$ be a sequence of Z_n -integrable functions on $[a,b]$ which converge uniformly there to a function f . Then f is Z_n -integrable in $[a,b]$ and $(Z_n) \int_a^b f = \lim (Z_n) \int_a^b f_k$.

Proof. Let $\epsilon > 0$. Let K be chosen so that $f_k - \epsilon \leq f \leq f_k + \epsilon$ for all $k \geq K$. Then f is Z_n -integrable. Moreover as $-\epsilon \leq f - f_k \leq \epsilon$ we see that



$$\lim(Z_n) \int_a^b (f-f_k) = 0$$

by the dominated convergence theorem. Now apply linearity. $\square$

Note also that $\{(Z_n) \int_a^x f_k\}$ converges uniformly to $(Z_n) \int_a^x f$ in $[a,b]$. Namely, $f - f_k$ is Lebesgue integrable and so $(L) \int_a^x (f-f_k)$ converges uniformly to zero on $[a,b]$.

Proposition 6.21. Let f be Z_n -integrable on $[a,b]$ and M an n -majorant. Let $P_{(n)} = M$. Then $P_{(n+1)}$ exists a.e. on $[a,b]$ and is Z_n -integrable there.

Proof. Let $F(x) = (Z_n) \int_a^x f$. Let $Q_{(n)} = F$ on $[a,b]$. (Such a Q exists by theorem 6.7). Then $(P-Q)_{(n)}$ is a continuous, non-decreasing function by propositions 6.6 and 6.5. But then $(P-Q)_{(n)} = (P-Q)^{(n)}$. Namely, there is a function R such that $R^{(n)} = (P-Q)_{(n)}$ which implies $(R-(P-Q))_{(n)} = 0$. Then R and $P-Q$ differ by a polynomial so that $(P-Q)^{(n)}$ exists. Since $(P-Q)^{(n)}$ is increasing and continuous we see that $(P-Q)^{(n+1)}$ exists a.e. in $[a,b]$. Thus

$$\begin{aligned} P_{(n+1)} &= (P - Q + Q)_{(n+1)} \\ &= (P-Q)^{(n+1)} + Q_{(n+1)} \\ &= (P-Q)^{(n+1)} + f \quad (\text{a.e. in } [a,b]). \end{aligned}$$

(We have applied theorem 6.9.) Since $(P-Q)^{(n+1)}$ is Lebesgue integrable we see that $P_{(n+1)}$ must be Z_n -integrable. $\square$

To finish this section we would like to present a theorem of Oliver [10]. We need some preliminaries.

Lemma 6.22. Let f be differentiable on $J = [a, b]$. Let $\alpha > 0$. Let $|f'| \geq \alpha$ on J . Then there is an interval $J_0 \subset J$ such that $m(J_0) \geq \frac{1}{4} m(J)$ and that $|f| \geq \frac{\alpha}{4} m(J)$ on J_0 .

Proof. By the intermediate value property for derivatives we may assume $f' \geq \alpha$. Let

$$c = \frac{a+b}{2}, \quad d = b-a.$$

If $f > 0$ on (c, b) , then $f(x) \geq \alpha(x-c)$ for $x \in [c, b]$ and we may choose $J_0 = [b - \frac{d}{4}, b]$. If $f(y) \leq 0$ for some $y \in (c, b)$, then $f(x) < \alpha(x-c)$ for $x \in [a, c]$ and we may choose $J_0 = [a, a + \frac{d}{4}]$. $\square$

Lemma 6.23. Let n be a natural number. Set $q = 1 + \dots + n = n(n+1)/2$. Let J be a compact interval. Let $\alpha > 0$ and $|F^{(n)}| \geq \alpha$ on J . Then there is an interval $K \subset J$ such that $m(K) \geq 4^{-n} m(J)$ and that $|F| \geq \alpha m(J)^n \cdot 4^{-q}$ on K .



Proof. For $n=1$ this is exactly lemma 6.22. Assume that $n > 1$ and that the assertion is true for $n-1$. Set $p = 1 + \dots + n - 1$. Then we have an interval $K_0 \subset J$ such that $m(K_0) \geq 4^{-n+1}m(J)$ and that

$$|F'| \geq \alpha m(J)^{n-1} 4^{-p} \text{ on } K_0.$$

By lemma 6.22 we have an interval $K \subset K_0$ such that $m(K) \geq \frac{1}{4} m(K_0) \geq 4^{-n}m(J)$ and that

$$\begin{aligned} |F| &\geq \alpha m(J)^{n-1} \cdot 4^{-p} \cdot \frac{1}{4} m(K_0) \geq \\ &\geq \alpha m(J)^{n-1} \cdot 4^{-p} \cdot 4^{-n} \cdot m(J) = \alpha m(J)^n 4^{-q} \text{ on } K. \quad \square \end{aligned}$$

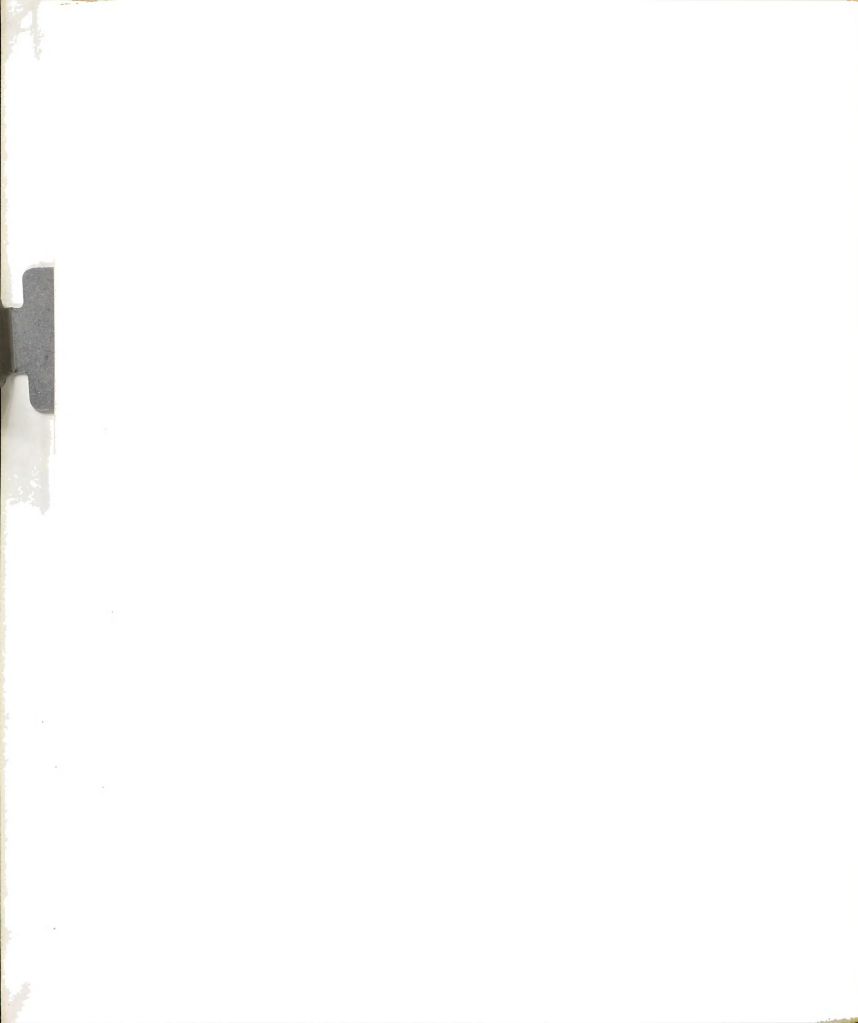
Theorem 6.24. Let F be a p -th indefinite integral of a function which is bounded below on some neighborhood of 0 ($p > 0$). Let $F_{(p)}(0) = 0$. Then $F^{(p)}(0) = 0$.

Proof. Let $\alpha > 0$, $\delta > 0$. Let $f > -\alpha$ on $(-\delta, \delta)$ and let F be a p -th indefinite integral of f . Let $g(x) = \int_0^x f$. We may assume $F(0) = \dots = F^{(p-1)}(0) = 0$. Then

$$(6.3) \quad F(x) = o(x^p) \text{ as } x \rightarrow 0.$$

Assume that $D^+g(0) > 0$. Then there is an ϵ , $0 < \epsilon < 2\alpha$, and numbers $a_n > 0$ such that

$$g(a_n) > \epsilon a_n \quad (n = 1, 2, \dots), \quad a_n \rightarrow 0.$$



For $a_n < x < a_n + a_n \epsilon / 2\alpha$ we have

$$\begin{aligned} g(x) &= g(a_n) + \int_{a_n}^x f > \epsilon a_n - (x - a_n)\alpha > \\ &> \epsilon a_n - \alpha a_n \epsilon / 2\alpha = \frac{1}{2} a_n \epsilon. \end{aligned}$$

Since $F^{(p-1)} = g$ we have, by lemma 6.23, an x_n in $(a_n, a_n + a_n \epsilon / 2\alpha)$ such that

$$|F(x_n)| \geq \frac{1}{2} a_n \epsilon (a_n \cdot \frac{\epsilon}{2\alpha})^{p-1} \cdot 4^{-r} = (a_n)^{pQ}$$

where

$$r = \frac{p(p-1)}{2}, \quad Q = \frac{\epsilon^p}{2^p \cdot \alpha^{p-1} \cdot 4^r}.$$

Since $\epsilon / 2\alpha < 1$, we have $x_n < 2a_n$ whence

$$|F(x_n)| \geq x_n^p \cdot \frac{Q}{2^p} \quad (n = 1, 2, \dots)$$

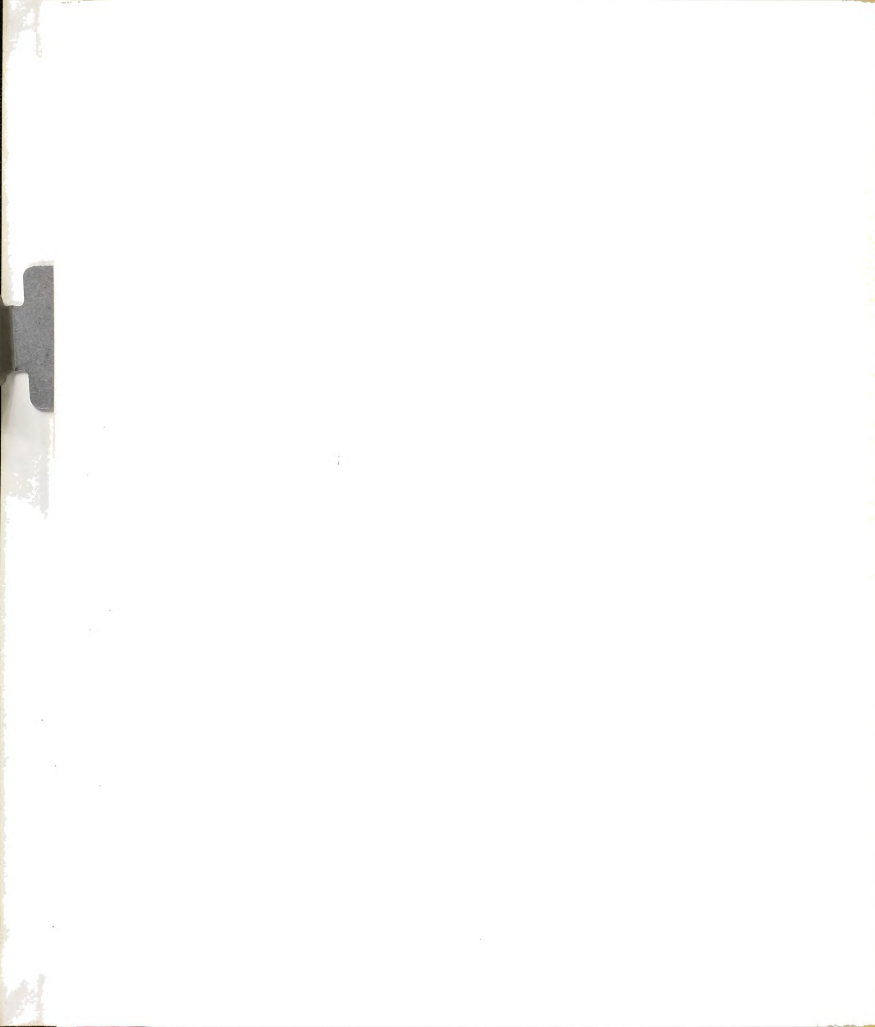
which contradicts (6.3). Thus $D^+g(0) \leq 0$. We may

show similarly that $D_+g(0) \geq 0$ so that $g'^+(0) = 0$.

Analogously $g'^-(0) = 0$ so that $g'(0) = F^{(n)}(0) = 0$. $\square$

Corollary 6.25. Suppose $M_{(n)}$ exists on $[a, b]$ and is bounded below there. Then it is an ordinary derivative.

Proof. According to theorem 5.1, $M_{(n)}$ is $Z_{(n-1)}$ -integrable on $[a, b]$. By proposition 6.12, it is thus Lebesgue integrable. Thus M is an n -th indefinite integral of $M_{(n)}$. It follows easily from theorem 6.24 that $M_{(n)}(x) = M^{(n)}(x)$ for any $x \in [a, b]$. $\square$



CHAPTER VII

INTEGRATION BY PARTS FOR THE Z_n -INTEGRAL

Notation. In what follows let $BV[a,b]$ signify the class of functions which have finite variation on the interval $[a,b]$. Let $(R) \int_a^b f dg$, for $g \in BV[a,b]$ and f continuous, be the Riemann-Stieltjes integral of f with respect to g in the sense that the infimum of the upper sums associated with f and the positive (negative) variation of g is the same as the supremum of the lower sums.

Fundamental to all that follows is the next theorem.

Theorem 7.1, Integration by Parts.

a) Let f be Perron integrable in $[a,b]$,

$$F(x) = (Z_0) \int_a^x f. \text{ Let } G \in BV[a,b].$$

Then

$$(Z_0) \int_a^b fG + (R) \int_a^b FdG = [FG]_a^b.$$

b) Let $n \geq 1$. Let f be Z_n -integrable on $[a,b]$. Let $F(x) = (Z_n) \int_a^x f$. Let G and γ be defined on $[a,b]$ such that $\gamma \in BV[a,b]$ and $G^{(n-1)}$ is an



indefinite integral of γ . Then

fG is Z_n -integrable and

$$(Z_n) \int_a^b fG + (Z_{n-1}) \int_a^b FG' = [FG]_a^b.$$

The proof is quite long and involves a chain of lemmas. Later we shall show that the strong conditions put on the function G are really necessary. This proof is due to J. Marik.

We assume that a) is well known but for completeness we present a proof.

Proof of Theorem 7.1, a). Assume G is positive and non-decreasing. (This is sufficient by the linearity of the integrals). Let $a = x_0 < x_1 < \dots < x_m = b$ be a partition of $[a, b]$. Let

$$\gamma_k = \min\{F(x) : x_{k-1} \leq x \leq x_k\}.$$

If M is a O -majorant of f on $[a, b]$ then one easily sees that $G(M - \gamma_k)$ is a O -majorant of fG on $[x_{k-1}, x_k]$. Thus

$$(Z_0^*) \int_{x_{k-1}}^{x_k} fG \leq [G(F - \gamma_k)]_{x_{k-1}}^{x_k}.$$

Thus

$$(Z_0^*) \int_a^b fG \leq [GF]_a^b - \sum_k [\gamma_k G]_{x_{k-1}}^{x_k}$$

for each such partition, and so

$$(Z_O^*) \int_a^b fG \leq [GF]_a^b - (R) \int_a^b FdG.$$

Now replacing F and f by $-F$ and $-f$ we see

$$(Z_{O*}) \int_a^b fG \geq [GF]_a^b - R \int_a^b FdG. \quad \square$$

Lemma 7.2. Let φ, ψ be functions on $[a, b]$.

Let q be a natural number. Let $\varphi^{(q)}$ be continuous on $[a, b]$ and let $\psi^{(q-1)}$ be absolutely continuous on $[a, b]$. Let R be an indefinite integral of $\varphi\psi^{(q)}$. Set

$$v = \sum_{j=0}^{q-1} (-1)^j \varphi^{(q-j-1)} \psi^{(j)} + (-1)^q R.$$

Then $v' = \varphi^{(q)} \psi$.

Proof.

$$\begin{aligned} \int_a^x \varphi^{(q)} \psi &= [\varphi^{(q-1)} \psi]_a^x - \int_a^x \varphi^{(q-1)} \psi' = \dots = \\ &= [\varphi^{(q-1)} \psi - \varphi^{(q-2)} \psi' + \dots + (-1)^{q-1} \varphi \psi^{(q-1)}]_a^x + (-1)^q \int_a^x \varphi \psi^{(q)} = \\ &= v(x) - v(a). \quad \square \end{aligned}$$

Definition 7.1. Let $n > 1$ be an integer. Let M be a continuous function on $[a, b]$. Let $\gamma \in BV[a, b]$. Let $g^{(n-2)}$ be an indefinite integral of γ . Let $K_0, \dots, K_n$ be functions on $[a, b]$ with the following properties:



- a) $K_k^{(k)} = M g^{(k)} \quad (k = 0, 1, \dots, n-2),$
- b) $K_{n-1}^{(n-2)}$ is an indefinite integral of $M \gamma,$
- c) there is a number c such that $K_n^{(n-2)},$
is an indefinite integral of $c + \int_a^x M d\gamma \quad (x \in [a, b]).$

Then we say that $K_0, \dots, K_n$ have property ω with respect to M, g on $[a, b].$

Lemma 7.3. Let a, b, n, g, γ be as above. Let M be a function on $[a, b]$ such that $M^{(n)}$ is continuous on $[a, b].$ Let $K_0, \dots, K_n$ have property ω with respect to $M, g.$ Set

$$S = \sum_{k=0}^n (-1)^k \binom{n}{k} K_k.$$

Then $S^{(n)} = M^{(n)} g.$

Proof. We first show that

$$(7.1) \quad S^{(r)} = \sum_{j=0}^r (-1)^j \binom{n+j-r-1}{j} M^{(r-j)} g^{(j)} + \sum_{j=r+1}^n (-1)^j \binom{n}{j} K_j^{(r)}$$

for $r = 0, \dots, n-2.$ The relation (7.1) is obvious for $r=0.$ If it is true for some $r \quad (0 \leq r < n-2),$ then $S^{(r+1)} = A + B + C + D + E$ where

$$A = \sum_{j=0}^r (-1)^j \binom{n+j-r-1}{j} M^{(r+1-j)} g^{(j)},$$

$$B = \sum_{j=0}^{r-1} (-1)^j \binom{n+j-r-1}{j} M^{(r-j)} g^{(j+1)},$$



$$C = (-1)^r \binom{n-1}{r} M_g^{(r+1)},$$

$$D = (-1)^{r+1} \binom{n}{r+1} K_{r+1}^{(r+1)},$$

$$E = \sum_{j=r+2}^n (-1)^j \binom{n}{j} K_j^{(r+1)}.$$

We can re-write B and D as

$$B = - \sum_{j=1}^r (-1)^j \binom{n+j-r-2}{j-1} M^{(r+1-j)}_g(j),$$

$$D = (-1)^{r+1} \binom{n}{r+1} M \cdot g^{(r+1)}.$$

Then, applying the relation $\binom{p+1}{q+1} - \binom{p}{q} = \binom{p}{q+1}$, we get

$$A+B = \sum_{j=0}^r (-1)^j \binom{n-j-r-2}{j} M^{(r+1-j)}_g(j)$$

and

$$C+D = (-1)^{r+1} \binom{n-1}{r+1} M_g^{(r+1)},$$

which completes the induction and proves (7.1). If we put $r = n-2$ into (7.1) we see

$$(7.2) \quad S^{(n-2)} = \sum_{j=0}^{n-2} (-1)^j (j+1) M^{(n-2-j)}_g(j) + (-1)^{n-1} n K_{n-1}^{(n-2)} + (-1)^n K_n^{(n-2)}.$$

Set

$$R = K_{n-1}^{(n-2)}, \quad T_p = \sum_{j=p}^{n-2} (-1)^j M^{(n-2-j)}_g(j) + (-1)^{n-1} R$$

$$(p = 0, 1, \dots, n-2).$$

Set

$$Z = R - K_n^{(n-2)}.$$

Then

$$\sum_{p=0}^{n-2} T_p = \sum_{j=0}^{n-2} (-1)^j (j+1) M^{(n-2-j)} g(j) + (-1)^{n-1} (n-1) R,$$

so that

$$(7.3) \quad S^{(n-2)} = \sum_{p=0}^{n-2} T_p + (-1)^{n-1} Z.$$

We also have

$$T_p = \sum_{i=0}^{n-2-p} (-1)^{i+p} M^{(n-2-i-p)} g(i+p) + (-1)^{n-1} R.$$

If we put $\varphi = (-1)^p M$, $\psi = g^{(p)}$, $q = n-1-p$ then we get from lemma 7.2 that

$$(7.4) \quad T'_p = (-1)^p M^{(n-1-p)} g^{(p)}.$$

Further

$$\begin{aligned} Z(x) &= (c_1 + \int_a^x M \gamma) - c_2 - \int_a^x (c_3 + \int_a^t M d\gamma) dt = \\ &= c_4 + \int_a^x (-c_3 + M(t) \gamma(t) - \int_a^t M d\gamma) dt = \\ &= c_4 + \int_a^x (c_5 + \int_a^t M' \gamma) dt \quad (\text{the } c_j \text{'s are constant}). \end{aligned}$$

Thus

$$(7.5) \quad Z'(x) = c_5 + \int_a^x M' \gamma.$$

Now from (7.3) and (7.4) we get

$$(7.6) \quad S^{(n-1)} = \sum_{p=0}^{n-2} (-1)^p M^{(n-1-p)} g^{(p)} + (-1)^{n-1} Z',$$

and, from (7.5), Z' is an indefinite integral of $M' \gamma$.

If we put $\varphi = M'$, $q = n-1$, $\psi = g$ we have $\psi^{(q-1)} = g^{(n-2)}$

on $[a, b]$ and $\psi^{(q)} = \gamma$ a.e. so the formula $S^{(n)} = M^{(n)}g$ follows at once from (7.6) and lemma 7.2. $\square$

Lemma 7.4. Let $n > 1$ be an integer. Let M, g, γ be functions defined on $I = [a, b]$. Let $M_{(n)}$ exist on I and let $\gamma \in BV(I)$. Let $g^{(n-2)}$ be an indefinite integral of γ . Then there is a function S such that

$$(7.7) \quad S_{(n)} = M_{(n)}g \quad \text{on } I.$$

For any such function S and for any $x \in I$ for which $g'(x)$ is finite we have

$$(7.8) \quad A_{n+1}(S, x, h) = M_{(n)}(x)g'(x) + g(x)\theta_{n+1}(M, x, h) + o(1).$$

Note. If $n > 2$ then $g'(x)$ is finite for every x . If $n=2$ then g' is finite except possibly on a countable set and even here we have, for $h > 0$ (respectively $h < 0$), that formula (7.8) holds if we replace $g'(x)$ by $g'^+(x)$ ($g'^-(x)$), the right (left) hand derivative.

Proof. Let $K_0, \dots, K_n$ have property ω with respect to M, g . Let $S = \sum_{k=0}^n (-1)^k \binom{n}{k} K_k$. Let $x \in I$.

We may assume $x=0$. Let P be a polynomial of degree $\leq n$ such that the function $\bar{M} = M - P$ fulfills

$$(7.9) \quad \bar{M}(t) = o(t^n) \quad \text{as } t \rightarrow 0.$$



Let $\bar{K}_k$ be functions such that $\bar{K}_k^{(k)} = \bar{M}g^{(k)}$ ($k = 0, \dots, n-2$),

$$\bar{K}_{n-1}^{(n-2)}(t) = \int_0^t \bar{M} \gamma, \quad \bar{K}_n^{(n-2)}(t) = \int_0^t \left(\int_0^s \bar{M} d\gamma \right) ds,$$

and $\bar{K}_k^{(j)}(0) = 0$ whenever $0 \leq j \leq k-1$ and $k \leq n-1$ or $0 \leq j \leq n-3$ and $k=n$. Obviously $\bar{K}_k(t) = o(t^{n+k})$ ($k = 0, \dots, n-1$) and $\bar{K}_n(t) = o(t^{2n-1})$. If we put

$$\bar{S} = \sum_{k=0}^n (-1)^k \binom{n}{k} \bar{K}_k$$

we have

$$(7.10) \quad \bar{S}(t) = \bar{M}(t)g(t) + o(t^{n+1}).$$

Set $H_k = K_k - \bar{K}_k$. It is easy to see that $H_0, \dots, H_n$ have property w with respect to P, g . Define

$$V = \sum_{k=0}^n (-1)^k \binom{n}{k} H_k = S - \bar{S}.$$

By lemma 7.3 we have $V^{(n)} = P^{(n)}g$. Let G satisfy $G^{(n)} = g$. Let $\alpha = M^{(n)}(0)$. Obviously $P^{(n)} = \alpha$ so that $V^{(n)} = \alpha g = \alpha G^{(n)}$. Thus there is a polynomial Q of degree $< n$ such that $S - \bar{S} = V = \alpha G + Q$ or

$$(7.11) \quad S = Q + \alpha G + \bar{S}.$$

There is a polynomial Q_1 of degree $< n$ such that

$$(7.12) \quad G(t) = Q_1(t) + \frac{t^n}{n!} g(0) + o(t^n).$$

It follows from (7.9)-(7.12) that

$$S(t) = Q_2(t) + \frac{t^n}{n!} \alpha g(0) + o(t^n),$$



where $Q_2 = Q + \alpha Q_1$ is a polynomial of degree $< n$.

Thus $S_{(n)}(0) = \alpha g(0)$. If $g'(0)$ is finite we have

$$g(t) = g(0) + o(t)$$

and it follows from (7.9) and (7.10) that

$$(7.13) \quad \bar{S}(t) = g(0)\bar{M}(t) + o(t^{n+1}) = g(0) \frac{t^{n+1}}{(n+1)!} \theta_{n+1}(M, 0, t) + o(t^{n+1}).$$

Further,

$$G(t) = Q_1(t) + \frac{t^n}{n!} g(0) + \frac{t^{n+1}}{(n+1)!} g'(0) + o(t^{n+1}),$$

so that from (7.11) and (7.13)

$$S(t) = Q_2(t) + \frac{t^n}{n!} \alpha g(0) + \frac{t^{n+1}}{(n+1)!} (\alpha g'(0) + g(0) \theta_{n+1}(M, 0, t)) + o(t^{n+1})$$

which completes the proof. $\square$

Theorem 7.5. Let n be an integer, $n > 1$.

Let f, G, γ be functions on $[a, b]$ such that f is Z_n -integrable on $[a, b]$, $\gamma \in BV[a, b]$ and $G^{(n-2)}$ is an indefinite integral of γ . Then for any indefinite Z_n -integral F of f we have

$$(Z_n) \int_a^b (FG' + fG) = [FG]_a^b.$$

Proof. Assume first that G, G'^+ and G'^- are positive functions. (If $n > 2$ there is no need to consider G'^+ and G'^- separately as G' exists in this case.) Let U be an n -majorant of f such that $U(a) = F(a)$. By lemma 7.4 there is a function S such that



$$S_{(n)} = P_{(n)} G = UG.$$

Moreover, by formula (7.8) and the note following lemma 7.4

$$\begin{aligned} \delta_{n+1} S(x) \geq \min \{ & U(x) G'^{+}(x) + G(x) \delta_{n+1} P(x), U(x) G'^{-}(x) + \\ & + G(x) \delta_{n+1} P(x) \}. \end{aligned}$$

Then $\delta_{n+1} S(x) > -\infty$ for every $x \in [a, b]$ and

$$\delta_{n+1} S(x) \geq F(x) G'(x) + G(x) f(x)$$

whenever $G'(x)$ exists (that is, except for a countable set when $n=2$, and everywhere if $n > 2$). Applying theorem 6.16 we see that

$$(Z_n^*) \int_a^b (FG' + fG) \leq [UG]_a^b$$

and so

$$(Z_n^*) \int_a^b (FG' + fG) \leq [FG]_a^b.$$

A similar consideration for n -minorants shows that

$$(Z_{n*}) \int_a^b (FG' + fG) \geq [FG]_a^b$$

and so the theorem is proved for this case.

Returning to the general case we see easily that G'^{+} is bounded on $[a, b)$ and G'^{-} is bounded on $(a, b]$. Namely, if $n > 2$ then G' is continuous and if $n=2$ then G is an indefinite integral of a function of bounded variation. Thus we can find a linear

function G_2 on $[a,b]$ such that $G_2 > 0$ on $[a,b]$, $G_2' > 0$ on $[a,b]$ and that the function $G_1 = G + G_2$ satisfies the conditions of the special case considered above. Let

$$\gamma_2 = G_2^{(n-1)} \quad \text{and} \quad \gamma_1 = \gamma + \gamma_2.$$

Then $\gamma_j \in BV[a,b]$ and $G_j^{(n-2)}$ is an indefinite integral of γ_j ($j = 1, 2$). Then we may apply the first part of the proof to each G_j , so

$$(Z_n) \int_a^b (FG_j' + fG_j) = [FG_j]_a^b \quad (j = 1, 2),$$

from which the assertion follows at once by linearity. $\square$

The analogues of theorems 7.4 and 7.5 for the case $n=1$ are slightly different.

Lemma 7.6. Let M, g be functions on $[0,1]$. Let M be Z_0 -integrable, let $g \in BV[0,1]$ and let $\text{var}(g, [0, h]) = O(h)$. Let P be an indefinite Z_0 -integral of M and let S be an indefinite Z_0 -integral of Mg . Assume $P'(0)$ is finite. Set

$$T(h) = \frac{2}{h^2} \int_0^h (g(t) - g(0)) dt,$$

$$\lambda = \limsup_{h \rightarrow 0} h^{-1} \text{var}(g, [0, h]).$$



Then $\lambda < \infty$ and

$$(7.14) \quad \begin{aligned} -\lambda &\leq \delta_1 g(0) \leq \liminf_{h \rightarrow 0} T(h), \\ \limsup_{h \rightarrow 0} T(h) &\leq \Delta_1 g(0) \leq \lambda, \end{aligned}$$

and

$$(7.15) \quad S(h) = S(0) + hg(0)P'(0) + \frac{h^2}{2}(T(h)P'(0) + g(0)\theta_2(P,0,h)) + o(h^2).$$

Proof. The inequalities $\Delta_1 g(0) \leq \lambda < +\infty$ are obvious. Let $\beta > \Delta_1 g(0)$. Then for sufficiently small h we have $g(h) - g(0) < h\beta$ and $T(h) \leq \frac{2}{h^2} \int_0^h t\beta dt = \beta$ so that

$$\limsup_{h \rightarrow 0} T(h) \leq \Delta_1 g(0).$$

This proves (7.14).

If $M \equiv 1$ then $P'(0) = 1$ and $\theta_2(P,0,h) \equiv 0$, $S(h) - S(0) = \int_0^h g = hg(0) + \int_0^h (g(t) - g(0))dt = hg(0) + \frac{h^2}{2} T(h)$ so that (7.15) holds in this case. If $g \equiv 1$ then $T(h) \equiv 0$, $S(h) - S(0) = P(h) - P(0) = hP'(0) + \frac{h^2}{2} \theta_2(P,0,h)$ so that (7.15) holds in this case also. If we hold either M or g fixed in formula (7.15) it is easy to see that the formula is linear with respect to the other so that we may assume $P'(0) = g(0) = 0$. Let $K(x) = P(x) - P(0)$. Then $K(t) = o(t)$, $g(t) = O(t)$ and since $\text{var}(g, [0, h]) = O(h)$ we have $\int_0^h Kdg = o(h^2)$ so that



$$S(h) - S(0) = \int_0^h Mg = K(h)g(h) - \int_0^h Kdg = o(h^2)$$

which completes the proof. $\square$

Theorem 7.7. Let f, G be functions on $[a, b]$. Let f be Z_1 -integrable on $[a, b]$. Let G be continuous and in $BV[a, b]$. Suppose $\text{var}(G, [x-h, x+h]) = O(h)$ for each $x \in (a, b)$. Let F be an indefinite Z_1 -integral of f . Then

$$(Z_1) \int_a^b (FG' + fG) = [FG]_a^b.$$

Proof. If G is constant the conclusion is obvious so applying a simple linearity argument we may assume that $G > 0$. Let $\epsilon > 0$. Let M be a 1-majorant of f on $[a, b]$ such that $M(a) = F(a)$ and $M(b) < F(b) + \epsilon$. Let S be an indefinite Z_0 -integral of MG and let $P' = M$ on $[a, b]$. Using theorem 7.1 a) and the well known mean value theorem for the Riemann-Stieltjes integral we see that $S' = MG$ on $[a, b]$. Moreover, from lemma 7.6 we have that $\delta_2 S(x) > -\infty$ for every x and that

$$\delta_2 S(x) = G'(x)M(x) + G(x)\delta_2 P(x)$$

whenever $G'(x)$ exists. Thus

$$\begin{aligned} \delta_2 S(x) &\geq G'(x)M(x) + G(x)f(x) \geq \\ &\geq G'(x)F(x) + G(x)f(x) - \epsilon |G'(x)| \quad \text{a.e. in } [a, b]. \end{aligned}$$



Thus

$$\begin{aligned}
 & -\epsilon \int_a^b |G'| + (Z_1^*) \int_a^b (FG' + fG) \\
 & = (Z_1^*) \int_a^b (-\epsilon |G'| + FG' + fG) \leq \\
 & \leq [S']_a^b = [MG]_a^b \leq [FG]_a^b + \epsilon [G]_a^b .
 \end{aligned}$$

It follows that

$$(Z_1^*) \int_a^b (FG' + fG) \leq [FG]_a^b .$$

We can show similarly that

$$(Z_{1*}) \int_a^b (FG' + fG) \geq [FG]_a^b$$

so that the theorem is proved. $\square$

Proof of Theorem 7.1, b). Let $n=1$, and suppose that F, f, G and γ satisfy the conditions of the theorem. Then F, f and G satisfy the conditions of theorem 7.7, so that

$$(Z_1) \int_a^b (FG' + fG) = [FG]_a^b .$$

Moreover, by corollary 6.8, F is Z_0 -integrable and $\gamma \in BV[a, b]$ so that $F\gamma$ is Z_0 -integrable and

$$(Z_0) \int_a^b F\gamma = (Z_0) \int_a^b FG' ,$$

by proposition 6.15, which in turn is $(Z_1) \int_a^b FG'$ by theorem 6.11. Thus



$$[FG]_a^b = (Z_1) \int_a^b (FG' + fG) = (Z_0) \int_a^b FG' + (Z_1) \int_a^b fG$$

as was to be shown.

Let $n > 1$. Let F, f, G and γ satisfy the conditions of the theorem. Then by theorem 6.7 there is a function P such that $P_{(n)} = F$. Moreover $(G')^{(n-2)}$ is the indefinite integral of a function in $BV[a, b]$ so, by lemma 7.4, there is a function S such that

$$S_{(n)} = P_{(n)} G' = FG'.$$

But then, by theorem 5.1, FG' is Z_{n-1} -integrable.

Thus, since F, f, G, γ satisfy theorem 7.5, we see that

$$\begin{aligned} [FG]_a^b &= (Z_n) \int_a^b (FG' + fG) = \\ &= (Z_n) \int_a^b fG + (Z_n) \int_a^b FG' = \\ &= (Z_n) \int_a^b fG + (Z_{n-1}) \int_a^b FG'. \quad \square \end{aligned}$$



CHAPTER VIII

RELATION BETWEEN THE Z_n -INTEGRAL AND THE $C_n P$ -INTEGRAL

Theorem 8.1. The Z_n -integral is identical to the $C_n P$ -integral ($n = 0, 1, 2, \dots$).

The proof is by induction. For $n=0$ it is certainly true as both the Z_0 -integral and the $C_0 P$ -integral are precisely the Perron integral. We assume that, for $0 \leq k \leq n-1$, the $C_k P$ -integral is identical to the Z_k -integral. We need some lemmas (which will also be useful later).

Lemma 8.2. If $P_{(n)}$ is finite in $[a, b]$ then for every $x \in [a, b]$ and for every h such that $x+h \in [a, b]$ we have

$$(8.1) \quad P(x+h) = P(x) + \sum_{k=1}^{n-1} \frac{h^k}{k!} P_{(k)}(x) + \frac{h^n}{n!} C_n(P_{(n)}, x, h).$$

Proof. According to the definition and the induction assumption

$$\frac{h^n}{n!} C_n(P_{(n)}, x, h) = \frac{1}{(n-1)!} (Z_{n-1}) \int_x^{x+h} (x+h-\xi)^{n-1} P_{(n)}(\xi) d\xi.$$



Applying theorem 7.1 and theorem 5.1 we see that this is

$$\begin{aligned}
& \frac{1}{(n-1)!} [P_{(n-1)}(\xi) (x+h-\xi)^{n-1}]_x^{x+h} + \\
& + \frac{1}{(n-2)!} (Z_{n-2}) \int_x^{x+h} (x+h-\xi)^{n-2} P_{(n-1)}(\xi) d\xi \\
& = - \frac{h^{n-1}}{(n-1)!} P_{(n-1)}(x) + \frac{1}{(n-2)!} (Z_{n-2}) \int_x^{x+h} (x+h-\xi)^{n-2} P_{(n-1)}(\xi) d\xi \\
& = \dots = \text{(repeating the first step)} \\
& = - \frac{h^{n-1}}{(n-1)!} P_{(n-1)}(x) - \frac{h^{n-2}}{(n-2)!} P_{(n-2)}(x) - \dots + \\
& \quad + \frac{1}{0!} (Z_0) \int_x^{x+h} P_{(1)}(\xi) d\xi \\
& = - \sum_{k=1}^{n-1} \frac{h^k}{k!} P_{(k)}(x) + P(x+h) - P(x) \quad \text{as was to be shown.} \quad \square
\end{aligned}$$

Lemma 8.3. Suppose $P_{(n)}$ is finite in $[a, b]$.

Then (a) $C_n D_* P_{(n)} = \delta_{n+1} P$,

(b) $C_n D^* P_{(n)} = \Delta_{n+1} P$ and

(c) $C_n D P_{(n)} = P_{(n+1)}$ provided one side exists.

Proof. We prove (a) ((b) is similar and (c) follows from (a) and (b)). According to lemma 8.2,



$$\theta_{n+1}(P, x, h) =$$

$$\begin{aligned} \frac{P(x+h) - \sum_{k=0}^n \frac{h^k}{k!} P^{(k)}(x)}{h^{n+1}/(n+1)!} &= \frac{\frac{h^n}{n!} C_n(P_{(n)}, x, h) - \frac{h^n}{n!} P_{(n)}(x)}{h^{n+1}/(n+1)!} = \\ &= \frac{C_n(P_{(n)}, x, h) - P_{(n)}(x)}{h/(n+1)}. \end{aligned}$$

Now simply take the lower limit of both sides to see (a). $\square$

Lemma 8.4. Let M be defined on $[a, b]$. Then M is C_n -continuous on $[a, b]$ if and only if there is a function G on $[a, b]$ such that $G_{(n)} = M$. That is, only n -th exact Peano derivatives are C_n -continuous on an interval.

Proof. Suppose $G_{(n)} = M$. Then according to lemma 8.2,

$$\begin{aligned} \lim_{h \rightarrow 0} C_n(M, x, h) &= \lim_{h \rightarrow 0} \frac{G(x+h) - \sum_{k=0}^{n-1} \frac{h^k}{k!} G^{(k)}(x)}{h^n/n!} \\ &= G_{(n)}(x) = M(x) \quad \text{so that } M \\ &\quad \text{is } C_n\text{-continuous.} \end{aligned}$$

Conversely suppose M is C_n -continuous on $[a, b]$. Then it is C_{n-1} -integrable and thus Z_{n-1} -integrable by the induction hypothesis. Let

$$g_{n-1}(x) = (Z_{n-1}) \int_a^x M.$$



By corollary 6.8, g_{n-1} is Z_{n-2} -integrable and so we may define successively

$$g_k(x) = (Z_k) \int_a^x g_{k+1} \quad (0 \leq k \leq n-2).$$

Put $G = g_0$. Then applying theorem 7.1 $n-2$ times we have

$$\begin{aligned} \frac{1}{(n-1)!} (Z_n) \int_x^{x+h} (x+h-\xi)^{n-1} M(\xi) d\xi &= \\ &= G(x+h) - G(x) - \sum_{k=1}^{n-1} \frac{h^k}{k!} g_k(x). \end{aligned}$$

Then, as M is C_n -continuous, we see that $G_{(n)}(x) = M(x)$ for each x . (Also $G_{(k)}(x) = g_k(x)$ ($k = 0, \dots, n-1$).) $\square$

Thus from lemmas 8.3 and 8.4 we see that the class of $(n+1)$ th e.P.d.'s is exactly the same as the class of exact C_n -derivatives.

We now finish the proof of theorem 8.1 by showing that the $C_n P$ -integral is identical to the Z_n -integral.

Let f be any function defined on $[a, b]$. Let M be a C_n -majorant of f on $[a, b]$. Then M is C_n -continuous so, by lemma 8.4, there is a function P such that $P_{(n)} = M$ and, by lemma 8.3,

$$\delta_{n+1} P(x) = C_n D_* P_{(n)}(x) = C_n D_* M(x)$$

so that $-\infty < \delta_{n+1} P(x) \leq f(x)$. Thus M is an n -majorant



as well and so

$$(Z_n^*) \int_a^b f \leq [M]_a^b$$

for each such M . It follows that

$$(Z_n^*) \int_a^b f \leq (C_n P^*) \int_a^b f.$$

If we consider C_n -minorants we see that

$$(Z_{n*}) \int_a^b f \geq (C_n P_*) \int_a^b f$$

as well, so that the Z_n -integral extends the $C_n P$ -integral.

On the other hand, if M is an n -majorant of f on $[a, b]$ then there is a function P such that $P_{(n)} = M$ so that M is C_n -continuous. Moreover $C_n D_* M(x) = \delta_{n+1} P(x)$ by lemma 8.3 and so M is a C_n -majorant of f . But then

$$(C_n P^*) \int_a^b f \leq [M]_a^b \quad \text{for each such } M$$

and so

$$(C_n P^*) \int_a^b f \leq (Z_n^*) \int_a^b f.$$

Similarly considering n -minorants we see that

$$(C_n P_*) \int_a^b f \geq (Z_{n*}) \int_a^b f.$$

Thus we see that if f is integrable in the sense of either Z_n or $C_n P$ then it is also integrable in the other sense and

$$(C_n P) \int_a^b f = (Z_n) \int_a^b f. \quad \square$$



CHAPTER IX

EXAMPLES

In this section we would like to give a few examples to illustrate the extent of generalization of the Z_n -integral and also the necessity of the strong assumptions in the integration by parts theorem.

Example 9.1. Let $n \geq 1$. Let $g(x) = x^{n+1} \sin x^{-n}$ ($x \neq 0$), $g(0) = 0$. Then $g(t) = o(t^n)$ as $t \rightarrow 0$ so that $g'(0) = g_{(2)}(0) = \dots = g_{(n)}(0) = 0$. Thus, as g is certainly n -times differentiable away from $x=0$, we see that $g_{(n)}(x)$ exists for every x . Thus, by theorem 5.1, $g_{(n)}$ is Z_{n-1} -integrable in $[-1,1]$. But if $n > 1$, $g_{(n)}$ is not Z_{n-2} -integrable there. Since if $g_{(n)}$ were Z_{n-2} -integrable, its indefinite integral, $g_{(n-1)}$, would then be Z_{n-3} -integrable, etc. Continuing this we see that an $n-1$ times iterated integral of $g_{(n)}$ would be continuous (since the last integral is a Perron integral) and also equal to g' plus a polynomial. But this is impossible as



$$g'(x) = \begin{cases} -n \cos x^{-n} + (n+1)x^n \sin x^{-n}, & x \neq 0 \\ 0 & x = 0 \end{cases}$$

which is not continuous.

It also follows from this that $g_{(n)}$ is not an $(n-1)$ th e.P.d. of any function, for if it were it would be Z_{n-2} -integrable. Thus the Z_n -integral is a proper generalization of the Z_{n-1} -integral and, for any n there is an n -th generalized derivative on $[-1,1]$ which is not Z_{n-2} integrable on $[-1,1]$.

Example 9.2. Let n be a natural number. There is a function G which is n times differentiable in a neighborhood of zero, and a function f which is an n -th exact Peano derivative and yet fG is not Z_m -integrable for any m in any neighborhood of 0.

Remark. Such a function f is of course Z_{n-1} -integrable. Thus it is essential to assume more than the n -fold differentiability of G to obtain an integration by parts theorem for the Z_{n-1} -integral.

We will show that G and F may be taken to be

$$G(x) = \begin{cases} x^{n^2+3n-2} \cos \frac{1}{x^{n+3}} & x \neq 0, \\ 0 & x = 0; \end{cases}$$



$$f(x) = \begin{cases} \frac{1}{x^{n^2+3n-1}} \cos \frac{1}{x^{n+3}} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

To establish this we shall need some lemmas.

Lemma 9.1. Let

$$M(x) = \frac{1}{x^r} \exp\left(\frac{i}{x^s}\right) \quad \text{where } x > 0, \quad r \text{ real,} \\ s > 0, \quad i^2 = -1.$$

Let $F(x) = \int_x^1 M(t) dt \quad (x > 0)$. Then

$$F(x) = O\left(\frac{1}{x^{r-s-1}}\right) + \text{constant}.$$

Proof. If $r = 1+s$ this is trivial, so assume otherwise. We integrate F by parts to see

$$\begin{aligned} F(x) &= \int_x^1 \frac{1}{t^{r-s-1}} \cdot \frac{1}{t^{s+1}} \exp\left(\frac{i}{t^s}\right) dt \\ &= A_0 + A_1 \frac{1}{x^{r-s-1}} \exp \frac{i}{x^s} + B_1 \int_x^1 \frac{1}{x^{r-s}} \exp\left(\frac{i}{t^s}\right) dt, \end{aligned}$$

where A_0, A_1, B_1 are constants. Repeating this for the last term we see that

$$\begin{aligned} F(x) &= A_0 + A_1 \frac{1}{x^{r-s-1}} \exp \frac{i}{x^s} + A_2 \frac{1}{x^{r-2s-1}} \exp \frac{i}{x^s} + \dots + \\ &+ A_k \frac{1}{x^{r-ks-1}} \exp \frac{i}{x^s} + \dots \\ &+ A_n \frac{1}{x^{r-ns-1}} \exp \frac{i}{x^s} + B_n \int_x^1 \frac{1}{t^{r-ns}} \exp \frac{i}{t^s} dt \end{aligned}$$



and if we choose n such that $r - ns < 0$ we see that each term but the last is

$$O\left(\frac{1}{x^{r-s-1}}\right). \quad \square$$

Lemma 9.2. If $r < ns$ and if

$$H(x) = \int_x^1 d\xi_1 \int_{\xi_1}^1 d\xi_2 \dots \int_{\xi_{n-1}}^1 M(t) dt$$

then

$$H(x) = O\left(\frac{1}{x^{r-ns-n}}\right) + P(x)$$

where P is a polynomial of degree $< n$. Thus

$$H(x) = o(x^n) + P(x),$$

and so $H_{(n)}(0)$ exists and is zero.

Proof. This is a trivial induction using lemma 9.1. $\square$

Lemma 9.3. We get exactly the same results as lemmas 9.1 and 9.2 if we replace M by one of

$$a) \quad \frac{1}{x^r} \exp \frac{-i}{x^s}$$

$$b) \quad \frac{1}{x^r} \sin \frac{1}{x^s}$$

$$c) \quad \frac{1}{x^r} \cos \frac{1}{x^s}$$



Proof. a) is obvious and b) and c) follow from a) and the above and well known trigonometric relations. $\square$

We now show that the functions f and G of example 9.2 have the desired properties. Let

$$r = n^2 + 3n - 1, \quad s = n + 3.$$

Then $r - ns < 0$ and so by lemma 9.2 (for cosine), f is an n -th exact Peano derivative in any neighborhood of 0. Moreover it is easily seen that $G^{(k)}(0) = 0$ ($k = 1, \dots, n-1$) and $G^{(n-1)}(x) = O(x^2)$ so that $G^{(n)}(0)$ exists. Thus $G^{(n)}(x)$ exists for all x .

But

$$(fG)(x) = \begin{cases} \frac{1}{x} \cos^2 \frac{1}{x^{n+3}} & x \neq 0, \\ 0 & x = 0, \end{cases}$$

and this function cannot be Z_m -integrable for any m . Namely, fG has constant sign in $[0, 1]$ and in $[-1, 0]$ so if it were Z_m -integrable there it would be Lebesgue integrable there. But it is not Lebesgue integrable in any interval of the form $[0, \alpha]$ or $[-\alpha, 0]$.

Remark. Even though $G^{(n)}$ exists, $G^{(n-1)}$ is not of finite variation. In fact it cannot be, according to theorem 7.1.



CHAPTER X

AN IMPROVEMENT IN THE DEFINITION OF THE Z_n -INTEGRAL

In this section we would like to show that it is possible to relax somewhat the requirement 3) in definition 5.1, the definition of an n -majorant.

Lemma 10.1. Let $\alpha_0 > a_1 \geq a_2 \geq \dots \geq a_n \rightarrow 0$.
Let $\beta_0 = 1 > \beta_1 > \beta_2 > \dots > \beta_n \rightarrow 0$ and $\beta_n < \frac{\beta_{n-1}}{2}$.
Then for each $k = 1, 2, \dots$, there is a function F^k on $[0, 1]$ such that

- 1) F^k has a non-negative continuous $(k+1)$ th derivative on $(0, 1]$,
- 2) $F^k(0) = F^{k'}(0) = \dots = F^{k(k)+}(0) = 0$,
 $F^{k(k)-}(\beta_0) = \alpha_0$,
- 3) $F^k(x) \geq \beta_n^k a_n / 2^{k+1}$ ($x \in [\beta_n, \beta_{n-1}]$;
 $n = 1, 2, \dots; k = 1, 2, \dots$).

Proof. Let φ be a continuous, non-decreasing function on $[0, 1]$ with a continuous derivative in $(0, 1]$ such that $\varphi(x) = a_n$ for $\beta_n/2 < x < \beta_n$. Let



$$\begin{aligned}
F^k(x) &= \int_0^x d\xi_1 \int_0^{\xi_1} d\xi_2 \dots \int_0^{\xi_{k-1}} \varphi(\xi_k) d\xi_k \\
&= \int_0^x \varphi(\xi) d(\xi, k-1), \quad \text{a } k\text{-fold indefinite} \\
&\quad \text{integral of } \varphi.
\end{aligned}$$

Then for $x \in [\beta_n, \beta_{n-1}]$, we have

$$F^1(x) = \int_0^x \varphi \geq \int_0^{\beta_n} \varphi \geq \int_{\beta_n/2}^{\beta_n} \varphi = \frac{\beta_n}{2} \alpha_n.$$

Suppose we have established the formula of 3) for the case $k-1$. Then for $x \in [\beta_n, \beta_{n-1}]$ we have

$$\begin{aligned}
F^k(x) &= \int_0^x F^{k-1} \geq \int_0^{\beta_n} F^{k-1} \geq \int_{\beta_n/2}^{\beta_n} F^{k-1} \geq \\
&\geq \frac{\beta_n^{k-1} \cdot \alpha_n}{2^k} \cdot \frac{\beta_n}{2} = \beta_n^k \cdot \alpha_n / 2^{k+1}.
\end{aligned}$$

The rest is obvious. $\square$

Lemma 10.2. Let $k \geq 1$ be an integer. Let M be a finite function on $[-1, 1]$ such that $M_{(k)}(0)$ is finite. Let $\epsilon > 0$ be given. Then there is a function F with the properties

- 0) $F(0) = F'(0) = \dots = F^{(k)}(0) = 0$,
- 1) $F^{(k)}$ is continuous and non-decreasing on $[-1, 1]$,
- 2) $[F^{(k)}]_{-1}^1 < \epsilon$,
- 3) $\delta_{k+1}(M+F)(0) \geq 0$.

Proof. Let ℓ be a polynomial of degree $\leq k$ such that $\ell(0) = M(0)$, $\ell'(0) = M'(0)$, $\dots$, $\ell^{(k)}(0) = M_{(k)}(0)$.



Let $Q = M-l$. Then $Q(t) = o(t^k)$. Let $\beta_n = 3^{-n}$ ($n = 0, 1, 2, \dots$). Define, for $n = 1, 2, \dots$,

$$c_n = \sup\left\{\frac{|Q(t)|}{t^k} \cdot 2^{k+1} : t \in [0, \beta_{n-1}]\right\}.$$

Then c_n is a sequence decreasing to zero. Let $d_n = c_n \cdot 3^k$. Then for $t \in [\beta_n, \beta_{n-1}]$,

$$|Q(t)| \leq \frac{c_n t^k}{2^{k+1}} \leq \frac{c_n \beta_{n-1}^k}{2^{k+1}} = \frac{d_n \beta_n^k}{2^{k+1}}.$$

Let $d_{n_0} < \frac{\epsilon}{2}$ and define a_n ($n = 1, 2, \dots$) by

$$a_n = \begin{cases} d_{n_0} & n \leq n_0 \\ d_n & n > n_0 \end{cases}.$$

Now define F on $[0, 1]$ as in lemma 10.1 for this choice of k , $\{a_n\}$, $\{\beta_n\}$. Then for some $\beta > 0$, $F(x) \geq |Q(x)|$ in $[0, \beta]$ and $F^{(k)+}(1) < \frac{\epsilon}{2}$. Repeat this construction in $[-1, 0]$ and thus define F on all of $[-1, 1]$ so that

$$F(0) = F'(0) = \dots = F^{(k)}(0) = 0,$$

so that F has a continuous, non-decreasing k -th derivative in $[-1, 1]$, $F^{(k)+}(-1) > -\frac{\epsilon}{2}$ and

$$F(x) \geq |Q(x)| \text{ on } [-\beta, 0] \text{ (k odd) or}$$

$$F(x) \leq -|Q(x)| \text{ on } [-\beta, 0] \text{ (k even), for some } \beta > 0.$$

Then $\delta_{k+1}(M+F)(0) = \delta_{k+1}(Q+F)(0) \geq 0$. The rest is obvious. $\square$



Corollary 10.3. A simple argument shows that we may replace $[-1,1]$ with any compact interval $[a,b]$, and instead of 0 we may take any point c in the interior. Moreover, if we allow for one-sided derivatives we may take c to be any point in $[a,b]$. $\square$

Theorem 10.4. Let M be a finite function on $[a,b]$. Let k be a natural number. Let $\epsilon > 0$. Let T be a countable set in $[a,b]$. Let $M_{(k)}$ be finite for points in T . Then there is a function F satisfying

- 1) $F^{(k)}$ is continuous and non-decreasing in $[a,b]$,
- 2) $[F^{(k)}]_a^b < \epsilon$,
- 3) $\delta_{k+1}(M+F)(x) \geq 0 \quad (x \in T)$.

Proof. Let $T = \{a_n\}_1^\infty$. Then, by lemma 10.2, for each $n = 1, 2, \dots$, there is a function F_n such that

- a) $F_n^{(k)}$ is continuous and non-decreasing,
- b) $[F_n^{(k)}]_a^b < \frac{\epsilon}{2^n}$,
- c) $\delta_{k+1}(M+F_n)(a_n) \geq 0$.

We may also suppose without loss of generality that

$$F_n(a) = F_n'(a) = \dots = F_n^{(k)}(a) = 0 \quad (n = 1, 2, \dots),$$

or else we simply add to F_n a polynomial of degree $\leq k$ and this leaves a), b) and c) unaffected. Then



$$|F_n^{(k)}(x)| < \frac{\epsilon}{2^n} \quad (n = 1, 2, \dots; x \in [a, b]),$$

and so if we define

$$f(x) = \sum_{n=1}^{\infty} F_n^{(k)}(x)$$

we see that f is continuous and non-decreasing on $[a, b]$, and $[f]_a^b < \epsilon$. Let

$$F(x) = \int_a^x f(\xi) d(\xi, k-1),$$

a k -fold indefinite integral of f . Then it is easily seen by the uniform convergence of the series defining f that

$$F(x) = \sum_{n=1}^{\infty} \int_a^x F_n^{(k)}(\xi) d(\xi, k-1) = \sum_{n=1}^{\infty} F_n(x).$$

Moreover $F^{(k)} = f$ and so 1) and 2) are obvious. To see 3) we let $x = a_n \in T$. Then

$$\begin{aligned} \delta_{k+1}^{(M+F)}(x) &= \delta_{k+1}^{(M+F_n + \sum_{m \neq n} F_m)}(x) \geq \\ &\geq \delta_{k+1}^{(M+F_n)}(x) \geq 0. \end{aligned}$$

The first inequality follows from the fact that

$$\sum_{m \neq n} F_m^{(k)} = \left(\sum_{m \neq n} F_m \right)^{(k)}$$

is non-decreasing, whence

$$\delta_{k+1} \left(\sum_{m \neq n} F_m \right) \geq 0. \quad \square$$



Theorem 10.5. Let f, M be finite functions on $[a, b]$. Let T be a countable set in $[a, b]$. Assume $M_{(k)}$ is finite on $[a, b]$ and that

$$-\infty < \delta_{k+1} M(x) \geq f(x) \quad (x \notin T).$$

Then

$$(Z_k^*) \int_a^b f \leq [M_{(k)}]_a^b.$$

Proof. Let $\epsilon > 0$. According to theorem 10.4 there is a function F on $[a, b]$ such that

$$[M_{(k)}]_a^b \leq [(M+F)_{(k)}]_a^b \leq [M_{(k)}]_a^b + \epsilon$$

and

$$\delta_{k+1}(M+F)(t) > -\infty \quad \text{everywhere and}$$

$$\delta_{k+1}(M+F) \geq f \quad \text{except on } T.$$

Then applying theorem 6.16 we see that

$$(Z_k^*) \int_a^b f \leq [(M+F)_{(k)}]_a^b \leq [M_{(k)}]_a^b + \epsilon.$$

As ϵ is arbitrary the result is immediate. $\square$

Applying theorem 6.16 to this case we easily obtain

Theorem 10.6. Let f, P be finite functions on $[a, b]$. Let T be a countable set in $[a, b]$. Assume $P_{(k)}$ is finite on $[a, b]$ and that



- a) $\delta_{k+1} P(x) > -\infty \quad (x \in [a, b] \setminus T),$
 b) $\delta_{k+1} P \geq f \quad \text{a.e. on } [a, b].$

Then

$$(Z_k^*) \int_a^b f \leq [P_{(k)}]_a^b. \quad \square$$

This result shows that if we take the following definition for n -majorant and the corresponding one for n -minorant, then the Z_n -integral is left unaffected.

Definition 10.1. Let M and f be defined in $[a, b]$. Then M is called an n -majorant of f in $[a, b]$ if there is a function P on $[a, b]$ such that

- 1) $M = P_{(n)}$ on $[a, b],$
- 2) $\delta_{n+1} P \geq f$ a.e. on $[a, b],$
- 3) $\delta_{n+1} P(x) > -\infty$ for all but at most countably many $x \in [a, b].$

To finish we illustrate how such a definition may be useful.

Theorem 10.7. Suppose f is Z_n -integrable on $[a, \beta]$ for each $\beta < b$ and suppose $\lim_{x \rightarrow b-} (Z_n) \int_a^x f$ exists and is finite. Then f is Z_n -integrable on $[a, b]$ and

$$(Z_n) \int_a^b f = \lim_{x \rightarrow b-} (Z_n) \int_a^x f.$$



Proof. Let $\epsilon > 0$. Let $a = b_0 < b_1 < \dots < b_k \rightarrow b$. Let M_1, m_1 be respectively an n -majorant and an n -minorant for f on $[a, b_1]$ such that $M_1(a) = m_1(a) = 0$ and $[M_1 - m_1]_a^{b_1} < \epsilon/2$. Suppose M_{k-1} and m_{k-1} have been chosen. Let M_k, m_k be respectively an n -majorant and an n -minorant of f on $[b_{k-1}, b_k]$ such that

$$M_k(b_{k-1}) = M_{k-1}(b_{k-1}), \quad m_k(b_{k-1}) = m_{k-1}(b_{k-1})$$

and

$$[M_k - m_k]_{b_{k-1}}^{b_k} < \epsilon/2^k.$$

Let

$$M(x) = M_k(x) \quad \text{when } x \in [b_{k-1}, b_k],$$

$$m(x) = m_k(x) \quad \text{when } x \in [b_{k-1}, b_k].$$

Then $[M - m]_a^{b_k} < \epsilon$ for every k . It is easy to see that M and m are respectively an n -majorant and an n -minorant on each interval $[a, \beta]$ ($\beta < b$). Thus there is a function P such that $P_{(n)}(x) = M(x)$ ($x \in [a, b)$), and

$$-\infty < \delta_{k+1} P(x) \geq f(x) \quad (x \in [a, b)).$$

We shall now define $M(b)$ and $P(b)$ in such a way that $P_{(n)}(b) = M(b)$. Let $F(x) = \int_a^x f$ ($x < b$), $F(b) = F(b-)$. Then it is easily seen that $M - F$ is bounded and non-decreasing in $[a, b)$ so that $M(b-) = P_{(n)}(b-)$ exists and is finite. We may prove analogously that $m(b-)$ is finite. Define $M(b) = M(b-)$, $m(b) = m(b-)$. Moreover



$P_{(n)} = M$ is bounded in some interval (α, b) so it is an ordinary derivative there by corollary 6.25. Thus $P_{(n)} = P^{(n)}$ and, since $P^{(n)}(b-)$ exists and is finite we must also have that

$$P^{(k)}(b-) \quad (k = 0, 1, \dots, n)$$

exist and are finite. Now we define $P(b) = P(b-)$.

It follows from L'Hopital's rule that $P^{(k)-}(b) = P^{(k)}(b-)$ ($k = 0, \dots, n$), so that $P^{(n)}(b) = M(b)$. Thus M is an n -majorant of f on $[a, b]$ using definition 10.1. Similarly m is an n -minorant of f on $[a, b]$ and $[M-m]_a^b < \epsilon$. Thus f is Z_n -integrable on $[a, b]$ and

$$\begin{aligned} -\epsilon &< m(b) - M(b) \leq (Z_n) \int_a^b f - F(b-) \leq \\ &\leq M(b) - m(b) < \epsilon, \end{aligned}$$

so that

$$(Z_n) \int_a^b f = \lim_{x \rightarrow b-} (Z_n) \int_a^x f. \quad \square$$



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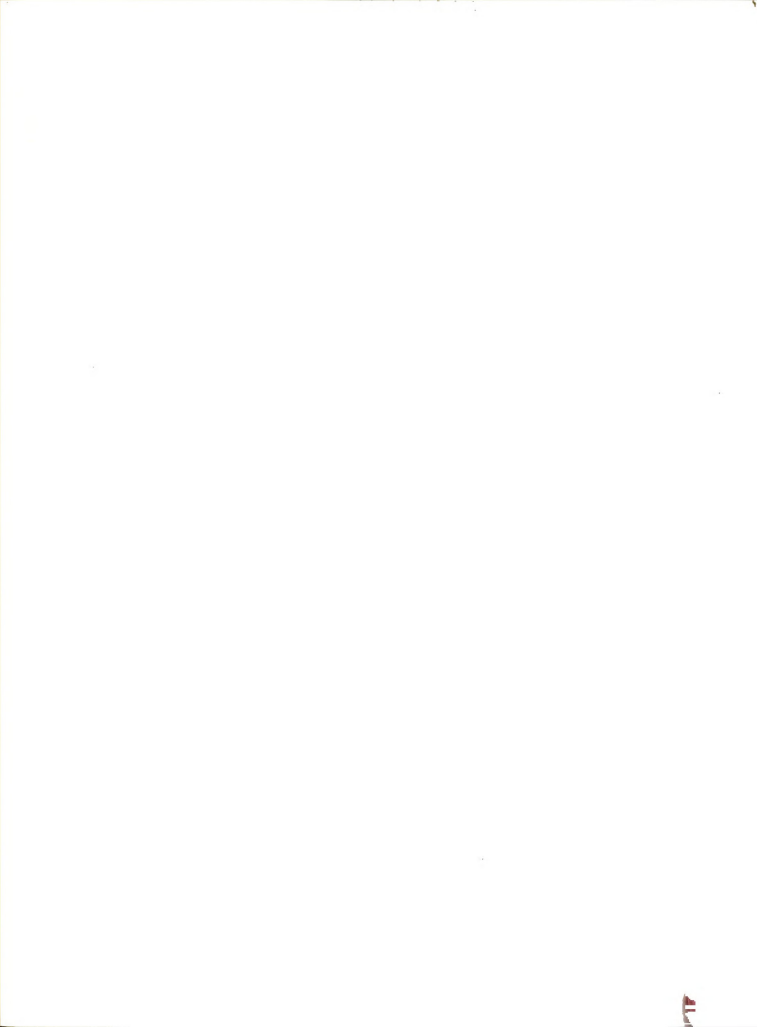


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