SOME STABILITY GROUPS OF FINITE GROUPS

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ABSTRACT

SOME STABILITY GROUPS OF FINITE GROUPS

By

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A topic of study in finite group theory is the group of automorphisms of a finite group. One method for studying automorphisms is to look at their effect on chains of subgroups, rather than individual elements.

Given a chain of subgroups $s: G = G_0 \geq G_1 \geq \ldots \geq G_n = 1$, we define Stab(s) by

$$\mathrm{Stab}(s) = \{\alpha \in \mathrm{Aut}\, G \mid (g_i G_{i+1})^\alpha = g_i G_{i+1} \text{ for all } g_i \in G_i,\ i = 0,1,2,\ldots,n-1\}.$$

P. Hall has shown that Stab(s) is a nilpotent group of class less than or equal to $\binom{n}{2}$.

If $A \leq \mathrm{Stab}(s)$, there is a canonical chain $\bar{s}: G \geq [G,A] \geq [G,A,A] \geq \ldots \geq 1$ in G and we define the closure of A, written $\bar{A}$, by

$$\bar{A} = \mathrm{Stab}(\bar{s}) \ .$$

A is said to be a closed stability group if $\bar{A} = A$.

In Chapter I we have:

(1) $\bar{\bar{A}} = \bar{A}$

(2) The prime divisors of $\bar{A}$ are the same as the prime divisors of A.

(3) $N_{\text{Aut } G}(\bar{A}) = \{\beta \in \text{Aut } G \mid \beta$ leaves each group in $\bar{s}$ invariant$\}$.

(4) If $A \lhd \text{Aut } G$, then $\bar{A} \lhd \text{Aut } G$.

Two questions arise

(1) Which nilpotent groups are stability groups?

(2) Which stability groups are closed?

Using the following

If $A \leq Z(G)$, the center of G, then

$$\text{Hom}(G/A, A) \simeq \text{Stab}(G \geq A \geq 1)$$
$$f \leftrightarrow \alpha_f : g \to g\, g^f$$

we showed that

Any abelian group is a closed stability group.

The search for stability groups may be simplified by $A \leq \text{Aut } G$ is a closed stability group if and only if for every p dividing $|A|$, a p-Sylow of A is a closed stability group.

Let $G = H \times K$ and

$$s: \; H \times K \geq H_1 \times K \geq \ldots \geq H_s \times K = 1 \times K \geq K_1 \geq \ldots \geq K_n = 1$$

then $\text{Stab}(s)$ is the semidirect product of $\text{Stab}(H \times K \geq K \geq 1)$ and a subgroup isomorphic to $\text{Stab}(H \geq H_1 \geq \ldots \geq 1) \times \text{Stab}(K \geq K_1 \geq \ldots \geq 1)$.

Let G be nilpotent and ψ the isomorphism from $\text{Aut } G$ onto the direct product of the automorphism groups of the p-Sylows of G. $A \leq \text{Aut } G$ is a stability group if and only if $A^{\psi} = \prod_{p \mid |A|} \text{Stab}(s_p)$ where s_p is a chain from the p-Sylow of G to 1.

This last theorem leads us to Chapter II and stability groups of p-groups which must be p-groups.

If G is a p-group then:

(1) Any p-Sylow of Aut G is a closed stability group.

(2) $O_p(\text{Aut } G)$ is a closed stability group.

If G is a p-group with $G' \leq Z(G)$ or $|Z(G)| = p$, then $\text{Stab}(G \geq Z \geq 1)$ is the group of central automorphisms.

If G is a p-group and $A \leq \text{Aut } G$ is of the form

(i) A is a p-group·

(ii) A is normal in every p-Sylow of Aut G that contains A.

(iii) A is the intersection of all p-Sylows of Aut G that contain A.

then A is said to be of K-type.

If G is a p-group and $A \leq \text{Aut } G$ is of K-type then A is a closed stability group.

Let G be an elementary abelian p-group. Then

(1) (Kaloujnine) A is a stability group if and only if A is of K-type.

(2) A is a minimal stability group if and only if $A = \text{Stab}(G \geq H \geq 1)$ for a subgroup H of G. A minimal stability group is one that contains no other nontrivial stability groups.

(3) If A and B are two stability groups in the same p-Sylow of Aut G, then $\langle A,B\rangle = AB$ is a stability group.

(4) $\text{Stab}(G \geq G_1 \geq \ldots \geq G_n = 1)$ is the product of the minimal stability groups $\text{Stab}(G \geq G_i \geq 1)$, $i = 1,2,\ldots,n-1$.

If G is a p-group for which every stability group is Kaloujnine, then G is elementary abelian or cyclic of order p^2.

In Chapter III we examine Fitt(Hol G), the Fitting subgroup of the holomorph of G, and Fitt(Aut G), the Fitting subgroup of Aut G.

If A is the product of all stability groups of characteristic series of G then Fitt(Hol G) = A·Fitt G.

If G is a p-group, not $\sigma(2) \times \sigma(2)$ or $\sigma(3) \times \sigma(3)$, then:

$$\text{Fitt(Aut G)} = \begin{cases} O_p(\text{Aut } G) \text{ when } G \text{ is nonabelian or} \\ O_p(\text{Aut } G) \times B \text{ where } B \text{ is a cyclic subgroup of} \\ \text{the center of Aut } G \text{ and } |B| = p\text{-}1. \end{cases}$$

If Fitt G is purely nonabelian or if Exp Z(G) divides Exp(G/ZG') then Fitt(Aut G) is a closed stability group.

In Chapter IV we have:

The quaternion group of order eight is a closed stability group but is not a closed stability group for a normal series in a 2-group.

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INDEX OF NOTATION

I. Relations:

$\leq$	Is a subgroup of
$\lneq$	Is a proper subgroup of
$\lhd$	Is a normal subgroup of
$\simeq$	Is isomorphic to
$\underset{\sim}{\subset}$	Is isomorphically contained in
$\in$	Is an element of

II. Operations:

$< >$	The subgroup generated by
X^f	The image of X under the mapping f.
S^x	$x^{-1}Sx$
$[x,y]$	$x^{-1}y^{-1}xy$
$[x,\alpha]$	$x^{-1}x^{\alpha}$
$[H,A]$	Subgroup generated by all $[h,\alpha]$, $h \in H$, $\alpha \in A$.
$[H,\overbrace{A,\ldots,A}^{n}]$	$[[H,\overbrace{A,\ldots,A}^{n-1}],A]$
$[H:K]$	The index of K in H.
H/K	The factor group, H mod K.
$H\backslash K$	The elements of H not in K.
$H \times K$	The direct product of H and K.
$\vert h\vert$	The order of the element h.
$\vert H\vert$	The number of elements in H.

$\pi(H)$	The set of prime divisors of $\lvert H \rvert$.
Exp H	The exponent of H.
π_h	The inner automorphism induced by h.
Π_H	The set of all π_h, $h \in H$.
1_H	The mapping $h \to h$ for all $h \in H$.
c_1	The mapping sending everything to 1.
$\alpha\vert_H$	The mapping α restricted to H.
Π	Product, not necessarily direct product.
$\oplus\ \Sigma$	Direct sum
$n \mid m$	n divides m

III. Groups:

Aut G	The automorphism group of G.
$C_H(K)$	The centralizer in H of K.
$C_{\mathrm{Aut}\ G}(H/K)$	$\{\alpha \in \mathrm{Aut}\ G \mid h^{-1}h^{\alpha} \in K \text{ for all } h \in H\}$.
Fitt G	The Fitting subgroup of G.
G'	The commutator group, $[G,G]$.
Hol G	The Holomorph of G.
Hom(H,K)	The set of homomorphisms from H into K.
Inn G	The inner automorphisms of G.
$\Phi(G)$	The Frattini subgroup of G.
$\sigma(n)$	The cyclic group of order n.
Sym(n)	The symmetric group on n symbols.
Z(G)	The center of G.

INTRODUCTION

This dissertation arose from an effort to characterize the stability groups of finite groups. In [10], Kaloujnine defined stability groups for normal chains and showed that they are nilpotent. Later, in [5], Hall defined stability groups for arbitrary chains and showed that they too are nilpotent. Thus the question, of which nilpotent subgroups of the automorphism group of a group can be stability groups, arises naturally. In [6], Hall and Hartley investigated what groups may be subgroups of stability groups of infinite length chains.

In Chapter I, we define the stability group of a chain s and state some of its elementary properties. If A is a subgroup of a stability group, we define $\bar{A}$, the closure of A, and develop some properties of those automorphism groups for which $\bar{A} = A$. Using a method of Schmid [14], we give some examples of stability groups. As a partial answer to the question, "What nilpotent groups may be stability groups?", we see that any abelian group may be a closed stability group.

Stability groups of p-groups are also p-groups. Hence, Chapter II deals with p-groups of automorphisms of a p-group that are stability groups. The beginning of the chapter contains a condition under which the centralizer in Aut G of a normal subgroup H of G is a stability group and two instances when the group of central automorphisms is a stability group. In [11], Kaloujnine characterized the stability groups of elementary abelian p-groups as those automorphism groups A such that

(i) A is a p-group.

(ii) A is normal in every p-Sylow of Aut G that contains A.

(iii) A is the intersection of all p-Sylows of Aut G that contain A.

We refer to those automorphism groups of p-groups which satisfy i, ii, and iii as K-type automorphism groups. A proof of Kaloujnine's characterization is given and it is shown that for a p-group those automorphism groups of K-type are stability groups. In general a p-group has stability groups that are not of K-type because if G is a p-group, all of whose stability groups are of K-type, then G is elementary abelian or cyclic of order p^2.

In Chapter III, we consider the Fitting subgroup of Hol G and of Aut G. Fitt(Hol G) is shown to be the product of Fitt G and the product of all stability groups of characteristic chains. Fitt(Aut G) is characterized when G is a p-group and we determine two conditions under which Fitt(Aut G) is a closed stability group.

Chapter IV begins with two examples, one of which is a 2-group G and a normal chain s such that Stab(s) is isomorphic to the quaternion group of order eight. The bulk of Chapter IV consists of a proof that the quaternion group of order eight is not a closed stability group for a normal chain in a 2-group.

Several examples are listed in the Appendix.

CHAPTER I

DEFINITIONS AND ELEMENTARY PROPERTIES OF STABILITY GROUPS AND CLOSED STABILITY GROUPS

In this chapter we introduce some elementary properties which will be used in later chapters. All groups considered are assumed to be finite.

Definition 1.1. Let $s: G = G_0 \geq G_1 \geq \ldots \geq G_n = 1$ be a chain of subgroups for an arbitrary group G. We define the stability group of s, written $\mathrm{Stab}(s)$ by:

$$\mathrm{Stab}(s) = \{\alpha \in \mathrm{Aut}\, G \mid (g_i G_{i+1})^\alpha = g_i G_{i+1} \text{ for all } g_i \in G_i, \\ i = 0,1,\ldots,n-1\} .$$

The process of finding stability groups may be simplified by:

Theorem 1.2. Let $A = \mathrm{Stab}(s)$ with s as above, and $\theta \in \mathrm{Aut}\, G$. If s^θ signifies the chain

$$s^\theta: G = G_0^\theta \geq G_1^\theta \geq \ldots \geq G_n^\theta = 1$$

then

$$A^\theta = \mathrm{Stab}(s^\theta) .$$

Proof: Let $\alpha \in \mathrm{Stab}(s)$, $B = \mathrm{Stab}(s^\theta)$, and $g_i \in G_i$, $i = 0,1,\ldots n-1$.

$$(g_i^{\theta}G_{i+1}^{\theta})^{\theta^{-1}\alpha\theta} = (g_iG_{i+1})^{\alpha\theta}$$

$$= (g_iG_{i+1})^{\theta}$$

since $\alpha \in \mathrm{Stab}(s)$. Thus $(g_i^{\theta}G_{i+1}^{\theta})^{\theta^{-1}\alpha\theta} = g_i^{\theta}G_{i+1}^{\theta}$, and $A^{\theta} \leq \mathrm{Stab}(s^{\theta}) = B$. The same argument gives that

$$B^{\theta^{-1}} \leq \mathrm{Stab}(s^{\theta\,\theta^{-1}}) = \mathrm{Stab}(s) = A .$$

Operating on the containment by θ gives $B \leq A^{\theta}$, and $A^{\theta} = B = \mathrm{Stab}(s^{\theta})$.

Using 1.2, we are able to simplify the process of finding all the stability groups of a group G. Any series is a subseries of at least one nonrefinable chain in G. Aut G induces a permutation group on the chains of G. By 1.2, the stability groups of any two chains in the same permutation orbit are conjugate in Aut G. We first pick a chain from each orbit of nonrefinable chains. Then, we find the stability groups of all their subchains. Conjugation of those stability groups gives all possible stability groups, because any chain s is a subseries of some nonrefinable chain s_1. We may permute s_1 to one of the nonrefinable chains s_2 for which we have computed the stability groups of its subchains, i.e. s^{α} is a subchain of s_2 for some $\alpha \in$ Aut G so we know $\mathrm{Stab}(s^{\alpha})$. By 1.2, $\mathrm{Stab}(s) = (\mathrm{Stab}(s^{\alpha}))^{\alpha^{-1}}$ is one of the specified conjugates.

Definition 1.3. Let s be the series $s\colon G = G_0 \geq G_1 \geq \ldots \geq G_n = 1$.

$$\hat{S}(s) = \{\alpha \in \mathrm{Aut}\ G \mid G_i^{\alpha} = G_i \quad \text{for all} \quad i = 0,1,\ldots,n\} .$$

Theorem 1.4. $\hat{S}(s) \leq N_{\mathrm{Aut}\ G}(\mathrm{Stab}(s))$.

Proof: Let $A = \mathrm{Stab}(s)$ and $\beta \in \hat{S}(s)$. By 1.2, $A^\beta = \mathrm{Stab}(s^\beta)$. $s^\beta = s$ since β fixes each subgroup of the chain. Thus $A^\beta = \mathrm{Stab}(s)$ and $\beta \in N_{\mathrm{Aut}\ G}(A)$.

In Example 1 of the Appendix with $s: D_4 \geq \langle x^2, y\rangle \geq \langle x^2\rangle \geq 1$ we see that $\hat{S}(s) \lneq N_{\mathrm{Aut}\ G}(\mathrm{Stab}(s))$. Thus we do not in general have equality in 1.4.

Corollary 1.5. If each G_i is characteristic in G then $\mathrm{Stab}(s) \lhd \mathrm{Aut}\ G$.

Proof: If each G_i is characteristic, then $\mathrm{Aut}\ G = \hat{S}(s)$. Thus, by 1.4 $\mathrm{Aut}\ G \leq N_{\mathrm{Aut}\ G}(\mathrm{Stab}(s))$, and $\mathrm{Aut}\ G$ normalizes $\mathrm{Stab}(s)$.

Definition 1.6. Let $A \leq \mathrm{Aut}\ G$. Set

$$\gamma\, G\, A^0 = G \ , \quad \text{and}$$

$$\gamma\, G\, A^{i+1} = [\gamma\, G\, A^i, A] \quad \text{for} \quad i \geq 0 \ .$$

For a group H, $\pi(H)$ will denote the prime divisors of H.

Theorem 1.7. Using the notation of 1.1 and $A \leq \mathrm{Stab}(s)$, we see:

(i) $\gamma\, G\, A^i \subseteq G_i$, $i = 0,1,\ldots,n$

(ii) $[G, A] \subseteq \mathrm{Fitt}\ G$

(iii) The prime divisors of A are the same as the prime divisors of $[G, A]$, i.e. $\pi(A) = \pi([G, A])$.

Proof: See Schmid [13].

Definition 1.8. Let $A \leq \mathrm{Aut}\ G$. A stabilizes a series s if $A \leq \mathrm{Stab}(s)$. Following Schmid [13], $A \in \mathcal{T}_G$ if and only if A stabilizes a chain. Hence $\mathcal{T}_G$ denotes the set of all subgroups of the stability groups of G.

Definition 1.9. For $A \in \mathcal{J}_G$, define $\bar{s}$ by $\bar{s} = \{\gamma\, G\, A^i\}_{i=1}^{n(A)}$ where $n(A)$ is the first integer such that $\gamma\, G\, A^{n(A)} = 1$. Define $\bar{A}$, the closure of A, by

$$\bar{A} = \mathrm{Stab}(\bar{s}) \ .$$

We say a stability group A is closed if $A = \bar{A}$.

We record the following properties of closure in:

Lemma 1.10. Let $A \in \mathcal{J}_G$. Then

(i) $A \leq \bar{A}$

(ii) If $A \leq B \leq \bar{A}$, then $\gamma\, G\, B^i = \gamma\, G\, A^i$ for all i.

(iii) $\bar{\bar{A}} = \bar{A}$

(iv) If $\beta \in N_{\mathrm{Aut}\ G}(A)$ then $(\gamma\, G\, A^i)^\beta = \gamma\, G\, A^i$.

(v) If $A \lhd \mathrm{Aut}\ G$ then $\bar{A} \lhd \mathrm{Aut}\ G$.

(vi) A and $\bar{A}$ have the same prime divisors.

(vii) $\hat{S}(\bar{s}) = N_{\mathrm{Aut}\ G}(\bar{A}) = N_{\mathrm{Aut}\ G}(\mathrm{Stab}(\bar{s}))$.

Proof: (i) Let $x \in \gamma\, G\, A^i$, and $\alpha \in A$.

$$\begin{aligned} x^\alpha &= x\, x^{-1}\, x^\alpha \\ &= x\, [x, \alpha] \in x\, \gamma\, G\, A^{i+1} \quad \text{and} \\ [G, A, \ldots, A]^\alpha &= [G^\alpha, A^\alpha, \ldots, A^\alpha] \\ &= [G, A, \ldots, A] \quad \text{so} \\ (x\, \gamma\, G\, A^{i+1})^\alpha &= x\, [x, \alpha]\, \gamma\, G\, A^{i+1} \\ &= x\, \gamma\, G\, A^{i+1} \ . \end{aligned}$$

Thus $\alpha \in \mathrm{Stab}(\bar{s})$.

(ii) Since $B \leq \bar{A} = \mathrm{Stab}(\bar{s})$, 1.7 says $\gamma\, G\, B^i$ is contained in the $i^{\underline{th}}$ term of the chain which is $\gamma\, G\, A^i$.

Induct on i for the other inclusion.

$$\gamma\, G\, B^0 = G = \gamma\, G\, A^0 .$$

A generator of $\gamma\, G\, A^i$, $i > 0$, is of the form $[x, \alpha]$ with $x \in \gamma\, G\, A^{i-1}$ and $\alpha \in A$. By induction $x \in \gamma\, G\, B^{i-1}$ and since $\alpha \in A \leq B$, $[x, \alpha] \in [\gamma\, G\, B^{i-1}, B] = \gamma\, G\, B^i$. By induction $\gamma\, G\, A^i = \gamma\, G\, B^i$ for $i \geq 0$, and consequently $\bar{A} = \mathrm{Stab}(\gamma\, G\, A^i) = \mathrm{Stab}(\gamma\, G\, B^i) = \bar{B}$.

(iii) Let $B = \bar{A}$. Then $\bar{\bar{A}} = \bar{B}$ and $\bar{B} = \bar{A}$ by (ii).

(iv) Induct on i. $\gamma\, G\, A^0 = G$ so for $\beta \in \mathrm{Aut}\, G$, $(\gamma\, G\, A^0)^\beta = G^\beta = G = \gamma\, G\, A^0$.

For $i > 0$, take a generator $[x, \alpha]$ of $\gamma\, G\, A^i$ with $x \in \gamma\, G\, A^{i-1}$, $\alpha \in A$. $[x, \alpha]^\beta = [x^\beta, \alpha^\beta]$ and by induction $x^\beta \in \gamma\, G\, A^{i-1}$. Since β normalizes A, we have

$$(\gamma\, G\, A^i)^\beta \leq [\gamma\, G\, A^{i-1}, A] = \gamma\, G\, A^i \text{ as required.}$$

(v) Let $\beta \in \mathrm{Aut}\, G$. By 1.2, $(\bar{A})^\beta = \mathrm{Stab}((\gamma\, G\, A^i)^\beta)_{i=1}^{n(A)}$. Since $\beta \in N_{\mathrm{Aut}\, G}(A)$, part iv says $(\gamma\, G\, A^i)^\beta = \gamma\, G\, A^i$. We now have

$$(\bar{A})^\beta = \mathrm{Stab}(\gamma\, G\, A^i)_{i=1}^{n(A)} = \bar{A} .$$

Hence $\bar{A} \lhd \mathrm{Aut}\, G$.

(vi) By ii, $[G, A] = [G, \bar{A}]$. Since $A, \bar{A} \in \mathcal{J}_G$, 1.7 gives $\pi(A) = \pi([G, A])$ and $\pi(\bar{A}) = \pi([G, \bar{A}])$. Thus $\pi(A) = \pi(\bar{A})$.

(vii) By iv, $N_{\mathrm{Aut}\, G}(\bar{A}) \leq \hat{S}(\bar{s})$. The opposite inclusion is 1.4.

We record the following important theorems for future reference.

Theorem 1.11. (P. Hall [5]). If s is defined by $s\colon G = G_0 \geq G_1 \geq \ldots \geq G_n = 1$ then Stab(s) is nilpotent of class less than or equal to $\binom{n}{2}$.

Theorem 1.12. (Kaloujnine [10]). If each $G_i \lhd G$ and $G_n = 1$ then the class of Stab(s) is less than or equal to $n-1$.

The following lemma and method were introduced by Schmid in [14].

Lemma 1.13. Let H and K be normal subgroups of G. If $K \leq H$ and $C = C_G(H/K)$ then $C_{\mathrm{Aut}\ G}(H/K) \leq C_{\mathrm{Aut}\ G}(G/C)$.

Schmid notes that if $H \lhd G$, $C_G(H) \leq H$ and $L \geq H$ then $C_{\mathrm{Aut}\ G}(L) = \mathrm{Stab}(G \geq L \geq 1)$. To see this let $K = 1$ in the above lemma. Then $C_{\mathrm{Aut}\ G}(H) \leq C_{\mathrm{Aut}\ G}(G/H)$. Since $L \geq H$,

$$C_{\mathrm{Aut}\ G}(L) \leq C_{\mathrm{Aut}\ G}(H) \leq C_{\mathrm{Aut}\ G}(G/H) \leq C_{\mathrm{Aut}\ G}(G/L) .$$

Thus $C_{\mathrm{Aut}\ G}(L) = \mathrm{Stab}(G \geq L \geq 1)$ as stated.

If G is solvable, by Gorenstein [9, 6.1.3], $C_G(\mathrm{Fitt}\ G) \leq \mathrm{Fitt}\ G$. Thus if $H = \mathrm{Fitt}\ G$ and $L \geq H = \mathrm{Fitt}\ G$, the above method gives that $C_{\mathrm{Aut}\ G}(L) = \mathrm{Stab}(G \geq L \geq 1)$.

For G a p-group, Thompson has proved (see Gorenstein [9, 5.3.11]) the existence of a characteristic subgroup C, called the critical subgroup, such that

(i) class of $C \leq 2$ and $C/Z(C)$ is elementary abelian

(ii) $[G, C] \leq Z(C)$

(iii) $C_G(C) \leq Z(C)$.

With $L = C$ in Schmid's method, $C_{\mathrm{Aut}\ G}(C) = \mathrm{Stab}(G \geq C \geq 1)$.

When G is supersolvable with a maximal abelian normal subgroup M, $C_G(M) = M$. If not, $C_G(M) \gneq M$, because $C_G(M) \geq M$. We could refine the normal series $G \geq C_G(M) \gneq M \geq 1$ to a chief series. Then there would be a normal subgroup N with $C_G(M) \geq N \gneq M$. Supersolvability forces the chief factor N/M to have order a prime p. Thus $N = \langle M, x \rangle$ where $x \in C_G(M)$ and xM generates N/M. Since x commutes with M, we get $G \rhd N \gneq M$ and N is abelian, a contradiction to the choice of M.

Thus if G is supersolvable and H is a maximal abelian normal subgroup $C_G(H) = H$. Schmid's method says that

$$C_{\mathrm{Aut}\ G}(L) = \mathrm{Stab}(G \geq L \geq 1)$$

for $L \geq H$.

Definition 1.14. For $H \geq K$,

$$C_{\mathrm{Aut}\ G}(H\backslash K) = \{\alpha \in \mathrm{Aut}\ G \mid H^{\alpha} = H,\ K^{\alpha} = K,\ h^{-1}h^{\alpha} \in K \ \text{ for all } \ h \in H\} .$$

The next theorem gives us our first example of a closed stability group.

Theorem 1.15. Let $H \leq \Phi(G)$, the Frattini subgroup of G. If $B = C_{\mathrm{Aut}\ G}(G\backslash H)$ then B is a closed stability group.

Proof: In [14], Schmid has shown that if $A = C_{\mathrm{Aut}\ G}(G/\Phi)$ then $\gamma\, G\, A^n = 1$ for some n. Since $[G, B] \leq H \leq \Phi$, we see that $B \leq A$ and thus $\gamma\, G\, B^n = 1$. $B \leq \bar{B} \leq C_{\mathrm{Aut}\ G}(G/[G, B])$. Since $[G, B] \leq H$, we have $C_{\mathrm{Aut}\ G}(G/[G, B]) \leq C_{\mathrm{Aut}\ G}(G\backslash H) = B$ and $B = \bar{B}$ is a closed stability group.

In 1.15 B may be the identity subgroup, which is always a closed stability group for the trivial series $G \geq 1$.

We will see in 1.21 that the question "Which nilpotent groups may be closed stability groups?" may be reduced to "Which p-groups may be closed stability groups?" First we must develop a few lemmas.

Lemma 1.16. Let $A \in \mathcal{T}_G$ and π be the prime divisors of A. If K is an A-admissible π'-subgroup of G, then A fixes K pointwise.

Proof: $\gamma\, K\, A^i \leq \gamma\, G\, A^i$ for all i, and $\gamma\, G\, A^n = 1$ for some n, so $A|_K \in \mathcal{T}_K$. By 1.7, $\pi(A|_K) = \pi([K, A|_K])$. $[K, A|_K]$ is contained in the π'-group K, and $\pi(A|_K) \leq \pi(A) = \pi$. Thus $A|_K = 1_K$, since it is both a π and π' group.

Lemma 1.17. If $A \in \mathcal{T}_G$ then $\gamma\, G\, A^i = \gamma\, G(A_\pi)^i \times \gamma\, G(A_{\pi'})^i$ for $i = 1,2,\ldots,n$. Here π and π' are any two disjoint sets of primes with $\pi \cup \pi' = \pi(A)$ and $A_\pi = \prod_{p\in\pi} S_p$, $A_{\pi'} = \prod_{q\in\pi'} S_q$ with S_t the t-Sylow subgroup of A.

Proof: Induct on i. $[G, A] \geq [G, A_\pi]\cdot[G, A_{\pi'}]$ since $A_\pi, A_{\pi'} \leq A$. Theorem 1.7 says $\pi = \pi([G, A])$ and $\pi([G, A_{\pi'}]) = \pi'$. The two normal subgroups $[G, A_\pi]$ and $[G, A_{\pi'}]$ must now have trivial intersection so the subgroup generated by the two is a direct product. A generator of $[G, A]$ must have the form $g^{-1}g^{\alpha\beta}$ with $g \in G$, $\alpha \in A_\pi$ and $\beta \in A_{\pi'}$ as A is nilpotent and thus the direct product of A_π and $A_{\pi'}$.

$$g^{-1}g^{\alpha\beta} = g^{-1}g^{\alpha}(g^{\alpha})^{-1}(g^{\alpha})^{\beta} \in [G, A_\pi] \cdot [G, A_{\pi'}] \cdot$$

Hence $[G, A] = [G, A_\pi] \times [G, A_{\pi'}]$.

For the case $i+1 \geq 2$ we again have $\gamma\, G\, A^{i+1} \geq \gamma\, G(A_\pi)^{i+1} \times \gamma\, G(A_{\pi'})^{i+1}$, because $A \geq A_\pi \times A_{\pi'}$. Since $\gamma\, G(A_\pi)^{i+1} \leq [G, A_\pi]$ and $\gamma\, G(A_{\pi'})^{i+1} \leq [G, A_{\pi'}]$ we again have that the subgroup generated by the two is a direct product.

Take an arbitrary element $(x,y) \in \gamma\, G(A_\pi)^i \times \gamma\, G(A_{\pi'})^i$, $\alpha \in A_\pi$, and $\beta \in A_{\pi'}$. Since $[G, A_{\pi'}]$ is a π' group that is A_π admissible, we use 1.16 to see $(xy)^{-1}(xy)^{\alpha\beta} = (xy)^{-1}(x^\alpha y)^\beta = y^{-1}x^{-1}(x^\alpha y)^\beta$, $x^\alpha \in [G, A_\pi]^\alpha = [G^\alpha, A_\pi^\alpha] = [G, A_\pi]$ which by 1.16 is fixed pointwise by β so $(xy)^{-1}(xy)^{\alpha\beta} = y^{-1}x^{-1}x^\alpha y^\beta$. Since $x^{-1}x^\alpha \in [G, A_\pi]$ and $y^{-1} \in [G, A_{\pi'}]$ they must centralize each other and $(xy)^{-1}(xy)^{\alpha\beta} = x^{-1}x^\alpha y^{-1}y^\beta \in \gamma\, G(A_\pi)^{i+1} \cdot \gamma\, G(A_{\pi'})^{i+1}$. Thus $\gamma\, G(A_\pi)^{i+1} \times \gamma\, G(A_{\pi'})^{i+1} = \gamma\, G\, A^{i+1}$ and by induction the theorem is proved.

Definition 1.18. If p is a prime then $O_p(G)$ or just O_p is the largest normal p-subgroup of G. If $\varkappa$ is a set of primes then $O_\varkappa(G)$ is the product of $O_p(G)$ for every $p \in \varkappa$.

Theorem 1.19. If A is a closed stability group with A_π and $A_{\pi'}$ as in 1.17 then $A_\pi = \mathrm{Stab}(\gamma\, G(A_\pi)^i)_{i=0}^{n(A_\pi)}$ and $A_{\pi'} = \mathrm{Stab}(\gamma\, G(A_{\pi'})^i)_{i=0}^{n(A_{\pi'})}$.

Proof: We know from 1.10.i that $A_\pi \le \mathrm{Stab}(\gamma\, G(A_\pi)^i)_{i=0}^{n(A_\pi)}$. If $\alpha \in B = \mathrm{Stab}(\gamma\, G(A_\pi)^i)_{i=0}^{n(A_\pi)}$ then $g^{-1}g^\alpha \in [G, A_\pi]$. By 1.16 α fixes the characteristic subgroup $O_{\pi'}(G)$ pointwise and the normal nilpotent π'-group $[G, A_{\pi'}]$ is contained in $O_{\pi'}(G)$. Let $(x,y) \in \gamma\, G\, A^i = \gamma\, G(A_\pi)^i \times \gamma\, G(A_{\pi'})^i$, $i \ge 1$.

$$\begin{aligned}(xy^{-1})(xy)^\alpha &= y^{-1}x^{-1}x^\alpha y^\alpha \\ &= y^{-1}x^{-1}x^\alpha y \\ &= x^{-1}x^\alpha\end{aligned}$$

because $x^{-1}x^\alpha \in [G, A_\pi]$, $y \in [G, A_{\pi'}]$ and the two groups centralize each other. Thus $B \le \mathrm{Stab}(\gamma\, G\, A^i) = A_\pi \times A_{\pi'}$. Since $\pi(B) = \pi([G, B]) \subseteq \pi[G, A_\pi] = \pi$, $B \le A_\pi$. We now have that $A_\pi = \mathrm{Stab}\ \gamma(G\, A_\pi^i)$

and since π was an arbitrary set of primes, $A_{\pi'} = \mathrm{Stab}(\gamma\, G(A_{\pi'})^{i})$.

Theorem 1.20. If A and B are closed stability groups with $\pi(A)$ and $\pi(B)$ disjoint, then A and B commute and $\langle A,B\rangle = A \times B$ is a closed stability group.

Proof: In [13], Schmid proves that $C = \langle A,B\rangle = A \times B$ stabilizes a series and thus $\gamma\, G\, C^{n} = 1$ for some n. Lemma 1.17 says $\gamma\, G\, C^{i} = \gamma\, G\, A^{i} \times \gamma\, G\, B^{i}$ for all $1 \leq i \leq n$. Let $\bar{C} = \mathrm{Stab}(\gamma\, G\, C^{i})_{i=0}^{n(C)}$. By 1.7, $\pi(A) = \pi([G, A])$, $\pi(B) = \pi([G, B])$ and $\pi(C) = \pi([G, C])$.

We calculate the prime divisors of $\bar{C}$ by:

$$\begin{aligned}\pi(\bar{C}) &= \pi(C)\\ &= \pi([G, C])\\ &= \pi([G, A] \times [G, B])\\ &= \pi([G, A]) \cup \pi([G, B])\\ &= \pi(A) \cup \pi(B).\end{aligned}$$

$\bar{C}$ stabilizes a series and is therefore nilpotent. We have $\bar{C} = \bar{C}_{\pi(A)} \times \bar{C}_{\pi(B)}$ where $\bar{C}_{\pi(A)}$ $(\bar{C}_{\pi(B)})$ is the $\pi(A)(\pi(B))$-Hall subgroup of $\bar{C}$. For $\alpha \in \bar{C}_{\pi(A)}$ and $i \geq 0$, $[\gamma\, G\, A^{i}, \langle\alpha\rangle] \leq [\gamma\, G\, A^{i}, \bar{C}_{\pi(A)}] \leq \gamma\, G\, C^{i+1}$ because $\gamma\, G\, A^{i} \leq \gamma\, G\, C^{i}$ and $\bar{C}_{\pi(A)}$ stabilizes $(\gamma\, G\, C^{i})_{i=0}^{n(C)}$. By 1.7 and 1.17, $[\gamma\, G\, A^{i}, \langle\alpha\rangle] \leq (\gamma\, G\, A^{i+1} \times \gamma\, G\, B^{i+1}) \cap O_{\pi(A)}(G)$ which is $\gamma\, G\, A^{i+1}$. Thus $\bar{C}_{\pi(A)} \leq \mathrm{Stab}(\gamma\, G\, A^{i})_{i=0}^{n(A)} = A$. Since $\bar{C} \geq A$ and $\bar{C}_{\pi(A)}$ is the $\pi(A)$-Hall subgroup of $\bar{C}$, $\bar{C}_{\pi(A)} = A$. Likewise $\bar{C}_{\pi(B)} = B$, and $A \times B = \bar{C} = \mathrm{Stab}(\gamma\, G\, C^{i})_{i=0}^{n(C)}$.

Corollary 1.21. A is a closed stability group if and only if for every $p \in \pi(A)$ a p-Sylow of A is a closed stability group.

"$\Rightarrow$" This direction is 1.19 with $\pi = \{p\}$.

"$\Leftarrow$" By 1.20, A is a closed stability group.

Theorem 1.22. Let G be the semidirect product of $H \triangleleft G$ and K, $A = \mathrm{Stab}(H \geq H_1 \geq \cdots \geq H_n = 1)$, and $B = \mathrm{Stab}(G = HK \geq G_1 \geq \cdots \geq G_s = H \geq H_1 \geq \cdots \geq H_n = 1)$. If the automorphisms of H induced by K are contained in $C_{\mathrm{Aut}\ H}(A)$, then B is the semidirect product of $\mathrm{Stab}(G \geq G_1 \geq \cdots \geq G_s = H \geq 1) \triangleleft B$ by $\{\phi_\alpha : kh \to kh^\alpha | \alpha \in A\} \simeq A$.

First note the following lemma:

Lemma 1.23. With the notation and conditions of 1.22, ϕ_α is an automorphism of $G = HK$.

Proof: ϕ_α is well defined since every element of G is a unique product of the form kh with $k \in K$ and $h \in H$.

ϕ_α is a homormophism. Let k_1h_1 and k_2h_2 be elements of $K \cdot H$. $k_1h_1k_2h_2 = k_1k_2h_1^{k_2}h_2$ so

$$\begin{aligned}(k_1h_1k_2h_2)^{\phi_\alpha} &= k_1k_2(h_1^{k_2}h_2)^\alpha \\ &= k_1k_2(h_1^{k_2})^\alpha h_2^\alpha \\ &= k_1k_2h_1^{\alpha k_2}h_2^\alpha\end{aligned}$$

because A commutes with automorphisms of H that are induced by K. Thus $(k_1h_1k_2h_2)^{\phi_\alpha} = k_1h_1^\alpha k_2h_2^\alpha = (k_1h_1)^{\phi_\alpha}(k_2h_2)^{\phi_\alpha}$.

ϕ_α is one-to-one and since G is finite ϕ_α is an automorphism. If $1 = (k_1h_1)^{\phi_\alpha} = k_1h_1^\alpha$ then $h_1^\alpha = k_1^{-1} \in H \cap K = 1$ and $k_1h_1 = 1$.

Proof of 1.22: Let $\alpha, \beta \in A$. Since $\phi_{\alpha\beta} = \phi_\alpha\phi_\beta$, "$\alpha \to \phi_\alpha$" is a homomorphism from A into Aut G. If $1_G = \phi_\alpha$ then $1_H = (\phi_\alpha)|_H = \alpha$. Thus "$\alpha \to \phi_\alpha$" is an isomorphism, i.e.

$A \simeq \{\phi_\alpha | \alpha \in A\}$.

Let $\gamma \in B$ and $\gamma|_H = \alpha$. Since $\gamma \in B$, $\alpha \in A$. $\phi_{\alpha^{-1}}$ fixes K pointwise so it centralizes G/H. $\phi_{\alpha^{-1}}$ was defined to have the action of α^{-1} on H so $\phi_{\alpha^{-1}} \in \mathrm{Stab}(G \geq H \geq H_1 \geq \ldots \geq H_n = 1) \leq B$. $\gamma\phi_{\alpha^{-1}}$ is an element of B that centralizes H so $\gamma\phi_{\alpha^{-1}} \in \mathrm{Stab}(G \geq G_1 \geq \ldots \geq H \geq 1)$ and $\gamma = (\gamma\phi_{\alpha^{-1}})\phi_\alpha$ has the required form. All that remains to show is that $\{\phi_\alpha | \alpha \in A\} \cap \mathrm{Stab}(G \geq G_1 \geq \ldots \geq H \geq 1)$ is trivial. For ϕ_α in the intersection $\alpha = (\phi_\alpha)|_H = 1_H$ and $\phi_{(1_H)} = 1_G$.

Theorem 1.24. If $C = \mathrm{Stab}(H \times K \geq H \times K_1 \geq \ldots \geq H \times K_s = H \geq H_1 \geq \ldots \geq H_n = 1)$, $A = \{\phi_\alpha | (h,k)^{\phi_\alpha} = (h^\alpha,k)$ for every $\alpha \in \mathrm{Stab}(H \geq H_1 \geq \ldots \geq H_n = 1)\}$ and $B = \{\psi_\beta | (h,k)^{\psi} = (h,k^\beta)$ for every $\beta \in \mathrm{Stab}(K \geq K_1 \geq \ldots \geq K_s = 1)\}$, then C is the semidirect product of $\mathrm{Stab}(H \times K \geq H \geq 1) \lhd C$ by $A \times B$.

Proof: By 1.22, $C = \mathrm{Stab}(H \times K \geq H \times K_1 \geq \ldots \geq H \geq 1) \cdot A$. Let $D = \mathrm{Stab}(H \times K \geq H \times K_1 \geq \ldots \geq H \times K_s = H \geq 1)$, $\gamma \in D$ and $\overline{\gamma^{-1}}$ the automorphism induced by γ^{-1} on $H{\times}K/H \simeq K$. The automorphism $\psi_{\overline{\gamma^{-1}}}$ fixes H pointwise and induces $\overline{\gamma^{-1}}$ on $H{\times}K/H$ so $\psi_{\overline{\gamma^{-1}}} \in D$. Examine the action of $\gamma\,\psi_{\overline{\gamma^{-1}}}$ on the cosets kH, $k \in K$.

$$(kH)^{\gamma\psi_{\overline{\gamma^{-1}}}} = (k^{\gamma}H)^{\psi_{\overline{\gamma^{-1}}}} = kH \ .$$

$\gamma\psi_{\overline{\gamma^{-1}}}$ centralizes $K{\times}H/H$ and H, so $\gamma\psi_{\overline{\gamma^{-1}}} \in \mathrm{Stab}(H \times K \geq H \geq 1)$ and $\gamma = \gamma\psi_{\overline{\gamma^{-1}}}\,\psi_{\overline{\gamma}} \in \mathrm{Stab}(H \times K \geq H \geq 1) \cdot B$. Thus $C = \mathrm{Stab}(H \times K \geq H \geq 1) \cdot B \cdot A$.

$B \cdot A = B \times A$ since $\phi_\alpha\psi_\beta = \psi_\beta\phi_\alpha$ and $A \cap B = 1$.

Let $\psi_{\beta}\phi_{\alpha} \in B\cdot A \cap \mathrm{Stab}(H \times K \geq H \geq 1)$. Then for every $h \in H$, $h = h^{\psi_{\beta}\phi_{\alpha}} = h^{\phi_{\alpha}} = h^{\alpha}$ and $\alpha = 1_H$. For every $k \in K$, $k^{\beta} = k^{\psi_{\beta}\phi_{1}} = kh_k$ for some $h_k \in H$, since $\psi_{\beta}\phi_{\alpha} \in \mathrm{Stab}(H \times K \geq H \geq 1)$. $\beta \in \mathrm{Aut}\ K$ so $k^{\beta} = k$ and $\beta = 1_K$. Thus $\psi_{\beta}\phi_{\alpha} = 1$ and C is the appropriate semidirect product.

The next theorem could possibly be used to build up arbitrary stability groups rather than just closed stability groups.

<u>Theorem 1.25</u>. Let G be nilpotent and ψ the isomorphism $\mathrm{Aut}\ G \xrightarrow{\psi} \prod_{p||G|} \mathrm{Aut}(S_p)$ where S_p is the p-Sylow of G. $A \leq \mathrm{Aut}\ G$ is a stability group if and only if $A^{\psi} = \prod_{p||G|} \mathrm{Stab}(s_p)$ where s_p is a series from the p-Sylow to the identity.

<u>Proof</u>: Since the theorem is trivial for p-groups, we induct on the number of primes dividing $|G|$. Let $s\colon G = G_0 \geq G_1 \geq \ldots \geq G_n = 1$ be the series and $G_i = H_i \times K_i$ where H_i is the p-Sylow of G_i and K_i the p'-Hall subgroup of G_i. We use the isomorphism $\mathrm{Aut}\ G \simeq \mathrm{Aut}\ H_0 \times \mathrm{Aut}\ K_0$. For $(\theta_1,\theta_2) \in \mathrm{Aut}\ H_0 \times \mathrm{Aut}\ K_0$, $h_i \in H_i$ and $k_i \in K_i$

$$\begin{aligned}
(\theta_1,\theta_2) \in \mathrm{Stab}(s) &\Leftrightarrow (h_i,k_i)^{-1}(h_i,k_i)^{(\theta_1,\theta_2)} \in H_{i+1} \times K_{i+1} \\
&\Leftrightarrow (h_i^{-1}h_i^{\theta_1},\ k_i^{-1}k_i^{\theta_2}) \in H_{i+1} \times K_{i+1} \\
&\Leftrightarrow \theta_1 \in \mathrm{Stab}(H_0 \geq H_1 \geq \ldots \geq H_n = 1), \\
&\qquad \theta_2 \in \mathrm{Stab}(K_0 \geq \ldots \geq K_n = 1) .
\end{aligned}$$

By induction $\mathrm{Stab}(K_0 \geq \ldots \geq K_n = 1) = \prod_{q||K_0|} \mathrm{Stab}(s_q)$. Thus

$$(\theta_1,\theta_2) \in \mathrm{Stab}(s) \Leftrightarrow (\theta_1,\theta_2) \in \mathrm{Stab}(s_p) \times \prod_{\substack{q||G| \\ q \neq p}} \mathrm{Stab}(s_q) .$$

The following theorem is very useful for producing automorphisms and stability groups.

Theorem 1.26. Let $A \leq Z(G)$ the center of G, $\bar{g} = gA$ and $\alpha \in \mathrm{Stab}(s)$ then

$$\mathrm{Stab}(G \geq A \geq 1) \simeq \mathrm{Hom}(G/A, A)$$

by the mapping $\alpha \leftrightarrow f_\alpha : \bar{g} \to g^{-1}g^\alpha$.

Proof: (i) f_α is well-defined. Let $\bar{g} = \overline{ga} \in G/A$. $f_\alpha(\overline{ga}) = (ga)^{-1}(ga)^\alpha = a^{-1}g^{-1}g^\alpha a$, since α fixes A pointwise. Because $a \in Z(G)$, $f_\alpha(\overline{ga}) = g^{-1}g^\alpha = f_\alpha(\bar{g})$.

(ii) f_α is a homomorphism. Let $\bar{g}_1, \bar{g}_2 \in G/A$.

$$\begin{aligned} f_\alpha(\bar{g}_1\,\bar{g}_2) &= (g_1g_2)^{-1}(g_1g_2)^\alpha \\ &= g_2^{-1}g_1^{-1}g_1^\alpha g_2^\alpha \\ &= g_1^{-1}g_1^\alpha g_2^{-1}g_2^\alpha \quad \text{since} \quad g_1^{-1}g_1^\alpha \in Z(G). \end{aligned}$$

Thus $f_\alpha(\bar{g}_1\,\bar{g}_2) = f_\alpha(\bar{g}_1)f_\alpha(\bar{g}_2)$.

(iii) The mapping $\alpha \to f_\alpha$ is a homomorphism. Theorem 1.11 says $\mathrm{Stab}(G \geq A \geq 1)$ is abelian. For $\alpha,\beta \in \mathrm{Stab}(G \geq A \geq 1)$ and $\bar{g} \in G/A$,

$$\begin{aligned} f_{\alpha\beta}(\bar{g}) &= g^{-1}g^{\alpha\beta} \\ &= g^{-1}g^\alpha(g^\alpha)^{-1}g^{\alpha\beta} \\ &= g^{-1}g^\alpha(g^{-1})^\alpha g^{\beta\alpha} \\ &= g^{-1}g^\alpha(g^{-1}g^\beta)^\alpha \\ &= g^{-1}g^\alpha g^{-1}g^\beta \end{aligned}$$

because $g^{-1}g^\beta \in A$, which is fixed pointwise by α. Hence

$$\begin{aligned} f_{\alpha\beta}(\bar{g}) &= f_{\alpha}(\bar{g})f_{\beta}(\bar{g}) \\ &= (f_{\alpha}f_{\beta})(\bar{g}) \ . \end{aligned}$$

(iv) $\alpha \to f_{\alpha}$ is a one-to-one mapping. When we use a multiplicative notation for G/A, A, and $\mathrm{Hom}(G/A, A)$, the identity of $\mathrm{Hom}(G/A, A)$ is c_1, the constant map to 1. If $f_{\alpha} = c_1$ then $1 = f_{\alpha}(g) = g^{-1}g^{\alpha}$ for every $g \in G$. Thus $g = g^{\alpha}$ for every $g \in G$, and $\ker(\alpha \to f_{\alpha})$ is the identity of $\mathrm{Aut}\ G$.

(v) $\alpha \to f_{\alpha}$ is onto $\mathrm{Hom}(G/A, A)$. For $\tilde{\beta} \in \mathrm{Hom}(G/A, A)$, define $\psi : G \to G$ by

$$g^{\psi} = gg^{\beta}$$

for every $g \in G$, where β is the composition of $G \to G/A \xrightarrow{\tilde{\beta}} A$.

$$\begin{aligned} (g_1g_2)^{\psi} &= g_1g_2g_1^{\beta}g_2^{\beta} \\ &= g_1g_1^{\beta}g_2g_2^{3} = g_1^{\psi}g_2^{\psi} \end{aligned}$$

because $g_1^{\beta} \in A \leq Z(G)$. To show that ψ is an automorphism, take $x \in G$ such that $1 = x^{\psi} = xx^{\beta}$. This implies $x^{-1} = x^{\beta}$ and is in the image of β. The image of β is in A and thus x^{-1} is in the kernel of β. Since $x^{-1} \in \ker \beta$, $1 = xx^{\beta} = x \cdot 1 = x$. Thus ψ is a monomorphism and since G is finite, ψ is an automorphism. For $g \in G$, $g^{-1}g^{\psi} = g^{-1}gg^{\beta} = g^{\beta} \in A$. If $g = a \in A$, the equation becomes $a^{-1}a^{\psi} = a^{\beta} = 1$ and $a^{\psi} = a$. These calculations show $\psi \in \mathrm{Stab}(G \geq A \geq 1)$.

$$\begin{aligned} f_{\psi}(\bar{g}) &= g^{-1}g^{\psi} \\ &= g^{-1}gg^{\beta} = (\bar{g})^{\tilde{\beta}}, \end{aligned}$$

so ψ maps onto $\tilde{\beta}$ as required.

It might be noted that $\mathrm{Hom}(G/A, A) = \mathrm{Hom}(G/G'A, A)$ in 1.26 because A is abelian and thus any homomorphism from G/A into A will have G' in its kernel.

Corollary 1.27. Let $A \leq Z(G)$ with $A = \oplus \sum\limits_{i=1} \langle y_i \rangle$ where each $\langle y_i \rangle$ is a cyclic group, and $G/G'A = \oplus \sum\limits_{j=1}^{m} \langle \bar{x}_j \rangle$ where each $\langle \bar{x}_j \rangle$ is a cyclic group. Then

$$\mathrm{Stab}\{G \geq A \geq 1\} \simeq \oplus \sum_j \sum_i \sigma((|\bar{x}_j|, |y_i|)).$$

Here $(|\bar{x}_j|, |y_i|)$ is the greatest common divisor of the orders $|\bar{x}_j|$, and $|y_i|$, and $\sigma(n)$ denotes a cyclic group of order n.

Proof: By 1.26, $\mathrm{Stab}\{G \geq A \geq 1\} \simeq \mathrm{Hom}(G/G'A, A)$, $\mathrm{Hom}(\oplus \sum\limits_j \langle \bar{x}_j \rangle, \oplus \sum\limits_i \langle y_i \rangle) = \oplus \sum\limits_j \sum\limits_i \mathrm{Hom}(\langle \bar{x}_j \rangle, \langle y_i \rangle)$ since $\mathrm{Hom}(H \oplus K, L) \simeq \mathrm{Hom}(H, L) \oplus \mathrm{Hom}(K, L)$ and $\mathrm{Hom}(S, T \oplus U) \simeq \mathrm{Hom}(S,T) \oplus \mathrm{Hom}(S,U)$. Let $|\bar{x}_j| = n$, $|y_i| = m$ and $k = (n,m)$. A homomorphism $f \in \mathrm{Hom}(\langle \bar{x}_j \rangle, \langle y_i \rangle)$ is determined by its action on the generator $\bar{x}_j$. $x_j^f = y_i^\lambda$ is a homomorphism if and only if $(y_i^\lambda)^n = y_i^{n\lambda} = 1$. This in turn is equivalent to $m \mid n\lambda$. Since $(m,n) = k$ there are integers a,b such that $am + bn = k$. Thus $am\lambda + bn\lambda = k\lambda$, and since $m \mid n\lambda$, we have $m \mid n\lambda$ is equivalent to $m \mid k\lambda$ or $(y_i^\lambda)^k = 1$. Since there is only one such set of elements, namely $\langle y^{m/k} \rangle$, we have: $f \in \mathrm{Hom}(\langle \bar{x}_j \rangle, \langle y_i \rangle)$ if and only if $f(\bar{x}_j) \in \langle y_i^{m/k} \rangle$. If $f(\bar{x}_j) = y^{m/k}$ we see that $f^s(\bar{x}_j) = f(\bar{x}_j) \ldots f(\bar{x}_j) = y^{sm/k}$. f has order m and generates $\mathrm{Hom}(\langle \bar{x}_j \rangle, \langle y_i \rangle)$.

Lemma 1.28. Let A be an abelian p-group and $X = \langle x \rangle$ a cyclic group of order $p^k > \mathrm{Exp}\, A$. If $G = A \times X$ then $B = \mathrm{Stab}(G \geq A \times \{1\} \geq 1)$ is a closed stability group isomorphic to A.

Proof: Let $A = \oplus \sum_{i=1}^{n} \langle y_i \rangle$ where each $\langle y_i \rangle$ is a cyclic group and $B = \mathrm{Stab}(G \geq A \times \{1\} \geq 1)$. By 1.27, $B \simeq \oplus \sum_{i} \sigma(|x|, |y_i|)$. Since $|X| > \mathrm{Exp}\, A \geq |y_i|$, and both $|x|$ and $|y_i|$ are powers of p $(|x|, |y_i|) = |y_i|$. Thus $\mathrm{Stab}(G \geq A \times \{1\} \geq 1) \simeq \oplus \sum_{i} \sigma(|y_i|) \simeq A$. In 1.26 one notices that $[G, B] = \langle \text{Image of } f \mid f \in \mathrm{Hom}(G/A, A) \rangle$. Since $|X| > |y_i|$ then $[G, B] = A \times \{1\}$ and B must be a closed stability group.

Theorem 1.29. Any abelian group A is a closed stability group for some abelian group G.

Proof: Let A_p be a p-Sylow of the abelian group A. By the previous lemma, A_p is a closed stability group for a p-group G_p. Theorem 1.25 gives that $\prod_p A_p$ is a stability group for $G = \pi G_p$, so $\gamma\, G(A_p)^n = 1$ for some n. Let $D = \mathrm{Stab}(\gamma\, G\, A_p^i)$ for $i = 0, 1, \ldots, n$. By 1.7,

$$\pi(D) = \pi([G, D]) \subseteq \pi([G, A_p]) = \pi(A_p) = \{p\} .$$

Since the q-Sylows of an abelian group are characteristic, 1.16 says that both D and A_p fix all q-Sylows pointwise for $q \neq p$. Thus $\gamma\, G(A_p)^i = \gamma\, G_p(A_p)^i$ and $\gamma\, G\, D^i = \gamma\, G_p D^i$ for all $i \geq 1$. Since $D = \mathrm{Stab}(\gamma\, G\, A_p^i)$,

$$\gamma\, G_p D^i = \gamma\, G\, D^i = \gamma\, G(A_p)^i = \gamma\, G_p(A_p)^i .$$

Thus the restriction of D to G_p, $D|_{G_p}$ is contained in $A_p = \mathrm{Stab}(\gamma\, G_p(A_p)^i)$. If $d \in D$ fixes G_p pointwise, $d = 1$ because D fixes the other Sylows pointwise. By definition, D is the closure of A_p in Aut G so $D \geq A_p$ and we have $A_p \geq D|_{G_p} \simeq D \geq A_p$. Since every Sylow of A is now a closed stability group 1.21 says A is a closed stability group.

CHAPTER II

STABILITY GROUPS OF p-GROUPS

This chapter deals with stability groups of p-groups, which we have seen must also be p-groups (1.7). By applying 1.25, we will know all possible stability groups for a nilpotent group G provided we know the stability groups for each p-Sylow subgroup of G.

Theorem 2.1. Let G be a p-group, p a prime. If P is a p-Sylow of Aut G then P is the stability group of a chief series.

Proof: The semidirect product GP is a p-group and therefore has a lower central series, $GP = \Gamma_0 \geq \Gamma_1 \geq \ldots \geq \Gamma_n = 1$ for some integer n. By intersecting this with G we obtain s: $G \geq \Gamma_1 \cap G \geq \Gamma_2 \cap G \geq \ldots \geq \Gamma_n \cap G = 1$. Since $\Gamma_i \lhd GP$ we have $\Gamma_i \cap G \lhd G$ for all i and s is a normal series. Since (Γ_i) is the lower central series of GP, $\Gamma_{i+1} \geq [\Gamma_i, GP] \geq [\Gamma_i, P]$ for all i. Because $[g, P] \leq G$ for any $g \in G$, we see that P stabilizes s and consequently any refinement of s. In particular, if $\hat{s}$ is a chief series that refines s, by 1.7, Stab($\hat{s}$) is a p-group containing P. Thus $P = \text{Stab}(\hat{s})$.

Corollary 2.2. Let G be a p-group of order p^n, and P a p-Sylow of Aut G. If c is the class of P then $c \leq n-1$.

Proof: The previous theorem says that P is the stability group of a chief series which has length n, because $|G| = p^n$. By 1.12, $c \leq n-1$.

Aut G induces a permutation group on the chief series of G by $\theta: s \to s^{\theta}$ for every $\theta \in \text{Aut } G$, where s^{θ} is defined in 1.2. By 2.1, a p-Sylow, P, of the automorphism group of a p-group G is the stability group of some chief series s. Let P_s be the set of chief series in the permutation orbit that contains s.

Theorem 2.3. Let G be a p-group. If k is the number of p-Sylow subgroups of Aut G, then k is less than or equal to the number of chief series in any P_s, where P_s is defined above.

Proof: Let P be a p-Sylow of Aut G and $s: G \geq G_1 \geq \ldots \geq G_n = 1$ be a chief series with $P = \text{Stab}(s)$. Since all p-Sylows of Aut G are conjugate and $P^{\theta} = \text{Stab}(G \geq G_1^{\theta} \geq \ldots \geq G_n^{\theta} = 1)$ for $\theta \in \text{Aut } G$, we see that each p-Sylow is the stability group for a chief series in P_s. Thus the number of p-Sylows of Aut G is less than or equal to the number of elements (chief series) in the orbit P_s.

The following definition was motivated by Kaloujnine's characterization of the stability groups for an elementary abelian p-group (See 2.20).

Definition 2.4. Let G be a p-group, and $A \leq \text{Aut } G$. A is said to be of K-type if

(1) A is a p-group.

(2) A is normal in every p-Sylow of Aut G that contains A.

(3) A is the intersection of all p-Sylows of Aut G that contain A.

Theorem 2.5. If G is a p-group and A is of K-type then A is a closed stability group.

<u>Proof</u>: Since $A \leq P$ for some p-Sylow of Aut G, $\gamma\, G\, A^{i} \leq \gamma\, G\, P^{i}$ for all i. By 2.1, $\gamma\, G\, P^{n} = 1$ for some n and thus $\gamma\, G\, A^{n_1} = 1$ for some $n_1 \leq n$. $\bar{A} = \mathrm{Stab}(\gamma\, G\, A^{i})$ is a p-group since $\pi(\bar{A}) = \pi([G, A]) = \{p\}$. We now have $A \leq \bar{A} \leq P_0$ where P_0 is a p-Sylow of Aut G. Let $B = \langle P_i \mid P_i \geq A$ and P_i is a p-Sylow of Aut G, $0 \leq i \leq r\rangle$. Since A is normal in each P_i, $A \lhd B$. By 1.10.iv $(\gamma\, G\, A^{i})^{\beta} = \gamma\, G\, A^{i}$ for every $\beta \in B$. According to 1.2,

$$\begin{aligned}\bar{A} &= \mathrm{Stab}(\gamma\, G\, A^{i})\\ &= \mathrm{Stab}((\gamma\, G\, A^{i})^{\beta})\\ &= (\bar{A})^{\beta}.\end{aligned}$$

Since each P_i that contains A is a p-Sylow of B, there exists $\beta_i \in B$ such that $P_0^{\beta_i} = P_i$ for each i. Hence $\bar{A} = (\bar{A})^{\beta_i} \leq P_0^{\beta_i} = P_i$. This forces $\bar{A} \leq \bigcap_{i=0}^{r} P_i = A$. Thus $\bar{A} = A$, which is equivalent to saying that A is a closed stability group.

<u>Corollary 2.6</u>. If P is a p-Sylow of Aut G and G a p-group, then P is a closed stability group.

<u>Proof</u>: P is trivially of K-type.

<u>Corollary 2.7</u>. Let $|G| = p^{n}$, and $A \leq$ Aut G be of K-type and of nilpotence class k. If the class of G is c then $c-1 \leq k \leq n-1$.

<u>Proof</u>: Since A is an intersection of p-Sylows of Aut G, and every p-Sylow contains Inn G, $A \geq$ Inn G. The lower bound comes from the fact that Inn G has class $c-1$, and the upper bound from the containment of A in a p-Sylow of Aut G which by 2.2 has class bounded by $n-1$.

By intersecting all p-Sylows of Aut G, we are led to the smallest possible K-type stability group, $O_p(\text{Aut } G)$, which is contained in every K-type stability group.

Corollary 2.8. If G is a p-group and $A = O_p(\text{Aut } G)$, then A is a K-type stability group for a characteristic series.

Proof: $O_p(\text{Aut } G) = \bigcap\{P \mid P$ is a p-Sylow of $\text{Aut } G\}$ is normal in Aut G. By 2.5, $A = \text{Stab}(\gamma\, G\, A^i)$. Since $A \lhd \text{Aut } G$, 1.10.iv gives $(\gamma\, G\, A^i)^\beta = \gamma\, G\, A^i$ for every $\beta \in \text{Aut } G$, $i = 0,1,\dots,n$. Thus each $\gamma\, G\, A^i$ is characteristic in G.

Let us note that if G is a p-group and A is a p-group contained in Aut G, then $\gamma\, G\, A^n = 1$ for some n. If A is also the set of automorphisms fixing pointwise some normal subgroup H, then A centralizes $H\cdot\gamma\, G\, A^i/H\cdot\gamma\, G\, A^{i+1}$, $i = 0,1,2,\dots,n$. This gives $A \leq \text{Stab}(G \geq [G, A]\cdot H \geq [G, A, A]\cdot H \geq \dots \geq H \geq 1) \leq C_{\text{Aut } G}(H) = A$, and A is a stability group. In particular, if G is a p-group and $C_{\text{Aut } G}(H)$ is p-group for some $H \lhd G$, then $C_{\text{Aut } G}(H)$ is a stability group.

Definition 2.9. If G is a p-group, $\Omega_i(G) = \langle x \in G \mid x^{p^i} = 1\rangle$. When G is understood $\Omega_i(G)$ may be indicated by Ω_i.

If G is an abelian p-group for an arbitrary prime p or if G is a nonabelian p-group with p an odd prime then (See Gorenstein [9, pages 178, 184]) $C_{\text{Aut } G}(\Omega_1(G))$ is a p-group and thus a stability group.

If G is a p-group and E is maximal with respect to being a normal abelian subgroup of exponent $p^n > 2$, $n = 1,2,\dots,$ then (See Blackburn [3]) $C_{\text{Aut } G}(E)$ is a p-group and thus a stability group.

In [2], Adney and Yen investigated $C_{\text{Aut } G}(G/Z(G))$ which they called A_c. There one may find the following definition and theorem.

Definition 2.10. G is purely nonabelian if it does not have an abelian direct factor.

Theorem 2.11. For a purely nonabelian group the correspondence $\sigma \to f_\sigma$ (See 1.26) is a one-to-one map of A_c onto $\mathrm{Hom}(G, Z)$.

Theorem 2.12. Let G be a p-group and Z be the center of G. If $G' \geq Z$ then

$$A_c = \mathrm{Stab}(G \geq Z \geq 1) \simeq \mathrm{Hom}(G/G', Z).$$

Proof: G is purely nonabelian for otherwise its abelian direct factor would be inside the center of G but not in the commutator G'. By 2.11, $|A_c| = |\mathrm{Hom}(G, Z)| = |\mathrm{Hom}(G/G', Z)|$. $A_c = C_{\mathrm{Aut}\ G}(G/Z) \geq \mathrm{Stab}(G \geq Z \geq 1) \simeq \mathrm{Hom}(G/Z, Z)$ by 1.26. Since $G' \geq Z$, $\mathrm{Hom}(G/Z, Z) = \mathrm{Hom}(G/G', Z)$. Thus $A_c \geq \mathrm{Stab}(G \geq Z \geq 1) \simeq \mathrm{Hom}(G/G', Z)$ and since the orders of both ends are equal, $A_c = \mathrm{Stab}(G \geq Z \geq 1)$.

Corollary 2.13. Let G be a p-group, and Z be the center of G. If $|Z| = p$ and $|G| > p$, then $A_c = \mathrm{Stab}(G \geq Z \geq 1)$.

Proof: Since $|G| > |Z|$, G is nonabelian. Thus $1 \neq G' \triangleleft G$ and $G' \cap Z \gneq 1$. Since $|Z| = p$, $G' \geq Z$ and the previous theorem gives the desired result.

Notice that if G is cyclic of order p then $A_c = \mathrm{Aut}\ G$ but $\mathrm{Stab}(G \geq Z \geq 1) = \mathrm{Stab}(G \geq 1) = 1$.

Now we will calculate the stability groups for cyclic p-groups.

Let $G = \langle x \rangle$ be a cyclic group of order p^t, where p is an odd prime. It is known (See Scott [15, page 117]) that $\mathrm{Aut}\ G$ is cyclic of order $p^{t-1}(p-1)$ where $\alpha: x \to x^{1+p}$ generates the p-Sylow of Aut G. The subgroups of G are $G = \langle x\rangle \geq \langle x^{p}\rangle \geq \langle x^{p^2}\rangle \geq \dots \geq 1$. The reader should note that $\alpha^k : x \to x^{(1+p)^k}$ and any stability group is contained in $\langle\alpha\rangle$.

Lemma 2.14. With the notation above, and $0 \leq i < i + k$, i an integer, $i + k = 1,2,\dots,t$,

$$C_{\langle\alpha\rangle} \frac{\langle x^{p^i}\rangle}{\langle x^{p^{i+k}}\rangle} = \langle\alpha^{p^{k-1}}\rangle .$$

Proof:

$$\alpha^j \in C_{\langle\alpha\rangle} \frac{\langle x^{p^i}\rangle}{\langle x^{p^{i+k}}\rangle} \Leftrightarrow x^{-p^i}(x^{p^i})^{\alpha^j} = x^{-p^i+p^i(1+p)^j}$$

$$= x^{p^i[(1+p)^j-1]} \in \langle x^{p^{i+k}}\rangle$$

$\Leftrightarrow p^i[(1+p)^j - 1] \equiv 0 \mod p^{i+k}$

$\Leftrightarrow (1+p)^j - 1 \equiv 0 \mod p^k$

$\Leftrightarrow (1+p)^j \equiv 1 \mod p^k$.

Since the multiplicative order of $1+p$ in the integers mod p^k is p^{k-1}, this last statement is equivalent to p^{k-1} divides j or $\alpha^j \in \langle\alpha^{p^{k-1}}\rangle$.

Theorem 2.15. For an odd prime p, let $G = \langle x\rangle$ be a cyclic group of order p^t, and $\alpha: x \to x^{1+p}$. If p^k is the largest index of two consecutive groups in a series $s: G = G_0 \geq G_1 \geq \dots \geq G_n = 1$, then $\mathrm{Stab}(s) = \langle\alpha^{p^{k-1}}\rangle$ is cyclic of order p^{t-k}.

Proof: By the above lemma, $C_{\langle\alpha\rangle}\left(\frac{G_i}{G_{i+1}}\right) = \langle\alpha^{m(i)} \mid m(i) = \frac{[G_i:G_{i+1}]}{p}\rangle$.

Thus

$$\mathrm{Stab}(s) = \bigcap_{i=1}^{n-1} C_{\langle\alpha\rangle}\left(\frac{G_i}{G_{i+1}}\right)$$

$$= \langle\alpha^n \mid n = \max\{\frac{[G_i:G_{i+1}]}{p}\}\rangle$$

$$= \langle\alpha^{p^{k-1}}\rangle .$$

Let $G = <x>$ be cyclic of order $2^t > 4$. It is known (See Scott [15, page 121]) that $\mathrm{Aut}\, G = <5I> \times <-I>$ where $5I: x \to x^5$, $-I: x \to x^{-1}$. The reader should note that $(5I)^j: x \to x^{5^j}$.

Lemma 2.16. With the above notation and $i = 0,1,\ldots,t-k$,

$$C_{\mathrm{Aut}\, G}\left(\frac{<x^{2^i}>}{<x^{2^{i+k}}>}\right) = \begin{cases} \mathrm{Aut}\, G & \text{if} \quad k = 1 \\ (5I)^{2^{k-2}} & \text{if} \quad k > 1 \end{cases} .$$

Proof: Case $k = 1$.

Since there is only one subgroup of any given order, all subgroups are characteristic. Thus $\mathrm{Aut}\, G$ induces automorphisms on $\frac{<x^{2^i}>}{<x^{2^{i+1}}>} \simeq \sigma(2)$, and $\frac{\mathrm{Aut}\, G}{C_{\mathrm{Aut}\, G}\left(\frac{<x^{2^i}>}{<x^{2^{i+1}}>}\right)} \underset{\sim}{\subset} \mathrm{Aut}\, \sigma(2) = 1$. Thus

$$\mathrm{Aut}\, G = C_{\mathrm{Aut}\, G} \frac{<x^{2^i}>}{<x^{2^{i+1}}>} .$$

Case $k > 1$.

$$(5I)^j \in C_{\mathrm{Aut}\, G}\left(\frac{<x^{2^i}>}{<x^{2^{i+k}}>}\right) \Leftrightarrow x^{-2^i}(x^{2^i})^{(5I)^j} = x^{-2^i}x^{2^i 5^j} = x^{2^i(5^j-1)} \in <x^{2^{i+k}}>$$

$$\Leftrightarrow 2^i(5^j - 1) \equiv 0 \quad \mathrm{mod}\ 2^{i+k}$$

$$\Leftrightarrow 5^j - 1 \equiv 0 \quad \mathrm{mod}\ 2^k$$

$$\Leftrightarrow 5^j \equiv 1 \quad \mathrm{mod}\ 2^k .$$

Since the multiplicative order of 5 in the integers $\mathrm{mod}\ 2^k$ is 2^{k-2}, this last statement is equivalent to $2^{k-2} | j$ or $(5I)^j \in <(5I)^{2^{k-2}}>$

(*) Thus $<(5I)^{2^{k-2}}> = C_{\mathrm{Aut}\, G}\left(\frac{<x^{2^i}>}{<x^{2^{i+k}}>}\right) \cap <5I>$.

The following shows that $-I \notin C_{\text{Aut }G}(\frac{\langle x^{2^i}\rangle}{\langle x^{2^{i+k}}\rangle})$, $k > 1$.

$$x^{-2^i}(x^{2^i})^{-I} = x^{-2^{i+1}} \notin \langle x^{2^{i+k}}\rangle .$$

Since $\langle 5I\rangle \le C_{\text{Aut }G}\frac{\langle x^{2^i}\rangle}{\langle x^{2^{i+2}}\rangle}$ and $-I \notin C_{\text{Aut }G}\frac{\langle x^{2^i}\rangle}{\langle x^{2^{i+2}}\rangle}$, $\langle 5I\rangle = C_{\text{Aut }G}\frac{\langle x^{2^i}\rangle}{\langle x^{2^{i+2}}\rangle}$. If $k \ge 2$, $C_{\text{Aut }G}\frac{\langle x^{2^i}\rangle}{\langle x^{2^{i+k}}\rangle} \le C_{\text{Aut }G}(\frac{\langle x^{2^i}\rangle}{\langle x^{2^{i+2}}\rangle}) = \langle 5I\rangle$.

From (*) we have: $C_{\text{Aut }G}(\frac{\langle x^{2^i}\rangle}{\langle x^{2^{i+k}}\rangle}) = \langle (5I)^{2^{k-2}}\rangle$.

Theorem 2.17. Let $G = \langle x\rangle$ be a cyclic group of order $2^t > 4$. If 2^k is the largest index of 2 consecutive groups in a series $s: G = G_0 \ge G_1 \ge \ldots \ge G_n = 1$ then

$$\text{Stab}(s) = \begin{cases} \langle 5I\rangle \times \langle -I\rangle = \text{Aut } G & \text{if } k = 1 \\ \langle (5I)^{2^{k-2}}\rangle & \text{if } k > 1 \end{cases}.$$

Proof: By lemma 2.16,

$$\begin{aligned} \text{Stab}(s) &= \bigcap_{i=0}^{n-1} C_{\text{Aut }G}(\frac{G_i}{G_{i+1}}) \\ &= C_{\text{Aut }G}(\frac{G_{i_0}}{G_{i_0+1}}) \\ &= \begin{cases} \text{Aut } G & \text{if } k = 1 \\ \langle 5I^{2^{k-2}}\rangle & \text{if } k > 1 \end{cases} \end{aligned}$$

where $[G_{i_0} : G_{i_0+1}] = 2^k$.

Of course when G is cyclic of order 2, Aut $G = 1$ and there are no nontrivial stability groups. When G is cyclic of order 4

"$x \to x^5$" is the identity mapping. In this case $\text{Aut } G = \langle -I \rangle = \text{Stab}(\langle \alpha \rangle \geq \langle \alpha^2 \rangle \geq 1)$.

The rest of this chapter deals with stability groups of elementary abelian p-groups.

Kaloujnine, in [11], found a very nice characterization for stability groups of elementary abelian p-groups. The proof given here is mine since his was unavailable. By examples we see that almost none of the interesting characteristics may be generalized to stability groups of arbitrary p-groups.

In [17], Suprunenko also deals with nilpotent subgroups of $GL(n,p)$ which is the automorphism group of an elementary abelian p-group of order p^n.

Before proceeding to Kaloujnine's theorem it is necessary to fix the notation to be used and to prove several technical facts.

Let G be elementary abelian and $s: G = G_n \geq G_{n-1} \geq \ldots \geq G_1 \geq G_0 = 1$ be a series of subspaces. Construct a basis for G by picking a basis $\{v_1^1, v_2^1, \ldots, v_{j(1)}^1\}$ for G_1. Extend this set to a basis of G_2 where the new vectors added are $\{v_1^2, v_2^2, \ldots, v_{j(2)}^2\}$. Continue this process so that $\{v_1^i, \ldots, v_{j(i)}^i\}$ are the vectors used to extend the basis of G_{i-1} to a basis of G_i.

Definition 2.18. For $A \leq \text{Aut } G$, let

$$F(A) = \{g \in G \mid g^{\alpha} = g \text{ for all } \alpha \in A\} .$$

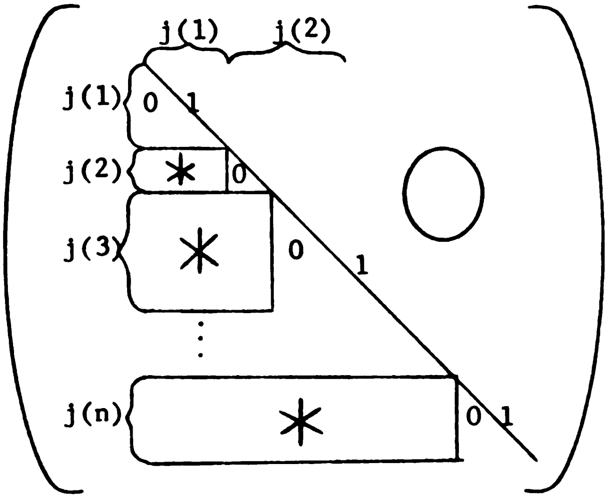
Lemma 2.19. Let G be elementary abelian with the above notation. If $A = \text{Stab}(s)$ then:

(i) $A = \{T \in \text{Hom}_{Z_p}(G,G) \mid v_j^iT - v_j^i \in G_{i-1}\}$

(ii) $\gamma\, G\, A^i = G_{n-i}$

(iii) $G_1 = F(A)$.

<u>Proof</u>: (i) Every automorphism of Stab(s) must have the required form. The matrix afforded such a T by the basis $\{v_j^i\}_{i,j}$ under the ordering $v_1^1, v_2^1, \ldots, v_{j(1)}^1, v_1^2, \ldots, v_{j(2)}^2, v_1^3, \ldots, v_{j(n)}^n$ has the form



Since $\det T = 1$, T is an automorphism. T induces an automorphism on each $\frac{G_{i+1}}{G_i}$ which fixes the generators. T therefore centralizes each $\frac{G_{i+1}}{G_i}$ and $T \in A$ as required.

(ii) A stabilizes s so $\gamma\, G\, A^i \leq G_{n-i}$. We prove the opposite containment by induction on i.

$$\gamma\, G\, A^0 = G = G_{n-0}.$$

Let $0 \neq w \in G_{n-i}$.

Define $T: \begin{cases} v_1^{n-i+1} \to v_1^{n-i+1} + w \\ v_j^i \to v_j^i \quad \text{for all other } i,j \end{cases}$

$$(v_\tau^s)T - v_\tau^s = \left\{ \begin{matrix} w & \text{if } s = n-i+1 \text{ and } \tau = 1 \\ 0 & \text{otherwise} \end{matrix} \right\} \in G_{s-1}\,.$$

By (i) $T \in A$ and $w \in [G_{n-(i-1)}, A]$ which by induction equals $[\gamma G A^{i-1}, A] = \gamma G A^{i}$ and we have $G_{n-i} \leq \gamma G A^{i}$ as required.

(iii) The last nontrivial term of a series s is always fixed pointwise by Stab(s).

For μ, ν, $2 \leq \mu \leq n$, $1 \leq \nu \leq j(\mu)$. Define T^{μ}_{ν} by:

$$(v^{i}_{j})T^{\mu}_{\nu} = \begin{cases} v^{i}_{j} + v^{1}_{1} & \text{if } i = \mu \text{ and } j = \nu \\ v^{i}_{j} & \text{otherwise} \end{cases} .$$

By (i) every $T^{\mu}_{\nu} \in \text{Stab}(s)$. If $\sum_{i,j} a^{i}_{j} v^{i}_{j}$ is fixed by Stab(s), then:

$$\sum_{i,j} a^{i}_{j} v^{i}_{j} = (\sum_{i,j} a^{i}_{j} v^{i}_{j})T^{\mu}_{\nu}$$

$$= (\sum_{i,j} a^{i}_{j} v^{i}_{j}) + a^{\mu}_{\nu} v^{1}_{1} .$$

Thus $a^{\mu}_{\nu} v^{1}_{1} = 0$, forcing $a^{\mu}_{\nu} = 0$. Since μ was an arbitrary integer with $2 \leq \mu \leq n$, the fixed point $\sum_{i,j} a^{i}_{j} v^{i}_{j}$ has nonzero coefficients only when $i = 1$. Thus $\sum_{i,j} a^{i}_{j} v^{i}_{j} \in G_1$ and G_1 is the set of fixed points of A.

Theorem 2.20 (Kaloujnine [11]). Let G be an elementary abelian p-group and $A \leq \text{Aut } G$. A is a stability group if and only if A is of K-type.

Proof: By 1.7.iii, A must be a p-group.

We next prove by induction on $|G|$ that $A = \text{Stab}(s) \lhd P$ for every p-Sylow P of Aut G that contains A.

If $|G| = p$, then $|\text{Aut } G| = p - 1$. Since every stability group of a p-group is a p-group the only possible stability group is $\{1\}$.

If P is a p-Sylow containing A, we may assume that P fixes v_1^1. Since $A \leq P$, the points fixed by P are also fixed by A. P must fix a nontrivial vector because P is a stability group by 2.1. Remember that v_1^1 was an arbitrary nontrivial point fixed by A, so we could just as well have chosen v_1^1 as a vector fixed by P.

Since $<v_1^1>$ is P and A-admissible, P and A induce automorphism groups P^* and A^* on $G/<v_1^1>$. Let $B^* = \mathrm{Stab}(G/<v_1^1> \geq G_{n-1}/<v_1^1> \geq \ldots \geq G_1/<v_1^1> \geq \tilde{1})$ and $\sim$ indicate an image in $G/<v_1^1>$. Since $G \simeq G/<v_1^1> \oplus <v_1^1>$, any automorphism in B^* may be extended to an automorphism of G in A. Consequently A maps onto B^*, i.e. $A^* = B^*$. By induction $A^* = B^* \triangleleft P^*$ because A^* is normal in the p-Sylow containing P^*. By lemma 1.10.iv, $G_i/<v_1^1>$, $0 \leq i \leq n$ is P^* admissible. Since A^* is a closed stability group $\hat{S}(\tilde{s}) = N_{\mathrm{Aut}\,\tilde{G}}(A^*)$, i.e. for $\psi \in P$, $G_i/<v_1^1> = (G_i/<v_1^1>)^{\psi^*} = (G_i^\psi + <v_1^1>)/<v_1^1>$. This forces $G_i = G_i^\psi + <v_1^1>$. G_i and G_i^ψ have the same dimension so $G_i^\psi = G_i$ and $\psi \in \hat{S}(s)$. By lemma 1.10.vi, $\hat{S}(s) = N_{\mathrm{Aut}\,G}(A)$ so P normalizes A.

Now we show that $A = \bigcap\{P \mid P \geq A,\ P$ a p-Sylow of $\mathrm{Aut}\,G\}$. Let P be the p-Sylow of $\mathrm{Aut}\,G$ of the form

$$P = \{T \in GL(G) \mid \text{matrix of } T = I + \sum_{i>j} a_{ij}e_{ij} \text{ for the basis}$$

$$\{v_1^1, v_2^1, \ldots, v_{j(1)}^1, v_1^2, \ldots, v_{j(2)}^2, v_1^3, \ldots, v_{j(n)}^n\}.$$

Note that $P \geq A$. For $1 \leq i \leq n$, $1 \leq j \leq j(i)$ let

$$\begin{aligned} \psi^i_j\colon\ & v^i_1 \to v^i_j \\ & v^i_j \to v^i_1 \\ & v^s_t \to v^s_t \quad (s,t) \neq (i,j) \ \text{ or } \ (i,1)\ . \end{aligned}$$

We see that $(\psi^i_j)^{-1} = \psi^i_j$ and $\psi^i_j \in \hat{S}(s) = N_{\mathrm{Aut}\ G}(A)$.

$$\begin{aligned} (v^i_j)P^{\psi^i_j} &= v^i_j\psi^i_j P\ \psi^i_j \\ &= v^i_1 P\ \psi^i_j \\ &= (v^i_1 + G_{i-1})\psi^i_j \\ &= v^i_j + G_{i-1}\ . \end{aligned}$$

If $\beta \in \bigcap_{i,j} P^{\psi^i_j}$ then $\beta\colon v^i_j \to v^i_j + G_{i-1}$, for all i,j. Thus $\bigcap_{i,j} P^{\psi^i_j} \le A$. Since $A = A^{\psi^i_j} \le P^{\psi^i_j}$ we have that $A = \bigcap_{i,j} P^{\psi^i_j}$. If we intersect more p-Sylows containing A then we still have $A = \bigcap\{P \mid P$ is a p-Sylow containing $A\}$.

The characterization of stability groups of p-groups as all those automorphism groups of K-type cannot be extended to abelian p-groups. In Example 3 of the Appendix, Aut G is a 2-group for an abelian 2-group G and thus the only stability group of K-type. There is a closed stability group, namely $\langle\eta\beta\rangle$, that is a proper nonnormal subgroup of Aut G, which is its own 2-Sylow.

If G is an elementary abelian p-group, any series is formed by adding a direct summand to the previous term in the series. The following theorem deals with similar series for groups which aren't necessarily even p-groups. For related results see Shoda [16].

Theorem 2.21. Let $G = H_1 \oplus \ldots \oplus H_n$ with H_i arbitrary groups, $G_i = H_1 \oplus \ldots \oplus H_i$ for $1 \leq i \leq n$, and $A_i = \mathrm{Stab}\{G \geq G_i \geq 1\}$. If $A = \mathrm{Stab}\{G = G_n \geq G_{n-1} \geq \ldots \geq G_1 \geq 1\}$ then,

(i) $A_i \lhd A$ and

(ii) $A = A_1 A_2 \ldots A_{n-1}$.

Proof: (i) Each G_i is A-admissible so $A \leq \hat{S}(G \geq G_i \geq 1)$. By 1.4, $A \leq \hat{S}(G \geq G_i \geq 1) \leq N_{\mathrm{Aut}\, G}(\mathrm{Stab}(G \geq G_i \geq 1))$.

(ii) To prove part (ii), induct on n, the length of the series. For the case $n = 2$, we have the trivial statement $A = \mathrm{Stab}(G \geq G_1 \geq 1) = A_1$. Let $\psi \in A$. By induction on n, $\psi|_{G_{n-1}} = \alpha_1 \ldots \alpha_{n-2}$ where $\alpha_i \in \mathrm{Stab}(G_{n-1} \geq G_i \geq 1) \leq \mathrm{Aut}\, G_{n-1}$. Extend each α_i to $\hat{\alpha}_i = \alpha_i \times 1_{H_n} : (g_1,\ldots,g_{n-1},g_n) \to ((g_1,\ldots,g_{n-1})^{\alpha_i},g_n)$. The following calculations show that $\hat{\alpha}_i \in \mathrm{Stab}(G \geq G_i \geq 1)$ for $i = 1,\ldots,n-2$.

$$(g_1,\ldots,g_i,1,\ldots,1)^{\hat{\alpha}_i} = ((g_1,\ldots,g_i,1,\ldots,1)^{\alpha_i},1) = (g_1,\ldots,g_i,1,\ldots,1).$$

Any $g \in G$ is of the form $g = (x,g_n)$ where $x \in G_{n-1}$ and $g_n \in H_n$.

$$g^{-1}g^{\hat{\alpha}_i} = (x^{-1},g_n^{-1})(x^{\alpha_i},g_n) = (x^{-1}x^{\alpha_i},1) \in G_i \quad \text{since}$$

$$\alpha_i \in \mathrm{Stab}(G_{n-1} \geq G_i \geq 1).$$

(*) Let $\hat{\psi} = \hat{\alpha}_1 \ldots \hat{\alpha}_{n-2} \in A_1 \ldots A_{n-2}$. $\hat{\psi}|_{G_{n-1}} = \alpha_1 \ldots \alpha_{n-2} = \psi|_{G_{n-1}}$. The previous calculation shows that $(\hat{\psi})^{-1}\psi$ fixes G_{n-1} pointwise. Since each $\hat{\alpha}_i \in C_{\mathrm{Aut}\, G}(G/G_i) \leq C_{\mathrm{Aut}\, G}(G/G_{n-1})$, $\hat{\alpha}_i^{-1} \in C_{\mathrm{Aut}\, G}(G/G_{n-1})$.

ψ centralizes G/G_{n-1}, therefore $(\hat{\psi})^{-1}\psi$ centralizes G/G_{n-1}. $(\hat{\psi})^{-1}\psi$ must now be in $\mathrm{Stab}(G \geq G_{n-1} \geq 1)$ and $\psi = \hat{\psi}((\hat{\psi})^{-1}\psi) \in A_1 \ldots A_{n-2}(A_{n-1})$ as required.

Corollary 2.22. If G is elementary abelian and $A_i = \mathrm{Stab}(G \geq G_i \geq 1)$ then $\mathrm{Stab}(G \geq G_{n-1} \geq \ldots \geq G_1 \geq 1) = A_1 \ldots A_{n-1}$.

Examination of $<x>$, a cyclic group of order 3^5, reveals that $\mathrm{Stab}(s)$ need not be the product of stability groups of proper subseries of s. For, if $\alpha: x \to x^4$, by 2.15

$$\mathrm{Stab}(<x> \overset{3^2}{\geq} <x^{3^2}> \overset{3}{\geq} <x^{3^3}> \overset{3^2}{\geq} 1) = <\alpha^3>$$

$$\mathrm{Stab}(<x> \overset{3^2}{\geq} <x^{3^2}> \overset{3^3}{\geq} 1) = <\alpha^{3^2}>$$

$$\mathrm{Stab}(<x> \overset{3^3}{\geq} <x^{3^3}> \overset{3^2}{\geq} 1) = <\alpha^{3^2}> \quad \text{and}$$

$$<\alpha^3> \neq <\alpha^{3^2}> \cdot <\alpha^{3^2}> .$$

Definition 2.23. $A = \mathrm{Stab}(s)$ is a minimal stability group if A contains no other nontrivial stability groups.

Since the stability group of a series contains the stability group of a subseries one might expect any $\mathrm{Stab}(G \geq H \geq 1)$ to be a minimal stability group. In this direction, we have

Theorem 2.24. If G is an elementary abelian p-group and H is a proper subgroup, then $\mathrm{Stab}(G \geq H \geq 1)$ is a minimal stability group.

Proof: Let $A = \mathrm{Stab}(G \geq H \geq 1)$ and B be some other stability group, $1 \neq B \leq A$, which by 2.22 may be assumed to have the form $B = \mathrm{Stab}(G \geq K \geq 1)$. By 2.19.ii, $H = [G, A]$ and $K = [G, B]$. Then $B \leq A$ implies $K = [G, B] \leq [G, A] = H$. 2.19.iii gives $F(A) = H$ and $F(B) = K$. Since $B \leq A$, $K = F(B) \geq F(A) = H$ and $K = H$, i.e. $B = \mathrm{Stab}(G \geq H \geq 1) = A$.

In Example 4 of the Appendix, Inn $G = \text{Stab}(G \ge \langle a^3 \rangle \ge 1)$ is not a minimal stability group.

There is an interesting relationship between the minimal stability groups contained in the same p-Sylow of Aut G when G is elementary abelian.

Lemma 2.25. Let G be an elementary abelian p-group, $A = \text{Stab}(G \ge H \ge 1)$ and $B = \text{Stab}(G \ge K \ge 1)$. If A and B are both in the same p-Sylow of Aut G then $K \ge H$ or $K \le H$.

Proof: $\langle A,B \rangle$ is contained in some p-Sylow P of Aut G, which by 2.20 must normalize A and B. Again by 2.20 every stability group is closed and according to 1.10.vii $\hat{S}(s) = N_{\text{Aut } G}(\text{Stab}(s))$. Thus $A \le N_{\text{Aut } G}(B) = \hat{S}(G \ge K \ge 1)$. Assume that there exist $0 \ne v_H \in H \backslash K$ and $0 \ne v_1^1 \in K \backslash H$. Extend $\{v_1^1\}$ to a basis of G by adding a basis of H and extending that to a basis of G. Define T by $T: v_1^1 \to v_1^1 + v_H$

$$v_j^i \to v_j^i \ , \ (i,j) \ne (1,1).$$

The computation $(\sum_{i,j} a_j^i v_j^i)T - \sum_{i,j} a_j^i v_j^i = a_1^1 v_H$ shows that T centralizes G/H. Also, if $\sum_{i,j} a_j^i v_j^i \in H$ then $a_1^1 = 0$ and T fixes H pointwise. Thus by 2.19.i, $T \in \text{Stab}(G \ge H \ge 1) = A$. $T \notin \hat{S}(G \ge K \ge 1)$ since $v_1^1 \to v_1^1 + v_H \notin K$. This contradicts $A \le \hat{S}(G \ge K \ge 1)$, so either $K \le H$ or $K \ge H$.

In Example 3 of the Appendix, $\langle \pi^3 \beta \rangle = \text{Stab}(G \ge \langle x^2 y \rangle \ge 1)$ and $\langle \pi \beta \rangle = \text{Stab}(G \ge \langle y \rangle \ge 1)$ are closed stability groups in the same 2-Sylow of Aut G but $\langle x^2 y \rangle \cap \langle y \rangle = 1$.

Theorem 2.26. If G is elementary abelian, the product of any two stability groups in the same p-Sylow of Aut G is again a stability group.

Proof: Let $A = \text{Stab}(G = G_n \geq G_{n-1} \geq \ldots \geq G_1 \geq 1)$ and $B = \text{Stab}(G = K_m \geq K_{m-1} \geq \ldots \geq K_1 \geq 1)$ be contained in some p-Sylow of Aut G. By Corollary 2.22, $A = A_1 \ldots A_{n-1}$ and $B = B_1 \ldots B_{m-1}$ where $A_i = \text{Stab}(G \geq G_i \geq 1)$ and $B_j = \text{Stab}(G \geq K_j \geq 1)$. By 2.25 $K_s \geq G_t$ or $K_s \leq G_t$ for all s,t, $0 \leq s \leq m$, $0 \leq t \leq n$. We insert the sets K_j into the series (G_i). Let (H_i) be the series after some of the K's have been inserted. If $H_i \leq K_j$ and $H_{i+1} \not\leq K_j$, then by 2.25 $H_{i+1} \geq K_j$. We insert this K_j, relabel the new series, and continue the process until we obtain a series s: (H_i) made up of all the G's and K's. By 2.22,
$\text{Stab}(s) = \text{Stab}(G \geq H_1 \geq 1) \ldots \text{Stab}(G \geq H_s \geq 1) \ldots$. Each $\text{Stab}(G \geq H_i \geq 1)$ normalizes the others because they are all in the same p-Sylow. We may, therefore, rearrange the product and repeat the same group if necessary to get $\text{Stab}(s) = A_1 \ldots A_{n-1} \cdot B_1 \ldots B_{m-1} = A \cdot B$.

In 2.29 we will show that the characterization of stability groups as those automorphism groups of K-type cannot be extended from elementary abelian p-groups to arbitrary p-groups.

Lemma 2.27. Let G be a p-group with $H \lneq G$. If H contains a nontrivial subgroup normal in G, then

$$\text{Stab}(G \geq H \geq 1) \gneq 1 .$$

Proof: H is contained in a maximal subgroup M which is therefore normal of index p. $H \cap Z(G) \gneq 1$ since H contains a subgroup $K \lhd G$ which must intersect the center nontrivially. Take a cyclic subgroup $\langle x \rangle$ of order p from $H \cap Z(G)$. There exists a homomorphism f from G onto $\langle x \rangle$ with kernel M. 1.26 says that $\alpha_f: g \to g(g)^f$ is an automorphism in $\text{Stab}(G \geq \langle x \rangle \geq 1)$, so it centralizes the cosets of H. If $g \in H \leq M$ then $g^{\alpha_f} = g\, g^f = g \cdot 1 = g$ and α_f fixes H pointwise. Thus $1 \neq \alpha_f \in \mathrm{Stab}(G \geq H \geq 1)$.

Lemma 2.28. Let G be a p-group. G has a unique minimal stability group if and only if G is cyclic.

Proof: If G is a cyclic p-group 2.15 and 2.17 show that G has a unique minimal stability group.

For the other implication let $1 \neq A$ be the unique minimal stability group, and let M_1 be a maximal subgroup of G, which is nontrivial unless $|G| = p$ and G is cyclic. By the previous lemma, $\mathrm{Stab}(G \geq M_1 \geq 1) \gneq 1$. If M_2 were some other maximal subgroup, then $1 \neq A \leq \mathrm{Stab}\{G \geq M_1 \geq 1\} \cap \mathrm{Stab}(G \geq M_2 \geq 1)$. This would say that A fixes $\langle M_1, M_2 \rangle = G$ pointwise, and would be a contradiction to $A \neq 1$. Thus G contains only one maximal subgroup M, and $M = \Phi$ the Frattini subgroup. Because M is maximal, $G = \langle M, x \rangle$. $M = \Phi$ is the set of nongenerators of G so $\langle M, x \rangle = \langle x \rangle$ as required.

Theorem 2.29. If G is a p-group for which every stability group is of K-type, then G is elementary abelian or cyclic of order p^2.

Proof: G is elementary abelian if and only if the Frattini $\Phi = 1$, so assume $\Phi \neq 1$. In a finite group $G \gneq \Phi$ so 2.27 gives $A = \mathrm{Stab}(G \geq \Phi \geq 1) \gneq 1$. Since the series is characteristic, $A \lhd \mathrm{Aut}\, G$. Since a normal p-group is contained in every p-Sylow, and A is of K-type, $A = \cap\{P \mid P$ is a p-Sylow of Aut G$\}$. A must be contained in any other nontrivial stability group since any other is of K-type and thus an intersection of p-Sylows of Aut G. Thus A is the unique minimal stability group for G, which by 2.28 is cyclic. A check of 2.15 and 2.17 shows that the only K-type stability groups of a cyclic group are the p-Sylows of Aut G and unless $|G| = p^2$, G has nontrivial stability groups that are not of K-type.

Along similar lines we have:

Theorem 2.30. If G is a p-group with Aut G containing no nonidentity normal stability groups then G is elementary abelian.

Proof: $\mathrm{Stab}(G \geq \Phi \geq 1) \lhd \mathrm{Aut}\ G$ since Φ is a characteristic subgroup. Unless $\Phi = 1$, 2.27 says that $\mathrm{Stab}(G \geq \Phi \geq 1) \gneq 1$. Thus $\Phi = 1$ and G must be elementary abelian.

If G is a p-group which isn't elementary abelian, 2.30 insures that Fitt(Aut G), the Fitting subgroup of Aut G, will be nontrivial. Fitt(Aut G) will be investigated more extensively in Chapter III.

Since K-type stability groups of a p-group G are intersections of the p-Sylow subgroups of Aut G, a K-type stability group must contain $O_p(\mathrm{Aut}\ G)$. Even though a homocyclic p-group G is very similar to elementary abelian p-group there is an example of a stability group A containing $O_p(\mathrm{Aut}\ G)$ but A is not of K-type. I did find that for a homocyclic p-group G and $A \leq \mathrm{Aut}\ G$, A is of K-type if and only if (i) $A \geq O_p(\mathrm{Aut}\ G)$ and (ii) A is a closed stability group. The proof has been omitted due to its length.

CHAPTER III

THE FITTING SUBGROUP OF AUT G AND HOL G

As was mentioned after 2.30 if G is a p-group, that is not elementary abelian, then $\mathrm{Fitt}(\mathrm{Aut}\ G) \gneq 1$. This chapter is an attempt to classify $\mathrm{Fitt}(\mathrm{Hol}\ G)$ and $\mathrm{Fitt}(\mathrm{Aut}\ G)$, using stability groups of characteristic series.

<u>Definition 3.1</u>. The holomorph of G, written $\mathrm{Hol}\ G$, is the semidirect product of G and $\mathrm{Aut}\ G$ where $\alpha^{-1}g\alpha = g^{\alpha}$ is the image of g under the automorphism α.

<u>Definition 3.2</u>. $\mathcal{T}^{c}_{G}$ is the set of stability groups of characteristic series of G.

<u>Theorem 3.3</u>. If $A = \prod_{A_i \in \mathcal{T}^{c}_{G}} A_i$ then $\mathrm{Fitt}(\mathrm{Hol}\ G) = A \cdot \mathrm{Fitt}\ G$.

<u>Proof</u>: In [14], Schmid has found that A is a closed stability group for a characteristic series and that $A \lhd \lhd \mathrm{Hol}\ G$. $\mathrm{Fitt}\ G \lhd \mathrm{Hol}\ G$ since $\mathrm{Fitt}\ G$ is characteristic in G. Because both are nilpotent, we see that $A \cdot \mathrm{Fitt}\ G \leq \mathrm{Fitt}(\mathrm{Hol}\ G)$, the subgroup of $\mathrm{Hol}\ G$ generated by all nilpotent subnormal subgroups.

Take $\alpha x \in \mathrm{Fitt}(\mathrm{Hol}\ G)$, $\alpha \in \mathrm{Aut}\ G$, $x \in G$. The natural homomorphism from $\mathrm{Hol}\ G$ onto $\mathrm{Aut}\ G$ maps αx onto α. Since the image of $\mathrm{Fitt}(\mathrm{Hol}\ G)$ is a normal nilpotent subgroup of $\mathrm{Aut}\ G$, $\alpha \in \mathrm{Fitt}(\mathrm{Aut}\ G)$. $\mathrm{Fitt}(\mathrm{Hol}\ G)$ stabilizes the series

$$s\colon \mathrm{Fitt}\ G \geq [\mathrm{Fitt}\ G, \mathrm{Fitt}(\mathrm{Hol}\ G)] \geq \ldots \geq \gamma\ \mathrm{Fitt}\ G(\mathrm{Fitt}(\mathrm{Hol}\ G))^{i} \geq \ldots \geq 1$$

which is normal in Hol G and therefore characteristic in G. s ends in the identity because Fitt(Hol G) is nilpotent and $\gamma(\text{Fitt } G)\text{Fitt}(\text{Hol}(G))^n \leq \gamma \text{ Fitt}(\text{Hol } G)(\text{Fitt}(\text{Hol } G))^n = 1$ for some n. Since $[g,\alpha\cdot x] = g^{-1}g^{\alpha\cdot x} = g^{-1}x^{-1}g^{\alpha}x = [g,\alpha\cdot \pi_x]$, $\alpha\, \pi_x$ stabilizes s. $[G, \text{Fitt}(\text{Hol } G)] \leq \text{Fitt}(\text{Hol } G) \cap G \leq \text{Fitt } G$, so $\alpha\, \pi_x$ stabilizes the characteristic series

s_2: $G \geq \text{Fitt } G \geq \ldots \geq \gamma \text{ Fitt } G(\text{Fitt}(\text{Hol } G))^i \geq \ldots \geq 1$. Thus

$\alpha\, \pi_x \in A \leq \text{Fitt}(\text{Aut } G)$ and because $\alpha \in \text{Fitt}(\text{Aut } G)$, $\pi_x \in \text{Fitt}(\text{Aut } G) \cap \text{Inn } G \lhd \text{Inn } G$. Let H be the preimage of Inn G $\cap$ Fitt(Aut G). H/Z(G) is nilpotent so x is an element of H which is a normal nilpotent subgroup of G, i.e. $x \in$ Fitt G. Since Fitt G $\leq$ Fitt(Hol G) we have that $\alpha \in \text{Fitt}(\text{Hol } G) \cap \text{Aut } G$. Now α stabilizes the characteristic series s_2 so $\alpha \in A$ as required.

<u>Corollary 3.4</u>. If Fitt G is a π-group, Fitt(Hol G) is a π-group.

<u>Proof</u>: By 3.3 Fitt(Hol G) = A·Fitt G. Thus Fitt(Hol G) is the product of the π-group Fitt G and A which by 1.7 is also a π-group. Fitt(Hol G) must therefore be a π-group.

The rest of the chapter deals with the question "When is Fitt(Aut G) a stability group?" It will first be necessary to develop some technical lemmas.

<u>Definition 3.5</u>. If $H \leq G$ let π_h be the inner automorphism of G induced by h and $\Pi_H = \{\pi_h | h \in H\}$.

<u>Lemma 3.6</u>. Let $H \leq G$, $A \leq \text{Aut } G$. Then $[\Pi_H, A] = \Pi_{[H,A]}$.

<u>Proof</u>: The calculation $x^{\alpha^{-1}\pi_h\alpha} = (h^{-1}x^{\alpha^{-1}}h)^{\alpha} = (h^{-1})^{\alpha}xh^{\alpha} = x^{\pi(h^{\alpha})}$, shows that $\alpha^{-1}\pi_h\alpha = \pi_{h^{\alpha}}$.

$[\pi_h, \alpha] = \pi_{h^{-1}} \alpha^{-1}(\pi_h)\alpha = \pi_{h^{-1}} \pi_{h^\alpha} = \pi_{h^{-1}h^\alpha} = \pi_{[h,\alpha]}$ gives the required conclusion.

Lemma 3.7. If $A \le \mathrm{Aut}\, G$, then $\Pi_{\gamma GA^i} = [\Pi_{\gamma GA^{i-1}}, A]$.

Proof: By 3.6 $[\Pi_{\gamma GA^{i-1}}, A] = \Pi_{[\gamma GA^{i-1}, A]} = \Pi_{\gamma GA^i}$.

Lemma 3.8. If $A \le \mathrm{Fitt}(\mathrm{Aut}\, G)$ then A stabilizes a series $G \ge \ldots \ge G_n = Z$.

Proof: If $A \le \mathrm{Fitt}(\mathrm{Aut}\, G)$ then $[B, \overbrace{A, \ldots, A}^{n}] = 1$ for every subgroup $B \le \mathrm{Aut}\, G$ and sufficiently large n. Therefore $[\mathrm{Inn}\, G, \overbrace{A \ldots A}^{n}] = 1$ for some large n. Repeated use of 3.7 gives

$$1 = [\mathrm{Inn}\, G, \overbrace{A, \ldots A}^{n}] = [\Pi_G, \overbrace{A \ldots A}^{n}] = [\Pi_{[G,A]}, \overbrace{A \ldots A}^{n-1}] = \ldots = \Pi_{[G, \overbrace{A \ldots A}^{n}]} .$$

Thus $\gamma\, G\, A^n \le Z(G)$ and A stabilizes the series $G \ge [G, A]Z \ge \ldots \ge (\gamma\, G\, A^n)Z = Z$.

Lemma 3.9. If $A \le \mathrm{Fitt}(\mathrm{Aut}\, G)$ and $\pi(A) \cap \pi(\mathrm{Fitt}\, G) = \phi$ then A centralizes $G/Z(G)$.

Proof: Let $Z = Z(G)$. By 3.8, A stabilizes a series $G \ge G_1 \ge \ldots \ge Z$ so A induces a subgroup $\tilde{A}$ of $\mathrm{Stab}(G/Z \ge G_1/Z \ge \ldots \ge Z/Z = \bar{1})$. By 1.7 $\pi(\tilde{A}) = \pi([G/Z, \tilde{A}])$ and $\mathrm{Fitt}(G/Z) \ge [G/Z, \tilde{A}] = [G,A]Z/Z$. Thus $[G,A]Z \le \mathrm{Fitt}\, G$ and $\pi(\tilde{A}) = \pi([G,A]Z/Z) \subseteq \pi(\mathrm{Fitt}\, G)$. Since $\tilde{A}$ is an image of A, $\pi(\tilde{A})$ is a subset of $\pi(A)$ which is disjoint from $\pi(\mathrm{Fitt}\, G)$. $\tilde{A}$ must therefore be the identity and A must centralize G/Z.

Lemma 3.10. Let G be a nonabelian p-group, and Z be the center of G. If $B = \mathrm{Stab}(G \ge Z \ge 1)$ then $[G,B] \ge \Omega_1(Z)$, the elements of order p in the center of G.

Proof: Let M be a maximal subgroup containing the proper subgroup Z, the center of G. If $1 \ne z \in \Omega_1(Z)$, we see that there is a homomorphism f from G onto $\langle z\rangle$ with kernel M. By 1.26, $\alpha: x \to x\, x^f$ is an automorphism and we see that $\alpha \in B$ with $[G,\alpha] = \langle z\rangle$. Thus $z \in [G,B]$ and $\Omega_1(Z) \leq [G,B]$.

Theorem 3.11. If G is a nonabelian p-group, then $\mathrm{Fitt}(\mathrm{Aut}\ G) = O_p(\mathrm{Aut}\ G) \geq \mathrm{Inn}\ G \neq 1$.

Proof: If H is a group, $\mathrm{Fitt}(H) = \oplus \sum_{p\,||H|} O_p(H)$. Consequently $\mathrm{Fitt}(\mathrm{Aut}\ G) \geq O_p(\mathrm{Aut}\ G)$, which contains the nontrivial normal p-group Inn G. Let $B = \mathrm{Stab}(G \geq Z \geq 1)$ and A the direct complement of $O_p(\mathrm{Aut}\ G)$ in $\mathrm{Fitt}(\mathrm{Aut}\ G)$. Since A and B commute $[B,A,G] = [1,G] = 1$. By 3.9 $[A,G,B] \leq [Z,B] = 1$. The Three Subgroup Lemma gives $[G,B,A] = 1$. That is, the p'-group A centralizes $[G,B]$ which contains $\Omega_1(Z)$. By Gorenstein [9, page 178], A must centralize Z. Because A is a p'-group contained in the p-group $\mathrm{Stab}(G \geq Z \geq 1)$, $A = 1$ and $\mathrm{Fitt}(\mathrm{Aut}\ G) = O_p(\mathrm{Aut}\ G)$.

Theorem 3.12. If G is an abelian p-group, not elementary abelian of order 4 or 9, then $\mathrm{Fitt}(\mathrm{Aut}\ G) = O_p(\mathrm{Aut}\ G) \times B$ where B is a cyclic subgroup of $Z(\mathrm{Aut}\ G)$ and $|B| = p - 1$.

Proof: In [7], Hightower has shown that if G is an abelian group of exponent p^t not elementary abelian of order 4 or 9, then $|\mathrm{Fitt}(\mathrm{Aut}\ G)| = (p-1)p^n$ for some integer n. Since G is abelian, $x \to x^i$, $(i,p) = 1$, gives $(p-1)(p^{t-1})$ automorphisms in the center of Aut G. By restricting to $\langle g\rangle$ of order p^t we may find that power automorphism of order $p-1$.

In 1.7 we saw that $[G,A] \leq \mathrm{Fitt}\ G$ whenever $A \in \mathcal{T}_G$. Thus it is not surprising that a condition about Fitt G might enable us to say something about $\gamma\, G\, A^i$ and closed stability groups.

Theorem 3.13. If Fitt G is purely nonabelian then Fitt(Aut G) is a closed stability group.

Proof: Let $p \neq q$, be two primes. By 3.8, Fitt(Aut G) stabilizes $G \geq \ldots \geq Z$. Since $O_q(G)$ is characteristic in G and $Z(O_q(G)) \geq O_q(G) \cap Z(G)$, Fitt(Aut G) must stabilize $O_q(G) \geq \ldots \geq Z(O_q(G))$. Thus $O_p(\text{Aut } G)$ induces a p-group of automorphisms on $O_q(G)/Z(O_q(G))$ that stabilizes the series $O_q(G)/Z(O_q(G)) \geq \ldots \geq Z(O_q(G))/Z(O_q(G)) = \bar{1}$. Since this stability group must be a q-group $O_p(\text{Aut } G)$ centralizes $O_q(G)/Z(O_q(G))$. $O_q(G)$ must be purely nonabelian because Fitt G is purely nonabelian. By [2, Corollary 2], $C_{\text{Aut } O_q(G)}(O_q(G)/Z(O_q(G)))$ is a q-group. $O_p(\text{Aut } G)$ is a p-group which induces a group of automorphisms on $O_q(G)$ which centralize $O_q(G)/Z(O_q(G))$. Thus $O_p(\text{Aut } G)$ must centralize $O_q(G)$. Since $O_{p'}(Z(G))$ the p-complement of Z(G) is contained in $\prod_{q \neq p} O_q(G)$, $O_p(\text{Aut } G)$ centralizes $O_{p'}(Z(G))$. $O_p(\text{Aut } G)$ induces a p-group of automorphisms on the p-group $Z(G)/O_{p'}(Z(G))$, which by 2.1 must stabilize a series $Z(G)/O_{p'}(Z(G)) \geq \ldots \geq O_{p'}(Z(G))/O_{p'}(Z(G)) = \bar{1}$. $O_p(\text{Aut } G)$ will then stabilize the preimage of that chain and consequently the chain, $G \geq \ldots \geq Z \geq \ldots \geq O_{p'}(Z) \geq 1$. Let $A = O_p(\text{Aut } G) \lhd \text{Aut } G$. By 1.10.v, $\bar{A} = \text{Stab}(\gamma\, G\, A^i) \lhd \text{Aut } G$. 1.10.vi says that $\pi(\bar{A}) = \pi(A)$. $\bar{A}$ is a normal p-group containing $A = O_p(\text{Aut } G)$ so $\bar{A} = O_p(\text{Aut } G)$ is a closed stability group. In 1.20 we showed that the product of closed stability groups of relatively prime order is a closed stability group, so $\text{Fitt}(\text{Aut } G) = \prod O_p(\text{Aut } G)$ is a closed stability group.

Lemma 3.14. Let $B = \mathrm{Stab}(G \geq O_{p'}(Z) \geq 1)$ and Z be the center of G. If $\mathrm{Exp}(O_{p'}(Z))$ divides $\mathrm{Exp}\ G/O_{p'}(Z)G'$ then $[G,B] = O_{p'}(Z)$, and B is a closed stability group.

Proof: Since $G/O_{p'}(Z)G'$ is abelian, $\mathrm{Exp}\ G/O_{p'}(Z)G'$ is the product of the orders of the largest cyclic summand of each Sylow of $G/O_{p'}(G)G'$. $\mathrm{Exp}\ O_{p'}(Z) | \mathrm{Exp}\ G/O_{p'}(Z)G'$ means that each cyclic direct summand $\langle y \rangle$ of a q-Sylow, $q \neq p$, of $O_{p'}(Z)$ has order less than or equal to a cyclic direct summand $\langle \bar{x} \rangle$ of the q-Sylow of $G/O_{p'}(Z)G'$. Thus there is a homomorphism f from $\langle \bar{x} \rangle$ onto $\langle y \rangle$ that may also be thought of as a homomorphism $f: G/O_{p'}(Z)G' \overset{\text{into}}{\rightarrow} O_{p'}(Z)$. As in 1.26 we let $\alpha: x \rightarrow xx^f$ for all $x \in G$ and see that $\alpha \in \mathrm{Stab}(G \geq O_{p'}(Z) \geq 1)$ and $[G,\alpha] = \langle y \rangle$. Thus $[G,B]$ contains any cyclic direct summand of the q-Sylow of $O_{p'}(Z)$, and consequently $[G,B] \geq O_{p'}(Z)$. $[G,B] \leq O_{p'}(Z)$ because $B = \mathrm{Stab}(G \geq O_{p'}(Z) \geq 1)$. Thus $[G,B] = O_{p'}(Z)$ and B is closed.

Lemma 3.15 (B.I. Plotkin). Let Γ act on G, $\Gamma = \langle A,B \rangle$, B stabilize $G \geq H \geq 1$, A normalize B, A stabilize a chain $G \geq \ldots \geq H$, $H \leq Z(G)$, $[B,A,\ldots,A] = 1$. Then A stabilizes a chain $[G,B] \geq \ldots \geq 1$.

Proof: See Plotkin [12].

Theorem 3.16. Let Z be the center of G. If $\mathrm{Exp}\ O_{p'}(Z)$ divides $\mathrm{Exp}\ G/O_{p'}(Z)G'$ then $O_p(\mathrm{Aut}\ G)$ is a closed stability group.

Proof: Let $A = O_p(\mathrm{Aut}\ G)$. By 3.8, A stabilizes a series $G \geq G_1 \geq \ldots \geq Z$. $Z/O_{p'}(Z)$ is a p-group acted upon by the p-group A. A therefore stabilizes a chain $Z \geq \ldots \geq O_{p'}(Z)$. Now we check the conditions of 3.15 with $B = \mathrm{Stab}(G \geq O_{p'}(Z) \geq 1)$ and

$H = O_{p'}(Z)$. Note that $B \lhd \text{Aut } G$ since B is the stability group of a characteristic series. A stabilizes a series $G \geq \ldots \geq H = O_{p'}(Z)$, $H = O_{p'}(Z) \leq Z$. $[B,A,\ldots,A] = 1$ because $A \leq \text{Fitt(Aut } G)$. Finally, 3.15 says that A stabilizes a chain $[G,B] \geq \ldots \geq 1$. By 3.14 $[G,B] = O_{p'}(Z)$. Thus A stabilizes $G \geq \ldots \geq Z \geq \ldots \geq O_{p'}Z \geq \ldots \geq 1$. By 1.10.v, $\bar{A} \lhd \text{Aut } G$. $\{p\} = \pi(A) = \pi(\bar{A})$ by 1.7. Thus $\bar{A}$ is a normal p-group of Aut G, i.e. $\bar{A} \leq O_p(\text{Aut } G)$. $O_p(\text{Aut } G) = A \leq \bar{A}$ by 1.10.i, so $O_p(\text{Aut } G)$ is a closed stability group.

<u>Corollary 3.17</u>. Let Z be the center of G. If $\text{Exp } Z$ divides $\text{Exp } G/Z\cdot G'$ then $\text{Fitt(Aut } G)$ is a closed stability group.

<u>Proof</u>: For an arbitrary prime p and $O_{p'}(Z)$, the p complement of Z we have: $\text{Exp } O_{p'}(Z)$ divides $\text{Exp } Z$ which divides $\text{Exp } G/Z\cdot G'$ which divides $\text{Exp}(G/O_{p'}(Z)G')$. By 3.16, $O_p(\text{Aut } G)$ is a closed stability group. Since p was arbitrary and the product of closed stability groups of relatively prime order is again a closed stability group, $\text{Fitt(Aut } G) = \prod_p O_p(\text{Aut } G)$ is a closed stability group.

CHAPTER IV

QUATERNIONS AS CLOSED STABILITY GROUPS

In my search for examples of stability groups, I had trouble finding a 2-group G and series s so that Stab(s) was isomorphic to Q, the quaternion group of order eight. I even conjectured that Q could not be a stability group. Since an arbitrary series s is difficult to work with, I first worked on the conjecture for a normal series (each $G_i \lhd G$). Toward the end of a long and rather messy proof, the following two examples arose. The second is an example where Q is a stability group but not a closed stability. With the addition of the condition of closed stability group to my conjecture I was able to prove 4.1.

Let G and s be defined by

$$G = \langle a, b \mid a^8 = b^2 = 1,\ a^b = a^5 \rangle$$

and

$$s: G \geq \langle a^2, b \rangle \geq \langle a^2 \cdot b \rangle \geq 1 \ .$$

We obtain 2 automorphisms $\psi, \phi \in$ Stab(s), namely

$$\begin{array}{llcll} \psi: & a \rightarrow a^3 & \qquad & \phi: & a \rightarrow a^7 \\ & b \rightarrow ba^4 & & & b \rightarrow ba^4 \ . \end{array}$$

Each one has order two. Stab(s) is not Q, because Q has only one element of order 2.

If $G = \langle a,b \mid a^{16} = b^2 = 1, a^b = a^9 \rangle$ and $s: G \geq \langle a^2,b \rangle \geq \langle a^2 \cdot b \rangle \geq 1$ then

(i) $Q \simeq \mathrm{Stab}(s)$

(ii) $\bar{s}: G \geq \langle a^4,b \rangle \geq \langle a^8 \rangle \geq 1$

(iii) $\bar{Q} \gneq Q$.

Theorem 4.1. If G is a 2-group, the quaternion group Q is not a closed stability group for a normal series of G.

Proof: Assume $Q = \mathrm{Stab}(G = G_0 \geq G_1 \geq \ldots \geq G_n = 1)$ where $G_i = \gamma\, G\, Q^i$ and each $G_i \lhd G$. Let $A = \mathrm{Stab}(G \geq G_1 \geq 1)$ and $B = \mathrm{Stab}(G \geq G_{n-1} \geq 1)$.

The proof is carried out in a series of steps and cases.

(i) $A \cap B = Z(Q)$.

Since $[G,B] \leq G_{n-1}$, $[G, A \cap B, Q] \leq [G_{n-1}, Q]$. Q fixes G_{n-1} pointwise so $[G, A \cap B, Q] = 1$. Since $[Q, G] \leq G_1$ and G_1 is fixed pointwise by A, we have $[Q, G, A \cap B] = 1$. The Three Subgroup Lemma gives us, $[A \cap B, Q, G] = 1$. Since $[A \cap B, Q]$ acts trivially on G, $[A \cap B, Q] = 1$ and $A \cap B \leq Z(Q)$. $Z(Q)$ is cyclic of order 2, so all that remains of (i) is to show that $A \cap B \gneq 1$. Let M be a maximal and therefore normal subgroup containing G_1. Since G_{n-1} is normal, we have $G_{n-1} \cap Z(G) \gneq 1$. Thus, there is a nontrivial $f \in \mathrm{Hom}(G/M, G_{n-1} \cap Z(G))$. By 1.26, $\alpha_f: x \to xx^f$ is an automorphism. α_f fixes M pointwise and $[G, \alpha_f]$ is the image of f which is contained in $G_{n-1} \cap Z(G)$. Thus $1 \neq \alpha_f \in A \cap B$.

(ii) $G/G_{n-1}G'$ and $G_1 \cap Z(G)$ are nontrivial cyclic groups.

By 1.12, A and B are abelian subgroups of Q which must, therefore, be cyclic. Theorem 1.26 says that $\mathrm{Hom}(G/G_1G', G_1 \cap Z(G))$ and $\mathrm{Hom}(G/G_{n-1}G', G_{n-1} \cap Z(G))$ are isomorphically contained in A

and B respectively. Since A and B are both cyclic, $\mathrm{Hom}(G/G_1G', G_1 \cap Z(G))$ and $\mathrm{Hom}(G/G_{n-1}G', G_{n-1} \cap Z(G))$ are both cyclic. By 1.27, $G/G_{n-1}G'$ and $G_1 \cap Z(G)$ are cyclic.

(iii) Either G/G_1G' or $G_{n-1} \cap Z(G)$ is cyclic of order two.

In (i) we saw that

$$1 \lneq \mathrm{Hom}(G/M, G_{n-1} \cap Z(G)) \leq \mathrm{Hom}(G/G_1G', G_{n-1} \cap Z(G)) .$$

By 1.26, $\mathrm{Hom}(G/G_1G', G_{n-1} \cap Z(G))$ is isomorphically contained in $A \cap B$. Since $A \cap B$ is cyclic of order two, 1.27 says that either G/G_1G' or $G_{n-1} \cap Z(G)$ is cyclic of order two.

(iv) G is nonabelian.

Assume G is abelian. G is not cyclic since the automorphism group of a cyclic group is abelian and thus could not contain Q. By (iii), G/G_1 or G_{n-1} is cyclic of order two. By (ii), G_1 and G/G_{n-1} are cyclic. If G/G_1 is cyclic of order 2, then G_1 is cyclic of maximal possible order and we see that $G = \langle y\rangle \times \langle x\rangle$, where $\langle x\rangle = G_1$. By 1.24,

$$\begin{aligned} Q &= \mathrm{Stab}(G \geq \langle x\rangle \geq \ldots \geq 1) \\ &\simeq \mathrm{Stab}(\langle x\rangle \times \langle y\rangle \geq \langle x\rangle \geq 1) \rtimes \mathrm{Stab}(G_1 \geq G_2 \geq \ldots \geq G_n = 1) . \end{aligned}$$

Since Q is nonabelian, $n \geq 3$ by 1.12. By 2.27,

$$\mathrm{Stab}(\langle x\rangle \times \langle y\rangle \geq \langle x\rangle \geq 1) \gneq 1$$

and

$$\mathrm{Stab}(G_1 \geq G_2 \geq \ldots \geq G_n = 1) \geq \mathrm{Stab}(G_1 \geq G_2 \geq 1) \gneq 1.$$

This is a contradiction since Q is not a semidirect product of nontrivial subgroups. If G_{n-1} is cyclic of order 2 we see that there is an $x \in G \backslash G_{n-1}$ that has maximum possible order because G/G_{n-1} is cyclic. $\langle x \rangle$ is therefore a direct summand and $G = \langle x \rangle \times \langle y \rangle$ where $\langle y \rangle = G_{n-1}$. Again apply 1.24 to obtain

$$Q = \mathrm{Stab}(G \geq \ldots \geq 1)$$
$$\simeq \mathrm{Stab}(\langle x \rangle \times \langle y \rangle \geq \langle y \rangle \geq 1)\,]\ \mathrm{Stab}(G/G_{n-1} \geq G_1/G_{n-1} \geq \ldots \geq G_{n-1}/G_{n-1} = \bar{1})$$

the desired contradiction. Thus we may assume that G is nonabelian.

(v) G/G_{n-1} is cyclic and either G/G_1 or $G_{n-1} \cap Z(G)$ is cyclic of order 2.

By (ii) $G/G_{n-1}G'$ is cyclic and since Φ, the Frattini subgroup, contains G', $G/G_{n-1}\Phi$ is cyclic. There exists $x \in G \backslash G_{n-1}\Phi$ such that $G = \langle x, G_{n-1}, \Phi \rangle$. Since Φ is the set of nongenerators, $G = \langle x, G_{n-1} \rangle$ and G/G_{n-1} is cyclic. Since G/G_{n-1} is cyclic, $G' \leq G_{n-1} \leq G_1$. The remaining conclusion follows from (iii).

(vi) $C_G(G_{n-1}) = [G, Q]Z(G)$ is an abelian group, and $|[G, Q]| \geq 8$.

Since $G' \leq G_{n-1}$, $C_G(G_{n-1})/Z(G)$ is isomorphic to a group of inner automorphisms in the cyclic group $\mathrm{Stab}(G \geq G_{n-1} \geq 1)$. $C_G(G_{n-1})$ mod its center is cyclic so $C_G(G_{n-1})$ is abelian. Since $G_{n-1} \lhd G$ and Q fixes G_{n-1} pointwise, 1.13 says that $[G, Q] \leq C_G(G_{n-1})$. Because $G/G_{n-1} = \langle \bar{x} \rangle$ for some $x \in G$ and G_{n-1} is fixed pointwise, every $\alpha \in Q$ is completely determined by $x^{\alpha} = x[x,\alpha]$. Thus there are at least eight different $[x,\alpha]$'s.

$$[G, C_G(G_{n-1}), Q] \leq [G', Q] = 1$$

and

$$[Q, G, C_G(G_{n-1})] = [[G, Q], C_G(G_{n-1})] \leq [C_G(G_{n-1}), C_G(G_{n-1})] = 1,$$

so the Three Subgroup Lemma says that $[C_G(G_{n-1}), Q, G] = 1$. By 3.6 $\Pi_{[C_G(G_{n-1}),Q]} = [\Pi_{C_G(G_{n-1})}, Q]$, so $[\Pi_{C_G(G_{n-1})}, Q] = 1$. This is equivalent to:

$C_G(G_{n-1})/Z(G)$ is isomorphically contained in the center of Q.

$[G,Q] \not\leq Z(G)$ or else we would have G/Z cyclic and G would be abelian. Thus

$$1 \lneq \frac{[G,Q]Z(G)}{Z(G)} \leq \frac{C_G(G_{n-1})}{Z(G)} \underset{\sim}{\subset} Z(Q) .$$

Since $Z(Q)$ is cyclic of order 2, $[G,Q]Z(G) = C_G(G_{n-1})$ and $Z(Q)$ is the set of inner automorphisms induced by $C_G(G_{n-1})$.

Case I. Suppose that one of the cyclic factors G_i/G_{i+1}, $i = 0,1,2,\ldots,n-2$, has order greater than two. Thus $|G/G_{n-1}| = 2^t > 4$.

Since $C_Q(G/G_{n-1}) = B = \mathrm{Stab}(G \geq G_{n-1} \geq 1)$ is abelian, we have that

$$1 \lneq Q/B \underset{\sim}{\subset} \mathrm{Stab}(G/G_{n-1} \geq G_1/G_{n-1} \geq \ldots \geq G_{n-1}/G_{n-1}) .$$

Since one of the factors is larger than 2 and G/G_{n-1} is cyclic, 2.17 says

$$1 \lneq Q/B \underset{\sim}{\subset} \langle 5I \rangle .$$

Any proper factor group of Q is elementary abelian, so

$$1 \lneq Q/B \underset{\sim}{\subset} \Omega_1(5I) = \langle (5I)^{2^{t-3}} \rangle$$

and B must be cyclic of order four. Let $B = \langle\beta\rangle$. If $\bar{Q}$ indicates the induced automorphisms, $[G/G_{n-1}, \bar{Q}] = [G,Q]/G_{n-1}$ is cyclic of order two because $(5I)^{2^{t-3}}$ centralizes G/G_{n-1} mod its cyclic group of order two. This coupled with the fact that the series is closed says that the series is $G \geq [G, Q] \geq [G, Q, Q] \geq 1$ with $|G/[G, Q]| \geq 4$ and $|[G, Q]/[G, Q, Q]| = 2$. From part vi and an isomorphism theorem we have

$$\frac{[G,Q]}{[G,Q] \cap Z(G)} \simeq \frac{[G,Q]Z(G)}{Z(G)} = \frac{C_G(G_{n-1})}{Z(G)}$$

is cyclic of order two. Since $|[G,Q]| \geq 8$, $|[G,Q] \cap Z(G)| \geq 4$. $\dfrac{[G,Q] \cap Z(G)}{[G,Q,Q] \cap Z(G)}$ is isomorphically contained in $[G,Q]/G_{n-1}$ which is cyclic of order two. $|G/G_1| \geq 4$ so v implies $|G_{n-1} \cap Z(G)| = 2$. This in turn forces $|[G,Q] \cap Z(G)| = 4$. By (ii) $G_1 \cap Z(G)$ is cyclic with some generator z. Since $A = \mathrm{Stab}(G \geq G_1 \geq 1)$ is an abelian subgroup of Q, $|A| \leq 4$. Let x be an element of G that maps onto a generator of G/G_{n-1}. By 1.26, $\alpha\colon x \to xz$

$g \to g$ for all $g \in G_1$, is a member of $\mathrm{Stab}(G \geq G_1 \geq 1)$ and since its order is four, $\langle\alpha\rangle = A$. Since $|A \cap B| = 2$, we see that $Q = \langle\alpha,\beta\rangle$. Since $\langle\beta\rangle = \mathrm{Stab}(G \geq [G,Q,Q] \geq 1)$ and $[G,Q] \not\leq Z(G)$,

$$x^{\beta^i} = x(x_\beta)^i \quad \text{for some } x_\beta \in [G,Q,Q] \backslash Z(G) .$$

Since $|\beta| = 4$,

$$[G, \langle\beta\rangle] = \langle x_\beta\rangle \text{ is a normal subgroup of order 4.}$$

$<z, x_\beta>$ is a normal subgroup. Since α and β centralize $G/<z, x_\beta>$, $<z, x_\beta> \geq [G,Q]$. The other inclusion is trivial so $<z, x_\beta> = [G,Q]$. We now have $G = <x, [G,Q]> = <x, z, x_\beta>$. $x^{2^{t-1}}$ is a member of the abelian group $[G,Q]$ because $|G/[G,Q]| = 2^{t-1}$. $x^{2^{t-1}} \notin [G,Q,Q]$ since $|G/[G,Q,Q]| = 2^t$. Since $\alpha^2 = \beta^2$, we see that

$$z^2 = x_\beta^2 .$$

Since $[G,Q] \cap Z(G) = <z>$ and $z^2 = x_\beta^2 \in [G,Q,Q]$,

$$x^{2^{t-1}} = z \text{ or } z^3 .$$

Because z was chosen as an arbitrary generator of $G_1 \cap Z(G)$, we may assume

$$x^{2^{t-1}} = z .$$

The inner automorphism π_{x_β} induced by x_β must have order two because $x_\beta \notin Z(G)$ but $x_\beta^2 \in Z(G)$. Since $[G,Q,Q]$ was assumed to be normal and $[G,Q,Q]$ is abelian,

$$\pi_{x_\beta} \in <\beta> = \mathrm{Stab}(G \geq G_{n-1} \geq 1) .$$

Because π_{x_β} has order two, $\pi_{x_\beta} = \beta^2$.

$$\begin{aligned} x_\beta^3 \, x \, x_\beta &= x^{x_\beta} \\ &= x^{\beta^2} \\ &= xz^2 \\ &= z^2 x \\ &= x_\beta^2 x . \end{aligned}$$

Cancelling x_β's we see, $x_\beta x x_\beta = x$ and

$$(xx_\beta)^2 = x^2 .$$

The following calculation shows that $\alpha\beta = \beta\alpha$, a contradiction to the fact that Q is nonabelian.

$$\begin{aligned}
x^{\alpha\beta} &= (xz)^\beta \\
&= (xx^{2^{t-1}})^\beta \\
&= xx_\beta(xx_\beta)^{2^{t-1}} \\
&= xx_\beta((xx_\beta)^2)^{2^{t-2}} \\
&= xx_\beta(x^2)^{2^{t-2}} \\
&= xx_\beta x^{2^{t-1}} \\
&= xx_\beta z
\end{aligned}$$

$$\begin{aligned}
x^{\beta\alpha} &= (xx_\beta)^\alpha \\
&= xzx_\beta \\
&= xx_\beta z .
\end{aligned}$$

Case II. Suppose that each of the factors G_i/G_{i+1}, $i = 0,1,2,\ldots,n-2$, has order two. G is nonabelian and by vi $C_G(G_{n-1}) = [G,Q]Z(G)$ is abelian. $|G/[G,Q]| = 2$ and $G \gneqq [G,Q]Z(G) \geq [G,Q]$ so

$$[G,Q] = [G,Q]Z(G) = C_G(G_{n-1})$$

and

$$[G,Q] \geq Z(G) .$$

By vi, $[C_G(G_{n-1}) : Z(G)] = 2$, so now

$$[[G,Q] : Z(G)] = 2 \ .$$

$[G,Q,Q] \gneq 1$ or else the series would have length two and Q would have to be abelian. $[G,Q,Q] \cap Z(G) \neq [G,Q,Q]$ or else we would have $G/Z(G)$ cyclic and G abelian. Thus

$$1 \lneq \frac{[G,Q,Q]}{[G,Q,Q] \cap Z(G)} \simeq \frac{[G,Q,Q]Z(G)}{Z(G)} \underset{\sim}{\subset} \frac{[G,Q]}{Z(G)} \ .$$

Since $[G,Q]/Z(G)$ has order two, $\frac{[G,Q,Q]}{[G,Q,Q] \cap Z(G)}$ has order two and $[G,Q,Q]Z(G) = [G,Q]$. By (ii), $Z(G) = [G,Q] \cap Z(G)$ is cyclic and

$$[G : Z(G)] = [G : [G,Q]][[G,Q] : Z(G)] = 4 \ .$$

Case II-a. Suppose G has no cyclic subgroup of index two. We know that there is a cyclic normal subgroup of index four, namely $\langle a\rangle = Z(G)$. Burnside, [4, page 138], says

$$G = \langle a,b,c \mid a \in Z(G),\ a^{2^{m-2}} = 1 = b^2 = c^2,\ b^c = ba^{2^{m-3}} \rangle \ .$$

One may calculate that μ and ν, defined by

$$\begin{aligned} \mu: a &\to a & \qquad \text{and} \qquad \nu: a &\to a \\ b &\to c & b &\to b \\ c &\to b & c &\to a^{2^{m-4}}bc \ , \end{aligned}$$

are automorphisms and

$$\langle a,b\rangle \xrightarrow{\mu} \langle a,c\rangle \xrightarrow{\nu} \langle a,bc\rangle \ .$$

From the Appendix we see that $m \geq 4$ because the quaternions are not a stability group for either the quaternion or dihedral group

of order eight. Since $G/Z(G)$ is elementary abelian and $[[G,Q] : Z(G)] = 2$, the only possibilities for $[G,Q]$ are $\langle a,b\rangle$, $\langle a,c\rangle$, and $\langle a,bc\rangle$. We may assume that $[G,Q] = \langle a,b\rangle$ or else $[G,Q^{\gamma}] = [G,Q]^{\gamma} = \langle a,b\rangle$ for some conjugate Q^{γ} of Q in Aut G.

$$\left|\frac{Z(G)}{Z(G) \cap [G,Q,Q]}\right| = \frac{\left|\frac{G}{Z(G) \cap [G,Q,Q]}\right|}{\left|\frac{G}{Z(G)}\right|}$$

$$= \frac{[G : [G,Q,Q]][[G,Q,Q] : [G,Q,Q] \cap Z(G)]}{4}$$

$$= \frac{4 \cdot 2}{4} = 2 \; .$$

$Z(G)$ is cyclic and therefore $Z(G) \cap [G,Q,Q] = \langle a^2\rangle$ is the only subgroup of index two. $[G,Q]/\langle a^2\rangle \simeq (\langle a\rangle \times \langle b\rangle)/\langle a^2\rangle$ is elementary abelian of order four. Since $\langle a,b\rangle \geq [G,Q,Q] \geq \langle a^2\rangle$ and has $[[G,Q,Q] : \langle a^2\rangle] = 2$, $[G,Q,Q]$ must be one of the three subgroups $\langle a\rangle$, $\langle a^2,b\rangle$ or $\langle ab\rangle$. $[G,Q,Q] = \langle a\rangle = Z(G)$ contradicts G non-abelian and $G/[G,Q,Q]$ cyclic. Neither $G/\langle ab\rangle$ nor $G/\langle a^2,b\rangle$ is cyclic contradicting $G/[G,Q,Q]$ is cyclic.

Case II-b. Suppose G has a cyclic subgroup of index two. $Z(G)$ has index four or else G is abelian and the Appendix again shows that $|G| \geq 16$. Thus $|Z(G)| \geq 4$. By Huppert [8, page 91], the only groups of order 2^{n+1} with a cyclic subgroup of index 2 are:

Dihedral, $\langle a,b \mid a^b = a^{-1},\ a^{2^n} = b^2 = 1\rangle$;

Generalized Quaternion, $\langle a,b \mid a^b = a^{-1},\ a^{2^n} = 1,\ b^2 = a^{2^{n-1}}\rangle$;

Quasidihedral, $\langle a,b \mid a^b = a^{-1+2^{n-1}},\ a^{2^n} = b^2 = 1\rangle$;

and $\langle a,b \mid a^b = a^{1+2^{n-1}},\ a^{2^n} = b^2 = 1\rangle$.

One may calculate that in all but the last case, $|Z(G)| = 2$.

Now $|G| = 2^{n+1} \geq 16$, so $n \geq 3$ and $n-1 \geq 2$. We see that $<a^2> \leq Z(G)$ and $Z(G)$ cannot be larger or G would be abelian. Thus $Z(G) = <a^2>$.

By Case II, we have the following diagram,

$$\begin{tikzcd}
 & G \arrow[d, dash] & \\
 & {[G,Q] = [G,Q,Q]Z(G)} \arrow[dl, dash, "2"'] \arrow[dr, dash, "2"] & \\
<a^2> = Z(G) \arrow[dr, dash] & & {[G,Q,Q]} \arrow[dl, dash, "2"] \\
 & {[G,Q,Q] \cap Z(G)} &
\end{tikzcd}$$

where the numbers indicate the indices. We notice that $[Z(G) : [G,Q,Q] \cap Z(G)] = 2$. Since $Z(G) = <a^2>$ has only one subgroup of index two

$$[G,Q,Q] \cap Z(G) = <a^4> .$$

Since $G/Z(G) = G/<a^2>$ is elementary abelian and $[G,Q] \geq <a^2>$, $[G,Q]$ is one of $<a^2,b>$, $<a>$, or $<ab>$. $[G,Q]$ is not cyclic or else $[G,Q,Q]$ and $Z(G)$ would both be of index two and therefore equal. Thus $[G,Q] = <a,b>$. Using the same elementary abelian argument again, we see that since $<a^2,b> \geq [G,Q,Q] \geq <a^4>$, $[G,Q,Q]$ must be one of $<a^2>$, $<b,a^4>$, $<a^2b>$. We rule out $<a^2>$ since $[G,Q,Q] \not\leq Z(G)$. Examination of $\mathrm{Stab}(G \geq <a^2,b> \geq <b,a^4> \geq 1)$ gives two automorphisms of order two:

$$\pi_b : \begin{array}{l} a \to a^{1+2^{n-1}} \\ b \to b \end{array} \qquad \text{and} \qquad \psi : \begin{array}{l} a \to ab \\ b \to b . \end{array}$$

Thus $\mathrm{Stab}(G \geq \langle a^2,b\rangle \geq \langle b,a^4\rangle \geq 1) \not\leq Q$ as would be the case if $\langle b,a^4\rangle = [G,Q,Q]$. The only possibility left is that

$$[G,Q,Q] = \langle a^2b\rangle .$$

If $[G,Q,Q,Q] \neq 1$, then $\langle a^4\rangle$ and $[G,Q,Q,Q]$ are two subgroups of index two in the cyclic group $\langle a^2b\rangle$. This gives $[G,Q,Q,Q] = \langle a^4\rangle$. Since $G/[G,Q,Q,Q]$ is cyclic and $\langle a^4\rangle \leq Z(G)$, G would have to be abelian. Since G is nonabelian
$[G,Q,Q,Q] = 1$.

Thus $\mathrm{Stab}(G \geq \langle a^2,b\rangle \geq \langle a^2b\rangle \geq 1) = Q$ and Q is closed. The two examples in the beginning of the chapter show that $n \neq 3$ or 4. If $\psi \in Q$ then

$$\begin{array}{lll} \psi: & a \to aa^{2i}b^{j} & \lambda, i = 0,1,2,\ldots,2^{n-1} - 1. \\ & b \to b(a^2b)^{\lambda} = b^{1+\lambda}a^{2\lambda} & j = 0,1. \end{array}$$

Calculation of the effect of ψ on the relations of G shows that ψ must have the form

$$\begin{array}{lll} \psi: & a \to aa^{2i}b^{j} & \lambda_1, j = 0,1 \\ & b \to b(a^{2^{n-1}})^{\lambda_1} & i = 0,1,2,\ldots,2^{n-1} - 1. \end{array}$$

$(a^2b)^{\psi} = a^2b$ limits us to eight possibilities.

$$\begin{array}{lll} 1_G, & a \to aa^{2^{n-1}} & a \to aa^{2^{n-2}} \\ & b \to b & b \to ba^{2^{n-1}} \\ a \to aa^{-2^{n-2}} & a \to aa^{2^{n-2}}b & a \to aa^{-2^{n-2}}b \\ b \to ba^{2^{n-1}} & b \to b & b \to b \\ a \to aa^{2^{n-1}}b & a \to ab & \\ b \to ba^{2^{n-1}} & b \to ba^{2^{n-1}} . & \end{array}$$

We see that each of these eight possibilities centralizes $G/\langle a^{2^{n-2}},b\rangle$ which has order

$$\frac{|\langle a\rangle|}{|\langle a^{2^{n-2}}\rangle|} = \frac{2^n}{2^2} = 2^{n-2} \quad .$$

Since $n \geq 5$ we see that $[G,Q]$ is contained in $\langle a^{2^{n-2}},b\rangle$ a subgroup of index $2^{n-2} > 2$.

Thus Q is not a closed stability group for a normal series.

Corollary 4.2. Let G be a 2-group and Q the quaternion group of order 8. Then

(i) $Q \not\simeq S_2(\text{Aut } G)$, the 2-Sylow of Aut G.

(ii) $Q \not\simeq C_{\text{Aut } G}(G/\Phi)$.

(iii) $Q \not\simeq O_2(\text{Aut } G)$ the 2-Sylow of Fitt(Aut G).

(iv) Q is not a K-type stability group.

Proof: By 2.1, 1.15, 2.8, and 2.5 respectively, the groups $S_2(\text{Aut } G)$, $C_{\text{Aut } G}(G/\Phi)$, $O_2(\text{Aut } G)$ and any K-type stability group are closed stability groups. Since Inn G normalizes each one, 1.10.vii gives that each of the four automorphism groups is a closed stability group for a normal series and therefore not isomorphic to Q.

Corollary 4.3. There is no group G with $\text{Aut } G \simeq Q$, the quaternion group of order eight.

Proof: Assume to the contrary that $\text{Aut } G \simeq Q$ for some group G. $G/Z \simeq \text{Inn } G \leq \text{Aut } G$, so G is nilpotent. Let S_p be a p-Sylow for an odd prime p. If p^2 divides $|G|$, we see in [1] that p divides $|\text{Aut } G|$. Thus $|S_p| = p$ or 1. If $|S_p| = p$, $\text{Aut}(S_p) \simeq \sigma(p-1)$, a cyclic group of order $p-1$, and is a direct summand of Aut $G \simeq Q$. Since Q has no cyclic direct summands, $S_p = 1$. We are now left with the 2-group G and $Q \simeq \text{Aut } G = S_2(\text{Aut } G)$ a contradiction to 4.2.i.

APPENDIX

APPENDIX

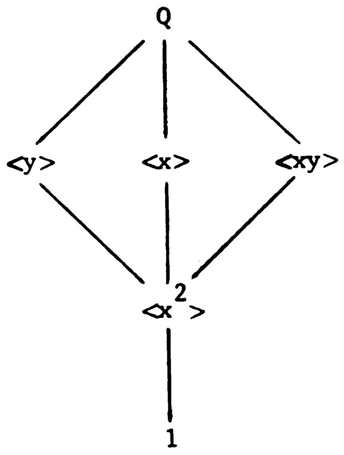
In the following examples $\boxed{A}$ will indicate that A is a closed stability group. $\textcircled{A}$ means A is a stability group but not closed. A series of arrows such as

$$\begin{tikzcd}
\boxed{C} & \\
& \textcircled{B} \arrow[ul] \\
A \arrow[ur] &
\end{tikzcd}$$

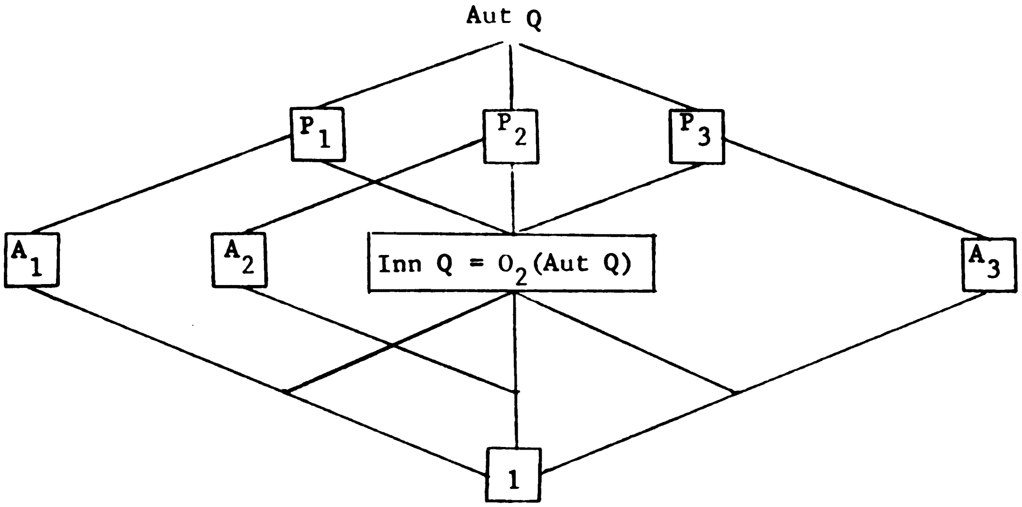
indicates that C is the closure of B and A.

Example 1

The quaternion group of order 8 is given by $Q = \langle x,y \mid x^2 = y^2, x^4 = 1, x^y = x^{-1} \rangle$, and has the following subgroup lattice.



Aut Q is the set of mappings taking x to one of the six elements of order four $(x, x^3, y\ \ y^3, xy, x^3y)$ and y to one of the four remaining elements not in the subgroup generated by the image of x. Aut Q $\simeq$ Sym (4).



Stability Groups

$P_1 = \mathrm{Stab}(Q \geq \langle x\rangle \geq \langle x^2\rangle \geq 1)$

$P_2 = \mathrm{Stab}(Q \geq \langle y\rangle \geq \langle x^2\rangle \geq 1)$

$P_3 = \mathrm{Stab}(Q \geq \langle xy\rangle \geq \langle x^2\rangle \geq 1)$

$\mathrm{Inn}\ Q = \mathrm{Stab}(Q \geq \langle x^2\rangle \geq 1)$

$A_1 = \mathrm{Stab}(Q \geq \langle x\rangle \geq 1)$

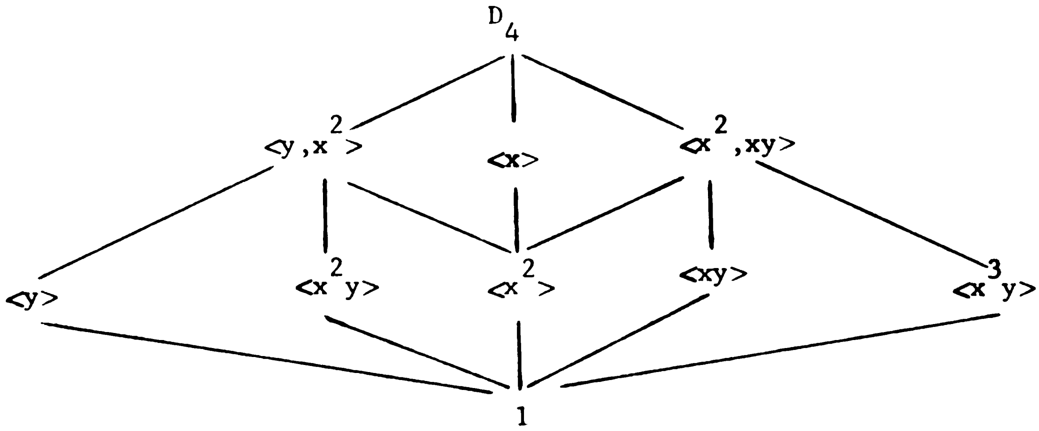
$A_2 = \mathrm{Stab}(Q \geq \langle y\rangle \geq 1)$

$A_3 = \mathrm{Stab}(Q \geq \langle xy\rangle \geq 1)$

P_i is dihedral of order eight and A_i is cyclic of order four for $i = 1,2,3$. Each stability group is closed with $\bar{s}$ being the series listed.

Example 2

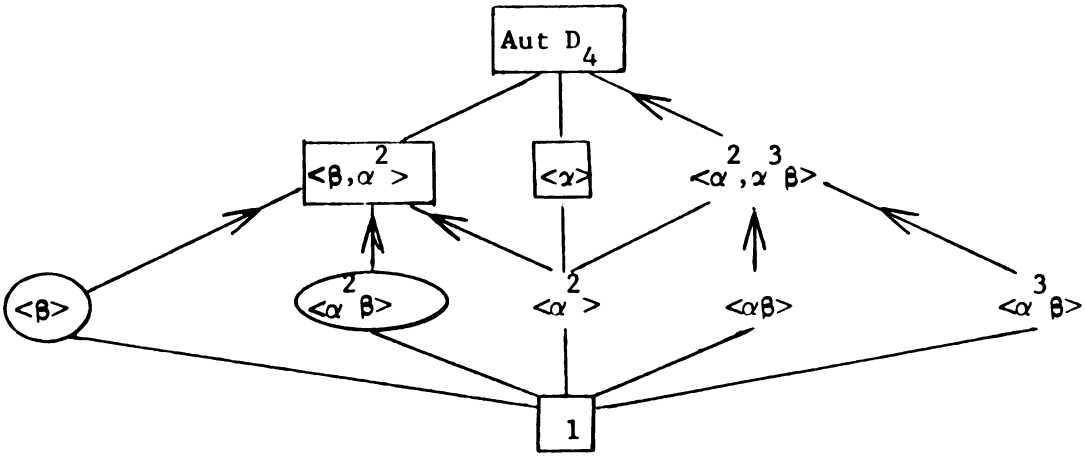
The dihedral group of order eight is given by $D_4 = \langle x,y \mid x^4 = y^2 = 1,\ yxy = x^3\rangle$ and has the following subgroup lattice.



By finding all relation preserving onto mappings we see that $\mathrm{Aut}\ D_4 = \langle \alpha,\beta \mid \alpha^4 = \beta^2 = 1,\ \alpha^\beta = \alpha^{-1}\rangle \simeq D_4$ where

$\alpha: x \to x$ $\qquad\qquad$ $\beta: x \to x^3$

$\quad\ \ y \to yx$ $\qquad\qquad\quad$ $y \to y$



Stability Groups

$\text{Aut } D_4 = \text{Stab}(D_4 \geq \langle x \rangle \geq \langle x^2 \rangle \geq 1)$

$\langle \beta, \alpha^2 \rangle = \text{Inn } G = \text{Stab}(D_4 \geq \langle x^2 \rangle \geq 1)$

$= \text{Stab}(D_4 \geq \langle x^2, y \rangle \geq \langle x^2 \rangle \geq 1)$

$= \text{Stab}(D_4 \geq \langle x^2, xy \rangle \geq \langle x^2 \rangle \geq 1)$

$\langle \beta \rangle = \text{Stab}(D_4 \geq \langle x^2, y \rangle \geq 1)$

$= \text{Stab}(D_4 \geq \langle x^2, y \rangle \geq \langle y \rangle \geq 1)$

$= \text{Stab}(D_4 \geq \langle x^2, y \rangle \geq \langle x^2 y \rangle \geq 1)$

$\langle \alpha^2 \beta \rangle = \text{Stab}(D_4 \geq \langle xy, x^2 \rangle \geq 1)$

$= \text{Stab}(D_4 \geq \langle xy, x^2 \rangle \geq \langle xy \rangle \geq 1)$

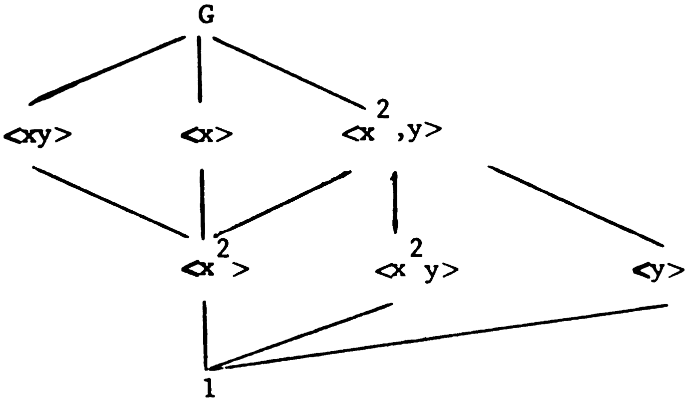
$= \text{Stab}(D_4 \geq \langle xy, x^2 \rangle \geq \langle x^3 y \rangle \geq 1)$

$\langle \alpha \rangle = \text{Stab}(D_4 \geq \langle x \rangle \geq 1)$

Closed Stability Groups	$\bar{s}$
$\text{Aut } D_4$	$D_4 \geq \langle x \rangle \geq \langle x^2 \rangle \geq 1$
$\langle \alpha^2, \beta \rangle$	$D_4 \geq \langle x^2 \rangle \geq 1$
$\langle \alpha \rangle$	$D_4 \geq \langle x \rangle \geq 1$

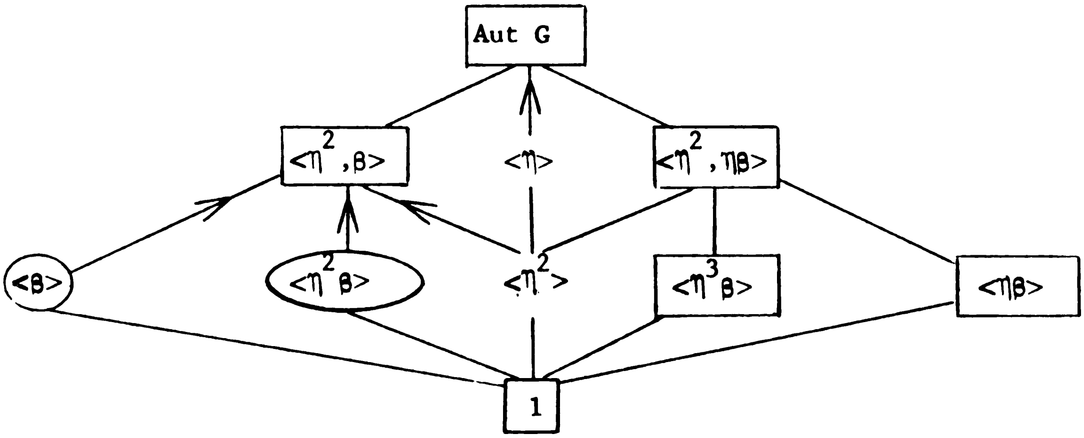
Example 3

Let $G = \langle x \rangle \times \langle y \rangle$ where $|x| = 4$ and $|y| = 2$.



An automorphism of G takes x to any one of the four elements (x, x^3, xy, x^3y) of order four and y to y or x^2y. Aut G is isomorphic to D_4 and is generated by

$$\eta: x \rightarrow xy \qquad\qquad \beta: x \rightarrow x^3$$
$$\quad y \rightarrow x^2y \qquad\qquad\quad y \rightarrow x^2y$$



Stability Groups

Aut G = Stab$(G \geq \langle x^2, y\rangle \geq \langle x^2\rangle \geq 1)$

$\langle \beta, \eta^2\rangle$ = Stab$(G \geq \langle x\rangle \geq \langle x^2\rangle \geq 1)$

= Stab$(G \geq \langle x^2\rangle \geq 1)$

= Stab$(G \geq \langle xy\rangle \geq \langle x^2\rangle \geq 1)$

$\langle \eta^2\beta\rangle$ = Stab$(G \geq \langle x\rangle \geq 1)$

$\langle \beta\rangle$ = Stab$(G \geq \langle xy\rangle \geq 1)$

$\langle \eta^2, \eta\beta\rangle$ = Stab$(G \geq \langle x^2, y\rangle \geq 1)$

= Stab$(G \geq \langle x^2, y\rangle \geq \langle x^2y\rangle \geq 1)$

= Stab$(G \geq \langle x^2, y\rangle \geq \langle y\rangle \geq 1)$

$\langle \eta\beta\rangle$ = Stab$(G \geq \langle y\rangle \geq 1)$

$\langle \eta^3\beta\rangle$ = Stab$(G \geq \langle x^2y\rangle \geq 1)$

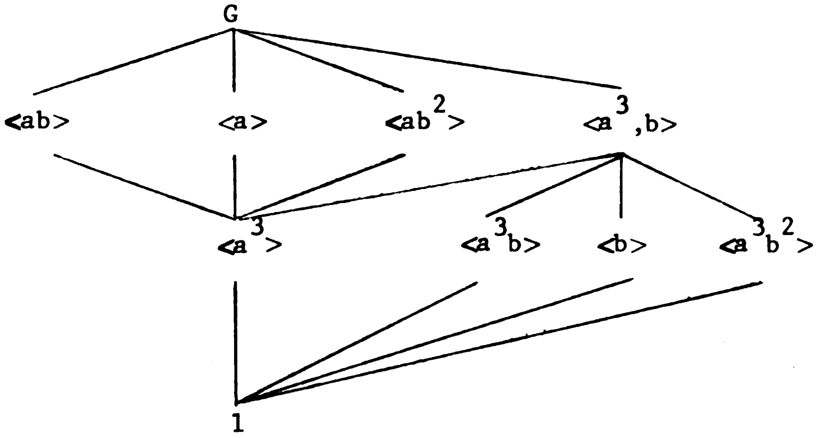
Closed Stability Groups	$\bar{s}$
Aut G	$G \geq \langle x^2, y\rangle \geq \langle x^2\rangle \geq 1$
$\langle \eta^2, \beta\rangle$	$G \geq \langle x^2\rangle \geq 1$
$\langle \eta^3 \beta\rangle$	$G \geq \langle x^2 y\rangle \geq 1$
$\langle \eta\beta\rangle$	$G \geq \langle y\rangle \geq 1$
$\langle \eta^2, \eta\beta\rangle$	$G \geq \langle x^2, y\rangle \geq 1$

<u>Example 4</u>

The nonabelian group G of order 3^3 and exponent 3^2, given by

$$\langle a, b \mid a^9 = b^3 = 1,\ a^b = a^4\rangle ,$$

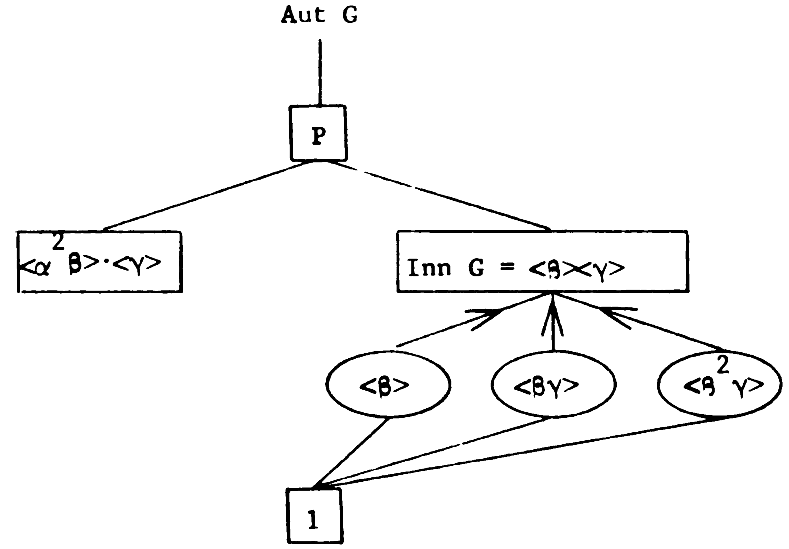
has the following subgroup lattice.



Aut G has a normal 3-Sylow generated by

$P = \langle \alpha, \beta, \gamma \mid \alpha^3 = \beta^3 = \gamma^3 = 1,\ \alpha\beta = \beta\alpha\gamma,\ \alpha\gamma = \gamma\alpha,\ \beta\gamma = \gamma\beta\rangle$ which is the nonabelian group of order 3^3 and exponent 3.

α: $a \to ab$	β: $a \to a$	γ: $a \to a^4$
$b \to a^3 b$	$b \to a^3 b$	$b \to b$



Stability Groups

$P = \mathrm{Stab}(G \geq \langle a^3, b\rangle \geq \langle a^3\rangle \geq 1)$

$\langle\beta\gamma\rangle = \mathrm{Stab}(G \geq \langle ab^2\rangle \geq 1)$

$\langle\alpha^2\beta\rangle\langle\gamma\rangle = \mathrm{Stab}(G \geq \langle a^3, b\rangle \geq 1)$

$= \mathrm{Stab}(G \geq \langle a^3, b\rangle \geq \langle b\rangle \geq 1)$

$= \mathrm{Stab}(G \geq \langle a^3, b\rangle \geq \langle a^3 b\rangle \geq 1)$

$= \mathrm{Stab}(G \geq \langle a^3, b\rangle \geq \langle a^3 b^2\rangle \geq 1)$

$\mathrm{Inn}\ G = \mathrm{Stab}(G \geq \langle a^3\rangle \geq 1)$

$= \mathrm{Stab}(G \geq \langle a\rangle \geq \langle a^3\rangle \geq 1)$

$= \mathrm{Stab}(G \geq \langle ab^2\rangle \geq \langle a^3\rangle \geq 1)$

$= \mathrm{Stab}(G \geq \langle ab\rangle \geq \langle a^3\rangle \geq 1)$

$\langle\beta\rangle = \mathrm{Stab}(G \geq \langle a\rangle \geq 1)$

$\langle\beta^2\gamma\rangle = \mathrm{Stab}(G \geq \langle ab\rangle \geq 1)$

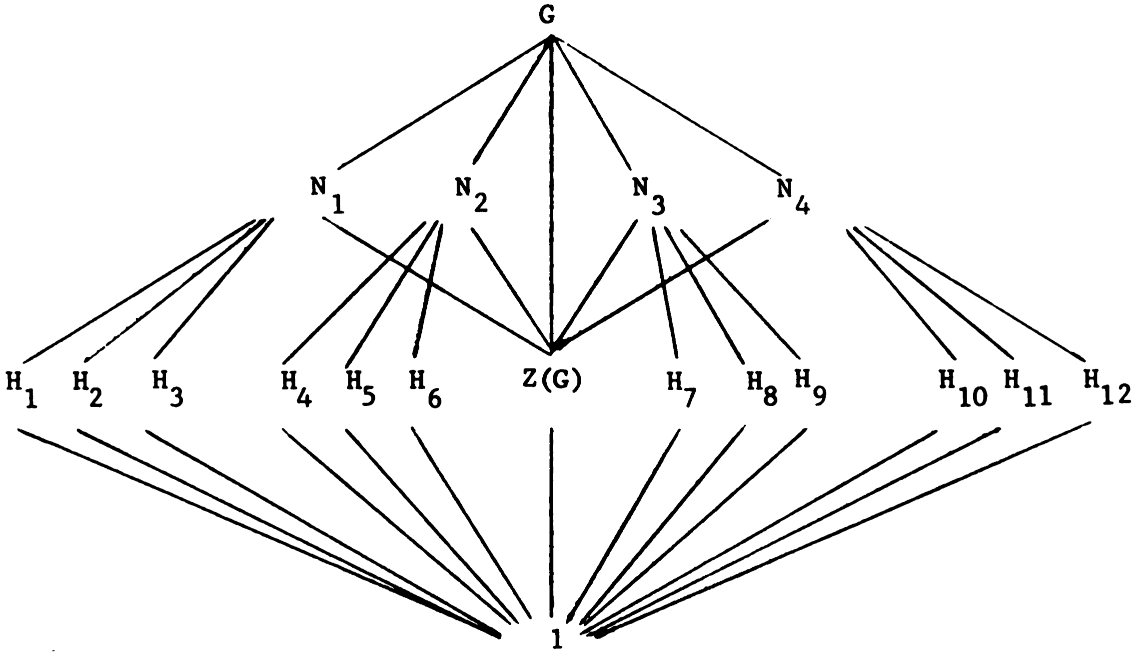
Closed Stability Groups	$\bar{s}$
P	$G \geq \langle a^3, b\rangle \geq \langle a^3\rangle \geq 1$
$\langle\alpha^2\beta\rangle\langle\gamma\rangle$	$G \geq \langle a^3, b\rangle \geq 1$
Inn G	$G \geq \langle a^3\rangle \geq 1$

Example 5

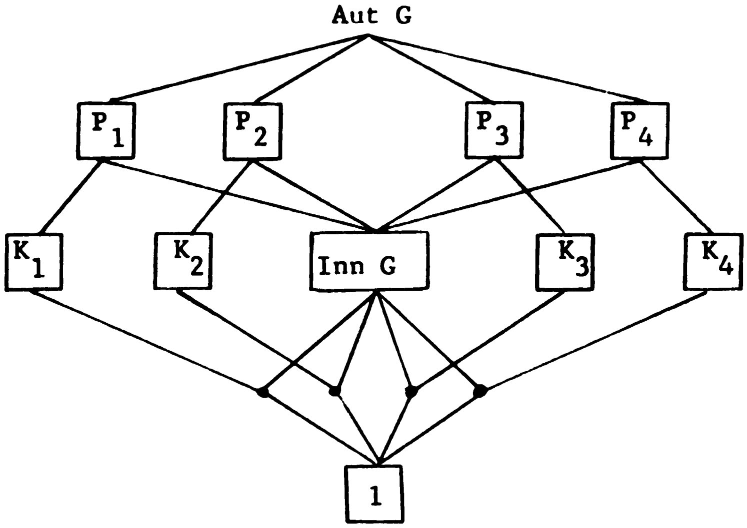
The nonabelian group of order 27 and exponent 3 is given by

$$G = \langle a,b,c \mid a^3 = b^3 = c^3 = 1,\ ac = ca,\ bc = cb,\ ab = bac\rangle$$

and has the following subgroup lattice.



Each N_i is elementary abelian of order nine and H_j is cyclic of order three, $i = 1,2,3,4$, $j = 0,1,2,\ldots,12$.



Each 3-Sylow P_i is isomorphic to G and K_i is elementary abelian of order nine, for $i = 1,2,3,4$.

Stability Groups

$P_i = \mathrm{Stab}(G \geq N_i \geq Z \geq 1)$ $\quad i = 1,2,3,4$

$\mathrm{Inn}\ G = \mathrm{Stab}(G \geq Z \geq 1)$

$K_i = \mathrm{Stab}(G \geq N_i \geq 1)$ $\quad i = 1,2,3,4$

$\quad = \mathrm{Stab}(G \geq N_i \geq H_j \geq 1)$ $\quad j = 3i-2,\ 3i-1,\ 3i$

Closed Stability Groups	$\bar{s}$	
P_i	$G \geq N_i \geq Z \geq 1$	$i = 1,2,3,4$
Inn G	$G \geq Z \geq 1$	
K_i	$G \geq N_i \geq 1$	$i = 1,2,3,4$

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