THE LINEAR GRAPH IN SYSTEM ANALYSIS

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By

William Allen Blackwell

AN ABSTRACT

Submitted to the School for Advanced Graduate Studies of
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ABSTRACT

None of the common system-analysis techniques are completely satisfactory for the analysis of mixed systems in general. Electromechanical analogies have been used with considerable success, but for complicated systems made up of multiterminal components, a more powerful formulation technique is needed.

In recent years, electrical network theory, using the notions of linear graph theory (studied formally under the mathematical designation of topology) has made significant advances in formulation techniques. In electrical network analysis, it has been found that networks, for which equations are virtually impossible to formulate using the conventional from-the-diagram node and mesh techniques, can be treated--using a systematic and simple procedure--by distinguishing between the equations characteristic of the components and those characteristic of the component connection pattern. The equations, which are characteristic of the connection pattern, are called circuit and segregate equations, and are written from a collection of oriented line segments, called a linear graph of the system.

The linear graph is useful in the analysis of any system in which one set of measurements sums to zero around closed circuits, and/or one set of measurements at points, areas or

regions, sums to zero. With a proper understanding of its role in the analysis of systems, the linear graph can be obtained for a particular system by an orderly, logical procedure.

The general pattern of formulation of equations, used in electrical network theory, can be extended to the analysis of mixed systems if mathematical forms, different from those encountered in formal electrical network theory, are admitted for the equations characteristic of the components. When these mathematical forms are used, questions relative to rank of equations arise in a manner not treated in electrical network theory.

The problem considered is defined in Chapter I. In Chapter II convenient terms are defined, and the background is set, with respect to terminology and concepts, for consideration of the system formulation problem.

In Chapter III an examination is made of the conditions on topological placement of the various component types considered, such that a unique solution to the system equations is possible.

A set of general procedures for the systematic reduction of equations to be solved simultaneously is presented in Chapter IV. In all cases, the equation-reduction procedures do not involve taking a matrix inverse--depending instead upon explicit forms of equations.

A set of procedures for systems containing specific component types is given in Chapters V and VI. The number of equations to be solved simultaneously for each situation is noted. In many of these procedures, the number of equations to be solved as a simultaneous set--under certain topological arrangements of particular components--is less than would be the case with conventional mesh or node formulation.

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CONTENTS

LIST OF SYMBOLS

A list of symbols, which are used repeatedly, is given below.

Symbol	Description
e	Number of elements of a graph.
v	Number of vertices of a graph.
p	Number of separate parts of a graph.
n_{j1}	Number of J terminal equations.
n_{k2}	Number of K terminal equations.
n_x	Number of specified x-variable equations.
n_y	Number of specified y-variable equations.
n_{ox}	Number of x-variables not related in the component equations.
n_{oy}	Number of y-variables not related in the component equations.
$\mathcal{X}_a$, $\mathcal{Y}_a$	Variables related by A terminal equations.
$\mathcal{X}_B$, $\mathcal{Y}_B$	Variables related by B terminal equations.
$\mathcal{Y}_j$	Variables related by J terminal equations.
$\mathcal{X}_k$	Variables related by K terminal equations.
$\mathcal{X}_d$	x-variable specified function.
$\mathcal{Y}_d$	y-variable specified function.

Symbol	Description
$\mathcal{X}_n$	x-variables not related in the component equations.
$\mathcal{Y}_n$	y-variables not related in the component equations.
$\mathcal{X}_t$, $\mathcal{Y}_t$	Tree variables, either the unknowns or the complete set, depending on context.
$\mathcal{X}_c$, $\mathcal{Y}_c$	Chord variables, either the unknowns or the complete set, depending on context.
$\mathcal{X}_b$, $\mathcal{Y}_b$	Tree variables not classified by some other criteria.
$\mathcal{X}_m$, $\mathcal{Y}_m$	Chord variables not classified by some other criteria.

I. INTRODUCTION

Historically, Lagrange's equations have served as the basis for formulating equations for the analysis of mechanical systems. Electrical network theory, using the notions of linear graph theory, developed along completely different lines. In electromechanical-system analysis and electronic-system analysis, signal-flow graphs [1,2] and block diagrams [3,4,5] have been utilized for formulation. Each technique is successful on a certain class of systems. No one technique is completely satisfactory in general for systems which contain subsystems of different types.

There are certain limitations in the Lagrangian formulation, which make it unsatisfactory for formulating equations of complex systems [6]. Firestone [7,8] and others recognized some years ago that the form of the equations, descriptive of a mechanical system, is identical to that used in electrical network theory. It was also recognized that the network-theory technique had some definite advantages over the Lagrangian technique for certain types of mechanical systems. Electrical analogs of mechanical systems have been used in an attempt to exploit these advantages.

It is possible to define the techniques of analysis used in electrical network theory in a manner such that they are equally applicable to systems of other types--mechanical, thermal, hydraulic, etc. [9,10,11].

Trent [12] has written of the usefulness of linear graph theory, and introduced the notion that the conventional node and mesh formulation procedures can be extended to include "perfect couplers"--components of the "ideal transformer" type.

The objective of this thesis is to extend the work of Trent with respect to the analysis of systems which include "direct couplers." In following this objective, the work of Reed [13,14,15,16] in electrical network theory is used and extended, and the work of Koenig [9,10,11], in establishing foundations of system analysis, is used and extended.

When components of the type called by Trent "perfect couplers" and "direct couplers" are included in a system, certain questions as to rank of equations arise. These questions have not occurred in electrical network theory, and hence have not been formally investigated until now. Chapter III is devoted to investigating conditions for which a unique solution to the equations of a system is possible.

Certain procedures in the writing of the equations, for a system containing "direct couplers", are very helpful in establishing a systematic formulation technique. Extensions of Reed's work in electrical network formulation [13] are required. This is the subject of Chapter IV.

The application of the general principles outlined in Chapter IV is carried out in Chapters V and VI for particular component types.

II. EQUATIONS OF PHYSICAL SYSTEMS

2.1 Introduction

The oriented linear graph, as used in system analysis, could be said to serve as a rack upon which is placed information relative to two-point observations made in the system. In electrical network theory, as expounded by Reed [13], two variables, v(t) and i(t), are associated with each element of the graph for the electrical system. These variables, as treated in [14,15], are postulated to satisfy the circuit and segregate equations of the graph, and are assumed to correlate with the instantaneous voltage and current measurements, respectively, associated with a pair of designated observation points.

In order to use the linear graph for a more general system analysis, it is desirable to associate with each element of the graph of the system a pair of variables, not carrying the connotation of voltage and current.

Definition 2.1.1: Across variable: The element variable, which is associated with an across measurement, is called the across variable.

The across variable of a graph element is also designated as the x-variable, and is postulated to satisfy the circuit equations of the linear graph.

Definition 2.1.2: Series variable: The element variable, which is associated with a series measurement, is called the series variable.

The series variable of a graph element is also called the y-variable, and is postulated to satisfy the segregate equations of the linear graph. A discussion of across and series measurements for various physical systems, and techniques for relating them to the variables of the graph, is included in [9,11,12]. The definitions and theorems of linear graph theory, which are used in this thesis without specific reference, are taken from Reed [13,16], Reed and Reed [15], and Reed and Seshu [14].

2.2 Components

Essential to the application of linear graph theory to the formulation of system equations is the notion of components. In electrical network theory, the "building block" is the two-terminal component. Devices with two terminals accessible for measurement (resistors, coils, etc.) are represented in the network graph by one element. However, the need for a graphical representation for larger subnetworks has resulted in such "equivalent circuits" as Thevenin's and Norton's two-terminal equivalent circuits, and the "tee" and "pi" three-terminal equivalent circuits. There is no question as to the usefulness of these equivalents. Based on experience, it can be reasonably assumed that any electrical subsystem, which is connected to the remainder of the system at two points, can be represented in a graph of the system by one element. Further, it seems apparent from developments in [10], that any subsystem, which has three terminals in common with its complement in the system, can be represented in the system graph by two elements, which are connected, and which do not form a circuit.

This notion can be extended to a subsystem with n terminals connected to the remaining system. A rigorous mathematical development to specify the necessary and sufficient conditions for a suitable subgraph for an n-terminal component is, as yet, lacking. The following postulate, based on experience, was suggested by Koenig [10]. It has proved useful, and in a wide variety of problems, a contradiction has not been found.

Postulate: A subsystem of a physical system, which is connected at n points to its complement in the system, and which involves p different kinds of measurements (electrical, hydraulic, rotational mechanical, etc.) can be represented in the system graph by some graph G of p parts, such that the subgraph of G in each part is connected, and contains no circuits.

The graph G thus contains for an n-terminal component with p different kinds of measurements, (n - p) elements and n vertices. The terminology "can be represented in the system graph by", used in the postulate, is intended to imply that an analysis of a system using a linear graph, which is made up of subgraphs of the type described for the subsystems, yields a solution which correlates with physical measurements made on the system.

<u>Definition 2.2.1</u>: Component subgraph: A graph for a component, which is sufficient to represent the component in a graph for the system, is defined to be a component subgraph.

The graph G, described in the postulate, is thus a special form of component subgraph.

In a physical system the components of the system are connected in some manner. The points of connection are logically the observation points, or terminals, of the components. Suppose a physical system to be composed of an arbitrary set of subsystems, for each of which, the component subgraph is known.

<u>Definition 2.2.2</u>: System graph: A collection of component subgraphs, such that the vertices common to two or more subgraphs correspond to coincidence of observation points of corresponding subsystems in the physical system, is called a system graph.

The equations for a system fall into two classes: (1) those equations peculiar to the components, and (2) those equations which result from the way the components are con-

nected. The equations of the first classification are designated as component equations. The component equations relate the variables of the component subgraph elements. These relations are assumed independent of the particular system in which the component is connected.

2.3 Component Equations

A convenient type of component is one such that the variables associated with the component subgraph are related only among themselves. Thus a convenient component in an electrical system might be a three-winding transformer, an operational amplifier, or some connected set of two-terminal components.

This restricted concept of a component is so useful that it is the only type discussed further. The following definitions were suggested by the formulation procedures outlined by Reed [13].

Definition 2.3.1: Component equations: Those mathematical relations between, and only between, the variables of a component subgraph, are called component equations.

Component equations can be conveniently divided into two types.

Definition 2.3.2: Specified-variable equations: The component equations, which equate element variables to specified functions, are called specified-variable equations.

Definition 2.3.3: Terminal equations: All those component equations, which are not specified-variable equations, are called terminal equations.

The only components known to have utility in system analysis, and hence the only ones considered here, are those for which there are exactly as many component equations as there are elements in the component subgraph.

In general, physical systems may be composed of a set of two-terminal components, a set of multiterminal components, or a combination of the two. As in electrical network theory, the terminal equations, which can be associated with the j'th two-terminal component, are limited to expressing x_j in terms of y_j, or vice versa. If the component equation takes the form of a specified-variable equation, either involving x or y, then the other element variable cannot be related in the component equations. Formulation of equations for systems containing two-terminal components is very thoroughly developed.

In contrast to the two-terminal component, there is no practical limit on the number of forms that might be encountered in the description of multiterminal components. However, there is a particular set of forms, which is of particular importance in electrical, mechanical and electromechanical systems. Forms of component equations which are encountered frequently, and which are considered here are shown in Fig. 2.3.1.

Type	Number	Description
A	n_a	$\mathcal{X}_a = \mathcal{Z}\mathcal{Y}_a$
B	n_b	$\mathcal{Y}_B = \mathcal{W}\mathcal{X}_B$
J	n_{j1}	$\mathcal{Y}_{j1} = \mathcal{J}\mathcal{Y}_{j2}$
K	n_{k2}	$\mathcal{X}_{k2} = \mathcal{K}\mathcal{X}_{k1}$
D_x	n_x	$\mathcal{X} = \mathcal{X}_d$
D_y	n_y	$\mathcal{Y} = \mathcal{Y}_d$
N_x	n_{ox}	Not related in component equations. ($\mathcal{X}_n$)
N_y	n_{oy}	Not related in component equations. ($\mathcal{Y}_n$)

Figure 2.3.1

Component-equation Types and Variable Designations

It should be noted that, since the system equations are formulated in the s-domain, $\mathcal{Z}$, $\mathcal{W}$, $\mathcal{J}$ and $\mathcal{K}$ contain elements which are functions of s. However, no functional notation is used, since t-domain equations do not appear except where specifically mentioned.

In this thesis there are no classifications of terminal equations in which specified functions appear. For example, measurements on a component may indicate terminal equations of the following types:

$$\mathcal{X} = \mathcal{Z}\mathcal{Y} + \mathcal{X}_d \tag{2.3.1}$$

$$\mathcal{Y} = \mathcal{W}\mathcal{X} + \mathcal{Y}_d \tag{2.3.2}$$

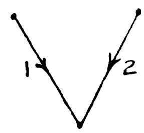
In the formulation procedures described later, (2.3.1) is handled in exactly the same manner as a set of A terminal equations, and (2.3.2) fits the same formulation pattern as the B terminal equations. The additional types of terminal equation classification are not shown because of the more cumbersome forms required in the formulation procedures examined later.

An alternate way of handling a component subgraph with terminal equations of the type of (2.3.1) is by an equivalent graph. In this equivalent graph, every element with an x_d associated with it, is replaced by two elements in series--an A-element and a D_x-element. An equivalent graph for a component subgraph, with terminal equations of the type of (2.3.2) is a graph in which each element of the component subgraph, which has a y_d associated with it, is replaced by two elements in parallel--a B-element and a D_y-element.

There is no reason to suppose that satisfactory correlation can be achieved with explicit terminal equations for all multiterminal components. In fact, it is easy to show that non-explicit t-domain differential terminal equations result for multiterminal components composed of simple combinations of two-terminal components. In present practice, the equivalent of explicit terminal equations is commonly assumed for the components used. The formulation procedure, which is taken up later, however, applies to non-explicit terminal equations as well as to explicit ones.

If an element has type A terminal equations, then it is called an A-element. If the variables associated with an element are related in terminal equations of type A, and only type A, it is called also an AA-element. Thus, a two-terminal resistor has as a component subgraph, one element which can be classified as an A-element, a B-element, an AA-element, or a BB-element. A two-terminal element with specified x-variable is a D_x-element, an N_y-element, and a D_xN_y-element.

As another example, suppose that the component equations for a three-terminal device are: $x_2 = k\ x_1$ and $y_1 = 0$, where the component subgraph might be:



A practical case corresponding to this example is an electronic amplifier, with grid current neglected in the analysis, and output voltage assumed proportional to input voltage. Both elements are K-elements. Element 1 is a KD_y-element. Element 2 is a KN_y-element. This same scheme is used in a later section to classify components.

2.4 Equations from the System Graph

Suppose a system graph to have e elements and v vertices. Using the fundamental circuit equations discussed by Reed and Seshu [14], (e - v + 1) independent circuit equations for a system graph can be written as

$$\begin{bmatrix} \mathcal{B}_{11} & \mathcal{U} \end{bmatrix} \begin{bmatrix} \mathcal{X}_t \\ \mathcal{X}_c \end{bmatrix} = 0 \qquad (2.4.1)$$

where subscripts t and c designate tree and chord respectively.

Using the fundamental segregate equations presented by Reed and Reed [15], (v - 1) independent segregate equations for a system graph of one part are written as

$$\begin{bmatrix} \mathcal{U} & \mathcal{S}_{12} \end{bmatrix} \begin{bmatrix} \mathcal{Y}_t \\ \mathcal{Y}_c \end{bmatrix} = 0 \qquad (2.4.2)$$

When the graph contains more than one part--as may well be the case in system analysis--these equations are formulated for each part by choosing a tree for each part. If the system graph contains p parts, there are exactly (e - v + p) independent circuit equations and (v - p) independent segregate equations for the graph. Thus, there is a total of e independent equations from the graph, regardless of the number of parts. The following definition is convenient for use with a system graph of more than one part:

<u>Definition 2.4.1</u>: Forest: A forest F, of a graph G of p parts, is a subgraph of G such that the elements of F in each separate part satisfy in that part the definition of a tree of a connected graph.

From the definition, it is clear that for a connected graph the terms "forest" and "tree" are synonomous. For a graph of p parts, the forest consists of a collection of p trees--one tree in each separate part. Since no ambiguity will result, the term "chord set" is used for both the complement of a tree and for the complement of a forest.

The properties of the fundamental segregate and circuit equations are so important that, unless otherwise stated, the symbols $\mathcal{S}$ and $\mathcal{C}$ are taken to indicate this particular matrix form in this thesis. A designation for the particular forest, for which the fundamental segregate and circuit equations are written, is often important.

Definition 2.4.2: Formulation forest: The forest for which a particular set of fundamental circuit and/or segregate equations are written, is called the formulation forest.

The question of permissible topological location of D-elements has been considered in [15] for the case of two-terminal components, where an element is a D-element if, and only if, it is also an N-element. The general theory in this development proceeds more smoothly with a slightly altered viewpoint.

The x-variables associated with D_x-elements in a system graph appear only in the specified-variable equations and in the circuit equations. If there exists a unique solution to the equations of the system, the rank of the circuit equations and the D_x-equations must be $(e - v + p + n_x)$.

Theorem 2.4.1: The circuit equations and the D_x equations for a system graph have rank $(e - v + p + n_x)$ if, and only if, the D_x-elements form a subset of some forest of the system graph.

Proof: The circuit equations and the D_x equations can be written as

$$\begin{bmatrix} \mathcal{B}_{11} & \mathcal{B}_{12} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \chi_3 \end{bmatrix}$$

in which there are $(e - v + p + n_x)$ rows in the coefficient matrix. If, and only if, $\mathcal{B}_{11}$ contains a set of $(e - v + p)$ columns, which correspond the complement of some forest, then there exists $(e - v + p)$ independent columns in $\mathcal{B}_{11}$, and a nonvanishing determinant of order $(e - v + p + n_x)$ in the coefficient matrix [14].

<u>Theorem 2.4.2</u>: The segregate equations and the D_y equations for a system graph have rank $(v - p + n_y)$ if, and only if, the D_y-elements form a subgraph of the complement of some forest of the system graph.

Proof: Follows the same pattern as the proof for Theorem 2.4.1.

The following theorem is taken from [17].

<u>Theorem 2.4.3</u>: Let G be a connected graph. Let S_1 and S_2 be disjoint subsets of elements of G such that there exists a tree T_1 with the elements of S_1 as chords, and there exists a tree T_2 with the elements of S_2 as branches. Then there

exists a tree T, with the elements of S_1 as chords and the elements of S_2 as branches.

In writing the segregate and circuit equations it is desirable, in general, to obtain a form which explicit for a set of unknowns in terms of other unknowns and specified functions. This necessitates using for $\mathcal{C}$ a formulation forest which includes the D_x-elements, and for $\mathcal{S}$ a formulation forest whose complement includes the D_y-elements.

In the cases considered, no element has both the x-variable and the y-variable specified. Therefore, Theorem 2.4.3 indicates that, if there is a satisfactory formulation forest for $\mathcal{C}$ and also one for $\mathcal{S}$, then there exists a formulation forest which is satisfactory for the writing of both sets.

2.5 System Equations

It is convenient to eliminate the specified-variable equations from the set of simultaneous equations for the system by substituting them into the segregate and circuit equations.

Definition 2.5.1: System equations: Let the circuit, segregate, and component equations for a physical system be known. Let all specified-variable equations be substituted into the circuit and segregate equations. The equations resulting are called the system equations.

III. INDEPENDENCE CRITERIA

3.1 Introduction

The system equations in the s-domain are all linear and algebraic. They can be classified in two groups, however, as to type of coefficients. The equations from the graph always have constant coefficients, while the component terminal equations may have coefficients which are polynomials in s. The questions of rank and independence may be explored for the circuit and segregate equations, using the standard definitions and theorems of the theory of linear algebraic equations with constant coefficients. However, some definitions are now presented in order to make clear what is meant by the terms, rank and independence, in the developments to follow, when applied to linear algebraic equations, which have coefficients which are polynomials in s.

Definition 3.1.1: S-matrix: A matrix with elements which are polynomials or ratios of polynomials in s is called an S-matrix.

Suppose that a system of Laplace-transformed ordinary linear differential equations, with constant coefficients, is written

$$\mathcal{A}(s)X(s) = \mathcal{B}(s)$$

where the order of $\mathcal{A}(s)$ is n x n, and $\mathcal{X}(s)$ and $\mathcal{S}(s)$ are column matrices. It is clear that a solution for $\mathcal{X}(t)$ does not exist if the determinant of $\mathcal{A}(s)$ vanishes for all s. Therefore the following definition is made.

Definition 3.1.2: Rank of an S-matrix: Let $\mathcal{A}(s)$ be an S-matrix. The rank of $\mathcal{A}(s)$ is the order of the highest-ordered determinant in $\mathcal{A}(s)$ which does not vanish identically for all s.

Definition 3.1.3: Independent equations: A set of m equations in n unknowns, $m \leq n$, is said to be independent if the coefficient matrix has rank m.

Theorem 3.1.1: The rank of the system equations (in the sense of Def. 3.1.2) must be equal to $(2e - n_x - n_y)$, the number of unknowns, if a unique solution is to exist for some value of s.

3.2 Topological Pattern of N-elements

The material in this section is an extension of the work of Reed and Reed [15]. Their development treats the case of two-terminal components for which a graph element is an N-element if and only if it is also a D-element. The multi-terminal component requires an extension to handle the situation where N-elements need not be associated with D-elements.

Theorem 3.2.1: Let $\mathcal{A}$ be a square n-order matrix which can be partitioned as follows:

$$\mathcal{A} = \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ 0 & \mathcal{A}_{22} \end{bmatrix}$$

where $\mathcal{A}_{11}$ is of order $m \times p$, $p \leq m \leq n$. $\mathcal{A}^{-1}$ exists only if $\mathcal{A}_{11}$ has rank p.

Proof: Use the first m rows to expand by Laplace's expansion [18] . If the p columns of $\mathcal{A}_{11}$ are not linearly independent the determinant of $\mathcal{A}$ vanishes.

Theorem 3.2.2: Given a matrix M of constant coefficients of order $p \times q$, $p \leq q$, which has rank p. Let any set of r independent columns, $r < p$, be designated by S_1. There exists a set of $(p - r)$ columns S_2 such that the union of S_1 and S_2 has rank p.

Proof: Follows from Theorem 4.22 of [18] .

Theorem 3.2.3: Given a $\mathcal{B}$ matrix for a connected graph, any p columns, $p \leq (e - v + 1)$, are independent if, and only if, the columns correspond to a subset of some chord set.

Proof: a) Sufficient: Follows from Theorem 14 of [14].

b) Necessary: By Theorem 14 of [14] , a set of $(e - v + 1)$ columns are independent only if they correspond to a chord set. Assume that at least one of some set of p independent columns S_1 does not correspond to a chord. By Theorem

3.2.2 there exists in $\mathcal{B}$ at least one set of (e - v + 1) independent columns S of which S_1 is a subset. But this is impossible, since by hypothesis at least one column in S does not correspond to a chord.

<u>Corollary 3.2.3</u>: Given a $\mathcal{B}$ matrix for a graph of p parts, any q columns, $q \leq (e - v + p)$, are independent if, and only if, the columns correspond to elements which form a subgraph of the complement of some forest.

<u>Theorem 3.2.4</u>: Given an $\mathcal{A}$ matrix for a connected graph, any q columns, $q \leq (v - 1)$, are independent if, and only if, the columns correspond to a subset of some tree.

Proof: It can be shown [16] that if, and only if, any (v - 1) columns of an $\mathcal{A}$ matrix correspond to some tree, the columns are independent. This proves the sufficiency aspect. The necessary proof follows the same pattern as for Theorem 3.2.3.

<u>Corollary 3.2.4</u>: Given an $\mathcal{A}$ matrix for a graph of p parts, any q columns, $q \leq (v - p)$, are independent if, and only if, the columns correspond to elements which form a subgraph of some forest.

<u>Definition 3.2.1</u>: Trivial segregate element: One element, which forms a segregate set, is called a trivial segregate element.

<u>Theorem 3.2.5</u>: A unique solution for the system equa-

tions exists only if the N_x-elements can be made a subset of some chord set, and the N_y-elements a subset of some forest.

Proof: The system equations can be written

$$\begin{bmatrix} \mathcal{B}_{11} & \mathcal{B}_{12} & 0 & 0 \\ 0 & \mathcal{A}_{22} & \mathcal{A}_{23} & 0 \\ 0 & 0 & \mathcal{S}_{33} & \mathcal{S}_{34} \end{bmatrix} \begin{bmatrix} x_n \\ {}^c x \\ {}^c y \\ y_n \end{bmatrix} + \begin{bmatrix} \mathcal{B}_{15} x_d \\ 0 \\ \mathcal{S}_{35} y_d \end{bmatrix} = 0$$

By Theorem 3.2.1 $\mathcal{B}_{11}$ must have rank n_{ox} and $\mathcal{S}_{34}$ must have rank n_{oy} if the inverse is to exist. The columns of $\mathcal{B}_{11}$ must, therefore, correspond to a subset of some chord set by Corollary 3.2.3. The columns of $\mathcal{S}_{34}$ must correspond to a subset of some forest by Corollary 3.2.4.

The following lemma is obtainable from developments in [15].

<u>Lemma 3.2.1</u>: If the fundamental segregate and circuit matrices, in the form of (2.4.1) and (2.4.2), are written for the same tree of a one-part graph then $\mathcal{B}_{11} = -\mathcal{S}'_{12}$.

<u>Theorem 3.2.6</u>: For a connected graph containing no trivial segregate elements, if a set of elements S forms a segregate set, the fundamental circuit equations for any tree of the graph can be written

$$\begin{bmatrix} 0 & \mathcal{B}_{12} & \mathcal{U} & 0 \\ \mathcal{B}_{21} & \mathcal{B}_{22} & 0 & \mathcal{U} \end{bmatrix} \begin{bmatrix} x_{s1} \\ \mathcal{X}_b \\ \mathcal{X}_m \\ \mathcal{X}_{s2} \end{bmatrix} = 0$$

where

$$\mathcal{X}_s = \begin{bmatrix} x_{s1} \\ \mathcal{X}_{s2} \end{bmatrix}$$

the across variables associated with S, $\mathcal{X}_m$ contains the chord variables except for $\mathcal{X}_{s2}$, and $\mathcal{B}_{21}$ is a column submatrix containing 1 or -1 in each position.

Proof: The fundamental segregate matrix for the same formulation tree is

$$\begin{bmatrix} 1 & 0 & 0 & \mathcal{S}_{14} \\ 0 & \mathcal{U} & \mathcal{S}_{23} & \mathcal{S}_{24} \end{bmatrix}$$

By Lemma 3.2.1 $\mathcal{B}_{11} = - \mathcal{S}'_{12}$. Therefore

$$\mathcal{B} = \begin{bmatrix} 0 & -\mathcal{S}'_{23} & \mathcal{U} & 0 \\ -\mathcal{S}'_{14} & -\mathcal{S}'_{24} & 0 & \mathcal{U} \end{bmatrix}$$

and the theorem follows.

<u>Theorem 3.2.7</u>: If in a system graph containing no trivial segregate elements, any segregate set contains only N_x-elements, the $\mathcal{X}_n$ of that set of elements is indeterminate.

Proof: By Theorem 3.2.6 the system equations can be written

$$\begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & 0 & 0 \\ 0 & \mathcal{B}_{22} & \mathcal{B}_{23} & \mathcal{U} \end{bmatrix} \begin{bmatrix} \mathcal{Y} \\ \mathcal{X}_a \\ x_{n1} \\ \mathcal{X}_{n2} \end{bmatrix}$$

where $$\mathcal{X}_n = \begin{bmatrix} x_{n1} \\ \mathcal{X}_{n2} \end{bmatrix}$$

and $\mathcal{X}_a$ contains all across variables not associated with the segregate of N_x-elements. Let the number of elements in the segregate set be p. With $\mathcal{X}_a$ known, there must exist a non-vanishing determinant of order p in the last p columns if a unique solution is to exist for $\mathcal{X}_n$. This is impossible since there are only (p-1) rows in $[\mathcal{B}_{23} \; \mathcal{U}]$.

If one N_x-element forms a segregate set the x-variable associated with that element does not appear in the system equations, and the y-variable is zero. The solution for the remainder of the system graph is unchanged if such an element is omitted from the graph.

<u>Theorem 3.2.8</u>: If a set of elements S form a circuit in a connected graph, the fundamental segregate equations for any tree of the graph can be written

$$\begin{bmatrix} \mathcal{U} & 0 & \mathcal{S}_{13} & \mathcal{S}_{14} \\ 0 & \mathcal{U} & \mathcal{S}_{23} & 0 \end{bmatrix} \begin{bmatrix} \mathcal{Y}_{s2} \\ \mathcal{Y}_b \\ \mathcal{Y}_m \\ y_{s1} \end{bmatrix}$$

where
$$\mathcal{Y}_s = \begin{bmatrix} y_{s1} \\ \mathcal{Y}_{s2} \end{bmatrix}$$

$\mathcal{Y}_b$ contains the tree variables not in $\mathcal{Y}_{s2}$, $\mathcal{Y}_m$ contains the chord variables except for y_{s1}, and $\mathcal{S}_{14}$ is a column submatrix with 1 or -1 entries.

Proof: Since the elements of S form a circuit, the complement of every tree contains at least one element of S. Furthermore a tree T can be chosen such that only one element of S is in its complement. The fundamental circuit matrix for T can be written as

$$\begin{bmatrix} \mathcal{B}_{11} & \mathcal{B}_{12} & \mathcal{U} & 0 \\ \mathcal{B}_{21} & 0 & 0 & 1 \end{bmatrix}$$

By Lemma 3.2.1 $\mathcal{B}_{11} = -\mathcal{S}'_{12}$. Therefore $\mathcal{S}$ can be written

$$\begin{bmatrix} \mathcal{U} & 0 & -\mathcal{B}'_{11} & -\mathcal{B}'_{21} \\ 0 & \mathcal{U} & -\mathcal{B}'_{12} & 0 \end{bmatrix}$$

<u>Theorem 3.2.9</u>: If in a system graph, any circuit contains only N_y-elements, the $\mathcal{Y}_n$ of that set of elements is indeterminate.

Proof: By Theorem 3.2.5 the system equations can be written

$$\begin{bmatrix} \mathcal{U} & \mathcal{S}_{12} & \mathcal{S}_{13} & 0 \\ 0 & 0 & \mathcal{A}_{11} & \mathcal{A}_{12} \end{bmatrix} \begin{bmatrix} \mathcal{Y}_{n2} \\ y_{n1} \\ \mathcal{Y}_a \\ \mathcal{X} \end{bmatrix} = 0$$

where

$$\mathcal{Y}_n = \begin{bmatrix} y_{n1} \\ \mathcal{Y}_{n2} \end{bmatrix}$$

and $\mathcal{Y}_a$ contains all through variables not associated with the circuit of N_y-elements. Let the number of elements in the N_y-element circuit be p. If all variables except $\mathcal{Y}_n$ are known, there must exist a non-vanishing determinant of order p in the first p columns if a unique solution is to exist for $\mathcal{Y}_n$. But there are only (p - 1) rows in $[\mathcal{U} \quad \mathcal{S}_{12}]$.

It should be noted, of course, that even though there is a circuit of N_y-elements or a segregate of N_x-elements in a system graph, Theorems 3.2.7 and 3.2.9 imply nothing about the existence or non-existence of a unique solution for the other variables in the system equations.

It has now been established that, if a unique solution exists for a set of system equations, the upper bound on no-relation elements is given by $n_{ox} \leq (e - v + p)$ and $n_{oy} \leq (v - p)$. If $(n_{ox} + n_{oy})$ is equal to e, the terminal equations may be solved independently of the remaining set. For this case the terminal equations consist of $(e - n_x - n_y)$ equations in $(e - n_x - n_y)$ unknowns. If the terminal equations are homogeneous, only a trivial unique solution is possible. Thus for a non-trivial solution for the system equations of a system with homogeneous terminal equations $(n_{ox} + n_{oy}) < e$.

3.3 <u>Rank of the System Equations</u>

The rank of the circuit and segregate equations is exactly e if the D-elements have an acceptable topological arrangement. The rank of the system equations must be $(2e - n_x - n_y)$ if a unique solution exists. Thus the rank of the terminal equations must be $(e - n_x - n_y)$.

The maximum rank of the circuit equations and the K terminal equations together fixes an upper limit on the number of K-elements permissible in a system graph.

<u>Theorem 3.3.1</u>: If the K terminal equations and the circuit equations form an independent set, then at most there can be $(v - p - n_x)$ K_2-elements in the system graph.

Proof: The rank cannot be greater than the number of variables related. Therefore, $(e - v + p + n_{k2}) \leq (e - n_x)$, from which $n_{k2} \leq (v - p - n_x)$.

Likewise, the maximum rank of the segregate equations and the J terminal equations limits the number of J-elements permissible in a system graph.

<u>Theorem 3.3.2</u>: If the J terminal equations and the segregate equations form an independent set, then there can be at most $(e - v + p - n_y)$ J_1-elements in the system graph.

Proof: Parallel to that of Theorem 3.3.1.
$(v - p + n_{j1}) \leq (e - n_y)$, from which $n_{j1} \leq (e - v + p - n_y)$.

It is, of course, desirable to be able to state necessary and sufficient conditions for which the segregate and circuit equations, and the terminal equations of type J and K, form an independent set. Necessary conditions are extremely difficult, if not impossible, to show in general. However, sufficiency criteria, of a nature general enough to be widely useful, can be established readily.

Theorem 3.3.3: If in a system graph, all K-elements and D_x-elements can be included in some forest, and all J-elements and D_y-elements can be included in the complement of some forest, then the set of circuit equations, segregate equations and terminal equations of type J and K have rank $(e + n_{j1} + n_{k2})$.

Proof:

a. Suppose that all K-elements and D_x-elements are included in the formulation forest of the system graph. The circuit equations and the K terminal equations can then be written as follows:

$$\left[\begin{array}{c|ccc} \mathcal{U} & \mathcal{B}_{12} & \mathcal{B}_{13} & \mathcal{B}_{14} \\ 0 & \mathcal{U} & -\mathcal{K} & 0 \end{array}\right] \left[\begin{array}{c} \mathcal{X}_c \\ \hline \mathcal{X}_{k2} \\ \mathcal{X}_{k1} \\ \mathcal{X}_b \end{array}\right] = 0$$

$\mathcal{X}_c$ contains the chord x-variables, and $\mathcal{X}_b$ contains the tree x-variables not associated with K-elements. This set of equations has rank $(e - v + p + n_{k2})$ because of the triangular submatrix in the leading position, with unity elements on the main diagonal.

b. Suppose that all J-elements and D_y-elements are included in the complement of the formulation forest of the system graph. The segregate equations and J terminal equations can be written:

$$\left[\begin{array}{c|ccc} \mathcal{U} & \mathcal{S}_{12} & \mathcal{S}_{13} & \mathcal{S}_{14} \\ 0 & \mathcal{U} & -\mathcal{G} & 0 \end{array}\right] \left[\begin{array}{c} \mathcal{Y}_t \\ \hline \mathcal{Y}_{j1} \\ \mathcal{Y}_{j2} \\ \mathcal{Y}_m \end{array}\right] = 0$$

These equations have rank $(v - p + n_{J1})$ by the same reasoning as that used in (a).

The hypotheses for Theorem 3.3.3 are relatively simple. For a given system graph the independence criteria outlined there are easy to apply, and thus may be quite useful. However, the sufficient conditions for independence can be made less restrictive, by sacrificing some of the simplicity of the hypotheses.

<u>Theorem 3.3.4</u>: If for some system graph,

1. The K-elements form a subset of some forest T_1,

2. The D_x-elements and K_2-elements form a subset of some forest T_2,

3. There are no circuits of K_1, K_2, and D_x-elements

(all three, and only all three, types),
then there exists a forest for which the fundamental circuits associated with the K_1-elements include no K_2-elements.

Proof: Take the forest T_2 which includes as a subset all elements of type D_x and K_2. If T_2 includes the K_1-elements, Theorem 3.3.3 holds. If not, each fundamental circuit involving a K_1-element either: (1) does not involve a K_2-element, or (2) involves at least one K_2-element. If (1) there is no problem. If (2), there must be at least one other element involved since T_1 includes all K-elements. If a K_2-element is involved, either: (a) only D_x-elements in addition to the K-elements are also involved, or (b) some other type element, say an A-element is included. If (a), then the hypothesis is violated. If (b), a forest T_3 can be chosen so as to include the K_1-element as a branch with the A-element in its complement.

The form of the proofs of Theorems 3.3.5 and 3.3.7 was suggested by Koenig [10]. From this form the hypothesis of Theorems 3.3.4 and 3.3.6 were devised.

<u>Theorem 3.3.5</u>: If the hypothesis of Theorem 3.3.4 is satisfied then the fundamental circuit equations and the K-terminal equations are independent for all $\mathcal{K}$.

Proof: By Theorem 3.3.4 there exists a forest for which the fundamental circuits for the K_1 chord elements involve no K_2-elements. Therefore the fundamental circuit equations for this tree and the K terminal equations can be written

$$\begin{bmatrix} 0 & -\mathcal{K}_1 & \mathcal{U} & -\mathcal{K}_2 & 0 \\ \mathcal{B}_{11} & \mathcal{B}_{12} & 0 & \mathcal{U} & 0 \\ \mathcal{B}_{21} & \mathcal{B}_{22} & \mathcal{B}_{23} & 0 & \mathcal{U} \end{bmatrix} \begin{bmatrix} \mathcal{X}_b \\ \mathcal{X}_{K1''} \\ \mathcal{X}_{K2} \\ \mathcal{X}_{K1'} \\ \mathcal{X}_m \end{bmatrix} = 0$$

where $\mathcal{X}_{K2} = \begin{bmatrix} \mathcal{K}_1 & \mathcal{K}_2 \end{bmatrix} \begin{bmatrix} \mathcal{X}_{K1''} \\ \mathcal{X}_{K1'} \end{bmatrix}$

Elementary operations will reduce the matrix to one with an identity submatrix in the trailing position.

Theorem 3.3.6: If for some graph G,

1. The J-elements form a subset of the complement of some forest T_1,

2. The J_1 and D_y-elements form a subset of the complement of some forest T_2,

3. There are no segregates of J_1, J_2, and N_y-elements (all three, and only all three, types), then there exists a forest for which the fundamental segregates for the J_2 branch elements include no J_1-elements.

Proof: Take the forest T_2 which has in its complement all elements of type J_1 and D_y. If the complement includes all the J_2-elements, Theorem 3.3.3 holds. If not, each fundamental segregate involving a J_2-element either: (1) does not involve a J_1-element, or (2) involves at least one J_1-element. If (1) there is no problem. If (2) there must be at least

one other element since the complement of T_1 includes all J-elements. If a J_1-element is involved either: (a) only D_y-elements in addition to J-elements are involved or (b) some other type element, say an A-element, is included. If (a) the hypothesis is violated, if (b) a forest T_3 can be chosen so as to include the A-element as a branch and the J_2-element in its complement.

<u>Theorem 3.3.7</u>: If the hypothesis of Theorem 3.3.6 is satisfied, then the fundamental segregate equations and the J terminal equations are independent for all $\mathcal{J}$.

Proof: By Theorem 3.3.6 there exists a forest for which the fundamental segregates for the J_2 branch elements include no J_1-elements. Therefore the fundamental circuit equations for this forest and the J terminal equations can be written

$$\left[\begin{array}{cc|ccc} 0 & -\mathcal{J}_1 & \mathcal{U} & -\mathcal{J}_2 & 0 \\ \mathcal{U} & 0 & \mathcal{S}_{13} & \mathcal{S}_{14} & \mathcal{S}_{15} \\ 0 & \mathcal{U} & 0 & \mathcal{S}_{24} & \mathcal{S}_{25} \end{array}\right] \begin{bmatrix} \mathcal{Y}_b \\ \mathcal{Y}_{j2'} \\ \mathcal{Y}_{j1} \\ \mathcal{Y}_{j2''} \\ \mathcal{Y}_m \end{bmatrix}$$

where

$$\mathcal{Y}_{j1} = \begin{bmatrix} \mathcal{J}_1 & \mathcal{J}_2 \end{bmatrix} \begin{bmatrix} \mathcal{Y}_{j2'} \\ \mathcal{Y}_{j2''} \end{bmatrix}$$

If the order of the variables is rearranged so as to bring the third column to the leading position, then elementary operations will reduce the leading submatrix to an

identity matrix.

A situation of considerable importance occurs when the system graph is in separate parts and when J-elements and K-elements are distributed such that: (1) the K_1-elements are in one set of parts--the K_2-elements in the other, and (2) the J_1-elements are in one set of parts--the J_2-elements in the other.

Theorems 3.3.5 and 3.3.7 apply, of course, to graphs of separate parts. However, a more general set of conditions can be given, for the particular distribution of J-elements and K-elements just stated, such that the segregate and circuit equations and the J and K terminal equations are independent.

Theorem 3.3.8: If for some graph G,

1. The graph is in two separate parts, such that K_1-elements are in part 1 and K_2-elements in part 2,

2. The D_{x1}-elements form a subset of some tree of part 1,

3. The K_2-elements and D_{x2}-elements form a subset of some tree of part 2,

then the fundamental circuit equations and the K terminal equations are independent for all $\mathcal{K}$.

Proof: The fundamental circuit equations and K terminal equations can be written as:

$$\left[\begin{array}{cccc|ccc} 0 & -\mathcal{K}_1 & 0 & \mathcal{U} & -\mathcal{K}_2 & 0 & 0 \\ \mathcal{B}_{11} & \mathcal{B}_{12} & 0 & 0 & \mathcal{U} & 0 & 0 \\ \mathcal{B}_{21} & \mathcal{B}_{22} & 0 & 0 & 0 & \mathcal{U} & 0 \\ 0 & 0 & \mathcal{B}_{33} & \mathcal{B}_{34} & 0 & 0 & \mathcal{U} \end{array}\right] \left[\begin{array}{c} \mathcal{X}_{b1} \\ \mathcal{X}_{K1''} \\ \mathcal{X}_{b2} \\ \mathcal{X}_{K2} \\ \hline \mathcal{X}_{K1'} \\ \mathcal{X}_{m1} \\ \mathcal{X}_{c2} \end{array}\right]$$

where $$\mathcal{X}_{K2} = \begin{bmatrix} \mathcal{K}_1 & \mathcal{K}_2 \end{bmatrix} \begin{bmatrix} \mathcal{X}_{K1''} \\ \mathcal{X}_{K1'} \end{bmatrix}$$

The K_1'-elements are in the chord set of part 1, and the K_1''-elements in the tree together with the D_x-elements. Thus the only restriction on the topological arrangement of the K_1-elements is that they be contained in part 1.

Theorem 3.3.9: If for some graph G,

1. The graph is in two separate parts, such that J_1-elements are in part 1 and J_2-elements in part 2,

2. The D_{y2}-elements form a subset of some chord set of part 2,

3. The J_1-elements and D_{y1}-elements form a subset of some chord set of part 1,

then the fundamental segregate equations and the J terminal equations are independent for all $\mathcal{J}$.

Proof: The fundamental segregate equations and J terminal equations can be written as:

$$\left[\begin{array}{ccc|cccc} \mathcal{U} & 0 & 0 & \mathcal{S}_{11} & \mathcal{S}_{12} & 0 & 0 \\ 0 & \mathcal{U} & 0 & 0 & 0 & \mathcal{S}_{23} & \mathcal{S}_{24} \\ 0 & 0 & \mathcal{U} & 0 & 0 & \mathcal{S}_{33} & \mathcal{S}_{34} \\ 0 & 0 & -\mathcal{J}_1 & \mathcal{U} & 0 & 0 & -\mathcal{J}_2 \end{array}\right] \left[\begin{array}{c} \mathcal{Y}_{t1} \\ \mathcal{Y}_{b2} \\ \mathcal{Y}_{j2'} \\ \hline \mathcal{Y}_{j1} \\ \mathcal{Y}_{m1} \\ \mathcal{Y}_{m2} \\ \mathcal{Y}_{j2''} \end{array}\right] = 0$$

where $$\mathcal{Y}_{j1} = \left[\mathcal{J}_1 \quad \mathcal{J}_2\right] \left[\begin{array}{c} \mathcal{Y}_{j2'} \\ \mathcal{Y}_{j2''} \end{array}\right]$$

The J_2'-elements form a subset of some tree of part 2, and the J_2''—elements together with the D_{y2}-elements form a subset of the chord set. If the third row is multiplied by $\mathcal{J}$ and added to the fourth row the result is a triangular submatrix, with 1 on the main diagonal, in the leading position. Thus the only requirement on the topological arrangement of the J_2—elements is that they be restricted to part 2.

As a result of Theorems 3.3.8 and 3.3.9, a theorem can be stated for the graph of two parts in which all K-elements are also J-elements, and vice-versa.

<u>Theorem 3.3.10</u>: If for some graph, containing no K-elements nor J-elements which are not JK-elements

1. The graph is in two separate parts such that the JK_1-elements are in part 1, and the JK_2-elements in part 2,

2. The JK_1-elements and the D_{y1}-elements form a subset of some chord set of part 1,

3. The D_{x1}-elements form a subset of some tree of part 2,

4. The JK_2-elements and the D_{x2}-elements form a subset of some tree of part 2,

5. The D_{y2}-elements form a subset of some chord set of part 2,

then the fundamental circuit and segregate equations, together with the JK terminal equations, form an independent set for all $\mathcal{J}$ and $\mathcal{K}$.

Proof: Follows from Theorems 3.3.8 and 3.3.9 since all conditions in the hypotheses of both theorems are satisfied.

Theorems 3.3.8 - 3.3.10 can all be extended to the case of a system graph of more than two parts by replacing the term "tree" by "forest" and the term "part" by "set of parts".

The proofs of independence have been based on the fact that fundamental $\mathcal{B}$ and $\mathcal{S}$ matrices are used, and that these matrices are written for the particular tree used to locate topologically the J-elements and the K-elements. The next step is to show that, if these equations are independent, then independence is assured for any full-rank set of circuit and segregate equations.

For this development let

$$\begin{bmatrix} \mathcal{B}_{11} & \mathcal{B}_{12} & \mathcal{B}_{13} \\ -\mathcal{K} & \mathcal{U} & 0 \end{bmatrix} \begin{bmatrix} \mathcal{X}_{K1} \\ \mathcal{X}_{K2} \\ \mathcal{X}_1 \end{bmatrix} = \begin{bmatrix} \mathcal{B}_1 \\ \mathcal{A} \end{bmatrix} \mathcal{X} = 0$$

and
$$\begin{bmatrix} \mathcal{S}_{11} & \mathcal{S}_{12} & \mathcal{S}_{13} \\ \mathcal{U} & -\mathcal{I} & 0 \end{bmatrix} \begin{bmatrix} \mathcal{Y}_{j1} \\ \mathcal{Y}_{j2} \\ \mathcal{Y}_{1} \end{bmatrix} = \begin{bmatrix} \mathcal{S}_1 \\ \mathcal{D} \end{bmatrix} \mathcal{Y} = 0$$

<u>Theorem 3.3.11</u>: For some graph of p parts let $\mathcal{B}_1$ and $\mathcal{B}_2$ be circuit matrices with rank (e - v + p).

$\begin{bmatrix} \mathcal{B}_1 \\ \mathcal{a} \end{bmatrix}$ and $\begin{bmatrix} \mathcal{B}_2 \\ \mathcal{a} \end{bmatrix}$ have the same rank.

Proof: Follows from the fact that $\mathcal{B}_1$ and $\mathcal{B}_2$ are related by a non-singular transformation.

<u>Theorem 3.3.12</u>: For a graph of p parts let $\mathcal{S}_1$ and $\mathcal{S}_2$ be segregate matrices with rank (v - p).

$\begin{bmatrix} \mathcal{S}_1 \\ \mathcal{D} \end{bmatrix}$ and $\begin{bmatrix} \mathcal{S}_2 \\ \mathcal{D} \end{bmatrix}$ have the same rank.

Proof: Parallel to that of Theorem 3.3.11.

Thus if, and only if, independence exists for some set of full-rank circuit and segregate equations, the equations are independent for any set of full-rank circuit and segregate equations.

IV. GENERAL PROCEDURE FOR FORMULATION

4.1 Introduction

If the graph of a physical system is known, together with the necessary number of component equations, all information required for simultaneous solution is available. However, for complex systems, the number of simultaneous equations is very large. Even when computers, either digital or analog, are to be used, a great saving in time can usually be gained by a reduction in the equations to be solved simultaneously.

With regard to formulation of the system equations, the system graph serves just one purpose--that of providing a systematic method of writing a set of independent equations in a convenient explicit form. When this has been accomplished, the graph has served its purpose and the analysis procedure is based only on the form of the system equations.

In general, the graph of a system may contain elements with all types of component equations. The formulation procedure is, of course, influenced by the type of elements present. However, a general procedure can be stated, utilizing the explicit form of at least some of the system equations, to effect a reduction in the number of equations to be solved simultaneously. There is no assurance that this procedure yields the smallest set of simultaneous equations, which it

is convenient to obtain, nor that the procedure is the simplest possible. It is, however, relatively simple to accomplish, and it handles systems containing the types of components discussed. In order to facilitate the reduction process, certain procedures in the writing of the circuit and segregate matrices are helpful. These procedures are examined next.

4.2 Circuit Equations for System Graph Containing N_x-Elements

The following is an extension of a formulation technique developed by Reed [13] for two-terminal components in which an element is an N_x-element if, and only if, it is also a D_y-element.

If there are N_x-elements in the system graph, then the only system equations involving these variables are the circuit equations. It has been shown that a complete solution to the system equations can only be obtained if the N_x-elements are a subset of some chord set. Discussion is limited to the case for which a complete solution exists. If a formulation forest is chosen, such that the N_x-elements are a subset of the chord set, and if the equations and variables are properly sequenced, then the circuit equations can be written

$$\begin{bmatrix} \mathcal{B}_{11} & \mathcal{B}_{12} & 0 \\ \mathcal{B}_{21} & \mathcal{B}_{22} & \mathcal{U} \end{bmatrix} \begin{bmatrix} \mathcal{X}_d \\ \mathcal{X}_T \\ \mathcal{X}_n \end{bmatrix} = 0 \qquad (4.2.1)$$

where the variables are: $\mathcal{X}_d$, specified across functions; $\mathcal{X}_T$, across variables related by some type of terminal equa-

tions; $\mathcal{X}_n$, no-relation across variables.

Since $\mathcal{X}_n$ is related only by the second row of Eq. 4.2.1, and the number of equations is equal the number of variables in $\mathcal{X}_n$, this set of equations can be "set aside" in that they need not be solved simultaneously with the other equations of the system. Thus, for simultaneous solution, the circuit equations are reduced to

$$\mathcal{E}_{11} \mathcal{X}_d + \mathcal{B}_{12} \mathcal{X}_T = 0 \tag{4.2.2}$$

The unknown variables in Eq. 4.2.2 number $(e - n_x - n_{ox})$.

4.3 The Segregate Equations for System Graph Containing N_y-Elements

The following is an extension of a formulation technique developed by Reed [13] for two-terminal components in which an element is an N_y-element if, and only if, it is also a D_x-element.

If the system graph contains N_y-elements, then the series variables $\mathcal{Y}_n$ are related only in the segregate equations. If a complete solution to the system equations exists, N_y-elements must form a subgraph of some forest. If a formulation forest is chosen such that the N_y-elements are included as a subset of the forest elements, and if the variables and equations are properly sequenced, then the segregate equations can be written

$$\begin{bmatrix} \mathcal{U} & \mathcal{S}_{12} & \mathcal{S}_{13} \\ 0 & \mathcal{S}_{22} & \mathcal{S}_{23} \end{bmatrix} \begin{bmatrix} \mathcal{Y}_n \\ \mathcal{Y}_T \\ \mathcal{Y}_d \end{bmatrix} = 0 \qquad (4.3.1)$$

where the variables are: $\mathcal{Y}_n$, no-relation series variables; $\mathcal{Y}_T$, series variables related by some type of terminal equations; $\mathcal{Y}_d$, specified through functions.

The only equations, in the entire set of system equations, that involve $\mathcal{Y}_n$, are those represented by the first row of Eq. 4.3.1. Thus, these equations can also be set aside. Therefore the contribution of the segregate equations to the set that must be solved simultaneously is:

$$\mathcal{S}_{22}\mathcal{Y}_T + \mathcal{S}_{23}\mathcal{Y}_d = 0 \qquad (4.3.2)$$

In this set the number of unknown variables is $(e - n_y - n_{oy})$.

4.4 The Substitution Procedure

In reducing a set of n simultaneous equations in n unknowns to a smaller simultaneous set, the equivalent of the following must be done: Some set of (n - m) equations are used to obtain an explicit relationship for some (n - m) unknown variables, in terms of the remaining m variables. Then the (n - m) variables are eliminated in the remaining m equations. If the n equations are independent, this can always be done for any subset of unknown variables. Therefore, whether the reduction process is a useful one or not, depends in general upon the degree of difficulty with which the explicit relations can be obtained. If matrix inverses must be taken before the substitution process can be carried out, it may well be that inverting the original set of n equations would be less tedious than reducing the set before inverting.

In the system formulation, as considered in this thesis, the terminal equations are explicit in form. Therefore, by substitution of the terminal equations into the circuit and segregate equations, the number of simultaneous equations can always be reduced to $(e - n_{ox} - n_{oy})$.

The circuit and segregate equations are not, in general, explicit forms. However, the fact that the fundamental circuit and segregate equations are explicit, make them very useful in any procedure involving reduction of simultaneous system equations. As noted earlier, because of this explicit relationship, the $\mathcal{X}_n$ and $\mathcal{Y}_n$ variables can be solved for, and since they appear in none of the remaining equations, the simultaneous set is reduced by $(n_{ox} - n_{oy})$. The system equations can be shown in the form:

$$\begin{bmatrix} \mathcal{U} & 0 & \mathcal{S}_{13} & 0 & 0 & 0 \\ \hdashline 0 & \mathcal{U} & \mathcal{S}_{23} & 0 & 0 & 0 \\ 0 & \mathcal{A}_1 & \mathcal{A}_2 & \mathcal{A}_3 & \mathcal{A}_4 & 0 \\ 0 & 0 & 0 & \mathcal{B}_{11} & \mathcal{U} & 0 \\ \hdashline 0 & 0 & 0 & \mathcal{B}_{21} & 0 & \mathcal{U} \end{bmatrix} \begin{bmatrix} \mathcal{Y}_m \\ \mathcal{Y}_b \\ \mathcal{Y}_c \\ \mathcal{X}_t \\ \mathcal{X}_m \\ \mathcal{X}_n \end{bmatrix} = 0 \qquad (4.4.1)$$

where $\mathcal{X}_m$ and $\mathcal{X}_n$ are the chord x-variables, and $\mathcal{Y}_b$ and $\mathcal{Y}_n$ the forest y-variables. These equations are not homogeneous, as the matrix form might seem to imply. The D_x-element x-variables are assumed to be contained in $\mathcal{X}_t$, and the D_y-element y-variables contained in $\mathcal{Y}_c$.

The equations represented by the rows between the partitioning lines in Eq. 4.4.1 are the $(2e - n_{ox} - n_{oy})$ simultaneous equations. The center row represents the various possible forms of terminal equations.

A second procedure, which could be used for reduction of the simultaneous equations, is to take the explicit relations from the second and fourth rows, and substitute them into the third row. This substitution can always be made, regardless of the form of the terminal equations (explicit or not). The number of simultaneous equations to be solved after such a substitution is $(e - n_x - n_y)$. If $(n_x + n_y) > (n_{ox} + n_{oy})$, this procedure would be preferable to that of substituting the terminal equations into the circuit and segregate equations.

4.5 Chord Formulation

A third reduction technique utilizes the explicit form of the second row of Eq. 4.4.1, and any terminal equations explicit in the x-variables. It is evident that the forest y-variables $\mathcal{Y}_b$, can be expressed in terms of the chord y-variables $\mathcal{Y}_c$. Let the second row be substituted into the terminal equations. Let m terminal equations be explicit in x. If these m terminal equations are then substituted into the fourth row, the set of simultaneous equations remaining are $(e - n_x - n_y - m) + (e - v + p - n_{ox}) = (2e - v + p - n_x - n_y - n_{ox} - m)$ in number.

If $m = (e - n_x - n_y)$, a common case for electrical circuits, this reduces to $(e - v + p - n_{ox})$ simultaneous equations, conventionally called "mesh equations" in electrical network theory. For this case the terminal equations are of type A and the problem of formulation has been treated by Reed [13].

4.6 Tree Formulation

A fourth reduction technique utilizes the explicit form of the fourth row of Eq. 4.4.1, and any terminal equations explicit in the y-variables. Using the fourth row, the chord x-variables, $\mathcal{X}_m$, can be expressed in terms of the forest x-variables, $\mathcal{X}_t$. Let the fourth equation be substituted into the terminal equations. If there are q equations explicit in the y-variables, substitution of these q equations into the second row results in $(e - n_x - n_y - q) + (v - p - n_{oy}) = (e + v - p - n_x - n_y - n_{oy} - q)$ equations.

If $q = (e - n_x - n_y)$, this reduces to $(v - p - n_{oy})$ simultaneous equations, sometimes referred in electrical network theory as the "branch equations". For this case the terminal equations are of type B and the problem of formulation has been treated by Reed [13].

If the graph is in separate parts, it is evident that there is an independent choice in each part as to what formulation technique is used.

After the various procedures of substitution have been carried out, as outlined, there still may be explicit relations, which allow of further systematic reduction without the necessity of inversion. Two cases are of sufficient importance to warrant examination.

Case 1: Terminal equations of type K:

1(a) Tree formulation: The first step in the tree formulation is to write from Eq. 4.4.1

$$X_m = -B_{11} X_t$$

and substitute into the terminal equations. If the K-elements are a subset of the formulation forest, then the K terminal equations can be written

$$\begin{bmatrix} -K & U \end{bmatrix} \begin{bmatrix} X_{K1} \\ X_{K2} \end{bmatrix} = \begin{bmatrix} -K & U & 0 \end{bmatrix} \begin{bmatrix} X_{t1} \\ X_{t2} \\ X_{t3} \end{bmatrix} = 0 \qquad (4.6.1)$$

This relation is explicit in X_{t2}, so that these variables may be eliminated by substitution into the remaining equations. It follows, of course, that this reduction can be made for any subset of the K terminal equations, which relate variables associated with the formulation forest.

1(b) Chord formulation: The general procedure, as outlined, covers this case, since the substitution is for terminal equations explicit in the x-variables, into the circuit equations.

Case 2: Terminal equations of type J:

2(a) Tree formulation: Covered by general procedure.

2(b) Chord formulation: The first step in the chord formulation procedure is to write from Eq. 4.4.1

$$\mathcal{Y}_b = -\mathcal{S}_{23} \mathcal{Y}_c \tag{4.6.2}$$

and substitute into the terminal equations. If the J-elements are a subset of the complement of the formulation forest, then the J terminal equations can be written:

$$\begin{bmatrix} \mathcal{U} & -\mathcal{G} \end{bmatrix} \begin{bmatrix} \mathcal{Y}_{j1} \\ \mathcal{Y}_{j2} \end{bmatrix} = \begin{bmatrix} \mathcal{U} & -\mathcal{G} & 0 \end{bmatrix} \begin{bmatrix} \mathcal{Y}_{c1} \\ \mathcal{Y}_{c2} \\ \mathcal{Y}_{c3} \end{bmatrix} = 0 \tag{4.6.3}$$

This relation is explicit in $\mathcal{Y}_{c1}$, so that a further reduction by substitution can be made. If only a part of the J terminal equations are associated with the chord set, that part can be used to reduce the number of simultaneous equations.

4.7 Relationship between $\mathcal{S}$ and $\mathcal{B}$ for the same Formulation Forest

Using Lemma 3.2.1 it can be shown that, for the same formulation forest,

$$\mathcal{B}_{11} = -\mathcal{S}_{12}'$$

[illegible] [illegible]

[illegible] [illegible]

It follows that the explicit relations from the segregate and circuit equations, respectively, can be written as:

$$\mathcal{Y}_t = -\mathcal{A}_{12} \mathcal{Y}_c = \mathcal{B}_{11}' \mathcal{Y}_c \quad (4.7.1)$$

$$\mathcal{X}_c = -\mathcal{B}_{11} \mathcal{X}_t = \mathcal{A}_{12}' \mathcal{X}_t \quad (4.7.2)$$

Therefore, regardless of the reduction procedure to be used, only the fundamental segregate or fundamental circuit equations need be written. It should be noted that, in general, the formulation forest most convenient for the equation-reduction procedure depends upon the particular reduction technique to be used. The formulation forest most convenient for reduction of equations, is certainly not necessarily the one used for independence criteria.

V. FORMULATION FOR SYSTEMS OF PARTICULAR COMPONENTS

5.1 Introduction

To examine specific formulation procedures, for system graphs containing all possible types of elements, is obviously not practicable. In this section, with two exceptions, the intention is to examine some formulation procedures for systems containing A-elements and/or B-elements, D_xN_y-elements, D_yN_x-elements, and components associated with at least one other type of terminal equation.

The objective in the formulation procedure is to arrive at the smallest set of equations, which can be obtained without the inversion of matrices. Consistent with this purpose, the A terminal equations have a convenient form for substitution into the circuit equations. Thus the chord formulation would usually be indicated where the system graph elements are predominantly of the A type. Likewise, the tree formulation is most effective where the B terminal equation is predominant. However, there are many cases where the terminal equations may be inverted by inspection, or at most by inverting a diagonal, or otherwise simple matrix form. In these cases, the terminal equations are assumed to be either type or A or B, depending on convenience.

In the special cases that follow, the chord and tree formulation procedures are presented for the most part. The procedures are valid for system graphs of more than one part, of course. However, the equation counts, unless otherwise specified, are based on a graph of one part. Essentially, this is to allow easy comparison of the number of simultaneous equations for the particular formulation procedure, to be made with the number of conventional mesh and node equations, which would result for the same graph and appropriate terminal equations. If the term "tree" is replaced by "forest", each procedure applies to a system graph of separate parts.

Each special formulation procedure is presented in outline form, as much as practicable. If the same formulation tree is used for both circuit and segregate matrices, one is obtainable from the other. Therefore in most of the cases examined, the only one written is the one into which the final substitution is made.

5.2 Types of Components Considered

In order to facilitate the presentation of formulation procedures for systems containing specific types of components, it is convenient to define some component types.

<u>Definition 5.2.1</u>: KDN-component: A KDN-component is one for which the component equations have the following form:

$$\chi_{k2} = K\chi_{k1}, \qquad y_{k1} = y_{d1}$$

y_{k2} not related in the component equations.

An electronic amplifier, with output voltage taken proportional to input voltage, and input current neglected for purpose of analysis, is an example of a KDN-component.

<u>Definition 5.2.2</u>: KBN-component: A KBN-component is one for which the component equations have the following forms:

$$\mathcal{X}_{k2} = \mathcal{K}\mathcal{Y}_{k1}, \qquad \mathcal{Y}_{k1} = \mathcal{H}\mathcal{X}_{k1}$$

$\mathcal{Y}_{k2}$ not related in the component equations.

An example of a KBN-component is an operational amplifier of an analog computer.

<u>Definition 5.2.3</u>: JDN-component: A JDN-component is one for which the component equations have the following forms:

$$\mathcal{Y}_{j1} = \mathcal{J}\mathcal{Y}_{j2}, \qquad \mathcal{X}_{k2} = \mathcal{X}_{j2}$$

$\mathcal{X}_{k1}$ not related in the component equations.

<u>Definition 5.2.4</u>: JAN-component: A JAN-component is one for which the component equations have the following forms:

$$\mathcal{Y}_{j1} = \mathcal{J}\mathcal{Y}_{j2}, \qquad \mathcal{X}_{j2} = \mathcal{Z}\mathcal{Y}_{j2}$$

$\mathcal{X}_{j1}$ not related in the component equations.

If, for a common-base transistor, the collector current

and the emitter voltage are assumed proportional to emitter current, the transistor can be classified as a JAN-component.

Definition 5.2.5: JK-component: A JK-component is one for which the component equations have the following forms:

$$\mathcal{X}_{k1} = \mathcal{K}\mathcal{X}_{k2}, \qquad \mathcal{Y}_{k2} = \mathcal{J}\mathcal{Y}_{k1}$$

5.3 Systems of A and KDN Components

Chord formulation is used:

In the formulation tree: All N_y-elements, which includes the K_2-elements.

In the chord set: All D_y-elements and N_x-elements, including the K_1-elements.

Circuit equations:

$$\left[\begin{array}{cccc} \mathcal{B}_{11} & \mathcal{B}_{12} & 0 & 0 \\ \mathcal{B}_{21} & \mathcal{B}_{22} & \mathcal{U} & 0 \\ \hdashline \mathcal{B}_{31} & \mathcal{B}_{32} & 0 & \mathcal{U} \end{array}\right] \begin{bmatrix} \mathcal{X}_{k2} \\ \mathcal{X}_{a} \\ \mathcal{X}_{k1} \\ \mathcal{X}_{n} \end{bmatrix} + \left[\begin{array}{c} \mathcal{B}_{15} \\ \mathcal{B}_{25} \\ \hdashline \mathcal{B}_{35} \end{array}\right] \mathcal{X}_{d} = 0 \qquad (5.3.1)$$

Explicit form of the segregate equations from the circuit equations:

$$\begin{bmatrix} \mathcal{Y}_{k2} \\ \mathcal{Y}_{a} \end{bmatrix} = \begin{bmatrix} \mathcal{B}'_{11} & \mathcal{B}'_{21} & \mathcal{B}'_{31} \\ \mathcal{B}'_{12} & \mathcal{B}'_{22} & \mathcal{B}'_{32} \end{bmatrix} \begin{bmatrix} \mathcal{Y}_{m} \\ \mathcal{Y}_{k1} \\ \mathcal{Y}_{d} \end{bmatrix} \qquad (5.3.2)$$

In (5.3.2) $\mathcal{Y}_{m}$ contains the unspecified tree y-variables.

Terminal equations after substitution from segregate equations:

$$\mathcal{X}_a = \mathcal{Z}_a \mathcal{Y}_a = \mathcal{Z}_a \begin{bmatrix} \mathcal{B}'_{12} & \mathcal{B}'_{22} & \mathcal{B}'_{32} \end{bmatrix} \begin{bmatrix} \mathcal{Y}_m \\ \mathcal{Y}_{K1} \\ \mathcal{Y}_b \end{bmatrix} \qquad (5.3.3)$$

$$\mathcal{X}_{K2} = \mathcal{K} \mathcal{X}_{K1} \qquad (5.3.4)$$

Circuit equations after substitution of terminal equations:

$$\begin{bmatrix} \mathcal{B}_{12} \\ \mathcal{B}_{22} \end{bmatrix} \mathcal{Z}_a \begin{bmatrix} \mathcal{B}'_{12} & \mathcal{B}'_{22} & \mathcal{B}'_{32} \end{bmatrix} \begin{bmatrix} \mathcal{Y}_m \\ \mathcal{Y}_{K1} \\ \mathcal{Y}_b \end{bmatrix} + \begin{bmatrix} \mathcal{B}_{11} \mathcal{K} \\ \mathcal{B}_{21} \mathcal{K} + \mathcal{U} \end{bmatrix} \mathcal{X}_{K1} + \begin{bmatrix} \mathcal{B}_{15} \\ \mathcal{B}_{25} \end{bmatrix} \mathcal{X}_d = 0 \qquad (5.3.5)$$

Unknowns: $\mathcal{Y}_m$ and $\mathcal{X}_{k1}$.

Number of equations: $(e - v + 1 - n_{ox})$.

Comments: n_{ox} is the number of $D_y N_x$-elements in the system graph for this case.

5.4 Systems of B and KDN Components

Tree formulation is used:

In the formulation tree: All D_y-elements, and all possible K-elements. If all K-elements cannot be put into the same formulation tree with the D_x-elements, then a formulation tree should be chosen such that the maximum number of

K terminal equations relate only tree variables.

In the chord set: All N_x-elements.

In this procedure all K-elements are assumed to be in the formulation tree.

Segregate equations:

$$\left[\begin{array}{cccc} \mathcal{U} & 0 & 0 & \mathcal{S}_{14} \\ 0 & \mathcal{U} & 0 & \mathcal{S}_{24} \\ \hline 0 & 0 & \mathcal{U} & \mathcal{S}_{34} \\ 0 & 0 & 0 & \mathcal{S}_{44} \end{array}\right] \left[\begin{array}{c} \mathcal{Y}_n \\ \mathcal{Y}_{K2} \\ \mathcal{Y}_{K1} \\ \mathcal{Y}_B \end{array}\right] + \left[\begin{array}{c} \mathcal{S}_{15} \\ \mathcal{S}_{25} \\ \hline \mathcal{S}_{35} \\ \mathcal{S}_{45} \end{array}\right] \mathcal{Y}_d = 0 \quad (5.4.1)$$

Explicit form of the circuit equations from the segregate equations:

$$\mathcal{X}_B = \left[\mathcal{S}'_{14} \;\; \mathcal{S}'_{24} \;\; \mathcal{S}'_{34} \;\; \mathcal{S}'_{44}\right] \left[\begin{array}{c} \mathcal{X}_d \\ \mathcal{X}_{K2} \\ \mathcal{X}_{K1} \\ \mathcal{X}_b \end{array}\right] \quad (5.4.2)$$

where $\mathcal{X}_b$ contains the unknown tree x-variables.

Terminal equations after substitution from segregate equations:

$$\mathcal{X}_{K2} = \mathcal{K}\mathcal{X}_{K1} \quad (5.4.3)$$

$$\mathcal{Y}_B = \mathcal{W}\mathcal{X}_B = \mathcal{W}\left[\mathcal{S}'_{14} \quad \mathcal{S}'_{24} \quad \mathcal{S}'_{34} \quad \mathcal{S}'_{44}\right]\begin{bmatrix}\mathcal{X}_d \\ \mathcal{X}_{K2} \\ \mathcal{X}_{K1} \\ \mathcal{X}_b\end{bmatrix} \qquad (5.4.4)$$

Terminal equations after elimination of $\mathcal{X}_{k2}$:

$$\mathcal{Y}_B = \mathcal{W}\left[\mathcal{S}'_{14} \quad (\mathcal{S}'_{24}K + \mathcal{S}'_{34}) \quad \mathcal{S}'_{44}\right]\begin{bmatrix}\mathcal{X}_d \\ \mathcal{X}_{K1} \\ \mathcal{X}_b\end{bmatrix} \qquad (5.4.5)$$

Segregate equations after substitution of (5.4.5):

$$\begin{bmatrix}\mathcal{U} \\ 0\end{bmatrix}\mathcal{Y}_{K1} + \begin{bmatrix}\mathcal{S}_{34} \\ \mathcal{S}_{44}\end{bmatrix}\mathcal{W}\left[\mathcal{S}'_{14} \quad (\mathcal{S}'_{24}K + \mathcal{S}'_{34}) \quad \mathcal{S}'_{44}\right]\begin{bmatrix}\mathcal{X}_d \\ \mathcal{X}_{K1} \\ \mathcal{X}_b\end{bmatrix} + \begin{bmatrix}\mathcal{S}_{35} \\ \mathcal{S}_{45}\end{bmatrix}\mathcal{Y}_d = 0 \qquad (5.4.6)$$

Unknowns: $\mathcal{X}_{k1}$ and $\mathcal{X}_b$.

Number of equations: $(v - 1 - n_{oy}) = (v - 1 - n_{k2} - n_x)$

5.5 Systems of A and KBN Components

Chord formulation is used.

In the formulation tree: All N_y-elements, which includes the K_2-elements.

In the chord set: All D_y-elements and N_x-elements.

Circuit equations:

$$\left[\begin{array}{c:ccc} \mathcal{U} & \mathcal{B}_{12} & \mathcal{B}_{13} & \mathcal{B}_{14} \\ \hdashline 0 & \mathcal{B}_{22} & \mathcal{B}_{23} & \mathcal{B}_{24} \end{array}\right] \begin{bmatrix} \mathcal{X}_n \\ \mathcal{X}_{K2} \\ \mathcal{X}_{K1} \\ \mathcal{X}_a \end{bmatrix} + \begin{bmatrix} \mathcal{B}_{15} \\ \hdashline \mathcal{B}_{25} \end{bmatrix} \mathcal{X}_d = 0 \qquad (5.5.1)$$

Explicit form of the segregate equations from (5.5.1):

$$\begin{bmatrix} \mathcal{Y}_{K2} \\ \mathcal{Y}_{K1} \\ \mathcal{Y}_a \end{bmatrix} = \begin{bmatrix} \mathcal{B}'_{12} & \mathcal{B}'_{22} \\ \mathcal{B}'_{13} & \mathcal{B}'_{23} \\ \mathcal{B}'_{14} & \mathcal{B}'_{24} \end{bmatrix} \begin{bmatrix} \mathcal{Y}_d \\ \mathcal{Y}_m \end{bmatrix} \qquad (5.5.2)$$

Terminal equations after substitution from (5.5.2):

$$\mathcal{X}_a = \mathcal{Z}_a \begin{bmatrix} \mathcal{B}'_{14} & \mathcal{B}'_{24} \end{bmatrix} \begin{bmatrix} \mathcal{Y}_d \\ \mathcal{Y}_m \end{bmatrix} \qquad (5.5.3)$$

$$\mathcal{X}_{K2} = \mathcal{K} \mathcal{X}_{K1} \qquad (5.5.4)$$

$$\begin{bmatrix} \mathcal{B}'_{13} & \mathcal{B}'_{23} \end{bmatrix} \begin{bmatrix} \mathcal{Y}_d \\ \mathcal{Y}_m \end{bmatrix} = \mathcal{W}_1 \mathcal{X}_{K1} \qquad (5.5.5)$$

Circuit equations after substitution of explicit terminal equations:

$$\left[\mathcal{B}_{22}\mathcal{K}+\mathcal{B}_{23}\right]\mathcal{X}_{K1}+\mathcal{B}_{24}\mathcal{Z}_a\left[\mathcal{B}'_{14}\ \mathcal{B}'_{24}\right]\begin{bmatrix}\mathcal{Y}_d\\ \mathcal{Y}_m\end{bmatrix}+\mathcal{B}_{25}\mathcal{X}_d=0 \tag{5.5.6}$$

Final equations:

$$\begin{bmatrix}\mathcal{W}_1\\ \mathcal{B}_{22}\mathcal{K}+\mathcal{B}_{23}\end{bmatrix}\mathcal{X}_{K1}+\begin{bmatrix}\mathcal{B}'_{13} & \mathcal{B}'_{23}\\ \mathcal{B}_{24}\mathcal{Z}_a\mathcal{B}'_{14} & \mathcal{B}_{24}\mathcal{Z}_a\mathcal{B}'_{24}\end{bmatrix}\begin{bmatrix}\mathcal{Y}_d\\ \mathcal{Y}_m\end{bmatrix}+\begin{bmatrix}0\\ \mathcal{B}_{25}\mathcal{X}_d\end{bmatrix}=0 \tag{5.5.7}$$

Unknowns: $\mathcal{X}_{k1}$ and $\mathcal{Y}_m$.

Number of equations: $(e - v + 1 - n_y + n_{k1})$.

If $\mathcal{W}_1^{-1} = \mathcal{Z}_1$ is known, or easily obtained, the K terminal equations can be written

$$\begin{bmatrix}\mathcal{X}_{K2}\\ \mathcal{X}_{K1}\end{bmatrix}=\begin{bmatrix}\mathcal{Z}_1\mathcal{K}\\ \mathcal{Z}_1\end{bmatrix}\mathcal{Y}_{K1} \tag{5.5.8}$$

For this form, the terminal equations after substitution from the segregate equations are:

$$\begin{bmatrix}\mathcal{X}_{K2}\\ \mathcal{X}_{K1}\\ \mathcal{X}_a\end{bmatrix}=\begin{bmatrix}\mathcal{Z}_1\mathcal{K} & 0\\ \mathcal{Z}_1 & 0\\ 0 & \mathcal{Z}_a\end{bmatrix}\begin{bmatrix}\mathcal{B}'_{13} & \mathcal{B}'_{23}\\ \mathcal{B}'_{14} & \mathcal{B}'_{24}\end{bmatrix}\begin{bmatrix}\mathcal{Y}_d\\ \mathcal{Y}_m\end{bmatrix} \tag{5.5.9}$$

Circuit equations after substitution of (5.5.9):

$$\begin{bmatrix} \mathcal{B}_{22} & \mathcal{B}_{23} & \mathcal{B}_{24} \end{bmatrix} \begin{bmatrix} \mathcal{Z}_1 \mathcal{K} & 0 \\ \mathcal{Z}_1 & 0 \\ 0 & \mathcal{Z}_a \end{bmatrix} \begin{bmatrix} \mathcal{B}'_{13} & \mathcal{B}'_{23} \\ \mathcal{B}'_{14} & \mathcal{B}'_{24} \end{bmatrix} \begin{bmatrix} \mathcal{Y}_d \\ \mathcal{Y}_m \end{bmatrix} + \mathcal{B}_{25} \mathcal{X}_d = 0 \qquad (5.5.10)$$

Unknowns: $\mathcal{Y}_m$.

Number of equations: $(e - v + 1 - n_{ox}) = (e - v + 1 - n_y)$.

5.6 Systems of B and KBN Components

Tree formulation is used.

In the formulation tree: All N_y-elements, and all K-elements possible. In any case the maximum number of K terminal equations possible should relate tree variables only.

In the chord set: All N_x-elements:

Segregate equations:

$$\begin{bmatrix} \mathcal{U} & 0 & 0 & \mathcal{S}_{14} \\ 0 & \mathcal{U} & 0 & \mathcal{S}_{24} \\ \hline 0 & 0 & \mathcal{U} & \mathcal{S}_{34} \\ 0 & 0 & 0 & \mathcal{S}_{44} \end{bmatrix} \begin{bmatrix} \mathcal{Y}_n \\ \mathcal{Y}_{K2} \\ \mathcal{Y}_{K1} \\ \mathcal{Y}_B \end{bmatrix} + \begin{bmatrix} \mathcal{S}_{15} \\ \mathcal{S}_{25} \\ \hline \mathcal{S}_{35} \\ \mathcal{S}_{45} \end{bmatrix} \mathcal{Y}_d = 0 \qquad (5.6.1)$$

Explicit form of the circuit equations from (5.6.1):

$$\mathcal{X}_B = \begin{bmatrix} \mathcal{S}'_{14} & \mathcal{S}'_{24} & \mathcal{S}'_{34} & \mathcal{S}'_{44} \end{bmatrix} \begin{bmatrix} \mathcal{X}_d \\ \mathcal{X}_{k2} \\ \mathcal{X}_{k1} \\ \mathcal{X}_b \end{bmatrix} \tag{5.6.2}$$

where $\mathcal{X}_b$ contains all tree x-variables not in $\mathcal{X}_{k1}$, $\mathcal{X}_{k2}$ and $\mathcal{X}_d$.

Terminal equations after substitution of (5.6.2):

$$\mathcal{Y}_B = \mathcal{W}_B \begin{bmatrix} \mathcal{S}'_{14} & \mathcal{S}'_{24} & \mathcal{S}'_{34} & \mathcal{S}'_{44} \end{bmatrix} \begin{bmatrix} \mathcal{X}_d \\ \mathcal{X}_{k2} \\ \mathcal{X}_{k1} \\ \mathcal{X}_b \end{bmatrix} \tag{5.6.3}$$

$$\begin{bmatrix} \mathcal{X}_{k2} \\ \mathcal{Y}_{k1} \end{bmatrix} = \begin{bmatrix} \mathcal{K} \\ \mathcal{W}_1 \end{bmatrix} \mathcal{X}_{k1} \tag{5.6.4}$$

Terminal equations after elimination of $\mathcal{X}_{k2}$:

$$\mathcal{Y}_B = \mathcal{W}_B \begin{bmatrix} \mathcal{S}'_{14} & (\mathcal{S}'_{24}\mathcal{K} + \mathcal{S}'_{34}) & \mathcal{S}'_{44} \end{bmatrix} \begin{bmatrix} \mathcal{X}_d \\ \mathcal{X}_{k1} \\ \mathcal{X}_b \end{bmatrix} \tag{5.6.5}$$

$$\mathcal{Y}_{k1} = \mathcal{W}_1 \mathcal{X}_{k1} \tag{5.6.6}$$

Segregate equations after substitution of (5.6.5) and (5.6.6):

$$\begin{bmatrix} \mathcal{U} \\ 0 \end{bmatrix} \mathcal{W}_1 \mathcal{X}_{k1} + \begin{bmatrix} \mathcal{S}_{34} \\ \mathcal{S}_{44} \end{bmatrix} \mathcal{W}_b \begin{bmatrix} \mathcal{S}'_{14} & (\mathcal{S}'_{24}\mathcal{K} + \mathcal{S}'_{34}) & \mathcal{S}'_{44} \end{bmatrix} \begin{bmatrix} \mathcal{X}_d \\ \mathcal{X}_{k1} \\ \mathcal{X}_b \end{bmatrix} + \begin{bmatrix} \mathcal{S}_{35} \\ \mathcal{S}_{45} \end{bmatrix} \mathcal{Y}_d = 0 \tag{5.6.7}$$

Unknowns: $\mathcal{X}_{k1}$ and $\mathcal{X}_b$.

Number of equations: $(v - 1 - n_{oy}) = (v - 1 - n_x - n_{k2})$.

The form of (5.6.7) is not appealing, but no good purpose is served by rewriting to get all variables together, since this greatly complicates the general form.

The case of $\mathcal{X}_{k1} = \mathcal{Z}_1 \mathcal{Y}_{k1}$ is not considered here. As might be expected, there is no appreciable change in the form of the final equations. If $\mathcal{W}_1 \mathcal{X}_{k1}$ is replaced by $\mathcal{Y}_{k1}$, and $\mathcal{X}_{k1}$ is replaced by $\mathcal{Z}_1 \mathcal{Y}_{k1}$ in (5.6.7) the alternate equations are obtained.

5.7 Systems of KBN Components

A special case, which is important enough to merit some attention, is the type which includes the analog-computer system. If each operational amplifier is represented by a Lagrangian-tree subgraph, with the common vertex corresponding to ground, the graph of the system is separable at the ground vertex. Each nonseparable part consists of two or more elements in parallel. In each separable part there is one, and only one, element of the following types: (1) D_xN_y-

element, or (2) an element representing the output of an operational amplifier. Thus, the specified-voltage elements and operational-amplifier output elements in general make up a Lagrangian tree of the system graph.

In the formulation tree: All K_2-elements, since $\mathcal{Y}_{k2}$ is not related in the component equations, and all D_x-elements.

In the chord set: All K_1-elements.

The complete set of system equations:

$$\left[\begin{array}{ccccc} \mathcal{U} & 0 & \mathcal{S}_{13} & 0 & 0 \\ 0 & \mathcal{U} & \mathcal{S}_{23} & 0 & 0 \\ \hline 0 & 0 & \mathcal{U} & 0 & -\mathcal{W} \\ 0 & 0 & 0 & \mathcal{U} & -\mathcal{K} \\ 0 & 0 & 0 & \mathcal{B}_{11} & \mathcal{U} \end{array}\right] \left[\begin{array}{c} \mathcal{Y}_{n} \\ \mathcal{Y}_{K2} \\ \mathcal{Y}_{K1} \\ \mathcal{X}_{K2} \\ \mathcal{X}_{K1} \end{array}\right] + \left[\begin{array}{c} 0 \\ 0 \\ \hline 0 \\ 0 \\ \mathcal{B}_{13}\mathcal{X}_{d} \end{array}\right] = 0 \qquad (5.7.1)$$

The first three rows may be set aside. The fourth row may be substituted into the fifth row, or vice versa. Therefore, the final equations are:

$$(\mathcal{U} + \mathcal{B}_{11}\mathcal{K})\mathcal{X}_{K1} + \mathcal{B}_{13}\mathcal{X}_{d} = 0 \qquad (5.7.2)$$

or

$$(\mathcal{K}\mathcal{B}_{11} + \mathcal{U})\mathcal{X}_{K2} + \mathcal{K}\mathcal{B}_{13}\mathcal{X}_{d} = 0 \qquad (5.7.3)$$

Unknowns: $\mathcal{X}_{k1}$ or $\mathcal{X}_{k2}$.

Number of equations: n_{k1} or n_{k2}. $(n_{k1} \geqslant n_{k2})$

Comments: The segregate equations, and the B terminal equations, do not affect the x-variable solution for this type of system.

5.8 Systems of A and JDN Components

Chord formulation is used.

In the formulation tree: All N_y-elements.

In the chord set: All D_y-elements, and all J-elements if possible. In any case, the maximum possible number of J terminal equations should relate only chord variables.

In this procedure, the J-elements are assumed to be a subset of the complement of the formulation tree.

Circuit equations:

$$\left[\begin{array}{cccc} \mathcal{B}_{11} & 0 & 0 & 0 \\ \mathcal{B}_{21} & \mathcal{U} & 0 & 0 \\ \hline \mathcal{B}_{31} & 0 & \mathcal{U} & 0 \\ \mathcal{B}_{41} & 0 & 0 & \mathcal{U} \end{array}\right] \begin{bmatrix} \mathcal{X}_a \\ \mathcal{X}_{j2} \\ \mathcal{X}_{j1} \\ \mathcal{X}_n \end{bmatrix} + \left[\begin{array}{c} \mathcal{B}_{15} \\ \mathcal{B}_{25} \\ \hline \mathcal{B}_{35} \\ \mathcal{B}_{45} \end{array}\right] \mathcal{X}_d = 0 \qquad (5.8.1)$$

Explicit form of the segregate equations from (5.8.1):

$$\mathcal{Y}_a = \begin{bmatrix} \mathcal{B}_{11}' & \mathcal{B}_{21}' & \mathcal{B}_{31}' & \mathcal{B}_{41}' \end{bmatrix} \begin{bmatrix} \mathcal{Y}_m \\ \mathcal{Y}_{j2} \\ \mathcal{Y}_{j1} \\ \mathcal{Y}_d \end{bmatrix} \qquad (5.8.2)$$

where $\mathcal{Y}_{m}$ contains the chord y-variables not in the remaining sets.

Terminal equations after substitution of (5.8.2):

$$\begin{bmatrix} \mathcal{X}_a \\ 0 \end{bmatrix} = \begin{bmatrix} \mathcal{Z}_a & 0 \\ 0 & \mathcal{U} \end{bmatrix} \begin{bmatrix} \mathcal{B}_{11}' & \mathcal{B}_{21}' & \mathcal{B}_{31}' & \mathcal{B}_{41}' \\ 0 & -\mathcal{J} & \mathcal{U} & 0 \end{bmatrix} \begin{bmatrix} \mathcal{Y}_{m1} \\ \mathcal{Y}_{j2} \\ \mathcal{Y}_{j1} \\ \mathcal{Y}_d \end{bmatrix} \qquad (5.8.3)$$

In general, the explicit form of the J terminal equations is retained, only if the J-elements are a subset of the chord set.

Terminal equations after eliminating $\mathcal{Y}_{j1}$:

$$\mathcal{X}_a = \mathcal{Z}_a \left[\mathcal{B}_{11}' \quad (\mathcal{B}_{21}' + \mathcal{B}_{31}'\mathcal{J}) \quad \mathcal{B}_{41}' \right] \begin{bmatrix} \mathcal{Y}_m \\ \mathcal{Y}_{j2} \\ \mathcal{Y}_d \end{bmatrix} \qquad (5.8.4)$$

Circuit equations after the substitution of (5.8.4):

$$\begin{bmatrix} \mathcal{B}_{11} \\ \mathcal{B}_{21} \end{bmatrix} \mathcal{Z}_a \left[\mathcal{B}_{11}' \quad (\mathcal{B}_{21}' + \mathcal{B}_{31}'\mathcal{J}) \quad \mathcal{B}_{41}' \right] \begin{bmatrix} \mathcal{Y}_m \\ \mathcal{Y}_{j2} \\ \mathcal{Y}_d \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{U} \end{bmatrix} \mathcal{X}_{k2} + \begin{bmatrix} \mathcal{B}_{15} \\ \mathcal{B}_{25} \end{bmatrix} \mathcal{X}_d = 0 \qquad (5.8.5)$$

Unknowns: $\mathcal{Y}_m$ and $\mathcal{Y}_{j2}$.

Number of equations: $(e - v + 1 - n_{ox}) = (e - v + 1 - n_y - n_{k1})$

5.9 Systems of B and JDN Components

Tree formulation is used:

In the formulation tree: The D_x-elements, including the J_2-elements.

In the chord set: The N_x-elements, including the J_1-elements.

Segregate equations:

$$\left[\begin{array}{cccc} \mathcal{U} & 0 & \mathcal{S}_{13} & \mathcal{S}_{14} \\ \hline 0 & \mathcal{U} & \mathcal{S}_{23} & \mathcal{S}_{24} \\ 0 & 0 & \mathcal{S}_{33} & \mathcal{S}_{34} \end{array}\right] \begin{bmatrix} \mathcal{Y}_n \\ \mathcal{Y}_{j2} \\ \mathcal{Y}_{j1} \\ \mathcal{Y}_B \end{bmatrix} + \left[\begin{array}{c} \mathcal{S}_{15} \\ \hline \mathcal{S}_{25} \\ \mathcal{S}_{35} \end{array}\right] \mathcal{Y}_d = 0 \qquad (5.9.1)$$

Explicit form of the circuit equations from (5.9.1):

$$\mathcal{X}_B = \begin{bmatrix} \mathcal{S}'_{14} & \mathcal{S}'_{24} & \mathcal{S}'_{34} \end{bmatrix} \begin{bmatrix} \mathcal{X}_d \\ \mathcal{X}_{j2} \\ \mathcal{X}_b \end{bmatrix} \qquad (5.9.2)$$

where $\mathcal{X}_b$ contains tree x-variables not in $\mathcal{X}_d$ and $\mathcal{X}_{j2}$.

Terminal equations after substitution from (5.9.2):

$$\mathcal{Y}_B = \mathcal{W}_B \begin{bmatrix} \mathcal{S}'_{14} & \mathcal{S}'_{24} & \mathcal{S}'_{34} \end{bmatrix} \begin{bmatrix} \mathcal{X}_d \\ \mathcal{X}_{K2} \\ \mathcal{X}_b \end{bmatrix} \qquad (5.9.3)$$

$$\mathcal{Y}_{j1} = \mathcal{J}\,\mathcal{Y}_{j2} \qquad (5.9.4)$$

Segregate equations after substitution of terminal equations:

$$\begin{bmatrix} \mathcal{U} + \mathcal{S}_{23}\mathcal{J} \\ \mathcal{S}_{33}\mathcal{J} \end{bmatrix} \mathcal{Y}_{j2} + \begin{bmatrix} \mathcal{S}_{24} \\ \mathcal{S}_{34} \end{bmatrix} \mathcal{W}_b \begin{bmatrix} \mathcal{S}_{14}' & \mathcal{S}_{24}' & \mathcal{S}_{34}' \end{bmatrix} \begin{bmatrix} \mathcal{X}_d \\ \mathcal{X}_{j2} \\ \mathcal{X}_b \end{bmatrix} + \begin{bmatrix} \mathcal{S}_{25} \\ \mathcal{S}_{35} \end{bmatrix} \mathcal{Y}_d = 0 \qquad (5.9.5)$$

Unknowns: $\mathcal{Y}_{j2}$ and $\mathcal{X}_b$.

Number of equations: $(v - 1 - n_{oy}) = (v - 1 - n_x)$.

5.10 Systems of A and JK Components

Chord formulation is used.

In the formulation tree: All N_y-elements.

In the chord set: All D_y-elements, and all JK-elements if possible. In any case, the maximum possible number of J terminal equations should relate only chord variables.

In this procedure, it is assumed that all JK-elements are in the chord set.

Circuit equations:

$$\begin{bmatrix} \mathcal{B}_{11} & 0 & 0 & 0 \\ \mathcal{B}_{21} & \mathcal{U} & 0 & 0 \\ \mathcal{B}_{31} & 0 & \mathcal{U} & 0 \\ \hline \mathcal{B}_{41} & 0 & 0 & \mathcal{U} \end{bmatrix} \begin{bmatrix} \mathcal{X}_a \\ \mathcal{X}_{K1} \\ \mathcal{X}_{K2} \\ \mathcal{X}_n \end{bmatrix} + \begin{bmatrix} \mathcal{B}_{15} \\ \mathcal{B}_{25} \\ \mathcal{B}_{35} \\ \hline \mathcal{B}_{45} \end{bmatrix} \mathcal{X}_d = 0 \qquad (5.10.1)$$

Explicit form of the segregate equations from (5.10.1):

$$\mathcal{Y}_a = \left[\mathcal{B}'_{11}\ \mathcal{B}'_{21}\ \mathcal{B}'_{31}\ \mathcal{B}'_{41}\right] \begin{bmatrix} \mathcal{Y}_m \\ \mathcal{Y}_{K1} \\ \mathcal{Y}_{K2} \\ \mathcal{Y}_d \end{bmatrix} \tag{5.10.2}$$

Terminal equations after substitution of (5.10.2):

$$\mathcal{X}_a = \mathcal{Z}_a \left[\mathcal{B}'_{11}\ \mathcal{B}'_{21}\ \mathcal{B}'_{31}\ \mathcal{B}'_{41}\right] \begin{bmatrix} \mathcal{Y}_m \\ \mathcal{Y}_{K1} \\ \mathcal{Y}_{K2} \\ \mathcal{Y}_d \end{bmatrix} \tag{5.10.3}$$

$$\mathcal{Y}_{K1} = \mathcal{J}\,\mathcal{Y}_{K2} \tag{5.10.4}$$

$$\mathcal{X}_{K2} = \mathcal{K}\,\mathcal{X}_{K1} \tag{5.10.5}$$

Terminal equations after elimination of $\mathcal{Y}_{K1}$:

$$\mathcal{X}_a = \mathcal{Z}_a \left[\mathcal{B}'_{11}\ \ (\mathcal{B}'_{21}\mathcal{J} + \mathcal{B}'_{31})\ \ \mathcal{B}'_{41}\right] \begin{bmatrix} \mathcal{Y}_m \\ \mathcal{Y}_{K2} \\ \mathcal{Y}_d \end{bmatrix} \tag{5.10.6}$$

$$\mathcal{X}_{K2} = \mathcal{K}\,\mathcal{X}_{K1} \tag{5.10.7}$$

Circuit equations after substitution of (5.10.6) and (5.10.7):

$$\begin{bmatrix} \mathcal{B}_{11} \\ \mathcal{B}_{21} \\ \mathcal{B}_{31} \end{bmatrix} \mathcal{Z}_a \left[\mathcal{B}_{11}' \;\; (\mathcal{B}_{21}'\mathcal{J}+\mathcal{B}_{31}') \;\; \mathcal{B}_{41}' \right] \begin{bmatrix} \mathcal{Y}_m \\ \mathcal{Y}_{k2} \\ \mathcal{Y}_d \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{U} \\ \mathcal{K} \end{bmatrix} \mathcal{X}_{k1} + \begin{bmatrix} \mathcal{B}_{15} \\ \mathcal{B}_{25} \\ \mathcal{B}_{35} \end{bmatrix} \mathcal{X}_d = 0 \tag{5.10.8}$$

Unknowns: $\mathcal{Y}_m$, $\mathcal{Y}_{k2}$ and $\mathcal{X}_{k1}$.

Number of equations: $(e - v + 1 - n_{ox}) = (e - v + 1 - n_y)$.

Comments: If it is not possible to put all JK-elements into the chord set for formulation, then the final set of simultaneous equations is larger by the number of J equations not associated with chord variables.

In (5.10.8) it is obvious that an explicit relationship exists for $\mathcal{X}_{k1}$ in the second row. Therefore, another substitution can be made which reduces the number of simultaneous equations by n_{k1}. A direct substitution procedure in the general form is very unwieldy. However, the equivalent of substitution for $\mathcal{X}_{k1}$ is premultiplication by

$$\begin{bmatrix} \mathcal{U} & 0 & 0 \\ 0 & -\mathcal{K} & \mathcal{U} \end{bmatrix}$$

The result of this premultiplication is:

$$\begin{bmatrix} \mathcal{B}_{11} \\ (\mathcal{B}_{31} - \mathcal{K}\mathcal{B}_{21}) \end{bmatrix} \mathcal{Z}_a \left[\mathcal{B}_{11}' \;\; (\mathcal{B}_{21}'\mathcal{J}+\mathcal{B}_{31}') \;\; \mathcal{B}_{41}' \right] \begin{bmatrix} \mathcal{Y}_m \\ \mathcal{Y}_{k2} \\ \mathcal{Y}_d \end{bmatrix} + \begin{bmatrix} \mathcal{B}_{15} \\ (\mathcal{B}_{35} - \mathcal{K}\mathcal{B}_{25}) \end{bmatrix} \mathcal{X}_d = 0 \tag{5.10.9}$$

Unknowns: $\mathcal{Y}_m$ and $\mathcal{Y}_{x2}$.

Number of equations: $(e - v + 1 - n_{ox} - n_{k1})$

Comments: If $\mathcal{J}' = -\mathcal{K}$ then

$$\begin{bmatrix} \mathcal{B}_{11} \\ (\mathcal{B}_{31} - \mathcal{K}\mathcal{B}_{21}) \end{bmatrix} = \left[\mathcal{B}_{11}' \quad (\mathcal{B}_{21}'\mathcal{J} + \mathcal{B}_{31}') \right]'$$

This is the case called "perfect coupling" by Trent [12]. For this situation, the matrix to be inverted is symmetrical if $\mathcal{Z}_a$ is symmetrical.

5.11 Systems of B and JK Components

Tree formulation is used.

In the formulation tree: All D_x-elements, and all JK-elements, if possible. In any case, the maximum possible number of K terminal equations should relate only tree x-variables.

In the chord set: All N_x-elements.

Segregate equations:

$$\begin{bmatrix} \mathcal{U} & 0 & 0 & \mathcal{S}_{14} \\ 0 & \mathcal{U} & 0 & \mathcal{S}_{24} \\ 0 & 0 & \mathcal{U} & \mathcal{S}_{34} \\ 0 & 0 & 0 & \mathcal{S}_{44} \end{bmatrix} \begin{bmatrix} \mathcal{Y}_n \\ \mathcal{Y}_{k1} \\ \mathcal{Y}_{k2} \\ \mathcal{Y}_B \end{bmatrix} + \begin{bmatrix} \mathcal{S}_{15} \\ \mathcal{S}_{25} \\ \mathcal{S}_{35} \\ \mathcal{S}_{45} \end{bmatrix} \mathcal{Y}_d = 0 \qquad (5.11.1)$$

Explicit form of the circuit equations from (5.11.1):

$$\mathcal{X}_B = \begin{bmatrix} \mathcal{S}'_{14} & \mathcal{S}'_{24} & \mathcal{S}'_{34} & \mathcal{S}'_{44} \end{bmatrix} \begin{bmatrix} \mathcal{X}_d \\ \mathcal{X}_{K1} \\ \mathcal{X}_{K2} \\ \mathcal{X}_b \end{bmatrix} \qquad (5.11.2)$$

Terminal equations after substitution of (5.11.2):

$$\mathcal{Y}_B = \mathcal{W}_B \begin{bmatrix} \mathcal{S}'_{14} & \mathcal{S}'_{24} & \mathcal{S}'_{34} & \mathcal{S}'_{44} \end{bmatrix} \begin{bmatrix} \mathcal{X}_d \\ \mathcal{X}_{K1} \\ \mathcal{X}_{K2} \\ \mathcal{X}_b \end{bmatrix} \qquad (5.11.3)$$

$$\mathcal{Y}_{K1} = \mathcal{J}\,\mathcal{Y}_{K2} \qquad (5.11.4)$$

$$\mathcal{X}_{K2} = \mathcal{K}\,\mathcal{X}_{K1} \qquad (5.11.5)$$

Terminal equations after elimination of $\mathcal{X}_{k2}$:

$$\mathcal{Y}_B = \mathcal{W}_B \begin{bmatrix} \mathcal{S}'_{14} & (\mathcal{S}'_{24} + \mathcal{S}'_{34}\mathcal{K}) & \mathcal{S}'_{44} \end{bmatrix} \begin{bmatrix} \mathcal{X}_d \\ \mathcal{X}_{K1} \\ \mathcal{X}_b \end{bmatrix} \qquad (5.11.6)$$

$$\mathcal{Y}_{K1} = \mathcal{J}\,\mathcal{Y}_{K2} \qquad (5.11.7)$$

Segregate equations after substitution of (5.11.6) and (5.11.7):

$$\begin{bmatrix} \mathcal{J} \\ \mathcal{U} \\ 0 \end{bmatrix} \mathcal{Y}_{k2} + \begin{bmatrix} \mathcal{S}_{24} \\ \mathcal{S}_{34} \\ \mathcal{S}_{44} \end{bmatrix} \mathcal{W}_B \left[\mathcal{S}_{14}' \quad (\mathcal{S}_{24}' + \mathcal{S}_{34}'\mathcal{K}) \quad \mathcal{S}_{44}' \right] \begin{bmatrix} \mathcal{X}_d \\ \mathcal{X}_{k1} \\ \mathcal{X}_b \end{bmatrix} + \begin{bmatrix} \mathcal{S}_{25} \\ \mathcal{S}_{35} \\ \mathcal{S}_{45} \end{bmatrix} \mathcal{Y}_d = 0$$

(5.11.8)

Unknowns: $\mathcal{Y}_{k2}$, $\mathcal{X}_{k1}$ and $\mathcal{X}_b$.

Number of equations: $(v - 1 - n_{oy}) = (v - 1 - n_x)$.

Premultiplication by

$$\begin{bmatrix} \mathcal{U} & -\mathcal{J} & 0 \\ 0 & 0 & \mathcal{U} \end{bmatrix}$$

effects a substitution for $\mathcal{Y}_{k2}$.

Final equations:

$$\begin{bmatrix} (\mathcal{S}_{24} - \mathcal{J}\mathcal{S}_{34}) \\ \mathcal{S}_{44} \end{bmatrix} \mathcal{W}_B \left[\mathcal{S}_{14}' \quad (\mathcal{S}_{24}' + \mathcal{S}_{34}'\mathcal{K}) \quad \mathcal{S}_{44}' \right] \begin{bmatrix} \mathcal{X}_d \\ \mathcal{X}_{k1} \\ \mathcal{X}_b \end{bmatrix} + \begin{bmatrix} (\mathcal{S}_{25} - \mathcal{J}\mathcal{S}_{35}) \\ \mathcal{S}_{45} \end{bmatrix} \mathcal{Y}_d = 0$$

(5.11.9)

Unknowns: $\mathcal{X}_{k1}$ and $\mathcal{X}_b$.

Number of equations: $(v - 1 - n_{oy} - n_{k2})$.

Comments: For the "perfect coupler" case

$$\begin{bmatrix} (\mathcal{S}_{24} - \mathcal{J}\mathcal{S}_{34}) \\ \mathcal{S}_{44} \end{bmatrix} = \left[(\mathcal{S}_{24}' + \mathcal{S}_{34}'\mathcal{K}) \quad \mathcal{S}_{44}' \right]$$

and the matrix to be inverted is symmetrical, if χ_B is symmetrical.

5.12 Systems of A, B and JK Components--Separate Graph

A formulation problem of considerable practical importance arises frequently in electromechanical systems, where the system graph consists of at least two separate subgraphs. The terminal equations for the electrical subgraph are most likely to be explicit in the x-variables. The terminal equations for the mechanical subgraph are more likely to be explicit in the y-variables. The coupling between the two subgraphs is usually of the "direct" type. The formulation procedure shown here is an extension of one proposed by Koenig [10].

If the electrical-subgraph variables are denoted by subscript 1, and the mechanical-subgraph variables are denoted by subscript 2, then the system equations can be written as: (5.12.1)

$$\begin{bmatrix} \mathcal{U} & \mathcal{S}_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{U} & \mathcal{S}_{2A} & 0 & 0 & 0 & 0 \\ \mathcal{Z}_{11} & \mathcal{Z}_{12} & 0 & 0 & -\mathcal{U} & 0 & \mathcal{K}_{11} & \mathcal{K}_{12} \\ \mathcal{Z}_{21} & \mathcal{Z}_{22} & 0 & 0 & 0 & -\mathcal{U} & \mathcal{K}_{21} & \mathcal{K}_{22} \\ \mathcal{G}_{11} & \mathcal{G}_{12} & -\mathcal{U} & 0 & 0 & 0 & \mathcal{H}_{11} & \mathcal{H}_{12} \\ \mathcal{G}_{21} & \mathcal{G}_{22} & 0 & -\mathcal{U} & 0 & 0 & \mathcal{H}_{21} & \mathcal{H}_{22} \\ 0 & 0 & 0 & 0 & \mathcal{B}_{11} & \mathcal{U} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{B}_{23} & \mathcal{U} \end{bmatrix} \begin{bmatrix} \mathcal{Y}_{1t} \\ \mathcal{Y}_{1c} \\ \mathcal{Y}_{2b} \\ \mathcal{Y}_{2c} \\ \mathcal{X}_{1t} \\ \mathcal{X}_{1m} \\ \mathcal{X}_{2t} \\ \mathcal{X}_{2m} \end{bmatrix} + \begin{bmatrix} \mathcal{S}_{15}\mathcal{Y}_{1d} \\ \mathcal{S}_{25}\mathcal{Y}_{2d} \\ 0 \\ 0 \\ 0 \\ 0 \\ \mathcal{B}_{15}\mathcal{X}_{1d} \\ \mathcal{B}_{25}\mathcal{X}_{2d} \end{bmatrix} = 0$$

where the equations involving no-relation variables are omitted.

If the first row and the last row are substituted into the terminal equations, the terminal equations remain explicit, and hence can be substituted into the second and seventh rows.

In this case, chord formulation in used in part 1, and tree formulation in part 2. In this procedure there is no particular advantage in partitioning to show the JK-element variables distinct from those remaining. Therefore, suppose the terminal equations to take the following form:

$$\mathcal{X}_{1a} = \mathcal{Z}_a \mathcal{Y}_{1a} + \mathcal{K}_a \mathcal{X}_{2B} \qquad (5.12.2)$$

$$\mathcal{Y}_{2B} = \mathcal{W}_B \mathcal{X}_{2B} + \mathcal{J}_B \mathcal{Y}_{1a} \qquad (5.12.3)$$

In the formulation forest: D_x-elements.

In the chord set: D_y-elements.

Circuit equations, part 1:

$$\left[\begin{array}{cc|c} \mathcal{B}_{11} & \mathcal{B}_{12} & 0 \\ \hline \mathcal{B}_{21} & \mathcal{B}_{22} & \mathcal{U} \end{array}\right] \begin{bmatrix} \mathcal{X}_{1d} \\ \mathcal{X}_{1a} \\ \mathcal{X}_{1n} \end{bmatrix} = 0 \qquad (5.12.4)$$

Explicit form of the segregate equations, part 1, from (5.12.4):

$$\mathcal{Y}_{1a} = \begin{bmatrix} \mathcal{B}'_{12} & \mathcal{B}'_{22} \end{bmatrix} \begin{bmatrix} \mathcal{Y}_{1m} \\ \mathcal{Y}_{1d} \end{bmatrix} \qquad (5.12.5)$$

Segregate equations, part 2:

$$\left[\begin{array}{c:cc} \mathcal{U} & \mathcal{S}_{12} & \mathcal{S}_{13} \\ \hdashline 0 & \mathcal{S}_{22} & \mathcal{S}_{23} \end{array}\right] \begin{bmatrix} \mathcal{Y}_{2n} \\ \mathcal{Y}_{2B} \\ \mathcal{Y}_{2d} \end{bmatrix} = 0 \qquad (5.12.6)$$

Explicit form of the circuit equations, part 2, from (5.12.6):

$$\mathcal{X}_{2B} = \begin{bmatrix} \mathcal{S}'_{12} & \mathcal{S}'_{22} \end{bmatrix} \begin{bmatrix} \mathcal{X}_{2d} \\ \mathcal{X}_{2b} \end{bmatrix} \qquad (5.12.7)$$

Terminal equations after substitution of (5.12.5) in part 1, and of (5.12.7) in part 2:

$$\begin{bmatrix} \mathcal{X}_{1a} \\ \mathcal{Y}_{2B} \end{bmatrix} = \begin{bmatrix} \mathcal{Z}_a & \mathcal{K}_a \\ \mathcal{J}_B & \mathcal{H}_B \end{bmatrix} \begin{bmatrix} \mathcal{B}'_{12} & \mathcal{B}'_{22} & 0 & 0 \\ 0 & 0 & \mathcal{S}'_{12} & \mathcal{S}'_{22} \end{bmatrix} \begin{bmatrix} \mathcal{Y}_{1m} \\ \mathcal{Y}_{1d} \\ \mathcal{X}_{2d} \\ \mathcal{X}_{2b} \end{bmatrix} \qquad (5.12.8)$$

Circuit equations, part 1, and segregate equations, part 2:

$$\begin{bmatrix} \mathcal{B}_{11} \\ 0 \end{bmatrix} \mathcal{X}_{1d} + \begin{bmatrix} 0 \\ \mathcal{S}_{23} \end{bmatrix} \mathcal{Y}_{2d} + \begin{bmatrix} \mathcal{B}_{12} & 0 \\ 0 & \mathcal{S}_{22} \end{bmatrix} \begin{bmatrix} \mathcal{X}_{1a} \\ \mathcal{Y}_{2B} \end{bmatrix} = 0 \qquad (5.12.9)$$

Equation (5.12.9) after substitution of (5.12.8):

$$\begin{bmatrix} \mathcal{B}_{11} \\ 0 \end{bmatrix} \mathcal{X}_{1d} + \begin{bmatrix} 0 \\ \mathcal{S}_{23} \end{bmatrix} \mathcal{Y}_{2d} + \begin{bmatrix} \mathcal{B}_{12} \mathcal{Z}_a \mathcal{B}_{22}' & \mathcal{B}_{12} \mathcal{K}_a \mathcal{S}_{12}' \\ \mathcal{S}_{22} \mathcal{G}_B \mathcal{B}_{22}' & \mathcal{S}_{22} \mathcal{W}_B \mathcal{S}_{12}' \end{bmatrix} \begin{bmatrix} \mathcal{Y}_{1d} \\ \mathcal{X}_{2d} \end{bmatrix}$$

$$+ \begin{bmatrix} \mathcal{B}_{12} \mathcal{Z}_a \mathcal{B}_{12}' & \mathcal{B}_{12} \mathcal{K}_a \mathcal{S}_{22}' \\ \mathcal{S}_{22} \mathcal{G}_B \mathcal{B}_{12}' & \mathcal{S}_{22} \mathcal{W}_B \mathcal{S}_{22}' \end{bmatrix} \begin{bmatrix} \mathcal{Y}_{1m} \\ \mathcal{X}_{2b} \end{bmatrix} = 0$$

(5.12.10)

Unknowns: $\mathcal{Y}_{1m}$ and $\mathcal{X}_{2b}$.

Number of equations: $(e_1 - v_1 + 1 - n_{ox1}) + (v_2 - 1 - n_{oy2})$.

5.13 Systems of A and B Components

If components of both A and B types are contained in the same connected system graph, neither the conventional mesh nor branch equations can be obtained. However, if the A-elements and B-elements of the system graph are located in certain topological patterns, it is possible to obtain a set of equations which are numerically the same as either the mesh or the branch equations.

In the formulation tree: All D_x-elements, and all A-elements possible.

In the chord set: All D_y-elements, and all B-elements possible.

System equations, with equations involving N-elements omitted:

$$\begin{bmatrix} \mathcal{U} & 0 & \mathcal{D}_{13} & \mathcal{D}_{14} & 0 & 0 & 0 & 0 \\ 0 & \mathcal{U} & \mathcal{D}_{23} & \mathcal{D}_{24} & 0 & 0 & 0 & 0 \\ \mathcal{Z}_t & 0 & \mathcal{Z}_{tc} & 0 & \mathcal{U} & 0 & 0 & 0 \\ \mathcal{Z}_{ct} & 0 & \mathcal{Z}_c & 0 & 0 & 0 & \mathcal{U} & 0 \\ 0 & \mathcal{U} & 0 & 0 & 0 & \mathcal{Y}_t & 0 & \mathcal{Y}_{tc} \\ 0 & 0 & 0 & \mathcal{U} & 0 & \mathcal{Y}_{ct} & 0 & \mathcal{Y}_c \\ 0 & 0 & 0 & 0 & \mathcal{B}_{11} & \mathcal{B}_{12} & \mathcal{U} & 0 \\ 0 & 0 & 0 & 0 & \mathcal{B}_{21} & \mathcal{B}_{22} & 0 & \mathcal{U} \end{bmatrix} \begin{bmatrix} \mathcal{I}_{1t} \\ \mathcal{I}_{2t} \\ \mathcal{I}_{1c} \\ \mathcal{I}_{2c} \\ \mathcal{V}_{1t} \\ \mathcal{V}_{2t} \\ \mathcal{V}_{1c} \\ \mathcal{V}_{2c} \end{bmatrix} + \begin{bmatrix} -\mathcal{D}_{15}\mathcal{I}_{1d} \\ -\mathcal{D}_{25}\mathcal{I}_{2d} \\ 0 \\ 0 \\ 0 \\ 0 \\ \mathcal{B}_{15}\mathcal{V}_{1d} \\ \mathcal{B}_{25}\mathcal{V}_{2d} \end{bmatrix} = 0 \qquad (5.13.1)$$

Subscripts 1 and 2 denote A-element and B-element variables, respectively.

Coefficient matrix of (5.13.1) after substitution of the first row into rows 3 and 4, and substitution of row 8 into rows 5 and 6:

$$\begin{bmatrix} \mathcal{U} & 0 & \mathcal{D}_{13} & \mathcal{D}_{14} & 0 & 0 & 0 & 0 \\ 0 & \mathcal{U} & \mathcal{D}_{23} & \mathcal{D}_{24} & 0 & 0 & 0 & 0 \\ 0 & 0 & (\mathcal{Z}_{tc} - \mathcal{Z}_t\mathcal{D}_{13}) & -\mathcal{Z}_t\mathcal{D}_{14} & \mathcal{U} & 0 & 0 & 0 \\ 0 & 0 & (\mathcal{Z}_c - \mathcal{Z}_{ct}\mathcal{D}_{13}) & -\mathcal{Z}_{ct}\mathcal{D}_{14} & 0 & 0 & \mathcal{U} & 0 \\ 0 & \mathcal{U} & 0 & 0 & -\mathcal{Y}_{tc}\mathcal{B}_{21} & (\mathcal{Y}_t - \mathcal{Y}_{tc}\mathcal{B}_{22}) & 0 & 0 \\ 0 & 0 & 0 & \mathcal{U} & -\mathcal{Y}_c\mathcal{B}_{21} & (\mathcal{Y}_{ct} - \mathcal{Y}_c\mathcal{B}_{22}) & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{B}_{11} & \mathcal{B}_{12} & \mathcal{U} & 0 \\ 0 & 0 & 0 & 0 & \mathcal{B}_{21} & \mathcal{B}_{22} & 0 & \mathcal{U} \end{bmatrix} \qquad (5.13.2)$$

Coefficient matrix after substitution in (5.13.2) of row 2 into row 5, and of row 7 into row 4:

$$
\left[\begin{array}{cc|cccc|cc}
\mathcal{U} & 0 & \mathcal{D}_{13} & \mathcal{D}_{14} & 0 & 0 & 0 & 0 \\
\hline
0 & \mathcal{U} & \mathcal{D}_{23} & \mathcal{D}_{24} & 0 & 0 & 0 & 0 \\
0 & 0 & (\mathcal{Z}_{tc} - \mathcal{Z}_{t}\mathcal{D}_{13}) & -\mathcal{Z}_{t}\mathcal{D}_{14} & \mathcal{U} & 0 & 0 & 0 \\
0 & 0 & (\mathcal{Z}_{c} - \mathcal{Z}_{ct}\mathcal{D}_{13}) & -\mathcal{Z}_{ct}\mathcal{D}_{14} & -\mathcal{B}_{11} & -\mathcal{B}_{12} & 0 & 0 \\
0 & 0 & -\mathcal{D}_{23} & -\mathcal{D}_{24} & -\mathcal{Y}_{tc}\mathcal{B}_{21} & (\mathcal{Y}_{t} - \mathcal{Y}_{tc}\mathcal{B}_{22}) & 0 & 0 \\
0 & 0 & 0 & \mathcal{U} & -\mathcal{Y}_{c}\mathcal{B}_{21} & (\mathcal{Y}_{ct} - \mathcal{Y}_{c}\mathcal{B}_{22}) & 0 & 0 \\
0 & 0 & 0 & 0 & \mathcal{B}_{11} & \mathcal{B}_{12} & \mathcal{U} & 0 \\
\hline
0 & 0 & 0 & 0 & \mathcal{B}_{21} & \mathcal{B}_{22} & 0 & \mathcal{U}
\end{array}\right]
$$

(5.13.3)

At this point, either of two substitutions may be made conveniently. Which is used would depend in general on the number of simultaneous equations to solved in each case.

1. If row 3 is substituted into rows 4, 5 and 6, the equation count is $(e - v + 1 - n_{ox} + n_{t2})$, where n_{t2} is the number of B-elements in the formulation tree. The variables in the simultaneous equations are the chord y-variables and the B-element x-variables. Therefore, if B-elements can be put into some chord set with the D_y-elements, only $(e - v + 1 - n_{ox})$ equations result.

2. If row 6 is substituted into rows 3, 4 and 5, the equation count is $(v - 1 - n_{oy} + n_{1c})$, where n_{1c} is the number of A-elements in the tree complement. The variables in the simultaneous equations are the tree x-variables and the

A-element y-variables. If, therefore, the A-elements can be put into the formulation tree with the D_x-elements, only $(v - 1 - n_{oy})$ equations result.

VI. SYSTEM GRAPHS FOR WHICH EACH PART INVOLVES EITHER ACROSS OR SERIES VARIABLES ONLY

6.1 System Graph of One Part

Consider first a system graph, with which is associated only x-variables. The number of unknowns is $(e - n_x)$, therefore there must be $(e - n_x)$ independent equations, if a unique solution exists. If the D_x-elements form a subset of some tree, the independent circuit equations number $(e - v + 1)$. The additional $(v - 1 - n_x)$ equations must be K terminal equations. Thus n_{k2} must be exactly $(v - 1 - n_x)$. From this, it follows that $n_{k1} = (e - v + 1 - n_{ox})$. This is so restrictive as to make it a trivial case. For the one-part graph involving only y-variables, a similar situation exists. Thus, because of a lack of practical use, the one-part graph does not seem to merit further consideration.

The following formulation procedures are written for a two-part system graph. They apply to a system graph of more than two parts if the term "part" is replaced by "set of parts."

6.2 System Graph of Two Parts--Across Variables

In the formulation forest: The D_x-elements.

In the chord set: N_x-elements.

System equations:

$$\begin{bmatrix} \mathcal{K}_1 & \mathcal{K}_2 & \mathcal{K}_3 & \mathcal{K}_4 & 0 & 0 \\ \mathcal{B}_{11} & \mathcal{U} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{B}_{23} & \mathcal{U} & 0 & 0 \\ \mathcal{B}_{31} & \mathcal{B}_{32} & 0 & 0 & \mathcal{U} & 0 \\ 0 & 0 & \mathcal{B}_{43} & \mathcal{B}_{44} & 0 & \mathcal{U} \end{bmatrix} \begin{bmatrix} \mathcal{X}_{t1} \\ \mathcal{X}_{m1} \\ \mathcal{X}_{t2} \\ \mathcal{X}_{m2} \\ \mathcal{X}_{n1} \\ \mathcal{X}_{n2} \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{B}_{17}\mathcal{X}_{d1} \\ \mathcal{B}_{27}\mathcal{X}_{d2} \\ \mathcal{B}_{37}\mathcal{X}_{d1} \\ \mathcal{B}_{47}\mathcal{X}_{d2} \end{bmatrix} = 0$$

(6.2.1)

The last two rows may be set aside. The terminal equations number $(v - 2 - n_x)$, since the system equations must number $(e - n_x)$. Whether they are explicit or not, the equations to be solved simultaneously can be reduced to $(v - 2 - n_x)$ by substituting the circuit equations into them, yielding

$$(\mathcal{K}_1 - \mathcal{K}_2\mathcal{B}_{11})\mathcal{X}_{t1} + (\mathcal{K}_3 - \mathcal{K}_4\mathcal{B}_{23})\mathcal{X}_{t2} + \mathcal{K}_2\mathcal{B}_{17}\mathcal{X}_{d1} + \mathcal{K}_4\mathcal{B}_{27}\mathcal{X}_{d2} = 0$$

(6.2.2)

If the variables of part 1 can be expressed explicitly in terms of the variables in part 2, then a different procedure may yield fewer equations. Suppose the terminal equations to have the form

$$\begin{bmatrix} \mathcal{U} & 0 & \mathcal{K}_{13} & \mathcal{K}_{14} \\ 0 & \mathcal{U} & \mathcal{K}_{23} & \mathcal{K}_{24} \end{bmatrix} \begin{bmatrix} \mathcal{X}_{t1} \\ \mathcal{X}_{m1} \\ \mathcal{X}_{t2} \\ \mathcal{X}_{m2} \end{bmatrix} = 0 \qquad (6.2.3)$$

Equation (6.2.3)--after the circuit equations of part 2 have been substituted in--together with the circuit equations of part 1:

$$\begin{bmatrix} \mathcal{U} & 0 & (\mathcal{K}_{13}-\mathcal{K}_{14}\mathcal{B}_{23}) \\ 0 & \mathcal{U} & (\mathcal{K}_{23}-\mathcal{K}_{24}\mathcal{B}_{23}) \\ \mathcal{B}_{11} & \mathcal{U} & 0 \end{bmatrix} \begin{bmatrix} \mathcal{X}_{t1} \\ \mathcal{X}_{m1} \\ \mathcal{X}_{t2} \end{bmatrix} = \begin{bmatrix} \mathcal{K}_{14}\mathcal{B}_{27}\mathcal{X}_{d2} \\ \mathcal{K}_{24}\mathcal{B}_{27}\mathcal{X}_{d2} \\ \mathcal{B}_{14}\mathcal{X}_{d1} \end{bmatrix} \qquad (6.2.4)$$

The first two rows of (6.2.4) may be substituted into the third row. The equations, to be solved simultaneously, number $(v_2 - 1 - n_{x2})$--the number of K-elements associated with the formulation tree in part 2.

For those systems in which the terminal equations may be expressed explicitly for the variables of either part 1 or part 2, the reduced set of equations may relate the tree variables of either part.

This formulation procedure is readily extended to system graphs of more than two parts.

6.3 System Graph of Two Parts--Series Variables

In the formulation forest: The N_y-elements.

In the chord set: The D_y-elements.

System equations:

$$\left[\begin{array}{cc|cccc} \mathcal{U} & 0 & \mathcal{S}_{13} & \mathcal{S}_{14} & 0 & 0 \\ 0 & \mathcal{U} & 0 & 0 & \mathcal{S}_{25} & \mathcal{S}_{26} \\ \hline 0 & 0 & \mathcal{U} & \mathcal{S}_{34} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{U} & \mathcal{S}_{46} \\ 0 & 0 & \mathcal{G}_1 & \mathcal{G}_2 & \mathcal{G}_3 & \mathcal{G}_4 \end{array}\right] \left[\begin{array}{c} \mathcal{Y}_{n1} \\ \mathcal{Y}_{n2} \\ \hline \mathcal{Y}_{b1} \\ \mathcal{Y}_{c1} \\ \mathcal{Y}_{b2} \\ \mathcal{Y}_{c2} \end{array}\right] + \left[\begin{array}{c} \mathcal{S}_{17}\mathcal{Y}_{d1} \\ \mathcal{S}_{27}\mathcal{Y}_{d2} \\ \hline \mathcal{S}_{37}\mathcal{Y}_{d1} \\ \mathcal{S}_{47}\mathcal{Y}_{d2} \\ 0 \end{array}\right] = 0 \qquad (6.3.1)$$

The first two rows may be set aside. The terminal equations number $(e - v + 2 - n_y)$, since the system equations must number $(e - n_y)$. The substitution procedures follow the same pattern as those discussed for a system involving x-variables only. The number of equations, and the variables involved, for three conditions on the terminal equations, are shown below.

1. Terminal equations not explicit:
   Variables: $\mathcal{Y}_{c1}$ and $\mathcal{Y}_{c2}$.
   Number of equations: $(e - v + 2 - n_y)$.

2. Terminal equations explicit for variables of part 1:
   Variables: $\mathcal{Y}_{c2}$.
   Number of equations: $(e_2 - v_2 + 1 - n_{y2})$.

3. Terminal equations explicit for variables of part 2:
   Variables: $\mathcal{Y}_{c1}$.
   Number of equations: $(e_1 - v_1 + 1 - n_{y1})$.

The two-dimensional static pin-type rigid truss, or bridge, is a type of system which has a system graph and terminal equations of this type. For this case, moreover, the terminal equations can be written explicitly for the variables of one part as easily as for the other. Therefore, a choice of either type of formulation is always possible.

6.4 System Graph of Two Parts--Across and Series Variables

Part 1: Series variables:

In the formulation tree: The N_y-elements.

In the chord set: The D_y-elements.

Part 2: Across variables:

In the formulation tree: The D_x-elements.

In the chord set: The N_x-elements.

System equations:

$$\begin{bmatrix} \mathcal{U} & 0 & \mathcal{S}_{13} & 0 & 0 & 0 \\ 0 & \mathcal{U} & \mathcal{S}_{23} & 0 & 0 & 0 \\ 0 & \mathcal{A}_1 & \mathcal{A}_2 & \mathcal{A}_3 & \mathcal{A}_4 & 0 \\ 0 & 0 & 0 & \mathcal{B}_{11} & \mathcal{U} & 0 \\ 0 & 0 & 0 & \mathcal{B}_{21} & 0 & \mathcal{U} \end{bmatrix} \begin{bmatrix} y_{n1} \\ y_{b1} \\ y_{c1} \\ x_{t2} \\ x_{m2} \\ x_{n2} \end{bmatrix} \begin{bmatrix} \mathcal{S}_{14} y_{d1} \\ \mathcal{S}_{24} y_{d1} \\ 0 \\ \mathcal{B}_{14} x_{d2} \\ \mathcal{B}_{24} x_{d2} \end{bmatrix} = 0 \qquad (6.4.1)$$

Equation (6.4.1) has the same form as (4.4.1). The difference between them lies in the fact that here the segregate equations and circuit equations are written for different parts of the system graph. Since the unknown variables number $(e - n_x - n_y)$, and the circuit and segregate equations together number $(v_1 + e_2 - v_2)$, there must be $(e_1 - v_1 + v_2 - n_x - n_y)$ terminal equations.

If the terminal equations are explicit for the variables of either part, a reduction procedure similar to that utilized in the chord or tree formulations can be carried out. The number of equations, and the variables involved, for three conditions on the terminal equations, are shown below:

1. Terminal equations not explicit:
   Variables: $\mathcal{Y}_{c1}$ and $\mathcal{X}_{t2}$.
   Number of equations: $(e_1 - v_1 + 1 - n_y + v_2 - 1 - n_x)$.

2. Terminal equations explicit for variables of part 1:
   Variables: $\mathcal{X}_{t2}$.
   Number of equations: $(v_2 - 1 - n_x)$.

3. Terminal equations explicit for variables of part 2:
   Variables: $\mathcal{Y}_{c1}$.
   Number of equations: $(e_1 - v_1 + 1 - n_y)$.

VII. CONCLUSION

The usefulness of the linear graph in system analysis stems, essentially, from the fact that a convenient set of equations for a system can be written by inspection from the system graph. The conditions for their independence are precisely known. The explicitness of the equations is useful in a general reduction procedure.

Given the system graph and component equations for a physical system, the problem of analysis is influenced only by the mathematical forms of the equations. By divorcing the equation formulation from the particular physical system, it becomes obvious that certain mathematical procedures apply equally well to a very large class of systems. This fact is particularly important, when systems include components of mixed types--electrical, thermal, hydraulic, etc.

In a system containing direct-coupled components, independence of the circuit and segregate equations, together with the direct-coupled terminal equations, is assured if the component subgraphs are located in certain topological patterns in the system graph. The conventional node and mesh equations may not be possible, but often a still smaller set of equations is obtainable without inversion of matrices.

A set of formulation procedures for particular systems of special interest are presented. However, system equations may contain, in almost any combination, component equations of all types discussed, and possibly some not explicitly discussed. Therefore, the notions contained in the discussion on a general formulation procedure are probably of more importance for application to systems which do not fit the particular patterns discussed.

An important aspect of this viewpoint on system analysis is the educational implications. Given a workable understanding of linear graph theory, and practice in the reduction of equations on an abstract basis, a student would have the necessary groundwork built for applications in particular fields. If this sequence were followed, courses in particular fields might well stress: (1) how the component equations are obtained, and (2) the properties of certain useful systems made up of the components. The unification of method, in this type of sequence, should allow of a great saving in time, otherwise devoted to particular methods of analysis.

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