A GENERALIZED FRATTINI SUBGROUP OF A FINITE GROUP

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ABSTRACT

A GENERALIZED FRATTINI SUBGROUP OF A FINITE GROUP

By

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The Frattini subgroup and its influence on a group have been the objects of study by group theorists for many years. In recent years several mathematicians have investigated various generalized Frattini subgroups. W. Gaschütz and more recently J. Rose and H. Bechtell have studied the family of non normal maximal subgroups of a finite group. W. Deskins has considered the family of maximal subgroups whose indices in the finite group are not divisible by a given preassigned prime.

In the present investigation we consider another family of maximal subgroups of a finite group. Let $\mathfrak{J}(G)$ be the family of maximal subgroups of nonprime index of the group G. Let L(G) denote the intersection of the members of the family $\mathfrak{J}(G)$. In case $\mathfrak{J}(G)$ is empty, define $L(G) = G$.

By a well-known theorem of B. Huppert, a finite group is supersolvable iff each of its maximal subgroups has prime index. It follows that $L(G) = G$ iff G is supersolvable.

Let $\mathcal{L}(G)$ be the family of all the non normal maximal subgroups of G and let $\Delta(G)$ denote the intersection of the

members of $\mathcal{L}(G)$. Since a group is nilpotent iff all its maximal subgroups are normal, it follows that $\Delta(G) = G$ iff G is nilpotent. The subgroup $\Delta(G)$ and its generalizations have also been studied by, among others, J. Beidleman, T. Seo [Pacific Jour. of Math. 23 (1967)] and D. Dykes [Pacific Jour. of Math. 31 (1969)].

Our investigations show that the two families $\mathcal{L}(G)$ and $\mathfrak{Z}(G)$ act in many respects as natural analogues of each other, the former with respect to the nilpotent structure and the latter with respect to the supersolvable structure of a group.

We also show that $L(G)$ is related to the hyperquasicenter [N. Mukherjee, Ph.D. Thesis, Michigan State University, 1968] in the same manner as $\Delta(G)$ is to the hypercenter. In Chapter I we also study the influence of $\mathfrak{Z}(G)$ on the solvability of G improving some results of J. Rose. The main results of Chapter I are: (1) $L(G)$ is supersolvable, (2) If G is solvable, $L(G) \cap G'$ is nilpotent, (3) The hyperquasicenter of G is contained in $L(G)$, and (4) If every maximal subgroup of G of nonprime index is nilpotent, then G is solvable.

In Chapter II some results of Beidleman, Seo and Dykes are generalized. It is shown that $L(G)$ satisfies some Frattini-like properties described below. Definition (a). Let π be the set of primes dividing $|G|$. Let $\pi_1 \subseteq \pi$ such that for all elements p and q of π_1 and $\pi \sim \pi_1$ respectively, $p > q$. Then π_1 is an upper set (UP-set) for

G. Definition (b). A proper normal subgroup is a special L-subgroup of G if for every normal subgroup N of G and A a Hall π-subgroup for π an arbitrary UP-set for N, $G = HN_G(A)$ implies $G = N_G(A)$. Definition (c). A proper normal subgroup H of G has property (θ) in G if for every $N \trianglelefteq G$ with $H \leq N$, N/H π-closed for π a UP-set for N implies N is π-closed. Following R. Baer we also define the concept of a weakly hyperquasicentral subgroup. The main results of Chapter II are: (1) L(G) has property (θ) in G, (2) If H is a special L-subgroup of G, then G/H has the Sylow tower property iff G has that property, (3) For a nilpotent normal subgroup H, the definitions (b) and (c) are equivalent to each other and also to the concept of weakly hyperquasicentral subgroups. So far we have not been able to confirm the necessity of the 'nilpotency' condition on the subgroup H above. We conjecture that this condition can be replaced by 'the Sylow tower property'.

In Chapter III we conclude the present investigation by obtaining some conditions for the group G to be supersolvable; and for $\Delta(G)$ and L(G) to coincide. Following Bechtell [Pacific Jour. of Math. 14 (1964)], we define L-series and relative L-series (relative to the commutator subgroup) of the group G. The latter series turn out to be far more interesting for our consideration. Two of the results of this chapter are:

(1) $\Delta(G) = L(G)$ if an L-series of G terminates in $\langle 1 \rangle$.

(2) G is supersolvable if and only if the upper relative L-series of G coincides with the descending central series of G'.

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TO MY FATHER

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INDEX OF NOTATIONS

I Relations:

$\subseteq$	Is a subset of
$\leq$	Is a subgroup of
$\nleq$	Is not a subgroup of
$<$	Is a proper subgroup of ($\lneq$ for emphasis)
Δ	Is a normal subgroup of
$\not\Delta$	Is not normal
$\in$	Is an element of
$\notin$	Is not an element of
$\cong$	Isomorphic to
$\overset{\subset}{\cong}$	Isomorphic to a subgroup of

II Operations:

$\sim$	Set difference
$< >$	Subgroup generated by
$\times$	Direct product of groups
$\|G\|$	The number of elements in G, the order of G
$\|x\|$	The order of the element x
π	Set of primes
$p(n)$	The prime divisors of the number n
$p(n_1) < p(n_2)$	The prime divisors of n_1 are smaller than the prime divisors of n_2

$(m,n) = 1$	m and n are two numbers relatively prime to each other
$[G:H]$	Index of H in G
$\frac{G}{H}$; G/H	Factor group
G^x	$x^{-1}Gx$
a^x	$x^{-1}ax$
$[h,k]$	$h^{-1}k^{-1}hk$
$[H,K]$	Subgroup generated by all $[h,k]$; $h \in H$, $k \in K$
$\{ \}$	Set whose elements are
$\cup$	Union
$\cap$	Intersection
$\forall$	For each
G^n	The subgroup generated by n^{th} powers of all the elements of G.

III Groups and Elements:

$<1>$	The identity of a group
$<\bar{1}>$	The identity of a factor group
π-element	An element whose order is divisible by only the primes in the set π
π'-element	An element whose order is not divisible by any prime in the set π
π-group	A group whose order is divisible by only the primes in the set π
π'-group	A group whose order is not divisible by any prime in the set π

$\mathrm{Core}_G K$	The largest normal subgroup of G contained in the subgroup K
$Z(G)$	The center of the group G
$Q(G)$	The quasicenter of the group G
$Z^*(G)$	The hypercenter of the group G
$Q^*(G)$	The hyperquasicenter of the group G
$\Delta(G)$	Defined on page 24
$L(G)$	Defined on page 12
S_n	The symmetric group of degree n
A_n	The alternating group of degree n
Φ, $\Phi(G)$	The Frattini subgroup of G
G'	The commutator subgroup of G
$\ell_p(G)$	The p-length of G

IV <u>Other</u>:

x,y permute means $\langle x\rangle$, $\langle y\rangle$ permute.

INTRODUCTION

In recent years a number of mathematicians have studied various generalized Frattini subgroups. W. Gaschütz [1] and more recently J. Rose [16, 17] and H. Bechtell [7] have considered in detail the family of non-normal maximal subgroups and their intersection. W. Deskins [8] has considered the family of maximal subgroups whose index in the group is not divisible by a given preassigned prime p, and studied the structure of the subgroup ϕ_p which is the intersection of the members of this family of maximal subgroups. More recently, J. Beidleman and T. Seo [5, 6] have generalized the work of H. Bechtell.

In the present work we investigate another family of maximal subgroups of a finite group G, namely that consisting of those maximal subgroups whose index in G is <u>not</u> a prime. Denote by $L(G)$ the intersection of all the maximal subgroups of G whose index in G is not a prime. In case G has no maximal subgroup of non-prime index, then as is customary, we set $L(G) = G$. It is well-known [13] that a finite group is supersolvable if and only if all its maximal subgroups have prime index. Thus $L(G) = G$ if and only if G is supersolvable. In a natural way therefore, $L(G)$ is a measure of how far G is from being supersolvable.

Since for a finite group G a normal maximal subgroup has a prime index, the family of non-normal maximal subgroups considered by Gaschütz contains all the maximal subgroups with non-prime index in G. If $\Delta(G)$ denotes the intersection of all the non-normal maximal subgroups, then $\Delta(G)$ is contained in $L(G)$. Thus in part our work is a direct extension of the ideas of Gaschütz.

It is well-known (see for example Gaschütz [11] and Deskins [8]) that $\Delta(G)$ is nilpotent. At the outset it is easy to see that $L(G)$ in general need not be nilpotent. This is so because for any supersolvable group G, $L(G)$ and G coincide and there are finite supersolvable groups which are not nilpotent as for example S_3, the symmetric group on three symbols. It seems reasonable however, to conjecture that $L(G)$ is supersolvable. The verification of this conjecture is one of the main results in Chapter I. It has been proved by, among others, Beidleman [6] that G is nilpotent if $G/\Delta(G)$ is nilpotent. In Chapter I we have been able to show that this result is still true when $\Delta(G)$ is replaced by $L(G)$ and 'nilpotent' by 'supersolvable'. Incidentally our result also generalizes a well-known result, namely: G is supersolvable iff $G/\Phi(G)$ is supersolvable. Note that $\Phi(G)$, the Frattini subgroup of a group G, is contained in $L(G)$.

The concept of the quasicenter as a generalization of the center was introduced by O. Ore [15] in the following manner:

An element $x \in G$ is quasicentral in G if for every element $g \in G$, the cyclic subgroups $\langle x\rangle$ and $\langle g\rangle$ permute. In other words $\langle x\rangle\langle g\rangle = \langle g\rangle\langle x\rangle$ forms a subgroup.

The subgroup generated by all the quasicentral elements of the group G is the quasicenter of G and is denoted by $Q(G)$. Ore proved that the quasicenter of the group is a characteristic subgroup. He studied the quasicenter as a quasinormal subgroup for the symmetric and alternating groups.

The structure of the quasicenter has been investigated in some detail by N. Mukherjee [14] who showed that the quasicenter of a group is nilpotent and that each of its Sylow subgroups is generated by quasicentral elements of the group.

Since every element in the center of a group commutes with every element of the group, it is in particular quasicentral in G. Thus the center of a group is part of the quasicenter. This leads to the definition of the hyperquasicenter [14] which is a generalization of the concept of the hypercenter and is defined in an analogous manner. The hyperquasicenter, denoted by $Q^*(G)$, is a normal supersolvable subgroup of G and contains every supersolvably embedded subgroup of G [14]. Also G is supersolvable iff $G = Q^*(G)$.

It is known [7, Theorem 2.2] that the hypercenter $Z^*(G)$ of a group is contained in $\Delta(G)$. In general $\Delta(G)$ does not contain the hyperquasicenter of G. However the analogy between $\Delta(G)$ and $L(G)$ suggests a relationship between the hyperquasicenter and $L(G)$. We are able to establish this in Chapter I by proving that the hyperquasicenter of a group G is contained in $L(G)$.

J. Rose [16] proved that a group is solvable if all its non normal maximal subgroups are nilpotent. As a last result in Chapter I, we show that this theorem of Rose can be improved by replacing 'non normal maximal subgroups' by only those 'maximal subgroups whose index in G is non-prime'. We have an example to show that our result is false if 'nilpotency' is replaced by 'supersolvability'.

J. Rose extended his result by proving that a finite group is solvable if each of its non normal maximal subgroups is supersolvable and has prime power index. Our example shows that this result of Rose is not true if 'non normal maximal subgroup' is replaced by 'maximal subgroup with non prime index'. This is not too surprising since maximal subgroups with non prime index may in general, form a very 'small' portion of all the non normal maximal subgroups of the group. However, we have so far not been able to obtain suitable conditions under which a finite group, with every maximal subgroup of non prime index supersolvable, is solvable.

In Chapter II we generalize the work of Beidleman and Seo [5, 6] and D. Dykes [10] and show that the subgroup $L(G)$ of a finite group G belongs to a class of normal subgroups called special L-subgroups of G. Following Beidleman and Seo, a proper normal subgroup H of a finite group G is called a generalized Frattini subgroup of G if for every normal subgroup N of G and a Sylow subgroup P of N, $G = H \cdot N_G(P)$ implies that $G = N_G(P)$. Generalized Frattini subgroups are normal, nilpotent subgroups of G. Also $\Delta(G)$ is a generalized Frattini subgroup of G. Furthermore, if H is a generalized Frattini subgroup of G such that G/H is nilpotent then G is nilpotent.

Generalizing the above definition we call a normal subgroup H of G an L-subgroup of G if H satisfies the above definition only for the Sylow subgroups of N corresponding to the largest prime divisor of the order of N.

We show that an L-subgroup is p-closed, for p the largest prime divisor of its order. Moreover if p is the largest prime divisor of the order of G, then G/H p-closed implies G is p-closed. Also $L(G)$ is an L-subgroup of G. Some of the results of [5, 6] we have not been able to generalize. The main difficulties in this respect seem to be the following:

(a) The concept of an L-subgroup seems to be too general to sufficiently restrict the behavior of Sylow subgroups for the various prime divisors.

(b) It is known (see for instance Gaschütz [11] and Deskins [8]) that $\Delta(G)/\Phi(G) = Z(G/\Phi(G))$. An analogous relation connecting $Q(G)$ and $L(G)$ however, does not hold.

A considerably restricted sub-class of the class of generalized Frattini subgroups, called special generalized Frattini subgroups was considered by Dykes [10]. He proved that $\Delta(G)$ is a special generalized Frattini subgroup of the group G. Moreover the concepts of special generalized Frattini subgroups and the weakly hypercentral subgroups defined by R. Baer [4] are equivalent.

Following these ideas we define special L-subgroups of the group G; and prove that L(G) is one such subgroup of G. In Chapter II we also define weakly hyperquasicentral subgroups and a certain property $(\mathcal{P})$.

One of the main results we prove in this direction states that:

For a normal nilpotent subgroup H of a group G, the following are equivalent:

(a) H is a special L-subgroup of G.

(b) H is weakly hyperquasicentral in G.

(c) H has property $(\mathcal{P})$ in G.

We also show that if H is a special L-subgroup of G, then G has the Sylow tower property if G/H has that property.

Our results on the above concepts are partial in the following two respects:

(1) The equivalence of the definitions of special L-subgroups and weakly hyperquasicentral subgroups is shown only for nilpotent normal subgroups. However we believe this to be true for any normal subgroup satisfying the Sylow tower property.

(2) We have not been able to justify the term 'weakly hyperquasicentral' as we cannot yet prove that a hyperquasicentral subgroup [14] is also weakly hyperquasicentral. However we have shown this to be true for the nilpotent case.

In Chapter III we conclude the present investigation by considering some conditions under which $\Delta(G)$ and L(G) coincide. Following Bechtell [7] we define an L-series and the upper and lower L-series. It is shown that if the upper L-series of the group G terminates in $\langle 1\rangle$, then $\Delta(G) = L(G)$.

We also define a relative L-series of G (relative to the commutator subgroup of G). These series yield some interesting information about the group G. Some of the results here are:

(1) The upper relative L-series coincides with the descending central series of G' iff G is supersolvable.

(2) The following are equivalent for a group G:

(i) $\hat{L}^*(G) = \langle 1\rangle$, where $\hat{L}^*(G)$ is the terminal member of the upper relative L-series.

(ii) $L(G) \cap G'' = Z^*(G') \cap G''$.

(iii) If S is any subgroup of G generated by 3 elements one of which belongs to $L(G)$, then $L(G) \cap S' \leq Z^*(S')$.

(3) If H is a solvable normal subgroup of G, then $\hat{L}^*(G) = \langle 1\rangle$ implies $\hat{L}^*(H) = \langle 1\rangle$.

We have not been able to decide whether the solvability condition on H in (3) is necessary.

CHAPTER I

Let $\mathfrak{Z}(G)$ be the family of those maximal subgroups of the finite group G which have nonprime index in G. Let $L(G)$ denote the subgroup which is the intersection of all the members of $\mathfrak{Z}(G)$. As is customary, define $L(G) = G$ in case the set $\mathfrak{Z}(G)$ is empty. It is obvious that $L(G)$ is a characteristic subgroup of G. In this chapter we investigate the structure of the subgroup $L(G)$ and study its influence on the structure of the group G. The main results proved here are the following:

(1) The subgroup $L(G)$ is supersolvable.

(2) If $G/L(G)$ is supersolvable, then G is supersolvable.

(3) The hyperquasicenter of the group G is contained in $L(G)$.

(4) If every maximal subgroup of G of nonprime index is nilpotent, then G is solvable.

For the sake of completeness and easy reference we include here some known results. Proofs or references are also given.

Throughout, only finite groups will be considered. An index of notations and special symbols appears on a previous page.

1.1 Basic Concepts and Preliminary Results.

We begin with the definition of the Frattini subgroup of a group and a lemma due to Frattini of which we shall make frequent use.

Definition 1.1.1: Let $\Phi(G) = \cap\{M | M$ is a maximal subgroup of $G\}$. $\Phi(G)$ is called the Frattini subgroup of G.

$\Phi(G)$ is a nilpotent characteristic subgroup of G.

The argument used in the following lemma is due to Frattini.

Lemma 1.1.2: Let H be a normal subgroup of a group G. If P is a Sylow subgroup of H, then $G = HN_G(P)$.

Proof: Since H is normal in G, for every $g \in G$, $g^{-1}Pg \leq H$. Therefore P and $g^{-1}Pg$ are both Sylow p-subgroups of H, for the prime p. Consequently both P and $g^{-1}Pg$ are conjugate in H. Let x be an element of H such that $x^{-1}Px = g^{-1}Pg$. Then $(gx^{-1})^{-1}P(gx^{-1}) = P$, so that gx^{-1} belongs to the normalizer of P in G. The element g therefore belongs to the set $N_G(P)x$ of $N_G(P)H$. This proves that $G = HN_G(P)$.

We shall refer to the above result as 'Frattini's lemma'.

Definition 1.1.3: A group G is p-closed if G has a normal Sylow p-subgroup.

Definition 1.1.4 (Baer): If the group G is p-closed, and if G/P is abelian of exponent dividing $p-1$, where P is the Sylow p-subgroup of G, then G is strictly p-closed.

It is easy to verify (see for example [1]) that G is strictly p-closed if, and only if G' and G^{p-1} are p-subgroups of G.

<u>Definition 1.1.5</u>: A group G has the <u>Sylow tower property of supersolvable groups</u>, if for every homomorphic image H of G and p the maximal prime divisor of the order of H, H is p-closed.

Note: We shall throughout refer to the above property as the Sylow tower property.

The above definition is easily seen to be equivalent to the following:

Let $p_1 > p_2 > \ldots > p_n$ be the natural ordering of prime divisors of the order of G, and let P_i be a Sylow p_i-subgroup of G. Then for each k, $P_1P_2\ldots P_k$ is a normal subgroup of G.

A group having the Sylow tower property is solvable.

Subgroups and homomorphic images of G have the Sylow tower property if G has that property.

<u>Definition 1.1.6</u>: A group G is <u>supersolvable</u> if every homomorphic image of G has a nontrivial normal subgroup which is cyclic.

It is well-known and easily verified that subgroups, homomorphic images and direct products of supersolvable groups are again supersolvable. An extension of a supersolvable group by a supersolvable group in general is not supersolvable. However, an extension of a cyclic group by a supersolvable group is supersolvable.

By a well-known theorem of B. Huppert [13], we have, Theorem 1.1.7: The following properties of a group G are equivalent:

(i) G is supersolvable.

(ii) $G/\Phi(G)$ is supersolvable.

(iii) Every maximal subgroup of G has index a prime.

By a result of Baer [1, Theorem 2.1], the above conditions are also equivalent to,

(iv) G has the Sylow tower property; and $N_G(P)/C_G(P)$ is, for every Sylow p-subgroup P of G, strictly p-closed.

If G is supersolvable, then its commutator subgroup G' is nilpotent; and G has the Sylow tower property [13, p. 415].

Definition 1.1.8: A subgroup M of G is supersolvably embedded (SSE) in G if for each homomorphism of G, the image of M contains a cyclic subgroup which is normal in the homomorphic image of G.

Remark 1. A supersolvably embedded subgroup is supersolvable. A proof of this may be found in [14].

Remark 2. Every SSE subgroup M of G is normal in G.

This can be easily proved by induction on the order of G.

Remark 3. If M is SSE in G and G/M is supersolvable, then G is supersolvable.

This is true because there is a normal series of G up to M with cyclic factor groups. By the supersolvability of G/M, this chain can be extended to a normal series of G

with cyclic factor groups, which implies the supersolvability of G.

Next we state a result of Baer [1, Theorem 4.1] that we shall need.

Theorem 1.1.9: The following properties of a normal subgroup K of a group G are equivalent:

(i) K is SSE in G.

(ii) $K/\Phi(K)$ is SSE in $G/\Phi(K)$.

(iii) K has the Sylow tower property; and if P is a Sylow p-subgroup of K, then $N_G(P)/C_G(P)$ is strictly p-closed.

1.2 The Subgroup $L(G)$.

Definition 1.2.1: Let G be a group. Define

$L(G) = \cap\{M|M$ is a maximal subgroup of G and $[G:M]$ is not a prime$\}$

$R(G) = \cap\{M|M$ is a maximal subgroup of G and $[G:M]$ is a prime$\}$.

In case G has no maximal subgroup of nonprime index, we set $L(G) = G$. Similarly $R(G) = G$ if G has no maximal subgroup of prime index.

It is obvious that both $L(G)$ and $R(G)$ are characteristic subgroups of G. Moreover $R(G) \cap L(G) = \Phi(G)$, where $\Phi(G)$ is the Frattini subgroup of G.

By Theorem 1.1.7, a group is supersolvable if and only if each of its maximal subgroups has prime index. It follows therefore that G is supersolvable if and only if $L(G) = G$.

Proposition 1.2.2: If K is a normal subgroup of G, then

(i) $L(G)K/K \leq L(G/K)$.

(ii) $R(G)K/K \leq R(G/K)$.

(iii) $\Phi(G)K/K \leq \Phi(G/K)$.

In particular, if $K \Delta G$ and $K \leq L(G)$, then $L(G/K) = L(G)/K$.

Proof: (i) Let $\overline{M}_i$ be a maximal subgroup of G/K of nonprime index. There exists a maximal subgroup M_i of G containing K such that $\overline{M}_i = M_i/K$. Also $[G/K\colon M_i/K] = [G:M_i]$, so that M_i has nonprime index in G. Then $\cap(M_i/K) = L(G/K)$. Let J be the intersection of all the M_i's corresponding to $\overline{M}_i$; then J contains $L(G)$ as well as K. Hence $L(G)K/K \leq J/K$. But now it is easy to show that $J/K \leq \cap(M_i/K) = L(G/K)$. For if not, suppose there exists $xK \in J/K$ such that $xK \notin \cap(M_i/K)$, where $x \in J$. Then there exists some $\overline{M}_i = M_i/K$ such that $xK \notin M_i/K$. This implies that $x \notin M_i$. But this is a contradiction since $x \in J \subseteq M_i$. Thus $L(G)K/K \leq J/K \leq L(G/K)$ and (i) is proved.

(ii) This is proved in the same manner as (i). Let $\overline{M}_i$ be any maximal subgroup of G/K of prime index. There exists a maximal subgroup M_i of G containing K such that $\overline{M}_i = M_i/K$, and $[G/K\colon M_i/K] = [G:M_i]$. Consequently $R(G) \leq M_i$ and therefore $R(G)K/K \leq T/K$, where T is the intersection of all those maximal subgroups of G that correspond to $\overline{M}_i$'s. Then as in (i) we can show that $R(G)K/K \leq T/K \leq \cap(M_i/K)$.

(iii) Duplicating the argument of (i), let J be the intersection of all those maximal subgroups of G that contain K. Then since $\Phi(G)$ is contained in every maximal subgroup of G, $\Phi(G)K/K \leq J/K$. The maximal subgroups of G/K are the images under the homomorphism $\theta: G \to G/K$, of those maximal subgroups of G which contain K. Consequently, $J/K \le (\bigcap M_i)/K \le \bigcap(M_i/K) = \Phi(G/K)$. The proof of (i), (ii) and (iii) is thus complete.

In the particular case, suppose $K \le L(G)$. By (i) $L(G)/K \le L(G/K)$. It only remains to show that $L(G/K) \le L(G)/K$. For this assume that xK is an element of $L(G/K)$ which does not belong to $L(G)/K$. Then $x \notin L(G)$ and so there exists a maximal subgroup M of G of nonprime index such that $x \notin M$. Since by hypothesis $K \le L(G) < M$, $xK \notin M/K$. But M/K is a maximal subgroup of G/K with $[G/K: M/K] = [G:M] =$ nonprime. We thus arrive at a contradiction since xK is an element of $L(G/K)$. Therefore $L(G/K) = L(G)/K$.

<u>Corollary 1.2.3</u>: (a) $L(G/\Phi(G)) = L(G)/\Phi(G)$.

(b) $L(G/L(G)) = \langle\bar{1}\rangle$.

<u>Remark</u>: In general $L(G)K/K < L(G/K)$. To confirm this consider $G = A_4$. The 4-group V_4 is a normal subgroup of G. The maximal subgroups of G of nonprime index are cyclic of order 3. Since these maximal subgroups are non normal, their intersection is the identity subgroup. Thus $L(G) = \langle 1\rangle$. Hence $L(G)V_4/V_4 = \langle\bar{1}\rangle$. On the other hand since $G/V_4 \cong C_3$, we have $L(G/V_4) \cong C_3$.

In general $L(G)$ is not nilpotent. For example let $G = S_3$. Since S_3 is supersolvable, $L(G) = S_3$. As is well-known S_3 is not nilpotent. We also know that a supersolvable group has nilpotent commutator subgroup. Furthermore $L(G) = G$ *iff* G is supersolvable. From these considerations it seems natural to ask what the nature of the common subgroup of $L(G)$ and G' would be. We answer this in the following result.

<u>Theorem 1.2.4</u>: If G is solvable, then $G' \cap L(G)$ is nilpotent.

Proof: Let T be any maximal subgroup of G of prime index, say p. Then G has p cosets, say $a_1T, a_2T, \ldots, a_pT$. Denote by S the set of these p cosets of G. If $g \in G$, then $g(a_iT) = ga_iT \in S$. Thus elements of G act as permutations of the set S. If a_iT and a_jT are any two arbitrary members of S, we can find an element $g \in G$ such that $a_i = ga_j$ and hence $a_iT = ga_jT = g(a_jT)$. Thus we see that G acts as a transitive permutation group of degree p, on the set S. Also by hypothesis G is solvable. By [13, Satz 8], a solvable transitive permutation group of prime degree is metabelian. Let T_G be the core of T in G, which is the intersection of all the conjugates of T in G. If $g \in T_G$, then g belongs to every conjugate of T and consequently induces the identity permutation on the set S. Thus the group of permutations representing G is isomorphic with G/T_G, which as remarked earlier is metabelian. This implies that $G'' \leq T_G \leq T$. Since T was chosen arbitrarily, we see that $G'' \leq R(G)$.

Now $G'' \leq R(G)$ implies that $G'' \cap L(G) \leq R(G) \cap L(G) = \Phi(G)$. Also $[G' \cap L(G), G'] \leq [G', G'] = G''$. Since $G' \cap L(G)$ and G' are both normal in G, we have $[G' \cap L(G), G'] \leq L(G) \cap G''$. Thus, $[G' \cap L(G), G'] \leq L(G) \cap G'' \leq \Phi(G)$.

Now consider the following two cases:

Case (1): Suppose $\Phi(G) = \langle 1 \rangle$. Then $[G' \cap L(G), G'] = \langle 1 \rangle$, which implies that $G' \cap L(G)$ commutes with every element of G'. Since $G' \cap L(G) \leq G'$, we have $G' \cap L(G) \leq Z(G')$. Hence $G' \cap L(G)$ is abelian and therefore nilpotent.

Case (2): Suppose $\Phi(G) \neq \langle 1 \rangle$ and let $\bar{G} = G/\Phi(G)$. Obviously $\Phi(G/\Phi(G)) = \langle \bar{1} \rangle$. Then by Case (1), $(\bar{G})' \cap L(\bar{G})$ is nilpotent. By Corollary 1.2.3, $L(\bar{G}) = \overline{L(G)}$. Moreover by [12, p. 18], $[G/\Phi(G), G/\Phi(G)] = \frac{[G,G]\Phi(G)}{\Phi(G)}$ and so $(\bar{G})' = (G/\Phi(G))' = G'\Phi(G)/\Phi(G)$. Then we have that $\frac{G'\Phi(G)}{\Phi(G)} \cap \frac{L(G)}{\Phi(G)} = \frac{G'\Phi(G) \cap L(G)}{\Phi(G)}$ is nilpotent. Now since $G'\Phi(G) \cap L(G)$ is a normal subgroup of G containing $\Phi(G)$ and $\frac{G'\Phi(G) \cap L(G)}{\Phi(G)}$ is nilpotent, by a well-known theorem of H. Wielandt (see for instance [7, Corollary 2.3.2]), we conclude that $G'\Phi(G) \cap L(G)$ and hence $G' \cap L(G)$ is nilpotent. The proof is thus complete.

Remark: As mentioned before, for any supersolvable group G the commutator subgroup G' is nilpotent. This result can be easily deduced from the above theorem. For when G is supersolvable, it is solvable and $L(G) = G$. Thus $G' \cap L(G) = G'$, which is nilpotent by Theorem 1.2.4.

One of our aims in this chapter is to prove that $L(G)$ is supersolvable. Our next result is a step in this direction. It also shows that $L(G)$ is solvable.

Proposition 1.2.5: In any group G, $L(G)$ has the Sylow tower property.

Proof: We proceed by induction on the order of G. Let p be the largest prime dividing the order of $L(G)$ and P a Sylow p-subgroup of $L = L(G)$. We claim that P is normal in G. For this, suppose $N_G(P) \neq G$ and let S be a maximal subgroup of G containing $N_G(P)$. By Frattini's lemma, $G = LN_G(P) = LS$. Since S is a proper subgroup of G, $L \nleq S$. So S has prime index in G, say q. Since $q = [G:S] = [L:L \cap S]$, we note that q divides $|L(G)|$. Moreover since $N_L(P) \leq L \cap S$, by the Sylow theorems and a result in Scott [18, 6.2.3] we have $[L:L \cap S] = 1 + kp$, where k is some integer. It follows that $q = 1 + kp$ and hence p divides $q-1$. But this is impossible since both q,p divide $|L(G)|$ and p is the largest such divisor. Hence we conclude that $P \Delta G$.

Now by Proposition 1.2.2, $L(G)/P = L(G/P)$ and by the induction hypothesis, $L(G)/P$ has the Sylow tower property. It is now easy to see that $L(G)$ itself has the Sylow tower property.

<u>Corollary 1.2.6</u>: $L(G)$ is solvable.

This is true since the Sylow tower property implies solvability.

<u>Corollary 1.2.7</u>: If $G/L(G)$ is solvable and $(G/L(G))''$ is nilpotent, then G'' is nilpotent.

Proof: (a) Obviously G is solvable. (b) By Theorem 1.2.4, $G'' \cap L(G) \leq \Phi(G)$. Then $(G/L(G))'' = \dfrac{G''L(G)}{L(G)} \cong G''/G'' \cap L(G)$ implies that G'' is nilpotent.

Before proceeding to prove the supersolvability of $L(G)$, we digress a bit to consider some generalizations and study the influence of $L(G)$ on the group G.

It is known that if $G/\Phi(G)$ has the Sylow tower property, then G has that property. In the following proposition we improve this result.

<u>Proposition 1.2.8</u>: If $G/L(G)$ has the Sylow tower property, then G has the same property.

Proof: We use induction on the order of G. First let K be any normal subgroup of G. We show that G/K satisfies the hypothesis. By Proposition 1.2.2, $L(G)K/K \leq L(G/K)$ and hence $G/K/L(G/K)$ is isomorphic to a homomorphic image of $G/K/L(G)K/K \cong G/L(G)K$ (by the isomorphism theorems). Since $G/L(G)$ and hence $G/L(G)K$ has the Sylow tower property, $G/K/L(G/K)$ has that property. Hence for any $K \Delta G$, G/K satisfies the hypothesis.

Now let p be the largest prime divisor of $|G|$ and P be a Sylow p-subgroup of G. If $P \leq L(G)$, then p is also the largest prime divisor of $|L(G)|$ and P its Sylow p-subgroup. By Proposition 1.2.5, P is normal in $L(G)$ and hence normal in G.

If $P \not\leq L(G)$, then $p/|G/L(G)|$ and p is its largest prime divisor. By [18, 6.1.16], $PL(G)/L(G) = \overline{P}$ is a Sylow p-subgroup of $G/L(G)$. Since by hypothesis $G/L(G)$ has the Sylow tower property, $\overline{P}$ is normal in $G/L(G)$ and hence $PL(G) \Delta G$. Now P is also a Sylow p-subgroup of $PL(G)$ and using Frattini's lemma, we have $G = L(G)PN_G(P) = L(G)N_G(P)$. If $P \not\Delta G$, then $G \neq N_G(P)$. Let S be a maximal subgroup of G containing $N_G(P)$. So $G = L(G)S$ and $L(G) \not\leq S$. Therefore S has prime index in G, say q. Since $N_G(P) \leq S$ and P

is a Sylow subgroup of G, by the Sylow theorems as in Proposition 1.2.5, $q = [G:S] = 1 + kp$, where k is some integer. This is impossible since p is the largest prime divisor of $|G|$. Thus we conclude that P is normal in G.

By the first part of the proof, G/P satisfies the hypothesis. This means that $G/P/L(G/P)$ has the Sylow tower property. By the induction hypothesis it follows that G/P has the same property. Consequently G itself enjoys the Sylow tower property. This completes the proof.

In the same vein we improve another well-known result. As remarked earlier (Theorem 1.1.7), a group G is supersolvable if and only if $G/\Phi(G)$ is supersolvable. We shall show that this result still holds when $\Phi(G)$ is replaced by $L(G)$. The fact that $L(G)$ is solvable (Corollary 1.2.6) will be used in the next theorem.

Theorem 1.2.9: A group G is supersolvable if and only if $G/L(G)$ is supersolvable.

Proof: (a) Since every homomorphic image of a supersolvable group is supersolvable, clearly $G/L(G)$ is supersolvable if G is supersolvable. (b) Conversely, assume that $G/L(G)$ is supersolvable. Since $L(G)$ and $G/L(G)$ are solvable, G is solvable. Moreover, since $G/L(G)$ and $G/G'\Phi(G)$ are supersolvable, $G/L(G) \times G/G'\Phi(G)$ is supersolvable. It follows that $G/L(G) \cap G'\Phi(G)$ which is isomorphic to a subgroup of $G/L(G) \times G/G'\Phi(G)$ is also supersolvable. Note that $\Phi(G) \leq L(G) \cap G'\Phi(G)$. If $\Phi(G) = L(G) \cap G'\Phi(G)$, then from the previous remark, $G/\Phi(G)$ is supersolvable. This by Theorem 1.1.7

implies the supersolvability of G. We may thus assume that $\Phi(G) < L(G) \cap G'\Phi(G)$. Next we show that $\frac{L(G) \cap G'\Phi(G)}{\Phi(G)}$ is SSE in $G/\Phi(G)$. Hereafter in the proof let $\Phi = \Phi(G)$ and $L = L(G)$.

As shown in the proof of Theorem 1.2.4 using the solvability of G, $\frac{L \cap G'\Phi}{\Phi}$ is nilpotent. Let P be the Sylow p-subgroup of $L \cap G'\Phi/\Phi$. Then P is normal in G/Φ. Let N/Φ be a minimal normal subgroup of G/Φ contained in P. Since $\Phi(G/\Phi) = \langle\bar{1}\rangle$, there exists a maximal subgroup S/Φ of G/Φ such that $N/\Phi \nleq S/\Phi$. Also S is a maximal subgroup of G. Then $G/\Phi = N/\Phi \cdot S/\Phi$ and since N/Φ is solvable minimal normal in G/Φ, we have $N/\Phi \cap S/\Phi = \langle\bar{1}\rangle$ [2, p. 118]. So $G = NS$, $N \cap S = \Phi$ and $N \nleq S$. Since $N \leq L(G)$, it is contained in every maximal subgroup of G of nonprime index. It follows that S has prime index in G and hence S/Φ has prime index in G/Φ. Therefore $|N/\Phi| = [G/\Phi : S/\Phi]$ is a prime and hence N/Φ is a subgroup of order p. Furthermore, since $\Phi(G/\Phi) = \langle\bar{1}\rangle$ and $P \,\Delta\, G/\Phi$, it follows [11] that P is elementary abelian. From this it follows that $P = C_1 \times C_2 \times \dots C_r$, where the C_i's are cyclic subgroups of order p and each C_i is normal in $G/\Phi(G)$. Thus there is a normal series of $G/\Phi(G)$ up to P with cyclic factor groups. By definition this implies that P is SSE in G/Φ. Hence every Sylow subgroup of $\frac{L \cap G'\Phi}{\Phi}$ is SSE in G/Φ. It can be easily shown that product of two SSE subgroups is again SSE. Since $\frac{L \cap G'\Phi}{\Phi}$ is the product of its Sylow subgroups, we conclude that $\frac{L \cap G'\Phi}{\Phi}$ is SSE in G/Φ.

Now since $G/\Phi/L \cap G'\Phi/\Phi \cong G/L \cap G'\Phi$ is supersolvable and $L \cap G'\Phi/\Phi$ is SSE in G/Φ, it follows that G/Φ is supersolvable (Remark 3, Definition 1.1.8). Consequently G is supersolvable and the proof is complete.

Corollary 1.2.10: If $N \Delta G$ and $N \leq L(G)$, then G/N supersolvable implies that G is supersolvable.

Proof: Since $G/L(G)$ is isomorphic to a homomorphic image of G/N, it is supersolvable. Hence by the theorem, G is supersolvable.

Corollary 1.2.11: Let $D(G) = \cap\{N \Delta G \mid G/N$ is supersolvable$\}$. Then either $D(G) = \langle 1 \rangle$ or $D(G) \nleq L(G)$.

Proof: Suppose $D(G) \neq \langle 1 \rangle$ is part of $L(G)$. By definition, $G/D(G)$ is supersolvable and by Corollary 1.2.10, G is supersolvable. But this implies that $D(G) = \langle 1 \rangle$ because $G/\langle 1 \rangle$ is supersolvable. This contradiction proves the Corollary.

1.3 L(G) Expressed as a Direct Product.

Next we study the relationship of $L(G)$ with $L(H)$ where $H \leq G$.

Let H be a subgroup of G. Gaschütz [11] showed that if $H \trianglelefteq G$, then $\Phi(H) \leq \Phi(G)$. We see that no such relation holds in the case of $L(G)$. For example, let $G = A_4$. As shown earlier (Proposition 1.2.2), $L(G) = \langle 1 \rangle$. But $L(V_4) = V_4$, where V_4 is the normal Sylow 2-subgroup of G.

In the case of $R(G)$ however, we do have such a relation.

Proposition 1.3.1: If $H \trianglelefteq G$, then $\Phi(H) \leq R(H) \leq R(G)$.

Proof: If $R(G) = G$, we have nothing to prove. So let S be a maximal subgroup of G of prime index. If $H \leq S$, then $R(H) \leq H \leq S$. If $H \not\leq S$, then $G = HS$ and so $[G:S] = [HS:S] = [H:H \cap S]$. Therefore $H \cap S$ is a subgroup of H of prime index, hence it is also a maximal subgroup of H. Thus $R(H) \leq H \cap S \leq S$. We see that in any case $R(H) \leq S$. Since S is arbitrary, $R(H) \leq R(G)$. Since $\Phi(H) \leq R(H)$, the assertion is proved.

The next two results concern $L(G)$ when the group G is the direct product of two groups.

If $G = AB$ is an extension of a group A by a group B, it is not true in general that $L(G) = L(A) \cdot L(B)$. This is easily seen from the example of A_4. However in case G is a direct product we have,

Proposition 1.3.2: If $G = A \times B$ and $(|A|,|B|) = 1$, then $L(G) = L(A) \times L(B)$.

Proof: First we show that $L(A) \times L(B) \leq L(G)$. Suppose $x \in L(A)$ but $x \notin L(G)$. Then there exists S, a maximal subgroup of nonprime index in G such that $x \notin S$. Since G is the direct product of two groups of relatively prime orders, any subgroup of G has that property. Thus $S = A_1 \times B_1$, where $A_1 \leq A$, $B_1 \leq B$. Since $x \in A$ and $x \notin S$, $A_1 < A$. Moreover if $B_1 < B$, then $S = A_1 \times B_1 < A_1 \times B < G$. This contradicts the maximality of S. Hence $B_1 = B$ and $S = A_1 \times B$. Then $S/B = A_1 \times B/B \cong A_1$ implies that A_1 is a maximal subgroup of A of nonprime index. But $x \in A_1 \leq S$. This is a contradiction. Therefore $L(A) \leq L(G)$. Similarly $L(B) \leq L(G)$.

Next to show that $L(G) \leq L(A) \times L(B)$, let $x = ab \in L(G)$, where a,b are unique elements of A,B respectively. We claim that $a \in L(A)$ and $b \in L(B)$. First suppose $a \notin L(A)$. Then there exists a maximal subgroup A_1 of A of nonprime index such that $a \notin A_1$. If $S = A_1 \times B$, then S is a maximal subgroup of G of nonprime index. For suppose S is not maximal in G and let T be a maximal subgroup of G containing S. Then $T = A_2 \times B$, which implies that $A_1 < A_2 < A$. This contradicts the maximality of A_1. Therefore S is maximal and of nonprime index in G. Since $x = ab \in L(G)$, $a \in A_1$. Therefore $a \in L(A)$ and similarly $b \in L(B)$. The result is thus proved completely.

<u>Remark</u>: We have as yet, not been able to confirm that the condition $(|A|,|B|) = 1$ is necessary.

The following is another useful condition for determining $L(G)$ when G is a direct product.

<u>Proposition 1.3.3</u>: If $G = A \times B$ and A or B is supersolvable, then $L(G) = L(A) \times L(B)$.

Proof: We proceed by induction on the order of G. Thus by the induction hypothesis if a group has order less than $|G|$ and is the direct product of two groups one of them supersolvable, then the conclusion holds for that group.

WLOG we may assume that A is supersolvable. Then there exists a minimal normal subgroup M of A which is cyclic of prime order. Since A,B centralize each other, in particular M is normalized by B. Hence M is normal in

G and being of prime order it is a minimal normal subgroup of G. Now if S is a maximal subgroup of G of nonprime index, then $M \leq S$. For otherwise $M \nleq S$ implies that $G = MS$ and hence $[G:S] = |M|$ is a prime. It follows therefore that $M \leq L(G)$. Now by Proposition 1.2.2, $L(G/M) = L(G)/M$ and also $L(A/M) = L(A)/M$. Note that $L(A) = A$, since A is supersolvable. By the induction hypothesis, since $G/M = A/M \times B/\langle 1\rangle$, we have $L(G/M) = L(A/M) \times L(B/\langle 1\rangle)$ and so $L(G)/M = L(A)/M \times L(B)/\langle 1\rangle$. This implies that $L(G) = L(A) \times L(B)$.

1.4 L(G): A generalization of $\Delta(G)$.

In this section we show that our work is a generalization of some of the ideas of Gaschütz. In [11] he considered the family $\mathcal{L}(G)$ of all the non normal maximal subgroups of a group G. Since in a nilpotent group all the maximal subgroups are normal, $\mathcal{L}(G)$ is empty for such a group. The family $\mathcal{L}(G)$ has also been studied from essentially two different standpoints by H. Bechtell [7] and J. Rose [16, 17]. Betchel has considered the basic structure of $\Delta(G)$ and its relation to other subgroups of G; and Rose has studied the influence of the family $\mathcal{L}(G)$ on the group G, in particular on the solvability of G.

We begin with a mention of some of the basic notions and known results.

Definition 1.4.1: For a group G define

$$\Delta(G) = \cap\{M \mid M \text{ is a non normal maximal subgroup of } G\}.$$

Define $\Delta(G) = G$ is case G has no non normal maximal subgroup.

It is obvious that $\Delta(G)$ is a characteristic subgroup of G and contains the Frattini subgroup $\Phi(G)$. Also notice that $\Delta(G)$ is contained in $L(G)$ since the maximal subgroups of G with nonprime index are non normal and hence are in the family $\mathcal{L}(G)$. The following theorem gives some of the results on $\Delta(G)$, due to Gaschütz [11], Deskins [8] and Bechtell [7].

Theorem 1.4.2: In a group G,

(a) $\Delta(G)$ is nilpotent.

(b) The hypercenter $Z^{*}(G)$ is contained in $\Delta(G)$.

(c) $\Delta(G)/\Phi(G) = Z(G/\Phi(G))$.

To these we also add some results due to Beidleman and Seo [5] and Rose [16],

(d) If $G/\Delta(G)$ is nilpotent, then G is nilpotent.

(e) If all the non normal maximal subgroups of G are nilpotent, then G is solvable.

(f) If all the non normal maximal subgroups of G are supersolvable and have prime power indices in G, then G is solvable.

From the definitions and some of the results we have obtained so far, it is apparent that there is a similarity in the nature and behaviour of the two families $\mathfrak{J}(G)$ and $\mathcal{L}(G)$; the former with respect to the supersolvable structure of G and the latter with respect to the nilpotent structure of G. Thus $L(G)$ acts in a natural way as a generalization of $\Delta(G)$. These considerations also motivate us to search for some suitable form of 'generalized hypercenter' - one that is relevant to our family $\mathfrak{J}(G)$. We show that this role is effectively played by the <u>hyperquasicenter</u> of the group G defined in the following pages.

It is now natural to ask: Can all the results of Theorem 1.4.2 be generalized? We successfully answer this question before closing Chapter I.

To begin in the chronological order then, we first prove the supersolvability of $L(G)$. For this we need the following lemma whose proof is analogous to that of Theorem 1.2.4, and uses the solvability of $L(G)$.

<u>Lemma 1.4.3</u>: In a group G, $L'(G)$ is nilpotent.

Proof: Since $L'(G) \leq L(G)$, it is contained in every maximal subgroup of G of nonprime index. If $L'(G)$ is also contained in every maximal subgroup of G of prime index, then clearly $L'(G)$ is contained in every maximal subgroup of G. This implies that $L'(G) \leq \Phi(G)$, hence it is nilpotent. We may therefore assume that there exists a maximal subgroup S of G of prime index not containing $L'(G)$. Observe that if S is normal in G, then G/S is a cyclic group and hence $L'(G) \leq G' \leq S$, which is a contradiction. Therefore S is not normal in G.

Now since S is maximal in G and $L'(G) \nleq S$, we have $G = L'(G)S = L(G)S$. Then, $p = [G:S] = [L(G)S:S] = [L(G):L(G) \cap S]$, for p some prime. It follows that $L(G) \cap S$ is a subgroup of $L(G)$ of prime index and consequently it is maximal in $L(G)$. Moreover by Corollary 1.2.6 $L(G)$ is solvable. Now by the same argument as in Theorem 1.2.4, it follows that $L(G)$ acting as a group of permutations on the cosets of $L(G) \cap S$

is represented by a metabelian group, and the kernel of the representation is the core of $L(G) \cap S$ in $L(G)$. Note: $L(G)$ is not necessarily metabelian. The permutation representation of $L(G)$ is metabelian. Hence $L''(G) \leq \mathrm{Core}_{L(G)}(L(G) \cap S) \leq L(G) \cap S < S$. Thus we conclude that $L''(G)$ is contained in every maximal subgroup of G. Note that $L''(G) = \langle 1 \rangle$ implies that $L'(G)$ is abelian and we are done. So $L''(G)$ is non trivial normal subgroup of G contained in $\Phi(G)$. Also $L'(G)/L''(G)$ is abelian. By a theorem of Wielandt (see Betchel [7]) we conclude that $L'(G)$ is nilpotent.

We are now ready to prove our main theorem.

<u>Theorem 1.4.4</u>: $L(G)$ is supersolvable, for any group G.

Proof: (By induction on the order of G). Let p be the largest prime dividing $|L(G)|$. Since $L(G)$ has the Sylow tower property (Proposition 1.2.5), the Sylow p-subgroup P of $L(G)$ is characteristic in $L(G)$ and hence normal in G. Since $L(G)/P = L(G/P)$, by the induction hypothesis $L(G)/P$ is supersolvable. Moreover, if G has a non trivial normal q-subgroup Q $(q \neq p)$ contained in $L(G)$, then once again by the induction hypothesis $L(G)/Q$ is supersolvable. This implies that $L(G) \cong L(G)/P \cap Q$ is isomorphic to a subgroup of $L(G)/P \times L(G)/Q$ and hence $L(G)$ is supersolvable. Thus we can assume that G has no non trivial normal q-subgroup contained in $L(G)$. Furthermore if $\Phi(L(G)) \neq \langle 1 \rangle$, then by the induction hypothesis $L(G)/\Phi(L(G))$ is supersolvable and hence by Huppert's theorem (Theorem 1.1.7) $L(G)$ is supersolvable.

Therefore we may also assume that $\Phi(L(G)) = \langle 1 \rangle$ and this in turn implies that $\Phi(P) = \langle 1 \rangle$. Hence P is elementary abelian. By a theorem of Gaschütz [11, Satz 7] since P is an abelian normal subgroup of $L(G)$ and $\Phi(L(G)) \cap P = \langle 1 \rangle$, it follows that P is completely reducible in $L(G)$. This means that every normal subgroup of $L(G)$ in P has a complement in P which is normal in $L(G)$. Since $L'(G)$ is a normal nilpotent subgroup of G, by an earlier remark $L'(G) \leq P$. Hence $L(G)/P$ is abelian. Also by the Schür-Zassenhaus theorem [18, 9.3.6], $L(G) = PD$ where D is a p-complement in $L(G)$; and D is abelian.

Now let Q be a Sylow q-subgroup of $L(G)$, for $q \neq p$. WLOG we may take $Q \leq D$. Since G has no normal q-subgroup in $L(G)$, $N_G(Q) \neq G$. However since D is abelian, $D \leq N_G(Q)$. By Frattini's lemma, $G = L(G)N_G(Q)$. If S is a maximal subgroup of G containing $N_G(Q)$ we have, $G = L(G)S$ and $L(G) \not\leq S$. Since $D \leq S$, $P \not\leq S$. Therefore $G = PS$. Also in G, S has a prime index which is p (observe that $q/p-1$). Let $K = L(G) \cap S$. Then $p = [G:S] = [L(G):L(G) \cap S] = [L(G):K]$. Since K has prime index in $L(G)$, it is a maximal subgroup of $L(G)$ not containing P. Therefore $L(G) = PK$ and $[L(G):K] = [P:P \cap K] = p$. Therefore $P_0 = P \cap K$ is normal in P and K and hence normal in $L(G)$. Also the index of P_0 in P is p. Since P is completely reducible in $L(G)$, there exists $P_1 \Delta L(G)$ such that P_1 is cyclic of order p.

Now let $P^* = \prod_{x \in G} P_1^x = P_1P_2 \dots P_n$. Each P_1^x is cyclic of order p and is contained in P. Therefore P^* is a normal subgroup of G contained in $L(G)$. Moreover each P_i is normal in $L(G)$, since P_1 is normal in $L(G)$ and $L(G) \Delta G$. For let $P_i = P_1^x$, then for every $g \in L(G)$, $xgx^{-1} \in L(G)$. So $P_1^{xgx^{-1}} = P_1$ and hence $P_1^{xg} = P_1^x$. Since P is elementary abelian, we have $P^* = P_1 \times P_2 \times \dots \times P_n$. Consequently there exists a normal series of $L(G)$ up to P^* with cyclic factor groups. Also by the induction hypothesis $L(G)/P^* = L(G/P^*)$ is supersolvable. Hence $L(G)$ is supersolvable and the theorem is proved.

<u>Corollary 1.4.5</u>: If $|L(G)|$ does not contain prime divisors p,q such that $q/p-1$, then $L(G)$ is nilpotent.

Proof: We use induction on $|G|$ and then essentially the method of the theorem can be duplicated to show that $q/p-1$ for some p, whenever a Sylow q-subgroup of $L(G)$ is not normal. Thus under the hypothesis, every Sylow subgroup of $L(G)$ is normal and hence it is nilpotent.

<u>Remark</u>: As mentioned before, in general $L(G)$ is not nilpotent. For example, in S_3 $L(S_3) = S_3$ is not nilpotent. However the hypotheses of the Corollary 1.4.5 are clearly not satisfied.

1.5 <u>Relationship of $L(G)$ with the Hyperquasicenter of G</u>.

Next we consider a generalization of the hypercenter and investigate its relationship with the subgroup $L(G)$.

The concept of the quasicenter as a generalization of the center of a group was introduced by O. Ore [15] as follows:

Definition 1.5.1: Let G be a group. An element $x \in G$ is called a quasicentral element (QC-element) in G if $\langle x \rangle$ permutes with $\langle y \rangle$ for every element $y \in G$.

The subgroup generated by all the quasicentral elements in G is the quasicenter of G, denoted by $Q(G)$.

The quasicenter is a characteristic subgroup of G. Since every element in the center of a group permutes with every element in the group, it follows that the center is contained in the quasicenter of the group. Note that if x is a QC-element in G, then $\langle x \rangle$ permutes with every subgroup of G.

The structure of the quasicenter and its generalizations has been investigated in more detail by N. Mukherjee [14]. In the next few theorems we list some of the results obtained by him. On occasions we shall use some of these.

Theorem 1.5.2: In a group G,

(a) If X is a QC-element of G, then X^r is a QC-element of G, for every integer r.

(b) The quasicenter of G is nilpotent and each of its Sylow subgroups is generated by QC-elements of G.

Proof: [14, Theorem 1.7, Lemma 1.9, Theorem 1.10].

Analogous to the ascending-central series we also have,

Definition 1.5.3 [14]: For a group G let, $\langle 1 \rangle \le Q(G) = Q_1 \le Q_2 \le \ldots \le Q_n = Q^*(G)$ be a normal series of G such that $Q(G/Q_i) = Q_{i+1}/Q_i$.

This series is the ascending-quasicentral series of G. The terminal member $Q^*(G)$ of the series is the hyperquasicenter of G.

Some properties of the hyperquasicenter are listed in the following:

Theorem 1.5.4 [14]: For a group G,

(a) The hyperquasicenter is supersolvable.

(b) If T is a normal subgroup of G contained in $Q^*(G)$, then $Q^*(G/T) = Q^*(G)/T$.

(c) If $G/Q^*(G)$ is supersolvable, then G is supersolvable.

Proof: [14, Theorems 2.5, 2.3]

Remark: Mukherjee [14] has shown that the hyperquasicenter of a group G contains the largest SSE subgroup of G. However, during a communication he pointed out that the hyperquasicenter is itself SSE in G and hence is the largest SSE subgroup of G.

In the following we shall prove that the hyperquasicenter of G is contained in $L(G)$ and thus slightly improve Theorem 1.5.4 in view of the fact that conclusions of that theorem have already been proved for $L(G)$ in place of $Q^*(G)$. Also by the remark above, we observe that in general the hyperquasicenter is not equal to $L(G)$. For this we argue as follows. It can be shown that in general the Frattini subgroup $\Phi(G)$ is not SSE in G. It is also easy to show that a normal subgroup of G contained in a SSE subgroup of G is SSE in G. Now if $L(G) = Q^*(G)$, then $L(G)$ is SSE in G and hence by the previous comment $\Phi(G)$ is SSE in G. This, as we remarked earlier is not in general true. Therefore $L(G) \neq Q^*(G)$ in general.

We now turn to the proof of our theorem. First we show the following:

<u>Proposition 1.5.5</u>: The quasicenter of a group G is contained in $L(G)$.

Proof: Assume $Q(G)$ is not contained in $L(G)$. Since $Q(G)$ is nilpotent (Theorem 1.5.2), there is a Sylow p-subgroup P of $Q(G)$ which is not contained in $L(G)$. Since P is generated by QC-elements of G, there is a QC-element x of G such that $x \notin L(G)$. Therefore there exists a maximal subgroup S of G of <u>nonprime</u> index such that $x \notin S$. Since x is a QC-element, $\langle x\rangle$ and S permute. Consequently $G = \langle x\rangle S$. Also $|x| = p^a$, for some integer a. Let $T = S \cap \langle x\rangle$ and $|T| = p^r$. Then T is contained in the unique subgroup $\langle x^{p^{a-r-1}}\rangle$ of $\langle x\rangle$ and $|\langle x^{p^{a-r-1}}\rangle| = p^{r+1}$. Since T is the intersection of S with $\langle x\rangle$, $\langle x^{p^{a-r-1}}\rangle \nleq S$. Moreover by Theorem 1.5.2(a), $x^{p^{a-r-1}}$ is again a QC-element of G. Therefore $G = \langle x^{p^{a-r-1}}\rangle S$ and the index of S in G is equal to the index of $T = \langle x^{p^{a-r}}\rangle$ in $\langle x^{p^{a-r-1}}\rangle$ which is p. But this is a contradiction since S was assumed to have nonprime index in G. Therefore we conclude that $Q(G) \leq L(G)$.

From the above result we now immediately deduce,

<u>Theorem 1.5.6</u>: The hyperquasicenter of the group G is contained in $L(G)$.

Proof: Let $\langle 1\rangle \le Q(G) = Q_1 \le Q_2 \le \ldots \le Q_n = Q^*(G)$ be the ascending-quasicentral series of G. By Proposition 1.5.5, $Q(G) \le L(G)$ and $Q(G/Q_1) \le L(G)/Q_1$. By definition, $Q_2/Q_1 = Q(G/Q_1)$ hence $Q_2 \le L(G)$. Continuing in this manner we see that the terminal member of the series namely $Q^*(G)$, is contained in $L(G)$.

As mentioned earlier (Theorem 1.4.2) in the case of $\Delta(G)$, we have $\Delta(G)/\Phi(G) = Z(G/\Phi(G))$. But no such relation holds between $L(G)$ and the quasicenter of G.

Example 1.5.7: Let $G = S_3$. Then $L(G) = S_3$ and $\Phi(G) = \langle 1\rangle$. But the quasicenter of S_3 is the cyclic subgroup of order 3.

The above example leads to the following observation:

Proposition 1.5.8: If a group G is solvable and $\Phi(G) = \langle 1\rangle$, then $L(G)/Q^*(G)$ is abelian.

Proof: As shown in Theorem 1.2.9 the solvability of G implies that $L(G) \cap G'$ is SSE in G. Since $Q^*(G)$ the hyperquasicenter of G contains every SSE subgroup, $L(G) \cap G' \le Q^*(G)$. Thus $L(G)/Q^*(G)$ is isomorphic to a homomorphic image of $L(G)/L(G) \cap G' \cong \frac{L(G)G'}{G'} \le G/G'$. This implies that $L(G)/Q^*(G)$ is abelian.

1.6 A Condition for Solvability of a Group.

J. Rose [16] has proved that if every non normal maximal subgroup of a group is nilpotent, then the group is solvable. In this section we shall improve this result slightly by showing that 'every non normal maximal subgroup' can be replaced by 'every maximal subgroup of nonprime index'. We begin with some well-known definitions and preliminary results that we shall need.

Definition 1.6.1: A group G is called p-solvable, for a prime p if there exists a series $\langle 1\rangle = G_0 \leq G_1 \leq \ldots G_n = G$ such that $G_i \Delta G$ and each factor G_i/G_{i-1} is either a p-group or a p'-group.

The p-length $\ell_p(G)$ of the p-solvable group G is the least number of p-factors appearing in the series of the kind specified above.

Definition 1.6.2: A group G is p-nilpotent, for a prime p if G has a normal p-complement.

Theorem 1.6.3 (Thompson [19]): Let p be an odd prime and P a Sylow p-subgroup of a group G. If for every non trivial characteristic subgroup P_0 of P, G induces in P_0 a p-group of automorphisms, then G has a normal p-complement.

Lemma 1.6.4 (Rose [16, Cor: Lemma 1]): A group G can have at most one conjugacy class of nilpotent non normal maximal subgroups.

Lemma 1.6.5 (Baer [2, p. 181]): If S is a maximal subgroup of a group G whose Core $S_G = \langle 1\rangle$ and whose index $[G:S]$ is a prime p, then $q < p$ for every prime divisor q of $|S|$.

Definition 1.6.6: A subgroup N of G is subnormal in G if there exists a series $N = N_0 \Delta N_1 \Delta \ldots \Delta G$.

We are now ready to prove our theorem.

Theorem 1.6.7: Suppose every maximal subgroup of a group G of nonprime index is nilpotent. Then

(i) G is solvable.

(ii) If p is the largest prime divisor of $|G|$, then either G is p-nilpotent or G has a normal p-subgroup P_0 such that G/P_0 is p-nilpotent. Moreover, $\ell_p(G) \leq 2$.

Proof: Let G be a minimal counter-example. Observe that if $K \Delta G$, then G/K satisfies the hypothesis.

(i) Let p be the largest prime divisor of $|G|$ and P a Sylow p-subgroup of G. If $P \Delta G$, then P and G/P are both solvable which implies that G is solvable. This contradicts the assumption that G is a minimal counter-example. Therefore $P \not\Delta G$. Let $N(P)$ be the normalizer of P in G and let S be a maximal subgroup of G containing $N(P)$. The index of S in G is not a prime. For if $q = [G:S]$ for some prime q, then by the Sylow theorems $q = 1 + kp$. This is impossible since $q < p$. Hence S has nonprime index in G and by hypothesis S is nilpotent. Since P is a Sylow subgroup of S, $N(P) = S$. Let P_0 be any non trivial characteristic subgroup of P. By the same argument used for P, it follows that $P_0 \not\Delta G$. Consequently $N(P) \leq N(P_0) < G$. Since $N(P) = S$ is maximal in G, $N(P_0) = N(P)$. In a nilpotent group the elements of relatively prime order centralize each other. Hence $N(P_0)$ induces only p-automorphisms in P_0. Moreover, p is an odd prime. By Theorem 1.6.3, G has a normal p-complement D. Now S is a maximal subgroup of G

and is nilpotent of nonprime index. By Lemma 1.6.4, all the nilpotent non normal maximal subgroups of G are conjugate. Therefore all the maximal subgroups of G of nonprime index are conjugate to S.

Since G is not simple, let M be a minimal normal subgroup of G. Since G/M is solvable, M cannot be solvable. Also M is the unique minimal normal subgroup of G. This is true because if N is another minimal normal subgroup of G distinct from M, then $M \cap N = \langle 1 \rangle$. Since G/M and G/N are solvable, $G/M \cap N \subseteqq G/M \times G/N$ is solvable. This contradicts the assumption that G is nonsolvable. Therefore M is unique.

If $M \leq S$, then M is nilpotent. So $M \not\leq S$ and hence $G = MS$. Let q be a prime divisor of $[M:M \cap S] = [G:S]$ and Q a Sylow q-subgroup of M. Obviously $Q < M$ since M is nonsolvable. Since M is minimal normal in G, $Q \ntriangleleft G$. Let T be a maximal subgroup of G containing $N_G(Q)$. Since D, the p-complement of G is normal in G, M is contained in D. This is so because of the uniqueness of M. Now by Frattini's lemma, $G = MN_G(Q) = MT$. So $M \not\leq T$ and hence $D \not\leq T$. It follows that $[G:T] = [D:D \cap T]$ is prime to p, since D is a p'-group.

Suppose $\text{Core}_G T \neq \langle 1 \rangle$. Since $M \not\leq T$, $M \cap \text{Core}_G T < M$. But this contradicts the minimality of M. Therefore $\text{Core}_G T = \langle 1 \rangle$. Again if $[G:T] = r$, a prime, then by Lemma 1.6.5, the prime divisors of $|T|$ are smaller than r. But $p/|T|$ and it is the largest prime divisor of $|G|$. This contradiction shows that T cannot have prime index. Consequently by an earlier remark, T is conjugate to S. But this leads to a contradiction. To see this notice that

$[G:S] = [M:M \cap S]$ is a multiple of q and

$[G:T] = [M:M \cap T]$ is prime to q since $Q \leq M \cap T$. But two conjugate subgroups must have the same index in the group. Thus we arrive at a contradiction. Therefore G is solvable.

(ii) If G does not possess a normal p-subgroup, then as in (i) by the application of Thompson's theorem, G has a normal p-complement and (ii) is proved. Assume therefore that G has a normal p-subgroup and let P_0 be the largest such subgroup. Then consider G/P_0. We note that G/P_0 does not have a normal p-subgroup. For suppose there exists $\overline{P}$, a non trivial normal p-subgroup of G/P_0. Then there exists a p-subgroup $P_1 \Delta G$ such that $\overline{P} = P_1/P_0$. This contradicts the maximality of P_0.

Now since G/P_0 satisfies the hypothesis and has no normal p-subgroup, by the induction hypothesis G/P_0 is p-nilpotent.

Also if T/P_0 is the normal p-complement of G/P_0, then there is a normal series $\langle 1 \rangle \leq P_0 \leq T \leq G$. Hence the p -length $\ell_p(G) \leq 2$. The theorem is thus proved.

To continue further with the above theorem, let $\overline{G} = G/P_0$ and let $\overline{D}$ be the normal p-complement of $\overline{G}$. Let $\overline{S}$ be a maximal subgroup of $\overline{G}$ containing $\overline{D}$. Then $\overline{S}/\overline{D}$ is maximal in $\overline{G}/\overline{D} \cong \overline{P}$. Since maximal subgroups of a nilpotent group are normal, $\overline{S} \Delta \overline{G}$ and $[\overline{G}:\overline{S}] = p$.

(a) Suppose now that every normal maximal subgroup of G of index p has nilpotent commutator subgroup. Consider $\overline{P}$ a Sylow p-subgroup of $\overline{G}$. If $|\overline{P}| = p$, then $\overline{P}$ is cyclic of order p. In the other case let $\overline{P}_1$ be any maximal subgroup of $\overline{P}$. Since $\overline{D} \Delta \overline{G}$, $\overline{P}_1\overline{D}$ is a subgroup of $\overline{G}$. Moreover it is a maximal subgroup in $\overline{G}$ containing $\overline{D}$. By previous discussion it follows that $\overline{P}_1\overline{D} \Delta \overline{G}$ and is of index p. By hypothesis (a), $(\overline{P}_1\overline{D})'$ is nilpotent. Now by [13, Satz 4] it follows that $\overline{P}_1' \Delta \overline{P}_1\overline{D}$ and hence is subnormal in $\overline{G}$. But it is easy to show (see for instance [16, Lemma 3]) that a group possessing a non trivial subnormal p-subgroup also possesses a non trivial normal p-subgroup. This is a contradiction since $\overline{G}$ has no non trivial normal p-subgroup. Hence $\overline{P}_1' = <\overline{1}>$ and therefore every proper subgroup of $\overline{P}$ is abelian. Thus we see that under the hypothesis (a), $\overline{P}$ is either cyclic of order p or is minimal nonabelian.

(b) Finally, suppose that every normal maximal subgroup of G of index p has the Sylow tower property. Let $\overline{P}_1$ be a maximal subgroup of $\overline{P}$, where $\overline{P}$ is a Sylow p-subgroup of $\overline{G}$. Then as before, $\overline{P}_1\overline{D}$ is a normal maximal subgroup of $\overline{G}$ of index p. Since p is the largest prime divisor of $|\overline{P}_1\overline{D}|$, by (b), $\overline{P}_1 \Delta \overline{P}_1\overline{D}$. This implies that $\overline{P}_1$ is subnormal p-subgroup of $\overline{G}$. As in (a) this is again a contradiction. Therefore we conclude that $\overline{P}$ has no proper maximal subgroup and consequently it is cyclic of order p.

The foregoing discussion can be summed up in the following proposition:

Proposition 1.6.8: Under the hypothesis of Theorem 1.6.7, if p is the largest prime divisor of $|G|$ and P_0 the largest normal p-subgroup of G, then either P_0 is the Sylow p-subgroup of G or,

(a) If every normal maximal subgroup of G of index p has nilpotent commutator subgroup, then a Sylow p-subgroup of G/P_0 is cyclic of order p or minimal nonabelian.

(b) If every normal maximal subgroup of G of index p has the Sylow tower property, then a Sylow p-subgroup of G/P_0 is cyclic of order p.

Rose [16, Theorem 4] extended his result by proving that if every non normal maximal subgroup of a group is supersolvable and has prime power index, then the group is solvable.

By way of an example, we show that this result cannot be improved by replacing 'non normal maximal subgroups' by 'maximal subgroups with nonprime index'.

Example 1.6.9: Let $G = PSL(2,7)$, the simple group of order 168. It is well-known (see Scott [18, p. 336]) that the maximal subgroups of G have index 7 or 8. A maximal subgroup H of G of index 8 has order 21. Since H has two prime divisors, it is solvable and clearly it is supersolvable. H is not nilpotent since it can be shown [18, p. 336] that the normalizer of a Sylow 3-subgroup of G does not contain H. Thus the maximal subgroups of nonprime index are supersolvable and have prime power index.

To see how good a generalization of the family $\mathcal{L}(G)$ is the family $\mathfrak{H}(G)$, we compare our results so far with

Theorem 1.4.2. We find that (a), (b) and (d) of that theorem have analogues for $\mathfrak{J}(G)$; (e) is slightly improved; while we have examples to show that (c) has no counterpart and (f) cannot be improved in our context. This is the conclusion of Chapter I.

CHAPTER II

In this chapter we investigate some additional properties of the subgroup $L(G)$. It is shown that $L(G)$ belongs to a class of normal subgroups we call the special L-subgroups of the group G. Following Baer [4] we define weakly hyperquasicentral subgroups of G and also a certain property $(\mathcal{P})$. Some of the work of Beidleman and Seo [5, 6] and Dykes [10] has also been generalized in this chapter. The main results here are:

(1) $L(G)$ is a special L-subgroup of the group G.

(2) If $H \Delta G$ is a special L-subgroup of G and G/H has the Sylow tower property, then G has the Sylow tower property.

(3) The following statements are equivalent for a nilpotent normal subgroup H of the group G:

   (a) H is a special L-subgroup of G.

   (b) H has property $(\mathcal{P})$ in G.

   (c) H is weakly hyperquasicentral in G.

2.1 L-subgroups of the Group G.

We begin with the following definition.

Definition 2.1.1 [5]: Let H be a proper normal subgroup of a group G. Then H is a generalized Frattini subgroup of G if the following holds for every normal subgroup

N of G:

If P is any Sylow subgroup of N, then $G = HN_G(P)$ implies $G = N_G(P)$.

It is well-known that if $G = \Phi(G)K$ for any $K \leq G$, then $G = K$. It follows therefore that the Frattini subgroup of a group is a generalized Frattini subgroup.

Some of the results on generalized Frattini subgroups are given in the next theorem.

Theorem 2.1.2 [5, 6]: Let H be a generalized Frattini subgroup of G. Then:

(a) H is nilpotent.

(b) If K is a normal subgroup of G containing H such that K/H is nilpotent, then K is nilpotent.

(c) $\Delta(G)$ is a generalized Frattini subgroup of G if G is not nilpotent.

As our investigations in Chapter I showed, the structure of L(G) and its related properties are very much influenced by the largest prime divisor of the order of the relevant subgroup. This leads naturally to the following generalization of Definition 2.1.1.

Definition 2.1.3: Let H be a proper normal subgroup of a group G. Then H is an L-subgroup of G if the following holds for every normal subgroup N of G:

If p is the largest prime divisor of $|N|$ and P is a Sylow p-subgroup of N, then $G = HN_G(P)$ implies $G = N_G(P)$ (i.e. $P \Delta G$).

First we obtain some elementary properties of L-subgroups.

Proposition 2.1.4: Let G be a finite group.

(i) If H is an L-subgroup of G, then

(a) H is p-closed, where p is the largest prime divisor of $|H|$, and

(b) If $H_1 \Delta G$, $H_1 \leq H$, then H_1 is also an L-subgroup of G.

(ii) $L(G)$ is an L-subgroup of G.

Proof: (i) (a) Let P be a Sylow p-subgroup of H for p the largest prime divisor of $|H|$. Since $H \Delta G$, by Frattini's lemma $G = HN_G(P)$. Since H is an L-subgroup of G, we have $G = N_G(P)$. Hence $P \Delta G$ and H is p-closed.

(b) Let $H_1 \leq H$ and $H_1 \Delta G$. Let N be a normal subgroup of G and p the largest prime divisor of $|N|$. Suppose $G = H_1N_G(P)$, where P is a Sylow p-subgroup of N. Since $H_1 \leq H$, $G = HN_G(P)$. Now since H is an L-subgroup of G, $G = N_G(P)$. Hence H_1 is an L-subgroup of G.

(ii) Let $N \Delta G$ and P a Sylow p-subgroup of N, where p is the largest prime divisor of $|N|$. Suppose $G = L(G)N_G(P)$. Then we must show that $G = N_G(P)$. Assume $G \neq N_G(P)$ and let M be a maximal subgroup of G containing $N_G(P)$. Then $G = L(G)M$ and hence $L(G) \not\leq M$. It follows that M has prime index in G. Let $q = [G:M]$. Now by Frattini's lemma since $N \Delta G$, $G = NN_G(P)$ and hence $G = NM$. This implies that $q = [G:M] = [N: N \cap M]$. Since $N_G(P) \leq M$, by the Sylow theorems we have $[G:M] = 1 + kp$, where k is some integer. This means that p divides $q - 1$, which is a contradiction since both p and q divide $|N|$ and p is the largest such divisor. Therefore $G = N_G(P)$ and $L(G)$ satisfies the definition of an L-subgroup of G.

Corollary 2.1.5: $Q(G)$ and $Q^*(G)$ are L-subgroups of G.

Remark: From the definition it is obvious that a generalized Frattini subgroup of a group is an L-subgroup. The converse of this however is not true, as the following example shows.

Example 2.1.6: Let $G = A_4 \times S_3$. By Proposition 1.3.3 since $L(A_4) = \langle 1 \rangle$ and S_3 is supersolvable, $L(G) = S_3$. By Theorem 2.1.2 however, a generalized Frattini subgroup is nilpotent. Hence $L(G)$ is not a generalized Frattini subgroup of G though it is an L-subgroup of G by Proposition 2.1.4(ii).

The next result shows the influence of L-subgroups on the group.

Proposition 2.1.7: If H is an L-subgroup of G and p is the largest prime divisor of $|G|$, then G/H is p-closed if and only if G is p-closed.

Proof: If G is p-closed, then G/H being a homomorphic image of G is also p-closed.

Conversely, assume G/H is p-closed. If $p \nmid |G/H|$, then a Sylow p-subgroup of G is also a Sylow p-subgroup of H. By 2.1.4, H is p-closed and since $H \Delta G$, G is p-closed. Suppose now that $p/|G/H|$ and let P_1 be the normal Sylow p-subgroup of G/H. Then there exists P a Sylow p-subgroup of G such that $P_1 = PH/H$. Since $P_1 \Delta G/H$, $PH \Delta G$. Moreover, P is also a Sylow p-subgroup of PH. By Frattini's lemma, $G = PH.N_G(P) = HN_G(P)$. Now since H is an L-subgroup of G, we have $G = N_G(P)$. Thus G is p-closed.

Unfortunately the concept of L-subgroup seems to be too general in that it does not sufficiently restrict the behavior of the Sylow subgroups for the various prime divisors. So in order to attain great control we specialize the above concept to a certain degree.

2.2 Special L-subgroups of a Group G.

We begin with a well-known definition.

Definition 2.2.1: A subgroup H of a group G is called a Hall-subgroup if $(|H|, [G:H]) = 1$.

The Sylow subgroups of a group are well-known examples of Hall-subgroups.

H is a Hall-subgroup of G and a π-subgroup for a set π of primes, then H is a Hall π-subgroup. Similarly a Hall subgroup which is also a π'-subgroup is a Hall π'-subgroup.

Definition 2.2.2: Let π be the set of primes dividing the order of a group G. Let π_1 be a subset of π such that every element in the set $\pi \sim \pi_1$ is smaller than every element in the set π_1. Then we call the set π_1 a set of upper primes for the group G.

As a notational convenience we denote by $p(n)$ the set of prime divisors of the number n. Finally, we shall call a set of upper primes for G a UP-set for G.

Now we are ready to make the following definition.

Definition 2.2.3: Let H be a proper normal subgroup of a group G. Then H is a special L-subgroup of G (sp-L-subgroup) if the following holds for every normal subgroup N of G:

If π is any UP-set for N and A any Hall π-subgroup of N, then $G = H.N_G(A)$ implies $A \Delta G$ (i.e., $G = N_G(A)$).

We note that every normal subgroup N of a group has at least one Hall π-subgroup for π a UP-set; namely when $\pi = \{p\}$, where p is the largest prime divisor of $|N|$.

It is obvious that $\Phi(G)$ is a sp-L-subgroup of G. Moreover, every sp-L-subgroup of G is also an L-subgroup of G. The converse of this, however, is not true as the following example shows:

Example 2.2.4: Let G be the group with the following properties:

(i) $|G| = 2^2 \cdot 3 \cdot 7$

(ii) G has a normal Sylow 7-subgroup P_7.

(iii) G has a normal Sylow 2-subgroup V_4, isomorphic to the 4-group.

(iv) G has a Sylow 3-subgroup C_3 which is self-normalizing in G.

By Scott [18, 9.2.14] such a group G exists.

Now G has a normal subgroup $H = P_7 \times V_4$ of order 28. The only other normal subgroups K of G are: $K = G, P_7, V_4$. It is easy to verify that for each of these normal subgroups K, and P a Sylow p-subgroup of K corresponding to the largest prime divisor of $|K|$, $G = HN_G(P)$ implies $G = N_G(P)$. In other words H satisfies the definition of an L-subgroup of G. However H is not a sp-L-subgroup of G. To see this let $N = G$ and $A = P_7 \cdot C_3$. Then A is a Hall $\{7,3\}$-subgroup of G and $\{7,3\}$ is a UP-set for G. A is a maximal self-normalizing subgroup of G not containing H. Consequently, $G = H \cdot N_G(A) = HA$. But $G \neq N_G(A)$. Therefore H is an L-subgroup but not a sp-L-subgroup of G.

Remark: In case the group G has the Sylow tower property then every proper normal subgroup of G is a sp-L-subgroup of G.

Next we have the following basic result on the structure of sp-L-subgroups.

Proposition 2.2.5: Let H be a sp-L-subgroup of a group G. Then:

(a) H has the Sylow tower property.

(b) If $H_1 \Delta G$ and $H_1 \leq H$, then H_1 is a sp-L-subgroup G.

Proof: (a) We prove by induction on n, where n is the number of distinct prime divisors of $|H|$.

Let $p_1 > p_2 > \dots p_k > \dots > p_n$ be the natural ordering of the prime divisors of $|H|$. Let P_i denote a Sylow p_i-subgroup of H. Since H is a sp-L-subgroup, it is also an L-subgroup of G. Therefore by Proposition 2.1.4, H is p_1-closed. So $P_1 \Delta H$ and hence $P_1 \Delta G$. Suppose now that $A = P_1 P_2 \dots P_k$ is a normal Hall subgroup of H. Consider $H/A \Delta G/A$. Let $\overline{P}_{k+1}$ be a Sylow p_{k+1}-subgroup of H/A. Then there exists P_{k+1} a Sylow p_{k+1}-subgroup of H such that $\overline{P}_{k+1} = P_{k+1}A/A$.

Now by Frattini's lemma, $G/A = H/A \cdot N_G/A(P_{k+1}A/A)$. We show that $G = HN_G(P_{k+1}A)$. For this let x be an arbitrary element of G. Then $xA = (hA)(yA)$, where $h \in H$ and $y \in G$ such that yA normalizes $P_{k+1}A/A$. So $(hy)^{-1}x \in A$. If $(hy)^{-1}x = t \in A$, then $x = hyt$. Also $(yt)^{-1}(P_{k+1}A)(yt) = t^{-1}(y^{-1}P_{k+1}Ay)t$. Now yA normalizes $P_{k+1}A/A$ hence y normalizes $P_{k+1}A$. Since $t \in A$, it is clear that $x \in HN_G(P_{k+1}A)$. Thus $G = HN_G(P_{k+1}A)$. Note that $\pi_{k+1} = \{p_1, p_2, \ldots, p_{k+1}\}$ is a UP-set for H and $P_{k+1}A$ is a Hall π_{k+1}-subgroup of H. Since H is a sp-L-subgroup of G it follows that $P_{k+1}A \Delta G$. Thus H has normal Hall π-subgroup for every UP-set π. Hence H has the Sylow tower property.

(b) Let $N \trianglelefteq G$, π a UP-set for N and A a Hall π-subgroup of N. Suppose $G = H_1N_G(A)$. Since $H_1 \leq H$, $G = HN_G(A)$. Since H is a sp-L-subgroup of G, $G = N_G(A)$. Thus H_1 is a sp-L-subgroup of G.

<u>Corollary 2.2.6</u>: If H is a sp-L-subgroup of G, then $H\Phi(G)$ and $HZ(G)$ are also sp-L-subgroups of G.

Proof: Since $G = \Phi(G)K$ implies $G = K$ for any $K \leq G$, it is clear that $\Phi(G)H$ is a sp-L-subgroup when H is a sp-L-subgroup of G. Next, let N be a normal subgroup of G, π a UP-set for N and A a Hall π-subgroup of N. Suppose $G = HZ(G).N_G(A)$. Since $Z(G)$ commutes with every element of G, it normalizes A. Hence $G = HN_G(A)$ and since H is a sp-L-subgroup, $G = N_G(A)$.

<u>Remark</u>: In general a sp-L-subgroup need not be supersolvable. This is so because there are groups with the Sylow

tower property which are not supersolvable (see for example [18, 9.2.13]). Let H be such a group and let $G = H \times K$, where K is any group having the Sylow tower property. Then G has the Sylow tower property. By an earlier remark, every proper normal subgroup of G is a sp-L-subgroup of G. Consequently H is a sp-L-subgroup of G but H is not supersolvable.

The next result gives a condition for a group G to have the Sylow tower property.

Proposition 2.2.7: Let H be a sp-L-subgroup of G. Then G has the Sylow tower property if and only if G/H has the Sylow tower property.

Proof: If G has the Sylow tower property, then G/H, being a homomorphic image of G, has that property.

Conversely, assume that G/H has the Sylow tower property. We shall use induction on $|G|$. Let p be the largest prime divisor of $|G|$ and P be a Sylow p-subgroup of G. If $P \leq H$, then H is p-closed since H has the Sylow tower property (Proposition 2.2.5). Hence $P \Delta G$.

Suppose $P \not\leq H$. Then $p/|G/H|$ and by hypothesis G/H is p-closed. By Proposition 2.1.7 since H is an L-subgroup, G is p-closed. Thus in any case $P \Delta G$.

Now we show that HP/P is a sp-L-subgroup of G/P. Suppose that $N/P \underline{\Delta} G/P$ and A/P is a Hall π-subgroup of N/P, where π is an arbitrary UP-set for N/P. Let $\pi_1 = \pi \cup \{p\}$. Then A/P is also a Hall π_1-subgroup of N/P. Moreover π_1 is a UP-set for N. Suppose $\frac{G}{P} = \frac{HP}{P} \cdot N_{G/P}(A/P)$.

Then as shown in Proposition 2.2.5, $G = HP.N_G(A) = H.N_G(A)$. Now $[N:A] = [N/P:A/P]$ is a π_1'-number (because p does not divide $|N/P|$). Also A is a π_1-number. Therefore A is a Hall π_1-subgroup of N and π_1 is a UP-set for N. Since H is a sp-L-subgroup of G, it follows that $G = N_G(A)$. So $A \Delta G$ and hence $A/P \Delta G/P$. Thus we have shown that HP/P is a sp-L-subgroup of G/P. Furthermore, $G/P/HP/P \cong G/HP$ has the Sylow tower property. By the induction hypothesis we conclude that G/P has the Sylow tower property. It is now clear that G itself has that property.

The above result does not hold if instead of being a sp-L-subgroup, H is merely an L-subgroup of G. To see this consider the following:

Example 2.2.8: Let G be the group defined in Example 2.2.4. Then $H = P_7 \times V_4$ is an L-subgroup of G but not a sp-L-subgroup. However, $G/H \cong C_3$ has the Sylow tower property. But G does not have that property since we have seen that $P_7 \cdot C_3$ is not normal in G.

Our objective in the following pages is to show that $L(G)$ is a sp-L-subgroup of G and also to attempt to characterize the sp-L-subgroups of the group. In order to achieve our purpose we shall proceed by first considering some definitions and intermediate results which are also of some independent interest.

2.3 Subgroups with Property $(\mathcal{P})$.

Definition 2.3.1: Let π be a set of primes. A group G is π-closed if G has normal Hall π-subgroup.

If the primes in the set π do not divide $|G|$, then G is trivially π-closed.

It is well-known and easily verified that every subgroup and homomorphic image of a π-closed group is again π-closed.

A π-group is obviously π-closed.

If H is a normal π-subgroup of G such that G/H is π-closed, then G is π-closed. If however, H and G/H are both π-closed then G is not necessarily π-closed. Even in the particular case when π is a UP-set for G and H is an L-subgroup of G such that both H and G/H are π-closed, G is not always π-closed. An example to illustrate this is the following:

<u>Example 2.3.2</u>: Let G be the group defined in Example 2.2.4. Let $H = P_7 \times V_4 \Delta G$, which is an L-subgroup of G. Hence V_4 is also an L-subgroup of G. Let $\pi = \{7,3\}$, which is a UP-set for G. Now G/V_4 is a π-group, hence π-closed. Also V_4 is trivially π-closed (note that V_4 is not a π-group). But G is not π-closed as we have shown before.

The above example leads us to make the following definition.

<u>Definition 2.3.3</u>: Let H be a proper normal subgroup of a group G. Then H has <u>property (θ) in G</u> if the following holds for every normal subgroup K of G containing H:

If π is a UP-set for K, then K/H π-closed implies K is π-closed.

The next two results establish the relationship between sp-L-subgroups of G and the subgroups with property (θ) in G.

Proposition 2.3.4: If $H \Delta G$ has property (θ) in G, then H is a sp-L-subgroup of G.

Proof: Let N be a normal subgroup of G, π be an arbitrary UP-set for N and A a Hall π-subgroup of N. Suppose $G = HN_G(A)$. We must show that $G = N_G(A)$. First observe that $HA \Delta G$. Next, $HA/H \cong A/A \cap H$ is a π-group since A is a π-group. Also $[N:A]$ is a π'-number. Now let $\pi_1 = \pi \cup \{$all primes dividing $|HA|$ larger than $\pi\}$. Then in particular, HA/H is a π_1-group and hence it is π_1-closed. Since π_1 is a UP-set for HA and H has the property (θ) in G, HA is π_1-closed. Hence $HA \cap N$ is π_1-closed. But π is a UP-set for N and the primes in $\pi_1 \sim \pi$ are larger than the primes in π. This means that $HA \cap N$ is π-closed. Since A is a Hall π-subgroup of N and hence a Hall π-subgroup of $HA \cap N$, it follows that $A \Delta HA \cap N$. So A is normal in G as we wanted to show.

The following is a partial converse of the above.

Proposition 2.3.5: If H is a nilpotent sp-L-subgroup of G, then H has the property (θ) in G.

Proof: Let K be a normal subgroup of G containing H and π be an arbitrary UP-set for K. Suppose that K/H is π-closed and let L/H be the Hall π-subgroup of K/H. Then $L/H \Delta G/H$ and $L \Delta G$. If H is a π-subgroup, then L is a π-subgroup. Also $[K:L] = [K/H:L/H]$ is a π'-number. Thus L is a normal Hall π-subgroup of K and this implies the π-closure of K.

Assume therefore that H is not a π-subgroup. By hypothesis H is nilpotent and hence it has normal Hall π'-subgroup A. Consider $[L:A] = [L:H][H:A]$. Since $[L:H]$ and $[H:A]$ are both π-numbers, $[L:A]$ is a π-number. Therefore L has normal Hall π'-subgroup A. By the Schür-Zassenhaus theorem [18, 9.3.6], L has a π'-complement C. Moreover since A is solvable, any two such complements are conjugate in L. Now since $L \Delta G$, for any $x \in G$ if $x \notin N_G(C)$, then $x^{-1}Cx \leq L$. This means that all conjugates of C in G are also conjugates in L. Therefore $G = LN_G(C) = AC.N_G(C) = AN_G(C)$. Since $A \Delta G$ and $A \leq H$, A is also a sp-L-subgroup of G. Moreover, C is a Hall π-subgroup of L and π is a UP-set for $L \Delta G$. So $G = N_G(C)$ i.e., $C \Delta G$. Now

$$\begin{aligned}[K:C] &= [K:L][L:C]\\ &= [K/H:L/H][L:C]\\ &= (\pi'\text{-number})(\pi'\text{-number}).\end{aligned}$$

Therefore C is normal Hall π-subgroup of K. Thus K is π-closed as we were required to show.

<u>Remark</u>: We are as yet unable to decide whether the condition that H be nilpotent in the above proposition is necessary. We know however, that 'nilpotent sp-L-subgroup' cannot be replaced by 'nilpotent L-subgroup'. Once again Example 2.3.2 confirms this.

The next two results will be useful in the sequel.

<u>Proposition 2.3.6</u>: Let K be a normal subgroup of G having property $(\mathcal{P})$ in G and H be a normal subgroup of G contained in K. Then,

(i) H has property $(\mathscr{P})$ in G.

(ii) K/H has property $(\mathscr{P})$ in G/H.

Proof: (i) Let N be a normal subgroup of G containing H. Let π be an arbitrary UP-set for N and suppose N/H is π-closed. Since $H \leq N \cap K$, $N/N \cap K$ is isomorphic to a homomorphic image of N/H and hence it is π-closed. Consequently $NK/K \cong N/N \cap K$ is π-closed. Let $\pi^* = \pi \cup \{$all primes dividing $|K|$ larger than $\pi\}$. Since π is a UP-set for N, there are no primes dividing $|N|$ and larger than the primes in the set π. Hence π^*-closure of NK/K is essentially π-closure of NK/K. Now K has property $(\mathscr{P})$ in G, π^* is a UP-set for $NK \Delta G$ and NK/K is π^*-closed. It follows that NK is π^*-closed. In particular N is π^*-closed, which in effect means that N is π-closed. Hence H has the property $(\mathscr{P})$ in G.

(ii) Let L/H be a normal subgroup of G/H such that $K/H \leq L/H$. Suppose L/H/K/H is π-closed, for π any UP-set for L/H. We must show that L/H is π-closed. Notice that $L/H/K/H \cong L/K$ implies that L/K is π-closed. Moreover the primes dividing $|L/H|$ also divide $|L|$. As before, let $\pi_1 = \pi \cup \{$all the primes dividing $|L|$ and larger than $\pi\}$. Clearly π_1 is a UP-set for L. Also, L/K being isomorphic to a homomorphic image of L/H implies that the primes dividing $|L/K|$ are not larger than the primes dividing $|L/H|$ hence no larger than the elements in π. This means that L/K π-closed implies it is π_1-closed. Now by hypothesis, K has property $(\mathscr{P})$ in G. Hence L is π_1-closed. Therefore L/H is

π_1-closed, which means it is π-closed. Thus we show that K/H has the property (θ) in G/H.

The following is a converse of the above.

Proposition 2.3.7: If H has property (θ) in G and K/H has property (θ) in G/H, then K has property (θ) in G.

Proof: Let L be a normal subgroup of G containing K. Suppose L/K is π-closed, π any UP-set for L. Then $L/K \cong L/H/K/H$ is also π-closed and π is a UP-set for L/H. By hypothesis, it follows that L/H is π-closed. Again H too has the property (θ) in G and therefore L is π-closed. This proves that K has the property (θ) in G.

We now show that the hyperquasicenter of a group G has the property (θ) in G and hence by Proposition 2.3.4 is a sp-L-subgroup of G. This will eventually enable us to prove that $L(G)$ is a sp-L-subgroup of G - our main objective in this section. We begin with the following:

Theorem 2.3.8: The quasicenter of a group G has property (θ) in G.

Proof: Let N be a normal subgroup G containing $Q = Q(G)$. Let π be an arbitrary UP-set for N and suppose N/Q is π-closed. Let A/Q be the normal Hall π-subgroup of N/Q. We must show that N has normal Hall π-subgroup. Note that $A \Delta G$. In case Q is a π-subgroup then A is a π-subgroup. Moreover, $[N:A] = [N/Q:A/Q]$ is a π'-number. Thus A is normal Hall π-subgroup and we are done. Assume therefore that Q is not a π-subgroup. Since the quasicenter of a group is nilpotent (Theorem 1.5.2), Q has normal Hall π'-subgroup Q_1. Consider $[A:Q_1] = [A:Q][Q:Q_1] = (\pi\text{-number}).(\pi\text{-number})$. Thus Q_1 is a normal Hall π'-subgroup of A. By the Schür-Zassenhaus theorem, A has a π'-complement B. So $A = Q_1B$. Since Q_1 is solvable and $A \Delta G$, all the conjugates of B in G are actually conjugates by elements of A. This implies that $G = AN_G(B) = Q_1N_G(B)$. Note that Q_1 is a π'-subgroup and B is a π-subgroup. Both Q_1, B lie in N and hence the prime divisors of $|Q_1|$ are smaller than the prime divisors of $|B|$. In the following discussion we show that $Q_1 \leq N_G(B)$.

Suppose $Q_1 \nleq N_G(B)$. Since Q is nilpotent and all its Sylow subgroups are generated by QC-elements of G (Theorem 1.5.2), we can choose an element $x \in G$ satisfying the following:

(a) x is a p-element for some p dividing $|G|$.

(b) x is a QC-element of G.

(c) $x \notin N_G(B)$, and

(d) $x \in Q_1$ and therefore p is smaller than the elements in π.

Let y be an arbitrary element in B. Then y is a π-element. Since x is a quasicentral element of G, it permutes with y. Hence $T = \langle x\rangle\langle y\rangle = \langle y\rangle\langle x\rangle$ is a subgroup. By Scott [18, 13.3.1], T is supersolvable. Also $\langle x\rangle$ is a Sylow p-subgroup of T and is contained in $Q(T)$, the quasicenter of T. Therefore $\langle x\rangle$ is a Sylow subgroup of $Q(T)$. Consequently by Theorem 1.5.2, $\langle x\rangle \Delta T$. On the other hand $\langle y\rangle$ is a Hall π-subgroup of T and is in fact a p-complement of T. Since p is the smallest prime divisor of $|T|$, T is p-nilpotent. Thus we conclude that $\langle y\rangle \Delta T$. Consequently we see that x,y centralize each other.

Since y was an arbitrary element of B, it follows in particular that $x \in N_G(B)$. But this is a contradiction since x was chosen such that $x \notin N_G(B)$. Therefore we conclude that $Q_1 \leq N_G(B)$. This implies that $B \Delta G$. Further, $[N:B] = [N:A][A:B] = (\pi'\text{-number})(\pi'\text{-number})$. So B is a normal Hall π-subgroup of N and hence N is π-closed. We have thus shown that $Q(G)$ has the property $(\mathcal{P})$.

Corollary 2.3.9: The hyperquasicenter $Q^*(G)$ has property $(\mathcal{P})$ in G.

Proof: Let $\langle 1\rangle \leq Q(G) = Q_1 \leq Q_2 \leq \ldots \leq Q_r = Q^*(G)$ be the ascending-quasicentral series of G. By definition $Q_2/Q_1 = Q(G/Q_1)$. Since Q_1 has property $(\mathcal{P})$ in G and $Q_2/Q_1 = Q(G/Q_1)$ has property $(\mathcal{P})$ in G/Q_1 by Proposition 2.3.7, Q_2 has property $(\mathcal{P})$ in G. By repeating this argument we eventually see that $Q^*(G)$, the terminal member of the above series has property $(\mathcal{P})$ in G. This proves the corollary.

A hyperquasicentral subgroup of a group is defined by Mukherjee [14] as follows:

Definition 2.3.10: A normal subgroup H of G is hyperquasicentral in G if for every $M \Delta G$ and $M \lneq H$, $H/M \cap Q(G/M) \neq \langle 1\rangle$.

It is shown in [14, Theorem 2.18] that every hyperquasicentral subgroup of G is contained in the hyperquasicenter

$Q^*(G)$. This leads to the following rather obvious corollary.

Corollary 2.3.11: A hyperquasicentral subgroup of a group G has the property $(\mathcal{P})$ in G. Moreover every SSE subgroup of G has property $(\mathcal{P})$ in G.

We are now in a position to derive the main result of this section.

Theorem 2.3.12: In a group G, $L(G)$ has the property $(\mathcal{P})$.

Proof: Case (1). Suppose $L'(G) \neq \langle 1 \rangle$, i.e. $L(G)$ is not abelian. We show that $L'(G/\Phi(G))$ is SSE in $G/\Phi(G)$. Notice that if G is solvable, then as in Theorem 1.2.9 $L(G/\Phi(G)) \cap (G/\Phi(G))'$ is SSE in $G/\Phi(G)$. This implies that $L'(G/\Phi(G))$ is SSE in $G/\Phi(G)$. In the general case, we know $L'(G)$ is nilpotent. Suppose $\Phi(G) = \Phi = \langle 1 \rangle$. Then $\Phi(L'(G)) \leq \Phi(G) = \langle 1 \rangle$. Let P be the Sylow p-subgroup of $L'(G)$, where p is an arbitrary prime divisor of $|L'(G)|$. Then $P \Delta G$ and P is elementary abelian. Let $N \leq P$ be a minimal normal subgroup of G. Since $\Phi(G) = \langle 1 \rangle$, there exists a maximal subgroup M of G not containing N. Since $N \leq P$ is solvable, $G = MN$ and $|N| = [G:M]$. But $N \leq P \leq L'(G) \leq L(G)$. Therefore M must have a prime index in G. Consequently $|N| = p$. Since P is abelian normal in G and $\Phi(G) = \langle 1 \rangle$, by [11, Satz 7] P is completely reducible in G. This means that $P = C_1 \times C_2 \times \ldots C_r$, where the C_i's are minimal normal subgroups of G of order p. Therefore G induces in P a strictly p-closed group of automorphisms. By Theorem 1.1.9 we conclude that $L'(G)$ is SSE in G.

If $\Phi(G) \neq \langle 1\rangle$, then $\Phi(G/\Phi) = \langle\bar{1}\rangle$ and from above, $L'(G/\Phi)$ is SSE in G/Φ. By Corollary 2.3.11 every SSE subgroup has property (θ). Hence $L'(G/\Phi) = \frac{L'(G)\Phi(G)}{\Phi(G)}$ has property (θ) in $G/\Phi(G)$. Since $\Phi(G)$ has property (θ) in G, by Proposition 2.3.7 $L'(G)\Phi(G)$ has property (θ) in G. Now $L'(G) \Delta G$ and $L(G/L'(G)) = L(G)/L'(G)$. By the induction hypothesis $L(G)/L'(G)$ has property (θ) in $G/L'(G)$.

Application of Proposition 2.3.7 now shows that $L(G)$ has the property (θ) in G.

Case (2). Suppose $L'(G) = \langle 1\rangle$, i.e. $L(G)$ is abelian. As in Case (1) it is not difficult to show that $L(G)/\Phi(G)$ is SSE in $G/\Phi(G)$ and hence has the property (θ) in $G/\Phi(G)$. This by Proposition 2.3.7 implies that $L(G)$ has property (θ) in G. The theorem is now completely proved.

Corollary 2.3.13: $L(G)$ is a sp-L-subgroup of G.

In an earlier remark we mentioned that if a group G has the Sylow tower property, then each of its proper normal subgroups is a sp-L-subgroup of G. The following result considers the case when G does not have the Sylow tower property but every proper subgroup of G has that property. It is seen that in such a group G the sp-L-subgroups are severely restricted.

Proposition 2.3.14: If a group G does not have the Sylow tower property but each of its proper subgroups has that property, then $\Phi(G)$ is the largest sp-L-subgroup of G. Moreover $\Phi(G) = L(G)$.

Proof: Let K be a maximal sp-L-subgroup of G (i.e., K is not properly contained in any sp-L-subgroup of G). By Corollary 2.2.6, $K\Phi(G)$ is a sp-L-subgroup of G. From the maximality of K, we have $\Phi(G) \leq K$. Suppose on the other hand that $K \not\leq \Phi(G)$. Then there exists M, a maximal subgroup of G such that $K \not\leq M$. Hence $G = KM$. By hypothesis M has the Sylow tower property. Hence $G/K \cong M/M \cap K$ has the Sylow tower property. Since K is a sp-L-subgroup of G, by Proposition 2.2.7 G has the Sylow tower property. This contradicts the hypothesis. So $\Phi(G) = K$. Finally since $L(G)$ is a sp-L-subgroup of G, we have $\Phi(G) = L(G)$.

We continue our investigation of special L-subgroups and give one more characterization of nilpotent sp-L-subgroups.

2.4 Weakly Hyperquasicentral Subgroups.

In [4] Baer introduced the concept of a weakly hypercentral subgroup as follows:

Definition 2.4.1: Let H be a proper normal subgroup of a group G. Then H is weakly hypercentral in G if the following holds for every normal subgroup K of G containing H:

For every pair of elements x,y belonging to H,K respectively if $(|x|,|y|) = 1$ and $(|x|,[K:H]) = 1$, then x and y commute.

Since a group is nilpotent if and only if elements of relatively prime orders permute, it follows by setting $K = H$ above that a weakly hypercentral subgroup is nilpotent.

Two characterizations of weakly hypercentral subgroups based on the concepts of generalized Frattini subgroups are given by D. Dykes [10]. We shall begin by first stating these.

Definition 2.4.2 [10]: A proper normal subgroup H of a group G is a special generalized Frattini subgroup of G provided that for every $N \trianglelefteq G$ and A any Hall subgroup of N, $G = HN_G(A)$ implies $G = N_G(A)$.

It is obvious that a sp-generalized Frattini subgroup is already a generalized Frattini subgroup and hence is nilpotent. Moreover a sp-generalized Frattini subgroup is also a sp-L-subgroup of the group. The converse of this however, is not necessarily true. This can be easily seen from the fact that sp-generalized Frattini subgroups are nilpotent; while sp-L-subgroups in general are not. For instance L(G) is a sp-L-subgroup of the group G which in general is not nilpotent (see Example 2.1.6).

Definition 2.4.3 [10]: Let H be a proper normal subgroup of a group G. Then H satisfies property (N_π) in G if the following holds for every normal subgroup K of G containing H:

If π is any set of primes, then K/H π-closed implies K is π-closed.

The following theorem shows the equivalence of the concepts defined above.

Theorem 2.4.4 [10]: The following statements for a normal subgroup H of the group G are equivalent:

(i) H is a special generalized Frattini subgroup of G.

(ii) H satisfies the property (N_π) in G, for any set of primes π.

(iii) H is a weakly hypercentral subgroup of G.

We shall attempt to generalize the above theorem in the present section. For this we consider the following generalization of the Definition 2.4.1:

Definition 2.4.5: A proper normal subgroup H of a group G is weakly hyperquasicentral (WHQC) in G if the following holds for every normal subgroup K of G containing H:

For every pair of elements x,y belonging to H and K respectively if $(|x|,|y|) = 1$ and $p(|x|) < p([K:H])$, then $\langle x \rangle$ and $\langle y \rangle$ permute.

It is obvious from the definition that every weakly hypercentral subgroup of G is already weakly hyperquasicentral in G.

Next, we list some of the elementary properties of weakly hyperquasicentral subgroups. The proofs of these are rather straightforward and therefore will be omitted.

Proposition 2.4.6: Let H be a WHQC subgroup of G. If $H_1 \Delta G$ and $H_1 \leq H$, then H_1 is WHQC in G and H/H_1 is WHQC in G/H_1.

Proposition 2.4.7: Let K be a WHQC subgroup of G. If $H \Delta G$ and $p(|K|) < p(|H|)$, then KH/H is WHQC in G/H.

It seems unlikely that in general the product of two WHQC subgroups of G is again WHQC in G. So let T^* be the intersection of all the maximal WHQC subgroups of G. Then T^* satisfies the following property.

<u>Proposition 2.4.8</u>: The following are equivalent for a normal subgroup N of a group G:

(a) $N \leq T^*$.

(b) If M is WHQC in G, then MN is WHQC in G.

We now turn to our main objective in this section and establish the relationship of WHQC subgroups with sp-L-subgroups and the subgroups satisfying property $(\mathcal{P})$.

<u>Proposition 2.4.9</u>: Let H be a normal subgroup of G such that,

(i) H has the Sylow tower property, and

(ii) H is WHQC in G. Then H is a sp-L-subgroup of G.

Proof: Let N be a normal subgroup of G and π a UP-set for N. Let A be any Hall π-subgroup of G such that $G = HN_G(A)$. Then we must show that $A \Delta G$. It is obvious that $HA \Delta G$. Let $\pi_1 = \pi \cup \{$all prime divisors of $|H|$ larger than primes in $\pi\}$. Then π_1 is a UP-set in HA. Since H has the Sylow tower property, it has normal Hall π_1-subgroup H_1. Moreover by the Schür-Zassenhaus theorem, H has a π_1-complement H_2 and $H = H_1 \cdot H_2$. Now let $x \in H_2$ and $y \in A \leq HA \Delta G$. Then $(|x|,|y|) = 1$. Also $p(|x|) \in \pi_1'$ and $p([HA:H] \in \pi \subsetneq \pi_1$. Since π_1 is a UP-set for HA, it follows that $p(|x|) < p([HA:H])$. Since $x \in H$, $y \in HA$ and H is WHQC in G, we conclude that x,y permute. Thus $\langle x\rangle\langle y\rangle = \langle y\rangle\langle x\rangle = T$. Since T is a product of two cyclic subgroups, it is supersolvable. Hence T has the Sylow tower property. Since

$\langle y\rangle$ is a Hall π_1-subgroup of T and π_1 is a UP-set for T, $\langle y\rangle \Delta T$. Furthermore if $y_1 \in \langle y\rangle \le HA$, then y_1 is a π_1-element. So $\langle x\rangle$, $\langle y_1\rangle$ permute. Thus x permutes with every element of T. This implies that x belongs to $Q(T)$, the quasi-center of T. Since $Q(T)$ is normal nilpotent and $\langle x\rangle$ is a Sylow subgroup of T, it follows that $\langle x\rangle \Delta T$. Hence x and y centralize each other. Since y is an arbitrary element of A, we see that $x \in N_G(A)$. So $H_2 \le N_G(A)$. Then $G = HN_G(A) = H_1H_2N_G(A) = H_1N_G(A)$. Therefore $H_1A \Delta G$. Also H_1 is a π_1-group and H_1A/H_1 is a π_1-group. Hence H_1A is π_1-closed. In particular $H_1A \cap N$ is π_1-closed. But π is a UP-set for N and hence for $H_1A \cap N$. This means that A is the characteristic Hall π-subgroup of $H_1A \cap N$. So $A \Delta G$, as we were required to show.

Proposition 2.4.10: If H is a nilpotent sp-L-subgroup of G, then H is WHQC in G.

Proof: Since H is a nilpotent sp-L-subgroup of G, by Proposition 2.3.5 H has the property $(\mathcal{P})$ in G. Let K be a normal subgroup of G containing H and let π be the set of primes dividing $[K:H]$. Then K/H is a π-group. Let $\pi_1 = \pi \cup \{$all the primes dividing $|K|$ larger than the smallest prime in $\pi\}$. It is evident that π_1 is a UP-set for K. By the property $(\mathcal{P})$ it follows that K is π_1-closed. Let A be the normal Hall π_1-subgroup of K. Since H is by hypothesis nilpotent, $H = H_1 \times H_2$, where H_1, H_2 are π_1 and π_1' Hall subgroups of H. Since A contains every π_1 hence π-element of K, we have $K = AH = A.(H_1 \times H_2) = A \times H_2$. Now let $x \in H$, $y \in K$ such that $(|x|,|y|) = 1$ and $p(|x|) < p([K:H])$. Then $p(|x|) < \pi \subseteq \pi_1$. So x is a π_1'-element of H, hence belongs to H_2. Also $y = ah = ha$, where $h \in H_2$, $a \in A$. Notice that $(|h|,|x|) = 1$. For if some prime q divides $(|h|,|x|)$, then since $|ha| = |a|\cdot|h|$, it follows that q divides $|ha| = |y|$. But this is a contradiction since $(|x|,|y|) = 1$. Thus $(|h|,|x|) = 1$. Now since h,x are elements of H_2 which is nilpotent, they commute. Thus $xy = xah = xha = (hx)a = ahx = yx$. So x,y commute. Therefore H is WHQC in G.

We derive from above one of our main results of this chapter.

Theorem 2.4.11: The following statements are equivalent for a nilpotent normal subgroup H of the group G:

(i) H is an sp-L-subgroup of G.

(ii) H has property $(\mathcal{P})$ in G.

(iii) H is weakly hyperquasicentral in G.

Moreover, every nilpotent hyperquasicentral subgroup of G is weakly hyperquasicentral in G.

CHAPTER III

In this chapter we investigate the conditions under which the subgroups $L(G)$ and $\Delta(G)$ coincide. Following Bechtell[7] we define an L-series of the group G and also the upper and the lower L-series of G. We also define the relative L-series (relative to the commutator subgroup of G) and study some properties of this series and also its relation to the group G. Some of the results obtained here are the following:

Let G be a group.

(1) If $L^*(G)$, the terminal member of the upper L-series, is the identity subgroup, then $L(G) = \Delta(G)$.

(2) The upper relative L-series of G coincides with the lower central series of G' if and only if G is supersolvable.

(3) The following are equivalent:

(i) $\hat{L}^*(G) = \langle 1\rangle$, where $\hat{L}^*(G)$ is the terminal member of the upper-relative L-series of G.

(ii) $L(G) \cap G'' = Z^*(G') \cap G''$.

(iii) If S is any subgroup of G generated by 3 elements one of which belongs to $L(G)$, then $L(G) \cap S' \leq Z^*(S')$.

(4) If $H \Delta G$ and H is solvable, then $\hat{L}^*(G) = \langle 1\rangle$ implies $\hat{L}^*(H) = \langle 1\rangle$.

3.1 The L-series of the Group G.

Definition 3.1.1: For a group G,

(i) an L-series of G is a series

$L(G) = C_0 \geq C_1 \geq \ldots$, where $C_i/C_{i+1} \leq Z(G/C_{i+1})$

or $[C_i, G] \leq C_{i+1}$

(ii) the upper L-series (UL-series) is the series

$L(G) = L_0 \geq L_1 \geq L_2 \geq \ldots$, where $L_{i+1} = [L_i, G]$

(iii) the lower L-series is defined as the ascending

central series of G, i.e.

$\langle 1 \rangle = Z_0 \leq Z_1 \leq Z_2 \leq \ldots$, where $Z_{i+1}/Z_i = Z(G/Z_i)$.

Let $L^*(G)$ denote the terminal member of the UL-series. The terminal member of the lower L-series is the hypercenter of G, $Z^*(G)$.

Some elementary properties of the above series are the following:

Proposition 3.1.2: For a group G:

(i) An L-series is a normal series of G.

(ii) The upper and lower L-series are characteristic.

(iii) If G is solvable, L_i's are all nilpotent, for $i \geq 1$.

(iv) There is no normal subgroup H of G contained in $L^*(G)$ such that $L^*(G)/H \leq Z(G/H)$.

(v) If θ is a homomorphism of G, then $L^*(G)\theta \leq L^*(G\theta)$.

Proof: (i) and (ii) are obvious by definition.

(iii). If G is solvable, by 1.2.4 $L(G) \cap G'$ is nilpotent. Now $L_1 = [L(G), G] \leq L(G) \cap G'$, since $L(G) \Delta G$. So

L_1 is nilpotent. Thus L_i is nilpotent for every $i \geq 1$.

(iv). Suppose there exists $H \Delta G$ and $H \leq L^*(G)$ such that $L^*(G)/H \leq Z(G/H)$. Then $[L^*(G),G] \leq H$. But $L^*(G)$ is the terminal member of the UL-series hence $[L^*(G),G] = L^*(G)$. Thus $L^*(G) \leq H$ and this implies that $H = L^*(G)$.

(v). Let θ be a homomorphism of G. By Proposition 1.2.3, $L(G)\theta \leq L(G\theta)$. Moreover, $[L(G),G]^{\theta} = [L(G)\theta,G\theta]$. From this it follows that $L_1\theta \leq [L(G\theta),G\theta] = L_1(G\theta)$. By the same argument it is easy to see that $L_i(G\theta) \leq L_i(G\theta)$, $\forall$ i.

As with many series, it is the terminal member of the series which yields some information about the group. We shall say that the group G possesses an L-series if the terminal member of the L-series is the identity subgroup. For such a group we have the following:

<u>Proposition 3.1.3</u>: In a group possessing an L-series, i.e. $L(G) = C_0 \geq C_1 \geq C_2 \geq \ldots C_k = \langle 1 \rangle$ we have $L_i \leq C_i$ for $i = 0,1,\ldots,k$, and $C_{k-j} \leq Z_j$, $j = 0,1,\ldots,k-1$.

<u>Remark</u>: The proof of the above is analogous to the proof of similar results for the ascending and descending central series of the group (see for example Scott [18, 6.4.1]). However for completeness we outline the proof here.

Proof: (i) Obviously $L_0 \leq C_0 = L(G)$. By the induction hypothesis we can assume that $L_i \leq C_i$. Since $[C_i,G] \leq C_{i+1}$, we have $[L_i,G] \leq C_{i+1}$. Thus $L_{i+1} = [L_i,G] \leq C_{i+1}$. So $L_i \leq C_i$, $\forall$ i.

(ii) To show $C_{k-j} \leq Z_j$ show that when $j = 0$, $C_k = \langle 1 \rangle \leq Z_0 = \langle 1 \rangle$. Now assume that for $j = r$, the assertion holds. Then $C_{k-r} \leq Z_r$. Let $T = G/Z_r$; then T is a homomorphic image of G/C_{k-r}, and the kernel of this homomorphism is Z_r/C_{k-r}. By definition, $C_{k-r-1}/C_{k-r} \leq Z(G/C_{k-r})$. Hence the image of C_{k-r-1}/C_{k-r} must lie in the center of T. Therefore, $C_{k-r-i}Z_r/Z_r \leq Z(T) = Z(G/Z_r) = Z_{r+1}/Z_r$. Thus $C_{k-r-i}Z_r \leq Z_{r+1}$ and in particular $C_{k-(r+1)} \leq Z_{r+1}$.

Corollary 3.1.4: If there exist integers j and k such that $L_j \leq Z_k$, then $L_0 \leq Z^*(G)$.

Proof: By exactly the same method as in the Proposition 3.1.3 we see that $L_j \leq Z_k$ implies that $L_{j-1} \leq Z_{k+1}$. Now if $j < k$, then by repeating the above process, we get $L_0 \leq Z_{k+1} \leq Z^*(G)$. If $j > k$, then $L_{j-k} \leq Z^*(G)$ and repeating the process we have $L_0 \leq Z^*(G)$.

Corollary 3.1.5: In a group G possessing an L-series, $L(G) = Z^*(G)$.

Proof: By 3.1.3, $L_i \leq C_i$; and in particular $C_0 \leq Z_k = Z^*(G)$. Then $L_0 \leq C_0 \leq Z_k = Z^*(G)$. On the other hand we know $Z^*(G) \leq \Delta(G) \leq L(G)$. Thus $Z^*(G) = L(G)$.

Proposition 3.1.6: In a group possessing an L-series, $\Delta(G) = L(G)$.

Proof: First we show that the existence of L-series implies the existence of Δ-series (defined by replacing $L(G)$ by $\Delta(G)$). Also if $\Delta_r \leq L_r$, then $\Delta_{r+1} = [\Delta_r, G] \leq [L_r, G] = L_r+1$. Since the terminal member of L-series is the identity subgroup of G, $\Delta^*(G) = 1$. Now by the same argument as used for $L(G)$, it is easily seen that $\Delta^*(G) = <1>$ implies $\Delta(G) = Z^*(G)$. Therefore when G possesses the L-series, it also possesses the Δ-series and $\Delta(G) = L(G)$. Also note that since $Z^*(G) \leq Q^*(G) \leq L(G)$, it follows that $Q^*(G) = L(G)$.

The above result is false if G does not possess an L-series. This can be easily confirmed from the following example.

Example 3.1.7: Let G be the group of order 84 described in Example 2.2.4. Then G has,

(i) 28 Sylow 3-subgroups

(ii) 1 Sylow 7-subgroup P_7

(iii) 1 Sylow 2-subgroup, V_4.

Obviously G is solvable. G has maximal subgroups of order 21 and hence index 4. Every such maximal subgroup has a normal Sylow 7-subgroup which must be P_7. Hence $P_7 \leq L(G)$. Since $L(G)$ is normal in G and $P_7 \cdot C_3$ is not normal in G, it follows that $L(G) = P_7$. It is not difficult to verify that $\Phi(G) = \Delta(G) = <1>$. But G does not possess an L-series. To see this first notice that if $L_1 = [L(G),G] = <1>$, then $L(G) \leq Z(G) = <1>$, which is a contradiction. So $L_1 \neq <1>$, hence $L_1 \leq L(G)$ implies $L_1 = P_7 = L(G)$. Thus we see that an L-series of G does not terminate in $<1>$.

The example also shows that $\Delta^*(G) = <1>$ does not guarantee that $L^*(G) = <1>$.

Proposition 3.1.8: If every proper subgroup of a group G has the Sylow tower property (but not the group G itself), then G has an L-series.

Proof: By Proposition 2.3.14, $\Phi(G) = L(G)$. Since $\Phi(G) \le \Delta(G) \le L(G)$, $\Delta(G) = L(G)$.

By J. Rose [17, p. 588], under the above hypothesis G is an SRI-group. An SRI-group has the form $G = PQ$, $P \Delta G$ and Q is a cyclic Sylow q-subgroup of G. Moreover $\Phi(Q) \le Z(G)$, so it is normal in G. By [7, Theorem 4.1], $\Delta(G) = Z^*(G)$ and so $\Phi(G) \le Z^*(G)$. Since $\Phi(G) = L(G)$ and $Z^*(G) = \Delta(G) \le L(G)$, it follows that $Z^*(G) = \Delta(G) = L(G)$. By Proposition 3.1.6, G possesses an L-series.

The converse of above is not true. This is easily seen by considering any simple group G. G cannot satisfy the hypothesis of Proposition 3.1.8, otherwise G is solvable. Moreover for a simple group, $L(G) = Z^*(G) = \langle 1 \rangle$ and an L-series exists trivially.

More interesting however, from our viewpoint is the concept of relative L-series of G defined in the following.

3.2 Relative L-series of the Group G.

We have noticed in Chapter I that the structure of the group G depends in large measure on the relation between G' and $L(G)$. For example we know that G is supersolvable if and only if $G' \le L(G)$. Also when G is solvable $G' \cap L(G)$ is nilpotent. This leads to the study of an L-series of G relative to the commutator subgroup of G.

Definition 3.2.1: For a group G define $B_0 = L(G) \cap G' \ge B_1 \ge B_2 \ge \ldots$, such that $B_i/B_{i+1} \le Z(G'/B_{i+1})$ $i = 0,1,\ldots$. This is a relative L-series (RL-series) of G.

The <u>upper relative L-series</u> (URL-series) of G is the series $L(G) \cap G' = \hat{L}_0 \geq \hat{L}_1 \geq \ldots$, where $\hat{L}_{i+1} = [\hat{L}_i, G']$.

The <u>lower relative L-series</u> (LRL-series) of G is the ascending central series of G', i.e.

$\langle 1\rangle = Z_0 \leq Z_1 \leq \ldots \leq Z_r \leq \ldots$, where $Z_i/Z_{i-1} = Z(G'/Z_{i-1})$.

The terminal members of URL-series and LRL-series are denoted respectively, by $\hat{L}^*(G)$ and $Z^*(G')$.

The following is easily verified.

<u>Proposition 3.2.2</u>: In a group G,

(i) RL-series are normal series of G';

(ii) the URL-series and LRL-series are characteristic series of G';

(iii) the URL-series coincides with the upper central series of G' if and only if G is supersolvable.

(iv) If G is solvable, then $\hat{L}_r \leq \Phi(G)$, for every $r \geq 1$. Moreover $\hat{L}^*(G) \leq \Phi(G)$.

(v) For any homomorphism θ of G, $\hat{L}^*(G)\theta \leq \hat{L}^*(G\theta)$.

Proof: (i) and (ii) are obvious from the definitions of these series.

(iii) If G is supersolvable, then $L(G) = G$ and $L(G) \cap G' = G'$ is nilpotent. Hence in this case URL-series defines precisely the upper central series for G' which terminates in $\langle 1\rangle$, since G' is nilpotent.

Conversely, suppose URL-series coincides with the upper central series of G'. This implies that $L(G) \cap G' = G'$ and so $G' \leq L(G)$. By Theorem 1.2.5 it follows that G is supersolvable.

(iv) $\hat{L}_1 = [L(G) \cap G',G'] \leq L(G) \cap [G',G']$. Since G is solvable, G'' is contained in every maximal subgroup of prime index (Proposition 1.2.4). Then $L(G) \cap G'' \leq \Phi(G)$ and consequently $\hat{L}_1 \leq \Phi(G)$. Since $\hat{L}_r \leq \hat{L}_1$ $\forall$ $r \geq 1$, we have $L_r \leq \Phi(G)$, $\forall$ $r \geq 1$. Now to show that $\hat{L}^*(G)$ is in $\Phi(G)$ first notice that if $\hat{L}^*(G) \neq \hat{L}_0$, then $\hat{L}^*(G) \leq \hat{L}_1 \leq \Phi(G)$. In case $\hat{L}^*(G) = \hat{L}_0$, we have $\hat{L}_0 = L(G) \cap G' = \hat{L}_1 = [L(G) \cap G',G'] \leq L(G) \cap G'' \leq \Phi(G)$. Thus in any case $\hat{L}^*(G)$ is contained in $\Phi(G)$.

(v) Let θ be any homomorphism of G, then $(L(G) \cap G')\theta \leq L(G)\theta \cap G'\theta$. Since $L(G)\theta \leq L(G\theta)$ and $G'\theta = (G\theta)'$, we have $\hat{L}_0(G)\theta \leq \hat{L}_0(G\theta)$. Similarly, $\hat{L}_1(G)\theta = [\hat{L}_0,G']\theta = [\hat{L}_0(G)\theta,G'\theta] \leq [\hat{L}_0(G\theta),G'\theta] = \hat{L}_1(G\theta)$. This argument leads to (v).

We shall say that a group possesses a relative L-series if the series terminates in the identity subgroup. For such a group we have the following.

<u>Proposition 3.2.3</u>: In a group G possessing a RL-series, i.e. $L(G) \cap G' = B_0 \geq B_1 \geq \ldots \geq B_k = \langle 1 \rangle$ we have (a) $\hat{L}_i \leq B_i$, $i = 0,1,\ldots,k$ and (b) $B_{k-j} \leq Z_j$, $j = 0,1,\ldots,k-1$.

Proof: The method of proof is essentially a duplication of the corresponding results for the L-series. Therefore we only outline the proof here.

(a) Notice that $\hat{L}_0 \leq B_0 = L(G) \cap G'$. Suppose now that $\hat{L}_i \leq B_i$ for a fixed $i < k$. Then by definition $B_i/B_{i+1} \leq Z(G'/B_{i+1})$. This implies that $[B_i,G'] \leq B_{i+1}$. Hence $\hat{L}_{i+1} = [\hat{L}_i,G'] \leq [B_i,G'] \leq B_{i+1}$. So we conclude that $\hat{L}_i \leq B_i$ for $i = 0,1,\ldots,k$.

(b) Now to show that $B_{k-j} \leq Z_j$, observe that when $j = 0$, $B_k = \langle 1 \rangle = Z_0$. Again assume that $B_{k-i} \leq Z_i$ for a fixed integer i. Let $T = G'/Z_i$ be the homomorphic image of G'/B_{K-i} with kernel Z_i/B_{k-i}.

By definition, $B_{K-i-1}/B_{K-i} \leq Z(G'/B_{K-i})$. Hence under the homomorphism the image of B_{K-i-1}/B_{K-i} must lie in the center of T. But this image is $B_{K-i}Z_i/Z_i$ which lies in $Z(T) = Z(G'/Z_i) = Z_{i+1}/Z_i$. From this it follows that $B_{K-(i+1)} \leq Z_{i+1}$. This proves (b).

As a consequence of the above results we have the following.

Proposition 3.2.4: In a group G, the following are equivalent:

(a) $\hat{L}^*(G) = \langle 1\rangle$.

(b) $\hat{L}_0(G) \leq Z^*(G')$.

(c) $L(G) \cap G'' = Z^*(G') \cap G''$.

Proof: (a) $\Rightarrow$ (b). Let $\hat{L}_r = \hat{L}^*(G) = Z_0 = \langle 1\rangle$. By definition and the hypothesis, $\hat{L}_r = [\hat{L}_{r-1}, G'] = \langle 1\rangle$. So $\hat{L}_{r-1} \leq Z(G') = Z_1$. If $Z_1 = \langle 1\rangle$, then $Z^*(G') = \langle 1\rangle$ and hence $\hat{L}_{r-1}, \hat{L}_{r-2}, \ldots, \hat{L}_0$ are all equal to $\langle 1\rangle$ and (b) is proved. So assume $\hat{L}_{r-1} \leq Z_1 \neq \langle 1\rangle$ and let $T = G'/Z_1$. T is a homomorphic image of $G'/\hat{L}_{r-1}$ with kernel $Z_1/\hat{L}_{r-1}$. Also $\hat{L}_{r-1} = [\hat{L}_{r-2}, G']$, so $\hat{L}_{r-2}/\hat{L}_{r-1} \leq Z(G'/\hat{L}_{r-1})$ [12, p. 18]. Consequently, the image of $\hat{L}_{r-2}/\hat{L}_{r-1}$ lies in the center of T. Thus $\hat{L}_{r-2}Z_1/Z_1 \leq Z(G'/Z_1) = Z_2/Z_1$ and hence $\hat{L}_{r-2} \leq Z_2$. By repeating this process we eventually have $\hat{L}_0 \leq Z_t \leq Z^*(G')$, if $r < t$ (t is the length of LRL-series). On the other hand if $\hat{L}_{r-t} \leq Z^*(G')$, then by repeating the above process we reach $\hat{L}_0 \leq Z^*(G')$.

(b) $\Rightarrow$ (c). Suppose $\hat{L}_0 \le Z^*(G')$. By [7, Theorem 2.2], $Z^*(G') \cap G'' \le \Phi(G')$. Therefore $Z^*(G') \cap G'' \le \Phi(G') \le \Phi(G) \cap G'' \le L(G) \cap G''$. Also by hypothesis $\hat{L}_0 = L(G) \cap G' \le Z^*(G')$ and so $L(G) \cap G'' \le Z^*(G') \cap G''$. Thus, $L(G) \cap G'' = Z^*(G') \cap G''$ and this is (c). Finally assume (c). Then since $[L(G) \cap G', G'] \le L(G) \cap G''$, $\hat{L}_1 \le L(G) \cap G'' = Z^*(G') \cap G''$. Thus $\hat{L}_1 \le Z^*(G') = Z_t$, where t is the length of LRL-series. Since $Z_{t-1} < Z_t$, we have $\hat{L}_1 \cdot Z_{t-1}/Z_{t-1} \le Z^*(G')/Z_{t-1} = Z(G'/Z_{t-1})$. This implies that $[\hat{L}_1 \cdot Z_{t-1}, G'] \le Z_{t-1}$. Hence $\hat{L}_2 = [\hat{L}_1, G'] \le Z_{t-1}$. This argument leads to $\hat{L}^*(G) \le Z_0 = \langle 1 \rangle$. Thus (a), (b) and (c) are equivalent.

Remark: In the above proposition $\hat{L}_0(G)$ may actually be smaller than $Z^*(G')$. For example consider $G = A_4$. As we know already, $L(G) = \langle 1 \rangle$ and so $\hat{L}_0(G) = L(G) \cap G' = \langle 1 \rangle$. However $G' \cong V_4$, the 4-group and $Z^*(G') \cong V_4$. Notice that $\hat{L}^*(G) = \langle 1 \rangle$.

Corollary 3.2.5: The conditions in Proposition 3.2.4 are also equivalent to the following: If S is any subgroup of G generated by 3 elements one of which belongs to $L(G)$, then $L(G) \cap S' \le Z^*(S')$, where $Z^*(S')$ is the hypercenter of S'.

Proof: This can be immediately deduced from a result of R. Baer [3, p. 177] which states that any normal subgroup N of G satisfies the hypothesis of the corollary (in place of $L(G)$) if and only if $G' \cap N \le Z^*(G')$.

Remark: The above may have some interesting consequences, and we hope to return to its investigation on a later occasion.

Corollary 3.2.6: If $\hat{L}^*(G) = \langle 1 \rangle$, then $Z^*(G')/\hat{L}_0 = Z(G'/L_0)$.

Proof: By Proposition 3.2.4, $L(G) \cap G'' = Z^*(G') \cap G''$. Then $[Z^*(G'),G'] \leq Z^*(G') \cap G'' = L(G) \cap G'' \leq L(G) \cap G' = \hat{L}_0$. Moreover since $\hat{L}_0 \leq Z^*(G')$ we have, $Z^*(G')/\hat{L}_0 \leq Z(G'/\hat{L}_0)$. On the other hand since $\hat{L}_0 \leq Z^*(G')$, $G'/Z^*(G')$ is a homomorphic image of $G'/\hat{L}_0$ with kernel $Z^*(G')/\hat{L}_0$. Under this homomorphism θ, $Z(G'/\hat{L}_0)$ is mapped into the center of $G'/Z^*(G')$. But $Z(G/Z^*(G')) = \langle \bar{1} \rangle$. Therefore $Z(G'/\hat{L}_0)$ is in the kernel of the homomorphism θ. Consequently, $Z(G'/\hat{L}_0) \leq Z^*(G')/\hat{L}_0$. Thus we conclude that $Z(G'/\hat{L}_0) = Z^*(G')/\hat{L}_0$.

We close the chapter and the present investigation with the following.

Proposition 3.2.7: If H is a solvable normal subgroup of G and $\hat{L}^*(G) = \langle 1 \rangle$, then $\hat{L}^*(H) = \langle 1 \rangle$.

Proof: If H is supersolvable, then by Proposition 3.2.2 the URL-series of H coincides with the descending central series of H' and hence $\hat{L}^*(H) = \langle 1 \rangle$. We may assume therefore, that H and hence G is not supersolvable. Since $\hat{L}^*(G) = \langle 1 \rangle$, by Proposition 3.2.4, $\hat{L}_0(G) = L(G) \cap G' \leq Z^*(G')$. Also since $H \Delta G$, $\Phi(H) \leq \Phi(G)$. Then $H' \cap \Phi(G) \leq G' \cap L(G) \leq Z^*(G')$. Suppose $H' \cap \Phi(H) = \langle 1 \rangle$. Since H is solvable, by Proposition 3.2.2 (iv), $\hat{L}^*(H) \leq H' \cap \Phi(H) = \langle 1 \rangle$. Therefore assume that $H' \cap \Phi(H) \neq \langle 1 \rangle$, in particular $\Phi(H) \neq \langle 1 \rangle$. It is known that the hypercenter of a group is the intersection of the normalizers of all the Sylow subgroups of the group. In the following discussion we shall make use of this definition.

If P_1 is the Sylow p-subgroup of $H' \cap \Phi(H)$, then $P_1 \Delta G'$. Furthermore from above $P_1 \leq H' \cap \Phi(H) \leq Z^*(G')$. Therefore $P_1 \leq N_{G'}(Q)$, where Q is any Sylow q-subgroup of G'. Now if $p \neq q$, then $P_1Q = P_1 \times Q$ and hence every p-element of $H' \cap \Phi(H)$ centralizes every p'-element of G'. In particular, every p-element of $H' \cap \Phi(H)$ centralizes every p'-element of H'. Let $\overline{P}$ be a Sylow p-subgroup of H' and P a Sylow p-subgroup of G' containing $\overline{P}$. Since $H' \Delta G'$, $\overline{P} = P \cap H'$. If $x \in P_1$, then $\overline{P}^x = (P \cap H')^x = P^x \cap H' = P \cap H' = \overline{P}$ since $x \in N_{G'}(P)$. This shows that x normalizes every Sylow p-subgroup of H'. Hence $x \in Z^*(H')$ and so $P_1 \leq Z^*(H')$. Since $H' \cap \Phi(H)$ is nilpotent, we conclude that $H' \cap \Phi(H) \leq Z^*(H')$. Since H is solvable, $L(H) \cap H'' \leq \Phi(H)$. Thus $L(H) \cap H'' \leq \Phi(H) \cap H' \leq Z^*(H')$. Also by [7, Theorem 2.2], $Z^*(H') \cap H'' \leq \Phi(H') \leq \Phi(H) \cap H''$. So we conclude that $Z^*(H') \cap H'' = L(H) \cap H''$. By Proposition 3.2.4 it follows that $\hat{L}^*(H) = \langle 1 \rangle$.

Remark: We are not able to decide whether the solvability condition in the above proposition is necessary.

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