PRESERVICE ELEMENTARY TEACHERS' CONCEPTIONS OF MATHEMATICAL DEFINITIONS

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ABSTRACT

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Definitions play a central role in the discipline of mathematics and in teaching and learning mathematics (Vinner 1991). Knowledge of mathematical definitions is an important component of Subject Matter Knowledge for elementary teachers (Ball, Thames, and Phelps, 2008). Teachers' conceptions of mathematical definitions may affect how they engage in or engage their students in the process of defining (Johnson, Blume, Shimizu, Graysay & Konnova, 2014). However, to date, little research has focused on US teachers' conceptions of mathematical definitions and even less has examined this issue at the elementary level.

In this study, 24 preservice elementary teachers who were in their senior year were interviewed about their understanding of what constitutes a mathematical definition, the distinction between good and bad definitions, and their understanding of the roles of mathematical definitions in mathematics. The interviews were audiotaped and transcribed. Data were analyzed for correctness and general themes were examined using theoretical perspectives found in the literature (e.g., Van Dormolen & Zaslavsky, 2003). This study compares the conceptions of preservice elementary teachers whose teaching major was mathematics (PSTs-M) with the conceptions of preservice teachers whose teaching major was not mathematics (PSTs-N) in order to reveal whether advanced mathematical training contributes to preservice

elementary teachers' understanding of mathematical definitions.

The study results indicate that preservice elementary teachers understand quite a few of the necessary and preferred features of mathematical definitions which are suggested in the literature (e.g., precise). However, they also demonstrated limited understanding of how to distinguish a necessary feature from a preferred feature (e.g., noncircularity). Moreover, preservice elementary teachers had a few misconceptions about mathematical definitions. The most common misconceptions include that mathematical definitions are expected to be written in the format "A is B" and that mathematical definitions could be used to name a property/procedure/theorem. Throughout the whole interview, only one preservice elementary teacher demonstrated the idea that no mathematical definition needed to be proved. PSTs' misconceptions of mathematical definitions contributed to their incorrect understanding of the relationship between mathematical definitions and proofs. Also, PSTs-M's idea of mathematical definitions as a meta level concept was not that different from the idea of PSTs-N.

The results of this study suggest that current instruction in mathematics courses for preservice elementary teachers is not very effective in preventing the formation of misconceptions regarding mathematical definitions. The finding that PSTs-M's and PSTs-N's conceptions of mathematical definitions were similarly problematic provides further evidence that simply going through advanced training in mathematics does not automatically improve preservice elementary teachers' understanding of mathematical definitions. This indicates that the nature and roles of mathematical definitions need to be addressed more directly and more often in teacher education programs and that assessment needs to be designed as well.

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CHAPTER 1: INTRODUCTION AND OVERVIEW

Definitions play a central role in the discipline of mathematics. For instance, mathematical definitions describe the meaning of the mathematical concepts and form the foundation for mathematical proofs (Zaslavsky and Shir, 2005). They are also crucial in teaching and learning mathematics (Tall & Vinner, 1981; Vinner 1991). As asserted by Zazkis and Leikin (2008), “The definition of a concept, once determined in a curriculum, influences the approach to teaching mathematics, the learning sequence, the set of theorems and proofs” (p. 133). In spite of the importance of mathematical definitions, the notion of definition as a meta level concept develops through students’ experience in learning specific mathematical concepts and usually is not discussed explicitly during mathematics instruction. For instance, McCrory and Stylianides (2014) reported that among 16 mathematics textbooks for preservice elementary teachers, only two explicitly discussed what mathematical definitions were and their important roles in mathematics, which indicates the inadequate opportunities for preservice elementary teachers to learn this important meta-level concept in mathematics courses. Thus, Wilson (1990) claimed that “Although we frequently use definitions, we rarely focus on the nature of definitions. There is little agreement on what constitutes a good definition” (p. 33). Similarly, Levenson (2012) stated that “Being aware that definitions exist does not necessarily coincide with knowledge of the nature of definition” (p. 211).

As argued by Vinner (1991), “Definition creates a serious problem in mathematics learning. It represents, perhaps, more than anything else the conflict between the structure of mathematics, as conceived by professional mathematicians, and the cognitive processes of concept acquisition” (p. 65). In fact, research suggests that students have difficulty of understanding the nature of definitions (e.g., Wilson, 1990). Others also point out how the lack of understanding of the roles of mathematical definitions affects students' performance in proving (e.g., Vinner, 1991; Edwards & Ward, 2004). Even at the elementary level, definitions are important. In a study conducted by Ball and Bass (2000), elementary students disagreed whether six was an even number, an odd number or both. Ball reminded students to consult the definition of an even number, students then realized that six being an odd number was inconsistent with the definition of an even number that was agreed upon in class. Similarly, Saxe et al. (2013) found that if teachers stressed enough the roles of mathematical definitions, when elementary students disagreed with each other's claims, they consulted mathematical definitions and used definitions to build their arguments to resolve the disagreement.

Understanding mathematical definitions is important for teachers as well. First, as argued by Ball, Thames, and Phelps (2008), knowledge of mathematical definitions is an important component of Subject Matter Knowledge. Ball (1990) further argued that teachers should have substantial knowledge *of* mathematics and knowledge *about* mathematics. She referred knowledge *of* mathematics to knowledge of "particular concepts and procedures" (Ball, 1990, p. 458) such as squares and the multiplication of fractions. Knowledge *about* mathematics includes "understandings about the nature of mathematical knowledge and of mathematics as a field" (Ball, 1990, p. 458). For instance, what is a mathematical definition? What is its role in mathematics? How does definition differ from mathematical proof? Edwards and Ward (2004) concurred with Ball (1990) and argued that "the special nature of mathematical definitions should be treated as a concept in its own right, one that should be understood at some level by all college mathematics students" (p. 419). Similarly, Seaman and Szydlik (2007) argued from the perspective of the mathematics community that elementary teachers should have mathematical sophistication and one of the norms that indicate mathematical sophistication is understanding the nature of mathematical definitions. Second, definitions in mathematics textbooks sometimes are not completely correct, as pointed out by Smith, Males, Dietiker, Lee and Mosier (2013) about the concept of "length"; thus "teachers must be able to choose or develop a definition that is mathematically appropriate also usable by students at a particular level (Ball, 2003, p.5)". In order to choose or develop a good definition, teachers should have a deep understanding of the issues related to mathematical definitions, such as what attributes of mathematical definitions are necessary (e.g., unambiguous, non-contradictory) for major consideration and what attributes are optional (e.g., elegant) which deserve minor attention. Third, one of the central components of mathematical knowledge is defining (e.g., Mariotti & Fischbein, 1997; Vinner, 1991) and teachers should be able to provide various opportunities to develop students' abilities to abstract and define mathematical concepts (de Villiers, 1998). Mathematics teachers' competence in enacting activities of defining requires their own deep understanding of the defining process and also their understanding of nature of mathematical definitions (Leikin, & Winicki-Landman, 2000). Through a case study with a beginning high school teacher, Johnson, Blume, Shimizu, Graysay & Konnova (2014) found that teachers' conception of mathematical definition may affect how they engage in or engage their students in the process of defining. Fourth, mathematical definitions draw attention to the role of mathematical language in teaching mathematics. As indicated by Ball (2003), "Using mathematical language with care, and understanding how definitions and precision shape mathematical problems solving and thinking is another element crucial to understanding how teachers must use - and therefore know – mathematics" (p. 7).

Furthermore, the Common Core State Standards for Mathematics (Common Core State Standards Initiative, 2010) explicitly mention mathematical definitions in two mathematical practices:

> Construct viable arguments and critique the reasoning of others: Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. (p. 6)
>
> Attend to precision: Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning… By the time they reach high school they have learned to examine claims and make explicit use of definitions. (p. 7)

As those who are going to implement CCSSM , preservice elementary teachers need to understand and be able to use mathematical definitions well because the mathematical practices are expected to be implemented across all grade levels.

The Conference Board of the Mathematical Sciences (2012) also points out the importance for teachers to recognize the need for definitions and their consequences. For instance, a trapezoid could be defined as either having at least one pair of parallel sides or exactly one pair. However, the choice made leads to different mathematical conclusions, such as if parallelograms are trapezoids or not. Failure to understand the connection between a definition and its consequence may cause confusion about why a mathematical conclusion sometimes holds or does not.

Borko and Putnam (1996) argued that teachers' subject matter knowledge and beliefs may have an impact on teachers' classroom practices. Thus, teachers' conceptions of definitions may affect their practice on enacting defining activities, teaching mathematical definitions, and even teaching mathematical proofs in the classroom. However, though some research done internationally has addressed teachers' conception of mathematical definitions (e.g., Zaslavsky &

Shir, 2005), to date, not too much research has focused on US teachers' conceptions of definitions and even less has examined this issue at elementary level. Seaman and Szydlik (2007) found that when encountering an unfamiliar definition, preservice elementary teachers tended to mimic the procedures given in the examples after the definitions rather than studying each condition given in that definition. This observation revealed the possibility that future elementary teachers did not fully understand the nature and roles mathematical definitions played in mathematics so they valued mimicking procedures more than understanding mathematical definitions. However, the study by Seaman and Szydlik did not reveal exactly what mathematical definitions entailed in preservice elementary teachers' minds. Therefore, the goal of my study is to develop a picture of preservice elementary teachers' conceptions of mathematical definitions. This study investigates the following research questions:

Research Questions

1. What are preservice elementary teachers' conceptions about the nature of mathematical definitions in mathematics?

   a. What constitutes a mathematical definition for preservice elementary teachers?

   b. Among the valid definitions, what types of mathematical definitions do preservice elementary teachers prefer?

   c. To what extent do preservice elementary teachers distinguish mathematical definitions from other meta level mathematical concepts such as mathematical proofs?

2. What are preservice elementary teachers' conceptions about the role of mathematical definitions in mathematics?

3. What opportunities do preservice elementary teachers have to explicitly learn about mathematical definitions during their teacher education programs?

As conceptions of mathematical definitions are part of the knowledge of the structure of mathematics, it is natural to ask if the amount of mathematics knowledge elementary teachers have affects their conceptions. Thus, when analyzing the data, I examine two groups of preservice elementary teachers (preservice elementary teachers whose teaching major is mathematics vs. preservice elementary teachers whose teaching major is not mathematics) separately according to the amount of advanced mathematical training they receive. I also compare the conceptions of mathematical definitions between these two groups. Details of how these two groups are defined are given in Chapter 3 when I explain the research methods.

Overview of the Study

Twenty four preservice elementary teachers who were in their senior year in their teacher education programs in a major research university volunteered for this study. Among the 24 participants, twelve had their teaching majors as mathematics and the other twelve had their teaching majors as non mathematics. Each participant was invited for a 60-90 minutes semi-structured interview. During the interviews, participants were asked general questions about mathematical definitions (e.g., From your perspective, what is a mathematical definition?) and were also asked to work on tasks relevant to mathematical definitions such as selecting or rating mathematical definitions. The interviews were audiotaped and transcribed.

Data were analyzed mainly in two ways. For questions which have right or wrong answers, I analyzed the data by examining the extent of correctness of participants' responses. Questions which reflect preservice elementary teachers' opinions and are subjective were analyzed by themes through the theoretical perspectives found in the literature. I report the findings as a group of preservice elementary teachers rather than reporting multiple cases studies of individual participant. A comparison was also made between preservice elementary teachers whose teaching majors were mathematics and whose teaching majors were not mathematics to explore if taking advanced mathematics courses contributed to preservice elementary teachers' understanding of mathematical definitions as a meta level concept.

Significance of the Study

This study may contribute to the field of mathematics education in the following ways. First, though the plethora of research on teachers' conceptions of mathematical definitions were done internationally (e.g., Zaslavsky & Shir, 2005; Levenson, 2012), little research has focused on US preservice elementary teachers' conceptions of mathematical definitions. This study fills the void in the mathematics educational research literature by describing what mathematical definitions mean for preservice elementary teachers, how they compare and contrast mathematical definitions and how they think about roles of mathematical definitions in mathematics. Understanding how preservice elementary teachers think about mathematical definitions may uncover their misconceptions. This information may inform mathematics educators and provide insight into design of instructional activities to eliminate these misconceptions.

In addition, studies on mathematical definitions typically involved specific mathematical topics. For instance, Vinner (1977), Dickerson and Pitman, (2012), and Levenson (2012) all investigated how undergraduates thought about defining concepts related to exponents. Johnson et al. (2014) investigated how a beginning high school teacher's conceptions of mathematical definitions affected how she engaged her students in the process of defining geometric concepts. As found by Knuth (2002b), practicing secondary teachers tended to associate mathematical proofs with geometry rather than with algebra. Similarly, conceptions of mathematical definitions could also depend on content areas. Different from the previous studies, this study involves concepts in multiple content areas and therefore the results from this study are potentially more generalizable.

Also, this study aims to develop frameworks to capture preservice elementary teachers' views regarding the nature and roles of mathematical definitions. Such frameworks may enable future researchers to examine how preservice elementary teachers think about mathematical definitions. Mathematics educators could also use these frameworks to discuss definitions in university mathematics courses or in professional development activities.

Furthermore, this study examines an understudied population, preservice elementary teachers whose teaching major is mathematics. Usually in US, preservice elementary teachers only need to take basic college mathematics classes and two or three mathematics courses specifically designed for them. Thus far, little research has reported mathematical knowledge of preservice elementary teachers who take more advanced mathematics courses. My study contributes to the literature by describing their thinking and compare them with typical preservice elementary teachers. This information may provide insight into whether advanced mathematical training has an impact on preservice elementary teachers' mathematical knowledge for teaching.

Organization of This Dissertation

This dissertation is organized into seven chapters. This first chapter is meant to provide an introduction to the issues surrounding mathematical definitions and motivate my study. Chapter 2 is a review of related literature. I present both theoretical perspectives and empirical studies about how students and teachers perceive the nature and roles of mathematical definitions in mathematics and in teaching mathematics.

In Chapter 3, I present the research design and methods. I describe the research site, participants, development of the instruments, the process of data collection, methods for data analysis and plans to report findings.

In Chapters 4, 5 and 6, I present my results. Chapter 4 aims to answer research question 1a, namely, what constitutes a mathematical definition for preservice elementary teachers. I first describe preservice elementary teachers' performance in generating and evaluating mathematical definitions. Though these two analyses revealed how well preservice elementary teachers wrote and evaluated mathematical definitions, the focus was on uncovering how they thought about mathematical definitions as a meta level concept through their performances. I then describe what constitutes a mathematical definition for preservice elementary teachers mainly from two aspects. I first present the necessary features preservice elementary teachers associated with mathematical definitions. I then describe the problematic thinking they held about mathematical definitions.

In Chapter 5, I answer research question 1b, namely, among the valid definitions, what types of mathematical definitions do preservice elementary teachers prefer? I first provide an overall description of preservice elementary teachers' performance in evaluating good and bad mathematical definitions. I then present a further analysis on the criteria preservice elementary teachers used to evaluate good and bad mathematical definitions. These criteria reveal features preservice elementary teachers prefer to include in mathematical definitions.

In Chapter 6, I answer research questions 1c and 2, namely to what extent do preservice elementary teachers distinguish mathematical definitions from other meta level mathematical concepts and what are preservice elementary teachers' conceptions about the roles of mathematical definitions in mathematics. I first present how and what criteria preservice elementary teachers used to determine whether a given statement needs a mathematical proof. I then describe how preservice elementary teachers perceived the relationship among mathematical definitions, mathematical proofs and counterexamples explicitly and implicitly. Finally, I discuss preservice elementary teachers' thinking about roles mathematical definitions play in mathematics.

In Chapter 7, I provide a summary and discussion. I first summarize the key results in each chapter. I then briefly discuss the findings relevant to research question 3 about preservice elementary teachers' self-reported opportunities to explicitly learn about mathematical definitions in their teacher education programs. Data collected for this research question is not very reliable because they are self-reported data and they are based on memory. Therefore, I only briefly discuss the findings in Chapter 7 rather than writing a separate chapter. I conclude the chapter by describing the contributions of the study, implications for teacher education, limitations of the study and future directions of research on definitions.

CHAPTER 2: LITERATURE REVIEW

The purpose of this chapter is to survey the research related to conceptions of mathematical definitions in the field of mathematics education. It consists of three sections: the nature of students' or teachers' understanding of mathematical definitions, the roles of mathematical definitions, and a summary that includes an explanation of the stance I take about the nature and role of definitions in this dissertation.

I use nature of mathematical definitions to express the the fundamental and essential characters of mathematical definitions and role of mathematical definitions to indicate the part played by mathematical definitions in mathematics. In other words, nature of mathematical definitions captures the characteristics of mathematical definitions such as what makes a statement a (good) mathematical definition as well as how mathematical definitions differ from other mathematical objects such as proofs, theorems, etc. Role of mathematical definitions is meant to capture the position of mathematical definitions in mathematics, namely, why mathematical definitions are important or necessary in mathematics.

Nature of Mathematical Definitions

What Is a Good Mathematical Definition?

Wilson (1990) asserted that "Although we frequently use mathematical definitions, there is little agreement on what constitutes a good definition" (p. 33). Different researchers proposed different features of a good mathematical definition. Borasi (1992) suggested that a good definition should have the following features:

a. Precision in terminology: all the terms employed in the definition should have been previously defined, unless they are one of the few undefined terms assumed as a starting point in the axiomatic system one is working with;

b. Isolation of the concept: all instances of the concept must meet all the requirements of its definition, while a noninstance will not satisfy at least one of them;
c. Essentiality: only terms and properties that are strictly necessary to distinguish the concept in question from others should be explicitly mentioned in the definition;
d. Noncontradiction: all the properties stated in a definition should be able to coexist;
e. Noncircularity: the definition should not use the term it is trying to define. (p. 17-18)

Winicki-Landman and Leikin (2000) distinguished two dimensions of the features of a mathematical definition in the context of teaching and learning a concept: mathematical characteristics and didactic considerations. Mathematical characteristics of a definition refer to "some logical principles that must be met when defining any mathematical concept" (Winicki-Landman & Leikin, 2000, p. 17). Winicki-Landman and Leikin (2000) summarized the literature and proposed several features about what constitutes a good mathematical definition. First, "defining is giving a name" (p. 17). This means that the statement used to define the concept should include the name of the concept and the name should appear only once in the statement. For instance, a statement "a polygon with four equal sides and four equal angles" meant to define a square but because the key term "square" does not appear in the sentence, it is not a mathematical definition. Second, only previously defined concept can be used to define a new concept. Third, a definition should include the necessary and sufficient conditions for the concept. Fourth, the conditions included in the statement as the definition should be minimal. Last, a definition is arbitrary.

Van Dormolen and Zaslavsky (2003) outlined the features of a good mathematical definition by distinguishing necessary features and frequently preferred features. The necessary features they include are criterion of hierarchy, criterion of existence, criterion of equivalence, and criterion of acclimatization. Criterion of hierarchy means “any new concept must be described as a special case of a more general concept. One or more properties must be used to describe this special case” (Van Dormolen & Zaslavsky, 2003, p. 94). For instance, in the statement “a rectangle with four equal sides is a square”, the new concept is “square”, the more general concept is “a rectangle” and the property is “four equal sides”. Criterion of existence means “it must be proven that at least one instance of newly defined concept exists in the current context” (p. 94). Van Dormolen and Zaslavsky (2003) used an example “circlesquare” to show that it is meaningless to define this concept in Euclidean geometry because circlesquare does not exist in Euclidean geometry. Criterion of equivalence means “when one gives more than one formulation for the same concept, one must prove that they are equivalent” (p. 95). For instance, there are multiple ways to define a square. A square could be defined as “a polygon with four equal sides and four equal angles”. A square could also be defined as “A parallelogram with diagonals that are equal and perpendicular.” Either statement could be chosen as a mathematical definition of a square, but once one of them is selected, the other statement cannot be a mathematical definition but only a theorem and needs to be proved based on the chosen definition. Criterion of acclimatization means “one must verify if it [a definition] fits in and is part of a deductive system. In practice this would mean that it has to be verified if all the concepts used in the definition are, in their turn, again defined within the same deductive system” (p. 96). Van Dormolen and Zaslavsky (2003) used a description of a circle “a figure that everywhere has the same roundness” to illustrate their point. They argued that this description is not a definition for a circle in Euclidean geometry because “everywhere” and “roundness” are not defined.

In addition to the necessary features, Van Dormolen and Zaslavsky (2003) also proposed other features they regarded as “part of a general culture” (p. 93) which included criterion of minimality, criterion of elegance, and criterion for generations. Criterion of minimality demands that “no more properties of a concept be mentioned unless it is necessary to establish the concept” (p. 93). For instance, the statement “a square is a polygon with four equal sides and four 90 degrees angles” is not a minimal definition because extra properties are involved in this statement. The statement “a square is a polygon with four equal sides and one 90 degrees angle” is enough to delimit the concept of a square and other properties are additional. Criterion of elegance means that a definition “looks nicer, needs fewer words or less symbols, or uses more general basic concepts from which the newly defined concept is derived” (p. 93). The criterion for degenerations means that definition allows instances that are inconsistent with our concept images and such instances are called degenerations.

Similar to the Criterion of acclimatization outlined by Van Dormolen and Zaslavsky (2003), Freudenthal (1973) raised a similar point “In mathematics definitions are links in deductive chains, but how can you forge such a link unless you know in which chain it could fit” (p. 416)?

Zaslavsky and Shir (2005) argued that the notion of definition has two types of features: imperative and optional. They defined the imperative features to include “noncontradicting (i.e., all conditions of a definition should coexist) and unambiguous (i.e., its meaning should be uniquely interpreted)” (p. 319). In addition, Zaslavsky and Shir also proposed some features that are imperative only when relevant. They argued that a mathematical definition must be “invariant under change of representation” (p. 319). and it should be hierarchical, namely, “it should be based on basic or previously defined concept in a noncircular manner” (p. 319). In addition to the imperative features mentioned above, they also proposed optional features that are not agreed to be imperative by all mathematicians. The most controversial feature is minimality.

While most textbooks assume that students understand the nature of mathematical definitions and no further explanation is needed, textbooks from The University of Chicago School Mathematics Project see the needs of the clarification and have a specific section on what are good mathematical definitions. Usiskin et al. (1997) proposed that a good definition must:

a. include only words either commonly understood, defined earlier, or purposely undefined;
b. accurately describe the idea being defined; and
c. include no more information than is necessary. (p. 81)

In addition to those features listed above, Pimm (1993) emphasized that a good definition is the one that "makes us think". He commented "The tag 'By definition' seems to be an invitation *not* to think - resulting in something that is automatic or perhaps 'for free' and consequently often not valued highly" (p. 273).

In response to elementary teachers' letters about which definition is a good definition and should be adopted in teaching, a Russian mathematician Khinchin (1968) pointed out that some of teachers' questions such as "Which way to define subtraction is better? An operation which consisted in seeking the second member of a summation given the sum and the first member? Or an operation which consisted in taking away from one number however many units were contained in the other number?" are not a choice between two competing definitions. Khinchin argued that the second "definition" actually is not a definition but an explanatory description. He explained:

> By no means every phrase uttered in the attempt to clarify the meaning of a newly introduced concept need pretend to the role of a definition of that concept....*The only formulations of a new concept that can serve as definitions of that concept (and which in*

fact are accepted as definitions in mathematics) are those which fully reduce the new concept to concepts already familiar in the given scientific field. (p. 47)

According to Khinchin, in the above example of subtraction, if we assume that the definitions of sum and addition have already been clearly presented before subtraction is introduced in the curriculum; then the first definition is a true definition. However, the second "definition" is defined by "taking away" which is based on "considerations taken from practical life, from other sciences, or from everyday experience, and not on concepts already established in the same scientific field, then that is an explanatory description, and for all its pedagogical value cannot be called a definition" (Khinchin, 1968, p. 47).

The main argument Khinchin made is the importance to distinguish between definitions and exploratory descriptions, especially at the elementary level while more concepts cannot be defined accurately and within the intellectual reach of children. However, he did not mean to say which one is better, but pointed out that they were different objects and needed to be recognized and treated differently. For example, he believes:

> A logical definition is a formula from which nothing can be dropped and to which not a word can be added, or otherwise the sense will be distorted....For you see this is no definition, in which every word has its irreplaceable logical weight, indispensable in all ensuing formal discussions." (Khinchin, 1968, p. 56-57)

In other words, from his perspective, definitions should be precise and minimal. He drew the conclusion that definition should be memorized word by word and presenting one definition is enough to students. However, if it is an explanatory description, understanding the description is more important than memorizing it word by word since the description may only reveal one aspect of the concept instead of all aspects; therefore, giving students multiple descriptions may help them see multiple perspectives of the concept better. He also gives several suggestions on which one between definition and exploratory description should be chosen based on the specific situation. The most essential point of the selection is the consideration of whether the choice can lead to "a proper understanding of the concept -- its real nature, its connections with other concepts, with life, with practice" (Khinchin, 1968, p. 52).

Research on students' thinking about the nature of mathematical definitions. An important part of the literature on definition focuses on studying students' or teachers' conceptions of mathematical definitions. If no consensus is ever reached about what is a good mathematical definition, why is research on definitions exploring this issue? According to Philipp (2007), conception is a type of belief that could filter the way people see an object. Students who do not have a deep understanding of what a definition is will not fully appreciate the value of a mathematical definition and may not apply it well in reasoning. For instance, if a student does not know a mathematical definition has to be a necessary and sufficient conditions for the concept, she/he may not realize that a definition could be used in two directions in problem solving or proving.

Though obvious to mathematicians that a mathematical definition should be a necessary and sufficient conditions, this fact may not be apparent to mathematics learners. Wilson (1990) found that sixth grade students were not consistent in performing four tasks which were supposed to give consistent results. The four tasks were (a) drawing a picture for geometric shapes (e.g., a triangle), (b) writing definitions for the shapes drawn before, (c) identifying examples of the shapes from a collection of figures, and (d) stating whether the student agrees or disagrees with the statements pertaining to properties of figures and relationship between figures. Wilson found that students did not understand necessary and sufficient conditions of mathematical definitions. For instance, one student claimed that it was not necessary to put right angles into the definition of a rectangle because everyone knew that a rectangle had right angles. He only chose figures with right angles which means he understood the concept of a rectangle, but he did not want to add it to the definition because it seemed unnecessary to him. Another student indicated that writing definitions was like what you do in spelling, "Sometimes...I won't put the whole thing...cause I don't want to write anymore. And just try to make it shorter if you can, and don't make it so long" (Wilson, 1990, p. 43).

In another study conducted by Zaslavsky and Shir (2005), four 12th-grade Israeli students' conceptions of what counted as a valid and good mathematical definition is revealed. Participants agreed that minimality was a preferred but not an imperative feature of mathematical definitions. Also, participants were hesitant to accept a procedural statement as a mathematical definition. They shared the opinion that a new mathematical concept should be defined through previously defined familiar and clear concepts. They also accepted that “there is no need to go back to concepts that are too basic” (Zaslavsky & Shir, 2005, p. 330). However, they had different opinions on which concepts are too basic. Furthermore, they claimed that not every statement that was a necessary and sufficient conditions of a concept could serve as a definition. They cannot reach an agreement that the latent parts (i.e., the parts that are usually invisible in the figure such as a diagonal) of the shapes could be included in its definition.

Research on teachers’ thinking about the nature of mathematical definitions. The conception of a mathematical definition is even important for teachers because it may affect their pedagogical decisions making, especially when teachers have to judge if a definition is valid to be used or needs modification or when teachers have to select a definition from multiple equivalent definitions for their teaching practice. If teachers are not sure about the criteria of choosing valid and good mathematical definitions, how can we expect them to make the correct decisions? What are the standards to count two mathematical definitions as equivalent is an issue under debate. Mathematically, if two definitions are equivalent, they describe the same set of mathematical objects. However, Pimm (1993) commented that "But logical equivalence may not be the same as functional equivalence. *Equivalence* can mean 'equivalence for our purposes'…. Definitions help when you want to prove something. But the help they offer will vary" (p. 274). Teachers need to be sensitive to these issues and their decisions on which definition to choose may greatly affect students' learning. For example, a teacher who only understands the logical or mathematical equivalence may define even number as "whole numbers that end with 0, 2, 4, 6, and 8 under common place value system (base 10)". In contrast, a teacher who holds a similar conception as Pimm (1993) and believes that definition should "make us think" and help with reasoning and proving would choose "2k where k is a whole number" or its graphical equivalence (i.e., drawing two columns of squares with each square paired with another) as the definition of an even number. It is obvious that these two definitions are logically equivalence because they define the same set of objects. However, they differ in their pedagogical potential in affecting students' learning of an even number.

Leikin and Winicki-Landman (2000) conducted a study to investigate the ways in which secondary teachers explore the equivalent or non equivalent definitions. They interviewed teachers about their preference to choose a definition for instruction among the equivalent definitions. Teachers identified the following attributes as the most important ones "(a) intuitiveness; (b) matching students' knowledge and needs; (c) clarity to the students or ease of understanding; (d) convenience for applying to problem solving; (e) enabling mathematical generalizations" (Leikin & Winicki-Landman, 2000, p.27). During a discussion of minimality, teachers regarded “a parallelogram has four right angles” as a better definition for a rectangle than the statement including three right angles. They agreed that though the minimality was in favored from the mathematical perspective, it did not make much sense to students when they had a visual image of a rectangle having four right angles instead of having three right angels. This argument indicated that when mathematical and pedagogical consideration were in conflict, teachers tended to favor the pedagogical consideration.

Another study conducted by Zazkis and Leikin (2008) echoes the same point. In their study, researchers asked secondary teachers to generate examples of a definition of a square and evaluate which one was better. They found that teachers automatically shifted their discussion about which definition was appropriate from the mathematical perspective to pedagogical perspective. Teachers’ discourse tended to be suited in teaching and they judged the validity of definition based on whether the definition was accessible to their students rather than from the perspective of mathematics, When the rigor was in conflict with pedagogical consideration, “there were strong voices advocating clarity of a definition and accessibility for students, at the expense of mathematical rigor” (Zazkis & Leikin, 2008, p. 145). For example, one teacher thought “figure” was much clearer for a student than “polygon” and she commented that “there is no need to confuse a student with extra words or unfamiliar words” (Zazkis & Leikin, 2008, p. 145).

Shir and Zaslavsky (2001) investigated 24 secondary mathematics teachers’ thinking about what constitutes a (good) definition. They asked teachers individually to complete a written questionnaire that included eight equivalent statements about a square by indicating whether they accepted the statement as a definition and reasons for their decisions. Then, teachers were divided into groups for discussions and were asked to report their group decisions in the whole class discussions. Shir and Zaslavsky found that although all eight statements were equivalent to the definition of a square, only five teachers accepted all eight statements as definitions. Moreover, for none of given statements, teachers reached a unanimous agreement for acceptance or rejection it as a mathematical definition. Teachers also had different reasons for acceptance or rejection of the statements as possible definitions of a square. The reasons were categorized into four main categories: mathematical, pedagogical, both (mathematical and pedagogical), and embodied cognition. Mathematical arguments were applied a lot when accepting a statement as a mathematical definition but were used rarely to reject a statement as a mathematical definition. Pedagogical considerations were applied a lot when accepting and rejecting a statement as a mathematical definition. The results also pointed out that teachers disagreed on if procedural statements could be mathematical definitions. Some teachers favored procedural statements, because they believed such statements supported the formation of mental images of the concepts. Others rejected them mainly based on mathematical grounds.

Though research discussed before has investigated secondary teachers' conceptions of mathematical definitions, little research has been done to directly examine the same issue at the elementary level. In a study, Seaman and Szydlik (2007) selected 11 preservice elementary teachers who failed to solve problems involving greatest common factors (GCF), least common multiples (LCM) and other mathematical topics. Seaman and Szydlik suspected that "the true underlying cause of students' lack of skill is not merely a need for knowledge refreshment, but rather is a paucity of 'accessing skills,' a profound lack of mathematical sophistication" (p. 175). Therefore, they gave participants access to website of relevant resources so the participants can look for information and refresh their knowledge. Seaman and Szydlik found that participants' performance did not change a lot before and after using the website and when exploring the resources, preservice elementary teachers typically ignored the definitions and focused almost exclusively on mimicking procedures for computing a GCF and an LCM. In other words, these preservice elementary teachers concentrated on "what to do, and not on sense making" (p. 177). This phenomenon revealed that preservice elementary teachers did not value mathematical definitions as mathematicians did, which indicated that preservice elementary teachers' lack of understanding of the nature of mathematical definitions. Because understanding mathematical definitions is one of the norms that indicate mathematical sophistication, along with other results found in the study, Seaman and Szydlik concluded that preservice elementary teachers "display a set of values and avenues for learning mathematics that is so different from that of the mathematical community and so impoverished, that their attempts to create fundamental mathematical understandings often meet with little success" (p. 179). The results of this study also suggested that in addition to knowing preservice elementary teachers' lack of understanding of mathematical definitions, more research needed to be done to examine preservice elementary teachers' thinking about mathematical definitions.

Differences Between Mathematical Definitions and Conclusions

Though mathematical definitions and mathematical conclusions share some similarities, they differ a lot. The validity of a mathematical conclusion is established by proving, and proving is validated or falsified based on whether it is against "syntactic structure" of mathematics (Shulman, 1986). The process of proving could be determined as valid or not by looking at the *logic*, namely, whether each step draws logically correct inference, based on the accepted knowledge and assumption, and by using correct representations (Stylianides, 2007). The validity of a mathematical definition, however, relies on if it is against the *conventional* knowledge of the meaning of the concept as established by the community of mathematicians.

For instance, if I call an even number "an odd number", mathematicians will correct me by saying that "You are wrong; we called it an even number and you will have to accept this convention in order to be able to communicate with us." For definitions, even if students are not pleased with whatever is agreed upon by mathematicians, they have to accept it; for the mathematical conclusions, students can reason through and decide on its validity based on logic by themselves. As pointed by Ball (1990), “mathematical knowledge is based on both *convention* and *logic”* (p. 458). Elementary teachers need to know “What establishes the validity of an answer?” Namely, “Which ideas are *arbitrary or conventional* and which are *logical*” (p. 458)? Students’ tendency to think that they should listen to and accept all that teachers tell them might due to their confusion about when mathematical knowledge is based on *convention* and when it is based on *logic*. Being able to distinguish convention and logic may affect how teachers establish classroom norms. Should students follow authorities such as mathematicians or should they follow their own thinking by using correct logic?

In addition, though conventional knowledge is arbitrary, sometimes there are reasons for why the concept is defined in such a way (Vinner, 1977). For instance, if 1 is taken as a prime number, the “uniqueness of prime decomposition” theorem would become invalid. However, this reason of excluding 1 from being a prime number is not a mathematical proof but a motivation for the definition of prime numbers. Failure to realize the distinctions between mathematical proofs and motivations for definitions may cause students’ confusion that a mathematical definition needs to be proved as reported in multiple studies in the literature (e.g., Vinner, 1977; Dickerson & Pitman, 2012; Levenson, 2012). Along with the idea that mathematical definitions are used to construct mathematical proofs, students may develop confusion about the incorrect circular relationship between mathematical definitions and proofs: mathematical definitions are used to construct proofs and mathematical definitions also need to be proved. This confusion may reinforce the bad attitude that mathematics does not make sense. Therefore, understanding the relationship among mathematical definitions, proofs and theorems are important for students to appreciate mathematics as a discipline which has a clear structure.

Research on undergraduates' understanding of mathematical definitions, mathematical proofs and mathematical conclusions. Though the knowledge to distinguish mathematical definitions and mathematical conclusions is important for mathematics learners, it is unclear that all mathematical learners understand that mathematical definitions and mathematical conclusions are two entirely different mathematical objects. As pointed by Gottlob Frege in a letter to David Hilbert and cited by Pimm (1993),

There is widespread confusion with regard to definitions in mathematics, and some seem to act according to the rule:

If you can't prove a proposition,

Then treat it as a definition. […] (p. 261)

Research also provides evidence that students lack deep understanding of these two mathematical objects. Edwards and Ward (2004) reported students' confusion between mathematical definitions and theorems. One student Andre claimed that "Once [a theorem] is proven, it becomes a definition .... At some point in time, we proved that 1 plus 1 is 2. Therefore, the definition is that always 1 plus 1 equals 2..." (p. 415). Edwards and Ward used the categorization of mathematical definitions (Rhobinson, 1962) to explain this phenomenon. According to Rhobinson, there are two types of definitions: *lexical* and *stipulative* definitions. A "Lexical definition is that sort of word-thing definition in which we are explaining the actual way in which some actual word has been used by some actual person" (Rhobinson, 1962, p. 35).

A key feature of lexical definition is “[It] is a form of history. It referred to the real past. It tells what certain persons meant by a certain word at a certain less or more specified time and place” (Rhobinson, 1962, p. 35). By contrast, Robinson uses stipulative definition to "mean the explicit and self-conscious setting up of the meaning-relation between some word and some object, the act of assigning an object to a name (or a name to an object), not the act of recording an already existing assignment" (p. 59). According to Rhobinson (1962), the key distinction between lexical and stipulative definition is that lexical definition reports the meaning of the word used in the past, and stipulative definition announces the meaning of the word which will be used in a new context. In their paper, Edwards and Ward (2004) adopted a different terminology used by Landau (2001) who uses the term extracted definitions in place of lexical definitions because they are "definitions that are based on examples of actual usage, definitions extracted from a body of evidence" (p. 412). Edwards and Ward argued that it was hard to understand Ardre’s point of view unless thinking in terms of the categorization of definitions. They claimed that for Andre mathematical definitions were not stipulative but reported facts. They said “Andre was a student for whom all properties of a mathematical object held ‘by definition’. His definitions were extracted, though not from common usage, but rather from the body of knowledge about the concept” (p. 415).

Vinner (1977) conducted a survey by giving undergraduate students a few algebraic questions and asked them to complete the sentence by selecting from the following words: a theorem, a law, a fact about numbers, a definition, or an axiom. For instance, one question is “the equality $a^{-m} = \frac{1}{a^m}$ is: ……”. He found that about 25% of the freshmen and 50% of the sophomores and juniors successfully identified all three mathematical definitions given in the form of equations. For those students who did not choose “a definition” as the correct answer, they tended to choose either “a theorem” or “a fact about numbers”. Vinner suspected that students tended to take the motivations of definitions as proofs of definitions. For instance, there is a reason why mathematicians define a^{-m} to mean $\frac{1}{a^m}$ when m is a whole number, but this motivation of defining is not a mathematical proof. Vinner conjectured that mixing proofs and motivations was the reason why students had the impression that mathematical definitions needed to be proved. However, because his study is a quantitative study, there is no data to support his conjecture.

Dickerson and Pitman (2012) conducted a qualitative study to further investigate students’ thinking of the same issue. Rather than asking students’ to complete the sentences by choosing from “a theorem, a law, a fact about numbers, a definition, or an axiom”, Dickerson and Pitman gave five undergraduate students the definition $x^{-a} = \frac{1}{x^a}$ and told them this was the definition of x^{-a}. Students were then asked “Does this definition need to be proved?” All five participants said yes. In addition to the reason proposed by Vinner (1977), Dickerson and Pitman found that if a participant felt that a definition was not immediately obvious to him/her, he/she would want that definition to be proved. Some participants also thought that all mathematical statements needed to be proved. Other students pointed out that in advanced mathematics courses, students were often asked to prove obvious mathematical statements and defining was a way to get around these difficult proofs.

Research on practicing teachers’ understanding of mathematical definitions, mathematical proofs and mathematical conclusions. Levenson (2012) further explored the similar issue about teachers’ understanding of zero exponent by studying three junior high school teachers in Israel through case studies. She found that only one teacher correctly identified the exponent operation $a^n = a \times a \times \ldots a$ (n times) as a mathematical definition. All three teachers regarded $a^0 = 1$ as a theorem and claimed that it can be proved either by induction or by the division rule. No teacher demonstrated understanding that $a^0 = 1$ extended the definition of $a^n = a \times a \times \ldots a$ (n times) even if the teacher correctly identified $a^n = a \times a \times \ldots a$ (n times) as a mathematical definition. In addition to teachers' understanding of zero exponent, Levenson also reported teachers' general understanding of the relationships among mathematical definitions, theorems and mathematical proofs. She found that only one teacher had the correct idea about the relationship between mathematical definitions and proofs. The other two either were uncertain about the relationship or insisted that some definitions can be proved but not others. Two of three teachers cannot explain what mathematical definitions meant in their minds. All three teachers knew that theorems had to be proved. No teachers indicated that definitions can be motivated.

Role of Mathematical Definitions

This section consists of two sections. In the first section, I explain why mathematical definitions are important. In the second section, I briefly explain the axiomatic system of mathematics and how mathematical definitions connect to other elements such as theorems in the system.

Why Definitions Are Important

It is important for mathematics learners to understand the role of definitions in mathematics because mathematics learners need to understand the "epistemological knowledge of the functional role of items" (Michener, 1978, p. 371) such as role of different examples, role of mathematics and role of proofs, in order to claim understanding mathematics. The necessity principle stated that "For students to learn the mathematics we intend to teach them, they must have a need for it, where 'need' refers to intellectual need." (Harel, 2008, p. 900). Similarly, for teachers to enact defining activities, they need to know why mathematical definitions are important in mathematics.

As summarized by Zaslavsky and Shir (2005), mathematical definitions played four important roles in mathematics. The first role is "introducing the objects of a theory and capturing the essence of a concept by conveying its characterizing properties" (Zaslavsky & Shir, 2005, p. 317). The process of defining is not only to assign a name to a concept, but also to precisely characterize the the concept. As explained by De Villiers (1998), "A posteriori defining is usually accomplished by selecting an appropriate subset of the total set of properties of the concept from which all the other properties can be deduced" (p. 2).

Second, mathematical definitions constitute "fundamental components for concept formation" (Zaslavsky & Shir, 2005, p. 317). Vinner (1991) pointed out there is a significant difference between "technical concept" (e.g., mathematics concept) and every day concept (e.g., house, orange, cat). Vinner argued that most time people acquired every day concepts through experience but not through definitions. For instance, people understand red color not from definition but through seeing many red objects. However, sometimes people do acquire every day concepts through definitions. For example, a person may introduce a laptop to her friend by describing it as "a portable computer which could be put on the top of lap" rather than showing several laptops to her friend. In this case, definitions support formation of concept images. Vinner argued that after the concept image was formed, "the definition becomes dispensable. It will remain inactive or even be forgotten when handling statements about the concept in consideration" (p. 69). He used the 'scaffolding metaphor' to describe how definitions supported concept formation. Namely, the scaffolding becomes useless after a building is constructed. In technical contexts, however, definitions play extremely important roles. Vinner emphasized that "They have the potential of saving you from many traps which are set by the concept image" (p. 69). For instance, without consulting mathematical definitions, people tend to think a constant sequence does not have a limit because it is not approaching closely to any number. Thus, mathematical definitions affect concept formation through generating "concept definition image" (Tall & Vinner, 1981), which counterbalances the commonly hold incorrect concept images.

Third, mathematical definitions establish "the foundation for proofs and problem solving" (Zaslavsky & Shir, 2005, p. 317). Weber (2002) argued that a proof that convinces and a proof that explains "begins with an accepted set of definitions and axioms and concludes with a proposition whose validity is unknown" (p. 14). Moore (1994) analyzed undergraduate students' difficulties in proving and found that limited understanding of definitions of concepts was a common difficulty undergraduate students expressed in solving proof related tasks. He revealed several factors that contributed to undergraduate students' low proficiency in proving and two of them were directly related to mathematical definitions. For instance, students had difficulties stating definitions which impeded their ability to use definitions in proofs. Even if students knew mathematical definitions, they did not know how to use definitions to develop proofs.

Fourth, mathematical definitions create "uniformity in the meanings of concepts, which allows us to communicate mathematical ideas more easily" (Zaslavsky & Shir, 2005, p. 317). As argued by Pimm (1993) "...in order to communicate, people must agree with one another about the meaning of words" (p. 272). According to Vinner (1991), mathematical definitions are stipulative in nature. Stipulative definition is like a new convention that people follow in the new context no matter how the word was used in the past (Rhobinson, 1962). For instance, "similarity" has specific meaning in geometry which is different from every day meaning.

Landau (2001) commented that such definitions "are imposed on the basis of expert advice" with the goal of "ease and accuracy of communication between those versed in the language of science" (p. 165). Robinson (1962) also noted that "the most obvious advantage we may hope to gain by stipulation is the removal of an ambiguity and the avoidance of an inconvenience caused by the ambiguity" (p. 66). The clarity is extremely important in the field of mathematics where "…a changed definition does affect conclusions" (Fawcett, 1938, p. 32). As argued by (Edwards & Ward, 2008), "when a term is defined by stipulation, it is to be free from connotation, that is, free from all the associations the term may have acquired in its non-technical use" (p. 224).

In addition to the four roles summarized by Zaslavsky and Shir (2005), there is another important role of mathematical definitions. As argued by Polya (1957) "The mathematicians are not concerned with the current meaning of his technical term....The mathematical definition *creates* the mathematical meaning" (p. 86), therefore, mathematical definitions create and bring a concept into live. de Villiers (1998) also distinguished two types of defining "descriptive defining" and "constructive defining". Constructive defining "takes places when a given definition of a concept is changed.... a new concept is defined '*into being'"*(p. 2). Pimm (1993) made a similar point "There is an important switch of field and ground when a definition produces the concept rather than the other way round" (p. 274). In advanced mathematics, sometimes the theory is the focus and mathematicians are searching for a concept and its definition to ensure that the new concept fits into the theory well (Weber, 2002). One famous example of defining a concept to fit into a theory is the modification of the definition of polyhedral during the proof process from zero-definitions to proof-generated definitions described in Lakatos' (1976) book. The original definition of polyhedral is based on intuition and common experience. However, in order to fit the definition of polyhedral into Euler's formula, mathematicians had to revise it and at the same time reconstruct the concept.

The Axiomatic System of Mathematics

In the textbook written for mathematics courses for preservice elementary teachers, Bennett, Burton and Nelson (2012) give the following description of a mathematical system.

> A **mathematical system** consists of *undefined terms, definitions, axioms,* and *theorems.* There must always be some words that are undefined. *Line* is an example of an undefined term in geometry. We all have an intuitive idea of what a line is, but trying to define it involves more words, such as *straight, extends indefinitely,* and has *no thickness.* These words would also have to be defined. To avoid this problem of *circularity*, certain basic words such as *point* and *line* are **undefined terms.** These words are then used in **definitions** to define other words. Similarly, there must always be some statements, called **axioms**, that we assume to be true and do not try to prove. Finally, axioms, definitions, and undefined terms are used together with deductive reasoning to prove statements called **theorems**. (p. 564)

A few important things are noticed in this description. First, “undefined terms”, “definitions”, “axioms” and “theorems” have special meanings and functions in a mathematical system. As argued by Dormolen and Zaslavsky (2003), “one must verify if it [a definition] fits in and is part of a deductive system” (p. 96). Terms such as mathematical definitions are associated with a mathematical system (Van Dormolen & Zaslavsky, 2003). For instance, all statements which give necessary and sufficient conditions of a mathematical concept are logically equivalent. Any of them could be chosen arbitrarily as a mathematical definition in a mathematical system and others become theorems in the same system. Therefore, there is no way to say whether a statement is a mathematical definition or a theorem without putting it into a mathematical system.

In other words, there is no absolute mathematical definition or theorem. This distinguishes a mathematical definition from a every day definition because every day definition does not depend on system. For instance, in daily life, both statements “a square is a polygon with four equal sides and four equal angles” and “a square is a parallelogram with diagonals that are equal and perpendicular” could be definitions of a square simultaneously because they both give necessary and sufficient conditions of the concept. But they cannot be mathematical definitions of a square simultaneously in one particular mathematical system. Once one of them is selected as a mathematical definition of a square, the other becomes a theorem.

Second, axioms and theorems share similarities that they both have true values. They either are assumed to be true (axioms), or proved to be true (theorems). Mathematical definitions and undefined terms are similar because they are both conventional knowledge and they do not have true values. There seems to be a standard we use to judge whether a definition is “correct”, but this standard is completely by convention. For instance, if I call a triangle a square, it is conventionally wrong because it is inconsistent with how others use the term triangle, but there is nothing wrong other than contracting to the convention. However, for the statement “For any natural numbers, a, b, and c, $(a+b)c=ac+bc$”, there is a true value of the statement which is not determined by convention. Even if all people on the earth reach an agreement that it is better for the statement to be wrong, the statement is still true.

Third, Bennett, Burton and Nelson’s (2012) description of a mathematical system suggests that mathematical definitions are the foundation of deductive reasoning and therefore, no deductive reasoning should be involved in any mathematical definitions. Otherwise, it would be a circular relation: mathematical proofs build on mathematical definitions and mathematical definitions need to be proved. This restriction suggests that not every defining statement in mathematics qualifies to be a mathematical definition. In other words, not every definition involving mathematics is a mathematical definition. For instance, on the surface level, the statement “distributive property is ‘For any natural numbers, a, b, and c, $(a+b)c=ac+bc$’” defines the distributive property. However, part of this statement “For any natural numbers, a, b, and c, $(a+b)c=ac+bc$” needs a mathematical proof, therefore the original statement is not a mathematical definition but only a definition in mathematics. Pimm (1993) warned people to use the word “define” more carefully. He argued that:

> Although it is common to say we 'define' $f(x)$ to be ..., in fact we are merely specifying which function we are referring to, selecting the focus of our attention and then assigning it a chosen name, one rich with past associations. The term' definition' might fruitfully be reserved for concepts. (p. 275)

Similarly here, a general property $(a+b)c=ac+bc$ was found and proved to be true for any natural numbers a, b, and c. Then a name is given to this general property. This process is more like a proving and then naming process rather than a defining process where common features of a class of objects are abstracted (e.g., “four equal sides and four equal angles” is abstracted based on several square-shaped objects) and no proof is involved.

Notice that in this example, I asserted that the distributive property needed to be proved to be true for natural numbers. This seems to contradict to that in advanced mathematics the distributive property is an axiom and no proof is needed. It is true that when defining ring as a set of objects with a binary operation that satisfies certain axioms, the distributive property is one of the axioms. However, in the natural number system which is defined through Peano axioms, the distributive property is not one of the axioms but a deduced result. Similar to the case of distributive property, the statement naming a theorem such as “The Pythagorean Theorem means that in a right triangle, the square of the hypotenuse (the side opposite the right angle) is equal to the sum of the squares of the other two sides" is not a mathematical definition because proof is involved.

Summary

In this section, I briefly summarize the literature and explain my stance, so the reader may understand how I will interpret and evaluate my data. In summary, when discussing what is a good mathematical definition, two main aspects are considered: mathematical characteristics and didactic considerations.

Mathematical characteristics of a definition refer to "some logical principles that need be met when defining any mathematical concept" (Winicki-Landman & Leikin, 2000, p. 17), whereas didactic considerations refer to pedagogical factors that teachers consider in their teaching practice (Winicki-Landman & Leikin, 2000). From the mathematical point of view, a logical equivalence class of defining statements could be established and every statement belonging to this class could be chosen arbitrarily as a definition (Winicki-Landman & Leikin, 2000). However, different choices of a definition may have an impact on students' learning of the relevant concepts, as argued by Zazkis and Leikin (2008), "Each didactic decision is based on the relationship between the nature of mathematics and the nature of the learning process" (p. 133).

Because the focus of this dissertation is to investigate preservice elementary teachers' thinking about mathematical definitions from a disciplinary perspective rather than from a teaching perspective, the *necessary features* I associate with mathematical definitions are mathematical characteristics identified in the literature. Specifically, I believe that the mathematical definition of a concept should provide a necessary and sufficient conditions of the concept and be consistent with the way the mathematical community defines the concept (Winicki-Landman & Leikin, 2000). It should include the name of the concept being defined. In addition, a mathematical definition needs to be precise, namely, "all the terms employed in the definition should have been previously defined, unless they are one of the few undefined terms assumed as a starting point in the axiomatic system one is working with" (Borasi, 1992, p. 17). Also, a mathematical definition needs to be unambiguous, namely, its meaning "should be uniquely interpreted" (Zaslavsky & Shir, 2005, p. 319). Furthermore, mathematical definitions should be noncontradicting, namely, "all properties stated in a definition should be able to coexist" (Borasi, 1992, p. 17) and noncircular, namely, "the definition should not use the term it is trying to define" (Borasi, 1992, p. 18). There are other mathematical characteristics mathematicians and mathematics educators have imposed as necessary features of mathematical definitions such as existence of an instance (Van Dormolen & Zaslavsky, 2003). However, these characteristics are applicable in advanced mathematics and are irrelevant to mathematics in elementary schools. Hence, they are not considered as necessary features in this study.

I regard other features mentioned in the literature review such as minimality, intuitiveness, ease to apply to problem solving (Winicki-Landman & Leikin, 2000) as *preferred features* of mathematical definitions rather than necessary features. These features could be taken as necessary features from the perspective of teaching, but because the focus of this dissertation is on preservice elementary teachers' thinking about mathematical definitions from a disciplinary perspective, I do not take these features as necessary features in this study.

Mathematical definitions are associated with one mathematical system (Van Dormolen & Zaslavsky, 2003). The same mathematical statement could be a mathematical definition in one mathematical system but could be a theorem in another system. It is meaningless to discuss mathematical definitions without specifying the system. Because mathematical definitions are the foundation of deductive reasoning, no deductive reasoning should be involved in any mathematical definitions. This suggests that not every defining statement in mathematics qualifies to be a mathematical definition. For instance, the statement which defines the Pythagorean Theorem is not a mathematical definition.

CHAPTER 3: METHODS

In this dissertation, I studied preservice elementary teachers' conceptions about mathematical definitions. Twenty four preservice elementary teachers who were in the senior year in their teacher education programs volunteered to participate in this study. A 60 - 90 minutes semi-structured interview was conducted with each preservice elementary teacher. Before coming to the interview, the preservice elementary teachers were asked to complete a short survey about their academic background. This before-interview survey also served the purpose to screen preservice elementary teachers to ensure that I recruited equal numbers of mathematics teaching majors and participants who were not mathematics teaching majors. Details of the selection of the participants are explained later in this chapter. During the interviews, preservice elementary teachers were asked to answer general questions about mathematical definitions and to solve tasks about mathematical definitions. I analyzed the data mainly in two ways. First, I examined the data based on the extent of correctness of preservice elementary teachers' responses to individual interview questions. Second, I examined the data by looking for themes through the theoretical perspectives described in the literature review. I will report distribution of preservice elementary teachers as a group rather than reporting multiple cases studies of individual participant.

This chapter is divided into five sections. I begin the chapter by describing the research site and participants. I then describe the instruments and data collection procedure. In the last section, I describe the data analysis methods including a description of how I report results in the following chapters.

Research Site

This study was conducted in a research university in the Midwest. At the time of the study, the university had approximately 47,000 students. Each year, about 200 undergraduates are enrolled in the Elementary Teacher Preparation Program. The Elementary Teacher Preparation Program at this university is a five-year program. Preservice elementary teachers complete the majority of their course work in the first four years. The fifth year is an internship year in which preservice elementary teachers take two graduate level courses while they are teaching in elementary schools.

I selected this university as the research site mainly because it offers the context for me to compare two groups of preservice elementary teachers who received different amount of mathematical training. Starting from Fall 2009, the state where the research site is located has mandated that preservice elementary mathematics teachers have to choose an elementary teaching major from Language Arts, Social Studies, Integrated Science, and Mathematics. Each year at the research site, approximately 25 preservice elementary teachers choose Mathematics as their teaching major. Similar number of preservice elementary teachers choose Social Studies or Integrated Science as their teaching majors. Language Arts is the teaching major most preservice elementary teachers choose. Preservice elementary teachers who choose the subject area as their teaching major have to take a few advanced courses in the corresponding subject area. At this university a series of mathematics and mathematics for teachers' courses are designed for preservice elementary teachers who choose mathematics as their teaching majors.

All preservice elementary teachers need to take two mathematics courses, one focusing on number and operations and the other on geometry and measurement for a total of 6 credits. Those who choose mathematics as their teaching major also have to take Calculus 1 and 2, Foundations of Higher Mathematics, Algebra for K-8 Teachers, Functions and Calculus for K-8 teachers, Higher Geometry, History of Mathematics and one of statistic courses for a total of 26 additional credits.

Participants

All 24 participants are senior students. I choose senior students because they have completed or are completing their mathematics courses at the time when they participated in my study. Therefore, after going through all the mathematical training, their understanding as revealed in my study reflects their best knowledge about mathematical definitions as a result of studying in their teacher education programs.

Among the 24 participants, twelve have their teaching major as Mathematics and the other twelve have their teaching majors spread out among the other three subject areas. I made the decision on the number of participants I want to recruit in three non-mathematics subjects according to the corresponding ratio in the population in the Elementary Teacher Preparation Program. The number of participants in my sample in each three subject area also depends on the availability of the participants who were willing to participate in my study. However, when deciding on the number of participants whose teaching major is mathematics, I did not follow the ratio in the population. I purposefully chose more Mathematics teaching majors (12 in total) because if I followed the ratio in the population, I would get a really small sample (about three participants) whose teaching major is Mathematics. This small size would prevent me from drawing any meaningful conclusions when comparing mathematics teaching majors to non-mathematics teaching majors. Table 1, describes the gender and teaching major of the participants. One participant is a Chinese female who is a mathematics teaching major; all other participants are American. I recruited the participants through sending emails, making classroom announcements and posting advertisements in the hallway. Generally, I selected the participants according to the order in which they indicated their interests to participate in my research.

However, I also considered their teaching majors to make sure I recruited enough participants to be approximately proportional to number of each teaching major in this university's population.

Table 1. Gender and Teaching Major of the Participants

Teaching Major	Female	Male
Mathematics	9	3
Social Studies	2	0
Integrated Science	2	1
Language Arts	7	0

Instruments

I used three instruments: a short survey instrument used before the subjects came to the interview (see Appendix A), a written handout with seven tasks for preservice elementary teachers to solve (see Appendix B), and an interview protocol associated with the written handout, but with some additional questions (see Appendix C). The before-interview survey asks participants about their academic background information such as their teaching major, the mathematics and mathematics education courses they took each year of their study in the teacher education program. The survey also ensures that expected numbers of participants were recruited.

The primary source of data came from semi-structured interviews. The protocol used in the interview includes 14 questions. Many of them include sub-questions. Some questions were adapted from existing research studies on either mathematical definitions or proofs (e.g., Knuth, 2002a, 2002b; Edwards, 1997; Dickerson, 2008; Zaslavsky & Shir, 2005). Other questions were created by me.

Questions 1-4, 12, 13 and 14 are general questions that focus on preservice elementary teachers' general conceptions about the nature and roles of mathematical definitions or their opportunities to learn about mathematical definitions during their teacher education programs. In particular, the participants were asked to comment about their familiarity with mathematical definitions, what is a mathematical definition in their minds, what needs to be in a statement for it to be a mathematical definition, how mathematical definitions are different from descriptions or properties, what criteria they use to evaluate good or bad mathematical definitions, how important are mathematical definitions in mathematics, why mathematics needs mathematical definitions, and opportunities they have had at their university to explicitly learn about mathematical definitions as a meta level concept.

Question 5 asks preservice elementary teachers to write mathematical definitions and offer explanations about factors that affect their output. Preservice elementary teachers' understanding of mathematical definitions was demonstrated through the way in which and what they chose to include in their written definitions. Preservice elementary teachers' comments on their thinking behind the scene also offered insights into their consideration of what qualified as a valid or good mathematical definition. Question 5 includes two components: preservice elementary teachers' free choices of any mathematical definitions they would like to write and two specific mathematical definitions (area of a rectangle and fraction multiplication) each preservice elementary teacher had to write.

Questions 7, 10 and 11 ask preservice elementary teachers to evaluate whether specific statements are valid or good mathematical definitions and provide justifications for their judgments. Question 7 focuses on how preservice elementary teachers distinguish mathematical definitions from non-mathematical definitional statements. Statements included in Question 7 include vague descriptions (e.g., 7c), deduced results (e.g., 7b, 7e, 7h), symbolic ways to define a concept (e.g., 7o) etc. Because mathematical definitions are arbitrary and are associated with a mathematical system, namely, the same mathematical definition in one mathematical system could become a theorem in another system, for advanced mathematical learners, the answers to some statements in Question 7 could be uncertain depending on which mathematical system is under discussion. However, Question 7 not only asks preservice elementary teachers' decision on if the statement is a mathematical definition but also asks the reasons for their decisions. If preservice elementary teachers were aware that mathematical definitions were associated with a mathematical system, they would articulate uncertainty and this information would be captured during the interview.

The purpose of Question 10 and Question 11 is to elicit how preservice elementary teachers distinguished good or bad mathematical definitions but the design of these two questions, asking preservice elementary teachers to first circle valid mathematical definitions and then rate them, offers insights into how preservice elementary teachers distinguished mathematical definitions vs. non-definitional statements as well. In addition, while most statements in Question 7 are not valid mathematical definitions, most statements in Question 10 and Question 11 could be regarded as valid mathematical definitions at the elementary level. This difference between the nature of statements in Question 7 and Question 10 and Question 11 offers different ways for preservice elementary teachers to evaluate mathematical definitions.

Preservice elementary teachers were asked to both generate definitions (Question 5) and evaluate whether given statements were valid or good mathematical definitions (Questions 7, 10 and 11) because Martin and Harel (1989) argued that preservice elementary teachers may act differently when producing and evaluating statements. Therefore, collecting preservice elementary teachers' responses under both production and evaluation modes provides a more complete picture of their understanding of mathematical definitions.

The concepts that I chose to include in Questions 5, 10 and 11 include area of a rectangle, fraction multiplication, an even number and a square. I purposefully chose these concepts based on the following criteria: (a) each concept is simple and familiar to preservice elementary teachers thus the focus is on the notion of definition rather than on the defined concept; (b) each concept is central to elementary school mathematics; (c) there are different ways to define or describe each concept, (d) definitions can be made non-equivalent and thus issues related to definitions discussed in Chapter 2 such as “precise” and “sufficient and necessary condition” may arise in participants’ responses, (e) the features of the concept may uncover different issues related to mathematical definitions. For instance, for the geometry concept, a square, the issues could be minimality. However, definitions of even numbers may evoke other issues such as revealing the essence of the concept and noncircularity.

Questions 7, 8, and 9 intend to elicit preservice elementary teachers’ responses to answer research question 1c, namely, preservice elementary teachers’ ability to distinguish definitions and other meta level mathematical concepts. Question 7 and 8 include statements of specific concepts for participants to comment on; a comparison of preservice elementary teachers’ responses to Question 7 and 8 revealed how preservice elementary teachers saw connections between mathematical definitions and mathematical proofs in an implicit way. Question 10 asks participants to judge a given list of general statements which targets the relationships among mathematical definitions, mathematical proofs, theorems, and counterexamples. This question intends to elicit preservice elementary teachers’ explicit understanding of the above relationships.

In summary, Table 2 establishes the correspondence between research questions and question numbers in the interview protocol (see Appendix C). Because each task is associated with an interview question, I only give the correspondence between interview questions and research questions.

Procedures

After preservice teachers indicated their willingness to participate in the research, I

Table 2. Correspondence Between Research Questions and Interview Questions in the Protocol

Research question	Question number in Interview Protocol
1. What are preservice elementary teachers' conceptions about the nature of mathematical definitions in mathematics?	All questions answering 1a-1c listed below
1a. What constitutes a mathematical definition for preservice elementary teachers?	1-7, 10, 11, 14
1b. Among the valid definitions, what types of mathematical definitions do preservice elementary teachers prefer?	2, 4, 5, 10, 11
1c. To what extent do preservice elementary teachers distinguish mathematical definitions from other meta level mathematical concepts such as mathematical proofs?	7, 8, 9
2. What are preservice elementary teachers' conceptions about the role of mathematical definitions in mathematics?	12
3. What opportunities do preservice elementary teachers have to explicitly learn about mathematical definitions during their teacher education programs?	13

invited them for a 60 - 90 minute interview. I audiotaped each interview and transcribed the conversation between the participant and the researcher afterwards. I also took field notes for two reasons. First the field notes captured participants' non-verbal actions which would be missed by the audiotape. In addition, the field notes kept a record of the key aspects relevant to mathematical definitions mentioned by preservice elementary teachers in the interview, which I could use to probe their thinking later. For instance, when a preservice elementary teacher indicated that an example should be included in the mathematical definition, I probed what kind of mathematical experience made she or he think in that way. Additionally, participants' written responses including the handouts were collected for analysis.

Data Analysis

This section divides into two subsections. In the first subsection, I describe general methods I used to analyze my data. In the second subsection, I explain specific issues and coding decisions I made in data analysis. For instance, I explain why I decided not to analyze responses to Question 6 in the Interview Protocol. I also describe how I plan to report results in the following three chapters.

General Coding and Data Analysis Methods

In the rest of this dissertation, I use PSTs, PSTs – M and PSTs – N to represent all preservice teachers, preservice teachers whose teaching major is mathematics and preservice teachers whose teaching major is not mathematics, respectively. To ease the reading, I used M1 to M12 to name PSTs – M who participated in my study and N1 – N12 to name PSTs - N. Data analysis was facilitated by using NVivo 10 (QSR International, 2012). Mainly two ways were used to analyze data depending on the nature of the interview questions and responses: analyses of responses to questions that were either right or wrong and analyses based on themes.

Analysis of PSTs' responses based on the extent of correctness. The first type of analysis is based on the extent of correctness of PSTs' responses. For Questions 7, 8, and 9, almost all statements have either a right or wrong answer. For these questions I counted the number of PSTs, PSTs - M and PSTs - N who gave the correct response to each statement and reported a distribution. However, the validity of two statements 8a and 8i is uncertain. As explained before, mathematical definitions and mathematical proofs are associated with a mathematical system. Whether a statement is a mathematical definition or needs a proof depends on the system the statement is in. Statement 8a states that "A rhombus is a special type of quadrilateral". Whether 8a needs a mathematical proof depends on how PSTs define a rhombus. If PSTs define a rhombus as a polygon having 4 equal sides. Then a proof is needed to show that a rhombus is a special type of quadrilateral because in PSTs' definition of a rhombus, quadrilateral is not specified even though it is apparent because of 4 sides. If PSTs define a rhombus as a quadrilateral having all sides equal, then the statement does not need a proof because it basically restates PSTs' definition for a rhombus. Because of this uncertainty, I cannot analyze Question 8a in terms of its correctness. Question 8i includes two parts. The first part (A Platonic solid is a regular, convex polyhedron with congruent faces of regular polygons and the same number of faces meeting at each vertex) is a mathematical definition of a Platonic solid and thus no proof is needed. The second part (There are five platonic solids in total. They are Tetrahedron, Cube, Octahedron, Dodecahedron and Icosahedron) is an additional property about Platonic solids, so a proof is needed to show these properties are true. Because there is no single answer for 8i as a whole, I will not report PSTs' responses to 8i in terms of its correctness. Similarly, Question 7 asks whether the given statement is a mathematical definition, there is no single answer for 7i as well. In the next section, I explain another way to analyze data based on themes. Even though I cannot report results of these two statements based on the extent of correctness, I could still examine them qualitatively based on themes.

Analysis of PSTs' responses based on themes. For questions which are opinion based (e.g., From your perspective, what is a mathematical definition?), I analyzed the data by examining themes through theoretical perspectives reported in the literature. I first made themes guided by research questions and used these themes to organize my codes. For instance, one theme of interest is the necessary features PSTs associated with mathematical definitions. For these themes, I began my coding with a set of external codes that were extracted from the research literature. When ideas emerged which cannot be coded by the external codes, I made new codes to capture this information. For instance, some PSTs indicated that examples should be included as part of mathematical definitions, so I made an "example" code to capture this type of responses. After these new codes were proposed, all transcripts were recoded incorporating these new codes.

After finishing coding all data, additional themes were identified. These themes were issues deemed important for the study and offered the insights into answering research questions from new perspectives. For instance, when examining data, I found that some PSTs thought describing procedures (e.g., describing how to solve a linear equation) was a mathematical definition. Some PSTs thought naming a theorem was a mathematical definition (e.g., naming the Pythagorean Theorem). Some PSTs thought naming a property was a mathematical definition (e.g., naming the distributive property). Because all these three codes were relevant to naming, I made a new theme “Name a property/procedure/theorem” to include all three cases. This new theme was one of PSTs’ misconceptions about mathematical definitions.

I report themes about preservice elementary teachers’ conceptions of mathematical definitions as a group rather than reporting multiple cases studies of individual participant. Though I also examined (in)consistency of each individual’s responses across each interview, I only report such (in)consistency for PSTs as a group. For instance, when examining PSTs’ understanding of the relationship between mathematical definitions and mathematical proofs, I found that many PSTs’ implicit and explicit understanding of this relationship was inconsistent. I then report the number of PSTs who demonstrated consistent vs. inconsistent understanding. Reporting frequency counts allows me to compare the importance of themes and also the possible different conceptions of PSTs - M and PSTs - N. When reporting data, I illustrated each theme by providing representative quotes from participants.

Within this type of thematic data analysis, two slightly different approaches were used. Sometimes the analysis was only conducted on responses to one question. For instance, I analyzed PSTs’ responses to Question 4 (Is it possible for a concept to have more than one mathematical definition?) and examined the reasons PSTs used to support their arguments. Other times the analysis was conducted on PSTs' responses to more than one question and themes were identified across multiple questions. For instance, one aspect to answer research question 1a was to analyze the necessary features PSTs associated with mathematical definitions. In order to describe the necessary features, I looked at PSTs' responses to multiple questions as a group including their responses to general interview questions (e.g., Question 1: From your perspective, what is a mathematical definition?) and their responses to questions involving specific mathematical statements (e.g., Question 7: You are given a list of statements; please circle those you think are mathematical definitions). In this case, no analysis was conducted on individual question because the group of questions intends to elicit the same information in multiple ways; therefore analyzing questions separately only gives partial results and is meaningless.

Because this dissertation involves multiple research questions and most of them are answered from multiple aspects, I will discuss relevant themes or codes to answer each research question in individual chapter. Results are reported in the order in which research questions are listed. Chapter 4 reports findings relevant to research question 1a. Chapter 5 reports results to answer research question 1b. Chapter 6 reports findings to answer research questions 1c and 2 as both research questions are relevant to PSTs' understanding of the position of mathematical definitions in mathematics. In each chapter, I compare the understanding of mathematical definitions of PSTs - M and PSTs - N.

Specific Coding Decisions

Decision on how to code a feature as a necessary or a preferred feature. In this section, I will explain how I differentiate data to answer research questions 1a and 1b. Research question 1a and 1b are related. Question 1a focuses on what constitutes a mathematical definition for preservice elementary teachers. To answer this research question, I identified both necessary features PSTs associated with mathematical definitions and their misconceptions relevant to mathematical definitions. Necessary features are features PSTs used to distinguish mathematical definitions from non-definitional statements. Research question 1b focuses on PSTs' preferred features of mathematical definitions, namely, how PSTs distinguished good or bad among valid mathematical definitions. Next I will explain how I determined whether a feature of mathematical definition was coded as a necessary feature (therefore reported in Chapter 4), a preferred feature (reported Chapter 5) or both (reported in both chapters).

First, the interview questions helped me identify which one of the two features (necessary or preferred) PSTs intended to convey. Usually I coded PSTs' responses to Question 2 (What needs to be in a statement for it to be a mathematical definition?) as necessary features and PSTs' responses to Question 4 (Are some definitions better than others? What makes them better?) as preferred features unless the context indicated the other way around. For instance, when responding to Question 4, M11 said that "Whatever definition is more concise." Here the interview question hinted that the feature "concise" in M11's responses was his preferred feature.

Second, the design of the interview questions, Questions 7, 10 and 11, allows me to see which statements PSTs regarded as valid mathematical definitions and what criteria they applied to evaluate good or bad definitions. Question 7 gives PSTs a list of statements and asks them to circle the ones they regard as mathematical definitions. This question intends to elicit necessary features because PSTs distinguish mathematical definitions from non-definitional statements. Question 10 gives PSTs a list of statements related to an even number and asks PSTs to circle those statements that they think are mathematical definitions and to rate those mathematical definitions by using 4 point scale. Question 11 is similar to Question 10 but is related to a different concept, a square. If PSTs mentioned that one statement did not possess one particular feature (e.g., concise) and therefore was not a mathematical definition, then I coded this feature as a necessary feature. In contrast, if PSTs indicated that the statement did not possess the feature but he or she still thought that the statement was a valid mathematical definition, then I coded it as a preferred feature instead of a necessary feature. For example, when N12 responded to statement 11l, she said:

> [Statement 11] l is not concise at all, and I don't know what the Euclidean Plane is. I kind of can use the context to figure it out, but it could be a lot of more concise. It is not clear and it is not concise, so I give it a 1.

Here, even if N12 pointed out that the statement 11l was not concise, she still gave it 1 point, which suggested that from her perspective, 11l was a valid mathematical definition of a square. Thus, I coded that "concise" was one of N12's preferred features of mathematical definitions.

There were times when PSTs used one particular feature to distinguish a definition verse a non-definitional statement, and also used the same feature to distinguish a good and bad definition. When this happened, the feature was coded as both necessary and preferred. For instance, when rating statement 11j (If a parallelogram is both a rectangle and a rhombus, then it is a square), M7 said:

> [Statement 11] j, I said it is a definition but I gave it 1 point... because I feel like you are thinking about two many different little pieces because you are thinking about all things make a rectangle and all the things make a rhombus and all of that makes a square. It is too much.

Here, M7 expressed the idea that using rectangle and rhombus together to define a square was not straightforward, namely, there was a need to unpack the properties of rectangles and rhombuses in order to reach properties of squares. Because M7 still gave the statement 1 point, which suggested that she accepted the statement as a mathematical definition, I coded “straightforward” as M7’s preferred feature for mathematical definitions. Meanwhile, when she responded to 11d (A parallelogram with diagonals that are equal, and perpendicular is called a square), she said:

> [Statement 11] d I thought it maybe is a definition but I crossed it out....Like it kinds of beat around the bush a little bit. It gets there that is a square, but it kind of does a really long ramp to get there. It could just [have] gone right to it.

Here, she rejected a statement as a mathematical definition because she thought it was not straightforward. Thus, I coded “straightforward” as a necessary feature for M7. Therefore, for M7, “straightforward” was coded as both necessary and preferred features. This coding decision was reasonable because adjectives such as “straightforward” or “concise” represented a continuum instead of a dichotomy. PSTs could think that mathematical definitions should be concise or straightforward (necessary feature), but they can still distinguish some definitions to be more concise or straightforward than others (preferred feature). Sometimes, PSTs used the word such as “very” or “most” to differentiate the extent.

Research Question 3. Data I collected to answer research question 3 which asks PSTs to recall the opportunities they had to learn about mathematical definitions as a meta level concept during their teacher education programs were based on PSTs’ memory about their past schooling in four years and data may not be an accurate description of what PSTs actually experienced. Thus, I do not conduct a detailed analysis as I did to answer other research questions. Instead, I briefly discuss findings relevant to this research question in Chapter 7 because my data analysis still reveals some interesting patterns, which are worth mentioning.

Interview Question 6. Question 6 intends to elicit preservice elementary teachers' understanding of the arbitrariness of mathematical definitions but was not successful in eliciting useful responses. Most PSTs only mentioned which one of the given two statements about trapezoid was the definition that they remembered from their schooling without discussing anything relevant to the arbitrariness of mathematical definitions. Therefore I decided to delete this question in my data analysis.

CHAPTER 4: PRESERVICE ELEMENTARY TEACHERS' UNDERSTANDING OF WHAT CONSTITUTES A MATHEMATICAL DEFINITION

This chapter aims to answer research question 1a, namely "what constitutes a mathematical definition for preservice elementary teachers"? I organize this chapter into two sections. In the first section, I discuss PSTs' understanding of specific mathematical definitions. I organize this section into two subsections. In the first subsection, I discuss PSTs' performance in generating mathematical definitions. This analysis is based on PSTs' responses to Question 5. In the second subsection, I present PSTs' performance in evaluating mathematical definitions, namely, how PSTs distinguished mathematical definitions from nondefinitional statements. This analysis is based on PSTs' responses to Question 7. Though analyses conducted in these two subsections focus on PSTs' understanding of specific mathematical definitions (e.g., fraction multiplication, area of a rectangle), PSTs' understanding of mathematical definitions as a meta level is revealed as well.

In the second section, I present PSTs' conceptions of mathematical definitions as a meta level concept. I divide this section into four subsections. First, I examine PSTs' first reactions to mathematical definition. This analysis is based on PSTs' responses to Question 1 when they were asked their familiarity to mathematical definitions and what is a mathematical definition in their minds. Second, I examine how PSTs thought about equivalent definitions. This analysis is based on PSTs' responses to Question 4a when PSTs were asked the number of mathematical definitions associated with one mathematical concept. Third, I discuss PSTs' overall understanding of mathematical definitions as a meta level concept. This analysis is based on PSTs' responses to questions 1-3, 5, 7, 10, and 11. Last, I examine PSTs' responses to questions 1-3 as a group. The purpose of this analysis is to describe PSTs' general understanding of mathematical definitions before scaffolding (i.e., specific mathematical statements which offer examples for PSTs to consider) is given in questions 7, 10 and 11. A comparison between results in subsection 3 and 4 reveals the differences on how PSTs perceived mathematical definitions with and without scaffolding.

PSTs' Understanding of Specific Mathematical Definitions

PSTs' Performance in Generating Mathematical Definitions

In this section, I present an analysis of PSTs' responses to Question 5. For this question, I first asked PSTs to choose a concept they felt comfortable with and then to write a definition of that concept. Each PST was asked to write 2-3 definitions. Later, I asked PSTs to write two specific mathematical definitions: a definition for the area of rectangle and a definition for fraction multiplication. I first discuss PSTs' free choice written definitions followed with an analysis of their written responses to two specific concepts about which I asked them to write definitions.

PSTs' performance in generating free choice mathematical definitions. Overall, PSTs generated a total of 50 definitions. Seven PSTs stated that it had been difficult for them to come up with a concept for which they could write a definition. N1 and N7 commented that they had never been asked to write a definition in the past. M4 asked what I meant by a concept. A list of concepts PSTs chose to write a definition for is given in Appendix D. Table 3 gives a summary of the number of the concepts PSTs chose to write a definition for in each content area. The concepts PSTs chose to write about show a good deal of variation. Examples of Geometry & Measurement concepts include squares, circles, parallel lines, platonic solids, congruent, the Pythagorean theorem, length and area. Examples of Number and Operations concepts include negative numbers, even numbers, addition, multiplication and division. Examples of Algebraic concepts include function, equation, expression, binary operation, exponent, and the distributive property. Other concepts include sequence, limit, geometry and algebra. Table 3 indicates that PSTs wrote more definitions in Geometry and Measurement than any other content areas. PSTs - N wrote more definitions of Geometry and Measurement concepts than other content areas, whereas PSTs - M wrote more definitions of Algebraic concepts than other content areas. In addition to writing about well-defined mathematical concepts such as squares, two PSTs chose to write about two general mathematical concepts, geometry and algebra, which do not have a universal definition in the community of mathematicians.

Table 3. Number of Mathematical Definitions Generated by PSTs in Each Content Area

Content area	Total	PSTs - M	PSTs - N
Geometry and measurement	19	7	12
Number and operations	11	5	6
Algebra	16	11	5
Others	4	3	1

Each definition was then coded by using the frameworks proposed by Leikin & Zazkis (2010) into three categories: appropriate rigorous examples of definitions, appropriate but not rigorous examples of definitions, and inappropriate examples. According to Leikin & Zazkis, appropriate rigorous examples of definitions include "necessary and sufficient conditions of the defined concept as well as accurate mathematical terminology and symbols, and are usually minimal" (p. 457). Appropriate but not rigorous examples of definitions usually "omit some constraint or use imprecise terminology because of a lack of attentiveness on the part of the PMTs or a lack of rigor in the mathematical language in the usual mathematics classroom" (p. 458). Inappropriate examples lack "either necessary or sufficient conditions, so that they represent mostly specific instances of the concepts" (p. 459). Slightly different from Leikin & Zazkis's definition of inappropriate examples, the inappropriate examples PSTs generated in this study also include elements which should not be in definitions such as examples or pictures. The following three tables 4, 5, 6 give the examples of definitions coded into each category.

The coding not only considered PSTs' written responses but also their verbal explanations of their thinking when they generated their definitions. For instance, M8 wrote a definition "Expression - a mathematical phrase that can include numerals and variables" First glance indicates that operation signs are missing in this definition, but during the conversation, M8 gave an example 5x+5 and tried to describe it as "a number, a variable but also a like add it together -". She indicated that she knew she should have operation signs in the definition, but she did not have the right terminology. Therefore, even though her written response missed an important component of the definition of expression and should be coded as inappropriate examples, her verbal explanation indicates that she understood the concept; therefore I coded this case as appropriate but not rigorous examples of definitions.

Table 4. Appropriate Rigorous Examples of Mathematical Definitions Generated by PSTs

Content area	Example	PST
Geometry and measurement	Parallel lines are any lines in a plane that do not intersect.	M3
Number and operations	Negative number, a number that is less than 0.	M1
Algebra	NA	
Others	NA	

Table 5. Appropriate But Not Rigorous Examples of Mathematical Definitions Generated by PSTs

Content area	Example	PST
Geometry and measurement	A circle is a geometric figure that all points are of equal distance from the center point.	N8
Number and operations	Multiplication - used to count a number of objects in a group; can be found by placing objects into equal groups and multiplying the total objects in a group by the total number of groups.	N5
Algebra	Expression - a mathematical phrase that can include numerals and variables	M8
Others	NA	

Table 6. Inappropriate Examples of Mathematical Definitions Generated by PSTs

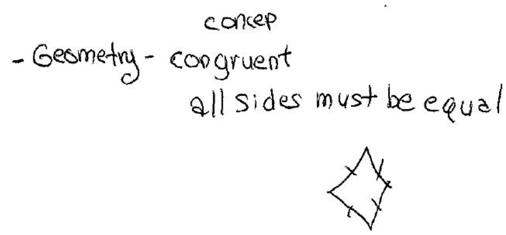
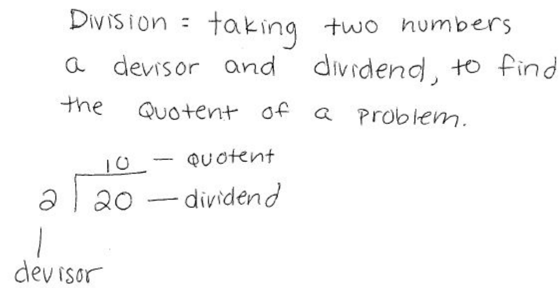
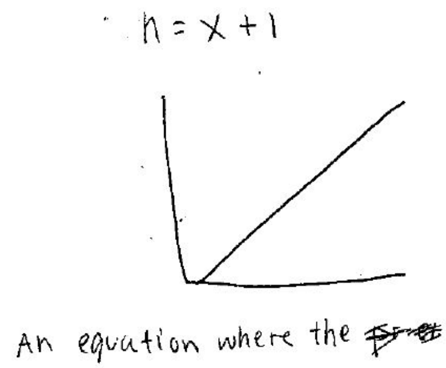
<table>
<tr><th>Content area</th><th>Example</th><th>PST</th></tr>
<tr><td>Geometry and measurement</td><td>
</td><td>N10</td></tr>
<tr><td>Number and operations</td><td>
</td><td>N2</td></tr>
<tr><td>Algebra</td><td>She tried to write a definition of linear equation and provided the following picture:

</td><td>N11</td></tr>
<tr><td>Others</td><td>Limit - the bound of a given expression</td><td>M10</td></tr>
</table>

Table 7. PSTs' Performance in Writing Mathematical Definitions

Category	Total	PSTs - M	PSTs - N
Appropriate and rigorous	7	6	1
Appropriate but not rigorous	13	8	5
Inappropriate	30	13	17

Overall, Table 7 shows that about 60% of definitions generated by PSTs were inappropriate. Only 15% of them were appropriate and rigorous. PSTs - M performed better than PSTs - N. Among all 27 examples of definitions generated by PSTs - M, six of them were appropriate and rigorous, which accounted 20% of the total number of definitions generated by

PSTs - M. But among all 23 examples of definitions generated by PSTs - N, only one of them was appropriate and rigorous. More than half of definitions generated by PSTs - M were appropriate (either rigorous or not rigorous) while about 25% of definitions generated by PSTs - N were appropriate. PSTs - N even made mistakes in generating definitions for very basic concepts such as a square.

Next, I performed a further examination of mathematical definitions produced by PSTs across different content areas. I only looked at three main content areas: Geometry and Measurement, Number and Operations and Algebra. The “Others” category was too small to report meaningful results. Also, general concepts such as geometry had no universal and precise mathematical definitions agreed upon by all mathematicians, so I did not examine that data. The results indicate that PSTs performed better in writing definitions of Geometry and Measurement concepts compared to Number and Operations and Algebraic concepts. For Geometry and Measurement concepts, about two thirds of definitions generated were appropriate (either rigorous or not rigorous), but for both concepts in Number and Operations and Algebra, roughly one third of definitions generated were appropriate. No PSTs were able to generate an appropriate and rigorous definition for any of the algebraic concepts. In all content areas, PSTs - M performed better than PSTs - N. All definitions in Geometry and Measurement generated by PSTs - M were appropriate but most definitions generated by PSTs - N in the same area were inappropriate. While roughly one half of definitions generated by PSTs - M in Algebra were appropriate, all definitions generated by PSTs - N in Algebra were inappropriate. For concepts in Number and Operations, PSTs - M performed only slightly better than PSTs - N.

PSTs’ performance in writing a definition for the area of rectangle. Overall, PSTs wrote four kinds of statements as their definitions for the area of rectangle: (a) the amount of

Table 8. Distribution of Definitions Generated by PSTs Based on Appropriateness in Three Content Areas

Content area	Category	Total	PSTs - M	PSTs - N
Geometry and measurement	Appropriate and rigorous	5	4	1
	Appropriate but not rigorous	7	3	4
	Inappropriate	7	0	7
Number and operations	Appropriate and rigorous	2	2	0
	Appropriate but not rigorous	1	0	1
	Inappropriate	8	4	4
Algebra	Appropriate and rigorous	0	0	0
	Appropriate but not rigorous	5	5	0
	Inappropriate	11	6	5

space inside the rectangle, (b) length times width (or its symbolic representation), (c) both the amount of space inside the rectangle and length times width (or its symbolic representation), and (d) the amount of space inside the rectangle or length times width. PSTs who wrote (c) as their definition included both the amount of space inside the rectangle and length times width in the definition, while PSTs who wrote (d) as their definition accepted either amount of space inside the rectangle or length times width individually as the mathematical definition.

Table 9. Four Statements PSTs Wrote as Definitions of the Area of a Rectangle

Type of statement	Total	PSTs - M	PSTs - N
(a) Amount of space inside the rectangle	5	3	2
(b) Length times width	11	8	3
(c) Amount of space inside the rectangle and length times width	7	1	6
(d) Amount of space inside the rectangle or length times width	1	0	1

Table 9 gives the distribution of PSTs who regarded one of the four statements as the definition of the area of a rectangle. The exact wording of PSTs' written definitions varied. The analysis here focused on whether the amount of space and/or the formula was chosen as the mathematical definition. PSTs' written examples of each type of statement will be given later with elaboration.

Around 20% of PSTs indicated that the amount of space inside the rectangle was the definition of the area of a rectangle. Two PSTs - M and one PSTs - N discussed the dichotomy - what it means vs. how to calculate it - as a way to distinguish a definition vs. a non-definitional statement. For instance, M5 said:

> Definition will be directly what does that phrase means. Area of a rectangle is the region that the shape of rectangle takes on a plane....when I say here usually measured by m^2, cm^2, etc and calculated by width x height. That is kind of how you get the area. It is not describing the word, defining the words or phrases.

One PST - M originally wrote both amount of space inside the rectangle and the formula in her definition. When probed, she indicated that the amount of space was more like a definition because the formula (length times width) can be discovered through the more basic statement "the amount of space inside the rectangle". However, M9 mentioned this idea in a teaching setting instead of in a pure mathematics setting:

> If I will still give them as base times height as it, then that would not give them anything to figure out.... If I were to give a definition to students I would not include that because I would like them to come up with their own conclusions....Like if they did multiple examples like that maybe something they figure out.

This PST demonstrated some knowledge about the axiomatic system of mathematics and she indicated to some extent that a statement that can be proved by other statements is not a mathematical definition. But because she only mentioned this in a teaching setting and she never clearly articulated this idea in a pure mathematics setting, the depth of her thinking was questioned.

Almost 50% of PSTs indicated that the formula (length times width) was the definition of the area of a rectangle. Other than a simple recall, four of them gave detailed reasons. M1 regarded the formula as the definition because “I thought it is more specific to a rectangle. I was going to write something out then I realized - a lot more shapes than just a rectangle.” Another PST M3 indicated that the reason why she thought the formula was the definition was that the formula was more clear. She said:

> I mean, I was thinking of things like the space contained within the perimeters of the rectangle and stuff like that but I didn’t, I feel like that wasn’t like clear enough, I guess. It might be like interpreted kind of loosely, I guess....Like the product of the length and width is, there’s not, I don’t think you can really get that confused with anything. Or it’s pretty straightforward.

Another two PSTs - N indicated that they thought the formula was the definition because in their mind the definitions should help calculate the concept. For instance, N10 said:

> This equation because that will show you how to find the area of a rectangle. I think that would be like the definition. So maybe not necessarily in words but more like what to use in order to get the area of that shape.

This is the opposite opinion compared to previous two PSTs - M and one PSTs - N (e.g., M5) who used the dichotomy - what it means vs. how to calculate it - as a way to distinguish a definition vs. a non-definitional statement.

Almost 30% of PSTs indicated that both the amount of space inside the rectangle and the formula (length times width) together formed the definition. Four of them gave detailed reasons. Three of them expressed that definitions should include both what the concept is and how to find or use it. Therefore, both the amount of space inside the rectangle and the formula should be in the definition. For instance, M7 said:

> I think the first sentence is more of a description so it helps you to more visualize like what is talking about. Then this [the second sentence] is somewhat like an equation because it tells you how without counting out boxes or something if you try to think how you can find the specific area....I think something that is a good factor to a definition is when it can give someone an understanding of the thing that is trying to define and I think that if you only have the 2nd sentence it would teach you how to do it but it would not like help you to understand what it is that you are looking at.

I further probed her by asking "Can it only be the 1st sentence without the 2nd sentence? If the 1st has already helps you understand what it is?" She responded "I suppose if you want the students to discover how to do it on their own then you don't want to give them how to do it." I asked her "for a definition, do you want to have both parts or one part is enough?" She said she wanted both parts. Noticed that in this quote, similar to M9, M7 also mentioned that the formula can be discovered by the students, but this knowledge did not affect her decision to include both amount of space and formula in the mathematical definition. Another PST, N9 indicated that definitions were audience dependent. She said:

> I mean, to me, what's more helpful is like thinking about it in terms of a formula and like visualizing the length times width but if I was explaining what area is to a student, I would probably say the area is the space that we're going to find within the corresponding sides. And then I would give them the formula.

Because different audiences' need were different, she wanted to include both amount of space and formula in the definition so everyone could access. She also explained another reason why she thought the formula was part of the definition. She said "I've seen it often in textbooks, it's like in parentheses at the end, length times width." A PST - N indicated that either the amount of space inside the rectangle or length times width was the definition but she did not provide any reason.

Overall, I found that PSTs - M tended to choose (b) as the definition. But PSTs - N tended to choose (c) as the definition. More PSTs - M than PSTs - N thought the formula was the definition and more PSTs - N than PSTs - M thought the combination of the amount of space and the formula together was the definition. In addition, eight PSTs used dichotomy, what it means vs. how to calculate it to support their claims. But they had different ideas about how this dichotomy related to mathematical definitions. Some thought how to calculate alone can not be counted as a definition; others thought it can be counted as a complete or part of definition. Another interesting observation was that two PSTs M9 and M7 pointed out that formula can be discovered by students in teaching contexts. But they made opposite arguments on if the formula should be included in the definition based on this observation. No PST has ever mentioned that formula can be deduced from the amount of space in pure mathematics setting and therefore as a deduced result it should not be the definition of area of a rectangle. This phenomenon suggests that PSTs' understanding of the difference between mathematical definitions and deduced results is very weak.

PSTs' performance in writing a definition for fraction multiplication. Overall, PSTs' written definitions for fraction multiplication were categorized into four types of statements: (a) a conceptual approach about grouping, (b) rephrasing "fraction multiplication", (c) formula (multiply denominators and numerators), and (d) uncertainty. Table 10 gives the distribution of PSTs who regarded each type of statement as their definitions of fraction multiplication. PSTs' written examples of each type of responses will be given later with elaboration.

Table 10. Four Statements PSTs Wrote as Their Definitions of the Fraction Multiplication

Type of statement	Total	PSTs - M	PSTs - N
(a) Conceptual approach about grouping	1	0	1
(b) Rephrase "fraction multiplication"	5	2	3
(c) Formula (multiply denominators and numerators)	14	9	5
(d) Uncertainty	4	1	3

Among all 24 PSTs, only one of them provided a definition which was somehow close to the conceptual approach to define fraction multiplication. This PST N1 first defined whole number multiplication through grouping. After clarifying that what I wanted her to do was defining fraction multiplication, she provided the following response (See Figure 1):

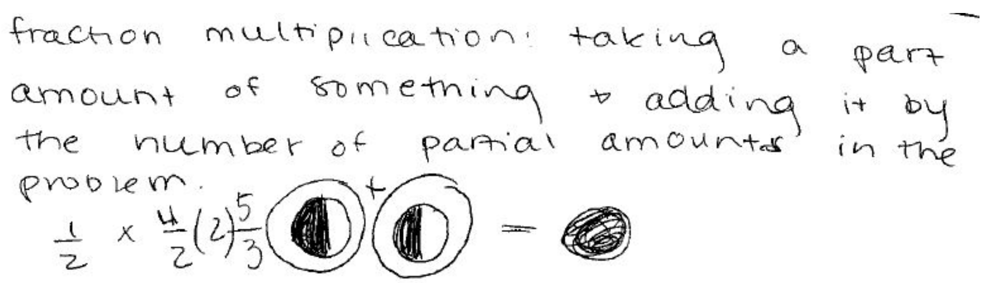


Figure 1. N1's written definition of fraction multiplication

She also provided further explanation:

> I guess I would show kind of like a similar one to this except saying that it's partial amounts and then doing like ½ x 4/2 which is obviously 2 so then you'd show like ½ two times and show like how adding them together would equal like one whole one, I guess....If you're looking at like a problem, you want to take like the first amount that they give you which is like a part of something, that's why I said that, because it's fractions. The part of something and then adding it by the number of like partial amounts they give you in the second part of the problem....I look at this as like being the group, like this is like two groups of this. Two groups of ½.

Here, N1 used an example 1/2 x 2 to illustrate that the meaning of multiplication was 2 groups of

1/2 in this example. Her written definition only applied to the cases when a fraction was multiplied by a whole number but not when a fraction was multiplied by a fraction because she used the idea of "grouping". I probed her by asking "what if the second number is also a fraction like 5/3?" She said:

> Obviously, that's like a more difficult problem so it just depends but in that situation, I guess… I don't know. I guess you'd just show how adding like, or multiplying the top amounts and then the bottom amounts.

Here, when she did not know how to define fraction multiplication in a conceptual way, she responded with multiplying the denominators and numerators. Even though she was unable to give a completely correct answer to define fraction multiplication in a conceptual approach, her idea of thinking about grouping distinguished her from the other PSTs.

About 20% of PSTs, almost evenly split between PSTs - M and PSTs - N only rephrased "fraction multiplication" without saying anything deeper in their definitions. For instance, M10 wrote "fraction multiplication: the multiplication of two fractions" and M9 wrote "fraction multiplication: the product of numbers written in a fraction form such as a/b." When I probed what they were thinking when they wrote the definitions, only one of the PSTs - N (N6) who offered a rephrased statement as the definition indicated her struggle because she was also considering the formula. Eventually she made the decision that the rephrasing statement was the definition because it told her what fraction multiplication was.

Almost 60% of PSTs indicated that formula was the definition of fraction multiplication. Two PSTs - M explained the reasons why they thought the formula was the definition were because for them no other options were available or the other options under consideration were problematic. For instance, M12 originally wrote both "fraction multiplication: timing fractions" and the formula. When probed, she said:

> Now I want to go this one [the formula]. Let us go this one.... I feel like this is just more in depth. I feel like if I am just saying time fractions, it is just like rewording what the question is.

Another PST - M (M3) who wrote the formula as her definition was also struggling with determining whether the formula was the definition. She said:

> I was kind of thinking the same thing. Like more like a conceptual definition about like, oh, like if you multiply something by a fraction like between 0 and 1, the product is gonna be less than like the original number you started with and stuff like that but I don't think, I think that's all like extra information. It's not something, it's more of like a property kind of thing than part of the definition so…

Here, she realized that the formula was not a "conceptual definition" and she attempted to come up with a conceptual one but she thought the one she made up was more like extra information but not necessary to be included in the definition.

Two PSTs gave clear reasons for why they thought the formula was the definition. M8 indicated that how to compute was the definition. She said "I did the same as the rectangle. I was trying to think of it in words but it was simplified in my head as symbols, so I just did how you would compute it again." N10 said "because I was thinking, like how, trying to remember like how to actually multiply fractions in order to properly define the steps to do that. Yes, that's what I was thinking as I was writing." Here, the quotes indicate that M8 associated how to compute with the definition and N10 thought that describing the steps of multiplying fractions was defining fraction multiplication.

About 20% of PSTs were uncertain of the definition of fraction multiplication. One of them (N7) did not generate any response to this question. The other three wrote something out, but when probed, they indicated uncertainty. One PST - M (M6) said "Kind of trying to explain the differences between just - I guess I did not really explain multiplication just what it means to do fraction multiplication". Here, M6 was clear that explaining fraction multiplication and doing fraction multiplication were different, but she did not know what was the meaning of fraction multiplication. Another PST (N3) did not make a final decision on what was her definition of fraction multiplication. She originally wrote "Definition of 2 fractions multiply: 2 fractions and multiplying it to find the answer." She commented:

> So I don't know. I kinda just wrote what the question was because I wasn't really quite sure how to write the definition of multiplying fractions. Like I know how to go, the equation and how to go about it.

Similar to M6, N3 was also aware of the differences between definitions and how to compute the concepts. She indicated uncertainty when probed whether an equation (e.g., formula) could be a definition.

Overall, more than half of the PSTs thought the formula was the definition. Other PSTs were roughly split between giving rephrased or uncertain responses. Only one PST attempted to give a conceptual approach to define fraction multiplication but did not succeed. This indicates that for concepts in Number and Operations, the conceptual approach to define a concept seemed to be hard for PSTs and was not well grasped. Compared to PSTs - N, PSTs - M were more likely to write formulas as definitions of fraction multiplication and actually 75% of PSTs - M wrote formulas as the definitions. This result is similar to PSTs' written definitions of the area of a rectangle.

When comparing to the results of PSTs' written definitions of the area of a rectangle, I found that for both concepts roughly half of PSTs wrote formulas as the definition. About one quarter of PSTs wrote a conceptual approach, amount of space inside the rectangle, as the definition of the area of a rectangle. However, only 4% of PSTs attempted to write a conceptual approach as the definition of fraction multiplication. Instead, quite a few PSTs chose to write rephrasing statements as their definitions of fraction multiplication because they disagreed with the formula as the definition and were unable to write a conceptual approach to define fraction multiplication. This results supported the conclusion I drew in the previous section that PSTs were better at writing definitions of Geometry and Measurement concepts than Number and Operations concepts.

PSTs' Performance in Evaluating Mathematical Definitions

Table 11 presents the numbers of total PSTs, PSTs - M and PSTs - N who gave correct judgment to each statement given in Question 7. Question 7 gives PSTs a list of statements and asks whether the statements are mathematical definitions and the reasons for their judgments. As explained in the Chapter 3, the statement 7i was excluded from this analysis based on the extent of correctness, but it is included in the analysis based on themes.

Overall, at least 70% of the PSTs gave correct responses to statements 7a, 7d, 7e, 7g, 7j, and 7o. All but one PST - N gave correct responses to statement 7g. For geometry statements 7a, 7d and 7j, PSTs correctly made judgments because they either had definitions in their minds which were different from the given statements or were able to point out the issues (e.g., too general, missing key term) of the statements. For instance, many PSTs said that the definition of the polyhedron should include things like a 3-d shape which iss made up of polygons. Having such a strong definition in their heads reminded them immediately that the given statement is not the definition of polyhedron. Statements 7e and 7g were the easiest; more than 80% of PSTs gave correct responses to these two statements. The reason was because for most PSTs the way the statements were worded did not make them read as if they were defining a concept. PSTs performed well in statement 7o and 70% of the PSTs gave correct responses to 7o. PSTs were able to identify an exponent, an equation/expression, or squaring something as the concept being defined.

Table 11. Numbers of PSTs Who Gave Correct Judgments to Whether Given Statements Are Mathematical Definitions

Statement in Question 7	Mathematical definition	Total	PSTs - M	PSTs - N
a (rhombus)	No	19	10	9
b (pythagorean theorem)	No	12	8	4
c (fraction)	No	14	8	6
d (polyhedron)	No	17	10	7
e (prime factorization)	No	20	10	10
f (area of triangle)	No	4	2	2
g (infinite primes)	No	23	12	11
h (fraction division)	No	9	5	4
j (square)	No	17	9	8
k (equivalent fraction)	No	14	8	6
l (prime number)	No	10	8	1
m (negative exponent)	Yes	14	8	6
n (distributive property)	No	10	5	5
o (exponent)	Yes	17	9	8

Statement 7f was the one where PSTs performed the worst; only four PSTs in total gave correct responses. Recall that in Question 5, I asked PSTs to write mathematical definitions for the area of a rectangle. Though in statement 7f the concept is a triangle instead of a rectangle, my hypothesis was that PSTs would give consistent responses. However, a comparison between PSTs' responses to these two questions suggested that only M6 gave consistent and correct answers to both questions, namely she wrote the amount of space inside the rectangles as the definition and also determined that statement 7f was not a definition of the area of a triangle. She identified that the amount of space was the definition and pointed out that statement 7f only gave a way to figure out the area and was not a definition from her perspective. Ten PSTs consistently wrote the formula as the definition across both questions. One PST who wrote either amount of space inside the rectangles or formula as the definition accepted formula as the definition in Question 7. Another PST wrote the formula as the definition rejected 7f as the definition. When responding to Question 5, five PSTs indicated combining the amount of space and the formula as the definition. These PSTs accepted formula as the definition when responding to statement 7f. When responding to Question 5, another two PSTs indicated combining the amount of space and the formula as the definition. These PSTs rejected statement 7f as the definition. Other four PSTs who indicated the amount of space as the definition when responding to Question 5 also accepted statement 7f as the definition. The main reason why PSTs accepted statement 7f as the definition was because the statement explained what the area was. Even though only a couple of PSTs explicitly pointed out the linguistic structure "A is B" played an important role in their decision making, quite a few PSTs used the language such as "it tells me like what it is" which suggested that the linguistic structure may have an implicit impact on their judgments. Another reason why PSTs thought statement 7f was a definition was because it also pointed out how to find the area.

For the rest of the statements, the percent of PSTs who gave correct responses to the statements ranged from 40% to 60%. Compared to previously discussed statements, PSTs also gave a wide variety of reasons for the ways they responded to these statements. Due to limited space, I only provide analysis of one statement 7b to exemplify this wide variety. The statement 7b stated "In a right triangle, the square of the hypotenuse (the side opposite the right angle) is equal to the sum of the squares of the other two sides." Overall, 12 PSTs (8 PSTs - M and 4 PSTs - N) indicated that the statement is a mathematical definition of a mathematical concept, but they had different opinions on the concept being defined. Table 12 gave the distribution of

PSTs about their thinking on what was the concept being defined.

Table 12. Defined Concepts Identified by PSTs in Statement 7b

Concept identified by PSTs	Total	PSTs - M	PSTs - N
Right Triangle	5	1	4
Pythagorean theorem	2	1	1
Hypotenuse or square of hypotenuse	6	2	4

Note. N8 said that the statement could be a definition of either the Pythagorean theorem or a right triangle. Therefore, even if the numbers in Total column adds up to 13, the responses were from only 12 PSTs.

Overall, five PSTs (1 PST - M and 4 PSTs - N) indicated that the statement was a mathematical definition for a right triangle and two types of explanations were offered. Three PSTs used their background information that the Pythagorean theorem is a “if and only if” statement to judge that if a triangle satisfies the equation in statement 7b, it is a right triangle. For instance, N6 said:

> If like the right triangle does not fit this property, the square of the hypotenuse is not equal to the sum of the squares of the other two sides, then it would not be a right triangle, so I think that would be a definition.

These three PSTs demonstrated this important knowledge about the Pythagorean Theorem, but the statement 7b only gave one direction of “if and only if” but not the other direction. Namely the statement in 7b did not state that if the triangle satisfies the given equation, then it is a right triangle.

Two PSTs - N indicated that the reason why they believed the statement 7b is a mathematical definition of a right triangle was because the statement includes important information about a right triangle. For instance, N8 said:

> There was not much more about right triangle than that. I mean you could say that the other two angles are cute, but it says that there is only one right angle....Because I think

that takes account into everything a right triangle is. Or as [statement]d doesn't account for everything that a polyhedron is.

N8 compared statement 7b to statement 7d which was the Euler's formula for polyhedron. He argued that statement 7b include everything important about a right triangle but statement 7d does not. For instance, statement 7d does not include that polyhedron is a 3-d shape and is made up of polygons. The quote suggests that though N8 understood that mathematical definitions should include key features of the concept, he did not know that definitions should only include necessary information of the concept and allow other information to be deduced rather than including all information. In addition, it also reveals one possible misconception PSTs had about mathematical definitions: any statement that contain enough information about the concept could be the definition even if the focus of the statement is not defining the concept. In this case, "right angle" is included in the parenthesis in statement 7b. However, the key point of statement 7b is not to tell that the right triangle is a triangle with a right angle, but in N8's mind as long as this key information "right triangle" is included in the statement, it could be a definition of a right triangle.

Two PSTs (one PST - M and one PST - N) indicated the statement 7b is a definition of the Pythagorean Theorem. For instance, M11 said "this is the definition of the Pythagorean theorem like verbally written out which is a definition in my mind."

Another six PSTs, mostly PSTs - N, demonstrated that they thought the statement is a mathematical definition of the hypotenuse or square of the hypotenuse mainly based on the linguistic feature of the statement. For instance, N4 said:

I think completely tells you what that is, in a right triangle. And it's defining, I guess… oh, shoot, what is it defining? The square of the hypotenuse is equal to the sum of the squares… okay, it's defining the square of the hypotenuse in a right triangle and it tells you exactly what that is. I think it's a definition.

When I probed her how she decided the concept being defined. She said:

> Based on, I guess, the grammatical structure of the sentence. Like it's just saying, like this is the subject, the square of the hypotenuse is… what is equal to the sum of the squares of the other two sides. It's not defining what a right triangle is. It's saying in a right triangle.

She noticed that another concept "right triangle" is an option for consideration, but she explained very clear that the grammatical structure of the sentence helped her decide the concept being defined. In addition to looking at the grammatical structure of the sentence, two PSTs - N commented that the phrase in the parenthesis "the side opposite the right angle" defines the hypotenuse. These comments supported my previous claim about PSTs' possible misconception. Even if the key function of the statement is not to define the hypotenuse, as long as enough information about the hypotenuse is given, some PSTs regarded it as a mathematical definition of the hypotenuse.

In total, twelve PSTs decided that statement 7b is not a definition for four types of reasons. Two PSTs - N wanted to add visual aid or picture to the statement. For instance, N11 said "you need the picture for sure." Two PSTs (1 PST - M and 1 PST - N) almost identified the statement to be the definition of the Pythagorean theorem, but they wanted to add the phrase "Pythagorean Theorem is" in the sentence to make it sound like a defining statement. For instance, M10 said "Or like Pythagorean theorem would be $a^2 + b^2 = c^2$ which that's what it's stating but it's not actually, like not saying this is what it's defining." Four PSTs (3 PSTs - M and 1 PST - N) had difficulty identifying the concepts being defined and therefore determined that the statement 7b is not a definition. Three of them specifically pointed out that statement 7b does not define a right triangle and the definition of a right triangle should look like "a right triangle is a shape where, and then the rest of it" (M6). The other four PSTs - M saw statement 7b as a property or description instead of as a definition. For example, M4 said "Because this is telling you something about a right triangle but it's not necessarily defining a right triangle."

Another interesting observation was that three PSTs - M who originally thought statement 7b is a definition changed their minds after being asked "what is the concept being defined" by the interviewer. All of them did not even think about this question when they made their first decision. For instance, M2 was shocked and pondered for a long time when she was asked this question. She gave an explanation about why she originally thought statement 7b is a definition. She said:

> I was just looking - was it precise enough, was it correct? So first I was like, oh yeah, this is right, then I thought OK, what is this actually defining? It is not really defining anything. It just gives an equation, a general equation that works for all right triangles, so maybe. Because I think it is telling you about the measures of the sides, but it is not telling you what a right triangle is. And you can know the measure of the size in any triangle and they might have relationship, but it is not telling you what a right triangle is.

Here, when she examined statement 7b, her original focus was applying the features of mathematical definitions (e.g., precise and correct) that she articulated at the beginning of the interview as the only criteria. She seemed to take these features as the sufficient conditions of mathematical definitions instead of necessary conditions of mathematical definitions.

I provide this detailed analysis to exemplify the wide variety PSTs demonstrated in their thinking. The divergence of PSTs' thinking indicated that when there was no clear instruction about what is a mathematical definition, the different generalizations PSTs could make in order to make sense of mathematical definitions by themselves. Notice that among the 12 PSTs who made correct judgments that statement 7b is not a definition, four of them gave incorrect reasons. This phenomenon was not unique for this particular statement 7b; PSTs gave different kinds of incorrect reasons when they explained their thinking to other statements as well. Therefore, a qualitative analysis focusing on how PSTs think about mathematical definition will provide additional insight and will be presented in the next section.

For two statements 7b and 7l, the number of PSTs - M who gave correct responses was at least four more than the number of PSTs - N who gave correct responses. For statement 7b, a comparison of the reasons for their thinking reveals that more PSTs - N thought statement 7b defined a right triangle and more PSTs - M identified statement 7b as a property or description. PSTs - M used the language "*about* a right triangle" and "what a right triangle is" to distinguish properties and definitions, but PSTs - N tended to accept information *about* a right triangle as a definition of a right triangle.

In order to see why a big performance difference existed between PSTs - M and PSTs - N on statement 7l, I conducted a further analysis which reveals an interesting pattern. The reason that most PSTs (3 PSTs - M and 9 PSTs - N) used to support that statement 7l is a mathematical definition was because statement 7l describes what a prime number is through stating how to test whether a given number is a prime or not. For instance, when I asked N1 why she thought statement 7l is a definition of a prime number, she said "it gives you like the, the means or the equations to find out whether it is or not and then it gives you like how to test it and stuff. " The reasons why PSTs thought statement 7l is not a mathematical definition vary. The reason mentioned by the largest number of PSTs was because even though statement 7l states how to test if a given number is prime or not, it does not define what a prime number is. For instance, M10 said:

> I think it's trying to get at the concept of like what, like a prime number, like when it is prime or when it's not prime but I think it's lacking some things.…I think it's even kinda lacking exactly what prime means maybe, potentially.

In other words, even if these PSTs realized that statement 7l states how to test whether a given number is prime, it was insufficient for them to accept statement 7l as a mathematical definition because these PSTs were concerned other missing features of mathematical definitions. However, for previously discussed twelve PSTs who used the same criterion (statement 7l states how to test whether a given number is a prime), they used this criterion to support their claim that statement 7l is the definition. Testing whether an instance fits the concept (e.g., if 4 is an even number) is an important feature of mathematical definitions (Borasi, 1992), but this feature is not a sufficient but only a necessary feature of mathematical definition. Data from statement 7l reveal that no PSTs - N noticed this point, but at least some PSTs - M realized it. Other reasons PSTs articulated to support their claims that statement 7l is a mathematical definition included that statement 7l defines a whole number or it defines a mathematical process to test a prime number. Other reasons PSTs articulated to reject statement 7l as a mathematical definition included that statement 7l is unclear, wordy, not the typical definition of prime numbers, categorizing whole numbers into prime vs. non-prime instead of defining prime numbers, and PSTs' incapacity to identify the concept being defined.

PSTs' General Notion of Mathematical Definitions

In this section, First I examine PSTs' responses to Question 1b in the interview, namely, "From your perspective, what is a mathematical definition?" The analysis of this question asked in the early interview is important because it captures PSTs' first impression of mathematical definitions before any probe. Second I analyze how PSTs responded to Question 4a "Is it possible for a concept to have more than one mathematical definition?" This analysis aims to elicit PSTs' understanding of equivalent definitions. Third, I present an analysis of PSTs' conceptions of mathematical definitions based on PSTs' responses to questions 1-3, 5, 7, 10, and 11. The aim of this analysis is to depict PSTs' overall understanding of mathematical definitions. Last, I examine PSTs' responses to questions 1-3 as a group. The purpose of this analysis is to describe PSTs' general understanding of mathematical definitions before scaffolding (e.g., specific mathematical statements which offer examples for PSTs to consider) is given in Questions 7, 10 and 11. A comparison between results in subsection 3 and 4 reveals the differences on how PSTs perceived mathematical definitions with and without scaffolding.

PSTs' First Impression of Mathematical Definitions

Overall, mathematical definitions seem to be a familiar concept to PSTs. When asked the first interview question "Are you familiar with the term mathematical definitions"? 22 PSTs said firmly "Yes". M2 said "Not off the top of my head" and M12 said "Maybe".

Question 1b in the interview asks "From your perspective, what is a mathematical definition?" Notice that at this time interviewers did not give any probes other than asking clarifying questions such as "what do you mean by this?" Analysis of PSTs' responses to Question 1b captured PSTs' very first impression about mathematical definitions. I categorized PSTs' various responses and displayed them in Table 13.

Six PSTs in total mentioned that mathematical definitions are related to mathematics or mathematical concepts or terms without specifying what is the exact relationship between mathematical definitions and mathematical concepts or terms. For instance, M10 said:

Table 13. Aspects PSTs Mentioned When Answering Question 1b

Aspect	Total	PSTs - M	PSTs - N
Related to mathematics in general ways	6	2	4
Roles in mathematics	13	7	6
Desired features	3	3	0
Problematic thinking	7	4	3
Involving non-relevant elements	6	1	5

Note. Because some PSTs mentioned multiple ideas in their responses, the numbers in Total column add up more than 24.

> I would say it would be a definition that deals with mathematical terms or like yeah, just normal terms you learn in math. So maybe being, yeah, maybe being like just like the concepts that are learned in math class formed in definitions.

Among the six PSTs, three of them only gave responses similar to M10's response which did not explain what is a mathematical definition at all other than saying they are related to mathematical concepts or terms. Two other PSTs mentioned additional information which is actually irrelevant to mathematical definitions. N5 said "I guess a definition; it would be words that....relate to math and different terms or practices or equations that we might use variables different things like that." Similar to N5, instead of mentioning equations and variables, N7 mentioned numbers and operations in her response. One PST N3 gave a response which makes no sense. She said:

> Mathematical definition would be like the definitions of what makes up math and like math is all around us so like from studying like different angles of how like the sun rises, like you're taking the angle, that's studying, I guess, what, figuring out what the angle is so you have to… I don't know. But I'm just trying to relate - how math is related to everyday life.

This phenomenon that one quarter of PSTs were unable to give a meaningful description about mathematical definitions seemed to be problematic.

About one half PSTs mentioned at least one of two important roles of mathematical definitions, namely describing the meaning of a mathematical concept and establishing the foundation for problem solving (Zaslavsky & Shir, 2005). Twelve PSTs, evenly split between PSTs - M and PSTs - N mentioned that mathematical definitions were descriptions or explanations of mathematical concepts. This indicated that half of PSTs had basic understanding of what a mathematical definition is and there was no difference between PSTs - M and PSTs - N in this aspect. For instance, N8 said "Mathematical definition clearly states the meaning of a term specific to mathematics as a subject." Two PSTs indicated that mathematical definitions are useful to solve problems. For instance, M11 said "Mathematical definition is the definition of some aspect of mathematics either like how to do a problem." While this response did not clearly reveal what M11 meant by "like how to do a problem". He provided an example about counting number of people by using the definition of multiplication and said "knowing the definition of multiplication will help you solve that problem."

Three PSTs - M brought out a few important and desired features of mathematical definitions when they answered Question 1b. Two PSTs - M indicated that mathematical definitions have to be necessary and sufficient conditions of the concept. For instance, M7 said "It is broad enough there covered all the cases that wants to but it is also specific enough that only cover what it wants to and it isn't too wordy so it is kind gets right to the point." Here, M7 also mentioned two additional desired features of mathematical definitions: concise and straightforward. In contrast, no PSTs - N mentioned any desired features of mathematical definitions when responding to Question 1b.

Seven PSTs, roughly split between PSTs - M and PSTs - N also demonstrated some problematic thinking about mathematical definitions. Three PSTs either thought defining properties or describing procedures to solve problems were mathematical definitions. For instance, N9 said “In math you also have definitions for specific concepts of how to solve problems or what something is, such as mathematical definition for a property, like commutative property or something like that.” Four PSTs (3 PSTs - M and 1 PST - N) brought incorrect relationship between mathematical definitions and proofs when they responded to question 1b. It seemed that PSTs - M had more tendency to mention proofs in their responses than PSTs - N when responding to Question 1b and this might due to PSTs’ more experiences in proofs. As M1 commented “I would say they [mathematical definitions] all require a proof, some type of proof to prove they are correct and nothing else can fit into that category, I guess.”

Six PSTs mentioned elements such as equations, variables, numbers or operations in their responses to Question 1b. Example of responses of this type was given previously. No PSTs insisted that any of the elements had to be in a statement for it to be a mathematical definition, but PSTs - N’s first responses to mathematical definitions involving these elements as options indicated that their understanding of mathematical definitions more focused on the appearance of the statements (e.g., equations) instead of the nature and the purpose of the statements (e.g., describing the meaning of concepts in mathematics) or features of mathematical definitions (e.g., concise). Only one PST - M but five PSTs - N mentioned these irrelevant elements.

In summary, 14 PSTs (8 PSTs - M and 6 PSTs - N) mentioned positive aspects of mathematical definitions such as different roles mathematical definitions play in mathematics or desired features suggested by mathematicians. Eleven PSTs (4 PSTs - M and 7 PSTs - N) mentioned either useless aspects of mathematical definitions (e.g., relating to mathematics) or problematic aspects of mathematical definitions. More PSTs - M than PSTs - N mentioned positive aspects and more PSTs - N than PSTs - M mentioned either useless or problematic aspects of mathematical definitions. This observation suggested PSTs - M’s first reaction to the phrase “mathematical definitions” were better than PSTs - N.

PSTs’ Understanding of the Number of Definitions Associated with a Mathematical Concept

This subsection aims to analyze PSTs’ responses to Question 4 which asks if it is possible for a concept to have more than one mathematical definition. The following Table 14 displays the distribution of PSTs in terms of their thinking of the existence of multiple mathematical definitions associated with one mathematical concept.

Table 14. PSTs’ Responses to the Number of Definitions Associated with a Mathematical Concept

Number of definitions	Total	PSTs - M	PSTs - N
One	5	4	1
More than one	19	8	11

Overall, about 20% of PSTs indicated that only one mathematical definition existed for one mathematical concept. Among those PSTs, only one gave a clear rationale for her thinking. At the beginning of the interview, M9 said “I think a statement does not have to include an entire definition. It just has to include the part that pertains to what you are discussing, but in a definition it would include all aspects.” Later when responding to Question 4, she gave the following response:

> No, I think they just have one definition. You may use particular parts of the definitions depending on what you are doing with that concept, but I think it probably just have one....like there is more of like in a complete definition I feel like would have everything about that. I feel like it is a complete definition.

She further clarified what she meant by “complete definition” when she was asked “Are definitions, properties and descriptions essentially the same thing”? She said “I think that the definitions contain properties and descriptions, but a property does not necessarily contain a definition. It is like a description does not necessarily contain a definition." Her responses indicate that she thought there is only one definition which is complete and encompasses all important features of the concept, properties or descriptions are part of the definition, and depending on the context, only certain parts are included in the definition. An interesting observation was that even if these five PSTs insisted that only one mathematical definition exists for one mathematical concept, later when they were asked to evaluate whether the given statements were mathematical definitions for an even number and a square in Questions 10 and 11, four PSTs selected at least two statements as valid definitions for both concepts (an even number and a square). M3 was the only PST who only selected one valid definition for a square, but she selected four valid definitions for an even number. This indicates that these PSTs' explicit statements about number of definitions associated with one mathematical concept (as articulated in Question 4) was inconsistent with their implicit understanding (as demonstrated in Questions 10 and 11).

About 80% of PSTs indicated that multiple mathematical definitions could exist for one mathematical concept. However, the reasons they gave were inconsistent with mathematicians' view about equivalent definitions for one mathematical concept. Seven PSTs (mainly PSTs - N) commented that different ways to describe the same idea led to different mathematical definitions. The following response was typical:

> I cannot necessarily think of one off my head but I know I think sometime they have like similar definitions but like kind of different forms of representing it. So I think sometimes it can have - people define things differently anyway, so a mathematical definition could have the same concept at the heart but the way you word it could be

different. (N6)

At first glance, N6's statement suggests that she understood that there were different ways to describe a concept which seemed to indicate that she understood "equivalent definitions". But without the support of specific examples (she was unable to give one), it is hard to draw conclusions about whether she understood equivalent definitions or only different ways to word the same mathematical definition. Actually, among all PSTs, only one PST (M2) was able to give an example of two equivalent mathematical definitions for a concept. M2 first gave one definition "So a definition of a square is like a quadrilateral with all right angles and all sides equal length." When asked if she can give another definition, she said "a square is formed by like one right triangle and a congruent right angle reflected across its hypotenuse or something like that. " Even though her definition was not completely correct, at least she had a different way to define a square while the other PSTs who indicated that multiple definitions exist for one mathematical concept were unable to give any example of two equivalent mathematical definitions for any mathematical concept.

Four PSTs (all PSTs -M) indicated that the same words or symbols may carry different meanings which led to different definitions of the same concept. M7 said:

> Because things could be used in different ways, there are different types of math. I think things like that. Kind of like a dash sign can be minus or take away or can be negative, so it is the same thing but it has different definitions. The same symbol.

Here, M7 mixed the function of a symbol with the definition of a symbol. Interestingly, no PSTs - N mentioned similar reason. Even if these four PSTs - M's general conclusion was incorrect, the advanced training PSTs - M received in mathematics seemed to broaden their minds and make them think more flexibly.

Another five PSTs who agreed that there are multiple mathematical definitions gave various interpretations of the phrase “multiple mathematical definitions”; however, their responses were inconsistent with how mathematicians think about equivalent definitions. For instance, when answering Question 4, N10 gave the following response:

> It’s possible [for a concept to have multiple definitions]....An example of that would be maybe like an algebra problem where you might have two variables with it and then have like more than one definition. This is the first variable, this is the second one. So maybe it’s like two separate definitions within one problem.

I probed her by asking that “Are the separate definitions for the same concept?” She said “yes. And you use the same concept because you’re gonna solve that one problem.” Here, she mixed up definition of variables with defining the meanings of two variables in a problem solving context and used that as an example to support the existence of multiple mathematical definitions. N7 gave another interesting example by saying “Some squares can also be rectangles so I think that would explain that, because the concept of a square can have the definition of a rectangle, too.” N7 mixed mathematical definitions with mathematical concepts. Her definition of a square included the concept of a rectangle instead of the definition of a rectangle. Other three PSTs did not give any reason about why they thought there were multiple mathematical definitions for one mathematical concept.

PSTs’ Overall Understanding of Mathematical Definitions

In this section, I will present an overall analysis of PSTs’ responses to questions 1-3, 5, 7, 10 and 11. Notice that though the main purpose of Questions 10 and 11 is to investigate how PSTs differentiated valid mathematical definitions, the design of these two questions, asking PSTs to circle the statements they thought were valid mathematical definitions offered insights into how PSTs distinguished definitions vs. non-definitions. By looking at a broad range of questions, the aim of this section is to answer research question 1a and describe what mathematical definitions entail in PSTs' minds. I will answer this research question from two aspects. First, I will describe from PSTs' perspective, what are necessary features of mathematical definitions. Notice that this analysis is different from the features PSTs mentioned when they were asked to evaluate which mathematical definitions were better than the others. The latter analysis targets to answer research question 1b, PSTs' preference for mathematical definitions and will be presented in Chapter 5. I explained in Chapter 3 about how I differentiated necessary and preferred feature when coding. Second, I will depict what are the problematic thinking PSTs had about mathematical definitions. I organize the section into two subsections according to these two aspects.

Necessary features of mathematical definitions. Table 15 gives the necessary features PSTs associated with mathematical definitions. Table 16 gives the distribution of PSTs mentioning specific features as the necessary features for mathematical definitions. Overall, all PSTs - M and all but one PSTs - N mentioned at least one feature in Table 16; PST N3 did not mention any feature in Table 16 as a necessary feature of mathematical definitions during the whole interview. Recall that in the previous section **PSTs' First Impression of Mathematical Definitions,** I reported that N3's response to the first interview question "From your perspective, what is a mathematical definition?" made no sense. The results here echoed the previous finding that N3 did not understand mathematical definition as a meta level concept.

Four features, unambiguous, correct, including the name of the concept and precise are reported in the literature as necessary features of mathematical definitions. For each of the first three features, about two thirds of PSTs reported it. However, for the feature "precise", only

Table 15. Necessary Features of Mathematical definitions Mentioned by PSTs

Feature	Meaning of the Feature
Unambiguous	The meaning of mathematical definitions should be uniquely interpreted (Zaslavsky & Shir, 2005).
Correct	Mathematical definitions should be necessary and sufficient conditions for the concept and be consistent with the way mathematical community defines the concept (Winicki-Landman and Leikin, 2000).
Name of the concept	The name of the concept is presented in the statement used as a definition and appears only once in the statement (Winicki-Landman and Leikin, 2000).
Minimal	Only the minimal number of properties necessary to reconstruct the concept should be mentioned (Van Dormolen & Zaslavsky, 2003).
Precise	"All the terms employed in the definition should have been previously defined, unless they are one of the few undefined terms assumed as a starting point in the axiomatic system one is working with" (Borasi, 1992, p. 17).
Understand the concept	Mathematical definitions should help beginning learners understand the concept.
Concise	Express much in few words.
Straightforward	Mathematical definitions should be explicit and take little effort to infer.

about one third of PSTs reported it. Minimality was reported by only one PST - M as a necessary feature. About one third of PSTs regarded helping beginning learners understand the concept, concise and straightforward as necessary features of mathematical definitions though they are actually only preferred features of mathematical definitions from the disciplinary perspective. I organize this section by first discussing the features mentioned by similar numbers of PSTs - M and PSTs - N. Then I discuss the features mentioned by different numbers of PSTs - M and PSTs - N.

Features mentioned by similar numbers of PSTs - M and PSTs - N. Overall, PSTs demonstrated a basic understanding of the features of mathematical definitions. For four features, unambiguous, name of the concept, minimal and understand the concept, similar numbers of PSTs - M and PSTs - N mentioned them. About three quarters of PSTs indicated that

Table 16. Distribution of PSTs Mentioning Specific Feature as the Necessary Feature of Mathematical Definitions

Feature	Total	PSTs - M	PSTs - N
Unambiguous	17	8	9
Correct	16	9	7
Name of the concept	17	8	9
Minimal	1	1	0
Precise	9	6	3
Understand the concept	9	5	4
Concise	7	2	5
Straightforward	7	6	1

Note. Because some PSTs mentioned multiple features in their responses, the numbers in Total column add up more than 24.

mathematical definitions should be unambiguous. For instance, when responding to statement 7c (A fraction is a part whole relationship), M2 said:

> It seems very general like part whole relationship what does that mean, like you have a part and then you have a whole, but to me it is not tell you like what you are doing with that. Or anything.

Here, M2 indicated that she was uncertain about the meaning of the phrase "part whole relationship"; therefore the statement was unambiguous to M2.

About three quarters of PSTs thought mathematical definitions should be correct. Namely, they should include necessary and sufficient conditions of the concept and be consistent with how mathematicians define the concept. For instance, when asked "From your perspective, what is a mathematical definition?", M1 answered "I think it is different from regular definition and a mathematical definition has to rule out all other possibilities." This response alone was not clear enough to draw the conclusion that M1 understood that mathematical definitions should include necessary and sufficient information. His understanding became more evident when he commented on statement 7a (A rhombus is a special type of quadrilateral). He said "It is just saying it is a type of quadrilateral and there are a lot of things that are quadrilaterals such as a square or a diamond or a rectangle or any other convex figure that has four sides." Here, he expressed that statement 7a did not provide sufficient information to rule out other shapes which were not rhombi.

Sixteen out of the 17 PSTs firmly indicated that the statement used as a mathematical definition should include the name of the concept for it to be a valid definition. For instance, when commenting on statement 7k (For any integer *a, b*, and *n, b* $\neq$ *0, n* $\neq$ *0*, $\frac{an}{bn} = \frac{a}{b}$), M3 said:

> I think you could say it is a definition of like equivalent fractions. But I think you might need, I think it is missing like a term like equivalent fractions or something like that....I think the word is very important in definitions. I think, yeah, like using a term, like equivalent fractions or something in there would be better.

About one third of the PSTs thought mathematical definitions should help beginning learners who has never been exposed to the concept understand the concept. For instance, M9 rejected statement 7c (A fraction is a part whole relationship) as a definition and said:

> I think to me still get the point across that a fraction is part of a whole amount of something, but if I did not have a lot of mathematics - if I was just beginning to learn what a fraction is - I don't think I would understand that.

Here, M9 herself understood the statement, but she was concerned for the people who is new to the concept.

Only one PST (M3) insisted that mathematical definitions should be minimal during the whole interview. When asked to explain why she thought statement 11f (If a quadrilateral in which all sides are equal and all angles are 90 degrees, it is a square), the most commonly seen definition of a square, is not a mathematical definition from her perspective, she responded with two reasons:

> So technically you only need to say that one angle is a right angle and that just like implies that the rest are right. So that was one reason. And the other one was that it's set up in this if/then statement kind of thing and that's, for me, is not really how you write definitions usually. This is almost like just asking to be proved, I feel like.

Because she used two reasons together to reject statement 11f as a valid mathematical, I probed her by eliminating the second reason and asked "if I rephrase [statement 11]f, if I say a square is a quadrilateral which has four equal sides and four equal angles, do you call it a definition?" She responded in the following way:

> So no, only just because it's not the most simple definition. It's still extra information. I think we usually say that just because, like we always say that one just because it like gives us a nice visual. But it's extra information. Like when we wrote like in class, our definitions of platonic solids, we were all adding all this extra information about it and you don't need that to define it. Mathematical definitions should be the most simple definition that's complete that we use.

Features mentioned by different numbers of PSTs - M and PSTs - N. There are three features, concise, straightforward, and precise, for which the differences between numbers of PSTs - M and PSTs - N who mentioned them were at least three. More PSTs - N than PSTs - M demonstrated that mathematical definitions should be concise. In fact, almost half of the PSTs - N but only two PSTs - M indicated that mathematical definitions should be concise. For instance, when asked "What needs to be in a statement for it to be a mathematical definition?" N1 responded "I'd say like a condensed statement." When probed what she meant by "condensed", she responded "I guess just explaining it in the least amount of words possible."

More PSTs - M than PSTs - N indicated that mathematical definitions should be straightforward and precise. Straightforward was mentioned by only one PST - N. Straightforward is different from unambiguous. Unambiguous was applied in coding when PSTs showed confusions or uncertainty about the meaning of a statement or if the meaning of the statement could be interpreted in multiple ways. Straightforward was applied in coding when PSTs had no difficulties in interpreting a statement but still wanted it to be expressed in a more explicit and direct way. Half of the PSTs - M but only one PST - N mentioned this feature. For instance, M4 rejected statement 11h (A polygon with four equal sides and three equal angles is a square) as a mathematical definition of a square and explained "Four equal sides and three equal angles, I know again you can infer that if there's three equal angles, then there's four but I don't think you should have to infer." Later in Chapter 5, the analysis will indicate that some PSTs thought straightforward is a preferred feature for mathematical definitions, but here seven PSTs indicated that straightforward is a necessary feature for mathematical definitions because they used straightforward as a criterion to reject the statements which need unpacking or inferring.

Half of the PSTs - M and a quarter of the PSTs - N thought mathematical definitions need to be precise. For instance, M2 mentioned "Just because it is not like precise enough.... Just like what we are saying, like skip counting by 2 like we do not really define what skip counting is except for this pattern." Here, she pointed out that skip counting was not a formally defined concept and therefore using it to define another concept made the statement imprecise.

Although noncircularity is reported in the literature as an necessary feature of mathematical definitions (Borasi,1992), no PSTs mentioned it as a necessary feature either when answering the general interview questions or commenting on specific statements. A few PSTs realized that several statements given in Questions 7 and 10 were circular, but they still accepted them as valid mathematical definitions and only gave them a lower rating. Therefore, instead of being reported here, noncircularity will be reported as a preferred feature in Chapter 5.

Problematic thinking about mathematical definitions. Table 17 presents the misconceptions PSTs associated with mathematical definitions.

Table 17. PSTs' Problematic Thinking about Mathematical Definitions

Problematic Thinking	Meaning
Format "A is B"	Definitions are expected to be written in a format of "A is B".
Format "If - then"	Whether a statement is a mathematical definition is associated with if it is written in the format of "if-then".
Presentation format	Uncertainty about if mathematical definitions could be written only in symbols.
Name a property/procedure/ theorem	Mathematical definitions could be used to name a property/procedure/theorem.
Examples/ pictures	It is necessary or better to include examples or pictures in definitions.

Table 18 gives the distribution of PSTs who demonstrated each type of problematic thinking about mathematical definitions. Overall, PSTs - M tended to associate linguistic structures with mathematical definitions. PSTs - N were more likely to include examples or pictures as part of mathematical definitions.

Table 18. Distribution of PSTs Who Demonstrated the Idea of Each Type of Problematic Thinking about Mathematical Definitions

Problematic Thinking	Total	PSTs - M	PSTs - N
"A is B" format	14	9	5
"If then" format	8	5	3
Presentation format	6	3	3
Name a property/procedure/theorem	20	10	10
Examples/pictures	10	3	7

Note. Because some PSTs mentioned multiple ideas in their responses, the numbers in Total column add up more than 24.

Problematic thinking relevant to mathematical language. In this section, I am going to discuss three types of problematic thinking PSTs associated with mathematical definitions: "A is B" format, "if then" format, and presentation format. Overall, more than half of PSTs associated linguistic structure “A is B” with mathematical definitions. PSTs - M had a stronger tendency than PSTs - N to attend to this structure when discussing mathematical definitions. Three quarters of PSTs - M indicated that mathematical definitions are expected to be written in the format of “A is B”. For instance, When asked “What needs to be in a statement for it to be a mathematical definition?” M5 responded that “Usually it will have something like ‘something is and something’ and try to explain in other words. So usually it will start with one word or a phrase and then a sentence.” When probed anything else needs to be in a mathematical definition, she said “That is only thing that I can think for now.” This indicated that M5 took the format of “A is B”as a very important feature of mathematical definitions.

Because many mathematical definitions are written in this standard way, using linguistic feature “A is B” to help judge whether a statement is a mathematical definition is valid most time in elementary mathematics, but relying on it as a standard causes problems. Nine PSTs (5 PSTs - M and 4 PSTs - N) incorrectly accepted at least one non-definition as a mathematical definition by relying only on the format of “A is B”. For instance, N6 accepted the statement 7c (A fraction is a part whole relationship) as a mathematical definition and supported her claim by saying “For [statement 7]c it is saying a fraction is a blank. That is typically how definitions are set up in my mind. So like a subject is blank - defining characteristics.” Here N6’s attention focused on the linguistic structure of the statement and she ignored other aspects of mathematical definitions which were also important to examine. Five PSTs - M incorrectly reject at least one mathematical definition by relying on the format of “A is B”. For instance, when commenting on statement 11c (A rectangle is a square if and only if it has four equal sides), M3 said:

> That’s something you can just prove, like based on… like that’s not saying what a square is. That’s saying what a rectangle is, kind of, in a special scenario. So that’s not really a

definition of square.

Here, she drew a conclusion that the statement 11c is not a mathematical definition of a square because the sentence starts with “a rectangle is” instead of “a square is”. Five PSTs (4 PSTs - M and 1 PST - N) made correct judgments on if a given statement is a mathematical definition, but they offered at least one incorrect reason by relying on the format of “A is B”. For example, N5 correctly determined that statement 7g is not a mathematical definition, but instead of focusing on the nature of mathematical definitions distinct from other meta level mathematical objects (e.g., theorem), N5 relied on the linguistic feature. She said:

> I feel like some of these are more like statements like there are infinitely prime numbers I feel like if it was worded differently, like prime numbers are infinite, that would be more of a mathematical definition, but saying that there are infinitely prime numbers I feel like that is more of a statement then a definition just by the wording of it.

Here, she even indicated that keeping the same content but only revising the language changed a non-definitional statement into a definition.

Although PSTs were very consistent with expecting mathematical definitions to be written in the format of “A is B”, PSTs demonstrated different thinking about writing mathematical definitions in the format of “if then”. Two PSTs - M indicated that they preferred “if then” as a way to write a mathematical definition. For example, in response to interview Question 2 “What needs to be in a statement for it to be a mathematical definition?” M1 commented that “One thing that is really popular in class I've gone through is if - then statement, you know if this and this is true, then this is true.” He also gave an example of a definition of a square to illustrate his point and he wrote “if a convex shape has all sides equal and all angles, all interior angles equal 90 degrees, then the convex shape is a square.” In contrast, one PSTs - M

(M3) demonstrated that “if then” statement invites a proof therefore mathematical definitions should not be written in this format. She said:

> It’s set up in this if then statement kind of thing and that’s, for me, is not really how you write definitions usually. This is almost like just asking to be proved, I feel like. So like if we have this as like our decided definition, then here, we can say, okay, well, we can start with these requirements and then prove that based on those requirements, it’s a square.

Later, when commenting on statement 11f (If a quadrilateral in which all sides are equal and all angles are 90 degrees, it is a square), she further explained her thinking:

> And the other one was that it’s set up in this if - then statement kind of thing and that’s, for me, is not really how you write definitions usually. This is almost like just asking to be proved.

Two PSTs - M and three PSTs - N pointed out another problem of using the format of if-then to define a concept. They felt using if - then is to transform another shape into a square instead of defining a square. For instance, M4 said:

> And [statement 11]b, I just, some of these were really confusing to me because I don’t think that they’re, it’s not… it’s not like defining a square. It’s saying if something else has this, then it’s a square but that’s not… it’s not how I would say the def, that’s not really a definition of a square. It’s just telling you something. I don’t know if that makes sense...Like certain cond, like I felt like these were listing conditions that make a square, not defining a square.

One quarter of PSTs - M and one quarter of PSTs - N were unsure if mathematical definitions could be written in mathematical symbols or had to be written in words. This uncertainty caused their difficulties in judging whether some statements are mathematical definitions. For instance, M5 struggled with determining if statement 7k, 7m, 7n and 7o are mathematical definitions. She agreed that those statements “defined expressions”. But she was not sure if she should circle them as mathematical definitions. She said:

> Debating....they are more like mathematical symbols, so I am debating between symbols vs. words, OK. If a mathematical definition can only be words, then these are not. If it can be both symbols and words, and these will be mathematical definition.

Daily life use of definitions and mathematical definitions. Most PSTs - M and PSTs - N indicated that mathematical definitions could be used to name a property/procedure/theorem. Six PSTs - M and eight PSTs - N indicated that mathematical definitions could be used to define a procedure. For instance, when asked “What needs to be in a statement for it to be a mathematical definition?” N10 said:

> For it to be a mathematical definition, it might say, in a statement, please solve the problem to get the answer to the equation possibly. So it needs to tell you to solve or define this or to get the answer.

I was confused about her response at first, so when I asked her to write a mathematical definition of a mathematical concept, I asked her to write something which could explain her previous thinking. N10 then wrote the following:

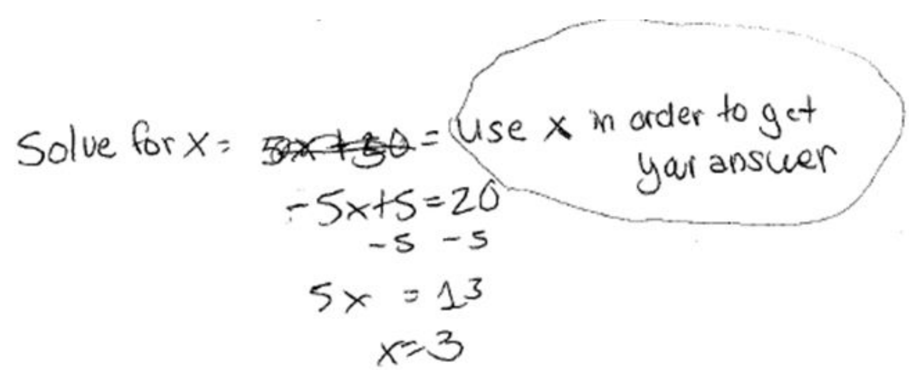


Figure 2. N10’s written definition of how to solve an algebraic problem

She explained:

> So I have an example, solve for x and then I said use x in order to get your answer so this problem is a 5x + 5 = 20 so you know, I showed how to subtract both sides. I gave another example. It was difficult trying to explain how to solve for x but this is an example of how to solve for something.

I probed her by asking "Which part you call definition?" She said:

> I would say probably using this part but it's not a very good definition I wrote here. Use x in order to get your answer....But probably something more descriptive like maybe. If you subtract, how to subtract from both sides.

I further probed her by asking her "This is a definition of what?" She said "The definition of how to solve like an algebraic problem such as like this, something like this I wrote here." It clearly showed that she took the description of a process of solving a problem as a mathematical definition.

Six PSTs - M and four PSTs - N demonstrated thinking that mathematical definitions could be used to define a mathematical property. For instance, when asked the first interview question, N9 said:

> Just like you have literacy definitions, in math you also have definitions for specific concepts of how to solve problems or what something is, such as mathematical definition for a property, like communicative property or something like that.

Here, N9 mixed up the mathematical definitions with daily life definitions (e.g., literacy definition) and thought mathematical definition could be used when *only* naming a mathematical idea. This misconception may cause confusion about the relationship between mathematical definitions and mathematical proofs which will be further elaborated in chapter 6.

Five PSTs, roughly split between PSTs - M and PSTs - N indicated that naming a theorem defines a mathematical concept and therefore the statement which names the theorem is a mathematical definition. For instance, when asked to write a definition for a concept that she felt comfortable with in Question 5, N12 said "Sure. We can do Pythagorean theorem." She then wrote the following as demonstrated in figure 3. Misidentifying a theorem as a mathematical definition may cause difficulties for PSTs to see the axiomatic structure of mathematics because now a statement could be a theorem and a mathematical definition simultaneously.

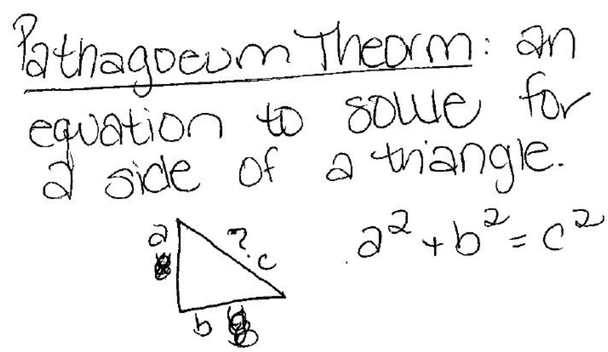


Figure 3. N12's written definition of the Pythagorean Theorem

Concept image and concept definition. As argued by Tall and Vinner (1981), "concept image is the total cognitive structure associated with the concept, which includes all the mental pictures, associated properties, and processes" (p. 151). Concept definition is "a form of words used to specify a concept" (p. 152). Tall and Vinner (1981) also found that mathematical learners may not recognize the differences between concept image and concept definition and may misapply concept image to solve mathematical problems when the concept definitions should be evoked to reach a correct answer. Being able to see differences between concept image and concept definition is important for future teachers. This subsection discusses how PSTs distinguished concept image and concept definition.

Overall all, two PSTs - M and seven PSTs - N indicated that it was either necessary or better to include pictures or examples in mathematical definitions. Four more PSTs - N than PSTs - M had this problematic thinking. Among the nine PSTs, two PSTs - M and five PSTs - N indicated that example is a necessary component of mathematical definitions. For instance, when I probed N3 “So which part of definition has the information that properties do not have which makes you feel you have to have mathematical definition?”, she responded with “Like an example.” When asked to write a mathematical definition of multiplication, N1 wrote the following:

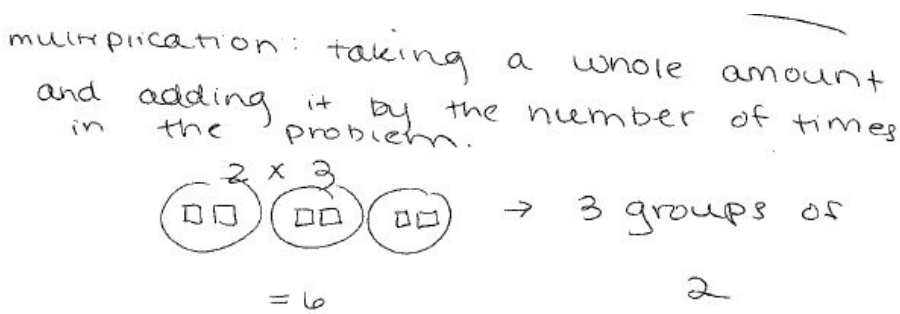


Figure 4. N1’s written definition of multiplication

N1 further explained the reasons for why she thought mathematical definitions should include examples and said:

> The written part might be kind of confusing but essentially, I think that this is why it’s important to use an example in your definition because here, like I would show like two times three is like three groups, like you do it three times of the two so then to show that that’s six. So I mean, for me, especially like I get tripped up on words a lot, I mean, regardless if I’m writing them or reading them. Like sometimes it’s confusing for me which is why I want the visual so that’s why I would always include it in a definition of something.

The motivation of her inclusion of examples is good because she thought examples help clarify confusions but it also showed her lack of understanding of the nature of mathematical definitions from disciplinary perspective.

However, the other two PSTs indicated that whether mathematical definitions should include an example vary across different situations. For instance, when asked to write a mathematical definition of the area of a rectangle, N12 wrote many things as indicated in the following figure.

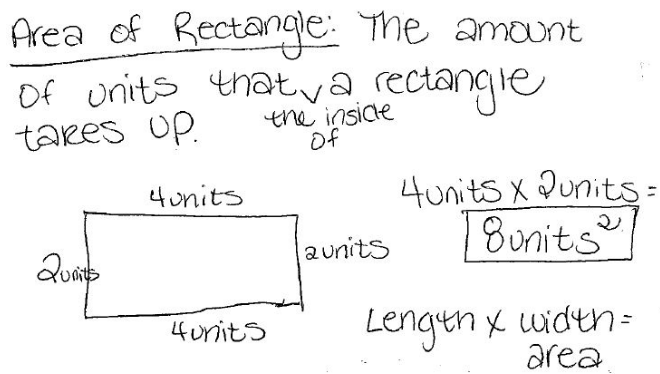


Figure 5. N12’s written definition of the area of a rectangle

When I probed her “Do you think all things you wrote are definitions?” She responded:

> No, I think this is more of an example (she pointed to the picture she drew) and then really you need this part and you need this part. This part being the equation and the purpose and the example it is nice to have with definition but it is not necessary.

This quote reveals that in her mind examples are not necessary to be included in mathematical definitions. However later when asked to write a mathematical definition of fraction multiplication, she wrote “Fraction multiplication: the act of multiplying two or more fractions.” She then continued saying:

> It is not a very good definition, but I will say fraction multiplication is the act of

> multiplying two or more fractions. So this one I would usually write an example because I don't feel like that definition is very strong. I could not think of another word for multiplying, so -

I probed her by asking “So you want to add an example to clarify that basically, but do you think examples belong to the definition?” She responded:

> If it is a really good definition writer, then I don't think so, but again I think definition needs to be very clear and the reader needs to understand it, so if the definition isn't clear then there should be an example to make it clear....If the definition is not clear, then example should be part of the definition, but if it is a very clear definition, then it should not need an example.

The conversation above indicates that N12 was flexible with including examples in the definition and the decision was based on if the statement was clear or not. We can see that both N1 and N12 had a reason for why they wanted to include examples in the definitions. And the reason they attended, namely making the statement clear, is an important feature of mathematical definitions. However, their over attention to this feature contributed to the fact they included non-relevant and extra information in mathematical definitions.

Two PSTs - N and two PSTs - M indicated that it was either necessary or better to include pictures in mathematical definitions. For instance, when asked “What needs to be in a statement for it to be a mathematical definition?”, M1 said:

> I would say lots of definitions include pictures to go along with what the if-then statement is. You know, for example, the square, if each line segment is equal and all angles are equal to 90 degrees, then the shape is a square and to include a picture with that, which most textbooks do and which professors do when teaching definitions.

I further confirmed with him that whether the picture is part of the definition, he said "Yes, it is." Here, M1 also mentioned an important reasons for why he thought pictures are part of mathematical definitions. He saw that pictures always follow definitions in textbooks and when professors presented definitions, they always drew pictures. Another PST S1 gave a similar response for why she thought mathematical definitions should include pictures:

> Yeah. I mean, a lot of them were like about, again, like how we went over like the shapes, like what, what, like what angles or whatever construct a certain shape. So in terms of those, he would like write on the board how they're constructed and then he would like draw a picture of it so I guess that's kinda like where my thinking about a definition including a visual comes from.

One PST M4 articulated that pictures could be an optional component of mathematical definitions. She chose to write a definition of a square in Question 5 as shown in the following figure:

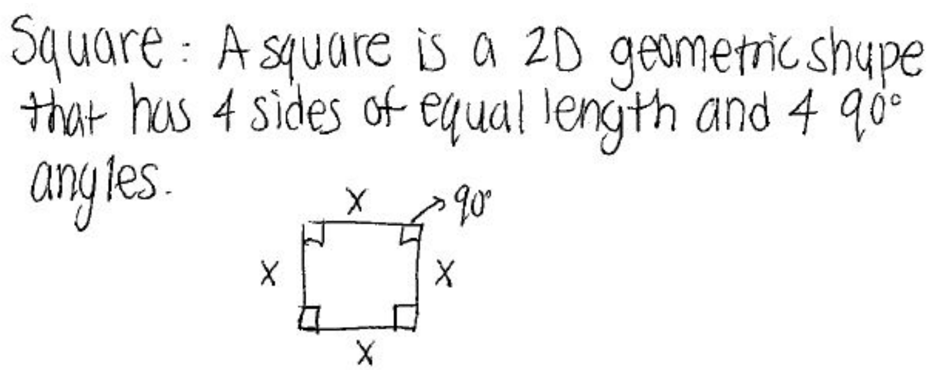


Figure 6. M4' written definition of a square

She then said "I think definitions are sometimes hard to word because you want to be as precise as possible which is why I drew the picture to make it even more clear." When probed "Is the picture part of a definition?" She said "No, not necessarily but it can be." Even though she did not insist including pictures in the definition, her thinking that pictures could be part of definitions is also problematic.

Notice that there is nothing wrong to include examples or pictures when teaching mathematical definitions. Actually elementary teachers are encouraged to include examples and pictures when teaching abstract mathematical concepts. However, the focus of this dissertation is to study preservice elementary teachers' thinking of mathematical definitions from the disciplinary perspective. Hence, it is problematic for preservice elementary teachers to think that mathematical definitions should include pictures or examples.

PSTs' Understanding of Mathematical Definitions When Responding to General Interview Questions

In this last section, I examine PSTs' general understanding of mathematical definitions by looking at their responses to general interview questions 1-3 as a group. The reason I examine questions 1-3 as a group is because all three questions target the same goal, but with different probes; therefore, different PSTs may give similar responses when answering different questions, which makes it meaningless to analyze these three questions separately. The analytical perspective in this section is the same one used in the previous section; the only difference is the portion of the data I used for analysis. In the previous section, I used a larger data set which included PSTs' responses to general questions and also to specific statements (Questions 7, 10, and 11), but in this section, I only look at PSTs' responses to general questions. The main purpose of this analysis is to explore to what extent PSTs are able to articulate necessary features of mathematical definitions without scaffolding. Table 19 gives distribution of necessary features PSTs articulated about mathematical definitions when they responded to interview Questions 1-3.

Table 19. Distribution of PSTs Who Articulated the Following Necessary Features When Responding to General Interview Questions

Feature	Total	PSTs - M	PSTs - N
Unambiguous	3	2	1
Correct	7	4	3
Precise	4	2	2
Concise	3	2	1
Straightforward	1	1	0

A comparison between Table 16 and Table 19 suggests a big difference in the number of PSTs' who mentioned necessary features of mathematical definitions with and without support from specific mathematical statements. Seven PSTs - M and nine PSTs - N did not mention any features listed in Table 19 when they responded to Questions 1-3. This is a big contrast with the fact that all PSTs - M and 11 PSTs - N mentioned at least one feature when responding to questions 1-3, 5, 7, 10 and 11. This observation indicates PSTs' difficulties in discussing mathematical definitions as a meta level concept when no scaffolding was given. This is an evidence that PSTs' understanding of mathematical definition was not very strong so they relied on specific statements to evoke certain features they would have missed if specific statements were not given. Similar numbers of PSTs - M and PSTs - N mentioned each of the features listed in Table 19.

CHAPTER 5: PRESERVICE ELEMENTARY TEACHERS' PREFERENCES FOR MATHEMATICAL DEFINITIONS

This chapter aims to answer research question 1b: Among the valid mathematical definitions, what types of definitions do preservice elementary teachers prefer? I organize this chapter by first presenting how PSTs rated statements in Question10 and Question 11 and then providing an analysis on what criteria PSTs used when they evaluated a potential mathematical definition. From these criteria, I extracted the features that PSTs preferred to include in mathematical definitions.

PSTs' Evaluation of Mathematical Definitions

Question 10 consists of a list of statements related to an even number and asks PSTs to circle those statements that they think are mathematical definitions. They then rated those circled mathematical definitions by using a 4 point scale. Question 11 gives the same instruction but is about a different concept, a square. Tables 21 and 22 provide average scores PSTs assigned to each statement in Question 10 and Question 11. If a statement was not selected as a mathematical definition (i.e., PSTs did not circle the statement), I assigned 0 points to the statement. Several PSTs were not sure about how to rate a couple of statements because they thought they did not have enough mathematical knowledge to support their judgments. In this case, when I calculated the average score, I excluded those PSTs.

Evaluation of Mathematical Definitions of an Even Number

Overall, the average rating given by all PSTs ranged from 1.2 to 3.1. The average difference of rating between PSTs - M and PSTs - N ranged from 0.1 to 0.9. The statements with the highest ratings were 10b, 10c, and 10f. The average rating for these three statements were about 3 points. These three statements are the formal ways to define an even number and are

Table 20. Distribution of Average Scores PSTs Assigned to Statements in Question 10

Statement in Question 10	Total	PSTs - M	PSTs - N
a. A number is called even provided it represents a number of objects that can be placed into two groups of equal size.	1.9	1.6	2.2
b. A number is called even if it is an integer multiple of 2.	3.1	3.2	3.0
c. A number is called even if it is divisible by 2.	2.9	2.9	2.8
d. A number is called even if it can be divided into 2 parts.	1.4	1.4	1.5
e. A number is called even if it ends with 0, 2, 4, 6, 8.	2.3	2.1	2.4
f. A number is called even if it can be written as 2k, where k is a whole number.	3.0	3.4	2.5
g. An even number is 1 more or 1 less than an odd number and an odd number is 1 more or 1 less than an even number.	1.3	1.1	1.4
h. A number which occurs as we skip count by two ("0, 2, 4, 6, 8, 10, 12, 14,…").	1.2	0.9	1.5
i. An even number of objects can be paired up (with none left unpaired).	1.3	1.2	1.5
j. An even number is a number that is twice a whole number.	2.2	2.5	1.9

often seen in mathematics textbooks. Even though statement 10j was very similar to statements 10b, 10c, and 10f, PSTs did not give it as high an average score as they gave to those statements. PSTs' main concern for statement 10j was the use of phrase "twice a whole number". Some PSTs thought it was unclear and confusing whereas others felt it took more effort to unpack this phrase. For statements 10b and 10c, PSTs - M and PSTs - N gave similar average scores. However, for the statement 10f, the average score PSTs - M gave was about 1 point higher than the score given by PSTs - N. A further analysis indicated that the reasons PSTs - M gave a high rating were due to the statement being formal, the use of mathematical language, and offering an easy way to check if a number was an even number. The reasons some PSTs - N gave a higher score to statement 10f were due to the statement being familiar to them and understandable. However, several PSTs - N gave low scores to this statement and commented that the disadvantage of using this statement as a mathematical definition was because the statement 10f involved equations and used variables which made the statement more complicated. Here, PSTs - M and PSTs - N's different mathematical levels played an important role when they rated the statements. PSTs - M who received more training in mathematics preferred mathematically sound statements and symbolic representation. It was easy to check if a given number was an even number. However, PSTs - N who were less fluent in symbolic representation were concerned about using equations in mathematical definitions.

The statements which got relatively lower average ratings (less than 1.5 points) were statements 10d, 10g, 10h and 10i. Many PSTs indicated that 10d was mathematically incorrect. Similarly, for the statement 10i, PSTs commented that odd numbers satisfied the statement as well. For instance, 1.5 and 1.5 can be paired up but the sum 3 was not an even number. PSTs' big concern for statement 10g was that it did not detail what an even number was and it was roundabout. PSTs commented that for the statement 10h the main problem was that it was hard to check large numbers through skip counting and also the statement was informal. For statements 10d, 10g and 10i, PSTs - M and PSTs - N gave similar average scores (the difference was at most 0.3 points), but for the statement 10h, the score PSTs - M gave was 0.6 points lower than the average score given by PSTs - N. One reason contributing to this rating difference was that three PSTs - N liked the example 0, 2, 4, 6, 8, 10, 12, 14 given in the parenthesis. Therefore PSTs-N gave 4 points to the statements whereas no PSTs - M thought the statement 10h was perfect and the highest scores assigned was 3 points.

For four statements 10a, 10f, 10h, and 10j, the difference of average scores given by PSTs - M and PSTs - N was at least 0.6 points. For statement 10a, PSTs - M gave lower average score compared to PSTs - N. PSTs - M's main concern about this statement was that it applied to odd numbers as well (e.g., 5 can be split into two equal groups 2. 5 and 2.5). A few PSTs - N also had similar concern but the number of them was much less than PSTs - M. Hence, the average score given by PSTs - N was higher than PSTs - M. I discussed the possible reasons for PSTs - M and PSTs - N's different ratings on statements 10f and 10h in the previous paragraph. For statement 10j, many more PSTs - N compared to PSTs - M were more concerned about the phrase "twice a whole number". This also indicates that the amount of mathematical training PSTs received impacted how they rated mathematical definitions. PSTs - M who were more familiar with "twice a whole number" tended to give a higher rating compared to PSTs - N who were less familiar with the phrase.

Evaluation of Mathematical Definitions of a Square

Overall, the average rating given by all PSTs ranged from 1.1 to 3.4. The average rating difference between PSTs - M and PSTs - N ranged from 0 to 1.4. The statements which received the highest average scores were the two statements that are commonly used as mathematical definitions of a square in textbooks, namely statements 11f and 11g. However, four PSTs gave 0 points to at least one of the two statements. For these PSTs, the statement did not include examples, the statement was not minimal or the statement used uncommon linguistic structure which did not look like a defining statement. Other concerns about using statements 11f and 11g as mathematical definitions of a square was using quadrilaterals and polygons to define a square. From PSTs' perspective, quadrilaterals and polygons were not commonly seen mathematical concepts. One important reason statement 11g got lower average score than statement 11f was because PSTs felt a need to reason. They contended a polygon with 4 equal angles implied that each angle had to be 90 degrees. There are two additional reasons. The first reason being the linguistic structure and phrasing of statement 11g using "if and only if." Another reason PSTs gave a lower rating was due to the phrase (i.e., a polygon is a square with certain properties vs. a square is a polygon with certain properties).

Table 21. Distribution of Average Scores PSTs Assigned to Statements in Question 11

Statement in Question 11	Total	PSTs - M	PSTs - N
a. A square is a rhombus with a right angle.	1.5	1.2	1.8
b. If a rhombus has 4 equal angles, then it is a square.	2.0	1.3	2.7
c. A rectangle is a square if and only if it has four equal sides.	2.1	2.0	2.3
d. A parallelogram with diagonals that are equal, and perpendicular is called a square.	2.4	2.4	2.4
e. A quadrilateral with diagonals that are equal, perpendicular, and bisect each other is a square.	2.5	2.5	2.4
f. If a quadrilateral in which all sides are equal and all angles are 90 degrees, it is a square.	3.4	3.3	3.5
g. A polygon is a square if and only if it has four equal sides and four equal angles.	2.8	2.6	2.9
h. A polygon with four equal sides and three equal angles is a square.	1.4	1.4	1.3
i. A square is the locus of points for which the sum of their distances from two given perpendicular lines is a positive constant.	1.4	1.1	1.6
j. If a parallelogram is both a rectangle and a rhombus, then it is a square.	1.1	1.3	1.0
k. A parallelogram with one right angle and two adjacent sides congruent is a square.	1.3	0.8	1.7
l. An object that can be constructed (in the Euclidean Plane) as follows: Draw a segment; from each edge erect a perpendicular to the segment, in the same length as the segment (both in the same direction). Connect the other 2 edges of the perpendiculars by a segment. The four segments form a quadrilateral that is a square.	1.5	1.5	1.4

For statements 11a, 11h, 11i, 11j, 11k and 11l, the average ratings were lower than 1.5 points. Among all statements, statement 11j received the lowest score. Noting that the statement was mathematically incorrect from their perspectives, ten PSTs assigned 0 points to statement 11j. Perhaps the PSTs were not familiar with rhombuses or they found it difficult to unpack the properties of rectangles and rhombuses in order to obtain the properties of squares. For statements 11a, 11h, and 11k, many PSTs' comments signaled that the statements did not fit the image of a square. PSTs either responded with the reasons of a square having a right angle, three equal angles, or two adjacent sides being equal. This was in contrast with their definition of, a square having four equal sides and four right angles. When probed, most PSTs realized that the properties in statements 11a, 11h and 11k indicated that the shape had four equal sides and angles. Therefore the shape was a square. For instance, PSTs were aware that a polygon with four equal sides and three equal angles forced the last angle to be equal too. However, they still preferred to have four equal angles and sides clearly listed in the statements because it was more consistent with the image of a square and took less effort to infer. This illuminates that when minimality was in conflict with other preferred features, PSTs preferred other features.

Statement 11i was rated the third lowest among all statements. Many PSTs did not know the meaning of locus point and did not understand the statement. Some PSTs also commented that the statement was "too complicated" and "there was no need to define an easy concept like a square in this complicated way". However, five PSTs gave statement 11i at least 3 points with three PSTs giving it 4 points. These PSTs demonstrated openness to accept new ideas and expressed that statement 11i showed another aspect of a square and reminded them of the mathematical definition of a circle. These five PSTs were roughly split between PSTs - M and PSTs - N.

The average rating statement 11l received was also lower than 1.5 points. PSTs thought how to construct a square was different from saying what a square was directly. PSTs also complained that statement 11l was confusing, wordy, and took too much effort to find the properties of a square from the description of the construction. Five PSTs gave statement 11l at least 3 points with two PSTs giving it 4 points. The reasons for the high rating were mainly due to the construction providing a visual aid to support understanding what a square is. Additionally, statement 11l involved no additional types of polygons (e.g., rectangles) to define a square. Thus, it was perceived as more straightforward.

Overall, the average scores PSTs - M assigned to most of the statements were either lower or roughly the same as the average scores PSTs - N assigned except for statement 11j. This finding suggests that PSTs - M tended to give lower scores than PSTs - N which indicates that when evaluating mathematical definitions PSTs - M were more strict than PSTs - N. For statement 11j, as discussed in the previous paragraph, quite a few PSTs had hard time seeing the relationship between rectangles, rhombuses and squares and were unable to sort out how parallelograms, rectangles and rhombuses fit together to define a square. Therefore, they gave statement 11j 0 points due to their uncertainty about the correctness of the statement. Due to this concern, more PSTs - N than PSTs - M assigned 0 points to statement 11j. Therefore, the average score PSTs - N assigned to statement 11j was lower than the scores assigned by PSTs - M. This finding suggests that PSTs' mathematical proficiency had an impact on how PSTs evaluated mathematical definitions.

For two statements 11b and 11k, the difference of rating between PSTs - M and PSTs - N was about 1 point. Further analysis indicated that for statements 11k, the criteria PSTs - M and PSTs - N used to judge the statements were quite similar. However, PSTs - M tended to assign lower scores to the statements even based on the same reasons PSTs - N applied. Hence, the average scores PSTs - M assigned to 11k were lower than PSTs - N. This finding echoed the previous claim that PSTs - M tended to be more harsh in evaluating statements than PSTs - N. For statement 11b, PSTs - M gave similar reasons as PSTs - N. They contended that the statement used an unfamiliar concept rhombus and took additional effort to infer that four equal angles implied four right angles. However, PSTs - M offered two additional reasons that no PSTs - N offered: mathematical definition needed to be minimal but statement 11b was not, and the linguistic structure used in statement 11b made the statement more like a property of a square instead of a mathematical definition of a square. Because four PSTs -M gave lower scores due to these two additional reasons, the average score PSTs - M assigned to statement 11b was lower than the average score PSTs - N assigned. This result provided further evidence that PSTs' mathematical knowledge, in this case their knowledge that mathematical definitions should be minimal and follow certain linguistic formats, had an impact on how they evaluated mathematical definitions.

PSTs' Preferred Features of Mathematical Definitions

Based on PSTs' responses to Questions 4, 10, and 11, I grouped the features PSTs mentioned into four categories based on different types of considerations: *mathematical features, communicative features, external features,* and *other features.* Table 22 presents the distribution of PSTs who preferred mathematical definitions to include specific feature. Later in the section, I explain the meaning of each category and feature by providing excerpts from transcripts.

Mathematical features are the features related to imperative and optional features of mathematical definitions as discussed in Chapter 2. Communicative features are mainly concerned whether a statement is clear or easy to understand. External features focus on non-content related features such as if a statement is familiar to PSTs, if a statement is mathematical sound or if a statement uses certain linguistic structure.

Overall, when comparing mathematical definitions, communicative features as a group were mentioned by almost all PSTs other than two PSTs - M. Further examination indicated that for each type of communicative feature, more PSTs - N compared to PSTs - M mentioned it. Sixteen PSTs considered mathematical features and similar number of PSTs considered external features. For each type of mathematical and external features, numbers of PSTs - M and PSTs - N who mentioned it were similar.

Table 22. Distribution of PSTs Who Preferred Mathematical Definitions to Include the Following Features

Feature	Total	PSTs - M	PSTs - N
Mathematical features			
Ease to test if an instance is an example of a concept	11	5	6
Essence of the concept	9	5	4
Ease to apply to construct a mathematical proof	2	2	0
Noncircularity	3	0	3
Communicative features			
Conciseness	9	3	6
Clarity	10	5	5
Ease to visualize	6	2	4
Straightforward	22	10	12
Precise	1	1	0
External features			
Familiarity	14	6	8
Mathematical sound	7	3	4
Linguistic features	4	3	1
Other Features			
Learning	4	3	1
Procedure	7	4	3

Mathematical Features

Almost one half of PSTs indicated that they preferred mathematical definitions which allowed them to easily check whether a given example was an instance of a mathematical concept. Similar numbers of PSTs - M and PSTs - N mentioned this feature. For example, when rating statement10h (A number which occurs as we skip count by two (“0, 2, 4, 6, 8, 10, 12, 14,…”)), M11 said:

> [Statement 10]h, I give this one the lowest rate because....if someone gives this to me and I know what an even number is and I need to check the number 2048, like it would be

> ridiculous to count up 2, 4, 6, 8 all the way to that number just to see if it is even.

Here, M11 expressed the idea that using skip counting to define an even number was not helpful because it would be hard to test if a number was even if the number was large.

One quarter of PSTs with similar number of PSTs - M and PSTs - N indicated that they preferred mathematical definitions which revealed the essence of the concept or helped understand the concept. For instance, when asked to rate statement 10e (A number is called even if it ends with 0, 2, 4, 6, 8), M11 said:

> It is very useful for checking if a number is even but it does not really give you any understanding of what it means to be even. So like this number 36 is even because I can see it ends with 6, but only if I divided by 2 like understand that oh, this number is a sum of two odd numbers, so I just there is more depth to these other definitions. This one is a little more shallow....[statement 10]f is similar to [statement 10]b and [statement 10]c, 2k one because like 36 we can write like k times 2 if we can find the k that is whole like it makes you analyze the number more. It makes you analyze what it means to be even.

Similarly, N2 commented on the same statement that "Obviously, if it ends with these numbers, we know that is an even number, but it does not make it. I mean it does not necessarily make it an even number." Both PSTs indicated that in addition to determining whether an example was an instance of a mathematical concept (even number in this case), the better mathematical definitions also shed light on the meaning and the essence of the concept. Namely, a good mathematical definition not only provided sufficient and necessary conditions for the concept, it also revealed the most significant features of the concept.

Two PSTs - M but no PSTs - N mentioned that they preferred mathematical definitions that allowed them to construct a mathematical proof or derive other mathematical results easily.

For instance, when commenting on statement 11g (A polygon is a square if and only if it has four equal sides and four equal angles), M11 said:

> Because - I just think like 4 equal sides and 4 equal angles like the most basic definition of what a square is and then from that you can derive everything else. Not it is that you cannot go backward. You can definitely derive [statement 11]c from [statement 11]e, it is just like - I feel it will be more difficult.

Here M11's consideration was that starting from which statement (as a mathematical definition) made it easier to derive other statements (deduced results). He understood the arbitrariness of the mathematical definitions, but he wanted to select a definition so other properties can be deduced easily.

Three PSTs - N but no PSTs - M pointed out a good mathematical definition should be noncircular. Namely, it is better for a mathematical definition not to use the term it is trying to define. For instance, when commenting on statement 10e (A number is called even if it ends with 0, 2, 4, 6, 8), N6 said:

> I think e is hard because it says a number is called even if it ends with even number essentially....like you are using an even number in a definition of an even number. So I guess I would say this also is one [mathematical definition], but it is not a very good one....Ok an even number is a number that ends with an even number, but what is an even number?

Here N6 indicated that using ending digits to define an even number was like using an even number to define an even number. Later, when commenting on another statement 10g (An even number is 1 more or 1 less than an odd number and an odd number is 1 more or 1 less than an even number), she gave a similar comment:

> [Statement 10]g also would be one [mathematical definition], but I think not a very good neither. Because is like using again it is like using the same language to define itself so it is saying an even number is 1 more or 1 less than an odd number and Ok if I was reading that I said Ok, what is an odd number then, and then it is saying oh an odd number is one more or one less than an even number so you have this big question of what is an odd and what is an even. It does not really tell you the difference. Because if you do not know what an odd number and an even number is, then you do not know what is one more or one less than an odd or even number.

However, all three PSTs - N who demonstrated that the circular definitions were not good mathematical definitions still regarded them as valid mathematical definitions. They gave the circular statements low scores, but they still selected them as a mathematical definition of the given concept. This finding suggests that even though some PSTs saw circular statements as problematic, they did not understand that this feature should be a necessary instead of a preferred feature for mathematical definitions.

Communicative Features

In total, almost one third of PSTs indicated that concise is a preferred feature of mathematical definitions. More PSTs - N than PSTs - M indicated this feature. For example, when commenting on Question 11 generally, N4 said “They’re all saying the same thing but with more or less language. In my opinion, less language but meaning the same thing is a smarter definition.” When I probed her what she meant by “less language”, she said “short and simple....Like you have to look to less things to understand what a square is than this.” Her comments here did not focus on the content of the statement, but on the length of the statement.

Less than one half of PSTs preferred more clear mathematical definitions. For instance, M11 stated that:

> I read [statement 10]b and it is a little confusing to me. What an integer multiple of something is. That was like a confusing term to me. I don't know if I have seen that before or I have not, it has not been for a while.

Here, the phrase “an integer multiple of something” is unclear to M11.

Almost one quarter of PSTs indicated that they preferred mathematical definitions which were easy to visualize or close to the concept image. PSTs who preferred this feature tended to give high ratings to statements that generated images. For instance, when commenting on statement 11i (A square is the locus of points for which the sum of their distances from two given perpendicular lines is a positive constant), N5 said:

> I think I am going based of the image wise. I feel like this [statement 11 i] is a mathematical definition but it's very confusing and uses a lot of....because to me saying that they’re just equal sides and four angles that are 90 like that's very basic. I can picture in my head what it looks like as a definition.

Similarly, when responding to statement 10a, another PST M2 said “I think of it I like relate to - saying like 2 parts, 2 groups, similar in my head, they gave that visual 2 groups.”

Almost one third of PSTs - M and one half of PSTs - N indicated that straightforward is a preferred feature for mathematical definitions. PSTs tended to rate definitions which involved more mathematical concepts and required more unpacking and reasoning lower than those that were more direct. For instance, when commenting on statement 11j (If a parallelogram is both a rectangle and a rhombus, then it is a square), M1 said:

> For [statement 11]j, I will say it is a definition, but I will give it a 3, because basically you have to know that a rectangle has all 4 angles 90 degrees and you have to know a

rhombus has 4 sides. So you have to know the definition of rectangles and rhombuses to know that this definition is a square. So I do not think it is very directly, I think it is - you have to know more definitions to...

Here, M1 was unhappy with the amount of unpacking or reasoning that had to be done on rectangles and rhombuses in order to obtain the properties of squares.

Among the 11 PSTs - N who preferred mathematical definitions to be straightforward, six of them directly pointed out that they preferred the mathematical definitions which used straightforward language. Instead of commenting on the content of the statement, their comments focused on how the statements were expressed. For instance, when commenting on statement 10j (An even number is a number that is twice a whole number.), N4 said:

> Twice a whole number is awkward wording for me to think about. I have to go twice a whole number, okay, so I have a whole number and then I have to go back and say twice… it was just two times, multiplied.

Here, she preferred using “multiplied by 2” instead of “twice” because the word “twice” took her time to think about. However, no PSTs - M indicated this preference of wording. The reason might be that PSTs - M had more chances being exposed to various ways of expressing mathematical ideas therefore they were more flexible with the wording compared to PSTs - N who only saw limited ways to convey mathematical ideas.

External Features

More than one half of PSTs indicated that familiarity was one criterion they applied to judge a mathematical definition. For 83.8% of the statements where PSTs applied familiarity as a criterion, PSTs were unable to to give other reasons for their ratings other than familiarity. Among the 14 PSTs who applied familiarity as their criterion of evaluating definitions, eight of them were unable to provide other reasons for their ratings of specific statements other than saying that the statement was familiar to them. This is not saying that the only criterion these eight PSTs used during the interviews was familiarity, but rather that when they rated a specific statement, instead of giving multiple reasons, the only reason they gave was the familiarity with the statement. For instance, when asked to explain why giving different ratings to two similar statements 10b (A number is called even if it is an integer multiple of 2) and 10f (A number is called even if it can be written as 2k, where k is a whole number), N9 said “Maybe because this [statement 10f] is what I’m most familiar with and I… I’m not exactly sure my reasoning more than just comfort in what I know. ” Similarly, when asked to explain why giving statement 10g (An even number is 1 more or 1 less than an odd number and an odd number is 1 more or 1 less than an even number) 3 instead of 4 points, M1 said “It was not how I was taught to define an even number” without giving other reasons. Four of the 14 PSTs who used familiarity as one rating criterion were able to provide other reasons in addition to familiarity when they rated some statements, but not all statements. Only two of the 14 PSTs were able to provide other criteria to all the statements they rated with the criterion of familiarity. Namely, these two PSTs used familiarity as one of their criteria for rating a statement, but they were also able to give others reasons for their ratings. In total, twelve PSTs who highly relied on familiarity when they rated a specific statement indicated PSTs’ lack of knowledge to evaluate mathematical definitions. Each PST may be familiar with certain mathematical definitions based on their own schooling and mathematical experience, but as a future teacher, she or he should have a better criterion to differentiate mathematical definitions other than solely relying on familiarity. Even though more PSTs - N compared to PSTs - M applied the criterion “familiarity” during their ratings, more PSTs - M compared to PSTs - N relied solely on familiarity when they rated a specific statement.

Almost one third of PSTs indicated that they preferred mathematical definitions that were mathematically sound. The PSTs who indicated this feature were roughly split between PSTs - M and PSTs - N. For instance, when commenting on statement 10c (A number is called even if it is divisible by 2), N7 said “It’s a much more mathematical sounding definition.”

Three PSTs - M and one PST - N indicated that they rated mathematical definitions based on the linguistic feature of the statements. These PSTs explicitly pointed out the linguistic structure of the sentence impacted their ratings. Two PSTs - M and one PST - N preferred to defining a concept where the name of the concept appeared the first followed with the properties. They tended to give lower ratings to the statements where the properties of the concepts appeared before the name of the concept. For instance, when commenting on statement 11j (If a parallelogram is both a rectangle and a rhombus, then it is a square), N6 commented:

> But the way the sentence is phrased is saying - if a parallelogram - the subject seems to be more a parallelogram turning into a square where like being a square rather than a square encompassing these 3 categories. So yes, but probably 1 or 2 [points] like the other ones.

However, she did not regard the latter case as an invalid mathematical definition but only a bad mathematical definition. Similarly, when commenting on the statement 11b (If a rhombus has 4 equal angles, then it is a square), M8 asked “Is it defining a square? Or a property of a rhombus?” Another PST - M paid attention to whether the statement was written in a definite way and used it as a criterion to evaluate the mathematical definitions. For instance, M9 had difficulty accepting the phrase “can be” in statement 11l and said “It did not say - Like to me that *can be* [emphasis added] constructed I guess I want it to be more explicitly state like the object that *is* [emphasis added] constructed is a square." When the interviewer probed "What is the problem of 'can be?" She said that "I would rather it to be definite like an object that is constructed." but she still accepted the statement as a definition. Similar to the results in Chapter 4 where more PSTs - M than PSTs - N attended to the linguistic features when evaluating mathematical definitions, here also more PSTs - M compared to PSTs - N rated mathematical definitions based on the linguistic features.

Other Features

In addition to the above three types of features, two additional features do not fall into any category, so I report them here separately. Learning feature is the feature PSTs considered when they attended to the learning sequence typically happening in school in terms of the order of concept building. Concepts used to define a new concept have to be acquired earlier in a typical school setting. At first glance, this feature is similar to the hierarchy feature of mathematical definitions (Van Dormolen & Zaslavsky, 2003). Namely, any new concept must be described as a special case of a more general concept (p. 94). However, learning feature may not always be consistent with the hierarchy feature and depend on the learning sequence. For instance, a polygon is a general concept compared to a square, so it is reasonable to think that a polygon is a good option to define a square, but a PST who attended to learning feature may have opposite opinions if she or he felt that a square was acquired earlier in the schooling than a polygon which made using a polygon to define a square unreasonable.

One sixth of PSTs mentioned learning feature when they rated mathematical definitions. More PSTs - M compared to PSTs - N mentioned this feature. For instance, when commenting on statement 10g (An even number is 1 more or 1 less than an odd number and an odd number is 1 more or 1 less than an even number), M2 said:

> And if you are learning about even numbers, or defining them, it would be hard to start with the definition of an odd number because I think the definition of an odd number - in my learning process I started even number and then I went to odd numbers because I used the definition of an even number in my definition for an odd number. Like 2k+1.

This PST remembered when she learned even and odd numbers, she learned even number first before she learned odd numbers. Thus she thought it was confusing to define an even number through an odd number.

Seven PSTs mentioned that a statement which gave a direction for constructing a concept was not a good mathematical definition mainly because it did not tell the property of the concept other than how to generate the concept. For instance, M2 gave statement 111 2 points and offered the following reason:

> It makes me want to do it. So I understand the definition once I do it.... it does not tell me properties of a square kind of clearly that I would want in definition. It tells me how to construct it but it does not really tell me like properties that I would want.

This finding is consistent with the report from Zaslavsky and Shir (2005) that high school students in Israel were not willing to accept a procedural statement as a mathematical definition.

CHAPTER 6: PRESERVICE ELEMENTARY TEACHERS' UNDERSTANDING OF AXIOMATIC SYSTEMS AND ROLE OF MATHEMATICAL DEFINITIONS

Mathematical definitions, as the foundation of mathematical proofs, play an important role in the axiomatic system of mathematics. Whether PSTs understand the important roles mathematical definitions play in mathematics and whether they understand that mathematical definitions are the beginning of mathematical proofs, rather than the deduced results from mathematical proofs, may affect their decisions on whether they request mathematical explanations from their future students. This chapter aims to investigate PSTs' thinking on these issues and address research questions 1c and 2: namely to what extent do PSTs distinguish mathematical definitions from other meta level mathematical objects, and what are PSTs' conceptions about the role of mathematical definitions in mathematics.

I organize this chapter into three sections. In the first section, I examine PSTs' performance in judging whether given mathematical statements need mathematical proofs and the criteria they used to make these judgments. The data source used in this section is PSTs' responses to Question 8. In the second section, I examined PSTs' knowledge of the axiomatic system of mathematics, namely, how they think about the relationship among mathematical definitions, theorems, mathematical proofs and counterexamples. Data used in this second section mainly are PSTs' responses to Questions 7, 8 and 9. In the last section, I look at how PSTs discussed the roles mathematical definitions play in mathematics. Data used in this last section are PSTs' responses to Question 12.

PSTs' Understanding of Needs for Mathematical Proofs

Balacheff (1991) claimed that if students had difficulties in constructing deductive reasoning. The first reason is "the lack of students' awareness of the necessity to give any proof"

(p. 175). The second reason is “their lack of logical maturity” (p. 175). Seeing the need for a mathematical proof is a prerequisite for PSTs to engage in proving. In this section, I depict how PSTs see the need for a mathematical statement to be proved. I divide this section into two subsections. In the first subsection, I report item level data from Question 8 on whether PSTs thought given statements needed mathematical proofs. In the second subsection, I report the criteria PSTs used in making their judgments in Question 8.

PSTs' Evaluation of Whether Statements Need Mathematical Proofs

Table 23 presents the numbers of total PSTs (PSTs - M and PSTs - N) who judged correctly whether a given statement needs a mathematical proof in Question 8. Because mathematical definitions are arbitrary and different mathematical systems may start with different mathematical definitions, the same statement could be a mathematical definition in one mathematical system, but could also be a theorem in another mathematical system. Therefore whether a statement needs a mathematical proof depends which system the statement is in. Here, I define the correct judgment to be consistent with PSTs' past mathematical experience. For instance, statement 8b (the Pythagorean Theorem) could be used to define a right triangle in a mathematical system and therefore no proof is needed. But defining a right triangle through the Pythagorean Theorem is not what PSTs typically learn in their mathematics courses, so I decided that the correct answer for statement 8b was that it needed a mathematical proof. In addition, in the interview, I also asked PSTs to provide rationales for their thinking. If they ever demonstrated any knowledge that mathematical definitions relied on mathematical systems, this information would be captured and reported. Because there is no single answer for either statement 8a or 8i, as explained in the Methods chapter, Table 23 excludes statements 8a and 8i. But these two statements will be analyzed based on themes in later sections.

Table 23. Number of PSTs Who Gave Correct Responses to Whether a Given Statement Needs a Mathematical Proof in Question 8

Statement in Question 8	Proof needed	Total	PSTs - M	PSTs - N
b (Pythagorean theorem)	Yes	20	11	9
c (fraction)	No	16	9	7
d (polyhedron)	Yes	20	12	8
e (prime factorization)	Yes	19	12	7
f (area of triangle)	Yes	19	11	8
g (infinite primes)	Yes	23	12	11
h (fraction division)	Yes	16	9	7
j (square)	No	15	9	6
k (equivalent fraction)	Yes	16	9	7
l (prime number)	Yes	15	9	6
m (negative exponent)	No	7	2	5
n (distributive property)	Yes	15	9	6
o (exponent)	No	8	3	5

In general, about 67% of the answers given for statements in Question 8 were correct. Overall, PSTs - M performed better than PSTs - N. About 75% of the judgments made by PSTs - M were correct, but only 59% of the judgments made by PSTs - N were correct. Actually, on all statements listed above, except for 8m and 8n, PSTs - M performed better than PSTs - N. For each statement other than 8m and 8n, at least 63% of the judgments made by total PSTs were correct. Note that statements (8c, 8j, 8m, & 8n) do not need mathematical proofs. Of these four statements, at least 63% of judgments made concerning statements 8c and 8j were correct, but for statements 8m and 8n, only about 33% were correct. Statements 8c and 8j involve familiar concepts and this might explain why PSTs performed better on these two statements; however statements 8m and 8n involve defining negative and positive exponents through equations which might be unfamiliar to PSTs. Another reason which could lead to PSTs' low performance on statements 8m and 8n is the tendency to request a proof when there is an equal sign.

About 76% of the judgments given for the statements that need mathematical proofs were correct. But only about 48% of the judgments given for the statements that do not need mathematical proofs were correct. It seems that PSTs were better at making correct decisions when the statements need mathematical proofs than when the statements do not need mathematical proofs.

For all statements listed in the Table 23, about 68% of the judgments made by PSTs were "a proof is needed". When considering PSTs - M and PSTs - N as two separate groups, about 76% of the judgments made by PSTs - M were "a proof is needed" while about 60% of the judgments made by PSTs - N were "a proof is needed". This comparison indicates that compared to PSTs - N, PSTs - M had more tendency to think that a proof was needed.

PSTs' Criteria to Determine the Need for Mathematical Proofs

In addition to the quantitative analysis on how PSTs judged the need for proofs for each statement in Question 8, this section examines the criteria PSTs used when they explained why they thought a statement needed or did not need a mathematical proof. Little research has directly examined criteria students or teachers use in judging the need for mathematical proofs, therefore no existing framework can be adopted in this analysis. However, ideas from the research in mathematical proofs and mathematical definitions inform the analytic framework used in the following analysis. Two important roles mathematical proofs play in mathematics are to verify that a statement is true and to explain why a statement is true (Harel & Sowder, 2007). These two roles inform two criteria PSTs may use when judging the need for mathematical proofs if they see the need to confirm or explain statements. Van Dormolen and Zaslavsky (2003) argued that the choice of mathematical definitions is arbitrary. For one mathematical concept, multiple mathematical definitions need to be proved equivalent. They argued that "...a definition fits in and is part of a deductive system" (p.95) and "In practice this means that one has to choose one of the formulations as the definition and consider the other formulations as theorems that have to be proved" (p. 95). Therefore, there are no absolute mathematical definitions or mathematical theorems; one statement could be a mathematical definition in one deductive system but could be a theorem in another deductive system. This informs another criterion that PSTs could use in making their judgment. Namely PSTs could put the statement in one deductive system and argue that the statement is conventional knowledge (a mathematical definition or an axiom) so no proof is needed or it is not conventional knowledge so a proof is needed. Different from secondary teachers, preservice elementary teachers typically receive much less training in advanced mathematics and therefore other interesting criteria emerged from data as well. Table 24 presents a list of the criteria used by PSTs to determine whether a mathematical statement needs a mathematical proof. The numeric data indicates the number of PSTs - M and PSTs - N using any particular criteria. The meaning of each criterion will be explained later with specific quotes.

Table 24. Criteria PSTs Mentioned When Judging the Need for a Mathematical Proof

Criterion	Total	PSTs - M	PSTs - N
Need to confirm or explain statements	18	10	8
Simplicity or clarity of the statements	14	6	8
Ability to remember or generate a proof	12	8	4
Knowledge related to the axiomatic system	12	8	4
External features of the statements	8	3	5

Note. Because PSTs may mention multiple criteria in their responses, the numbers in Total column add up more than 24.

Need to confirm or explain. Overall, 18 PSTs made their decisions about whether a proof is needed based on whether there was a need to confirm or explain statements. Among the 18 PSTs, 17 of them (10 PSTs - M & 7 PSTs - N) judged whether a mathematical statement needed a proof by referring to the need to verify the validity of the statement. PSTs stated that a mathematical proof was needed in order to make sure the statement was true for all cases or numbers involved in the statement. Slightly more PSTs - M than PSTs - N referred to this criterion. However, three PSTs (1 PST - M and 2 PSTs - N) misapplied this criterion to at least one statement and asserted that some mathematical definitions needed to be proved. For instance, when commenting on statement 8m (For any real number $x \neq 0$, and whole number n, $x^{-n}=1/x^{n}$), M10 mentioned that:

> And then [statements]m, n and o, all kinda the same thing. If you're stating that, or to show that x to the negative n actually is 1/x to the n, you kinda need to go through the steps to show that it works every time and not just like for certain numbers again.

Here, M10 was uncertain that $x^{-n}=1/x^{n}$ was true, so he suggested having a mathematical proof to "go through the steps to show it works". Another PST - N, regarded drawing a picture as a way to confirm the validity of the statement whenever she encountered a geometric object. When asked to judge if statement 8j (A polygon with 4 equal sides and 4 equal angles) needed a mathematical proof, N10 said:

> I think this also needs a proof, just a picture, to show that four sides are equal and so are the angles because it's like a, this is like a statement but a proof would be like just showing the statement is true.

What N10 said revealed two misconceptions she had. First, N10 thought drawing a picture of a square to show that four sides and four angles are equal was a mathematical proof. Second, N10 overgeneralized "proof can confirm" to any case when uncertainty arises.

Thirteen PSTs (6 PSTs - M and 7 PSTs - N) determined that a mathematical proof is needed if further explanations about how the statement works and why it works are needed. Similar numbers of PSTs - M and PSTs - N mentioned this reason. However, three PSTs (1 PST - M and 2 PSTs - N) misused this criterion. For instance, when commenting on 8m (For any real number $x \neq 0$, and whole number n, $x^{-n}=1/x^{n}$), M6 mentioned that:

> This is one of things for me I think I tried to prove before and I did not know how and could not figure out so I just accept the fact....But I think there is a proof that actually shows why that is true.

Here, M6 expected that why the fact of changing a negative exponent to an inverse of an exponent works could be explained by a proof. Two roles proof plays in mathematics are to confirm and explain, but a proof does not need to be applied in every situation when uncertainty arises or a reason is needed.

Among the 18 PSTs (10 PSTs - M & 8 PSTs - N) who used these two criteria, seven PSTs (2 PSTs - M & 5 PSTs - N) misused one or both of the two criteria at least once in Question 8. An interesting observation of this finding is that even if more PSTs - M than PSTs - N used these two criteria in their evaluation, less PSTs - M misused them than PSTs - N.

Simplicity or clarity of the statements. Overall, fourteen PSTs (6 PSTs - M & 8 PSTs - N) made their decisions about whether a proof is needed based on whether the statement is clear or simple. Among the 14 PSTs, six PSTs indicated that they judged if a statement needs a proof by the simplicity of the statement. They demonstrated the ideas that if the statement is hard to understand, then a mathematical proof is needed. Similarly, if a statement is simple and easy to understand, then no proof is needed. Different from previous code "The need to confirm or explain statements", the code "clarity or simplicity of the statement" applied only when PSTs expressed that the statement was hard but did not explicitly mention the need to explain why. Same number of PSTs - M and PSTs - N used this criterion in their judgment. However, two PSTs - M and two PSTs - N misapplied this criterion and incorrectly judged at least one statement. For example, N10 thought no proof is needed for statement 8f, the area of triangle "because as long as you do know the base and the height....it is simple....because this tells you, like how to solve it to get the area of a triangle. So it's pretty simple." When commenting on statement 8c (A fraction is a part whole relationship), N2 said "It is the part whole relationship that can be shown through using tangible objects....It is like easier to prove to people without using a mathematical proof. You can show visually, show real things. " N2 made the correct judgment that statement 8b did not need a mathematical proof, but the reason she offered was problematic. She thought that the statement was easy and it can be shown through a picture, so no mathematical proof was needed. In her mind, it seems that mathematical proof is only needed to prove hard statements. It is true that proofs can explain why hard to understand statements make sense, however, using whether the statements are hard as a criterion to judge if proofs are needed is incorrect. For instance, the sum of two even numbers is an even number is an easily understood statement, but a proof is still needed.

Ten PSTs (4 PTSs - M & 6 PSTs - N) judged if a statement needs a proof based on the level of clarity of the statement. However, PSTs hold different and even opposite opinions within this category. Seven PSTs (2 PSTs - M & 5 PSTs - N) either demonstrated that clear statement does not need a proof or unclear statement needs a proof. More PSTs - N than PSTs - M had this idea. Among these seven PSTs, one PST - M and three PSTs - N specifically pointed out that during the proof process, an unclear meaning of a statement can be clarified. For instance, when asked if the statement 8a (A rhombus is a special type of quadrilateral) was a mathematical definition, M9 said:

> Because if they are going to prove it, they can prove what a special type is in their definition because there is what I was unclear before. But they proved what a special type was because in order to be able to prove a rhombus is a special type of quadrilateral they have to state what a special type is, so I think that would make it more clear.

Here, M9 indicated that the phrase "special type" was vague. When a proof is generated to prove that a rhombus is a special type of quadrilateral, the meaning of "special type" has to be explained. Different from the above seven PSTs, another two PSTs - M and one PSTs - N hold opposite ideas. They indicated that unclear statement does not need a proof. For instance, when commenting on the same statement 8a, N8 said "That one I don't think you can prove it. It kinds of said what a fraction is but it is not very clear, so I would have a hard time proving something that isn't very clear." Compared to previous quote from M9, N8 held a different view about the relationship between unclear statements and mathematical proofs.

PSTs' mathematical experience or ability. Overall, twelve PSTs (8 PSTs - M and 4 PSTs - N) made their decisions about whether a proof is needed based on whether they have seen or done proofs before or whether they felt they were able to generate a proof. Ten PSTs (8 PSTs - M and 2 PSTs - N) determined that a mathematical proof is needed by recalling their previous mathematical experience. Six more PSTs - M than PSTs - N used this criteria. This may be due to the amount of mathematical training received by PSTs - M and their confidence that they remember mathematical knowledge correctly. Six PSTs (5 PSTs - M and 1 PST - N) reached at least an incorrect conclusion by using this criteria when responding to Question 8. For instance, when commenting on statement 8m (For any real number $x \neq 0$, and whole number n, $x^{-n}=1/x^n$), M6 said:

> This is one of things for me I think I tried to prove before and I did not know how and could not figure out so I just accept the fact....I just said that was true. But I think there is a proof that actually shows why that is true....I am pretty sure it was my freshman year. I took a class at my freshman. My professor wanted us to do it. I think I got really frustrated and I just ended up Ok I know it is true, so I am just going to live with that.

Here, M6 cannot remember the details of the proof for this statement, but she insisted on the existence of the proof and even claimed that she remembered when she learned the proof. She also mentioned the proof can show “why that is true”, so this quote was also coded as “The need to confirm or explain statements.”

Seven PSTs (5 PSTs - M and 2 PSTs - N) determined whether the statement needs a proof by considering their ability to produce a proof. Five PSTs indicated that the statement does not need a proof because they were unable to produce a proof. For instance, one PST (M12) indicated that the statement needs a proof because she can envision a proof. Another PST (M8) said she was able to prove some statements but not others and used this as a standard to judge if a proof is needed for a statement. More PSTs - M than PSTs - N used this criteria. This could also due to the different mathematical confidence PSTs - M have compared to PSTs - N. Most PSTs who applied this standard made correct judgments other than one PST - N (N2) who incorrectly claimed that the statement 8e “Every integer greater than 1 can be written as the product of prime numbers” does not need a proof because “I guess I don't know how that could be proved....I would not know how you would do a mathematical proof for that.”

Knowledge related to the axiomatic system of mathematics. Overall, a half of PSTs (8 PSTs - M & 4 PSTs - N) made their decisions about whether a proof is needed by using their knowledge about meta-level concepts (e.g., definitions, theorems, axioms) in the axiomatic system of mathematics. Twelve PSTs (8 PSTs - M & 4 PSTs - N) referred to mathematical definitions or naming when they judged whether a statement needs a mathematical proof. More PSTs - M than PSTs - N used this criterion. Eight of the 12 PSTs explicitly mentioned that because the statement is a mathematical definition, no proof is needed. Among the eight PSTs, six are PSTs - M and two are PSTs - N. For instance, when commenting on statement 8j (A polygon with 4 equal sides and 4 equal angles), N7 said:

> No, you wouldn't need to prove that because that's just a definition of the square....You don't need to prove because you're just saying, you're basically saying which of these things, or what is a polygon with four equal sides and four equal angles. It's just basically an identification question.

Most PSTs who used this criteria correctly made the judgments other than one PST - M. When commenting on the statement 8f (The area of a triangle with base *b* and height *h* is equal to 1/2 bh), M5 responded that "F will be like I said it is the definition of area, so for definition we don't need to prove." M5 made a mistake by misidentifying a non-mathematical definition as a mathematical definition and claimed that the statement does not need to be proved because it is a mathematical definition. However, this quote indicates that she had a clear understanding of the relationship between mathematical definitions and mathematical proofs. Among the 12 PSTs who referred to mathematical definitions or naming when making their judgments, one PST - M (M9) made an opposite claim and stated that for the statement she regarded as a mathematical definition, she thought a proof was needed. Right after M9 finished Question 8, I asked her general criterion to decide if a proof is needed for statements in Question 8, she said:

> To me like a mathematical statement or definition there has to be proof for it, in order to be able to use it, so I went through and looked at what I thought was like clearly defined as a mathematical statement or definition and then those I thought need to be proved.

Similar percentage (about 25%) of PSTs - M and PSTs - N referred to naming when asked to judge if a statement needs a mathematical proof. They regarded naming as an arbitrary process and any name could be selected for the concept; therefore, no proof is needed. For example, when asked to judge if statement 8i (A Platonic solid is a regular, convex polyhedron with congruent faces of regular polygons and the same number of faces meeting at each vertex. There are five platonic solids in total. They are Tetrahedron, Cube, Octahedron, Dodecahedron and Icosahedron) is a mathematical definition, N6 said:

> Just reading the first sentence - that is the definition. I just had a hard time understanding of how you would prove because it is basically like saying that some mathematicians somewhere thought that something with these properties would be called the platonic solid so it is someone giving a name to some sort of thing, but it is just like specific naming. I do not think that you would need to prove why he called something just a platonic solid.

When PSTs mentioned definitions or naming and used these criteria to help them judge if a statement needs a proof or not, they tended to apply the criteria to geometric concepts instead of concepts in algebraic or number and operations. In total, PSTs mentioned mathematical definitions or naming 19 times when they responded to Question 8, only three times they applied these criteria to number and operations concepts (e.g., fractions and exponents) and 16 times they applied to geometric concepts (e.g., rhombus, squares, platonic solids, area of triangles). There seem to be a tendency that they associated mathematical definitions more closely to geometric concepts than concepts in algebraic or numbers and operations. This finding echos the findings from Chapter 4 that PSTs had better performance in writing mathematical definitions of geometric concepts than writing number and operation concepts.

Two PSTs - M referred to meta-level concepts such as axioms and theorems when they determined if the statement needs a mathematical proof. For instance, when asked to judge if statement 8b (In a right triangle, the square of the hypotenuse (the side opposite the right angle) is equal to the sum of the squares of the other two sides) needs a mathematical proof, one PST -

M (M5) indicated that she thought statement 8b needs a mathematical proof because:

> I was thinking back to one of the math history class I took. So basically there are a lot of geometry questions here. Upon the Euclidean geometry, there are 5 fundamental theorems and upon those five, the other thing you need to prove them. That is for the geometry part, so for example, b is not one of the fundamental.

Here she tried to express the idea “axiom”, but she used an incorrect term “theorem”.. Later in the interview, she tried to articulate the idea “axiom” again, but she admitted that she did not have the correct terminology. She wrote “theorem” in a quotation mark and said “Not theorem, but I don't know the word. It is something some concept that we take for granted.” I asked her if she tried to say “axiom”. She said yes.

Even though M5 did not have the correct vocabulary to describe this important meta level concept “axiom”, at least she had the idea that there are “5 fundamental theorems and upon those five, the other thing you need to prove them”. She also implicitly expressed that if a statement is not one of the five (in the case of geometry), the statement needs to be proved. However, her understanding of axioms was not complete. She knew that axioms were fundamental and were the basis for proofs, but her responses did not show clearly that she knew that axioms were previously accepted conventional knowledge.

Three PSTs - M made their judgments based on the usefulness of statements to do other things in mathematics. This indicated some understanding of the axiomatic system of mathematics, but their responses showed that their understanding was limited. Two PSTs - M indicated that if a statement would be used in the future especially to prove other statements, then it needs to be proved. For instance, when explaining why she thought statement 8g (There are infinitely many prime numbers) needs a mathematical proof, M9 said:

Because I think any property or statement if you are going to use it to prove other things it has to be proved. Like you are going to use like there are infinitely many prime numbers to go ahead and like prove a different statement then this one has to have justification in order to be able to use it.

Another PST - M held an opposite opinion compared to the above two PSTs - M. The following conversation happened during the interview when she explained why she thought statement 8n (For any real numbers, *a, b,* and *c, (a+b)c=ac+bc*) does not need a mathematical proof:

For [statement 8]n, again, it’s like [statement 8]h. It’s just something you use....same thing for [statement 8]o....you can use those within a proof....Same thing as like [statement 8]h and all of these....those are all things that you use all the time in math and I think those are some of the things that are like, you’re just supposed to take as true.

I tried to clarify that I did not intend to ask if she needed to prove the statements but if there is a need for mathematicians to prove it. She said no. Here, she judged whether statements need mathematical proofs based on if the statements are used “all the time in math”. If so, she took it as true and saw no need to prove it.

External features of the statements. Overall, eight PSTs (3 PSTs - M and 5 PSTs - N) made their decisions about whether proofs are needed based on external features of the statements. Among the eight PSTs, three PSTs - M mentioned the linguistic feature “if then” and used it as a clue to determine whether a mathematical proof is needed. For instance, when explaining why he thought statement 8o (Let *a* and *m* be nonzero whole numbers, then $\underbrace{a \cdot a \cdot \ldots \cdot a}_{m \text{ times}} = a^m$) needs a mathematical proof, M11 said:

I know the basic form of proof, if P, then Q. So I recognized that if a and m are nonzero whole numbers, then this holds true. There is a lot of stuff missing right there that proves

> it. Because this is like the given conditions and this is like what is true based on the given conditions but there is nothing saying why.

No PSTs - N used this criterion to help them make the judgments. Two PSTs - M thought statement 8o needs a proof because the statement is written in “Let - then” format which is similar to “if then” format as illustrated in the previous quote from M11.

Five PSTs - N but no PSTs - M mentioned that whether the statement involves an equation is a criterion they used to judge if a proof is needed. Four of them indicated that if a statement has an equation then a proof is needed. For instance, when commenting on statement 8m (For any real number $x \neq 0$, and whole number n, $x^{-n}=1/x^n$), N7 said: “It’s the same as all of the equations. You can’t just say that....It would have to work.” One PST - N (N1) held an opposite opinion and she indicated that the equation makes the statement clear so no further proof is needed. When commenting generally on Question 8, she said “So I think essentially it’d be all the ones that don’t have any equations.” Later, she further explained her thinking by saying:

> I said, I think the ones that I underlined, [statements 8]a, c ,e, g and j need proofs because they’re just so like vague and such partial definitions, I think they need more information so I think those ones could benefit from having a mathematical proof. A lot of the other ones have equations in them so I guess like that helps a lot but the other ones are just like very vague statements.

The quote indicates that she thought the equations make the statements clear so no proofs are needed. Just a side note here, this quote was also coded by the previous code “clarity or simplicity of the statements” because this PST (N1) judged whether a proof is needed based on whether the statement is clear or not as well.

PSTs' Knowledge of Mathematical Definitions and Axiomatic System

This section aims to investigate PSTs' knowledge about the relationships among mathematical definitions, mathematical proofs, theorems and counterexamples. It is divided into two subsections. The first subsection reports PSTs' responses to Question 9. Question 9 asks PSTs to judge if the given general statements are correct (e.g., 9b: Mathematical definitions have to be proved before use). This question directly asks above relationships. The second subsection focuses on PSTs' understanding of the relationships among mathematical definitions, mathematical proofs and counterexamples.

PSTs' Responses to Question 9

I first report the number of PSTs who provided correct judgments to statements in Question 9 in Table 25. All statements in Question 9 are incorrect.

Table 25. Number of PSTs Who Provided Correct Judgments in Question 9

Statement in Question 9	Total	PSTs - M	PSTs - N
a. When something mathematical is defined, the result is a theorem.	21	10[a]	11
b. Mathematical definitions have to be proved before use.	11	6	5
c. If a theorem can't be proved, then it can be treated as a definition.	18	9	9
d.The process of defining is only to naming a concept.	21	11	10
e. A definition can be disproved by providing a counterexample.	11	6	5

Note. [a]One PSTs - M felt uncertain and did not give definite answer to statement 9a; so he was excluded from the analysis of statement 9a.

About 80% of PSTs gave correct responses to statements 9a and 9c which involve the relationship between mathematical definitions and theorems. However, a qualitative analysis revealed that even though PSTs' judgments to statement 9a and 9c were correct, the reasons they offered to support their arguments were not always correct. For Statement 9a, PSTs demonstrated weak understanding of theorems. Only four PSTs (2 PSTs - M and 2 PSTs - N) were able to articulate the key feature of a theorem in terms of its relationship with mathematical proofs: a theorem needs to be proved. Four PSTs (3 PSTs - M and 1 PST - N) thought a theorem used “multiple definitions to tell another true statement” (M2). This finding indicates that these PSTs cannot distinguish definitions and concepts. A theorem involves multiple concepts rather than multiple definitions. Eight PSTs (two PSTs - N and six PSTs - M) used examples or non examples of a theorem to support their judgments. For instance, N8 used the example that a definition of a square was not a theorem to support his argument that statement 9a was incorrect. PSTs’ other inadequate thinking about theorems includes that a theorem is a more official statement than a mathematical definition, a theorem goes deeper, it is a theory, a rule, a practice, a process or it is the same as a mathematical definition. Several PSTs were unable to articulate what a theorem is in their mind but they felt definitions and theorems were different because they had different names.

Similarly, for statement 9c, the reasons PSTs used to support their arguments were not always correct. Ten PSTs (3 PSTs - M and 7 PSTs - N) indicated that statement 9c was incorrect because “if you can’t prove something, then how can you say that it’s true?” (N3). This quote suggested that N3 thought “true knowledge” had to be established by proving and therefore revealed her inadequate understanding of two types of “true knowledge”, one established by convention (e.g., definition) and the other established by logic or proving (e.g., theorem) as argued by Ball (1990). Two PSTs - M pointed out statement 9c was self contradictory because a theorem needed to be proved and but statement 9c states that “if a theorem cannot be proved....” One PST - M and one PST - N used counterexamples to support their judgments. For instance, M3 argued that statement 9c was incorrect because before it was proved in the history, the

Fermat’s last Theorem was not a definition because it did not define anything. Five PSTs - M used relationship between mathematical definitions and proofs to support their judgments in statement 9c. Among the five PSTs - M, two gave correct judgments to statement 9c and three gave incorrect judgments. For instance, M5 gave incorrect judgment to statement 9c and developed an incorrect argument by saying “In Venn Diagram a theorem is included in a mathematical definition....for a definition we don't have to prove it any more and if a theorem cannot be proved, then it could be treated as a definition. ” It is interesting to see how she used knowledge of the relationship between theorems and definitions as well as relationship between definitions and proofs to build her argument. This is an evidence that PSTs tried to make sense of the axiomatic system of mathematics, but with limited understanding, M5 drew an incorrect conclusion. Different from M5, M6 gave correct judgment to statement 9c, but her reason was incorrect. She said “definitions have to be already been proved, so if a theorem cannot be proved, then it cannot be definitions.” Other PSTs either based their decisions on memory or were unable to offer arguments to support their decisions.

About 90% of PSTs gave correct judgments to statement 9d and most of them pointed out that defining also involves describing what a concept is or include the properties of the concept. About one half of PSTs correctly indicated that statement 9b was incorrect. Same number of PSTs correctly indicated that statement 9e was incorrect also. Even though the numbers of PSTs - M and PSTs - N who made the correct judgments on these two statements happen to be the same, they are not the same PSTs. Only eight PSTs (5 PSTs - M and 3 PSTs - N) gave correct responses to both statements. Because research (e.g., Dickerson & Pitman, 2012) found that undergraduates’ understanding of the relationships among mathematical definitions, proofs and disproofs were problematic, in the next section, I conducted further analysis on how PSTs thought about these relationships.

PSTs' Understanding of Relationships Among Definitions, Proofs and Counterexamples

This section is organized into three subsections. The first subsection reports PSTs' *explicit* expression about the above relationship. Different from the analysis done in the previous section, the data source for this analysis is not restricted to PSTs' responses to Question 9 but also includes PSTs' responses to other relevant interview questions. Because PSTs' responses at other parts of the interview may not be completely consistent with what they demonstrated in statements 9b and 9e, the results reported in this subsection may look different to results reported in the previous section. The data used in this subsection are PSTs' *explicit* statements about the relationships between mathematical definitions, mathematical proofs, and counterexamples. For instance, at the beginning of the interview, when asked if there were other things he wanted to say about mathematical definition, M1 said that "I would say they [mathematical definitions] all require a proof, some type of proof to prove they are correct and nothing else can fit into that category." Statements like this one are *explicit* statements of the relationship between mathematical definitions and proofs and are analyzed in this subsection. Sometimes PSTs responded with a general statement even when they commented on a specific statement. For instance, when explaining why she thought statement 8c (A fraction is a part whole relationship) does not need a mathematical proof, N6 said "So I kind of noticing even I just finished three [statements] is that the ones that I said were not mathematical definitions might need proofs...But for definitions, I think they do not necessarily need proofs." Here, she made a generalization about the relationship between mathematical definitions and mathematical proofs based on her observation of three statements. Even if data are from responses to specific statements, they are analyzed in this subsection because she drew a general conclusion.

In the second subsection I compare PSTs' responses to Question 7 and Question 8. Recall that Question 7 asks PSTs to select statements which are mathematical definitions and Question 8 asks PSTs to select statements which need mathematical proofs. A comparison between PSTs' responses to Question 7 and Question 8 reveals how PSTs see the relationship between mathematical definitions and proofs *implicitly*. In the last subsection, I investigate if PSTs' *explicit* statements (as shown in the first subsection) are consistent with their *implicit* expressions (as shown in the second subsection) by comparing results in the first two subsections.

PSTs' explicit expression of relationship among definitions, proofs and counterexamples. Table 26 gives the distribution of PSTs' *explicit* expression about the above relationships.

Table 26. PSTs' Explicit Expression about Relationships Among Definitions, Proofs and Counterexamples

Category	Total	PSTs - M	PSTs- N
Mathematical definitions and mathematical proofs			
All definitions need to be proved	12	6	6
Some definitions need to be proved but some do not	6	2	4
No definition needs to be proved	6	4	2
Mathematical definitions, disproof and counterexamples			
Definitions can be disproved by counterexamples	16	6	10
Definitions cannot be disproved by counterexamples	8	6	2

Overall, 50% of PSTs *explicitly* expressed that all mathematical definitions need to be proved. Same percentage of PSTs - M and PSTs - N held this idea. One quarter of PSTs demonstrated the idea that some mathematical definitions need to be proved but others do not. Slightly more PSTs - N than PSTs - M held this idea. The other one quarter of PSTs correctly indicated that no mathematical definition needs to be proved. Slightly more PSTs - M than PSTs - N held this idea.

Two thirds of PSTs indicated that mathematical definitions can be disproved by counterexamples. One third of PSTs correctly indicated that all mathematical definitions cannot be disproved by counterexamples. Slightly more PSTs - M than PSTs - N correctly identified the relationship between definitions and counterexamples.

Table 27 provides a finer level analysis of the reasons for PSTs' thinking of the relationships between mathematical definitions and mathematical proofs. Different from Table 26 in which I have three categories (all definitions need to be proved, some definitions need to be proved but some do not, and no definition needs to be proved), I collapsed reasons for PSTs' thinking into only two categories (reasons why PSTs thought definitions need to be proved and reasons why PSTs thought definitions do not need to be proved).

Table 27. Reasons Why PSTs Thought Definitions Need to be Proved

Reason	Total	PSTs - M	PSTs - N
Definitions need to be proved			
Need to confirm or explain statements	11	6	5
Misunderstand the meanings of mathematical definitions or proofs	7	2	5
Others	3	1	2
Definitions do not need to be proved			
Nature of mathematical definitions	7	4	3
Others	5	2	3

Note. PSTs may mention multiple reasons. Also, PSTs who indicated that some mathematical definitions need to be proved but others do not fall into both categories, so the column total is not 24.

The reason mentioned by the largest number of PSTs for why mathematical definitions need to be proved is the need to confirm or explain statements. About 40% of PSTs thought the truth of mathematical definitions need to be established by mathematical proofs. For instance, when asked to judge statement 9b (Mathematical definitions have to be proved before use), N3 said: "For this one, mathematical definitions have to be proved before use, yes, because you can't just make up mathematical definitions...Like someone has to prove that it's true". This result indicates that PSTs did not fully understand the nature of two types of mathematical knowledge, conventional knowledge (determined by the community of mathematicians) and conceptual knowledge (determined by mathematical proofs). Similar numbers of PSTs - M and PSTs - N indicated this idea. Two PSTs - M but no PSTs - N also thought that mathematical definitions need to be proved because an explanation of why the statement is true or why defining a concept in certain way is needed. For example, at the beginning of the interview when I asked M9 "From your perspective, what is a mathematical definition", she said "mathematical definition is the meaning of something in terms of like a math concept or typically like things are like proven and for like definitions - been backed up by proofs of some sort. " When probed to explain more, she said:

> I mean - I think I - basically like - like most definitions they did not just like make up they have evidence supporting them....Like they define parallel lines is like lines that don't intersect, same slope that kind of things but they went through and prove all. But they choose to define parallel lines that way like they set up, the criteria and proved like that.

When asked if she knew how to prove the way we define parallel lines is the correct way, she responded that "not really, the way they define things has a reason". Even though she did not know how to prove the definition of parallel lines, she held a strong conception that there is a reason.

About one third of PSTs explicitly expressed that mathematical definitions need to be proved due to a lack of understanding of the meanings of mathematical definitions or mathematical proofs. Three PSTs thought drawing pictures to show a mathematical concept is a mathematical proof. For instance, when explaining why he thought statement 8i (the statement involves the platonic solid) needs a proof, N8 said "The first part [A Platonic solid is a regular, convex polyhedron with congruent faces of regular polygons and the same number of faces meeting at each vertex], you could just show that which I guess it is proving by representing the shapes or drawing the models." Four PSTs misidentified non mathematical definitions as mathematical definitions and then thought proofs are needed to prove the "mathematical definitions". N5 and M4 thought the area formula for a triangle is a mathematical definition of area of a triangle. For instance, M4 said "I think some definitions need to be proved or can be proved". When probed for an example, she said "the area of a triangle." N11 believed that some mathematical definitions need to be proved, but some did not. When probed to give an example of one mathematical definition which needs to be proved, she said that the theorem of prime factorization is a mathematical definition and she knew that it needs to be proved.

When commenting on why they thought mathematical definitions did not need to be proved, about one third of PSTs referred to the nature of mathematical definitions. Five PSTs indicated that naming a concept does not need to be proved. For instance, when commenting on statement 9b (Mathematical definitions have to be proved before use), M3 said:

> And I think this is false just because like with what I think of as a mathematical definition, I think it's literally just us naming something. There's no proof involved in that. That's just our language for something.

Two PSTs indicated that the nature of mathematical definition as stipulated determines that no proof is needed. For instance, when responding to the same statement 9b, M5 said "[statement 9]b is false because when we define something, we won't try to judge whether it is true or false after we define them." She demonstrated thinking that mathematical definitions are stipulated, so there is no way to say it is correct or incorrect because it is just a decision made by the community of mathematicians. Another PST - M (M3) indicated that mathematical proofs build on mathematical definitions, so mathematical definitions cannot be proved. For instance, in addition to her previous comment that “naming a concept does not need to be proved”, M3 also added that “Right, and I think that we use definitions to make proofs. So how could you prove a definition?”

Five PSTs provided various other reasons to back up their claim that mathematical definitions do not need to be proved. Two PSTs indicated that they did not know how to prove a mathematical definition. For instance, when explaining why she thought statement 9b (Mathematical definitions have to be proved before use) is incorrect, M8 said:

> I am trying to think of a definition that you can prove. I define function and expressions I don't know how you would prove those. I don't know if it is just my level of math that I cannot think of anything....because a proof you like take statement and you take what you start with like givens, and you take other properties or definitions to lead up to the statement to show it is true. Like in a definition of expression it is just kind of general form of how you define it. I do not see how you could take steps to prove it.

Another PST - M (M7) misidentified a non-mathematical definition as a mathematical definition and used it as an example to show that mathematical definitions do not need to be proved before its use. When commenting on statement 9b (Mathematical definitions have to be proved before use), she said “For a long time, Fermat's last theorem is unprovable and until Andrew Weils but it was still something that you could tell like what he was communicating and different things like that. But it wasn't proved till recently. ” Here, she misidentified Fermat’s last theorem as a mathematical definition and she thought the Fermat’s last theorem is in use before it is proved, so she used this as an example to support her claim that statement 9b was false, namely, mathematical definitions [Fermat’s last theorem in this example] do not need to be proved before it is in use. She made correct judgment in terms of the validity of the statement but the reason she offered was incorrect. Another PST - N (N6) indicated that the reason that mathematical definition does not need to be proved is because it is clear enough so no further explanation is needed. When explaining why she thought statement 8c (A fraction is a part whole relationship) does not need a mathematical proof, N6 said “And By reading this you could tell what a fraction is so it is something that has a part related to a whole...I think that just reading it you could understand what it is.” In addition to commenting on specific statement 8c, N6 even made a generalization that “So I kind of noticing even I just finished three [statements] is that the ones that I said were not mathematical definition might need proof...But for the definitions, I think they do not necessarily need proofs.” Similar to M7, her judgment on the correctness of statement 9b was correct, but the reason she offered was incorrect.

Table 28 provides a finer level analysis of the reasons for PSTs’ thinking of the relationships between mathematical definitions, disproofs and counterexamples. The reason mentioned by the largest number of PSTs is PSTs’ incorrect understanding of the meaning of mathematical definitions.

Almost one third of PSTs fell into this category. Five more PSTs - N than PSTs - M fell into this category. About 20% of PSTs incorrectly identified a non-mathematical definition as a mathematical definition and used it to back up their claims. For instance, N1 indicated that statement 9e (A definition can be disproved by providing a counterexample) was correct by saying that “I’d say that’s true because if you find a correct counterexample for a definition, then you’re proving it is not true.” When probed to give an example about how to disprove a definition, N1 said:

Multiplying a number, that it’s always gonna be more than the original but I mean, in the case of 1x1, that's not true because 1x1 is 1 so it's not bigger...so I guess you could use

Table 28. PSTs' Descriptions of the Relationship Between Definitions, Disproofs and Counterexamples

Reasons	Total	PSTs - M	PSTs - N
Mathematical definitions can be disproved by counterexamples			
Misunderstand the meaning of mathematical definitions	7	1	6
Overgeneralize disprovable mathematical ideas to mathematical definitions	2	2	0
False impression	2	0	2
Reasons not specified	6	3	3
All mathematical definitions cannot be disproved by counterexamples			
Nature of mathematical definitions	4	2	2
The validity of mathematics definition is established	4	4	0

Note. Some PSTs's responses fall into multiple categories, so the column total is not the same as the sum of each individual cell in the same column.

that as like a counterexample. If someone gave a definition, saying any time you multiply a number by any other number, it's gonna be bigger.

When asked the given example was a definition of what mathematical concept, N1 said: "I mean, that would just be the definition of if you were defining, like if someone said… I don't know. If someone used that as an example of multiplication, I guess." Here, she misidentified a property of multiplication as a definition of multiplication and reached an incorrect conclusion that a mathematical definition can be disproved by a counterexample. Another two PSTs - N but no PSTs - M indicated that some mathematical definitions could be "vague", "bad" or "incorrect"; thus they can be disproved by a counterexample. For instance, when responding to statement 9e (A definition can be disproved by providing a counterexample), N9 said:

> And a definition can be disproved by a counterexample. I'd say that's true. If you have a false definition, like saying a square is… two adjacent sides, or two corresponding sides that aren't parallel, you could just provide a counterexample.

Here, she seemed to imply that when a mathematical definition was a bad or incorrect one, a counterexample could be provided. Obviously, she understood that the role of counterexamples is to be evidence and shows the statement is incorrect, but she did not understand that this process is not called “disprove”.

Two PSTs - M but no PSTs - N overgeneralized other disprovable mathematical ideas and incorrectly applied them to mathematical definitions. For instance, when responding to statement 9e (A definition can be disproved by providing a counterexample), M2 said:

> E is similar to like anything could be disproved by providing a counterexample. So I think definition would just fall same lines. So if you have a definition and there is a counterexample then a definition does not work any more.

In addition to responding to specific statement 9e, M2 made a generalization that “anything could be disproved by providing a counterexample” and “anything” includes mathematical definitions.

Two PSTs -N but no PSTs - M based decisions on their impression in mathematics. For instance, when probed how to disprove the definition of addition, N12 said:

> I am trying to think of a way you can disprove it. I don't know. I had a friend who actually is like a mathematicians and he is trying to prove that something can be right and something can be wrong at the same time. He was saying that some people may think 2+1 is 21 and he wants to prove that that can be true. So maybe in some crazy world, if he can prove that addition is not always true, then maybe, but right now I don't - I think that -

Here, even though she had no idea of how to disprove addition, she still believed that an expert is able to do it. It is just her level of mathematics which prevented her from disproving it. Another PST- N (N3) based her judgment on her memory of previous school experience in mathematics.

She said:

> And then the definition can be disproved by providing a counterexample. So yeah, like I mean that's what I remember doing in my classes. If it can't be true, then prove that isn't true by writing something for an example for it.

Four PSTs who demonstrated the idea that "all mathematical definitions cannot be disproved by counterexamples" made their arguments based on the nature of mathematical definitions. Two PSTs indicated that naming cannot be disproved, thus mathematical definitions cannot be disproved. For instance, when explaining why she thought statement 9e (A definition can be disproved by providing a counterexample) is incorrect, M3 said:

> You can't just say like, like if our definition of a triangle is a polygon with three sides, you can't just disprove that. Like that's just our name that we give that. There's nothing to prove or disprove there.

Another two PSTs indicated that the nature of mathematical definition as stipulated determines that no proof is needed. For example, when commenting on the same statement 9e, M5 said "E, A definition will not be judged or proved when you stated as a definition, it is already there we won't doubt whether it is true or not. There is no existence of those. "

Another four PSTs - M but no PSTs -N correctly articulated the relationships among mathematical definitions, disproofs and counterexamples, but the reasons they gave were incorrect. They thought that no mathematical definition can be disproved because the validity of mathematical definitions have already been established and they are true statements. For instance, when explaining why she thought statement 9e is incorrect, M4 said:

> Well, because the definitions are, should always be true so you shouldn't be able to disprove them. I think you can disprove false statements using a counterexample but I

> don’t think, definitions aren’t supposed to be wrong.

M4 had some good understanding about mathematical definitions and she knew that mathematical definitions have to be true, but she did not understand how axiomatic system works and she was unclear that the validity of mathematical definitions is not determined by the mathematical proof. Another PST - M (M6) gave further reason for how she thought the validity of mathematical definitions is established. She said:

> A theorem can be disproved by providing a counterexample but I think once you have - I don't think a theorem can become a definition until it is proved. So once it is proved, it cannot be disproved by counterexample unless the person who proved it did a very bad job. So I do not think a definition can be disproved.

She clearly indicated that she thought a mathematical statement becomes a mathematical definition only after someone proves it, so then of course it cannot be disproved by counterexamples. All four PSTs - M who offered the reason that no mathematical definition can be disproved because the validity of mathematical definitions are established did not give other reasons about their thinking when they made the claims. This indicates that even though these four PSTs gave correct judgments to statement 9e, they did not understand the relationships among mathematical definitions, disproofs and counterexamples. Therefore, in total only four PSTs, evenly split between PSTs - M and PSTs - N demonstrated correct understanding of the relationships among mathematical definitions, disproofs and counterexamples by giving the correct reasons to support their claims.

PSTs’ implicit expression of relationships between mathematical definitions and proofs. In this subsection I compare and contrast PSTs’ responses to Question 7 and Question 8. Recall that Question 7 asks PSTs to select statements which are mathematical definitions and Q8 asks PSTs to select statements which need mathematical proofs. A comparison between PSTs' responses to Question 7 and Question 8 reveals how PSTs see the relationship between mathematical definitions and mathematical proofs *implicitly*. This implicit relationship as manifested in the comparison is also categorized into the same three categories as I did in Table 26: (1) all definitions need to be proved, (2) some definitions need to be proved but some do not, (3) no definition needs to be proved. I will start this subsection by first giving explanations and three examples to show how I categorize PSTs' responses into these three categories.

Generally, I looked through PSTs' responses to Question 7 and pulled out all statements PSTs regarded as mathematical definitions. Then I checked their responses to the same statements in Question 8 and see if they indicated that proofs are needed for these statements. If PSTs thought mathematical proofs are needed for all statements they identified as mathematical definitions, I coded it as "all definitions need to be proved". If PSTs thought some statements identified as definitions need mathematical proofs, but others do not, I coded it as "some definitions need to be proved but some do not." If PSTs thought all statements they identified as definitions do not need mathematical proofs, I coded it as "no definition needs to be proved." For instance, N8 selected statements 7b, 7f, 7h, 7i, 7l, 7o as mathematical definitions and later he thought all of the above statements need to be proved; thus, his case was coded as "all definitions need to be proved". N7 selected statements 7b, 7f, 7h, 7i, 7j, 7k, 7l, 7m, 7n as mathematical definitions. However later when asked which statements need mathematical proofs, he only selected statements 7b, 7f, 7h, 7k, 7m, 7n but not other statements. Her case was coded as "some definitions need to be proved but some do not." M5 selected statements 7c, 7f, 7i, 7k, 7m, 7n, 7o as mathematical definitions and later when working on Question 8, she did not think any of them need mathematical proofs. Her case was coded as "no definition needs to be proved." Table 29 gives the distribution of PSTs' *implicit* expression of the relationship between mathematical proofs and mathematical definitions. About half of PSTs indicated some mathematical definitions need mathematical proofs, but not others. A quarter of PSTs indicated that all mathematical definitions need proofs and the rest indicated that no mathematical definition needs proofs.

Table 29. PSTs' Implicit Expression of the Relationship Between Definitions and Proofs

Category	Total	PSTs - M	PSTs - N
All definitions need to be proved	6	4	2
Some definitions need to be proved but some do not	13	5	8
No definition needs to be proved	5	2	3

Comparison between PSTs' explicit and implicit expression of the relationship between definitions and proofs. As shown in Table 26 and Table 29, the separate distributions of PSTs' *implicit* and *explicit* expressions of the relationship between mathematical definitions and mathematical proofs are quite different. For instance, half of PSTs who indicated *explicitly* that all mathematical definitions need mathematical proofs; however, only a quarter of PSTs who indicated the same idea *implicitly*. Table 30 gives detailed distribution across PSTs' *explicit* and *implicit* expressions of the relationship between mathematical definitions and proofs.

Table 30. PSTs' Explicit and Implicit Expressions of the Relationship Between Definitions and Proofs

		Explicit expression		
		All definitions need to be proved	Some definitions need to be proved but some do not	No definition needs to be proved
Implicit expression	All definitions need to be proved	6	0	0
	Some definitions need to be proved but some do not	4	4	5
	No definition needs to be proved	2	2	1

Only 11 PSTs' *explicit* and *implicit* expressions of the relationship between mathematical

definitions and mathematical proofs are consistent. Among the 11 PSTs, six PSTs consistently indicated that all mathematical definitions need to be proved; four PSTs consistently indicated that some mathematical definitions need to be proved but some do not; only one PST consistently indicated that no definition needs to be proved. Even though six PSTs *explicitly* expressed that mathematical definitions do not need to be proved and five PSTs *implicitly* expressed the same idea, only one PST - M consistently and correctly indicated that no mathematical definition needs to be proved through out the whole interview. This reveals PSTs' inadequate understanding of the relationship between mathematical definitions and proofs, so they did not even realize that their inconsistency through out the interview.

Six PSTs both *explicitly* and *implicitly* indicated that all mathematical definitions need to be proved. Among the six PSTs, when working on Question 8, four PSTs applied the criterion that all mathematical statements need to be proved to make decisions and said every statement (including those they identified as mathematical definitions) given in Question 8 needs to be proved. Two PSTs held similar ideas to the previous four PSTs but they added one restriction that only clear mathematical statements need to be proved but not those unclear mathematical statements. Another interesting observation from Table 30 is that more PSTs demonstrated the thinking that some mathematical definitions need to be proved but others do not *implicitly* than *explicitly*.

PSTs' Conceptions of Role of Mathematical Definitions in Mathematics

This last section aims to investigate the second research question proposed: what are PSTs' conceptions about the roles mathematical definitions play in mathematics. This section identify reasons why PSTs think mathematical definitions are important in mathematics. The data source for this analysis is mainly from PSTs' responses to Question 12 in the interviews but also include a few PSTs' responses from other interview questions when they commented on why mathematics needs mathematical definitions. In the following Table 31, I present an analytical framework to categorize different roles mentioned by PSTs. Some roles were reported in the literature but others emerged from the data. Later in the section, I will provide quotes to demonstrate the meaning of each code.

Among the roles mentioned by PSTs, describing the meaning of the concept was the one mentioned by the largest number of PSTs; about 85% of PSTs mentioned this role. More than one half of PSTs perceived mathematical definition as bonds to connect mathematical concepts. About one half of PSTs saw mathematical definition as the foundation for problem solving. About one third of PSTs mentioned the role of creating uniform meaning and forming the foundation for problem solving. Determining if an example is an instance of a mathematical concept was the role mentioned by the least amount of PSTs; only three PSTs mentioned this role.

Table 31. Roles of Mathematical Definitions in Mathematics

Role	Meaning
Describe the meaning of the concept	Mathematical definitions describe the meaning of the concept (Zaslavsky & Shir, 2005).
Create uniform meaning and ease communication	Mathematical definitions create uniformity in the meanings of concepts, which allows us to communicate mathematical ideas more easily (Zaslavsky & Shir, 2005, p. 317).
Determine if an example is an instance of a mathematical concept	Whether an instance is an example of a mathematical concept is determined by mathematical definitions (Zaslavsky & Shir, 2005).
Form the foundation for proofs	Mathematical definitions can be used to construct mathematical proofs (Moore, 1994).
Form the foundation for problem solving	Mathematical definitions can be used to solve problems.
Build connections among concepts	Through mathematical definitions, concepts build on previously defined concepts and are related to each others.

Table 32. Distribution of PSTs Mentioning Specific Role of Mathematical Definitions

Role	Total	PSTs - M	PSTs - N
Describe the meaning of the concept	20	10	10
Create uniform meaning and ease communication	8	7	1
Determine if an example is an instance of a mathematical concept	2	1	1
Form the foundation for proofs	7	4	3
Form the foundation for problem solving	9	4	5
Build connections among concepts	14	7	7

Note. Some PSTs mentioned multiple roles mathematical definitions play in mathematics, so the number in Total column adds up to more than 24.

More than 80% of the PSTs indicated that one of the important roles mathematical definitions play is to describe meanings of the concepts. For instance, when asked why mathematical definition is important, M1 said "Basically every concept we learned - every concept that has been introduced to me they come along with a definition, a way to define it - a way to give it a meaning." M1 also indicated that mathematical definitions contributed to the concept formation. He said "They give meaning to everything. They help describe and help visualize what a concept is. "

Almost one quarter of PSTs - M but only one PST - N demonstrated the idea that mathematical definitions "create uniformity in the meanings of concepts, which allows us to communicate mathematical ideas more easily" (Zaslavsky & Shir, 2005, p. 317). For instance, M11 said:

> They help make mathematics the same culture to culture so like the definition of a circle I wouldn't think would change from one culture to another from one mathematician to the other from one textbook to the other. I think that is really important.

Another PST - M (M7) said:

> I think it would just be a little bit crazy because without the definitions people would not be able to communicate with each other well like you could not go to the store and be like

I have a square cake pan because that square can mean different things to different people. And you would not get what you need it.

One PSTs - M and one PST - N also mentioned that mathematical definition was the authority to determine whether an example was an instance of a mathematical concept. For instance, M6 said "like if we don't have a definition for a square, I could draw this and say that is a square, but there are specific guidelines and definitions for what a square is."

Slightly less than one third of PSTs indicated that mathematical definitions were foundation of mathematical proofs. Similar number of PSTs - M and PSTs - N mentioned this role. For instance, N6 said "Through those definitions you are even able to do math like set up an equation and prove things to be true or things like that." Another PST (N3) offered more details on how mathematical definitions support proof construction. She said:

> Like knowing like the definitions of, like in our proofs, knowing definitions of how to prove equivalence class, like knowing the equivalents, like relations, knowing that has to be reflective, symmetric and transitive, having to know those definitions of those, and then when I went through the proofs, like I had to define it was reflective and symmetric and transitive.

Here she specifically mentioned that knowing the definition of equivalence class which includes reflective, symmetric and transitive, guides her to prove that an mathematical object is an equivalence class. In addition to understanding that mathematical definitions are used to construct mathematical proofs, one PST - M (M3) clearly indicated that mathematical definitions are the starting point of mathematics. She said:

> I mean, I guess part, a big part is like I just said, as a building block for things. If you don't have a definition to start with, then where do you start? I mean, I know when I first

> took geometry in high school, like definitions were a huge thing because you can't really prove anything without knowing what, you know, different shapes are, parallel lines and stuff like that. If you don't have a definition, then you can't really build on it. On different ideas and stuff.

Same number of PSTs indicated that mathematical definitions were the foundation of problem solving as those who indicated that mathematical definitions were the foundation of mathematical proofs. For instance, N10 mentioned:

> In order to like do a math problem, you need to be able to know what is an exponent or what is a variable because if you see it in like a word problem or something, if you don't know what it is, you can't like do the problem.

One PST - M and two PSTs - N misidentified non-mathematical definitions as mathematical definitions and therefore thought mathematical definitions can be used to solve problem. For instance, when asked "How important are mathematical definitions in mathematics?", M5 regarded the Pythagorean theorem as a mathematical definition and she said "Solving problems like here a squared plus b squared equals c squared, we can use that to solve a lot of geometry problem. "

Almost one third of PSTs - M and same number of PSTs - N indicated that mathematical definitions are important in mathematics because through mathematical definitions, new mathematical concepts build on previously defined concepts and are connected to each other. This role as seen from PSTs' perspectives was not reported in the literature. For instance, one PST - N (N2) said:

> Because I know a lot of topics in mathematical terms build on each other when you learn one topic you are going to learn more on topic. If you don't have definitions of something

and you learned more on it, you will just be even more confused .... Because if you learned addition and you don't know what the definition of addition is and you are doing, and you are adding fractions, you still don't know what adding is, you will be even more confused for adding fractions.

CHAPTER 7: SUMMARY, DISCUSSION AND CONCLUSION

As discussed in Chapter 1, this study investigated preservice elementary teachers' understanding of the nature and roles of mathematical definitions. Though there have been studies done internationally which addressed teachers' conceptions of mathematical definitions (e.g., Zaslavsky and Shir, 2005), little research has focused on US teachers' conceptions of mathematical definitions and even less has examined this issue at elementary level.

In this study, 24 preservice elementary teachers' who were in their senior year were interviewed about their understanding of what constitutes a mathematical definition, what makes a good definition, and their understanding of the role of mathematical definitions. The interviews were audiotaped and transcribed. Data were analyzed mainly in two ways. Questions that have correct or incorrect answers were scored, and distributions of the number of PSTs who gave correct responses were reported. After item analyses were completed, other more general themes were examined using theoretical perspectives found in the literature. Categories were developed for questions that were based on opinions and distributions of the numbers of PSTs whose responses illustrated each category were reported. Conceptions of preservice elementary teachers whose teaching major was mathematics (PSTs - M) and whose teaching major was not mathematics (PSTs - N) were compared in order to reveal if advanced mathematical training contributed to PSTs' understanding of mathematical definitions.

This Chapter is organized into three sections. The first section summarizes the key findings in each of Chapters 4 - 6 and also gives a brief description about the opportunities PSTs had to learn the nature and roles of mathematical definitions in their teacher education programs. The second section focuses on connecting and discussing the results across chapters. The last section describes the contributions of the study, implications for teacher education, limitations of the study and future directions of research on definitions.

Summary

This section is organized into four subsections. The first three summarize the findings in Chapter 4 to 6. The last describes PSTs' self-reported opportunities to learn about mathematical definitions in their teacher education programs.

PSTs' Thinking about What Constitutes a Mathematical Definition

Overall, to answer research question 1a (what constitutes a mathematical definition for preservice elementary teachers?) I identified necessary features PSTs associated with mathematical definitions, namely, the features PSTs used to distinguish mathematical definitions from non-definitional statements. In addition, I identified problematical thinking PSTs had about mathematical definitions. These two aspects together gave an overall description of what constitutes a mathematical definition for PSTs. Further analyses of specific items revealed other interesting patterns about PSTs' thinking.

The most common necessary features of mathematical definitions mentioned by PSTs include correctness, clarity, including the name of the concept, and helping beginning learners understand the concept. Similar number of PSTs - M and PSTs - M mentioned the above features. Other necessary features of mathematical definitions such as precise, concise, and straightforward were only mentioned by a few PSTs. At least three more PSTs - M than PSTs - N mentioned above three features. Only one PST - M mentioned minimality as a necessary feature. Because PSTs did not understand all important features of mathematical definitions, when evaluating statements, some PSTs may focus on certain features they were familiar with but ignored other important features and hence making incorrect judgments. For instance, one PST misidentified a non-definitional statement as a definition because she focused exclusively on if the statement was precise and correct and completely forgot to check if the statement intended to define a mathematical concept.

The most common misconceptions about mathematical definitions held by PSTs include that mathematical definitions are expected to be written in the format “A is B” and mathematical definitions could be used to name a property/procedure/theorem. About 85% of PSTs believed that mathematical definitions could be used to name a property/procedure/theorem; therefore they misjudged non mathematical definitions such as distributive property and the Pythagorean Theorem as mathematical definitions. Other problematic thinking include PSTs’ uncertainty about if mathematical definitions could be written only in symbols, PSTs’ preferences to include examples or pictures in the definitions, and PSTs’ association of mathematical definitions with “if-then” format to write the statement. PSTs - M tended to associate linguistic structures with mathematical definitions while PSTs - N were more likely to include examples or pictures in mathematical definitions.

Additional analyses of specific items revealed other interesting aspects about PSTs’ thinking about mathematical definitions. First, PSTs generally performed poorly in writing mathematical definitions. They performed better in writing definitions of Geometry and Measurement concepts than Number and Operations and Algebraic concepts. For Geometry and Measurement concepts, two thirds of definitions generated were appropriate (either rigorous or not rigorous), but for both concepts in Number and Operations and Algebra, roughly one third of definitions generated were appropriate. This finding is consistent with the finding reported by Leikin & Zazkis (2010) on prospective mathematics teachers. Second, PSTs - M performed better than PSTs - N in generating appropriate mathematical definitions. About 25% of definitions generated by PSTs - M were appropriate and rigorous whereas about 8% of definitions generated by PSTs - N were appropriate and rigorous. More than half of definitions generated by PSTs - M were appropriate (either rigorous or not rigorous) while less than 30% of definitions generated by PSTs - N were appropriate.Third, when commenting on their written definitions, one third of PSTs used the dichotomy - what it means vs. how to calculate it - as a way to distinguish a definition from a non-definitional statement. On the surface level, this dichotomy seemed to parallel to the dichotomy of conventional knowledge vs. deduced results (Ball, 1990). However, the big variety of PSTs' opinions on whether how to calculate the concept belonged to the definition indicates that many PSTs did not truly understand the relationship between conventional knowledge and deduced results. Last, though most PSTs indicated that multiple mathematical definitions could exist for one mathematical concept, the reasons they provided to support their claims were inconsistent with mathematicians' view about equivalent definitions for one mathematical concept.

PSTs' Preferences for Mathematical Definitions

To answer research question 1b (Among the valid mathematical definitions, what types of definitions do preservice elementary teachers prefer?) I categorized features PSTs preferred to include in mathematical definitions into four categories: mathematical features, communicative features, external features, and other features. Overall, I found that when comparing mathematical definitions almost all PSTs attended to communicative features, which included concise, clear, easy to visualize, straightforward. About two thirds of PSTs considered mathematical features which included ease of testing if an instance is an example of a concept, explaining the essence of the concept, ease of applying the definition in a mathematical proof and noncircularity. Among these four mathematical features, the first two were mentioned by at least one third of PSTs and the last two were attended by only two to three PSTs. About two thirds of PSTs also considered external features such as familiarity and linguistic features. Among these features, familiarity was the one attended the most. Half of PSTs highly relied on familiarity when rating a specific statement and were unable to offer additional rating criteria. This phenomenon was a sign of PSTs' inadequate knowledge of using other more important criteria to evaluate mathematical definitions. No noticeable differences existed for each type of features mentioned by PSTs - M and PSTs - N.

PSTs' Understanding of Axiomatic System and Role of Mathematical Definitions

To answer research question 1c (to what extent do PSTs distinguish mathematical definitions from other meta level mathematical objects), I first examined the criteria PSTs used to determine whether a proof is need for a statement. I then explored PSTs' thinking about the relations among mathematical definitions, proofs, theorems and counterexamples. Overall, when judging whether a given statement needed a mathematical proof, the criteria PSTs used included the need to confirm or explain statements, simplicity or clarity of the statements, PSTs' mathematical experience or ability, knowledge related to the axiomatic system of mathematics and external features of the statements. About 85% of PSTs applied the criterion "the need to confirm or explain statements". However, 40% of PSTs who used this criterion made mistakes when they applied these criteria. For instance, PSTs overgeneralized "proof can confirm" to any case when uncertainty arose even though the uncertainty was associated with conventional knowledge (e.g., uncertainty about if "A polygon with 4 equal sides and 4 equal angles" defines a square). It is true that two important roles of proofs are to confirm and explain why deduced results are true (Harel & Sowder, 2007), but a need for confirmation or explanation is not always associated with a need for proving. Applying the criterion "the need to confirm or explain statements" to all cases when uncertainty arises is an indicator that PSTs did not fully understand the differences between conventional knowledge and deduced knowledge. Another interesting observation was that when PSTs argued that “naming” is conventional knowledge and a statement involving naming does not need a proof, they tended to apply this criterion to geometric concepts instead of algebraic or numbers and operations concepts.

Three quarter of PSTs explicitly expressed that either all mathematical definitions need to be proved or some mathematical definitions need to be proved. The reasons given to support their claims included a need to confirm or explain mathematical definitions and misunderstanding of the meanings of mathematical definitions or proofs. Three quarter of PSTs expressed that mathematical definitions could be disproved by counterexamples. PSTs’ reasons for this claim included misunderstanding the nature of definitions or proofs, overgeneralizing disprovable mathematical ideas to mathematical definitions, and false impressions based on past mathematical experience. One quarter of PSTs correctly identified the relationship between mathematical definitions and mathematical proofs explicitly. One third of PSTs correctly articulated the relationship between mathematical definitions and counterexamples explicitly. The main reason to support this correct conception was PSTs’ awareness of the nature of mathematical definitions as stipulated. A comparison of PSTs’ responses to Question 7 (identify statements which is a mathematical definition) and Question 8 (identify statement which needs a mathematical proof) suggests that roughly one quarter of PSTs implicitly indicated no mathematical definition needs to be proved. However, a comparison between PSTs’ explicit articulation and their implicit expression of the relationship between mathematical definitions and proofs indicates that only one PST - M consistently and correctly showing her thinking that no mathematical definition needs to be proved through out the whole interview. This reveals PSTs’ inadequate understanding of the relationship between mathematical definitions and proofs, so they did not even realize that their inconsistency during the interview.

To answer research question 2 (what are PSTs' conceptions about the roles mathematical definitions play in mathematics), I identified reasons why PSTs thought mathematical definitions were important in mathematics. The roles identified by PSTs included describing the meaning of the concept, creating uniform meaning and easing the communication, determining if an example was an instance of a mathematical concept, being the foundation for proofs and problem solving, and building connections among concepts. About 85% of PSTs mentioned that one role was describing the meaning of the concept. About half of PSTs mentioned the role of building connections among concepts. Though about one third of PSTs mentioned that mathematical definitions were foundations of proofs, only one PST - M clearly stated that mathematical definition was the starting point of the deductive system.

Opportunities to Learn Mathematical Definitions in Teacher Education Programs

In this section, I will briefly discuss PSTs' self-reported opportunities to learn mathematical definitions during their teacher education programs. Overall, three quarters of PSTs reported having opportunities to learn mathematical definitions of specific mathematical concepts. About one third of PSTs reported learning how to teach mathematical definitions in schools. Similar numbers of PSTs reported learning process of defining a mathematical concept or learning how to use a mathematical definition to construct a proof. No big differences were found between self reported opportunities between PSTs - M and PSTs - N from the above aspects.

When explicitly asked if they were given opportunities to learn about the nature and roles of mathematical definitions in their teacher education programs, half of PSTs reported having some opportunities, but the other half reported having no opportunities. Three quarters of PSTs -

M and one quarter of PSTs - N reported having some opportunities to learn the nature and roles of mathematical definitions and had been exposed to tasks, discussions or thinking similar to what they experienced during the interview in this study. However, PSTs' responses to this interview question may be unreliable because PSTs may say they had opportunities only because they were asked this question.

PSTs also admitted that many of their understanding of mathematical definitions was based on their own mathematical experiences. For instance, N1 said:

> He [instructor] would like write on the board how they [shapes] are constructed and then he [instructor] would like draw a picture of it [the shape] so I guess that's kinda like where my thinking about a definition including a visual comes from.

M11 had a very strong idea that everything needs to be proved; therefore definitions need to be proved. He said:

> Probably in my classes about proofs like even in like Calculus 1 & 2, if I've taken those and not taking any proof classes, like whenever they teach us a topic, I always remember they teaches us. They will go through this whole long thing - they give us the equations and they go through this whole long thing about deriving equation like why it is true, where it came from.

These two quotes suggest PSTs' attempts to make sense of what they learned and make a generalization. Obviously they drew incorrect conclusions, but is this a mistake of PSTs or a lack of clarification during the instruction?

Discussion

In this section, I connect findings across chapters and highlight found in this study.

Necessary Features vs. Preferred Features

As discussed in the previous sections, PSTs demonstrated understanding of many features of mathematical definitions reported in the literature. However, some features that should be necessary features were misplaced into preferred features or vice versa. For instance, noncircularity is a necessary feature of mathematical definitions but all three PSTs - N who mentioned this feature only used it to distinguish good and bad definitions. All three PSTs regarded circular definitions as bad mathematical definitions but still accepted them as valid definitions. In addition, helping beginners to understand mathematical concepts is considered by experts to be a preferred feature of mathematical definitions, but PSTs in this study took it as a necessary feature of mathematical definitions. That is, PSTs tended to reject a statement as a mathematical definition when they felt the statement did not help beginners to understand mathematical concepts. Being able to distinguish necessary features from preferred features is important for future teachers because they need to be aware of what features deserve major considerations and what features are only optional.

Nature of Mathematical Definitions and Proofs

Findings in Chapter 6 revealed that misunderstanding of the meaning of mathematical definitions contributed to PSTs' incorrect understanding of the relationship between mathematical definitions, proofs and counterexamples. As reported in Chapter 4, PSTs demonstrated problematic thinking of mathematical definitions such as mathematical definitions are expected to be written in a way of "A is B". This incorrect understanding of mathematical definitions caused PSTs to misidentify non-definitional statements as mathematical definitions or vice versa and therefore contributed to the formation of incorrect understanding of the relationship between definitions and proofs. For instance, PSTs thought that the area formula of a rectangle was a definition of area of a rectangle; therefore they concluded that mathematical definitions need to be proved. Also, PSTs did not realize that the defining statement of the Pythagorean Theorem included two parts. One part asserts a relationship among lengths of three sides through an equation and the other part assigns a name “Pythagorean Theorem” to this property of right triangle. The equation part needs a proof to confirm or explain why the equation always holds in right triangles and PSTs have seen proofs before in their mathematics courses for future elementary teachers. However, the naming part does not need a proof. Lack of awareness of the differences in these two parts misled PSTs to think that “mathematical definitions need to be proved” because they saw proofs for the Pythagorean Theorem before and they saw defining (i.e., naming the statement as the Pythagorean Theorem) in the statement as well.

Similarly, PSTs’ misunderstanding of mathematical proofs (e.g., regarding drawing pictures as mathematical proofs) also caused their incorrect understanding of the relationship between mathematical definitions and mathematical proofs because PSTs thought drawing pictures to show a mathematical definition was proving it. These observations indicated that knowing exactly what was mathematical definitions and proofs was important for PSTs to build other knowledge relevant to the axiomatic system of mathematics.

Mathematical Definitions and Content Areas

There seem to be a tendency for PSTs to associate mathematical definitions more closely with geometric concepts than concepts in algebra or numbers and operations. This claim was supported by findings from Chapter 4 that PSTs had better performance in writing mathematical definitions of geometric concepts than writing number and operations concepts. In addition, in Chapter 6 when PSTs argued that mathematical definitions did not need to be proved, they tended to draw on the knowledge of the nature of mathematical definitions as stipulated when judging geometric concepts compared to concepts in algebraic or number and operations. For instance, PSTs argued that the definition of a Platonic solid was basically a naming process; therefore no proof was needed. This observation echoes the findings from Knuth (2002b) where practicing secondary teachers tended to associate mathematical proofs with geometry rather than with algebra.

Advanced Training in Mathematics

One issue I pursued in this study was to investigate whether PSTs - M who received more advanced training in mathematics had better understanding of mathematical definitions than PSTs - N who received limited training in mathematics. Results indicated that knowledge of specific mathematical definitions among PSTs - M was better than that of PSTs - N. For instance, the performance of mathematics teaching majors in generating appropriate mathematical definitions were better than that of teaching majors in other subjects. Also, when evaluating whether the given statement was a mathematical definition in Question 7 and Question 8, for all but two statements, more PSTs - M than PSTs - N gave correct responses.

However, PSTs - M's knowledge of mathematical definitions as a meta level concept was not that different compared to PSTs - N. This is not saying that for all aspects, PSTs - M demonstrated no better understanding than PSTs - N. Results in Chapter 4 indicated that for some problematical thinking such as "it is better or necessary to include examples in mathematical definitions", fewer PSTs - M than PSTs - N held this misconception. But for other misconceptions such as "mathematical definitions are expected to be written in the format "A is B", more PSTs - M than PSTs - N held this misconception. Therefore, generally speaking, PSTs - M did not demonstrate a clearly better understanding than PSTs - N as they did on specific mathematical definitions.

This observation indicated that taking advanced mathematics courses may contribute to improvement of PSTs' knowledge of mathematical definitions of specific concepts, but taking advanced courses may not have a big impact on their knowledge of mathematical definitions as a meta level concept.

Conclusion

Implications for Teacher Education

As discussed in the previous section, even though PSTs reported having some opportunities to explicitly learn mathematical definitions as a meta-level concept during their teacher education programs, they also admitted that their understanding of mathematical definitions came from generalization of their past mathematical experiences. As a result, their understanding of mathematical definitions included good features agreed upon by the community of mathematicians and also misconceptions. This finding suggests that PSTs are unable to gain a completely correct understanding of the nature and roles of mathematical definitions only through their mathematical experience with specific mathematical definitions. The finding also suggests that current instruction in mathematics courses for preservice elementary teachers may not very effective in preventing the formation of misconceptions regarding mathematical definitions. The finding that PSTs - M's conceptions of mathematical definitions were similarly problematic provides further evidence that simply going through advanced training in mathematics does not automatically improve PSTs' understanding of mathematical definitions. This finding is consistent with the findings from the literature (e.g., Vinner, 1977; Edwards & Ward, 2004; Dickerson, & Pitman, 2012) in which researchers found that even undergraduate students who took quite a few advanced mathematical courses demonstrated weak understanding of mathematical definitions.

Edwards and Ward (2004) argued that “it is particularly important that undergraduate mathematics students who plan to teach should have experiences in their college courses that help them build robust understandings of the role and use of mathematical definitions” (p. 422). Thus mathematicians and mathematics educators need to design instructional activities and appropriate mathematical experiences to help PSTs grasp the nature and roles of mathematical definitions. Echoing Edwards and Ward (2004), the nature and roles of mathematical definitions need to be addressed more directly and more often. In the following I suggest several possible instructional activities.

First, the instrument in my study seemed to trigger PSTs’ reflection on their thinking of mathematical definitions. For instance, when striving to give consistent answers across the interview, M8 commented that “I think I had a lot of contradicting views in here.” Therefore, the instrument adopted in this study could serve as a starting point to design more instructional activities. University mathematics instructors could give PSTs definitions and non-definitional statements to sort out. Similarly, they could also give PSTs good or bad definitions to sort out. Instructors could give PSTs general statements (e.g., the statements in Question 9) and specific statements (e.g., the statements in Question 7 and 8) and ask PSTs to compare their thinking demonstrated in different settings. One finding in Chapter 6 is that PSTs’ explicit and implicit expressions about the relationship between mathematical definitions and mathematical proofs are inconsistent. Therefore, asking PSTs to compare thinking in different settings may trigger their reflections on the inconsistency.

In addition, instead of asking PSTs to finish the instrument individually as I did in the interview, the instrument could be first given to individuals to finish and then follow up with small group or whole class discussions. Though my study found clear trends for PSTs’ common misconceptions, there were also a big variation in PSTs' thinking. For instance, as illustrated in Chapter 4, when PSTs commented on statement 7b, PSTs offered many reasons why they thought statement 7b was or was not a mathematical definition. Even among PSTs who agreed that statement 7b was a mathematical definition, there was also a big variation on what was the concept being defined (e.g., the Pythagorean Theorem, a right triangle, or hypotenuse). This big variation indicated that if statement 7b were discussed in a group setting, PSTs would be exposed to various and even conflicting ideas. This environment could provide a great opportunity for PSTs to practice giving rationales to support their arguments and refine their understanding of mathematical definitions. Zaslavsky and Shir (2005) also found that participants' thinking about mathematical definitions changed a lot after group discussions. After group discussion, a whole class discussion could be held and the instructor could pull out different ideas generated by groups. Eventually, ideas suggested by the class could be compared to what is suggested in the literature as necessary and good features of mathematical definitions to summarize the discussion.

Moreover, in order to handle the problematic thinking relevant to mathematical definitions, instructors of mathematics courses could explicitly discuss theoretical perspectives such as concept image and concept definition (Tall and Vinner, 1981). My study found that some PSTs included examples as part of their definitions. A discussion about what is a concept image and how it connects to but is different from a concept definition could deepen PSTs' understanding of mathematical definitions and help them see a strong need for mathematical definitions.

Furthermore, in the domains of number and operations and algebra, PSTs had worse understanding of mathematical definitions. This claim is supported by the finding that PSTs performed better in writing mathematical definitions in the domain of geometry and they were better at judging whether a geometry concept (e.g., platonic solid) needs a proof than an algebraic concept (e.g., exponent). PSTs' uncertainty about if mathematical definitions can be given solely in symbolical representation (e.g., defining an exponent) further supports the claim that defining statements were not emphasized enough in the domain of algebra. Similar to what was reported in the literature of proofs (e.g., Wu, 1996; Knuth, 2002b) that teachers tended to associate mathematical proofs with the domain of geometry, my finding suggests that more instructional efforts need to be made to address mathematical definitions in the domains of number and operations and algebra. In other words, university mathematics courses should put more emphasis on building connections among all content areas to achieve a more uniform understanding of mathematical definitions.

In addition to adding instructional activities to explicitly address the nature and roles of mathematical definitions in mathematics courses, more assessment should be designed to monitor PSTs' learning of mathematical definitions. Students will not learn well if there is no appropriate assessment to evaluate their learning. My finding that PSTs articulated difficulties in coming up with a concept to write a definition about and commented that they had never been asked to write a definition in their past mathematical experience supports my claim that less assessment regarding mathematical definitions were given in university mathematics courses. My own experience as an instructor of mathematics courses for preservice elementary teachers also indicates that instructors tend to focus on assessing PSTs' knowledge of specific mathematical topics (e.g., draw a picture to explain distributive property), but ignore assessing metal level knowledge such as the nature and roles of mathematical definitions which is equally important for future teachers as argued by Ball (1990). Sample assessment questions could include comparing and contrasting two definitions and providing reasons for the judgments.

Contributions of the Study

This study contributes to the literature in multiple ways. First, little research has focused on US preservice elementary teachers' conceptions of mathematical definitions. This study fills the void in the mathematics educational research literature by depicting what constitutes a mathematical definition for preservice elementary teachers, how they compare and contrast mathematical definitions, and how they think about the position of mathematical definitions in the axiomatic system of mathematics. In addition, many research studies on mathematical definitions focused on specific topics. For instance, Vinner (1977), Dickerson and Pitman, (2012), and Levenson (2012) focused on defining exponents; Johnson et al. (2014) focused solely on geometric concepts. The concepts included in this study have a wider range, which made the findings potentially more generalizable.

Moreover, this study confirms findings from the previous studies and also provides additional explanations for phenomena reported in the literature. For instance, research has reported that students have a tendency to think that mathematical definitions need to be proved. Reasons for this misconception include failure to understand definitions as stipulated rather than extracted (Edwards & Ward, 2004), mixing the motivation of a definition with a proof of a theorem (Vinner, 1977), believing that all mathematical statements needed to be proved (Dickerson & Pitman, 2012) and believing a possible way to get around difficult proofs is to define (Pimm, 1993). This study identified additional reasons that could explain why mathematical learners tended to think mathematical definitions should be proved. As discussed in Chapter 4, two misconceptions PSTs held about mathematical definitions were that PSTs used the linguistic structure “A is B” to judge if a statement was a mathematical definition and PSTs thought the statement naming a theorem or a property was a mathematical definition. Due to these two misconceptions, PSTs thought that the area formula of a triangle and the Pythagorean Theorem were mathematical definitions. Because PSTs also have seen proofs for these two statements before, they reached the conclusion that mathematical definitions need to be proved. The newly identified misconceptions and reasons for these misconceptions provide insights for mathematics educators and may support their efforts to better design instructional activities to eliminate these misconceptions and build correct understanding of mathematical definitions.

Also, the multiple frameworks developed from chapter 4 to 6 captured preservice elementary teachers' views regarding the nature and role of mathematical definitions. These frameworks provide a theoretical lens for future researchers to examine how preservice elementary teachers think about the necessary and desired features of mathematical definitions and how mathematical definitions are positioned in the axiomatic system of mathematics. These frameworks also provide tools for mathematics educators to use in university mathematics courses and in professional development activities.

Furthermore, this study also took an important step in studying an understudied population, preservice elementary teachers whose teaching major is mathematics. Typically in the United States, preservice elementary teachers only need to take very basic college mathematics classes such as College Algebra and two or three mathematics courses specifically designed for future elementary teachers. Thus far, little research has reported mathematical knowledge about preservice elementary teachers who receive more advanced mathematical training; my study made an effort to describe their thinking and compared the results to typical preservice elementary teachers.

Limitations of the Study

This study has several limitations. First, characterizing preservice elementary teachers' conceptions of mathematical definitions in one 60 - 90 minute interview provides only a single snapshot of PSTs' conceptions. It is possible that with additional sources of data (e.g., observing PSTs' engagement in activities of defining in university mathematics courses or collecting artifacts produced in the class), I may find that PSTs show different understanding of mathematical definitions compared to what was reported in the interview setting in this study.

Furthermore, 24 participants is a small sample, and when comparing PSTs - M and PSTs - N, the sample size for each group is even smaller. Moreover, PSTs who were the subjects of this study were not representative of the U.S. population because they all came from the same institution and volunteered to participate in the study. This limitation prevented further generalization of my results to the population.

Also, even though before starting the interviews, I reminded the participants that that they should answer the questions from the perspective of mathematical learners rather than as a potential mathematical teacher because the focus of this study is about their thinking of mathematical definitions from the disciplinary perspective, PSTs tended to answer questions as if they were teaching elementary students. This is not saying that it is incorrect for PSTs to think from the perspective of being a mathematics teacher; it is actually very important for PSTs to consider students' need because this has been an important focus of their teacher education programs. However, this created a concern that some responses given by PSTs may reflect their views as mathematics teachers rather than as mathematics learners. This may cause that the collected data did not completely mismatch the research questions.

Another limitation of the study is the lack of inter-coder reliability for the data analysis. I am the only person who reviewed and coded the whole data set, and I am not a native speaker of English. The way I interpreted a response may be different from how a native speaker would interpret the data.

Future Directions of Research on Definitions

As a follow-up study of my dissertation research, I aim to design instructional activities to promote elementary teachers' understanding of the nature and roles of mathematical definitions. The instrument I used in my dissertation research is a starting point for me to design additional instructional activities. As I have seen during the interviews, the instrument led preservice elementary teachers to reflect on their inconsistent thinking about definitions and proofs and in some cases this led PSTs to refine their thinking. It may take me a few rounds of trials to improve the current instrument. After the instrument is well constructed, I plan to disseminate the instrument to mathematics educators who are interested in improving preservice elementary teachers' knowledge about mathematical definitions. I will follow these mathematics educators into their university mathematics classrooms and collect data about how they implement the tasks and how their students' understanding of mathematical definitions changes. I will disseminate materials to a broader audience if the materials prove to be successful. I also plan to produce case study reports to document the difficulties mathematics educators encounter and the ways they overcome the difficulties in order to to help their students understand the nature and role of mathematical definitions.

A second area worth examining is to explore the opportunities to learn mathematical definitions in the university mathematics courses that preservice elementary teachers are required to take. As reported by McCrory and Stylianides (2014), very little opportunities to address the nature and roles of mathematical definitions were found in mathematics textbooks for preservice elementary teachers. My study found similar results based on PSTs' self reported data. However, the self reported data were based on PSTs' memory over the past four years and may not be an accurate reflection of their actual experience. Therefore, in order to better characterize the nature of the opportunities to learn mathematical definitions in teacher education programs, I plan to conduct classroom observations in university mathematics and methods courses that PSTs are required to take. Many interesting and important questions might be asked such as to what extent university instructors address the necessary and preferred features of mathematical definitions and to what extent university instructors address the role of mathematical definitions in the axiomatic system of mathematics? Furthermore, one finding of this study is PSTs' incorrect understanding of the relationships between mathematical proofs and mathematical definitions. Thus, in a future study that involves classroom observations, I will pay specific attention to how mathematics teacher educators use the words such as "prove", "explain", "show", "define", "describe" in classroom discourse. My hypothesis is that the inconsistent use of these words may contribute to pre-service teachers' confusion about the relationship between mathematical definitions and mathematical proofs.

A third area of pursuit would be to investigate similar research questions on practicing elementary teachers. Practicing teachers are different from preservice teachers because they are engaged in real teaching practices and their conceptions and beliefs may change a lot after they start teaching. In addition, this study only investigated preservice elementary teachers' conceptions of mathematical definitions from disciplinary perspective. Future research could examine how practicing teachers understand the nature and roles of mathematical definitions from the perspective of the mathematics teacher. Knuth (2002a, 2002b) found that practicing secondary teachers hold quite different conceptions of mathematical proofs when they consider proofs in secondary school setting. For instance, he found that for many teachers, proofs seemed to be interpreted more narrowly in secondary school mathematics than in the discipline of mathematics. Similarly, practicing elementary teachers could demonstrate different understanding of mathematical definitions when they consider themselves as mathematical teachers as opposed to as mathematics learners. Studying practicing elementary teachers may shed lights on how teachers think about mathematical definitions in elementary school mathematics when students are considered.

Finally, I am interested in investigating how Chinese elementary teachers think about mathematical definitions. Chinese elementary teachers have been reported to have rich knowledge of teaching specific mathematical topics such as multiplication of multi-digit whole numbers or division of fractions (e.g., Ma, 1999). However, little research has investigated their understanding of metal level concepts such as mathematical definitions or mathematical proofs. A comparison could be made to explore in what ways US and Chinese elementary teachers are similar or different in their conceptions of mathematical definitions and what factors (e.g., instructional or institutional factors) may contribute to these differences.

APPENDICES

APPENDIX A: Survey of Student's Background

For question 1-4, please highlight your choice.

1. What is your gender? Female Male

2. What is your year of your study at MSU?
1st year 2nd year 3rd year 4th year

3. What is your teaching major? (check all that apply)
Language Arts Integrated Science Social Studies Mathematics

4. What is your teaching minor? (check all that apply)
Language Arts Integrated Science Social Studies Mathematics

The following is a list of coursework related to either **mathematics** or **mathematics education** that you've taken at Michigan State, please put a X in the cell for the year when you took it. If there are courses that you've taken but are not listed here, please specifying them in others.

For the courses that you are required to take, but you have not taken yet, please put a circle in the box and select the year that you plan to take it. You only need to put the courses relevant to **mathematics** and **mathematics education**.

Table 33. PSTs' Course Work

	1st year	2nd year	3rd year	4th year
MTH 132 Calculus 1				
MTH 133 Calculus 2				
MTH 201 Elementary Math for Teachers 1				
MTH 202 Elementary Math for Teachers 2				
MTH 301 Foundations of Higher Math				
MTH 304 (290 FS10) Algebra for K-8 Teachers				
MTH 305 (490 SS11) Functions and Calculus for k-8				
MTH 330 Higher Gometry				
MTHE 430 History of Mathematics				
STT 250 Statistics and Probability for K-8 teachers				
STT 201 Statistical Methods				
STT 290 Topics in Statistics and Probability				
TE402 Crafting Teaching Practice Elementary				
TE801 Professional Roles & Teaching Practice 1				
Others please fill in				

APPENDIX B: Handout for Preservice Elementary Teachers

Question 5: Choose a concept you feel comfortable with and write a definition of it.

Question 6: Here are two definitions of trapezoid, what do you think about them?

a) A trapezoid is a quadrilateral that has <u>at least</u> one pair of parallel sides.

b) A trapezoid is a quadrilateral that has <u>exactly</u> one pair of parallel sides.

Question 7: You are given a list of statements; please circle those you think are mathematical definitions.

a. A rhombus is a special type of quadrilateral.

b. In a right triangle, the square of the hypotenuse (the side opposite the right angle) is equal to the sum of the squares of the other two sides.

c. A fraction is a part whole relationship.

d. For a polyhedron, the number of vertices minus the number of edged plus the number of faces is always equal to 2.

e. Every integer greater than 1 can be written as the product of prime numbers.

f. The area of a triangle with base b and height h is equal to $\frac{1}{2}bh$.

g. There are infinitely many prime numbers.

h. When you divide two fractions, you invert and multiply them, namely, for integers a, b, c, and d, and b, d $\neq 0$,
$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$.

i. A Platonic solid is a regular, convex polyhedron with congruent faces of regular polygons and the same number of faces meeting at each vertex. There are five platonic solids in total. They are Tetrahedron, Cube, Octahedron, Dodecahedron and Icosahedron.

j. A polygon with 4 equal sides and 4 equal angles.

k. For any integer *a, b*, and *n, b* $\neq$ *0, n* $\neq$ *0*, $\frac{an}{bn} = \frac{a}{b}$.

l. A whole number *N>1* is prime unless it has a prime factor $p \le \sqrt{N}$. Thus to test whether N is prime one need only check divisibility by the primes *p=2, 3, 5,….* satisfying $p^2 \le N$.

m. For any real number $x \neq 0$, and whole number n, $x^{-n} = \frac{1}{x^n}$.

n. For any real numbers, *a, b,* and *c, (a+b)c=ac+bc.*

o. Let *a* and *m* be nonzero whole numbers, then $\underbrace{a \cdot a \cdot \ldots \cdot a}_{m \text{ times}} = a^m$.

Question 8: This is the same list as before, I would like you to underline those statements that you think need a mathematical proof.

a. A rhombus is a special type of quadrilateral.

b. In a right triangle, the square of the hypotenuse (the side opposite the right angle) is equal to the sum of the squares of the other two sides.

c. A fraction is a part whole relationship.

d. For a polyhedron, the number of vertices minus the number of edged plus the number of faces is always equal to 2.

e. Every integer greater than 1 can be written as the product of prime numbers.

f. The area of a triangle with base *b* and height *h* is equal to $\frac{1}{2}bh$.

g. There are infinitely many prime numbers.

h. When you divide two fractions, you invert and multiply them, namely, for integers a, b, c, and d, and b, d $\neq 0$,
$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$.

i. A Platonic solid is a regular, convex polyhedron with congruent faces of regular polygons and the same number of faces meeting at each vertex. There are five platonic solids in total. They are Tetrahedron, Cube, Octahedron, Dodecahedron and Icosahedron.

j. A polygon with 4 equal sides and 4 equal angles.

k. For any integer *a, b*, and *n, b* $\neq 0$, $n \neq 0$, $\frac{an}{bn} = \frac{a}{b}$.

l. A whole number $N>1$ is prime unless it has a prime factor $p \leq \sqrt{N}$. Thus to test whether N is prime one need only check divisibility by the primes $p=2, 3, 5, \ldots.$ satisfying $p^2 \leq N$.

m. For any real number $x \neq 0$, and whole number n, $x^{-n} = \frac{1}{x^n}$.

n. For any real numbers, *a, b,* and *c*, $(a+b)c=ac+bc$.

o. Let *a* and *m* be nonzero whole numbers, then $\underbrace{a \cdot a \cdot \ldots \cdot a}_{m \text{ times}} = a^m$.

Question 9: Here are a few statements for you to judge. Please circle True or False beneath each statement and explain your reasons to me while you go through the list.

a. When something mathematical is defined, the result is a theorem.

True False

b. Mathematical definitions have to be proved before use.

True False

c. If a theorem can't be proved, then it can be treated as a definition.

True False

d. The process of defining is only to naming a concept.

True False

e. A definition can be disproved by providing a counterexample.

True False

Question 10: Here is a list of statements about an even number. Please circle those that you think are mathematical definitions. For those you think are mathematical definitions, please rate. Use a 4 point scale to rate, where a 4 indicates that you consider it to be a good mathematical definition and 1 indicates that you do not consider it to be a good mathematical definition. Write the rating in front of the definition. When you are working on this task, I want you to forget about your teacher hat, but think yourself as a mathematics learner. You can write a brief note to record your thinking.

a. A number is called even provided it represents a number of objects that can be placed into two groups of equal size.

b. A number is called even if it is an integer multiple of 2.

c. A number is called even if it is divisible by 2.

d. A number is called even if it can be divided into 2 parts.

e. A number is called even if it ends with 0, 2, 4, 6, 8.

f. A number is called even if it can be written as 2k, where k is a whole number.

g. An even number is 1 more or 1 less than an odd number and an odd number is 1 more or 1 less than an even number.

h. A number which occurs as we skip count by two ("0, 2, 4, 6, 8, 10, 12, 14,…").

i. An even number of objects can be paired up (with none left unpaired).

j. An even number is a number that is twice a whole number.

Question 11: Here is a list of statements about a square. Please circle those that you think are mathematical definitions. For those you think are mathematical definitions, please rate. Use a 4 point scale to rate, where a 4 indicates that you consider it to be a good mathematical definition and 1 indicates that you do not consider it to be a good mathematical definition. Write the rating in front of the definition. When you are working on this task, I want you to forget about your teacher hat, but think yourself as a mathematics learner. You can write a brief note to record your thinking.

a. A square is a rhombus with a right angel.

b. If a rhombus has 4 equal angles, then it is a square.

c. A rectangle is a square if and only if it has four equal sides.

d. A parallelogram with diagonals that are equal, and perpendicular is called a square.

e. A quadrilateral with diagonals that are equal, perpendicular, and bisect each other is a square.

f. If a quadrilateral in which all sides are equal and all angles are 90 degrees, it is a square.

g. A polygon is a square if and only if it has four equal sides and four equal angles.

h. A polygon with four equal sides and three equal angles is a square.

i. A square is the locus of points for which the sum of their distances from two given perpendicular lines is a positive constant.

j. If a parallelogram is both a rectangle and a rhombus, then it is a square.

k. A parallelogram with one right angle and two adjacent sides congruent is a square.

l. An object that can be constructed (in the Euclidean Plane) as follows: Draw a segment; from each edge erect a perpendicular to the segment, in the same length as the segment (both in the same direction). Connect the other 2 edges of the perpendiculars by a segment. The four segments form a quadrilateral that is a square.

APPENDIX C: Interview Protocol

Table 34. Interview Protocol

Question number	Main interview question	Probe
1.	a. Are you familiar with the term "mathematical definitions"? b. From your perspective, what is a mathematical definition?	
2.	a. What needs to be in a statement for it to be a mathematical definition?	• What are the necessary features of mathematical definitions? The features definitions have to have in order to be qualified as mathematical definitions. • How can you identify a mathematical definition from a paragraph of narrative? Like if you have a paragraph in front of you, how can you know which part of it is a mathematical definition?
3.	a. Are definitions, properties and descriptions essentially the same things? If not, what makes them different?	*Probe the pair wise difference if one of the three concepts did not get enough attentions.*
4.	a. Is it possible for a concept to have more than one mathematical definition? *If they agree multiple definitions may exist, ask:* b. Are some better than others? What makes them better?	
5.	a. Choose a concept you feel comfortable with and write a definition of it.	• What do you consider when you are writing? (Or what factors affect your final output?) • Is the factor you consider a necessary feature or optional feature of mathematical definitions? • *Ask PETs to write definitions of absolute value, square, area of rectangle, fraction multiplication.*

Table 34 (cont'd)

6.	a. Here are two definitions of trapezoid, what do you think about them?	• *Look for if they use the word "correct" or "accepted".* • Which one do you think is correct? Or do you even think about if there is a correct one? • Do you think if there is an absolutely correct one? Or is it just a matter of choice? • *If they think either one is fine, probe the consequence of the selection.* What will happen if you choose one definition vs. the other? • *If they insist one of them is correct,* ask: Have you heard about the other one some where? In 202 or is this the first time you heard about it?
7.	a. You are given a list of statements; please circle those you think are mathematical definitions. b. My question is to circle definitions from this list, do you think my question is answerable? *If the participant says yes, repeat the question,* so you think you are able to choose definitions from the list? c. *If he/she said "Yes", ask:* Could you please explain the reasons while you are circling?	• *When they say definition, always ask:* a definition of what? • *Look for the words they use to describe other statements ___________ (e.g., "fact", "theorem", etc.) and probe:* I heard you call those aren't circled ___________ (e.g., "facts", "theorems"), what do you mean by ___________ ? • How do you distinguish definitions from ___________ (e.g., "facts", "theorems")? Could you please go over the list and explain to me why do you call it ___________ (e.g., "facts", "theorems")?
8.	a. This is the same list as before, I would like you to underline those statements that you think need a mathematical proof. b. How do you decide if it needs a proof or not?	• Are you familiar with the word "theorem"? • *If they say "yes", ask*: What are the similarities & differences between mathematical definitions and theorems? • Are mathematical definitions & theorems completely different or somehow related? • *If they say they do not know what theorems are, then ask:* Do you know why is Pythagorean Theorem called theorem?

Table 34 (cont'd)

9.	a. Here are a few statements for you to judge. Please circle True or False beneath each statement and explain your reasons to me while you go through the list.	• *Probe specific examples.* • What are the differences and similarities between defining and proving? • If they said b is correct, ask: how about square, what kinds of proof do you need to generate to prove it is a "square"?
10.	a. Here is a list of statements about an even number. Please circle those that you think are mathematical definitions. For those you think are mathematical definitions, please rate. Use a 4 point scale to rate, where a 4 indicates that you consider it to be a good mathematical definition and 1 indicates that you do not consider it to be a good mathematical definition. Write the rating in front of the definition. When you are working on this task, I want you to forget about your teacher hat, but think yourself as a mathematics learner. You can write a brief note to record your thinking. b. *After they finish, ask:* How do you judge if it is a mathematical definition or not? I'd like you to explain your rationale regarding the ratings you gave each statement.	• *If multiple get 4pts, ask:* Which one do you think is the best? • *If any statements are given the same rating, ask if they are as good as others.* • *For different ratings, ask what makes them different.* • *For a low ranked statement, ask what it might take to make it better.* • You mentioned following criterors when you are rating, which of those you think are the most important ones? How do you order them?
11.	a. Here is a list of statements about a square. Please circle those that you think are mathematical definitions. For those you think are mathematical definitions, please use a 4 point scale to rate, where a 4 indicates that you consider it to be a good mathematical definition and 1 indicates that you do not consider it to be a good mathematical definition. Write the rating in front of the definition. When you are working on this task, I want you to forget about your teacher hat, but think yourself as a mathematics learner. b. *After they finish, ask:* How do you judge if it is a mathematical definition or not? I'd like you to explain your rationale regarding the ratings you gave each statement.	• *If multiple get 4pts, ask:* Which one do you think is the best? • *Minimality: if three right angels can infer 4 right angels, why do you need 4 there?* • *Hierarchy: there are many concepts for you start with, which do you prefer to start for a definition of a square?* • *If any statements are given the same rating, ask if they are as good as others.* • *For different ratings, ask what makes them different.* • *For a low ranked statement, ask what it might take to make it better.*

Table 34 (cont'd)

12.	a. How important are mathematical definitions in mathematics? b. Why do we have definitions in mathematics?	• Why does mathematics as a discipline need mathematical definitions? • What purpose does it mathematical definitions serve in mathematics? • To what purposes have mathematical definitions been presented to you in *your* mathematics courses? • *If they use the word, "build", ask:* Could you please elaborate what do you mean by "build"? An example? • How do definitions differ from a description/property of its role in mathematics?
13.	a. What courses at MSU have provided you opportunities to learn about mathematical definitions?	• *If it is explicit discussion or implicit exposure to definitions* • *Go over the list of courses and probe specific examples. Probe:* What about mathematical definitions do you learn in that activity? • During our conversation, I found some interesting points you demonstrated about mathematical definitions: ____________________Are you able to recall why do you think in this way and if that connects to any of your course experience? Or it is just something you figured it out by yourself during your schooling?
14.	a. Now I want to revisit a question at the beginning of the interview. From your perspective, what is a mathematical definition? I bet you may have more to say after you see so many examples.	

APPENDIX D: A List of Concepts PSTs Wrote in Question 5

Table 35. A List of Concepts PSTs wrote in Question 5

Content area	Topic	Concept	PSTs - M	PSTs - N
Geometry	Polygon	Square	1	4
		Right triangle		1
		Polygon	1	
	Circle	Circle		1
	Angle and line	Reference angle		1
		Ray		1
		Parallel lines	2	
		Perpendicular lines	1	
	Polyhedron	Platonic solid	1	
	Congruent	Congruent	1	1
	Pythagorean theorem	Pythagorean theorem		1
Measurement	Length	Length		1
	Area	Area		1
Number and operations	Number	Negative number	1	
		Even number		1
	Whole number operation	Adding	1	
		Addition	2	2
		Whole number multiplication		2
		Whole number Division		1
	Fraction operation	Division of fraction	1	
Algebra	Function	Function	2	1
	Equation	Algebraic equation		1
		Equation	3	1
		Linear equation		1
		How to solve a linear equation		1
	Expression	Expression	2	
	Binary operation	Binary operation	1	
	Exponents	Exponents	1	
	Arithmetic property	Distributive property	1	
		Associative property	1	
Calculus		Sequence	1	
		Limit	1	
General concepts		Geometry		1
		Algebra	1	

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