INFERIOR INPUTS AND EXTERNAL EFFECTS

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ABSTRACT

INFERIOR INPUTS AND EXTERNAL EFFECTS

by Roger Blair

This thesis explores the implications of inferior inputs on the solutions to the resource misallocation problem created by external effects. To this end, the concepts of external effects and inferior inputs are systematically developed in separate chapters. Then a situation is hypothesized in which an inferior input is the cause of an external diseconomy. The solutions for removal of the Pareto relevance of this external diseconomy are subsequently analyzed. This analysis reveals that the solutions are unaffected by the influence of input inferiority.

INFERIOR INPUTS AND EXTERNAL EFFECTS

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CHAPTER I

INTRODUCTION

Although the early discussions of external effects were filled with errors, inconsistencies, and semantic difficulties, as with most theoretical concepts, a clearer picture finally emerged. In 1959, F. M. Bator[1] cleared away most of the rubble and summarized the theory that had been developed. Since that time, there has appeared a host of articles on the subject. Some of these have represented attempts to define externalities more precisely. Some discussed various solutions to the resource misallocation problem caused by external effects, i.e., the achievement of Pareto optimality. Still others have introduced further complications into this resource misallocation problem. This dissertation will also introduce a further complication: the influence of inferior inputs.

While the notion of an inferior good, i.e., one whose consumption decreases as an individual's income increases, is certainly not new, the analogous concept of an inferior input was not clearly defined until D. V. T. Bear's

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[1] F. M. Bator, "Anatomy of Market Failure," Quarterly Journal of Economics, LXXII (1959), 351-79.

article[2] appeared. There have been but two subsequent developments of this concept. While Charles Plott's contribution[3] was not very explicit, that of C. E. Ferguson[4] was.

I shall combine this concept with that of external effects by hypothesizing a case where one firm's use of an inferior input causes another firm to suffer an external diseconomy. The solutions that will "correct" the externality problem, i.e., that make the attainment of Pareto optimality possible, will then be analyzed. The purpose of this analysis is to determine whether these solutions still apply in this special case.

To accomplish this, I shall carefully develop the definition and effects of externalities along with their solutions in Chapter II. The concept of inferior inputs and some of their consequences will be developed in Chapter III. After these careful surveys we shall have all the tools at hand to complete the investigation. This will be done in the concluding Chapter IV.

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[2] D. V. T. Bear, "Inferior Inputs and the Theory of the Firm," Journal of Political Economy, LXXIII (1965), 287-89.

[3] Charles Plott, "Externalities and Corrective Taxes," Economica, XXXIII (1966), 84-87.

[4] C. E. Ferguson, "'Inferior Factors' and the Theories of Production and Input Demand," Economica, to appear May 1968.

CHAPTER II

THE CONCEPT OF EXTERNALITIES

1. The Definition of Externality

Mathematically, welfare maximization involves the solution of a constrained extremum problem. Solving this problem by Lagrange methods yields a set of first-order optimality conditions that include the familiar Lagrange multipliers. These multipliers represent the costs of the constraints. Thus, as Davis and Whinston note,[1] they also represent the implied costs of the constraints on technology and the market prices of factors and commodities. Under the usual assumptions concerning tastes, technology, and profit maximization, the equilibrium quantities of inputs and commodities that result from pure competitors responding to these prices will satisfy the conditions for Pareto efficiency. In other words, pure competition will put society on its "bliss" frontier. If the competitively imputed incomes could be redistributed without cost in some lump-sum

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[1] O. A. Davis and A. B. Whinston, "Welfare Economics and the Theory of Second Best," The Review of Economic Studies, XXXII (1965), p. 4.

way to achieve the "correct" income distribution, the social welfare function could then be maximized.

Achieving the "bliss" frontier can be thwarted by many real world phenomena: imperfect information, inertia, non-profit maximizing behavior, risk and uncertainty, etc. But these foils can be ignored in this discussion as "they have to do with the efficiency of 'real life' people in a nonstationary world of uncertainty, miscalculation, etc."[2] This thesis is concerned with the phenomena that disrupt Pareto-efficient resource allocation under the assumptions of individual profit- and utility-maximization in a stationary world. These phenomena are labeled externalities. The discussion of these externalities will be largely in partial equilibrium terms; therefore, it must be understood that the rest of the economy is, and remains, organized so that the Pareto optimum conditions are fulfilled. As will be emphasized later, this does not involve a second-best situation.

As Bator pointed out,[3] the modern formulation of externalities is embedded in the idea of direct interaction. This interaction is a result of nonindependence of some utility and/or production functions, i.e., the nonindependence may be between producers, between consumers, or between

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[2] F. M. Bator, "The Anatomy of Market Failure," Quarterly Journal of Economics, LXXII (1958), p. 352.

[3] F. M. Bator, "Market Failure," p. 358.

producers and consumers. Such nonindependence causes some Paretian costs and benefits to be omitted from the decentralized, private cost-revenue calculations. In other words, it causes social costs and social benefits to diverge. Although Bator gives an example of this nonindependence concept of externality, his verbal definition is not wholly satisfactory. Buchanan and Stubblebine,[4] however, have developed a precise set of definitions formulated in utility terms. These have been transformed into productivity terms because this thesis is concerned with production.

Assume that there are two firms, i and j, with the following twice differentiable production functions:

$$Q_j = G(x_{j1}, x_{j2}, \ldots, x_{jn}) \tag{1.1}$$

and

$$Q_i = F(x_{i1}, x_{i2}, \ldots, x_{in}, z) \tag{1.2}$$

where x_{jk} and x_{ik} are the amounts of input k used by firms j and i respectively. We also assume that $\partial G/\partial x_{jk} > 0$ over the range considered as is $\partial F/\partial x_{ik}$ where $k = 1,2,\ldots,n$. The element z is defined as

$$z = g(x_{j1}), \tag{1.3}$$

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[4] J. M. Buchanan and W. C. Stubblebine, "Externality," Economica, N.S. XXIX (1962), 371-384. The concepts developed in equations (1.1) through (1.12) are simply re-statements in productivity terms of their concepts.

i.e., j's use of input x_{j1} yields some output z that enters i's production function. To avoid confusion, I must make it quite clear that z is some accidental by-product or condition that j creates. We must emphasize that z has no market price. It is not the sort of by-product which is commonly sold. To clarify this point, consider the classic case of air pollution caused by a factory's operation where z represents the smoke emitted, g represents the burning process, and x_{j1} represents the fuel. Thus equation (1.2) implies that the output of i is a function of the inputs under its control and an output of j that is directly related to j's use of input x_{j1}. This condition constitutes the presence of an externality.

Input x_{j1} was chosen as the externality-causing input merely for expositional convenience. Moreover, i's production function could be written to include other inputs under j's control; but without loss of generality, we may direct our attention to the effects of the single input x_{j1}. In addition, since it serves no useful purpose to consider mutual, or reciprocal, externalities when discussing definitions, we assumed that j's production includes only inputs under its own control. One further comment requires mention: as a matter of notation we let $\partial F/\partial x_{ik} = \partial F(x_{i1}, x_{i2}, \ldots, x_{in}, z)/\partial x_{ik}$, i.e., $\partial F/\partial x_{ik}$ is, in general, a function of all the x_i's and z. The abbreviated form will be used throughout, but its precise meaning should not be forgotten.

Since i has no control over the level of $z = g(x_{j1})$, z enters i's production function parametrically. Thus, we assume that i attempts to maximize profits in the usual way, subject to given values of z. Whether i's optimum output must be modified to account for various values of z is a question considered later.

Assuming i and j maximize profits independently and the product and factor markets are competitive, the problems are to

$$\text{Max } \pi_i = P_i \; F(x_{i1}, x_{i2}, \ldots, x_{in}, z) - \sum_{k=1}^{n} p_k \cdot x_{ik} \qquad (1.4)$$

and

$$\text{Max } \pi_j = P_j \; G(x_{j1}, x_{j2}, \ldots, x_{jn}) - \sum_{k=1}^{n} p_k \cdot x_{jk} \qquad (1.5)$$

where π is profit, P_i and P_j are the prices of the products i and j produce, p_k is the price of input k, and the x_{ik} and x_{jk} are the quantities of the inputs used.

The first-order conditions for profit maximization are

$$P_i \cdot \partial F/\partial x_{ik} - p_k = 0 \quad \text{if } x_{ik} > 0 \qquad (k = 1,2,\ldots,n) \qquad (1.6)$$

and

$$P_j \cdot \partial G/\partial x_{jk} - p_k = 0 \quad \text{if } x_{jk} > 0. \qquad (k = 1,2,\ldots,n) \qquad (1.7)$$

These are the usual Pareto conditions: in equilibrium the value of the marginal product of each input must equal its parametrically given market price. But i cannot take account of z because there is no price attached to it even though it enters i's production function. In all cases where

$$\partial F/\partial z = \partial F/\partial g(x_{j1}) \cdot dg(x_{j1})/dx_{j1} \neq 0, \qquad (1.8)$$

there exists a <u>marginal externality</u>. This concept can be used to define explicitly external economies and diseconomies. A <u>marginal external diseconomy</u> exists when

$$\partial F/\partial z < 0,^{5} \qquad (1.8a)$$

i.e., a small change in the quantity of x_{j1} used by j will change z, which in turn will change the output level of i in the opposite direction. Similarly, a <u>marginal external economy</u> exists when

$$\partial F/\partial z > 0. \qquad (1.8b)$$

An <u>infra-marginal externality</u> exists at all points where

$$\partial F/\partial z = 0 \text{ and equation (1.2) holds.} \qquad (1.9)$$

An <u>infra-marginal external diseconomy</u> exists when, for any given set of values of $x_{i1}, x_{i2}, \ldots, x_{in}$,

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[5] Although I shall use $\partial F/\partial z$ to represent the longer expression $\partial F/\partial g(x_{j1}) \cdot dg(x_{j1})/dx_{j1}$, its precise meaning should not be forgotten.

$$\partial F/\partial z = 0, \int_0^Z \partial F/\partial z \, dz < 0, \text{ and equation (1.2) holds.} \quad (1.9a)$$

This means that although small changes in z do not affect the total output of i, the total effect of j's use of x_{j1} is to decrease i's output. Analogously, an infra-marginal external economy exists when, for any given set of values of

$x_{i1}, x_{i2}, \ldots, x_{in}$,

$$\partial F/\partial z = 0, \int_0^Z \partial F/\partial z \, dz > 0, \text{ and equation (1.2) holds.} \quad (1.9b)$$

Now small changes in $z = g(x_{j1})$ do not change i's total output, but the total effect of j's using x_{j1} increases i's output.[6]

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[6]The meaning of an infra-marginal externality can be clarified by considering an example. Let i's production function be

$$Q_i = (a^2-(x-1)^2 - (z-1)^2)^{1/2} \text{ where } a > 0.$$

Then the first-order conditions require

$$\partial Q_i/\partial x = (1-x)/Q_i = 0$$

and

$$\partial Q_i/\partial z = (1-z)/Q_i = 0.$$

This implies that an extremum is found where x = 1 and z = 1. Since

$$\partial^2 Q_i/\partial x^2 = - a^2/a^3 < 0,$$

$$\partial^2 Q_i/\partial z^2 = - a^2/a^3 < 0,$$

and

$\partial^2 Q_i/\partial x \partial z = 0$ at the point where x = 1 and z = 1, then

$$(\partial^2 Q_i/\partial x^2)\ (\partial^2 Q_i/\partial z^2) > (\partial^2 Q_i/\partial x \partial z)^2.$$

Therefore, the necessary and sufficient conditions for a maximum of Q_i are fulfilled. It is clear that infinitesimal changes in z will not change i's output. But evaluation of

The classifications so far introduced resulted from evaluating the partial derivatives of i's production function with respect to z over the whole range of z. Further concepts of relevance and irrelevance require considering the extent to which the externality-causing factor is used by the firm that has control over it, i.e., j. For an externality to be potentially relevant, the use of the externality-causing factor must create a desire on the part of i to change j's level of use. If an externality creates no such desire, it may be termed irrelevant. Formally, a potentially relevant marginal external diseconomy exists when

$$\partial F/\partial z < 0. \qquad (1.10)$$

In this case, i would like j to decrease its use of x_{j1} because that would decrease z and, consequently, increase i's output. Similarly, a potentially relevant marginal external economy exists when

$$\partial F/\partial z > 0. \qquad (1.11)$$

$\int_0^1 \partial F/\partial z \, dz$ will show that j's use of x_{j1} does have an effect on i's output:

$$\int_0^1 (1-z)/Q_i \, dz = (a^2-(x-1)^2)^{1/2} - (a^2-(x-1)^2-1)^{1/2} > 0.$$

Thus we have an example of an infra-marginal external economy. The quantity of z would not normally be equal to one if j ignored its effect on i. Hence we must suppose that either j made a mistake that resulted in z = 1 or that it was done purposely to increase social welfare. If the latter is the case, j was misguided because, as we shall see later, increasing z until $\partial F/\partial z = 0$ will not, in general, maximize social welfare.

Firm i would like j to increase its use of x_{j1} for analogous reasons.

The concepts of relevance and irrelevance may also be applied to infra-marginal externalities. Infra-marginal externalities are clearly irrelevant for small changes in the quantity of z. But when discrete changes are introduced, i will want to alter the quantity j employs in all cases except when

$$\begin{aligned} &\partial F/\partial z = 0 \text{ and} \\ &F(x_{i1}, x_{i2}, \ldots, x_{in}, \overline{x}_{j1}) \geq F(x_{i1}, x_{i2}, \ldots, x_{in}, x_{j1}) \end{aligned} \tag{1.12}$$

for all $x_{j1} \neq \overline{x}_{j1}$ where $\overline{x}_{j1}$ is the equilibrium quantity of x_{j1}. When equality holds in (1.12), i is getting the most "good" or the least "bad" from j's use of input x_{j1}.

Although potential relevance depends upon i's desire to alter j's behavior, this does not imply that it is possible to do so. But <u>Pareto relevance</u> of an externality does depend upon this possibility. Specifically, an externality is Pareto relevant when the quantity of z can be changed such that i is better off without making j worse off. In other words, if the externality is Pareto relevant, there are mutual benefits available. More formally, a marginal externality is Pareto relevant whenever

$$\left| P_i \cdot \partial F/\partial z \right| > \left| P_j \cdot \partial G/\partial x_{j1} - p_1 \right| . \tag{1.13}$$

This means that for i to be in a position to alter the quantity of x_{j1} used, the value of the effect on i's operations must exceed the value of the benefit j receives less the cost of purchasing the input, i.e., the net increased benefit to i must exceed the net cost to j consequent upon j's reducing x_{j1} from the present employment level. This difference is available to i and j; so a change can be made that will make at least one better off without making the other worse off.

From equation (1.7), if j is maximizing profits, $P_j \cdot \partial G/\partial x_{j1} = p_1$ when $x_{j1} = \bar{x}_{j1}$. Clearly then, when j is maximizing profits, a potentially relevant marginal externality must also be Pareto relevant because the right-hand side of inequality (1.13) vanishes. Thus there must be room for mutual benefits, i.e., both i and j can gain from some adjustment on j's part.

From condition (1.13), it follows that for the production sector the condition for <u>Pareto equilibrium when externalities are present</u> is

$$\left|P_i \cdot \partial F/\partial z\right| = \left|P_j \cdot \partial G/\partial x_{j1} - p_1\right|. \tag{1.14}$$

An extremely important implication of equation (1.14) is that Pareto equilibrium does <u>not</u> require the removal of the externality.[7] The opportunity for mutual benefit, however, is removed, i.e., the marginal externality is no longer Pareto relevant; the interests of the two firms are exactly offsetting.

Instead of dealing with only two firms, we can also include situations where j's action affects a group of other firms. This modification really does not change anything except the conditions for Pareto relevance and Pareto equilibrium. In this case, Pareto relevance requires

$$\left|\sum_{i=1}^{s} P_i \cdot \partial F^i/\partial z\right| > \left|P_j \cdot \partial G/\partial x_{j1} - p_1\right| \tag{1.15}$$

where F^i is the production function of the i-th firm. Of course, we are still assuming that all firms are pure competitors in both the product and factor markets.

Again, if j is in equilibrium, all marginal externalities must be Pareto relevant since the right-hand side of inequality (1.15) will vanish. Under these circumstances, the condition for Pareto equilibrium is

$$\left|\sum_{1}^{s} P_i \cdot \partial F^i/\partial_z\right| = \left|P_j \cdot \partial G/\partial x_{j1} - p_1\right|. \tag{1.16}$$

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[7] This demonstrates that the policy-maker cannot merely focus on the existence of an externality. He must determine whether it is Pareto relevant before he can make any decision.

The same comments that applied to equations (1.13) and (1.14) apply to these conditions.

An additional point worth emphasizing is the joint-supply nature of externalities.[8] As noted previously, z is supplied to i without charge. The reason no price is attached to z is that z is an accidental by-product of producing Q_j. In many cases, by-products are sold by the firm that produces them; but in this instance, such is not the case because independent operation precludes j from knowing of its effect on i.

Since joint-supply characterizes j's operation, its production function should be re-stated:

$$Q_j' = Q_j + z = G(x_{j1}, x_{j2}, \ldots, x_{jn}). \tag{1.17}$$

In general, j could produce Q_j independently; but when joint-supply occurs with a zero price for z, we may assume that this is simply because it is more efficient than separate supply. To demonstrate this, define j's alternative cost functions as functions of output:

$$C_1 = h_1(Q_j') \quad \text{where } Q_j' = Q_j + z, \tag{1.18}$$

or

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[8]This relation was pointed out by J. M. Buchanan in "Joint Supply, Externality and Optimality," Economica, N.S. XXXIII (1966), 404-415. I have merely adapted his developments for my purposes.

$$C_2 = h_2(Q_j) \quad \text{and} \quad C_3 = h_3(z). \tag{1.19}$$

Then the condition for the efficiency of joint-supply is given by

$$\partial C_1/\partial Q_j' < \partial C_2/\partial Q_j + \partial C_3/\partial z, \tag{1.20}$$

i.e., the marginal cost of producing Q_j and z together is less than the sum of the marginal costs of producing them separately. Since equation (1.20) is not inconsistent with

$$\partial C_1/\partial Q_j' > \partial C_2/\partial Q_j, \tag{1.21}$$

z will be supplied without charge if, and only if, j finds it more efficient to produce Q_j' than Q_j, i.e., when

$$\partial C_1/\partial Q_j' \leq \partial C_2/\partial Q_j. \tag{1.22}$$

This follows from the necessary marginal conditions for exchange equilibrium under joint-supply:

$$\partial C_1/\partial Q_j' = P_j + P_z. \tag{1.23}$$

Clearly, when $P_z = 0$ and inequality (1.21) holds, there will be no joint-supply. In addition, we may note that any situation satisfying condition (1.22) automatically satisfied inequality (1.20). That is to say, the existence of an externality implies joint-supply. But we should also note that joint-supply does not necessarily imply the existence of an externality because joint-supply may exist wh

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when (1.20) holds and the price of z is non-zero, even if (1.21) also holds.

Finally, it is worthy of mention that when j supplies z to i without charge, input x_{j1} becomes collective in a sense.[9] When j buys and employs input x_{j1}, the output z that is directly related to x_{j1} becomes available to i. Of course, in the case of externalities, the factor z is imposed upon i. This is analogous to some of the standard examples of collective goods, e.g., society decides it wants a certain amount of National Defense and whether or not I want any of it I am forced to consume it. The equilibrium condition for a collective input is the same as that which results from joint-profit maximization, viz.,

$$P_i \cdot \partial F/\partial z + P_j \cdot \partial G/\partial x_{j1} = p_1. \qquad (1.24)$$

This condition merely states that the sum of the values of marginal products must equal the price of the input. Inspection of equation (1.14), the condition for Pareto equilibrium when externalities are present, reveals that these two conditions are the same.

These definitions are not strictly in accord with all the literature. For example, Bator would find them too

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[9]The collective nature of such a factor was mentioned by Charles Plott in "Externalities and Corrective Taxes," Economica, N.S. XXXIII (1966), 86.

restricted.[10] He defines externality in a much broader sense as the existence of any phenomenon that precludes decentralized pricing from sustaining Pareto optimal outputs. While this discussion is limited to his first type, ownership externalities, Bator also includes technical externalities and public good externalities. Technical externalities are a consequence of indivisibilities or smoothly increasing returns to scale, which cause non-convexity of the set of feasible input-output points. The result is the natural monopoly case where decentralized competitive pricing cannot sustain Pareto optimal outputs because perpetual losses would be incurred. On the other hand, public goods supposedly preclude the existence of a set of prices associated with the point of maximum social welfare that would sustain the Pareto optimal output configuration, i.e., the exclusion principle fails to be operative. We have seen that this concept is not wholly at variance with our own. In fact, there must be some element of "publicness" in any instance of direct interaction in production because of the externality-causing input's collective nature.

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[10] F. M. Bator, "Market Failure."

2. The Effect of an Externality

In the absence of externalities, the transformation relation between inputs and outputs for society may be given in implicit form as

$$T(Y_1, Y_2, \ldots, Y_n; X_1, X_2, \ldots, X_m) = 0, \qquad (2.1)$$

where the Y_i represent the total amounts of society's n outputs at full employment and the X_j represent the total amounts of society's m inputs. This transformation function is a surface in n-space that shows the maximum amount of any one Y_k given the values of the other Y_i's and the X_j's. Thus to increase Y_1, for example, we must decrease some other output, or outputs, if we hold constant the amounts of inputs employed.[11]

Let us now introduce the externality discussed in the previous section, viz., the one specified in equation (1.2). The externality will change the transformation function to

$$T^*(Y_1, Y_2, \ldots, Y_n, Y_{n+1}; X_1, X_2, \ldots, X_m) = 0, \qquad (2.2)$$

where Y_{n+1} is the output z. Assuming that i produces only Y_1, that the externality is a diseconomy, and that its effects are confined to i, the transformation surface will

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[11]See P. Samuelson, Foundations of Economic Analysis (Cambridge: Harvard University Press, 1963), p. 230.

be lowered in the Y_1-direction. In the case of an external economy, it will be raised in the Y_1-direction. Thus when there is a change in the technical relation between inputs and outputs, our frame of reference concerning Pareto efficiency is different.

After introducing the externality, Pareto efficiency in production requires that society operate on surface T*; and we can ignore surface T as it is no longer relevant. When the condition for Pareto equilibrium is fulfilled, we shall be on surface T* even though a marginal externality exists; but if a Pareto relevant marginal externality exists, we shall be operating below the surface T*. Clearly, resources are not allocated properly when there exists a Pareto relevant marginal externality. It should be stressed that whatever the optimal adjustment is for an external diseconomy, its existence implies that society is worse off than it would be without the diseconomy in the sense of there being less output.[12]

We should also note that introducing an externality does not also introduce a second-best situation. The theory of the second-best involves cases where society is operating below the relevant transformation surface and for some

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[12]The effect of an externality on the production possibility surface was noted by E. J. Mishan in "Reflections on Recent Developments in the Concept of External Effects," The Canadian Journal of Economics and Political Science, XXXI (1965), 105, 113, 114.

reason cannot remove the impediment to Pareto optimality. Satisfaction of equation (1.14), the condition for Pareto equilibrium, implies that some method has been found for removing the obstacle from society's path to Pareto efficiency; society is doing the best that it can. Simply because surface T* is not the same as surface T does not mean that we are in the foggy realm of the second-best.

What it does mean is that there has been a change in the technical conditions underlying production. With such a change, it no longer makes any sense to talk about Pareto optimality in the absence of externalities because the marginal externality does not disappear in equilibrium as equation (1.14) shows. The externality-free transformation surface T is no longer relevant. Society must live with the technically feasible surface T*. On the other hand, if, for some reason, the Pareto relevance of the marginal externality cannot be removed, then society will operate below surface T*, and we have a second-best problem.

3. Solutions for the Externality Problem

We know that an efficient allocation of resources requires operation on the transformation surface T*. What prevents the attainment of surface T* is the Pareto relevance of the externality caused by z. Any solution to this problem must involve removing the Pareto relevance found in equation (1.13), i.e., a solution must result in Pareto equilibrium. As the previous sections pointed out, this does not require removal of the externality. In fact, we must live with it. But this does not mean that simply ignoring the externality constitutes a solution because that would not lead us to a Pareto optimum. Such a second-best approach is appropriate only when the costs of achieving efficiency exceed the gains.[13]

Assume that the problem is defined by equations (1.1), (1.2), (1.8a), and (1.13), i.e., there exists an external diseconomy and independent profit maximization demonstrates that it is Pareto relevant. We can discuss the following types of solutions: bargaining, taxes and subsidies, and mergers.

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[13] I might point out here that we can avoid introducing the costs of adjusting to the presence of the externality only if we assume that the pricing system works smoothly, i.e., without cost. This assumption is not strictly legitimate as it is clear that some resources must be expended in making the adjustments. If the costs of adjustment vary with output, we must add them to the social marginal cost. Their effect will be to reduce optimal output further. On the other hand, if they are lump sum, the decrease in social loss from removing the Pareto relevance of the external diseconomy must exceed the lump sum cost for the adjustment to be worthwhile. (On the costs of solution, see Mishan, "Reflections," p. 111, and R. H. Coase, "The Problem of Social Cost," Journal of Law and Economics, October 1960, pp. 2, 15-19.) In the subsequent discussion of external diseconomies we shall assume that the costs are lump sum and that the adjustment is worthwhile. Further, the discussion will primarily deal with external diseconomies as the treatment of external economies is quite symmetrical.

Bargaining

One approach to solving the problem is through direct bargaining between i and j. Three comments on bargaining are in order: first, bargaining is most feasible when the number of firms involved is not too large. In our case of two firms this presents no difficulty, but the results of this analysis cannot be taken to apply in all cases without recognizing the problems inherent in large-group decisions. Second, the form that the bargaining process will take depends upon the property rights defined exogenously by law, i.e., the direction of payment depends upon who is liable to whom. We will take the law as given and discuss the problem around it. Third, throughout this entire thesis, I shall exclusively deal with the production sector. Because bargaining involves confrontations of producers, we must assume that these discussions concerning the employment of resources in no way affects the markets for final output.

Since i is suffering from an external diseconomy imposed upon it by j, from i's point of view, it will appear that j has chosen the "incorrect" quantity of x_{j1}. Recall that fulfilling condition (1.13) means i desires a change in the behavior of j and it is possible to induce such a change, possible in the sense that there is a mutually advantageous alternative.

We can see the effect the law will have on the bargaining process by considering several legal arrangements separately. Let us begin with the assumptions that the law imposes no legal constraints on j, the firm that creates the externality, and that we are concerned with an external diseconomy, i.e., $\partial F/\partial z < 0$. The presumption is that i offers to pay j \$B for each unit of x_{j1} that j does not use, i.e., j will receive $\$B(\overline{x}_{j1}-x_{j1})$ from i where $\overline{x}_{j1}$ is the amount of input x_{j1} that j would otherwise use. Since i cannot know the precise form of j's production function, the offer must only be a tentative one. If the offer of \$B per unit does not result in an optimum for i, the offer will be withdrawn and further offers will be made until an optimum is reached.

Formally, the offer of a bribe changes the profit functions for i and j to

$$\pi_j = P_j \cdot G(x_{j1}, x_{j2}, \ldots, x_{jn}) + B(\overline{x}_{j1}-x_{j1}) - \sum_{k=1}^{n} p_k \cdot x_{jk} \qquad (3.1)$$

and

$$\pi_i = P_i \cdot F(x_{i1}, x_{i2}, \ldots, x_{in}, z) - B(\overline{x}_{j1}-x_{j1}) - \sum_{k=1}^{n} p_k \cdot x_{ik}. \qquad (3.2)$$

Because of the change to conditions (3.1) and (3.2), the first-order conditions for a profit maximum also change. These conditions for j are now

$$P_j \cdot \partial G/\partial x_{jk} - p_k = 0 \text{ if } x_{jk} > 0 \qquad (k = 2, 3, \ldots, n) \qquad (3.3)$$

and

$$P_j \cdot \partial G/\partial x_{jl} - (p_l + B) = 0 \quad \text{if } x_{jl} > 0. \tag{3.4}$$

Since $p_l < (p_l+B)$, $x_{jl} < \overline{x}_{jl}$, i.e., since the bribe offer increases the "effective" market price of x_{jl}, the quantity that j employs will decrease. This is the direction of change i desired, but the magnitude of the change may not be sufficient for the attainment of surface T*.

To determine this we must investigate the effect on i's first-order conditions:

$$P_i \cdot \partial F/\partial x_{ik} - p_k = 0 \quad \text{if } x_{ik} > 0 \qquad (k = 1,2,\ldots,n) \tag{3.5}$$

and

$$P_i \cdot \partial F/\partial z + B \begin{Bmatrix} \leq \\ = \\ > \end{Bmatrix} 0. \tag{3.6}$$

If "<" holds, a further decrease in j's use of x_{jl} is desired by i and a new offer $\overline{B}$ will be made such that $\overline{B} > B$. If ">" holds, the first offer made was too large and j decreased its use of x_{jl} by more than the optimum amount. Firm i will then make a new offer $\hat{B}$ such that $\hat{B} < B$.

Finally, if "=" holds, there is no incentive for i to make a new offer because he has already made the offer most profitable to him. We should recognize that the first offer may not be the correct one because, although i may know the exact value of $P_i \cdot \partial F/\partial z$, it may not know the exact change that will occur in the quantity of z as a result of a decrease in the usage of x_{jl}. In other words, i may not know the precise form of the relation $z = g(x_{jl})$. Thus this bargaining process will continue until equality holds in equation (3.6) and equilibrium is attained. When equations (3.3), (3.4), and (3.5) hold and there is equality in equation (3.6), we have a Pareto optimum. Neither firm can be made better off without making the other worse off.

More formally, we can solve equation (3.6) for B and substitute into equation (3.4) to get

$$P_j \cdot \partial G/\partial x_{j1} + P_i \cdot \partial F/\partial z - p_1 = 0. \tag{3.7}$$

This will be recognized as exactly the same as equation (1.24) and essentially the same as equation (1.14). Thus bargaining has led us to Pareto equilibrium in the presence of a marginal externality. Note, however, that the externality has not been removed.

Now let us consider the same problem, except that the law does not allow j to impose an externality upon i in the absence of i's consent. That is, the law prescribes $\overline{x}_{j1} = 0$. For a problem to exist under these circumstances we must have

$$P_j \cdot \partial G/\partial x_{j1} - p_1 > 0 \text{ for } x_{j1} < \hat{x}_{j1} \tag{3.8}$$

where $\hat{x}_{j1}$ is the quantity of x_{j1} that would make an equality hold in expression (3.8). Consequently, j has an incentive to bribe i for permission to use x_{j1}. The offer of a bribe to i changes the profit functions to

$$\pi_i = P_i \cdot F(x_{i1}, x_{i2}, \ldots, x_{in}, z) + Bx_{j1} - \sum_{k=1}^{n} p_k \cdot x_{ik} \tag{3.9}$$

and

$$\pi_j = P_j \cdot G(x_{j1}, x_{j2}, \ldots, x_{jn}) - (B+p_1)\, x_{j1} - \sum_{k=2}^{n} p_k \cdot x_{jk}. \quad (3.10)$$

Since i will suffer the inconvenience of j's use of x_{j1}, i will presumably specify indirectly the quantity of x_{j1} that j may use for a payment of B per unit. Firm i's specification must be indirect because the precise relation $z = g(x_{j1})$ may not be known to i. At any rate, offers will be made and rejected until the first-order conditions are satisfied:

$$P_i \cdot \partial F/\partial x_{ik} - p_k = 0 \text{ if } x_{ik} > 0, \quad (k = 1,2,\ldots,n) \quad (3.11)$$

$$P_i \cdot \partial F/\partial z + B = 0, \quad (3.12)$$

$$P_j \cdot \partial G/\partial x_{jk} - p_k = 0 \text{ if } x_{jk} > 0, \quad (k = 2,\ldots,n) \quad (3.13)$$

and

$$P_j \cdot \partial G/\partial x_{j1} - (B+p_1) = 0 \text{ if } x_{j1} > 0. \quad (3.14)$$

Since all we changed was the legal constraint, we can compare this equilibrium with the previous one. Interestingly, equations (3.3), (3.4), (3.5), and (3.6) correspond to equations (3.13), (3.14), (3.11), and (3.12). Moreover, we can solve equation (3.12) for B, substitute into equation (3.14), and derive a relation exactly like equation (3.7). Of course, the same comments that applied to equation (3.7) also apply to this condition.

An intermediate case could be analyzed, one in which the law permits j some use of x_{j1} and, therefore, some output z. Davis and Whinston handle this case and find that although both firms may attempt bribes initially, there will come a stage in the bargaining process when both will realize which firm must pay in order to reach an equilibrium.

In conclusion, when bargaining is feasible, i.e., when the number of parties is not too large, a Pareto optimum can be attained without outside interference. The question of what the legal constraints ought to be is a question of equity and does not have a bearing on the question of efficiency. But once the legal constraint for liability is specified, no further legal constraints should be imposed because they might prevent the firms from reaching an optimal solution. In other words, all the law should do is make clear the liability for externalities since bargaining can then move society to a Pareto optimum position.[14]

Taxes and Subsidies

An alternative to bargaining is the tax-subsidy approach. In general, this solution involves taxing the firm that causes the externality and compensating the firm

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[14]This entire discussion of bargaining depends heavily on O. A. Davis and A. B. Whinston, "Some Notes on Equating Private and Social Cost," The Southern Economic Journal, October 1965, pp. 113-126. R. H. Coase in "Social Cost" also deals with bargaining and the effect of legal constraints on income distribution.

suffering the externality in like amounts. But as Plott points out, it is important to levy the tax on the correct thing.[15] In our example, j causes the externality when it produces Q_j, but Q_j is not the culprit. The source of difficulty is z, which is a joint- or by-product of Q_j. Since z is a function of input x_{j1}, the tax should be placed on z or on the use of input x_{j1}. In fact, levying the tax on Q_j will result in an increase in z when x_{j1} is an inferior input. Plott demonstrated this result graphically for a two-factor production function, but it can be generalized to n inputs. Input inferiority will be dealt with in detail in the next chapter.

Obviously, taxing the offending firm inherently presumes a legal constraint which places the burden on that firm. The result of such a tax-subsidy scheme is the identical resource allocation and income distribution that private bargaining yields when the law specifies $\overline{x}_{j1} = 0$ in the absence of i's permission for it to be otherwise. To show this formally, all we must do is let B represent both the tax on j and the payment to i in equations (3.9) through (3.14). Now equation (3.12) states that the compensation is exactly equal to the damage done by j to i. In addition, by solving equation (3.12) for B and substituting into equation (3.14) we get condition (3.7):

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[15] C. Plott, "Externalities and Corrective Taxes," pp. 84-86.

$$P_j \cdot \partial G/\partial x_{j1} + P_i \cdot \partial F/\partial z - p_1 = 0.$$

Rearranging this equation we get

$$P_j \cdot \partial G/\partial x_{j1} = p_1 - P_i \cdot \partial F/\partial z. \qquad (3.15)$$

In words, j must equate the value of the marginal product of x_{j1} with its price plus the value of the damage it does to i's operation. Now the social value of the marginal product of x_{j1} is equated with its price.

In addition to the factor inferiority objection to levying the tax on Q_j, we can now see another objection. Levying the tax on Q_j so that condition (3.15) is satisfied will render j's choices for all other inputs non-optimal. Taxing Q_j changes j's profit function to

$$\pi_j = P_j \cdot G(x_{j1}, x_{j2}, \ldots, x_{jn}) - \sum_{k=1}^{n} p_k \cdot x_{jk} - tQ_j, \qquad (3.16)$$

where t is the per unit tax. The first-order conditions now become

$$\partial \pi_j/\partial x_{jk} = P_j \cdot \partial G/\partial x_{jk} - p_k - t = 0.$$

At the margin, the tax is levied on each input; therefore, j does not equate the social values of the marginal products with their respective input prices except for factor x_{j1}.

Thus equation (3.15) is fulfilled, but all the other input conditions are violated.[16]

In the absence of a good reason for supposing the government has some special knowledge of the precise forms of the production functions, we may assume that it must arrive at the appropriate tax or subsidy through some iterative procedure much like that used in the bargaining solution. Thus the difference between the private bargaining and the tax-subsidy approaches is that the government is an intermediary.

One very important point should be stressed: if j is to be taxed, an amount equal to the tax must be paid to i. When i is not so compensated, there remains a Pareto relevant marginal externality, i.e., there is room for further bargaining.[17] We can easily show this by supposing that a tax is levied on j and no compensation is made to i. Firm j's decision calculus changes because its profit function is altered to

$$\pi_j = P_j \cdot G(x_{j1}, x_{j2}, \ldots, x_{jn}) - (B+p_1)\, x_{j1} - \sum_{k=2}^{n} p_k\, x_{jk} \qquad (3.17)$$

where B is the tax on j's use of x_{j1}. Now for each value that B takes on, the marginal condition for x_{j1} is

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[16] C. Plott, "Corrective Taxes." Plott touched on this point, but did not demonstrate it explicitly.

[17] This was most clearly shown by Buchanan and Stubblebine, "Externality" and R. Turvey, "On Divergences between Social Cost and Private Cost," Economica, N.S. XXX (1963) 309-313.

$$\partial\pi_j/\partial x_{j1} = P_j \cdot \partial G/\partial x_{j1} - (B+p_1) = 0. \qquad (3.18)$$

Recall the definition of Pareto relevance and modify it to account for the tax:

$$\left|P_i \cdot \partial F/\partial z\right| > \left|P_j \cdot \partial G/\partial x_{j1} - p_1 - B\right|. \qquad (3.19)$$

Firm j will select quantities of x_{j1} to fulfill condition (3.18), but i cannot maximize its profits because $\partial F/\partial z < 0$ for all $x_{j1} > 0$, i.e., condition (3.19) will always hold until the externality is completely removed. Of course, removing the externality requires levying a prohibitive tax on the use of x_{j1}. Until $x_{j1} = 0$, room for bargaining will exist; therefore, Pareto equilibrium will not be reached unless $x_{j1} = 0$ so long as i is not compensated.[18]

Mergers

A third solution is the merger alternative. So far, the desired change in resource allocation has been accomplished through market transactions: directly via bargaining and indirectly via government tax-subsidy decisions. A merger will accomplish the same result by substituting an entrepreneurial decision for a market transaction. That the

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[18]This tax-subsidy alternative is dealt with or touched upon by Davis and Whinston, "Some Notes"; Coase, "Social Cost"; O. A. Davis and A. B. Whinston, "Externalities, Welfare, and the Theory of Games," Journal of Political Economy, June 1962, pp. 241-262; Buchanan and Stubblebine, "Externality"; Turvey, "On Divergences"; and Mishan, "Reflections."

result will be the same is fairly easy to demonstrate. We have seen that Pareto relevant marginal externalities permit gains from trade to exist. And we have also seen that reaping these gains through either the proper imposition of taxes and subsidies or bargaining yields a Pareto optimum resource allocation. Since this final equilibrium is exactly the same as the joint-profit maximizing solution, a merger clearly offers the same resource allocation. But recall that bargaining resulted in the two firms' sharing the gains from trade. In this important respect, division of the spoils, the merger approach more closely resembles the bargaining solution than the tax-subsidy solution. Of course, this will appear in the terms of the merger agreement.

The effect of the merger is to internalize the externality so that account is taken of its existence when output decisions are made. Therefore, whenever the administrative costs of the new, single firm are less than the costs of bargaining that it replaces, we should expect a merger to occur.

Continuing to deal with a unilateral, or non-reciprocal, externality, we must realize that where the incentive to merge lies depends upon the legal framework. When the law specifies that the offending firm is liable for damages, the offending firm will be the one interested in a merger. On the other hand, if the law specifies no such liability, the damaged firm will be anxious to merge. If we consider a mutual, or reciprocal, externality,[19] i.e., i imposes an externality on j and j imposes an externality on i, the likelihood of a merger is increased. The reason for the increased likelihood is obvious: a merger is now beneficial to both firms. And so long as the market structure remains competitive, the merger is beneficial to society because optimal resource allocation is ensured.[20]

4. Cost Functions and Reciprocal Externalities

The existence of externalities can be represented by including an output of another firm in the cost function of the affected firm.[21] This alternative view is worth discussing because it sheds some light on a few difficulties that have yet to be mentioned. The amount of difficulty we shall encounter depends primarily upon whether the externalities are unilateral or reciprocal and whether the cost functions

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[19] Mutual externalities are explored in more detail in the next section.

[20] The merger solution was suggested by Coase, "Social Cost"; Davis and Whinston, "Theory of Games"; and Mishan, "Reflections."

[21] On the derivation of cost functions, see J. M. Henderson and R. E. Quandt, Microeconomic Theory (New York: McGraw-Hill Book Company, 1958), pp. 55-62, 66-67.

are separable or non-separable.[22] Let us begin with the easier cases and proceed to the more difficult.

Unilateral Externalities

When the externality is unilateral the cost function of i includes an output of j, but j's cost function depends only upon its own output. This is essentially the case that has been discussed so far; however, by inspecting the effect on the cost function, we can see a little more clearly the effects of an externality on the firm's operation.

Consider an external diseconomy. Firm i's cost function is

$$C_i = C_i(Q_i, z), \tag{4.1}$$

and j's cost function is

$$C_j = C_j(Q_j') \tag{4.2}$$

where Q_i is the output of i, z is the externality-causing by-product of firm j, and Q_j' represents the joint products Q_j and z. Since i cannot control the quantity of z, the first-order conditions for independent profit maximization are

$$P_i = \partial C_i/\partial Q_i \text{ and } P_j = \partial C_j/\partial Q_j'. \tag{4.3}$$

[22]Reciprocal externalities and the distinction between separable and non-separable functions are handled in detail by Davis and Whinston, "Theory of Games."

For Pareto equilibrium, this resource allocation must correspond to that of joint-profit maximization. It is quite obvious that this is not the case since the joint profit function in the competitive case is

$$\pi = P_i \cdot Q_i + P_j \cdot Q_j - C_i(Q_i, z) - C_j(Q_j'), \tag{4.4}$$

and the first-order conditions are

$$\partial\pi/\partial Q_i = P_i - \partial C_i/\partial Q_i = 0, \tag{4.5}$$

and

$$\partial\pi/\partial Q_j' = P_j - \partial C_j/\partial Q_j' - \partial C_i/\partial z = 0.$$

If $\partial C_i/\partial z \neq 0$, conditions (4.5) are not the same as conditions (4.3), and non-optimal output decisions are made because the deleterious effect of z is ignored.

Any function, $f(y_1, y_2, \ldots, y_n)$, is termed "separable" if and only if

$$f(y_1, y_2, \ldots, y_n) = f_1(y_1) + f_2(y_2) + \ldots + f_n(y_n). \tag{4.6}$$

In our case, $C_i = C_i(Q_i, z)$ is separable if and only if

$$C_i(Q_i, z) = C_{i1}(Q_i) + C_{i2}(z). \tag{4.7}$$

When this is the case, the marginal cost of producing Q_i is unambiguously defined as $\partial C_{i1}/\partial Q_i$, i.e., as a function solely of its own output. The consequence of separability is that, although the total cost of i is a function of two variables, the marginal cost for i is unaffected by changes in z. In graphical terms, the height of the total cost curve varies with changes in z, but the slope is the same at all levels of output. In other words, if TC_i is the total cost curve for $Q_j' = 0$, $TC_i^* = TC_i + k$ for some $Q_j' > 0$ where k is constant. Of course, as Q_j' varies, the value of k will vary. Since the marginal costs are not affected in this case, there exists a unique output which will maximize i's profit regardless of the level of z. The only relevance that the level of z has lies in its effect on i's total profit. The greater is z, the smaller is π_i. But all this does not mean that the externality has no allocative effects.[23] Clearly, if j takes account of the effect that z has on the profits of i there will be a different resource allocation.

When the externality enters i's cost function in a multiplicative way the separability condition (4.7) is not satisfied and the cost function is said to be "non-separable."[24] The effect of non-separability is that when z changes the total cost curve is not vertically displaced

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[23]O. A. Davis and A. B. Whinston, "On Externalities, Information and the Government-Assisted Invisible Hand," Economica, N.S. XXXIII (1966), 304-305.

[24]In Appendix A, I derived the cost function of a firm with a Cobb-Douglas production function and of one with a CES production function. Neither production function yields a separable cost function.

by a constant amount. The total cost curve will be altered in some way such that the marginal cost will be affected. Firm i can no longer define its marginal cost unambiguously without first knowing the value of z since marginal cost is now a function of its own output and the output of j. Therefore, i must know the value of z before it can make the correct allocative decisions. But once it knows the value of z, i can proceed to maximize its profits as best it can.

So far, the cost function alternative has no real impact on anything done before this section. But this exercise has pointed out that when the cost functions are separable, the external effects do not affect i marginally. On the other hand, non-separable cost functions plus externalities will give rise to changes at the margin. As long as we retain the assumption of unilateral external effects, no new problem arises and all the externality solutions apply.

Mutual Externalities

When externalities are mutual, j's output affects i's cost function and vice versa, i.e., the cost functions become

$$C_i = C_i(Q_i', z) \tag{4.8}$$

and

$$C_j = C_j(Q_j', y) \tag{4.9}$$

where y is an externality-causing by-product of i. Here, again, independent profit maximization yields

$$P_i = \partial C_i/\partial Q_i' \quad \text{and} \quad P_j = \partial C_j/\partial Q_j' \tag{4.10}$$

as first-order conditions because each firm can only maximize profits with respect to the variables under its control. Comparing conditions (4.10) with the joint profit maximizing conditions

$$P_i = \partial C_i/\partial Q_i' + \partial C_j/\partial y \tag{4.11}$$

and

$$P_j = \partial C_j/\partial Q_j' + \partial C_i/\partial z$$

reveals that non-optimal decisions are made.

The effect of separability in the mutual externalities case is to leave the marginal costs of both firms unchanged regardless of the quantities of y and z. Thus i and j can unambiguously define their respective marginal costs in terms of their own outputs, and, therefore, there exist unique outputs which will maximize the profits of i and j individually. This means that one firm's output decision is wholly independent of the other firm's decision.

Since non-optimal decisions are made, we are interested in solutions to these problems. For a tax-subsidy scheme to constitute a solution, the government must be able to find the correct outputs for i and j. It can do this by solving the necessary conditions (4.11) for Q_i' and Q_j'. If we let t represent the per unit tax and $\overline{Q}_i$ and $\overline{Q}_j$ the optimal outputs, the correct tax is given by

$$P_i - t_i = \partial C_i/\partial Q_i \tag{4.12}$$

and

$$P_j - t_j = \partial C_j/\partial Q_j.$$

Inspection of equations (4.11) makes it clear that the tax on each firm should equal the damage done to the other firm. As before, the tax collected must be paid to the damaged firm. In a similar manner, the bargaining procedure can be carried out. Optimal output decisions will now follow because each firm is made aware of its effects on the profits of the other firm. We should recognize that, when the cost functions are separable and there are mutual externalities, the optimal solution simply involves removing the Pareto relevance of each externality separately. In other words, we can deal with one external effect at a time.

When the cost functions are non-separable and the externalities are reciprocal we encounter a bit of a problem. Since the marginal cost of each firm depends upon the output decision of the other firm, each firm would like to wait for the other firm to commit itself before making its own decision. It is fairly easy to appreciate this fact when one considers that i's output decision changes whenever j's output decision changes and the same is true for j. Such a situation introduces a measure of uncertainty into each firm's decision calculus. This certainly can be removed by internalizing the externalities through a merger of the firms. So long as the post-merger market structure remains competitive, a merger would prove to be mutually beneficial to the two firms and socially desirable since optimal output decisions would be ensured. Moreover, this solution may very well be the most practical.

A tax-subsidy scheme might also be devised if the government knows the cost functions of the two firms. The solution involves finding new cost functions c_i and c_j for the firms such that these new cost functions are single-valued functions of "own" output and they account for all social costs.[25] Then the use of these new cost functions is supposed to solve all the problems. But Davis and Whinston have pointed out some rather serious difficulties with this method.[26]

For the new cost functions to account for all social costs,

$$dc_i/dQ_i = \partial Ci/\partial Q_i + \partial C_j/\partial y \qquad (4.13)$$

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[25]This proposal was suggested in S. Wellisz, "On External Diseconomies and the Government-Assisted Invisible Hand," Economica, N.S. XXXI (1964), 358-359.

[26]Davis and Whinston, "On Externalities," pp. 307-312.

and

$$dc_j/dQ_j = \partial C_i/\partial z + \partial C_j/\partial Q_j \qquad (4.14)$$

must be satisfied. But if the cost functions are non-separable, the terms on the right-hand side of (4.13) are functions of Q_i and Q_j. Therefore, the domain of Q_j would have to be restricted for the function to be single-valued. There is, however, no a priori reason why the domain of Q_j should be restricted.

If we ignore this problem, what remains is to find Q_1 as a function of Q_j and Q_j as a function of Q_i. This involves solving partial differential equations where no truly general method of solution exists. But assuming the equations can be solved, we would have

$$Q_i = h_j(Q_j) \quad \text{and} \quad Q_j = h_i(Q_i). \qquad (4.15)$$

Each of these functions involves a constant of integration. Since h_i and h_j represent the taxes or subsidies, correct values for these constants must be found. Finding these values requires solving the joint profit maximization problem for the optimal outputs $\overline{Q}_i$ and $\overline{Q}_j$ and substituting into equations (4.15). Then the new cost functions are, by substitution,

$$c_i = C_i(Q_i, h_i(Q_i)) \quad \text{and} \quad c_j = C_j(Q_j, h_j(Q_j)). \qquad (4.16)$$

Supposing that we have assumed away all the problems or have overcome them in some way and the new cost functions (4.16) have been found, the coup de grace may now be applied: there is no way that the government can force the firms to use these new cost functions. In all the previous cases it was in each firm's interest to move to the socially optimal output, but in this case the firm may not believe that using the prescribed cost function will not hurt it. If each firm was previously aware of the other firm's influence on its cost function, they may not believe that the costs are truly independent now. Moreover, if, e.g., i does not believe this and produces some $Q_i \neq \overline{Q}_i$, j will experience costs that the new function, c_j, does not reflect. This experience would certainly lend credence to any skepticism j previously held with respect to the efficacy of using the new cost function.

From this discussion, one can readily appreciate the difficulties inherent in the case of mutual externalities with non-separable cost functions. Davis and Whinston go on to discuss an iterative procedure which purports to take care of this case. But the authors admit that they know nothing about the speed with which the procedure will converge to an equilibrium. Although I do not intend to go into this problem any further, it is worth recognizing the difficulties that this case presents; especially when a merger is not permitted by the requirement of maintaining a competitive market structure.[27]

5. Second-Order Conditions[28]

To this point, we have focused our attention on the first-order conditions for a profit maximum and have neglected the second-order conditions. But these second-order conditions can be ignored no longer since satisfaction of the first-order conditions does not ensure a maximum. We can approach this problem through the firm's cost function. Because profit is the difference between total revenue (R) and total cost (C), the profit function for i is

$$\pi_i = R(Q_i) - C_i(Q_i, p_1, p_2, \ldots, p_n). \qquad (5.1)$$

Of course, optimum output occurs when profit is maximized. Assuming the functions are twice differentiable, this requires

$$\partial\pi_i/\partial Q_i = \partial R/\partial Q_i - \partial C_i/\partial Q_i = 0 \qquad (5.2)$$

and

$$\partial^2\pi_i/\partial Q_i^2 = \partial^2 R/\partial Q_i^2 - \partial^2 C_i/\partial Q_i^2 < 0. \qquad (5.3)$$

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[27] Mishan pointed out in "Reflections" that the crux of this problem lies as much in the assumption of reciprocal externalities as in the assumption of non-separable cost functions.

[28] Most of this discussion is based on Samuelson, Foundations, pp. 57-62, 76-78 and Appendix A, pp. 357-379.

Equation (5.2) says that marginal revenue equals marginal cost at equilibrium. But this is not enough because equating marginal revenue and marginal cost will yield minimum profits when the slope of the marginal revenue curve exceeds the slope of the marginal cost curve. This event is ruled out by requiring that condition (5.3) be satisfied. Then, the optimum output, $\overline{Q}_i$, is found by solving equation (5.2).

Using cost functions assumes that they express the least total cost for each level of output. For the total costs of producing $\overline{Q}_i$ to be a minimum the marginal productivity of the last dollar spent must be equal in all uses, i.e., that[29]

$$1/\lambda = \frac{\partial F/\partial x_{ik}}{p_k} = \frac{\partial F/\partial(-z)}{B} . \qquad (k = 1,2,\ldots,n) \qquad (5.4)$$

These are the first-order conditions found in the constrained cost minimization problem. Since we assume that costs are a minimum for each output level, these equations must hold when $\overline{Q}_i$ is produced. Samuelson[30] has shown that $\lambda = MC$, i.e.,

$$MC = \lambda = \frac{p_k}{\partial F/\partial x_{ik}} = \frac{B}{\partial F/\partial(-z)} \qquad (k = 1,2,\ldots,n) \qquad (5.5)$$

It seems that all we must add to the assumption of perfectly competitive pricing is diminishing marginal productivity and

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[29]See Samuelson, Foundations, p. 60. On the sign of z, see my Appendix B.

[30]Samuelson, Foundations, pp. 65-66.

we will then ensure satisfaction of the second-order condition (5.3). This, however, is not true. We must have diminishing marginal productivity, but we need something more. The "something more" did not show up because we assumed that the second-order conditions were satisfied in equations (5.4) and (5.5), but these equations are not sufficient to ensure a minimum.

Consider i's profit function where all the inputs are considered independent variables. Let

$$\begin{aligned}\pi_i &= P_i \cdot F(x_{i1}, x_{i2}, \ldots, x_{in}, (\bar{z}-z)) \\ &\quad - \sum_{k=1}^{n} p_k \cdot x_k - B(\bar{x}_{j1} - x_{j1}).\end{aligned} \tag{5.6}$$

[31]

As before, the first-order conditions are

$$\partial\pi_i/\partial x_{ik} = P_i \cdot \partial F/\partial x_{ik} - p_k = 0 \qquad (k = 1,2,\ldots,n) \tag{5.7}$$

and

$$\partial\pi_i/\partial(-z) = P_i \partial F/\partial(-z) - B = 0.$$

Since P_i is marginal revenue, this again says that marginal revenue equals marginal cost in equilibrium. But it is well known that "a regular relative maximum requires that the quadratic form whose coefficients are the second partial

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[31] For this slight re-formulation of i's profit function, see Appendix B.

derivatives be negative definite."[32] It can be shown that the negative definiteness of this quadratic form implies that the principal minors of the Hessian determinant of the profit function must alternate in sign beginning with negative. Since $P_i > 0$ and it appears in every term, we may ignore it and write the Hessian as

$$H = \begin{vmatrix} F_{11} & F_{12} & \cdots & F_{1n} & F_{1z} \\ F_{12} & F_{22} & \cdots & F_{2n} & F_{2z} \\ \vdots & \vdots & & \vdots & \vdots \\ F_{1n} & F_{2n} & \cdots & F_{nn} & F_{nz} \\ F_{1z} & F_{2z} & \cdots & F_{nz} & F_{zz} \end{vmatrix} \qquad (5.8)$$

where $F_{hk} = \partial^2 F/\partial x_{ih}\partial x_{ik}$ $(h,k = 1,2,\ldots,n,z)$. Because the first principal minor, F_{11}, must be negative, we can conclude that the marginal productivity of factor x_{i1} must be decreasing. Since the numbering of inputs is wholly arbitrary, this condition must be invariant under any renumbering of inputs. Therefore, all F_{kk} must be negative. Thus, we have diminishing marginal productivity again. But this is still not enough. There could be a situation where increases in all factors will yield an increase in profits. Therefore, we also require that the decrease in a factor's "own" marginal productivity outweighs the positive effects on the marginal products of the other factors. For clarity, consider a two-input case. The Hessian will then be

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[32] Samuelson, Foundations, p. 360.

$$H = \begin{vmatrix} F_{11} & F_{12} \\ F_{12} & F_{22} \end{vmatrix}. \qquad (5.9)$$

For a maximum, we need $F_{11} < 0$ and $F_{11} \cdot F_{22} - (F_{12})^2 > 0$. The second inequality implies $F_{11} \cdot F_{22} > (F_{12})^2$. We can see that the "cross" effects of increasing both factors x_{i1} and x_{i2} must be outweighed by the "own" effects. When these conditions are satisfied the marginal cost curve will intersect the marginal revenue curve from below and the solution will represent a maximum of profit. For our purposes, we will assume that these second-order conditions are satisfied in all final equilibria. This is not an unreasonable assumption since we are just assuming away saddle points and perpetual losses. Saddle points are assumed away by convention and perpetual losses by common sense.

CHAPTER III

THE CONCEPT OF INFERIOR INPUTS

In the preceding chapter we encountered the concept of inferior inputs. We shall examine this concept in more detail in the present chapter. Although Professor Hicks alluded to something akin to inferior inputs, D. V. T. Bear first defined inferior inputs in a formal manner. Bear developed his definition under the assumption of competition in the commodity and factor markets. But he did not explore the concept much beyond the definition. In an article soon to appear in Economica, C. E. Ferguson generalized Bear's work to include imperfect competition in the commodity market. In addition, Ferguson extended the concept by investigating the consequences that input inferiority has on other relations. Charles Plott's contribution was to demonstrate graphically the significance of inferior inputs with respect to measures taken to correct externality-caused resource misallocation.[1]

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[1] J. R. Hicks, Value and Capital (2d ed.; Oxford: Clarendon Press, 1962), pp. 93-94. D. V. T. Bear, "Inferior Inputs and the Theory of the Firm," Journal of Political Economy, LXXIII (1965), 287-89. C. E. Ferguson, "'Inferior Factors' and the Theories of Production and Input Demand," Economica, to appear May 1968. Charles Plott, "Externalities and Corrective Taxes," Economica, XXXIII (1966), 84-87.

To explore these developments, we shall first set out a model of jointly-derived demand functions for inputs under general competitive conditions. Then we shall introduce the definition of inferior inputs and investigate some of the consequences of input inferiority.

1. A Basic Model of Input Demand

Assume that a firm sells its output under competitive conditions and produces its output according to the twice differentiable production function

$$Q = f(x_1, x_2, \ldots, x_n), \qquad (1.1)$$

where Q is total output and x_i is the quantity of input i. Competition in the commodity market implies that the firm's total revenue is

$$R = P \cdot Q = P \cdot f(x_1, x_2, \ldots, x_n), \qquad (1.2)$$

where P is the commodity price.

Competition in the factor markets implies that the firm accepts the input prices as given. Defining profit as the difference between total cost and total revenue, the firm's profit function may be written as

$$\pi = P \cdot f(x_1, x_2, \ldots, x_n) - \sum_{i=1}^{n} p_i x_i, \qquad (1.3)$$

where p_i is the price of input i. The firm attempts to maximize profit by selecting appropriate quantities of the n inputs.

The first-order conditions for a profit maximum are obtained by differentiating (1.3) with respect to x_i:

$$\partial\pi/\partial x_i = P\cdot f_i - p_i = 0, \quad (i = 1,2,\ldots,n) \tag{1.4}$$

where $f_i = \partial f/\partial x_i$. That is, in equilibrium the price of each input must equal the value of its marginal product. Equations (1.4) represent the n jointly-derived input demand functions in implicit form.

The second-order conditions for a regular relative maximum to exist at the point in n-space where the n first partial derivatives vanish require

$$d^2\pi = \sum_i \sum_j (\partial^2\pi/\partial x_i\cdot\partial x_j)\, dx_i\, dx_j < 0, \tag{1.5}$$

i.e., the quadratic form must be negative definite. The determinant of (1.5) is

$$N = \begin{vmatrix} Pf_{11} & Pf_{12} & \ldots & Pf_{1n} \\ Pf_{12} & Pf_{22} & \ldots & Pf_{2n} \\ \vdots & \vdots & & \vdots \\ Pf_{1n} & Pf_{2n} & \ldots & Pf_{nn} \end{vmatrix} = P^n\, F^*, \tag{1.6}$$

where $f_{ij} = \partial^2 f/\partial x_i\cdot\partial x_j$. Since P appears in every term, we may ignore P and consider F^*. It is well known that the negative definiteness of (1.5) implies that the principal minors of F^* must alternate in sign beginning with

$f_{11} < 0$. Without loss of generality, we may assume n is even. Thus $F^* > 0$.

Since the numbering of the inputs obviously should not matter, assume that the price of input 1 changes while the prices of the other n - 1 inputs are held constant. To observe the effect of changes in p_1, differentiate the first-order conditions (1.4) with respect to p_1. In matrix form, the result is

$$\begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{12} & f_{22} & \cdots & f_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ f_{1n} & f_{2n} & \cdots & f_{nn} \end{bmatrix} \begin{bmatrix} \partial x_1/\partial p_1 \\ \partial x_2/\partial p_1 \\ \cdot \\ \cdot \\ \cdot \\ \partial x_n/\partial p_1 \end{bmatrix} = \begin{bmatrix} 1/P \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}. \tag{1.7}$$

The determinant of the coefficient matrix in (1.7) is precisely the F^* of equation (1.6).

Equations (1.7) can be solved by Cramer's Rule:

$$\partial x_1/\partial p_1 = F^*_{11}/P \cdot F^*, \tag{1.8}$$

where F^*_{11} is the cofactor of the 1 - 1 element in F^*. Since F^* and P are positive,

$$\text{sign } \partial x_1/\partial p_1 = \text{sign } F^*_{11}. \tag{1.9}$$

But

$$\text{sign } F^*_{11} = - \text{ sign } F^* \tag{1.10}$$

because F^* is negative definite. Thus

$$\partial x_1/\partial p_1 = F^*_{11}/P \cdot F^* < 0, \tag{1.11}$$

i.e., the quantity of input 1 demanded is inversely related to its price. Moreover, because the selection of input 1 was wholly arbitrary we may conclude that this holds for all inputs:

$$\partial x_j/\partial p_j = F^*_{jj}/P \cdot F^* < 0. \quad (j = 1,2,\ldots,n) \tag{1.12}$$

Similarly, solving for $\partial x_j/\partial p_1$ yields

$$\partial x_j/\partial p_1 = F^*_{1j}/P \cdot F^*. \tag{1.13}$$

Further,

$$\partial x_j/\partial p_1 \gtrless 0 \quad \text{according as } F^*_{1j} \gtrless 0. \tag{1.14}$$

Thus, *a priori* we cannot say anything about the sign of $\partial x_j/\partial p_1$.[2]

2. Inferior Inputs

In words, an input is termed inferior if and only if an increase in its price leads to an increase in the optimal output of the firm.[3] From the production function (1.1), input 1 is inferior if and only if

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[2]This entire section is simply an adaptation of a model developed by C. E. Ferguson in a manuscript entitled "Neo-Classical Theory of Production and Distribution," to be published by Cambridge University Press, January 1969.

[3]Bear, "Inferior Inputs . . .," p. 287.

$$\partial Q/\partial p_1 = \sum_{j=1}^{n} f_j \cdot \partial x_j/\partial p_1 > 0. \qquad (2.1)$$

This result can be expressed in more familiar terms, however. First, differentiate the first-order conditions (1.4) with respect to P, the commodity price. The result is

$$f_i + P \sum_j f_{ij}\ \partial x_j/\partial P = 0. \quad (j = 1,2,\ldots,n) \qquad (2.2)$$

Equations (2.2) may be written in matrix form as

$$\begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{12} & f_{22} & \cdots & f_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ f_{1n} & f_{2n} & \cdots & f_{nn} \end{bmatrix} \cdot \begin{bmatrix} \partial x_1/\partial P \\ \partial x_2/\partial P \\ \cdot \\ \cdot \\ \cdot \\ \partial x_n/\partial P \end{bmatrix} = -1/P \begin{bmatrix} f_1 \\ f_2 \\ \cdot \\ \cdot \\ \cdot \\ f_n \end{bmatrix} \qquad (2.3)$$

The negative definite determinant of the coefficient matrix in (2.3) is

$$F^* = \left| f_{ij} \right| > 0. \quad (i,\ j = 1,2,\ldots,n) \qquad (2.4)$$

Clearly, the F^* of (2.4) is identically the F^* of (1.6).

Using (2.4) and Cramer's Rule, the solutions of equations (2.3) are

$$\partial x_j/\partial P = -\frac{\sum_i f_i\, F^*_{ji}}{P \cdot F^*} \qquad (i,j = 1,2,\ldots,n) \qquad (2.5)$$

where F^*_{ji} is the cofactor of the j - i element in F^*.

We have seen that differentiation of the first-order conditions (1.4) with respect to p_1 resulted in

$$\sum_{j=1}^{n} f_{ij} \; \partial x_j/\partial p_1 = 1/P \; \delta_{1i}, \tag{2.6}$$

where δ_{1i} is the Kronecker delta. Further, the solutions to (2.6) are

$$\partial x_j/\partial p_1 = F^*_{1j}/P \cdot F^*. \quad (j = 1,2,\ldots,n) \tag{2.7}$$

Substitute equation (2.7) into definition (2.1):

$$\partial Q/\partial p_1 = \frac{\sum_{j=1}^{n} f_j \cdot F^*_{1j}}{P \cdot F^*} . \tag{2.8}$$

Since P and F* are positive, we may conclude that input 1 is inferior if and only if

$$\sum_{j=1}^{n} f_j \cdot F^*_{1j} > 0. \tag{2.9}$$

By using equation (2.5) in equation (2.8), we shall find

$$\partial Q/\partial p_1 = - \partial x_1/\partial P. \tag{2.10}$$

In words, the change in optimal output resulting from a change in the price of input 1 equals the negative of the change in the optimal usage of input 1 that results from a change in the commodity price. In the case of inferior inputs, the change in optimal output due to a change in p_1 is positive by definition. Consequently, the change in the quantity of x_1 when commodity price changes is negative, i.e., when commodity price increases, the quantity of input 1 employed decreases.[4]

3. An Alternate View of Inferior Inputs

Equation (2.10) provides an equivalent definition of inferior inputs, viz., one whose use declines as commodity price, and hence output under perfect competition, increases. Consideration of this alternative makes the analogy between an inferior good in consumer demand theory and an inferior input in the theory of the firm easier to appreciate. But it will also point out that the analogy is not quite complete. In the theory of consumer behavior, a commodity is inferior if the quantity demanded varies inversely with the consumer's income level. But the consumer is solving a constrained maximum problem. Thus changing his money income and re-computing his optimal expenditure pattern will reveal successive positions of equilibrium.

In the theory of the firm, however, the firm determines optimal inputs by solving an unconstrained maximum problem rather than a constrained maximum problem. Total cost is found by evaluating the cost equation after substituting this vector of optimal inputs. The firm simply spends this amount in order to maximize profits. Although the firm's expansion path is analogous to the consumer's

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[4]This section depended heavily upon work done by C. E. Ferguson in an unpublished manuscript.

income-consumption curve, the expansion path is not a locus of profit-maximizing points. But it is a locus of cost-minimization points and, therefore, every profit-maximizing point must also lie on the expansion path. Thus we can direct our attention to these specific profit-maximizing points for analytical purposes. But the slope of the expansion path can be examined for conceptual purposes.

The firm's response to a demand led increase in commodity price, <u>ceteris paribus</u>, is an expansion of output. Since the firm was in a profit-maximizing position before the price change, its iso-output surface was tangent to an iso-cost surface. To increase its output, the firm must move to a higher iso-cost surface. When one compares the two profit-maximizing positions in n-space, one will see that $\partial x_1/\partial C < 0$ where C is total cost, i.e., the tangency of the new, higher iso-cost surface to the new, higher iso-output surface involves a diminished employment of the inferior input, x_1. Thus the two concepts of inferiority are more closely related than the first definition of an inferior input might have indicated.

4. <u>Inferior Inputs and the Level of Profits</u>

We have seen that an increase in the price of input 1 calls for an expansion of output when input 1 is inferior. But this expansion of output does not affect profits favorably. In fact, the increase in optimum output is accompanied by a decrease in profits. This can readily be demonstrated by differentiating the profit function (1.3) with respect to p_1:

$$\frac{\partial\pi}{\partial p_1} = \sum_i (P \cdot f_i - p_i) \cdot \partial x_j / \partial p_1 - x_1. \quad (i = 1, 2, \ldots, n) \qquad (4.1)$$

From the first-order conditions, i.e., in equilibrium, the terms in parentheses are identically zero. Thus we have

$$\partial\pi/\partial p_1 = - x_1. \qquad (4.2)$$

Since x_1 cannot be negative, $- x_1$ must be negative. Therefore, we may conclude that profit always varies inversely with input prices regardless of whether the input is inferior or not.[5]

5. Input Inferiority and Restrictions on the Production Function

Input inferiority places certain restrictions on the form that the production function can take. It can easily be shown that the production function cannot be homogeneous of degree one. To this end, suppose it is homogeneous of degree one. The bordered Hessian determinant of the production function is

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[5]Bear proved this result in an elegant fashion. But he asserted that this proved that inputs could not be inferior at all levels of output because if they were, profits would vary directly with input price. Since he proved the opposite was the case in *any* event, I find his argument rather unconvincing.

$$F = \begin{vmatrix} 0 & f_1 & f_2 & \cdots & f_n \\ f_1 & f_{11} & f_{12} & \cdots & f_{1n} \\ f_2 & f_{12} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ f_n & f_{1n} & f_{2n} & \cdots & f_{nn} \end{vmatrix}. \tag{5.1}$$

As a matter of notation, number the rows and columns $0, 1, 2, \ldots, n$ and let F_i represent the cofactor of the i-th element in column 0, F_{ij} the cofactor of the $i - j$ element in the body of F, and F_{0ij} the cofactor of the $i - j$ element in F_0.

Expansion of F_1, the cofactor of the 1 element in column 0 yields

$$F_1 = -\sum_{j=1}^{n} f_j \cdot F_{01j}. \tag{5.2}$$

But linear homogeneity of the production function implies

$$-\sum_{j=1}^{n} f_j\ F_{01j} = (x_1 \cdot F/Q^2) \sum_{j=1}^{n} f_j\ x_j. \tag{5.3}$$

Since Euler's Theorem applies, we may write (5.3) as

$$-\sum_{j=1}^{n} f_j\ F_{01j} = (x_1 F/Q^2) \sum_{j=1}^{n} f_j\ x_j = x_1 F/Q \tag{5.4}$$

Noting that x_1, F, Q > 0, linear homogeneity implies

$$-\sum_{j=1}^{n} f_j\ F_{01j} > 0. \tag{5.5}$$

Inspection of F and F* reveals

$$F_0 = F^* \tag{5.6}$$

and

$$F_{01j} = F^*_{1j}. \tag{5.7}$$

Thus

$$-\sum_{j=1}^{n} f_j F_{01j} = -\sum_{j=1}^{n} f_j F^*_{1j}. \tag{5.8}$$

Inequality (5.5) then implies that linear homogeneity requires

$$-\sum_{j=1}^{n} f_j F^*_{1j} > 0, \tag{5.9}$$

or

$$\sum_{j=1}^{n} f_j F^*_{1j} < 0. \tag{5.10}$$

Therefore, by (2.9) linear homogeneity precludes input inferiority.[6]

Moreover, if the production function is such that an increase in any input increases the marginal products of all other inputs, input inferiority is also precluded. To demonstrate this proposition, suppose that the production function satisfies this condition and that input 1 is inferior. Differentiating the first-order conditions for a profit maximum (1.4) with respect to P yields

$$[f_{ij}]\ [\partial x_i/\partial P] = -\ 1/P\ [f_i] \quad (i,j = 1,2,\ldots,n) \tag{5.11}$$

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[6]This point was mentioned as being obvious by Bear and was proved by Ferguson.

in matrix form. Under the hypothesis,

$$f_{ij} > 0 \quad \text{for } i \neq j. \tag{5.12}$$

It has been proven that all the elements of $[f_{ij}]^{-1}$ are negative. Equation (5.11) may be solved by matrix methods to obtain

$$[\partial x_i/\partial P] = -\,1/P\,[f_{ij}]^{-1}\,[f_i]. \tag{5.13}$$

Since f_i and P are positive, the right-hand side of (5.13) must be positive and this implies that

$$\partial x_i/\partial P > 0. \tag{5.14}$$

But substitution of (5.14) into (2.10) implies

$$\partial Q/\partial p_1 < 0. \tag{5.15}$$

We clearly have a contradiction of definition (2.1). Thus a production function in which all marginal products are increasing functions of all other inputs effectively precludes input inferiority.[7]

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[7]This result was proved by Bear, "Inferior Inputs . . .," p. 288.

6. Input Inferiority and the Cross-Elasticity of Input Demand

We can derive the cross-elasticities of input demand functions from inequality (1.14) and relate this to the concept of inferior inputs. From the first-order conditions, $f_j = p_j/P$. Multiply both inequalities (1.14) by f_j and sum over j:

$$1/P \sum_j p_j \, \partial x_j/\partial p_1 \gtrless 0 \quad \text{according as} \quad \sum_j f_j F^*_{1j} \gtrless 0. \tag{6.1}$$

The proportion of total cost spent on input j can be expressed as

$$\kappa_j = p_j \, x_j/\sum_i p_i x_i. \quad (i = 1,2,\ldots,n) \tag{6.2}$$

The price cross-elasticity of input demand between inputs i and j is defined as

$$\theta_{ij} = \partial x_j/\partial p_i \cdot p_i/x_j. \tag{6.3}$$

To introduce these definitions, multiply the left-hand inequality in (6.1) by

$$\frac{\Sigma p_i x_i}{\Sigma p_i x_i} \cdot \frac{p_i}{x_j} \cdot \frac{x_j}{p_i}.$$

Then condition (6.1) can be expressed as

$$\frac{\Sigma p_i x_i}{P \cdot p_i} \cdot \sum_j \kappa_j \, \theta_{1j} \gtrless 0 \quad \text{according as} \quad \sum_j f_j \, F^*_{1j} \gtrless 0. \tag{6.4}$$

Since $\frac{\Sigma p_i x_i}{P \cdot p_i} > 0$, the weighted sum of the cross-elasticities of input demand is negative if input 1 is normal. But when input 1 is inferior, this sum will be positive.[8]

7. Inferior Inputs and Output and Substitution Effects

So far, we have been dealing with a profit-maximizing firm. This, of course, requires taking cognizance of total revenue. But prior to any revenue considerations, it is implicitly, if not explicitly, assumed that each competitive firm has chosen optimal quantities of inputs such that total output is maximized for each level of total cost, i.e.,

$$Q = f(x_1, x_2, \ldots, x_n) \tag{7.1}$$

is maximized subject to a given cost constraint

$$\sum_i p_i x_i = \overline{C} \quad (i = 1, 2, \ldots, n) \tag{7.2}$$

where it is still assumed that input prices are parametrically given by the market.

To find these optimal input quantities, form the Lagrange expression

$$L = f(x_1, x_2, \ldots, x_n) - \lambda(\Sigma\, p_i x_i - \overline{C}), \tag{7.3}$$

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[8]This entire section and the following section are mere adaptations or restatements of work done by Ferguson in his unpublished manuscript.

where λ is an undetermined Lagrange multiplier. The first-order conditions are

$$\begin{aligned} \partial L/\partial x_i &= f_i - \lambda p_i = 0, \\ \partial L/\partial \lambda &= \sum_i p_i x_i - \overline{C} = 0. \quad (i = 1,2,\ldots,n) \end{aligned} \tag{7.4}$$

A regular relative maximum requires

$$d^2f = \sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij}\, dx_i\, dx_j < 0 \tag{7.5}$$

for

$$df = \sum_{i=1}^{n} f_i\, dx_i = 0, \tag{7.6}$$

where not all dx_i are zero. This is equivalent to requiring the bordered Hessian determinant,

$$D = \begin{vmatrix} 0 & p_1 & p_2 & \cdots & p_n \\ p_1 & f_{11} & f_{12} & \cdots & f_{1n} \\ p_2 & f_{12} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ p_n & f_{1n} & f_{2n} & \cdots & f_{nn} \end{vmatrix}, \tag{7.7}$$

to have principal minors that alternate in sign, the first being positive. From the first-order conditions (7.4)

$$p_i = f_i/\lambda. \quad (i = 1,2,\ldots,n) \tag{7.8}$$

Thus

$$D = 1/\lambda^2 \begin{vmatrix} 0 & f_1 & f_2 & \cdots & f_n \\ f_1 & f_{11} & f_{12} & \cdots & f_{1n} \\ f_2 & f_{12} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ f_n & f_{1n} & f_{2n} & \cdots & f_{nn} \end{vmatrix} = 1/\lambda^2 \, F. \qquad (7.9)$$

Since it can be shown that $1/\lambda$ represents marginal cost, $1/\lambda$ and consequently $1/\lambda^2$ are positive. Thus

$$\text{sign } D = \text{sign } F \qquad (7.10)$$

and the signs of the corresponding principal minors of D and F must also be the same. Further, assuming that n is even renders the sign of F positive.

In order to find the output and substitution effects, introduce a change in p_1. The result may be expressed in matrix form:

$$\begin{bmatrix} 0 & p_1 & p_2 & \cdots & p_n \\ p_1 & f_{11} & f_{12} & \cdots & f_{1n} \\ p_2 & f_{12} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ p_n & f_{1n} & f_{2n} & \cdots & f_{nn} \end{bmatrix} \begin{bmatrix} -\partial\lambda/\partial p_1 \\ \partial x_1/\partial p_1 \\ \partial x_2/\partial p_1 \\ \vdots \\ \partial x_n/\partial p_1 \end{bmatrix} = \begin{bmatrix} -x_1 \\ \lambda \\ 0 \\ \vdots \\ 0 \end{bmatrix} . \qquad (7.11)$$

The solutions to equations (7.11) may be found by using Cramer's Rule and (7.8):

$$-\partial\lambda/\partial p_1 = \frac{-\lambda^2 \, x_1 \, F_0}{F} + \frac{\lambda^2 \, F_1}{F} , \qquad (7.12)$$

and

$$\partial x_j/\partial p_1 = \frac{-\lambda^2 \ x_1 \ F_j}{F} + \frac{\lambda^2 \ F_{1j}}{F} \ . \qquad (7.13)$$

These solutions can be expressed in a more meaningful form by introducing the definition of the partial elasticity of substitution,

$$\sigma_{1j} \equiv \frac{\sum_i f_i \ x_i}{x_1 \ x_j} \cdot \frac{F_{1j}}{F} \ , \qquad (i,j = 1,2,\ldots,n) \qquad (7.14)$$

and the identities

$$F_j = (-1)^j \sum_k f_k \ F_{0jk} = (-1)^j \sum_k f_k \ F^*_{jk} \cdot \qquad (k = 1,2,\ldots,n) \ (7.15)$$

Using these in equations (7.12) and (7.13) reveals that

$$-\partial\lambda/\partial p_1 = \frac{-\lambda^2 \Sigma \ f_k \ F^*_{1k}}{F} - \frac{\lambda^2 \ x_1 \ F^*}{F} , \qquad (7.16)$$

and

$$\partial x_1/\partial p_1 = \frac{\lambda^2 x_1 \ \Sigma \ f_k \ F^*_{1k}}{F} + \frac{\lambda^2 \ x_1{}^2}{\sum_i f_i \ x_i} \sigma_{11} \qquad (7.17)$$

Because λ and F are positive,

$$\partial\lambda/\partial p_1 \gtrless 0 \quad \text{according as} \quad \Sigma \ f_k \ F^*_{1k} + x_1 \ F^* \lessgtr 0. \qquad (7.18)$$

If input 1 is inferior, $\Sigma \ f_k \ F^*_{1k}$ is positive while $x_1 \cdot F^*$ is also positive. But we know that λ is the reciprocal of marginal cost. Thus marginal cost varies inversely with the price of input 1. Now it should be much easier to appreciate why the optimal output is higher when the price of an inferior input rises. One certainly should expect a firm to increase its output when marginal cost falls. On the other hand, if input 1 were normal, $\Sigma f_k F^*_{1k}$ would be negative while $x_1 \cdot F^*$ is again positive. But when the price of a normal input rises, optimal output decreases. Therefore marginal cost must vary directly with p_1. Hence $x_1 \cdot F^*$ must be less than the absolute value of $\Sigma f_k F^*_{1k}$.

Because F is negative definite, F_{11} is negative when n is even. Thus σ_{11} is negative. The second term on the right-hand side of equation (7.17) may be called the <u>substitution effect</u>, which is clearly negative in all cases. This simply means that the quantity of input 1 demanded varies inversely with changes in input 1's price when output is held constant. If input 1 is normal, the <u>output effect</u>, which is the first term on the right-hand side of (7.17), will also be negative. In this case of normal inputs, when input price changes both the output and the substitution effects operate to move quantity demanded in the opposite direction.

On the other hand, if input 1 is inferior, we shall have a positive output effect. Whether the total effect will result in an increase or decrease in the employment of input 1 depends upon the relative sizes of the two effects. It is possible for $\partial x_1/\partial p_1$ to be positive if the positive output effect is larger (in absolute value) than the negative substitution effect. It must be emphasized that this does not imply that the slope of the firm's input demand function is positive. We have seen in equations (1.11) and (1.12) that the slope of the true demand function is always negative regardless of the type of input considered. Yet we now find that $\partial x_1/\partial p_1$ may be positive. This apparent inconsistency can easily be resolved: in equation (7.17) we are not considering an input demand function. The demand functions for inputs are derived from the profit maximizing conditions. Since revenue considerations never entered the calculations underlying (7.17), an input demand function cannot be derived from the calculus of maximizing output for given levels of total cost.

CHAPTER IV

THE INFLUENCE OF INFERIOR INPUTS UPON EXTERNALITY SOLUTIONS

In the previous chapters, the concepts of external effects and inferior inputs were developed in some detail. We are now in a position to examine the effects of inferior inputs on the solutions to the resource misallocation problem caused by externalities. As previously seen, an externality in production is caused by one firm's use of an input having an effect on another firm's production function. When this situation is further complicated by the externality-causing input's being inferior, we should like to know whether the remedies analyzed in Chapter II must be altered. In order to accomplish this, we shall pose a general problem of externality in which the input causing the trouble is inferior. The various solutions to the problem will then be examined to determine the influence, if any, of input inferiority. Since the analysis will be conducted in partial equilibrium terms, it must be emphasized that the rest of the economy is, and remains, organized so that the Pareto optimum conditions are fulfilled.

1. The General Problem

Suppose there are two firms, i and j, that produce according to the production functions:

$$Q_i = F(x_{i1}, x_{i2}, \ldots, x_{in}, z), \tag{1.1}$$

and

$$Q_j = G(x_{j1}, x_{j2}, \ldots, x_{jn}), \tag{1.2}$$

where x_{ik} and x_{jk} are the amounts of input k used by i and j respectively. We assume that these production functions are continuous and at least twice differentiable. The argument z in (1.1) is defined by

$$z = g(x_{j1}) \tag{1.3}$$

as in equation (1.3), Chapter II. This means that j's use of input x_{j1} yields some accidental by-product or condition that enters i's production function parametrically.

Further, we shall postulate that i and j operate in competitive factor and commodity markets and that each attempts to maximize profits independently. Thus each firm accepts commodity and input prices as given. Moreover, the behavioral assumption of independent profit maximization indicates that each tries to maximize the following profit functions:

$$\pi_i = P_i \cdot F(x_{i1}, x_{i2}, \ldots, x_{in}, z) - \sum_{k=1}^{n} p_k \, x_{ik}, \tag{1.4}$$

and

$$\pi_j = P_j \cdot G(x_{j1}, x_{j2}, \ldots, x_{jn}) - \sum_{k=1}^{n} p_k \, x_{jk}. \tag{1.5}$$

While there is no problem for j, i cannot satisfy all its first-order conditions so long as $\partial F/\partial z \neq 0$.

Employing the definitions established in Chapter II, we shall assume that z creates a marginal externality, i.e.,[1]

$$\partial F/\partial z < 0. \tag{1.6}$$

Further, we shall assume that the external diseconomy is Pareto relevant, i.e.,

$$\left|P_i \cdot \partial F/\partial z\right| > \left|P_j \cdot \partial G/\partial x_{j1} - p_1\right|, \tag{1.7}$$

where P_i and P_j are product prices and p_1 is the price of input x_{j1}. This simply means that not only would i like to change j's use of input x_{j1}, but that it is also possible for i to do so. A change is desirable because j's use of x_{j1} has a deleterious effect on i's operations. Moreover, a change is possible because the damage done to i exceeds the net marginal benefit that j receives from its profit-maximizing usage of x_{j1}.

So long as inequality (1.7) holds, i.e., so long as the externality is Pareto relevant, there is a misallocation

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[1]We must keep in mind that $\partial F/\partial z = \partial F/\partial g(x_{j1}) \cdot dg(x_{j1})/dx_{j1}$.

of resources that prevents society from being on its transformation surface. Only by removing the Pareto relevance of the externality, i.e., by changing inequality (1.7) to

$$\left|P_i \cdot \partial F/\partial z\right| = \left|P_j \cdot \partial G/\partial x_{j1} - p_1\right|, \tag{1.8}$$

can society attain its transformation surface. We might note again that this does not require removal of the marginal externality, but <u>only</u> the removal of its Pareto relevance.

Finally, assume that x_{j1} is an inferior input. As we have seen, this requires

$$\partial Q_j/\partial p_1 = \sum_{k=1}^{n} G_k \ \partial x_{jk}/\partial p_1 > 0, \tag{1.9}$$

where $G_k = \partial G/\partial x_{jk}$. By introducing this additional influence into the externality problem, analysis of the solutions set out in Chapter II, "Solutions for the Externality Problem," will reveal whether any complications result.[2]

2. <u>Solutions</u>

As in Chapter II, we shall only consider procedures that permit society to attain its transformation surface. In particular, we shall again examine the bargaining,

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[2]We shall retain the assumptions made when we first dealt with these solutions, viz., the costs of adjustment are lump sum and the adjustment is worthwhile. See footnote 13 p. 21 for a discussion of these costs and assumptions.

tax-subsidy, and merger solutions under the additional influence of inferior inputs.

Bargaining

As in Chapter II, we must consider the legal prescription for liability as given and analyze the problem of external effects within this context. We shall begin with the case where the law prescribes no liability for j, the source of the externality, and then proceed to the case where j suffers full liability.

No liability for j.--In this situation i is interested in j's reducing the quantity of x_{j1} employed. Because of the inequality in condition (1.7), i.e., because the external diseconomy is Pareto relevant, it is possible for i to induce j to decrease its use of x_{j1} with a resultant mutual advantage. Since there is no legal restriction on j's actions, i must make a monetary appeal to j. Assuming that j's optimal use of x_{j1} is $\overline{x}_{j1}$, the presumption is that i will offer an amount $B(\overline{x}_{j1}-x_{j1})$ to j.[3] In other words, B is the side payment or bribe that i offers j per unit decrease in j's employment of x_{j1}. Recall that x_{j1} is the input that is directly related to z, the externality-causing by-product or condition, which enters i's production function parametrically.

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[3] Of course, no payment will occur if the term in parentheses is negative, i.e., if j increases its use of x_{j1}.

The fact that x_{j1} is inferior has no effect on i because i is merely interested in removing the deleterious influence that x_{j1} creates. If the inferiority of x_{j1} has any influence at all it will appear in j's response to i's offer. We can examine the profit function that j now attempts to maximize:

$$\pi_j = P_j \cdot G(x_{j1}, x_{j2}, \ldots, x_{jn}) + B(\overline{x}_{j1} - x_{j1}) - \sum_{k=1}^{n} p_k \, x_{jk}. \qquad (2.1)$$

The first-order conditions for a regular relative maximum are

$$P_j \cdot \partial G/\partial x_{jk} - p_k = 0 \quad \text{if } x_{jk} > 0, \quad (k = 2, 3, \ldots, n) \qquad (2.2)$$

and

$$P_j \cdot \partial G/\partial x_{j1} - (p_1 + B) = 0 \quad \text{if } x_{j1} > 0. \qquad (2.3)$$

Since $B > 0$,

$$p_1 < (p_1 + B). \qquad (2.4)$$

Thus i's offer is equivalent to an increase in the price of x_{j1}. This is what one should expect. The side payment from i to j is exactly equivalent, from j's point of view, to an increase in the market price of x_{j1}. That is just to say, when externalities exist, some nonmarket mechanism must accomplish what the market would otherwise do. Whether such a price increase will induce the desired change in j's employment of x_{j1} is now the relevant question.

But the answer to this question has already been obtained in Chapter III, equation (1.11). There it was found that the quantity demanded of any input is inversely related to its price. This relation holds whether the input is inferior or not. Therefore, we may conclude that through an iterative procedure, the correct B can be found such that not only will equations (2.2) and (2.3) hold, but the first-order conditions for i will also be fulfilled, i.e.,

$$P_i \cdot \partial F/\partial x_{ik} - p_k = 0 \quad \text{if } x_{ik} > 0, \qquad (k = 1,2,\ldots,n) \qquad (2.5)$$

and

$$P_i \cdot \partial F/\partial z + B = 0.^{4} \qquad (2.6)$$

By solving equation (2.6) for B and substituting into (2.3), we shall find that the bargaining scheme will result in satisfaction of the requirement for Pareto optimality. Thus the inferiority of input x_{j1} has no effect on the bargaining solution in regard to this policy for achieving Pareto optimality when the law prescribes no liability for j's actions. But this inferiority does have an effect upon the resultant configuration of final prices and output.

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[4]These first-order conditions follow from i's altered profit function:

$$\pi_i = P_i \cdot F(x_{i1}, x_{i2}, \ldots, x_{in}, z) - B(\overline{x}_{j1} - x_{j1}) - \sum_{k=1}^{n} p_k\, x_{ik}.$$

From equation (2.6),

$$B = - P_i \cdot \partial F/\partial z. \tag{2.7}$$

Substituting into (2.3) and rewriting yields

$$P_j \cdot \partial G/\partial x_{jl} = p_l - P_i \cdot \partial F/\partial z. \tag{2.8}$$

Since $\partial F/\partial z < 0$ by condition (1.6),

$$P_j \cdot \partial G/\partial x_{jl} = p_l + \left| P_i \cdot \partial F/\partial z \right|. \tag{2.9}$$

Thus j must adjust its usage of x_{jl} such that the value of x_{jl}'s marginal product exceeds its market price by the value of the damage done to i. In other words, when the side payment or bribe is offered, the "effective market price" of x_{jl}, from j's point of view, becomes $p_l + \left| P_i \cdot \partial F/\partial z \right|$. Firm j will then reduce its employment of x_{jl}. Normally, reducing x_{jl} would cause a reduction of output and, if all entrepreneurs followed suit, the market price of commodity j would increase. But in the special case of x_{jl} being an inferior input, output varies directly with the price of the input by definition. Thus the increase in the "effective market price" of x_{jl} causes an expansion of j's output even though the employment of x_{jl} is still reduced. Now, if all entrepreneurs follow suit, the market price of commodity j will fall.

Full liability for j.--Now assume that the law prohibits j's imposing any externality upon i unless i is willing to accept it. It is quite obvious that i will require some compensation in return for its permission to suffer the existence of z. It is also quite obvious that if $\hat{x}_{j1}$ is the optimal quantity of x_{j1} and

$$P_j \cdot \partial G/\partial x_{j1} - p_1 > 0 \quad \text{for} \quad x_{j1} < \hat{x}_{j1}, \tag{2.10}$$

j will find it profitable to compensate i for permission to use some positive amount of x_{j1}.

Suppose j offers i a bribe of B per unit of x_{j1} for i's permission to employ x_{j1}. Firm j's offer will change their profit functions to

$$\pi_i = P_i \cdot F(x_{i1}, x_{i2}, \ldots, x_{in}, z) + B\ x_{j1} - \sum_{k=1}^{n} p_k\ x_{ik}, \tag{2.11}$$

and

$$\pi_j = P_j \cdot G(x_{j1}, x_{j2}, \ldots, x_{jn}) - (B+p_1)\ x_{j1} - \sum_{k=2}^{n} p_k\ x_{jk}. \tag{2.12}$$

Since these profit functions are essentially the same as those where no liability existed, we shall find that the first-order conditions are the same, i.e.,

$$P_j \cdot \partial G/\partial x_{jk} - p_k = 0 \quad \text{if } x_{jk} > 0, \quad (k = 2,3,\ldots,n) \tag{2.13}$$

$$P_j \cdot \partial G/\partial x_{j1} - (B+p_1) = 0 \quad \text{if } x_{j1} > 0, \tag{2.14}$$

$$P_i \cdot \partial F/\partial x_{ik} - p_k = 0 \quad \text{if } x_{ik} > 0, \quad (k = 1,2,\ldots,n) \tag{2.15}$$

and

$$P_i \cdot \partial F/\partial z + B = 0. \tag{2.16}$$

In light of the results (2.13) - (2.16), we can see that the effect of j's offer of a bribe is equivalent to an increase in the price of input x_{jl}. Because we have shown that, irrespective of the type of input under consideration, an increase in input price causes a decrease in the quantity demanded, this legal arrangement will also lead to Pareto optimal input combinations. The direction of payment is reversed in this instance; but this is irrelevant to questions of efficiency.[5]

Here again, however, the inferiority of input x_{jl} will affect final prices and output. In order for j to employ x_{jl}, j must make a side payment to i in addition to paying the usual market price for the input. Thus the "effective market price" is increased from j's point of view. Of course, this results in j's decreasing its use of x_{jl} below the level that would prevail in the absence of any legal liability. But, because x_{jl} is inferior, output and input price vary directly. Therefore, the increase in the "effective market price" causes an expansion of j's output. Again, if all entrepreneurs follow suit, the market price of commodity j will fall.

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[5]The intermediate case mentioned in Chapter II, "Bargaining," could also be analyzed as these polar cases have been. Clearly, the result would be the same: Pareto optimality would ensue.

Taxes and Subsidies

As we have seen, the tax-subsidy approach represents an alternative to the bargaining scheme.[6] Since j's use of input x_{j1} is causing the external diseconomy, this alternative solution requires the levy of an appropriate tax on j's use of x_{j1} and an equal subsidy payment to i. Although the result of this approach is fairly obvious, it is worth demonstrating.

Let T represent both the tax on j's use of x_{j1} and the subsidy to i. The profit functions of i and j now become

$$\pi_i = P_i \cdot F(x_{i1}, x_{i2}, \ldots, x_{in}, z) + Tx_{j1} - \sum_{k=1}^{n} p_k\, x_{ik}, \qquad (2.17)$$

and

$$\pi_j = P_j G(x_{j1}, x_{j2}, \ldots, x_{jn}) - (T+p_1)\, x_{j1} - \sum_{k=2}^{n} p_k\, x_{jk}. \qquad (2.18)$$

Thus the first-order conditions are

$$P_i \cdot \partial F/\partial x_{ik} - p_k = 0 \quad \text{if } x_{ik} > 0, \quad (k = 1, 2, \ldots, n) \qquad (2.19)$$

$$P_i \cdot \partial F/\partial z + T = 0, \qquad (2.20)$$

$$P_j \cdot \partial G/\partial x_{jk} - p_k = 0 \quad \text{if } x_{jk} > 0, \quad (k = 2, 3, \ldots, n) \qquad (2.21)$$

and

$$P_j \cdot \partial G/\partial x_{j1} - (T+p_1) = 0 \quad \text{if } x_{j1} > 0. \qquad (2.22)$$

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[6] See Chapter II, "Taxes and Subsidies."

Equation (2.20) shows that the appropriate tax to be levied on j is one that equals the value of the damage done to i.

Clearly, these first-order conditions are identical to those obtained by private bargaining. If we solve equation (2.20) for T and substitute into equation (2.22), we obtain

$$\left|P_j \cdot \partial G/\partial x_{j1} - p_1\right| = \left|P_i \cdot \partial F/\partial z\right|, \tag{2.23}$$

i.e., we obtain the condition for Pareto equilibrium. Rearranging this equation to

$$P_j \cdot \partial G/\partial x_{j1} = p_1 + \left|P_i \cdot \partial F/\partial z\right| \tag{2.24}$$

shows that the effect of the tax is to induce j into equating the value of x_{j1}'s marginal product with the price of x_{j1} plus the value of the damage its use does to i's operation. Again, input inferiority has no effect on this policy for achieving Pareto optimality because the price of x_{j1} is raised by the tax, and this causes a decrease in the quantity of x_{j1} demanded.

But from equation (2.20),

$$T = - P_i \cdot \partial F/\partial z \tag{2.25}$$

Thus the tax and subsidy are precisely the same as the bribe or side payment in the bargaining case. As one might expect, the results will be the same: the increased "effective market price" of x_{j1} will cause a reduction in the employment of x_{j1} and a consequent increase in j's output. As before, if all entrepreneurs follow suit, this will lead to a reduction in the market price of commodity j.

The result of taxing j's output when x_{j1} is inferior can now be demonstrated.[7] If, despite all objections by economists that taxing output will render j's choices for all inputs but x_{j1} non-optimal, the government decides to tax output to remove the Pareto relevance of the external diseconomy caused by j's operation, the direct opposite of the desired result will occur when x_{j1} is an inferior input. To demonstrate this, suppose a per unit tax t is levied on j's output Q_j. Firm j's profit function will then be

$$\pi_j = P_j \cdot G(x_{j1}, x_{j2}, \ldots, x_{jn}) - \sum_{k=1}^{n} p_k\, x_{jk} - t\, Q_j. \qquad (2.26)$$

Because

$$Q_j = G(x_{j1}, x_{j2}, \ldots, x_{jn}), \qquad (2.27)$$

equation (2.26) can be re-written as

$$\pi_j = (P_j - t) \cdot G(x_{j1}, x_{j2}, \ldots, x_{jn}) - \sum_{k=1}^{n} p_k\, x_{jk}. \qquad (2.28)$$

Clearly, the effect of the per unit tax is to decrease the "effective market price" of j's output rather than to increase the "effective market price" of the input that causes the externality. We have seen in equation (2.10), Chapter III, that

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[7]This was mentioned in Chapter II, "Taxes and Subsidies," without explanation. The result was demonstrated graphically by C. Plott in "Externalities and Corrective Taxes," <u>Economica</u>, XXXIII, 84-86.

$$\partial Q_j/\partial p_1 = -\ \partial x_{j1}/\partial p_j \tag{2.29}$$

in all cases. But the definition of input inferiority requires that $\partial Q_j/\partial p_1$ be positive. Thus expression (2.29) implies that the quantity of the inferior input x_{j1} must vary inversely with commodity price. Since x_{j1} causes the externality and the tax is supposed to remove this effect, we can now see the perverse result that Plott's graph demonstrated, viz., a tax on Q_j, by decreasing its effective market price, causes j to increase its employment of x_{j1}. This is obviously the opposite of the desired result.

Mergers

We can now turn to the third solution: merger of the firms involved. The effect of the merger will be to internalize the external diseconomy. This occurs because the new firm's profit function is

$$\begin{aligned}\pi = \pi_i + \pi_j = {} & P_i \cdot F(x_{i1}, x_{i2}, \ldots, x_{in}, z) - \sum_{k=1}^{n} p_k\, x_{ik} \\ & + P_j \cdot G(x_{j1}, x_{j2}, \ldots, x_{jn}) - \sum_{k=1}^{n} p_k\, x_{jk}.\end{aligned} \tag{2.30}$$

The first-order conditions for profit maximization are

$$P_i \cdot \partial F/\partial x_{ik} - p_k = 0 \quad \text{if } x_{ik} > 0, \quad (k = 1, 2, \ldots, n) \tag{2.31}$$

$$P_j \cdot \partial G/\partial x_{jk} - p_k = 0 \quad \text{if } x_{jk} > 0, \quad (k = 2, 3, \ldots, n) \tag{2.32}$$

and

$$P_i \cdot \partial F/\partial z + P_j \cdot \partial G/\partial x_{j1} - p_1 = 0 \quad \text{if } x_{j1} > 0. \tag{2.33}$$

It is obvious that condition (2.33) can be rearranged to read

$$\left|P_i \cdot \partial F/\partial z\right| = \left|P_j \cdot \partial G/\partial x_{j1} - p_1\right|, \tag{2.34}$$

which is the condition for Pareto equilibrium.

While the other proper solutions involved raising the price of the externality-causing input, the merger solution yields the desired result for a slightly different reason. The new, merged firm fully appreciates all the costs of employing x_{j1}, i.e., the decrease in Q_i for any given vector of inputs $x_{i1}, x_{i2}, \ldots, x_{in}$ as well as the price of x_{j1}. Thus the optimal amount of x_{j1} will be used. Since there are no longer any external effects, the fact that x_{j1} is inferior can have no bearing on questions of Pareto optimality. However, as in the bargaining and tax-subsidy cases above, the fact that x_{j1} is inferior leads to a different output configuration than would otherwise obtain.

3. Conclusion

In Chapter III, we saw the effects of input inferiority on the theory of production and the theory of derived demand. But in this chapter we found that input inferiority does not alter any of the solutions to the resource misallocation problem created by external effects. Since the possibility of the existence of inferior inputs cannot be ignored and their empirical significance is quite difficult to assess, this is a rather encouraging result. Moreover, since external effects cause enough complications by themselves, it may be a blessing that inferior inputs do not further complicate matters.

In the case of a non-inferior input, which is the cause of a Pareto relevant externality, full employment of all resources leaves society short of the transformation surface because the full utilization of the externality-causing input has a deleterious effect upon the operation of some firm or firms. When the Pareto relevance of the externality is removed by bargaining, tax-subsidy manipulations, or internalization through merger, the offending firm employs less of the externality-causing input and consequently produces less output. The inputs released by the offending firm will then be employed by other firms whose usage of the inputs does not lead to external effects. This will expand their outputs and permit society to attain the transformation surface.

But if the externality-causing input is inferior to j and non-inferior to other firms, the result is somewhat different. Removal of the externality's Pareto relevance through bargaining, tax-subsidy manipulations, or internalization will still cause the offending firm to decrease its usage of the inferior, externality-causing input; however, the firm's output will now increase as we have seen above. The released inputs will now be transferred to firms whose usage of the inputs does not give rise to external effects. Since the externality-causing input is not inferior to these firms, their outputs will also increase. Thus, in this case, removal of the externality's Pareto relevance actually causes an outward shift of the transformation surface.

APPENDIX A

DERIVATION OF COST FUNCTIONS WITH EXTERNALITIES PRESENT

While the term cost equation refers to cost expressed in terms of quantities of inputs and their respective prices, the term cost function denotes cost as a function of output. Cost functions can be used in the profit maximization problem by assuming that the firm employs optimum input combinations for all levels of output. Then the profit function becomes:

$$\pi = P \cdot Q - f(Q) \tag{A.1}$$

where P is the product price, Q is the quantity of output, and f(Q) is the cost function. The first-order condition for profit maximization is now

$$d\pi/dQ = P - df(Q)/dQ = 0. \tag{A.2}$$

Since df(Q)/dQ is obviously marginal cost, we have the usual condition for profit maximization: price equals marginal cost. The derivation of the cost functions, assuming the presence of an externality, can be revealing for our purposes. Therefore, I shall derive the cost function for a firm operating in the presence of an externality under the assumptions of (1) a Cobb-Douglas-type production function and (2) a CES-type production function.

Cobb-Douglas

Let the production function be

$$Q = A\, x_1^a\, x_2^b\, z^c \tag{A.3}$$

where x_1, x_2 are factors of production under i's control and z is the externality-causing output of j that enters i's production function parametrically. Differentiating the function partially with respect to the inputs yields the following marginal products:

$$\text{MP of } x_1 = \partial Q/\partial x_1 = aQ/x_1, \tag{A.4}$$

$$\text{MP of } x_2 = \partial Q/\partial x_2 = bQ/x_2,$$

and

$$\text{MP of } z = \partial Q/\partial z = cQ/z.$$

In accord with the definitions set out in Chapter II, "The Definition of Externality," we have an external diseconomy if $\partial Q/\partial z < 0$ which requires $c < 0$. On the other hand, $c > 0$ implies $\partial Q/\partial z > 0$, and we would have an external economy.

Since the value of z is given parametrically, the necessary conditions for profit maximization are

$$a \cdot x_2/b \cdot x_1 = p_1/p_2, \tag{A.5}$$

where p_1, p_2 are the factor prices. Solving equation (A.5) for x_2 and substituting into equation (A.3) yields

$$Q = A \cdot k^b \cdot z^c \cdot x_1^{a+b}, \tag{A.6}$$

where $k = p_1 \cdot b/p_2 a$. Then

$$x_1 = (Q/Ak^b \cdot z^c)^{1/a+b} \tag{A.7}$$

and similarly

$$x_2 = (Q/A \cdot k^{-a} \cdot z^c)^{1/a+b}. \tag{A.8}$$

Substituting these results into the cost equation gives us the cost function:

$$C = p_1 \cdot (Q/A \cdot k^b \cdot z^c)^{1/a+b} + p_2 \cdot (Q/A \cdot k^{-a} \cdot z^c)^{1/a+b}. \tag{A.9}$$

An interesting point to note is that this cost function is not separable. Let us next consider a CES-type production function.

CES-Type

Let the production function be

$$Q = A(a_1 \cdot x_1^b + a_2 \cdot x_2^b + a_3 \cdot z^d)^{1/g}. \tag{A.10}$$

In this case, the marginal products are

$$\text{MP of } x_1 = \partial Q/\partial x_1 = Qa_1 \cdot b \cdot x_1^{b-1}/g(Z), \tag{A.11}$$

$$\text{MP of } x_2 = \partial Q/\partial x_2 = Q \cdot a_2 \cdot b \cdot x_2^{b-1}/g(Z),$$

and

$$MP \text{ of } z = \partial Q/\partial z = Q \cdot a_3 \cdot d \cdot z^{d-1}/g(Z)$$

where Z represents the term in parentheses in equation (A.10). Again, if $d < 0$, we have an external diseconomy; if $d > 0$, we have an external economy. Introducing z parametrically, we have as necessary conditions

$$a_1 \cdot x_1^{b-1}/a_2 \cdot x_2^{b-1} = p_1/p_2. \qquad \text{(A.12)}$$

Solving (A.12) for x_1 yields

$$x_1 = (p_1 \cdot a_2 \cdot x_2^{b-1}/p_2 \cdot a_1)^{1/b-1}. \qquad \text{(A.13)}$$

Substituting into equation (A.10) and letting W = $(p_1 a_2/p_2 a_1)^{b/b-1}$, we have

$$Q = A(a_1 \cdot W \cdot x_2^b + a_2 \cdot x_2^b + a_3 \cdot z^d)^{1/g}. \qquad \text{(A.14)}$$

Then

$$x_2 = ((Q/A)^g - a_3 \cdot z^d)^{1/b} \cdot (a_1 \cdot W + a_2)^{-1/b} \qquad \text{(A.15)}$$

and similarly

$$x_1 = ((Q/A)^g - a_3 \cdot z^d)^{1/b} \cdot (a_1 + a_2 \cdot W^{-1})^{-1/b}. \qquad \text{(A.16)}$$

Now, substituting these results into the cost equation gives us the cost function:

$$C = p_1 \cdot \left[\frac{(Q/A)^g - a_3 \cdot z^d}{a_1 + a_2 \cdot W^{-1}}\right]^{1/b} + p_2 \cdot \left[\frac{(Q/A)^g - a_3\ z^d}{a_1 \cdot W + a_2}\right]^{1/b}. \qquad \text{(A.17)}$$

Note that in this case we also do not have separable cost functions. Further, we can see the effect of the externality in equations (A.4) and (A.11): the marginal products of the firm's own inputs are decreased in the diseconomy case and increased in the external economy case.

An Example of a Separable Cost Function

Let the production function be

$$Q = x_1^{\alpha} + x_2^{\alpha} + z^{\beta}. \tag{A.18}$$

The marginal products are

$$\begin{aligned} \text{MP of } x_1 &= \partial Q/\partial x_1 = \alpha x_1{}^{\alpha-1}, \\ \text{MP of } x_2 &= \partial Q/\partial x_2 = \alpha x_2{}^{\alpha-1}, \\ \text{MP of } z &= \partial Q/\partial Z = \beta Z^{\beta-1}. \end{aligned} \tag{A.19}$$

If $\partial Q/\partial Z < 0$, Z causes a marginal external diseconomy. The necessary conditions for a profit maximum are

$$\frac{x_1{}^{\alpha-1}}{x_2{}^{\alpha-1}} = \frac{p_1}{p_2}\ . \tag{A.20}$$

Solving (A.20) for x_1 yields

$$x_1 = \left(\frac{p_1\ x_2{}^{\alpha-1}}{p_2}\right)^{1/\alpha-1} . \tag{A.21}$$

Substituting into the production function gives

$$Q = \left(\frac{p_1 \; x_2{}^{\alpha-1}}{p_2}\right)^{\alpha/\alpha-1} + x_2^{\alpha} + z^{\beta} \; . \tag{A.22}$$

Thus

$$x_2 = \left\{\frac{Q + z^{\beta}}{(p_1/p_2)^{\alpha/\alpha-1} + 1}\right\}^{1/\alpha} , \tag{A.23}$$

and, similarly,

$$x_1 = \left\{\frac{Q + z^{\beta}}{1 + (p_2/p_1)^{\alpha/\alpha-1}}\right\}^{1/\alpha} .$$

These values for x_1 and x_2 can be substituted into the cost equation to obtain the cost function:

$$C = p_1 \left\{\frac{Q + z^{\beta}}{1 + (p_2/p_1)^{\alpha/\alpha-1}}\right\}^{1/\alpha} + p_2 \left\{\frac{Q + z^{\beta}}{(p_1/p_2)^{\alpha/\alpha-1} + 1}\right\}^{1/\alpha} . \tag{A.24}$$

Since this cost function can be written as

$$\begin{aligned} C = {} & p_1 \left\{\frac{Q}{1 + (p_2/p_1)^{\alpha/\alpha-1}}\right\}^{1/\alpha} + p_1 \left\{\frac{z^{\beta}}{1 + (p_2/p_1)^{\alpha/\alpha-1}}\right\}^{1/\alpha} \\ & + p_2 \left\{\frac{Q}{(p_1/p_2)^{\alpha/\alpha-1} + 1}\right\}^{1/\alpha} + p_2 \left\{\frac{z^{\beta}}{(p_1/p_2)^{\alpha/\alpha-1} + 1}\right\}^{1/\alpha} , \end{aligned} \tag{A.25}$$

this cost function is separable. That this result occurs is apparent from the form the marginal products take, i.e., they are independent. It appears that so long as the externality-causing input enters the production function in a purely additive way, the resultant cost function will be separable.

APPENDIX B

AN ALTERNATIVE VIEW OF THE EXTERNALITY-CAUSING INPUT

I have been assuming that successive increases in z will cause increasing diminutions of output for i. The inclusion of z in i's production function makes it difficult to visualize the corresponding isoquants, or more correctly, the corresponding iso-surfaces. We may be able to gain greater insight by altering some of our concepts. First, let us consider i's production function:

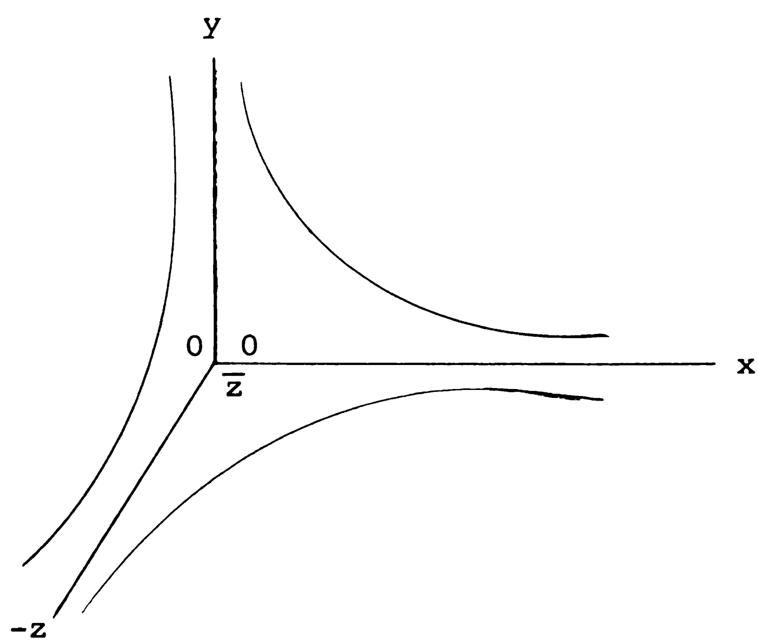
$$Q_i = F(x_{i1}, x_{i2}, \ldots, x_{in}, \bar{z}). \tag{B.1}$$

Note that I have replaced z with $\bar{z}$ to indicate that j has decided upon its own optimum output and, therefore, upon the quantity of its inputs and, in particular, the quantity of x_{j1}. Recall that $z = g(x_{j1})$. When the production function has a specific form and a definite value is assigned to z we know the output possibilities that confront i. Any of these possibilities may be attained by employing certain quantities of the n inputs. Naturally, the particular output and the corresponding vector of inputs decided upon will depend upon the price of the product and the prices of the inputs. The conceptual problem arises when we allow z to vary

because a positive z imposes a negative benefit on i. Although it is natural for one to recognize that the x_{ik} $(k = 1,2,\ldots,n)$ represent positive changes from zero, it is somewhat unusual to think of $\bar{z}$ as the origin for changes in z from i's point of view. But I am suggesting we do this so that firm i's production function becomes

$$Q_i = F(x_{i1}, x_{i2}, \ldots, x_{in}, \bar{z}-z). \tag{B.2}$$

All the constructs that were previously developed could easily be reworked in these terms. The advantage of using this formulation is that when i undertakes to change the value of z, we can differentiate equation (B.2) with respect to -z. The marginal product of i's increased employment of negative units of z is then positive. This alteration will render convex to the origin any isoquant's projection onto the relevant plane in (n+1)-space. In a three-dimensional example, all we need do is label the axes correctly:



Note, in particular, the labeling of the $-z$ axis and the definition of the z origin. Units of negative z are now "proper" inputs and the firm's iso-surfaces are concave from above. In addition, the assumption that increases in z yield increasing diminutions of i's output implies that the marginal productivity of negative units of z is decreasing, i.e., $\partial^2 F/\partial(-z)^2 < 0$. It is now clear why we previously treated z as a nearly "normal" input: negative units of z are normal inputs.

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