TESTING TWO FAILURE RATES WITH RANDOMLY RIGHT CENSORED DATA AND UNCENSORED DATA

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ABSTRACT

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We study the problem of testing H_0: $h_1(t) = h_2(t)$ almost everywhere Lebegue measure (μ) versus the alternatives H_1': $h_2(t) \geq h_1(t)$ almost everywhere (μ) and $h_2(t) \neq h_1(t)$ almost everywhere (μ) with randomly right censored data and uncensored data where h_i is the failure rate function for the right sided distribution function F_i, $i = 1,2$. We show that the functional

$$\Delta(\varphi;\ F_1,\ F_2) = \int_0^\infty \int_0^t \{F_1(t)F_2(s) - F_1(s)F_2(t)\}d\varphi(s)d\varphi(t)$$

where φ is a strictly increasing continuous function has the property that for $(F_1, F_2) \in H_0 \cup H_1$ then $\Delta(\varphi; F_1, F_2) = 0$ if and only if $(F_1, F_2) \in H_0$ and $\Delta(\varphi; F_1, F_2) > 0$ if and only if $(F_1, F_2) \in H_1$.

In the uncensored case, we give sufficient conditions so that the estimator $\hat{\Delta}(\varphi) = \Delta(\varphi; F_m, F_n)$ is asymptotically normal where F_m, F_n are the right sided empirical distributions of F_1, F_2. In the randomly right censored case, we give sufficient conditions so that the estimator $\tilde{\Delta}(\varphi) = \Delta(\varphi; \hat{F}_m, \hat{F}_n)$ is asymptotically normal where $\hat{F}_m$, $\hat{F}_n$ are the Susarla-Van Ryzin estimators of F_1, F_2. We also derive a consistent estimator for the asymptotic variance of $\tilde{\Delta}(\varphi)$ under H_0.

Kochar (1979) studied the above problem only for the uncensored case where he used the functional

$$\delta(F_1, F_2) = \int_0^\infty \int_0^t \{F_1(t)F_2(s) - F_1(s)F_2(t)\}d(\lambda\bar{F}_1(s) + (1-\lambda)\bar{F}_2(s)d(\bar{F}_1(t) + (1-\lambda)\bar{F}_2(t))$$

where $0 < \lambda < 1$ and $\bar{F}_i = 1-F_i$, $i = 1,2$. In this paper, we show that the estimator $\delta(\hat{F}_m, \hat{F}_n)$ is asymptotically normal under appropriate conditions. We also derive a consistent estimator for its asymptotic variance under H_0.

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INTRODUCTION AND SUMMARY

Let $X_1,\ldots,X_m$ be iid F_1; $X_1',\ldots,X_m'$ be iid G_1; $Y_1,\ldots,Y_n$ be iid F_2; and $Y_1',\ldots,Y_n'$ be iid G_2, where F_1, F_2, G_1, G_2 are right sided absolutely continuous distribution functions, i.e., $F_1(t) = P(X_1 > t)$, such that $F_1(0) = F_2(0) = G_1(0) = G_2(0) = 1$. All the random variables are independent. Also the life distributions F_1, F_2 are on $[0,\infty)$ with densities $-f_1$, $-f_2$ respectively.

We get to observe $U_i = \min(X_i, X_i')$, $\delta_i = [X_i < X_i']$ for $1 \leq i \leq m$ and $V_j = \min(Y_j, Y_j')$, $\gamma_j = [Y_j < Y_j']$ for $1 \leq j \leq n$, where $[A]$ is the indicator of the set A.

This model is called the two sample randomly censored model. In it the random variables X_j' and Y_j' are the censoring random variables. The G_i's may have all their mass at ∞ which corresponds to the uncensoring model.

On the basis of the above observations, we wish to test H_0: $h_1 = h_2$ a.e. Lebesgue measure (μ) vs. H_1: $h_2(t) \geq h_1(t)$ a.e. (μ) and $h_2(t) \neq h_1(t)$ a.e. (μ) where h_1, h_2 are the failure rates of F_1, F_2 respectively, i.e., for $i = 1, 2$,

$$(1.1) \qquad h_i(t) = \begin{cases} f_i(t)/F_i(t) & \text{if } F_i(t) > 0 \\ \\ +\infty & \text{if } F_i(t) = 0 \end{cases}$$

Our plan of constructing a test statistic is the following. Let $\mathcal{F} = \{(F_1, F_2)$: F_1, F_2 are right sided absolutely continuous distribution functions and $F_1(0) = F_2(0) = 1\}$. For $(F_1, F_2) \in \mathcal{F}$, let

(1.2) $$D(s,t) = F_1(t)F_2(s) - F_1(s)F_2(t) \quad \text{for} \quad t \geq s \geq 0$$

and

(1.3) $$T = \sup\{t: \ F_2(t) > 0\}.$$

If $D(s,t) \geq 0$ for all $t \geq s \geq 0$ then $F_2(t) \leq F_1(t)$ for all t since $F_1(0) = F_2(0) = 1$. Thus if $D(s, t) \geq 0$ for all $t \geq s \geq 0$ then $h_2(t) \geq h_1(t)$ for all $t \geq T$. Now for $t < T$ and $D(s, t) \geq 0$ for all $t \geq s \geq 0$, we have that

(1.4) $$\frac{F_1(t)}{F_2(t)} \geq \frac{F_1(s)}{F_2(s)} \quad \text{for all} \quad T > t \geq s \geq 0.$$

But (1.4) implies that $\frac{F_1(t)}{F_2(t)}$ is a nondecreasing function for $t < T$. Since F_1, F_2 are absolutely continuous and $\frac{F_1}{F_2}$ is nondecreasing for $t < T$, we have that

(1.5) $$\frac{f_2(t)F_1(t) - f_1(t)F_2(t)}{F_2^2(t)} \geq 0$$

for almost all $t < T$ where almost all is with respect to Lebesgue measure. But (1.5) is equivalent to $h_2(t) \geq h_1(t)$ for almost all $t < T$. Thus, if $(F_1, F_2) \in \mathcal{F}$ and $D(s, t) \geq 0$ for all $t \geq s \geq 0$ then $(F_1, F_2) \in H_0 \cup H_1$.

Now let $(F_1, F_2) \in H_0 \cup H_1$, then for $t \geq T$ we have that

$D(s, t) = F_1(t)F_2(s) \geq 0$ where $s \leq t$. Also notice that if $(F_1, F_2) \in H_0 \cup H_1$, then $F_1(t) = 0$ implies that $F_2(t) = 0$. Thus if $t < T$ then $F_1(t) > 0$ and $F_2(t) > 0$. Since $(F_1, F_2) \in H_0 \cup H_1$, $h_2(t) \geq h_1(t)$ for almost all $t < T$ then implies that (1.5) holds for almost all $t < T$. Since F_1, F_2 are absolutely continuous and (1.5) holds for almost all $t < T$, we have that $\frac{F_1}{F_2}$ is a nondecreasing function for almost all $t < T$. But F_1, F_2 are continuous, therefore, $\frac{F_1}{F_2}$ is nondecreasing for all $t < T$. Hence (1.4) holds for all $T > t \geq s \geq 0$. Therefore, if $(F_1, F_2) \in H_0 \cup H_1$ then $D(s, t) \geq 0$ for all $t \geq s \geq 0$. Thus for $(F_1, F_2) \in F$, we have that

$$(1.6) \qquad (F_1, F_2) \in H_0 \cup H_1 \quad \text{iff} \quad D(s, t) \geq 0 \quad \text{for all} \quad t \geq s \geq 0.$$

Now let $\varphi\colon [0, \infty) \to [0,\infty)$ be a nondecreasing right continuous function such that $\varphi(0) = 0$ and let

$$(1.7) \qquad \Delta(\varphi) = \Delta(\varphi; F_1, F_2) = \int_0^\infty \int_0^t D(s, t)d\varphi(s)d\varphi(t).$$

If $(F_1, F_2) \in H_0$ then $\Delta(\varphi) = 0$ since $D(s, t) \equiv 0$. But if $\Delta(\varphi) = 0$ and $(F_1, F_2) \in H_0 \cup H_1$ then $D(s, t) = 0$ a.e. with respect to $\mu_\varphi \times \mu_\varphi$ where μ_φ is the measure induced by φ. Suppose $D(s, t) = 0$ a.e. with respect to $\mu_\varphi \times \mu_\varphi$ and there exist a point (x_0, y_0) such that $x_0 < y_0$ and $D(x_0, y_0) > 0$. Then D continuous implies there exists $(a, b) \times (c,d)$ such that $(x_0, y_0) \in (a, b) \times (c, d)$, and for all $(x, y) \in (a, b) \times (c, d)$, $D(x, y) > 0$. Thus if φ is strictly increasing and continuous then $\mu_\varphi \times \mu_\varphi [(a, b) \times (c, d)] = (\varphi(b)-\varphi(a)) \times (\varphi(d)-\varphi(c)) > 0$. Therefore, $D(x, y) = 0$ for all $y \geq x \geq 0$ if $\Delta(\varphi) = 0$ and φ is a strictly increasing continuous function. Hence for $(F_1, F_2) \in H_0 \cup H_1$ and for those φ which are strictly increasing and continuous we have that $\Delta(\varphi) = 0$ iff $(F_1, F_2) \in H_0$ and $\Delta(\varphi) > 0$ iff $(F_1, F_2) \in H_1$. Therefore, $\Delta(\varphi)$ is a reasonable measure of departure of H_1 from H_0 for a given pair (F_1, F_2).

Observe that H_1 implies that $\bar{F}_2 \geq \bar{F}_1$ where $\bar{F} = 1-F$. From Doksum (1969) we recall that $\bar{F}_2$ is tail ordered with respect to $\bar{F}_1$ $(\bar{F}_2 <_t \bar{F}_1)$ if and only if, $\bar{F}_1^{-1}\, \bar{F}_2(x) - x$ is nondecreasing in $x \geq 0$. Observe that $\bar{F}_2 <_t \bar{F}_1$ and $\bar{F}_1^{-1}\, \bar{F}_2(0) = 0$ imply $\bar{F}_1^{-1}\, \bar{F}_2(x) \geq x$ for all $x \geq 0$ which is equivalent to saying $\bar{F}_2(x) \geq \bar{F}_1(x)$ for all $x \geq 0$. On the other hand $\bar{F}_2 <_t \bar{F}_1$ and h_1 nondecreasing and $f_1 > 0$ a.e. imply $h_2 \geq h_1$. To see this observe that $\bar{F}_2 <_t \bar{F}_1$ implies $f_2(x) \geq f_1(\bar{F}_1^{-1}\, \bar{F}_2(x))$ for all $x \geq 0$ which in turn yields $h_2(x) \geq f_1(\bar{F}_1^{-1}\, \bar{F}_2(x))/\{1-\bar{F}_1(\bar{F}_1^{-1}\, (\bar{F}_2(x))\} = h_1(\bar{F}_1^{-1}\, \bar{F}_2(x)) \geq h_1(x)$ for all $x \geq 0$. Thus the alternatives H_1 are smaller than the slippage alternatives H_a: $\bar{F}_2 \geq \bar{F}_1$ with strict inequality somewhere, and larger than the alternatives $\bar{F}_2 <_t \bar{F}_1$, $\bar{F}_1$ I. F. R and $f_1 > 0$ a.e.

In Chapter 1, we study the uncensored model where we use $\hat{\Delta}_N(\varphi) = \Delta(\varphi\, F_m, F_n)$ as an estimator of $\Delta(\varphi)$ where F_m, F_n are the right sided empirical distribution function for F_1, F_2 and $N = m + n$. We obtain the asymptotic normality of $\sqrt{N}\,(\hat{\Delta}_N(\varphi) - \Delta(\varphi))$ and obtain a consistent estimator of its asymptotic variance. Also a member of this class of tests, one based on $\varphi(x) = x$, is found to have asymptotic Pitman efficency relative to the locally most powerful rank test equal to .84375 at $F_\theta(x) = e^{-px} \exp\{-\frac{\theta}{2} p^2 x^2\}$, $\theta > 0$, equal to .7875 at $F_\theta(x) = (1 - \theta)e^{-x} + \theta e^{-x}(1 - (1-e^{-x})^3)$, $\theta > 0$, and equal to 1 at $F_\theta(x) = 1 - \exp(-\{x + \theta(x + e^{-x}-1)\})$, $\theta > 0$, the Makeham distribution.

Whereas, the asymptotic Pitman efficeny of the Wilcoxon test relative to the locally most powerful rank test (L.M.P.R.) are .0938, .35 and .25 at the above alternatives. Also the asymptotic Pitman efficency of the Savage test relative to the L.M.P.R. test are .5, .7292 and .75 at the above alternatives.

In Chapter 2, we study in the randomly censored model the asymptotic normality of $\sqrt{N}\,(\tilde{\Delta}_N(\varphi) - \Delta_N(\varphi))$ where $\Delta_N(\varphi) = \int_0^{M_N} \int_0^t D(s, t)\, d\,\varphi(s) d\varphi(t)$, $\tilde{\Delta}_N(\varphi) = \int_0^{M_N} \int_0^t \{\hat{F}_m(t)\hat{F}_n(s) - \hat{F}_m(s)\hat{F}_n(t)\}\, d\,\varphi(s) d\,\varphi(t)$; $\hat{F}_m$ and $\hat{F}_n$ are the modified Susarla and Van Ryzen estimators of F_1 and F_2; and $M_n \uparrow T = \sup\{t: F_2(t) > 0\}$ at an appropriate rate, under the assumption that $T \leq \sup\{t: G_2(t) > 0\}$ and $\sup\{t: F_1(t) > 0\} \leq \sup\{t: G_1(t) > 0\}$. We also obtain a consistent estimator for the asymptotic null variance of $\sqrt{N}\,\tilde{\Delta}_N(\varphi)$.

Kochar (1979) studied the above problem for the uncensored case using

$$\delta(F_1, F_2) = \int_0^{\infty} \int_0^t D(s, t) d(\lambda\overline{F}_1(s) + (1-\lambda)\overline{F}_2(s)) d(\lambda\overline{F}_1(t) + (1-\lambda)\overline{F}_2(t))$$

and $\hat{\delta}(F_m, F_n)$ instead of $\Delta(\varphi)$ and $\hat{\Delta}(\varphi)$ where $\overline{F}(t) = 1 - F(t)$. He found that the asymptotic Pitman efficency of the $\hat{\delta}$ test relative to the L.M.P.R. test is equal to .8203 at $F_\theta(x) = e^{-(\theta+1)x}$, $\theta > 0$, equal to .9843 at $F_\theta(x) = (1-\theta)e^{-x} + \theta\, e^{-x}(1-e^{-x})^2)$, $\theta > 0$ and equal to .7813 at $F_\theta(x) = (1-\theta)e^{-x} + \theta\, e^{-x}(1-(1-e^{-x})^3)$, $\theta > 0$. Whereas, the asymptotic Pitman efficency of the Wilcoxon test relative to the L.M.P.R. test is equal to .75, .625, and .35 at the above alternatives. In Chapter 3, we will study the asymptotic normality of $\sqrt{N}\,(\tilde{\delta}_N - \delta_N)$ for the randomly censored case where

$$\delta_N = \delta_N\,(F_1,\ F_2) = \int_0^{M_N}\int_0^{t} D(s,\ t)d(\lambda_N\overline{F}_1(s) + (1-\lambda_N)\overline{F}_2(s)d(\lambda_N\overline{F}_1(t)$$

$$+ (1-\lambda_N)\overline{F}_2(t)),$$

$\tilde{\delta}_N = \delta_N(\hat{F}_m,\ \hat{F}_n)$, $\lambda_N = \frac{m}{N}$, and $M_N \uparrow T$ at an appropriate rate.

In what follows "$\underset{P}{\to}$" means "converges in probability to" and "$\underset{\mathcal{D}}{\to}$" means "converges in distribution to".

CHAPTER I

$\hat{\Delta}_N(\varphi)$ IN THE UNCENSORED CASE

1. Asymptotic Normality of $\sqrt{N}(\hat{\Delta}_N(\varphi) - \Delta(\varphi))$.

In this section, we will restrict the model of Chapter 1 to the uncensored case, i.e., G_1 and G_2 put all their mass at ∞. Hence $U_i \equiv X_i$, $\delta_i \equiv 1$, $V_j \equiv Y_j$ and $\gamma_j \equiv 1$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.

Let $N = m + n$, $F_m(t) = \frac{1}{m} \sum_{i=1}^{m} [X_i > t]$, $F_n(t) = \frac{1}{n} \sum_{j=1}^{n} [Y_j > t]$, and $h_\varphi(x; y) = (\varphi(x) - \varphi(y))(\varphi(y)[y < x] + \varphi(x)[y \geq x])$. Also assume that $0 < \lambda < 1$, where $\lambda = \lim_{N\to\infty} \frac{m}{N}$.

Observe that $\hat{\Delta}_N(\varphi) = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} h_\varphi(X_i; Y_j)$. Thus $\hat{\Delta}_N(\varphi)$ is a two sample U-statistic of degree (1,1) with symmetric kernel h_φ and $E(\hat{\Delta}_N(\varphi)) = \Delta(\varphi)$. So by the theory of U-statistics (c.f. Puri and Sen), $\sqrt{N}(\hat{\Delta}_N(\varphi) - \Delta(\varphi)) \xrightarrow[D]{} N(0, \frac{\sigma_1^2}{\lambda} + \frac{\sigma_2^2}{1-\lambda})$ where

$\sigma_1^2 = \mathrm{Var}(\varphi(X) \int_0^X \varphi\, d\bar{F}_2 + \varphi^2(X)\, F_2(X) - \int_0^X \varphi^2\, d\bar{F}_2 - \varphi(X) \int_X^\infty \varphi\, d\bar{F}_2)$ and

$\sigma_2^2 = \mathrm{Var}(\varphi(Y) \int_0^Y \varphi\, d\bar{F}_1 + \varphi^2(Y) F_1(Y) - \int_0^Y \varphi^2\, d\bar{F}_1 - \varphi(Y) \int_Y^\infty \varphi\, d\bar{F}_1)$, provided $E\, h^2_\varphi(X; Y) < \infty$. The following lemma gives a necessary and sufficent condition so that $E\, h^2_\varphi(X; Y) < \infty$.

<u>Lemma 1.1:</u> $\int_0^\infty \varphi^2\, d\bar{F}_1 < \infty$ and $\int_0^\infty \varphi^2\, d\bar{F}_2 < \infty$ if and only if $E\, h^2_\varphi(X; Y) < \infty$.

<u>Proof:</u> $E\, h_\varphi^2(X;Y) = \int_0^\infty \varphi^2\, d\bar{F}_1 \int_0^\infty \varphi^2\, d\bar{F}_2 + \int_0^\infty \varphi^4\, F_1\, d\bar{F}_2 + \int_0^\infty \varphi^4\, F_2\, d\bar{F}_1$

$- 2 \int_0^\infty \varphi(x) \int_0^x \varphi^3\, d\bar{F}_2\, d\bar{F}_1 - 2 \int_0^\infty \varphi(x) \int_0^x \varphi^3\, d\bar{F}_1\, d\bar{F}_2$. Thus if

$E\, h_\varphi^2(X;\, Y) < \infty$, then $\int_0^\infty \varphi^2\, d\bar{F}_1 < \infty$ and $\int_0^\infty \varphi^2\, d\bar{F}_2 < \infty$.

Conversely, if $\int_0^\infty \varphi^2\, d\bar{F}_1 < \infty$ and $\int_0^\infty \varphi^2\, d\bar{F}_2 < \infty$ then

$\varphi^2(x)F_1(x) \to 0$ and $\varphi^2(x)F_2(x) \to 0$ as $x \to \infty$. So letting

$A = \sup_{0 \le x < \infty} \{\varphi^2(x)F_1(x)\}$ and $B = \sup_{0 \le x < \infty} \{\varphi^2(x)F_2(x)\}$, then $0 \le A < \infty$,

$0 \le B < \infty$ and

$$E\, h_\varphi^2\, (X;\, Y) \le \int_0^\infty \varphi^2\, d\bar{F}_1 \int_0^\infty \varphi^2\, d\bar{F}_2 + A \int_0^\infty \varphi^2\, d\bar{F}_2 + B \int_0^\infty \varphi^2\, d\bar{F}_1 < \infty.$$

Thus if $\int_0^\infty \varphi^2\, d\bar{F}_1 < \infty$ and $\int_0^\infty \varphi^2\, d\bar{F}_2 < \infty$ then $E\, h_\varphi^2(X;\, Y) < \infty$. □

So by the theory of U-statistics and lemma 1.1, we arrive at the following.

<u>Thoerem 1.1</u>: If $\int_0^\infty \varphi^2\, d\bar{F}_1 < \infty$ and $\int_0^\infty \varphi^2\, d\bar{F}_2 < \infty$, then

$\sqrt{N}(\hat{\Delta}_N(\varphi) - \Delta(\varphi)) \xrightarrow[D]{} N(0, \frac{\sigma_1^2}{\lambda} + \frac{\sigma_2^2}{1-\lambda})$ where

$$\sigma_1^2 = \mathrm{Var}(\varphi(X) \int_0^X \varphi\, d\bar{F}_2 + \varphi^2(X)F_2(X) - \int_0^X \varphi^2\, d\bar{F}_2 - \varphi(X) \int_X^\infty \varphi\, d\bar{F}_2)$$

and

$$\sigma_2^2 = \mathrm{Var}(\varphi(Y) \int_0^Y \varphi\, d\bar{F}_1 + \varphi^2(Y)F_1(Y) - \int_0^Y \varphi^2\, d\bar{F}_1 - \varphi(Y) \int_Y^\infty \varphi\, d\bar{F}_1).$$

Now let $V_{N,1}(X_i,\, \varphi) = \frac{1}{n} \sum_{j=1}^{n} h_\varphi(X_i;\, Y_j)$,

$V_{N,2}(Y_j;\, \varphi) = \frac{1}{m} \sum_{i=1}^{m} h_\varphi(X_i;\, Y_j)$, $S_{N,1}^2(\varphi) = \frac{1}{m-1} \sum_{i=1}^{m} \{V_{N,1}(X_i;\, \varphi) - \hat{\Delta}_N(\varphi)\}^2$

and

$S^2_{N,2}(\varphi) = \frac{1}{n-1} \sum_{j=1}^{n} \{V_{N,2}(Y_j;\ \varphi) - \hat{\Delta}_N(\varphi)\}^2$. By Proposition 4 of Sen (1960), we have that $S^2_{N,1}(\varphi) \xrightarrow[p]{} \sigma^2_1$ and $S^2_{N,2}(\varphi) \xrightarrow[p]{} \sigma^2_2$ provided $E\, h^2_{\varphi}(X;\ Y) < \infty$. Thus we have the following theorem.

Theorem 1.2: If $\int_0^\infty \varphi^2\, d\bar{F}_1 < \infty$ and $\int_0^\infty \varphi^2\, d\bar{F}_2 < \infty$,

then

$$\frac{\hat{\Delta}_N(\varphi) - \Delta(\varphi)}{\sqrt{\frac{1}{m} S^2_{N,1}(\varphi) + \frac{1}{n} S^2_{N,2}(\varphi)}} \xrightarrow[D]{} N(0,\ 1).$$

Corollary 1.1: Under H_0: $F_1 = F_2 = F$,

$$\frac{\hat{\Delta}_N(\varphi)}{\sqrt{\frac{1}{m} S^2_{N,1}(\varphi) + \frac{1}{n} S^2_{N,2}(\varphi)}} \xrightarrow[D]{} N(0,\ 1) \text{ provided } \int_0^\infty \varphi^2\, d\bar{F} < \infty.$$

2. Pitman A.R.E.

In this section, we will study the Pitman A.R.E. of the $\hat{\Delta}_N(x)$ test relative to the W_1 test of Kochar, the Savage test, the Wilcoxon test and the L.M.P. rank test for the following alternatives belonging to H_1.

$$H_2:\ h_{\bar{F}_\theta}(x) = (\theta + 1)\, h_{\bar{F}_0}(x), \text{ i.e., } F_\theta(x) = [F(x)]^{1+\theta}.$$

For $F_0(x) = e^{-x}$, this leads to $F_\theta(x) = e^{-(\theta+1)x}$ which is the scale alternative in the exponential case.

$$H_3:\ h_{\bar{F}_\theta}(x) = h_{\bar{F}_0}(x)(1 + \theta\, \ln F(x)), \text{ i.e., } F_\theta(x) = F_0(x)\exp\{-\tfrac{\theta}{2} \ln^2 F(x)\}.$$

When $F_0(x) = e^{-px}$, then $h_{\overline{F}_\theta}(x) = p + p^2\,\theta\,x$, the linearly increasing hazard rate.

$$H_4:\ \ F_\theta(x) = (1-\theta)\,e^{-x} + \theta\,e^{-x}\,[1-(1-e^{-x})^k],\ k > \frac{1}{2}\,.$$

$$H_5:\ \ h_{\overline{F}_\theta}(x) = 1 + \theta(1-e^{-x}),\ \text{i.e.,}$$

$\overline{F}_\theta(x) = 1 - \exp(-\{x + \theta\,(x + e^{-x}-1)\})$, the Makeham distribution.

The alternatives H_2, H_3 and H_4 have been considered by Chikkagoudar and Shuster (1974) where they have obtained the L.M.P. rank test for these alternatives. The L.M.P. rank test for H_5 can be similarly obtained.

Let $\dot{\Delta} = \dfrac{d\,\Delta(x;\,F_0,\,F_\theta)}{d\,\theta}\Big|\,\theta = 0$

$$\sigma_N^2 = \frac{\int_0^\infty\{x\int_0^\infty F_0 d_y - 2\int_0^x\int_t^\infty F_0(s)ds\,dt\}^2\,d\,\overline{F}_0(x)}{N\ \ \lambda(1-\lambda)}\ ,$$

$C(\hat{\Delta}_N) = \lim\limits_{N\to\infty}\ \dfrac{\dot{\Delta}}{\sqrt{N}\,\sigma_N}$, C(L.M.P.) = the efficacy of the L.M.P. rank test for the alternative being considered, and let $e(\hat{\Delta}_N,\ \text{L.M.P.}) = \left(\dfrac{C(\hat{\Delta}_N)}{C(\text{L.M.P.})}\right)^2$, the Pitman A.R.E. of $\hat{\Delta}_N$ to the L.M.P. rank test.

For H_2, $\dot{\Delta} = \frac{1}{2}$, $\sigma_N = (3\lambda(1-\lambda)N)^{-\frac{1}{2}}$, $C(\hat{\Delta}_N) = \frac{1}{2}\sqrt{3\lambda(1-\lambda)}$, $C(\text{L.M.P.}) = \sqrt{2\lambda(1-\lambda)}$ and $e(\hat{\Delta}_N,\ \text{L.M.P.}) = .375$.

For H_3, $\dot{\Delta} = \dfrac{3}{4P^2}$, $\sigma_N = P^{-2}(3\lambda(1-\lambda)N)^{-\frac{1}{2}}$, $C(\hat{\Delta}_N) = \frac{3}{4}\,(3\lambda(1-\lambda))^{\frac{1}{2}}$, $C(\text{L.M.P.}) = \sqrt{2\lambda(1-\lambda)}$, and $e(\hat{\Delta}_N,\ \text{L.M.P.}) = .84375$.

For H_4, $\dot{\Delta} = \dfrac{k}{(K+1)(k+2)}$, $\sigma_N = (3\lambda(1-\lambda)N)^{-\frac{1}{2}}$, $C(\hat{\Delta}_N) = \dfrac{k\sqrt{3\lambda(1-\lambda)}}{(k+1)(k+2)}$, $C(\text{L.M.P.}) = \{\lambda(1-\lambda)(\dfrac{k}{4k^2-1})\}^{\frac{1}{2}}$, and

$$e(\hat{\Delta}_N,\ \text{L.M.P.}) = \frac{3k(4k^2-1)}{(k+1)^2(k+2)^2}\ .$$

For H_5, $\dot{\Delta} = \frac{1}{3}$, $\sigma_N = (3\lambda(1-\lambda)N)^{-\frac{1}{2}}$ $C(\hat{\Delta}_N) = \frac{\sqrt{3\lambda(1-\lambda)}}{3}$.

Using Theorem 2.1 of C-S (1974), we have $G(x;\,0) = 1-e^{-x}$, $G^{-1}(x) = -\ln(1-x)$, $g(x;\,0) = e^{-x}$, $H(x;\,\theta) = \exp\{-\theta(x + e^{-x}-1)\}$, $\Phi(u) = \frac{\partial}{\partial\theta} H(G^{-1}(u);\,\theta)|_{\theta=0}$ $- \frac{(1-u)}{g(G^{-1}(u);\,0)} \frac{\partial^2}{\partial x\,\partial\theta} H(G^{-1}(u);\,\theta)|_{\theta=0} = (1-u) + 2u$, and $Var(\Phi(U)) = \frac{1}{3}$, where U is uniformly distributed over $(0, 1)$. Therefore, $C(L.M.P.) = \frac{\sqrt{\lambda(1-\lambda)}}{3}$ and $e(\hat{\Delta}_N, L.M.P.) = 1$.

The following table gives the Pitman A.R.E.'s of the Wilcoxon test, the Savage test, the W_1-test of Kochar, and the $\hat{\Delta}_N(x)$ test with respect to the corresponding L.M.P. rank test for the above alternatives.

Table 1. Pitman Asymptotic Relative Efficiencies with Respect to the L.M.P. Rank Tests.

Alternative Hypothesis		Wilcoxon	Savage	W_1	$\hat{\Delta}_N(x)$
H_2		.75	1.0	.8203	.375
H_3		.0938	.5	.2307	.84375
H_4	$k = 1$	1.0	.75	.70	.25
	$k = 2$	.625	.8333	.9843	.625
	$k = 3$	.35	.7292	.7813	.7875
H_5		.25	.75	.5353	1.0

CHAPTER II

THE ESTIMATOR $\tilde{\Delta}_N(\varphi)$

1. Notation and Preliminaries.

In the previours Chapter, we defined an estimator $\hat{\Delta}_N(\varphi)$ of $\Delta(\varphi) = \int_0^M \int_0^t \{F_1(t)F_2(s) - F_1(s)F_2(t)\}d\varphi(s)d\varphi(t)$, for the uncensored case. In this Chapter, we will define an estimator $\tilde{\Delta}_N(\varphi)$ of $\Delta(\varphi)$ for the censored model. To define $\tilde{\Delta}_N(\varphi)$, we first need to introduce some notation. Let

$$(1.1) \quad N = m + n$$

$$(1.2) \quad N_m^+(t) = \sum_{i=1}^{m} [U_i > t]$$

$$(1.3) \quad N_n^+(t) = \sum_{j=1}^{n} [V_j > t]$$

$$(1.4) \quad \hat{G}_m^{-1}(t) = \prod_{\ell=1}^{m} \left\{\frac{2 + N_m^+(U_\ell)}{1 + N_m^+(U_\ell)}\right\}^{[\delta_\ell = 0,\ U_\ell \leq t]}$$

$$(1.5) \quad \hat{G}_n^{-1}(t) = \prod_{\ell=1}^{n} \left\{\frac{2 + N_n^+(V_\ell)}{1 + N_n^+(V_\ell)}\right\}^{[\gamma_\ell = 0,\ V_\ell \leq t]}$$

$$(1.6) \quad \hat{F}_m(t) = \frac{N_m^+(t)\hat{G}_m^{-1}(t)}{m}$$

and

$$(1.7) \quad \hat{F}_n(t) = \frac{N_n^+(t)\hat{G}_n^{-1}(t)}{n}$$

The estimator $\tilde{\Delta}_N(\varphi)$ is defined by

$$(1.8)\quad \tilde{\Delta}_N(\varphi) = \int_0^{M_N}\int_0^{t} \{\hat{F}_m(t)\hat{F}_n(S) - \hat{F}_m(S)\hat{F}_n(t)\}\, d\varphi(S)d\varphi(t)$$

where $M \equiv M_N \uparrow T = \sup\{t:\ F_2(t) > 0\}$. Also let

$$(1.9)\quad \Delta_N(\varphi) = \int_0^{M}\int_0^{t} \{F_1(t)F_2(S) - F_1(S)F_2(t)\}\, d\varphi(S)\, d\varphi(t)$$

The following notation is used throughout this Chapter and Chapter 3:

$$(1.10)\quad H_i = F_iG_i \quad \text{for} \quad i = 1, 2$$

$$(1.11)\quad \tilde{H}_1(S) = P(\delta_1 = 0,\ U_1 \le s) = \int_0^s F_1\, d\bar{G}_1,\ s \ge 0$$

$$(1.12)\quad \tilde{H}_2(S) = P(\gamma_1 = 0,\ V_1 \le s) = \int_0^s F_2\, d\bar{G}_2,\ s \ge 0$$

$$(1.13)\quad \approx{H}_1(S) = P(\delta_1 = 1,\ U_1 \le s) = \int_0^s G_1\, d\bar{F}_1,\ s \ge 0$$

$$(1.14)\quad \approx{H}_2(S) = P(\gamma_1 = 1,\ V_1 \le s) = \int_0^s G_2\, d\bar{F}_2,\ s \ge 0$$

$$(1.15)\quad H_m(t) = \frac{N_m^+(t)}{m}$$

$$(1.16)\quad H_n(t) = \frac{N_n^+(t)}{n}$$

$$(1.17)\quad m\, \tilde{H}_m(t) = \sum_{i=1}^{m} [\delta_i = 0,\ U_i \le t]$$

and

$$(1.18)\quad n\, \tilde{H}_n(t) = \sum_{j=1}^{n} [\gamma_j = 0,\ V_j \le t]\ .$$

Also F_i^{-1}, G_i^{-1}, and H_i^{-1} stand for $1/F_i$, $1/G_i$, and $1/H_i$ respectively.

2. Asymptotic Normality of $\sqrt{N}\,(\tilde{\Delta}_N(\varphi) - \Delta_N(\varphi))$.

We start this section by quoting some inequalities from Koul, Susarla and Van Ryzin (1981) and state them as Lemmas for convenience.

Lemma 2.1:

(2.1) $E\{(1 + N_n^+(V_i))^{-r} | (V_i, \gamma_i)\} \le C\, n^{-r} H_2^{-r}(V_i)$

(2.1)' $\{E\, (1 + N_m^+(U_i))^{-r} | (U_i, \delta_i)\} \le C\, m^{-r} H_1^{-r}(U_i)$

(2.2) $E\{(1 + N_n^+(V_j))^{-r} | (V_i, \gamma_i), (V_j, \gamma_j)\} \le C\, n^{-r} H_2^{-r}(V_j)$

(2.2)' $E\{(1 + N_m^+(U_j))^{-r} | (U_i, \delta_i), (U_j, \delta_j)\} \le C\, m^{-r} H_1^{-r}(U_j)$

$1 \le i, j \le n$ and where C depends only on r.

Lemma 2.2:

(2.3) $$E\{(\hat{G}_n(V_i) - G_2(V_i))^{2d} | (V_i, \gamma_i)\} \le C\{n^{-d} \int_0^{V_i} H_2^{-4d} F_2\, d\bar{G}_2 + n^{-2d}(1 - \gamma_i)\, H_2^{-4d}(V_i)\}$$

(2.3)' $$E\{(\hat{G}_m(U_i) - G_1(U_i))^{2d} | (U_i, \delta_i)\} \le C\{m^{-d} \int_0^{U_i} H_1^{-4d} F_1\, d\bar{G}_1 + m^{-2d}(1 - \delta_i)\, H_1^{-4d}(U_i)\}$$

(2.4) $$E\{(\hat{G}_n(t) - G_2(t))^{2d}\} \le C\, n^{-d} \int_0^t H_2^{-4d} F_2\, d\bar{G}_2$$

whenever $H_2(t) > 0$

(2.4)' $$E\{(\hat{G}_m(t) - G_1(t))^{2d}\} \le C\, m^{-d} \int_0^t H_1^{-4d} F_1\, d\bar{G}_1$$

whenever $H_1(t) > 0$

for $d = 1, 2, \ldots,$ and where C depends only on d.

Lemma 2.3:

(2.5) $$E\{(\hat{G}_n^{-1}(V_i) - G_2^{-1}(V_i))^{2d} | (V_i, \gamma_i)\} \le C\, n^{-d} H_2^{-2d}(V_i)\{\int_0^{V_i} H_2^{-4d} F_2\, d\bar{G}_2 + (\int_0^{V_i} H_2^{-8d} F_2\, d\bar{G}_2)^{1/2} + n^{-d}(1 - \gamma_i)\, H_2^{-4d}(V_i)\}$$

$$(2.5)' \quad E\{(\hat{G}_m^{-1}(U_i) - G_1^{-1}(U_i))^{2d} | (U_i, \delta_i)\} \leq C\, m^{-d}\, H_1^{-2d}(U_i)\{\int_0^{U_i} H_1^{-4d} F_1 d\bar{G}_1 + (\int_0^{U_i} H_1^{-8d} F_1 d\bar{G}_1)^{\frac{1}{2}} + m^{-d}(1 - \delta_i)\, H_1^{-4d}(U_i)\}$$

$$(2.6) \quad E(\hat{G}_n^{-1}(t) - G_2^{-1}(t))^{2d} \leq C\, n^{-d}\, H_2^{-2d}(t)\{\int_0^t H_2^{-4d} F_2 d\bar{G}_2 + (\int_0^t H_2^{-8d} F_2\, d\bar{G}_2)^{\frac{1}{2}}\}$$

$$(2.6)' \quad E(\hat{G}_m^{-1}(t) - G_2^{-1}(t))^{2d} \leq C\, m^{-d}\, H_1^{-2d}(t)\{\int_0^t H_1^{-4d} F_1\, d\bar{G}_1 + (\int_0^t H_1^{-8d} F_1\, d\bar{G}_1)^{\frac{1}{2}}\}$$

whenever $H_1(t) > 0$, $H_2(t) > 0$ and for $d = 1, 2, \ldots,$ where C depends only on d.

<u>Lemma 2.4:</u>

$$(2.7) \quad E\{(\hat{F}_n(V_i) - F_2(V_i))^{2d} | (V_i, \gamma_i)\} \leq C\, n^{-d}\, H_2^{-6d}(V_i)$$

$$(2.7)' \quad E\{(\hat{F}_m(U_i) - F_1(U_i))^{2d} | (U_i, \delta_i)\} \leq C\, m^{-d}\, H_1^{-6d}(U_i)$$

$$(2.8) \quad E\{\hat{F}_n(t) - F_2(t)\}^{2d} \leq C\, n^{-d}\, H_2^{-6d}(t) \quad \text{whenever} \quad H_2(t) > 0$$

$$(2.8)' \quad E(\hat{F}_m(t) - F_1(t))^{2d} \leq C\, m^{-d}\, H_1^{-6d}(t) \quad \text{whenever} \quad H_1(t) > 0,$$

for $d = 1, 2, \ldots,$ and C depends only on d.

We will henceforth assume that $T = \sup\{t: F_2(t) > 0\} \leq \sup\{t: G_2(t) > 0\}$ and $\sup\{t: F_1(t) > 0\} \leq \sup\{t: G_1(t) > 0\}$. Also without loss of generality, we will assume that $T = +\infty$. Now let F_1, F_2, G_1, G_2, λ_N and $M = M_N$ satisfy the following conditions:

$$(2.9) \quad \int_0^\infty F_i\, d\,\varphi < \infty \quad \text{for} \quad i = 1, 2$$

(2.10) $N^{-\frac{1}{2}}(\int_0^M H_1^{-3}\, d\,\varphi)^2 \to 0$ as $N \to \infty$

(2.11) $N^{-\frac{1}{2}}(\int_0^M H_2^{-3}\, d\,\varphi)^2 \to 0$ as $N \to \infty$

(2.12) $N^{-1}(\int_0^M G_i^{-1}\, H_i^{-4}\, d\,\varphi)^2 \to 0$ as $N \to \infty$ for $i = 1, 2$,

(2.13) $N^{-1}\, \varphi^2(M)\, H_i^{-2}(M) \to 0$ as $N \to \infty$ for $i = 1, 2$

(2.14) $\lambda_N = \frac{m}{N} \to \lambda$ as $N \to \infty$ with $0 < \lambda < 1$, and $M = M_N \uparrow \infty$.

Observe that

$$
\begin{aligned}
\tilde{\Delta}_N(\varphi) - \Delta_N(\varphi) = &\int_0^M \int_0^t (\hat{F}_m(t) - F_1(t))(\hat{F}_n(s) - F_2(s))\, d\varphi(s) d\,\varphi(t) \\
&- \int_0^M \int_0^t (\hat{F}_m(s) - F_1(s))(\hat{F}_n(t) - F_2(t))\, d\,\varphi(s)\, d\,\varphi(t) \\
&- \int_0^M g_{1,m}(t)(\hat{F}_n(t) - F_2(t))\, d\,\varphi(t) \qquad (2.15) \\
&+ \int_0^M g_{2,n}(t)(\hat{F}_m(t) - F_1(t))\, d\,\varphi(t)
\end{aligned}
$$

where $g_{1,n}(t) = (\int_0^t F_1(s)\, d\,\varphi(s) - \int_t^M F_1(s)\, d\,\varphi(s))[t \leq M]$

and $g_{2,M}(t) = (\int_0^t F_2(s)\, d\,\varphi(s) - \int_t^M F_2(s)\, d\,\varphi(s))[t \leq M]$.

We will show that under conditions (2.9) - (2.14), $\sqrt{N}(\tilde{\Delta}_N(\varphi) - \Delta_N(\varphi))$ converges in distribution to a $N(0, \sigma^2)$ random variable in three steps. The first step is to show that the first two terms on the right hand side of (2.15) converges to zero in L_2. This step is done in Lemma 2.5. The second step is to show that the last two terms on the right hand side of (2.15) can be approximated in L_2 by a generalized U-statistic. This is done in Theorem 2.1. Finally, we show that this generalized U-statistc converges in distribution to a $N(0, \sigma^2)$ random variable.

Lemma 2.5: If conditions (2.10), (2.11) and (2.14) hold then

$$N\, E\{(\tilde{\Delta}_N(\varphi) - \Delta_N(\varphi)) - (\int_0^M g_{2,M}(\hat{F}_m - F)\, d\varphi - \int_0^M g_{1,M}(\hat{F}_n - F_2)\, d\varphi)\}^2$$

$$\to 0 \quad \text{as} \quad N \to \infty.$$

Proof: By (2.15), we have

$$N\, E\{(\tilde{\Delta}_N(\varphi) - \Delta_N(\varphi)) - \int_0^M g_{2,M}(\hat{F}_m - F_1)\, d\varphi + \int_0^M g_{1,M}(\hat{F}_n - F_2)\, d\varphi\}^2$$

$$\le 2\, N\{E(\int_0^M \int_0^t (\hat{F}_m(t) - F_1(t))(\hat{F}_n(s) - F_2(s))\, d\varphi(s)\, d\varphi(t))^2$$

$$+ E(\int_0^M \int_0^t (\hat{F}_m(s) - F_1(s))(\hat{F}_n(t) - F_2(t))\, d\varphi(s)\, d\varphi(t))^2\}.$$

However,

$$N\, E(\int_0^M \int_0^t (\hat{F}_m(t) - F_1(t))(\hat{F}_n(s) - F_2(s))\, d\varphi(s)\, d\varphi(t))^2$$

$$= 2NE\{\int_0^M (\hat{F}_m(t) - F_1(t)) \int_0^t (\hat{F}_n(s) - F_2(s))\, d\varphi(s)(\int_0^t (\hat{F}_m(x) - F_1(x)) \int_0^x (\hat{F}_n(y)$$

$$- F_2(y))\, d\varphi(y)\, d\varphi(x))\, d\varphi(t)\}$$

$$\le \frac{C\,N}{m\,n} (\int_0^M \int_0^t H_1^{-3}(t)\, H_2^{-3}(s)\, d\varphi(s)\, d\varphi(t))^2 \quad \text{by Lemma 2.4} \to 0 \quad \text{by conditions}$$

(2.10), (2.11) and (2.14).

Similarly,

$$N\, E\{\int_0^M \int_0^t (\hat{F}_m(s) - F_1(s)(\hat{F}_n(t) - F_2(t))\, d\varphi(s)\, d\, \varphi(t)\}^2$$

$$\le \frac{C\,N}{m\,n} (\int_0^M \int_0^t H_1^{-3}(s)\, H_2^{-3}(t)\, d\, \varphi(s)\, d\, \varphi(t))^2 \to 0$$

by conditions (2.10), (2.11) and (2.14). □

Thus $\sqrt{N}\, (\tilde{\Delta}_N(\varphi) - \Delta_N(\varphi))$ can be approximated in L_2 by

$\sqrt{N}(\int_0^M g_{2,M}(\hat{F}_m - F_1)\, d\varphi - \int_0^M g_{1,M}(\hat{F}_n - F_2)\, d\varphi)$. We will now proceed as in section 3 of Susarla and Van Ryzin (1980) with some modifications to show that

$$N\, E\{\int_0^M g_{1,M}(\hat{F}_n - F_2)\, d\varphi - n^{-\frac{1}{2}}(S_n - E(S_n))\}^2 \to 0$$

where

$$(2.16)\quad S_n = n^{+\frac{1}{2}}\{\int_0^M (2\, H_2^{-1} - H_n\, H_2^{-2})\mu_{2,M} d\, \tilde{H}_n + \int_0^M g_{1,M}\, G_2^{-1}\, H_n\, d\varphi\}$$

and $\mu_{2,M}(\cdot) = [\cdot \le M] \int_{\cdot}^M g_{1,M}\, F_2\, d\varphi.$

First observe that

$$\int_0^M g_{1,M}(\hat{F}_n - F_2)\, d\varphi = \int_0^M g_{1,M}\{H_n(\hat{G}_n^{-1} - G_2^{-1}) + G_2^{-1}(H_n - H_2)\}\, d\varphi\ .$$

Now $\left| \int_0^M g_{1,M}\{H_n(\hat{G}_n - G_2^{-1}) - H_n\, G_2^{-1}(\ \hat{G}_2^{-1} -\ \ G_2^{-1})\}\, d\varphi \right|$

$$\le 2\, (\int_0^\infty F_2\, d\varphi) \int_0^M |H_n(\hat{G}_n^{-1} - G_2^{-1}) - H_n\, G_2^{-1}(\ \hat{G}_2^{-1} -\ \ G_2^{-1})|d\varphi$$

$$\le 4\, (\int_0^\infty F_2\, d\varphi) \int_0^M G_2^{-1}(\ \hat{G}_n^{-1} -\ \ G_2^{-1})^2\, d\varphi$$

by equation (3.5) and Lemma 3.1 of Susarla and Van Ryzin (1980).

Therefore,

$$N\, E\{\int_0^M g_{1,M}(\hat{F}_n - F_2)\, d\varphi - \int_0^M g_{1,M}(H_n\, G_2^{-1}(\ \hat{G}_2^{-1} -\ \ \hat{G}_n) +$$

$$G_2^{-1}(H_n - H_2))\, d\varphi\}^2$$

$$\le 16\, N\, (\int_0^M F_2\, d\varphi)^2\, E\{\int_0^M G_2^{-1}\, (\ \ G_2 -\ \ \hat{G}_n)^2\, d\varphi\}^2$$

$$\le \frac{C\, N}{n^2}\, (\int_0^M F_2\, d\varphi)^2\, (\int_0^M G_2^{-1}\, H_2^{-4}\, d\varphi)^2 \to 0$$ by condition (2.12). But

$$\int_0^M g_{1,M} H_n G_2^{-1} (G_2 - \hat{G}_2) d\varphi = \int_0^M g_{1,M} H_2 G_2^{-1}(G_2 - \hat{G}_n)d\varphi$$

$$+ \int_0^M g_{1,M} G_2^{-1} (H_n - H_2)(G_2 - \hat{G}_n) d\varphi.$$

Since $N\ E\{ \int_0^M g_{1,M}(H_n - H_2) G_2^{-1}(G_2 - \hat{G}_2) d\varphi\}^2$

$\leq \frac{C\ N}{n^2} (\int_0^M g_{1,M} G_2^{-1} H_2^{-4} d\varphi)^2 \to 0$ by condition (2.12) we have that

$$N\ E\{ \int_0^M g_{1,M}(\hat{F}_n - F_2) d\varphi - \int_0^M g_{1,M}(G_2^{-1}(H_n - H_2) + F_2(G_2 - \hat{G}_n)) d\varphi\}^2$$

$\to 0$ if condition (2.12) holds. It is also easily seen that

$$N\ E\ \{\int_0^M g_{1,M}F_2(G_2 - \hat{G})d\varphi - \int_0^M g_{1,M}(t)\ F_2(t)(\int_0^t \frac{n}{1+N_n^+} d\tilde{H}_n - \int_0^t H_2^{-1}d\tilde{H}_2)d\varphi\}^2$$

converges to zero provided (2.12) holds. Now writing $n(1 + N_n^+)^{-1}$ - $(2\ H_2^{-1} - H_2^2\ H_n)$ as $(1 + N_n^+)^{-1}\ H_2^{-2}\ (H_n - H_2) - H_2^{-1}(1 + N_n^+)^{-1} + n(1 + N_n^+)^{-1}$ $H_2^{-2}\ (H_n - H_2)^2$, it can be shown that

$$N\ E\{\int_0^M g_{1,M}(t)\ F_2(t) \int_0^t (\frac{n}{1+N_n^+} - 2\ H_2^{-1} + H_2^{-2}\ H_n)d\ \tilde{H}_n\ d\varphi(t)\}^2 \to 0$$

provided condition (2.11) holds.

Therefore, if conditions (2.11), (2.12) and (2.14) hold then

$$N\ E\{\int_0^M g_{1,M}(\hat{F}_n - F_2)d\ \varphi - n^{-\frac{1}{2}}(S_n - E(S_n))\}^2 \to 0.$$

Similarly, we have by conditions (2.10), (2.12) and (2.14) that

$$N\ E\{\int_0^M g_{2,M}(\hat{F}_m - F_1)d\varphi - m^{-\frac{1}{2}}(S_m - E(S_m))\}^2 \to 0,$$

where

(2.17) $S_m = m^{\frac{1}{2}}\{\int_0^M (2\ H_1^{-1} - H_m\ H_1^{-2})\mu_{1,M}d\ \tilde{H}_m + \int_0^M g_{2,M}G_1^{-1}\ H_m d\varphi\}$ and

$\mu_{1,M}(\cdot) = \int_{\cdot}^M g_{2,M}\ F_1\ d\varphi[\cdot \leq M]$. Hence we have the following theorem.

<u>Theorem 2.1:</u> If conditions (2.9) - (2.14) hold, then

$$N\ E\{(\tilde{\Delta}_N(\varphi) - \Delta_N(\varphi)) - m^{-\frac{1}{2}}(S_m - E(S_m)) + n^{-\frac{1}{2}}(S_n - E(S_n))\}^2 \to 0$$

where S_n and S_m are defined by (2.16) and (2.17) respectively.

Now let

$$\begin{aligned}
\Phi_n((V_1, \gamma_1), (V_2,\ _2)) = {} & \mu_{2,M}(V_1)\ H_2^{-1}(V_1)[\gamma_1 = 0,\ V_1 \le M] \\
& + \mu_{2,M}(V_2)\ h_2^{-1}(V_2)[\gamma_2 = 0,\ V_2 \le M] \\
& + \frac{1}{2}\Big(\int_0^{V_1 \wedge M} g_{1,M} G_2^{-1}\ d\varphi + \int_0^{V_2 \wedge M} g_{1,M} G_2^{-1}\ d\varphi\Big) \qquad (2.18) \\
& - \Big(\frac{n-1}{2n}\Big)\{[V_1 > V_2]\mu_{2,M}(V_2)\ H_2^{-2}(V_2)[\gamma_2 = 0,\ V_2 \le M] \\
& + [V_2 > V_1]\mu_{2,M}(V_1)\ H_2^{-2}(V_1)[\gamma_1 = 0,\ V_1 \le M]\}.
\end{aligned}$$

<u>Lemma 2.6:</u> $\mathrm{Var}(\Phi_n) = O(\varphi^2(M)\ H_2^{-2}(M))$.

<u>Proof:</u> As will be shown below, each term in the right hand side of (2.18) has a second moment bounded by a constant multiple (C_1, C_2,... are constants) of $\varphi^2(M)\ H_2^{-2}(M)$. In particular, the following inequalities hold,

$$E([\gamma_1 = 0,\ V_1 \le M]\mu_{2,M}^2(V_1)\ H_2^{-2}(V_1)) \le C_1\ H_2^{-2}(M),$$

$$\begin{aligned}
E\Big(\int_0^{V_1 \wedge M} g_{1,n}\ G_2^{-1}\ d\varphi\Big)^2 & \le C_2\ E\Big(\int_0^{V_1 \wedge M} G_2^{-1}\ d\varphi\Big)^2 = 2\ C_2 \int_0^M F_2(v) \int_0^v g_2^{-1}(u)d\varphi\ \ d\varphi(v) \\
& \le C_3\ \varphi^2(M)\ H_2^{-1}(M),
\end{aligned}$$

and

$$E([V_1 > V_2]\mu^2_{2,M}(V_2)\, H_2^{-4}(V_2)[\gamma_2 = 0,\, V_2 \le M]) \le C_4 \int_0^M H_2^{-3}(u)\, F_2(u) d\, \bar{G}_2(u)$$
$$\le C_5\, H_2^{-2}(M).$$

All three inequalities complete the proof of the lemma. □

<u>Theorem 2.2:</u> $S_n - E(S_n) \underset{D}{\to} N(0, \sigma_2^2)$ where

$$(2.19) \quad \sigma_2^2 = \int_0^\infty H_2^{-2}(t)\, (\int_t^\infty g_1\, F_2\, d\, \varphi)^2\, d\, \tilde{\tilde{H}}_2(t)$$

and $g_1(u) = \int_0^u F_1\, d\, \varphi - \int_u^\infty F_1\, d\, \varphi$, provided $\sigma_2^2 < \infty$ and $N^{-1}\, \varphi^2(M)\, H_2^{-2}(M) \to 0$.

<u>Proof:</u> First notice that

$$n^{-\frac{1}{2}}\, S_n = \frac{2}{n} \sum_1^n \mu_{2,M}(V_j)\, H_2^{-1}(V_j)[\gamma_j = 0,\, V_j \le M] + \frac{1}{n} \sum_1^n \int_0^{V_j \wedge M} g_{1,M}\, G_2^{-1}\, d\varphi$$
$$- n^{-2} \sum_{j=1}^n \sum_{k=1}^n \mu_{2,M}(V_j)\, H_2^{-2}(V_j)[\gamma_j = 0,\, V_j \le M][V_k > V_j]$$
$$= \binom{n}{2}^{-1} \sum{}' \, \Phi_n\, ((V_j, \gamma_j), (V_k, \gamma_k))$$

where $\sum'$ stands for summation over all (j, k) such that $1 \le j < k \le n$.

Letting $\psi_{n,1}((V_1, \gamma_1)) = E(\Phi_n | (V_1, \gamma_1))$, $\psi_{n,2} = \Phi_n$, and $\theta_n = E(n^{-\frac{1}{2}}\, S_n)$, we then have

$$(2.20) \quad 2\, \psi_{n,1}((V_1, \gamma_1)) = \int_0^{V_1 \wedge M} g_{1,M}\, G_2^{-1}\, d\varphi + 2\, \mu_{2,M}(V_1)\, H_2^{-1}(V_1)[\gamma_1 = 0, V_1 \le M]$$
$$- (\frac{n-1}{n})\mu_{2,M}(V_1)\, H_2^{-1}(V_1)[\gamma_1 = 0,\, V_1 \le M] +$$
$$E(\int_0^{V_2 \wedge M} g_{1,M}\, G_2^{-1}\, d\varphi)$$

$$+ 2\ E(\mu_{2,m}(V_2)\ H_2^{-1}(V_2)[\gamma_2 = 0,\ V_2 \le M])$$

$$- (\tfrac{n-1}{n}) \int_0^{V_1 \wedge M} \mu_{2,M}\ H_2^{-2}\ d\ \tilde{H}_2$$

$$= \int_0^{V_1 \wedge M} g_{1,M}\ G_2^{-1}\ d\varphi + (\tfrac{n-1}{n})\mu_{2,M}(V_1)\ H_2^{-1}(V_1)[\gamma_1 = 0,\ V_1 \le M]$$

$$- (\tfrac{n-1}{n}) \int_0^{V_1 \wedge M} \mu_{2,M}\ H_2^{-2}\ d\ \tilde{H}_2 + E(\int_0^{V_2 \wedge M} g_{1,M}\ G_2^{-1}\ d\varphi)$$

$$+ 2\ E(\mu_{2,M}(V_2)\ H_2^{-1}(V_2)[\gamma_2 = 0,\ V_2 \le M]).$$

From Hoeffding (1948),

$$\mathrm{Var}(n^{-\frac{1}{2}}\ S_n) = \frac{4(n-2)}{n(n-1)}\ \mathrm{Var}(\psi_{n,1}) + \frac{2}{n(n-1)}\ \mathrm{Var}(\psi_{n,2}),$$

and

$$E\{(S_n - E(S_n)) - 2\ n^{-\frac{1}{2}} \sum_1^n (\psi_{n,1}((V_j,\ \gamma_j)) - \theta_n)\}^2$$

$$= 4(\tfrac{n-2}{n-1})\ \mathrm{Var}(\psi_{n,1}) + \frac{2}{n-1}\ \mathrm{Var}(\psi_{n,2}) + 4\ \mathrm{Var}(\psi_{n,1}) - 8\ \mathrm{Var}(\psi_{n,1})$$

$$\le \frac{2}{n-1}\ \mathrm{Var}(\psi_{n,2}) = \frac{2}{n-1}\ \mathrm{Var}(\Phi_n) \to 0 \quad \text{as} \quad N \to \infty$$

by lemma (2.6), since $N^{-1}\ \varphi^2(M)\ H_2^{-2}\ (M) \to 0$. Therefore, $S_n - E(S_n)$ can be approximated in L_2 by $2\ n^{-\frac{1}{2}} \sum_1^n \{\psi_{n,1}((V_j,\ \gamma_j)) - \theta_n\}$. Since

(2.21) $4\ \mathrm{Var}(\psi_{n,1}) \to \sigma_2^2$ (see appendix A) and $\sigma_2^2 < \infty$,

$2\ n^{-\frac{1}{2}} \sum_1^n \{\psi_{n,1}((V_j,\ \gamma_j)) - \theta_n\}$ can be shown to be asymptotically normal. Thus $(S_n - E(S_n)) \xrightarrow[D]{} N(0,\ \sigma_2^2)$. □

<u>Theorem 2.3:</u> $S_m - E(S_m) \xrightarrow[D]{} N(0,\ \sigma_1^2)$ where

(2.22) $$\sigma_1^2 = \int_0^\infty H_1^{-2}(t)\ (\int_t^\infty g_2\ F_1\ d\varphi)^2\ d\ \tilde{\tilde{H}}_1(t)$$

and $g_2(u) = \int_0^u F_2 \, d\varphi - \int_u^\infty F_2 \, d\,\varphi$ provided $\sigma_1^2 < \infty$ and $N^{-1} \varphi^2(M) \, H_1^{-2}(M) \to 0$.

Proof: By a similar arguement as in Theorem (2.2).

Theorem 2.4: $\sqrt{N}\,(\tilde{\Delta}_N(\varphi) - \Delta_N(\varphi)) \underset{D}{\to} N(0, \frac{\sigma_1^2}{\lambda} + \frac{\sigma_2^2}{1+\lambda})$, if $\sigma_1^2 < \infty$, $\sigma_2^2 < \infty$ and conditions (2.9) - (2.14) hold.

Proof: By Theorems (2.1), (2.2), (2.3) and independence.

Corollary 2.1: Under H_0: $F_1 = F_2$, $\sqrt{N}\,\tilde{\Delta}_N(\varphi) \underset{D}{\to} N(0, \frac{\sigma_1^2}{\lambda} + \frac{\sigma_2^2}{1-\lambda})$ provided $\sigma_1^2 < \infty$, $\sigma_2^2 < \infty$ and conditions (2.9) - (2.14) hold.

Remark: Notice that if G_1 and G_2 put all their mass at $+\infty$, i.e., there is no censoring, then

$$\sigma_2^2 = 2\{\int_0^\infty \int_0^y g_1(s)\, F_2(s) g_1(y) d\varphi(y) d\varphi(s) - \int_0^\infty \int_0^y g_1(s) F_2(s) g_1(y) F_2(y) d\varphi(y) d\varphi(s)\}$$

$$= \int_0^\infty (\int_0^y g_1(t) d\varphi(t))^2 \, d\bar{F}_2(y) - (\int_0^\infty \int_0^y g_1(t) d\varphi(t) \; d\,\bar{F}_2(y))^2$$

$$= \mathrm{Var}(\int_0^{Y_1} g_1(t)\, d\,\varphi(t)) = \mathrm{Var}(\varphi(Y_1) \int_0^{Y_1} \varphi \, d\bar{F}_1 + \varphi^2(Y_1) F_1(Y_1)$$

$$- \int_0^{Y_1} \varphi^2 \, d\bar{F}_1 - \varphi(Y_1) \int_{Y_1}^\infty \varphi \, d\bar{F}_1).$$ Also

$$\sigma_1^2 = \mathrm{Var}(\varphi(X_1) \int_0^X \varphi \, d\bar{F}_2 + \varphi^2(X_1) F_2(X_1) - \int_0^{X_1} \varphi^2 \, d\bar{F}_2 - \varphi(X_1) \int_{X_1}^\infty \varphi \, d\bar{F}_2).$$

So the asymptotic variance of $\sqrt{N}\,(\tilde{\Delta}_N(\varphi) - \Delta_N(\varphi))$, in this case will be

$$(1-\lambda)^{-1} \, \mathrm{Var}(\varphi(Y_1) \int_0^{Y_1} \varphi \, d\bar{F}_1 + \varphi^2(Y_1) F_1(Y_1) - \int_0^{Y_1} \varphi^2 \, d\bar{F}_1 - \varphi(Y_1) \int_{Y_1}^\infty \varphi \, d\bar{F}_1)$$

$$+ \lambda^{-1} \operatorname{Var}(\varphi(X_1) \int_0^{X_1} \varphi \, d\bar{F}_2 + \varphi^2(X_1)F_2(X_1) - \int_0^{X_1} \varphi^2 \, d\bar{F}_2 - \varphi(X_1) \int_{X_1}^{\infty} \varphi \, d\bar{F}_2)$$

which is the asymptotic variance in the uncensored case.

3. Consistent Estimators of σ_1^2 and σ_2^2 under H_0.

Under H_0: $F_1 = F_2 = F$, we have that

$\sigma_2^2 = \int_0^\infty \{H_2^{-1}(t) \int_t^\infty F(s)d\varphi(s) \int_0^t F(S)d\varphi(s)\}^2 \, d\,\tilde{\tilde{H}}_2(t)$. So let

$$(3.1) \quad \hat{\sigma}_{N,2}^2 = \int_0^M \{\frac{1+n}{1+N_n^+(t)} \int_t^M \hat{F}_n(s)\, d\varphi(s) \int_0^t \hat{F}_n(s)d\varphi(s)\}^2 \, d\,\tilde{\tilde{H}}_n(t)$$

where $\tilde{\tilde{H}}_n(t) = \frac{1}{n} \sum_{j=1}^n [V_j \leq t, \gamma_j = 1]$. We will now show that

$\hat{\sigma}_{N,2}^2 \underset{p}{\to} \sigma_2^2$ under H_0 provided conditions (2.9), (2.11), (2.12) and (2.13) hold.

<u>Lemma 3.1:</u> If conditions (2.9), (2.11) and (2.13) hold then

$$(3.2) \quad \frac{1}{n} \sum_j \{\frac{1+n}{1+N_n^+(V_j)} \int_0^{V_j} (\hat{F}_n(t) - F(t))d\varphi(t) \int_{V_j}^M F \, d\varphi\}^2 \gamma_j [V_j \leq M] \underset{p}{\to} 0$$

$$(3.3) \quad \frac{1}{n} \sum_j \{\frac{1+n}{1+N_n^+(V_j)} \int_0^{V_j} F \, d\varphi \int_{V_j}^M (\hat{F}_n(t) - F(t))d\varphi(t)\}^2 \gamma_j \, [V_j \leq M] \underset{p}{\to} 0$$

and

$$(3.4) \quad \frac{1}{n} \sum_j \{\frac{1+n}{1+N_n^+(V_j)} \int_0^{V_j} (\hat{F}_n - F)d\varphi \int_{V_j}^M (\hat{F}_n - F)d\varphi\}^2 \gamma_j \, [V_j \leq M] \underset{p}{\to} 0.$$

<u>Proof:</u>

$$E\{\gamma_j \, (\frac{1+n}{1+N_n^+(V_j)} \int_0^{V_j} (\hat{F}_n(t) - F(t))d\varphi(t) \int_{V_j}^M F \, d\varphi)^2 [V_j \leq M]\}$$

$$\leq \left(\int_0^M F d\varphi\right)^2 E\{\gamma_j [V_j \leq n] E_j^{\frac{1}{2}} \left(\frac{1+n}{1+N_n^+(V_j)}\right)^4 E_j^{\frac{1}{2}} \left(\int_0^{V_j} |\hat{F}_n(t) - F(t)| d\varphi\right)^4\}$$

where $E_j(\cdot) = E(\cdot|(V_j, \gamma_j))$,

$$\leq C\, n^{-1} \left(\int_0^M F\, d\varphi\right)^2 \int_0^M H_2^{-2}(t) \left(\int_0^t H_2^{-3}\, d\,\varphi\right)^2 G_2(t) d\,\overline{F}_2(t)$$

where we have used Lemma 2.1, Lemma 2.4, and the equality

$$\left(\int_0^t b(x) d\varphi(x)\right)^2 = 2 \int_0^t b(x) \int_0^x b(y) d\varphi(y) d\,\varphi(x).$$

But $n^{-1} \left(\int_0^M F d\,\varphi\right)^2 \int_0^M H_2^{-2}(t) \left(\int_0^t H_2^{-3}\, d\varphi\right)^2 G_2(t)\, d\, \overline{F}_2(t)$

$$\leq \left(\int_0^M F\, d\varphi\right)^2 \left(n^{-\frac{1}{2}} H_2^{-1}(M)\right) n^{-\frac{1}{2}} \left(\int_0^M H_2^{-3}\, d\varphi\right)^2.$$

By conditions (2.9) and (2.10), we have that $n^{-\frac{1}{2}} \left(\int_0^M F\, d\varphi\right)^2 \left(\int_0^M H_2^{-3}\, d\varphi\right)^2 \to 0.$

Since φ is strictly increasing, $\varphi(0) = 0$ and $M_N \uparrow \infty$, there is an N_0, such that $\forall\, N \geq N_0$ we have $\varphi(M_N) \geq \varphi(1) > 0$. Thus

$$n^{-\frac{1}{2}} H_2^{-1}(M) = \varphi^{-1}(1)\, n^{-\frac{1}{2}} \varphi(1)\, H_2^{-1}(M) \leq \varphi^{-1}(1)\, n^{-\frac{1}{2}} \varphi(M)\, H_2^{-1}(M) \to 0$$

by (2.13). Therefore,

$$E\{\gamma_j \left(\frac{1+n}{1+N_n^+(V_j)} \int_0^{V_j} (\hat{F}_n(t) - F(t)) d\varphi(t) \int_{V_j}^M F d\varphi\right)^2 [V_j \leq M] \to 0$$

and so we have that

$$\frac{1}{n} \sum_j \left\{\frac{1+n}{1+N_n^+(V_j)} \int_0^{V_j} (\hat{F}_n(t) - F(t)) d\varphi(t) \int_{V_j}^M F\, d\varphi\right\}^2 \gamma_j\, [V_j \leq M] \underset{P}{\to} 0.$$

By similar arguements, we have that (3.4) $\underset{P}{\to} 0$ and (3.3) $\underset{P}{\to} 0$ provided conditions (2.9), (2.11) and (2.13) hold. $\square$

Lemma 3.2: If condition (2.12) hold then

$$\frac{1}{n}\sum_j \left(\frac{1+n}{1+N_n^+(V_j)} - H_2^{-1}(V_j)\right)^2 \left(\int_0^{V_j} Fd\varphi \int_{V_j}^{M} Fd\varphi\right)^2 \gamma_j\,[V_j \le M] \xrightarrow[P]{} 0.$$

Proof:

$$E\{\gamma_j \left(\frac{n+1}{1+N_n^+(V_j)} - H_2^{-1}\,(V_j)\right)^2 \left(\int_0^{V_j} Fd\varphi \int_{V_j}^{M} Fd\varphi\right)^2 [V_j \le M]\}$$

$$\le \left(\int_0^M Fd\varphi\right)^3 E\{\gamma_j\, H_2^{-2}(V_j)(1 + N_n^+(V_j))^{-2}((n-1)H_2(V_j) - N_n^+(V_j)$$

$$+ (2H_2(V_j) - 1))^2 \int_{V_j}^{M} Fd\varphi\,[V_j \le M]\}$$

$$\le n^{-1}\, C \int_0^M H_2^{-4}\, G_2 \int_t^M Fd\varphi\, d\bar{F}$$

$$\le n^{-1}\, C \int_0^M H_2^{-4}\, d\varphi \to 0 \quad \text{by (2.12).}$$

Thus

$$\frac{1}{n}\sum_j \left(\frac{n+1}{1+N_n^+(V_j)} - H_2^{-1}(V_j)\right)^2 \left(\int_0^{V_j} Fd\varphi \int_{V_j}^{M} Fd\varphi\right)^2 \gamma_j\,[V_j \le M] \xrightarrow[P]{} 0. \qquad \square$$

Lemma 3.3: If conditions (2.9) and (2.13) hold then

$$\frac{1}{n}\sum_j H_2^{-2}(V_j)\gamma_j\left(\int_0^{V_j} Fd\varphi\right)^2 \left(\int_{V_j}^{M} Fd\varphi\right)^2 [V_j \le M] \xrightarrow[P]{} \sigma_2^2\,.$$

Proof: Since $\sigma_2^2 < \infty$, we have that $\sigma_{N,2}^2 \to \sigma_2^2$ where

$$\sigma_{N,2}^2 = \int_0^M \{H_2^{-1}(t) \int_t^M F(s)d\varphi(s) \int_0^t F(s)d\varphi(s)\}^2 d\,\tilde{\tilde{H}}_2(t) \tag{3.5}$$

Thus we need only show that

$$(3.6)\quad \frac{1}{n}\sum_j H_2^{-2}(V_j)\gamma_j \left(\int_0^{V_j} Fd\varphi\right)^2 \left(\int_{V_j}^{M} Fd\varphi\right)^2 [V_j \leq M] - \sigma_{N,2}^2 \underset{P}{\to} 0.$$

But

$$\begin{aligned} E\{\text{L.H.S. } (3.6)\}^2 &\leq n^{-1} \left(\int_0^M Fd\varphi\right)^7 E\left(H_2^{-4}(V_j)\left(\int_{V_j}^{M} Fd\varphi\right)\gamma_j[V_j \leq M]\right) \\ &= n^{-1} \left(\int_0^M Fd\varphi\right)^7 \int_0^M H_2^{-4}(t) \left(\int_t^M Fd\varphi\right) G_2\, d\overline{F}_2 \\ &\leq n^{-1} \left(\int_0^M Fd\varphi\right)^7 \int_0^M H_2^{-4}\, d\varphi \to 0 \end{aligned}$$

by conditions (2.9) and (2.13). Therefore, statement (3.6) holds which completes the proof. □

<u>Theroem 3.1:</u> If conditions (2.9), (2.11), (2.12) and (2.13) hold then $\hat{\sigma}_{N,2}^2 \underset{P}{\to} \sigma_2^2$.

<u>Proof:</u> We have that

$$\frac{1}{n}\sum_j \gamma_j\left\{ \frac{1+n}{1+N_n^+(V_j)} \int_0^{V_j} \hat{F}_n\, d\varphi \int_{V_j}^{M} \hat{F}_n\, d\varphi - H_2^{-1}(V_j) \int_0^{V_j} F_2\, d\varphi \int_{V_j}^{M} F_2\, d\varphi\right\}^2 [V_j \leq M]$$

$$\begin{aligned} = \frac{1}{n}\sum_j \gamma_j \Bigg\{ &\frac{1+n}{1+N_n^+(V_j)} \int_0^{V_j} (\hat{F}_n - F)d\varphi \int_{V_j}^{M} (\hat{F}_n - F)d\varphi \\ &+ \frac{1+n}{1+N_n^+(V_j)} \int_0^{V_j} (\hat{F}_n - F)d\varphi \int_{V_j}^{M} Fd\varphi \\ &+ \frac{1+n}{1+N_n^+(V_j)} \int_0^{V_j} Fd\varphi \int_{V_j}^{M} (\hat{F}_n - F)d\varphi \\ &+ \left(\frac{1+n}{1+N_n^+(V_j)} - H_2^{-1}(V_j)\right) \int_0^{V_j} Fd\varphi \int_{V_j}^{M} Fd\varphi\Bigg\}^2 [V_j \leq M] \underset{P}{\to} 0 \end{aligned}$$

by Lemma 3.1 and Lemma 3.2, and

$$\frac{1}{n} \sum_j \gamma_j H_2^{-2}(V_j) \left(\int_0^{V_j} F d\varphi\right)^2 \left(\int_{V_j}^{M} F d\varphi\right)^2 [V_j \le M] \xrightarrow[p]{} \sigma_2^2$$

by Lemma 3.3. Thus $\sigma_{N,2}^2 \xrightarrow[p]{} \sigma_2^2$ by Lemma 1 of Sen (1960). □

Now define $\hat{\sigma}_{N,1}^2$ as follows,

$$(3.7) \qquad \hat{\sigma}_{N,1}^2 = \int_0^M \left\{\frac{m+1}{1+N_m^+(t)} \int_t^M \hat{F}_m d\varphi \int_0^t \hat{F}_m d\varphi\right\}^2 d\,\tilde{\tilde{H}}_m(t)$$

where $\tilde{\tilde{H}}_m(t) = \frac{1}{m} \sum_i [u_i \le t, \delta_i = 1]$. Then proceeding as above we have the following result.

<u>Theorem 3.2:</u> If conditions (2.9), (2.10), (2.12) and (2.13) hold then $\hat{\sigma}_{N,1}^2 \xrightarrow[p]{} \sigma_1^2$.

Combining the results of Theorem 3.1, Theorem 3.2 and Corollory 2.1, we arrive at the following theorem.

<u>Theorem 3.3:</u> Under H_0: $F_1 = F_2$

$$\frac{\tilde{\Delta}_N(\varphi)}{\sqrt{\dfrac{\hat{\sigma}_{N,1}^2}{m} + \dfrac{\hat{\sigma}_{N,2}^2}{n}}} \xrightarrow[D]{} N(0, 1)$$

provided $\sigma_1^2 < \infty$, $\sigma_2^2 < \infty$ and conditions (2.9) - (2.14) hold.

4. Computational Aspects:

Let $V_{(1)} \le \cdots \le V_{(n)}$ be the ordered $V_1, \ldots, V_n$ and denote by $d_1, \ldots, d_n$ the corresponding $\gamma_1, \ldots, \gamma_n$. Thus $d_j = \gamma_i$ i.f.f. $V_{(j)} = V_i$.

Observe that for $V_{(j-1)} \le x < V_{(j)}$ one has

$$N_n^+(x) = n-j+1 \tag{4.1}$$

and

$$\hat{G}_n^{-1}(x) = \prod_{i=0}^{j-1} \{(2 + n-i)/(1 + n-i)\}^{[d_i = 0]}, \; j = 1,\ldots,n \tag{4.2}$$

where $V_{(0)} = 0$ and $d_0 = 1$.

Observe that $\hat{F}_n$ jumps only at $V_{(1)},\ldots,V_{(n)}$, and

$$\hat{F}_n(V_{(j)}) = n^{-1}(n-j) \prod_{i=0}^{j} \{(2 + n-i)/(1 + n-i)\}^{[d_i = 0]}, \; j = 1,\ldots,n. \tag{4.3}$$

Now

$$\begin{aligned} \int_0^{V_{(i)}} \hat{F}_n(S)d\varphi(S) &= \sum_{j=0}^{i-1} \hat{F}_n(V_{(j)})(\varphi(V_{(j+1)}) - \varphi(V_{(j)})) \\ &= n^{-1} \sum_{j=0}^{i-1} (n-j) \prod_{\ell=0}^{j} \{\frac{2+n-\ell}{1+n-\ell}\}^{[d_\ell=0]} (\varphi(V_{(j+1)}) - \varphi(V_{(j)})) \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} \int_{V_{(i)}}^{M} \hat{F}_n(S)d\varphi(S) = n^{-1} \sum_{k=i}^{n-1} (n-k) \prod_{\ell=0}^{k} \{\frac{2+n-\ell}{1+n-\ell}\}^{[d_\ell=0]} \{\varphi(M \wedge V_{(k+1)}) \\ - \varphi(V_{(k)})\}[V_{(k)} \le M]. \end{aligned} \tag{4.5}$$

Thus

$$\begin{aligned} \hat{\sigma}_{N,2}^2 &= \int_0^M \{\frac{1+n}{1+N_n^+(t)} \int_0^t \hat{F}_n d\varphi \int_t^n \hat{F}_n d\varphi\}^2 \, d\,\tilde{\tilde{H}}_n(t) \\ &= \frac{(1+n)^2}{n^5} \sum_{i=1}^{n} d_i(1+n-i)^{-2} \{\sum_{j=0}^{i-1} \sum_{k=i}^{n-1} (n-j)(n-k) \\ &\quad \times \prod_{\ell=0}^{j} (\frac{2+n-\ell}{1+n-\ell})^{[d_\ell=0]} \prod_{\ell=0}^{k} (\frac{2+n-\ell}{1+n-\ell})^{[d_\ell=0]} \end{aligned} \tag{4.6}$$

$$\times\ (\varphi(V_{(j+1)}) - \varphi(V_{(j)}))(\varphi(M\wedge V_{(k+1)}) - \varphi(V_{(k)}))[V_{(k)} \le M]\}^2$$

$$\times\ [V_{(i)} < M].$$

Similarly, let $U_{(1)} \le \cdots \le U_{(m)}$ be the ordered $U_1,\ldots,U_n$ and denote by $b_1,\ldots,b_m$ the corresponding $\delta_1,\ldots,\delta_m$. Thus $b_j = \delta_i$ i.f.f. $U_{(j)} = U_i$. Then

$$(4.7)\quad \hat{\sigma}^2_{N,1} = \frac{(1+m)^2}{m^5} \sum_{i=1}^{m} b_i(1+m-i)^{-2}\{\sum_{j=0}^{i-1}\sum_{k=i}^{m-1}(m-j)(m-k)$$

$$\times \prod_{\ell=0}^{j}\left(\frac{2+m-\ell}{1+m-\ell}\right)^{[b_\ell=0]}\prod_{\ell=0}^{k}\left(\frac{2+m-\ell}{1+m-\ell}\right)^{[b_\ell=0]}$$

$$\times\ (\varphi(U_{(j+1)}) - \varphi(U_{(j)}))(\varphi(M\wedge U_{(k+1)}) - \varphi(U_{(k)}))[U_k < M]\}^2$$

$$\times\ [U_{(i)} < M]$$

where $U_{(0)} = 0$ and $b_0 = 1$.

Now observe that

$$(4.8)\quad \int_0^M \hat{F}_n(t)\int_0^t \hat{F}_m(s)\,d\varphi(s)d\varphi(t)$$

$$= m^{-1}\sum_{j=0}^{m-1}(m-j)\prod_{\ell=0}^{j}\left(\frac{2+m-\ell}{1+m-\ell}\right)^{[b_\ell=0]}\left(\int_0^M \hat{F}_n(t)[U_{(j)} < t]\varphi(t\wedge U_{(j+1)})d\varphi(t)\right.$$

$$\left. - \varphi(U_{(j)})\int_0^M \hat{F}_n(t)[U_{(j)} < t]\,d\varphi(t)\right)$$

$$= \frac{1}{mn}\sum_{j=0}^{m-1}\sum_{i=0}^{n-1}(m-j)(n-i)\prod_{\ell=0}^{j}\left(\frac{2+m-\ell}{1+m-\ell}\right)^{[b_\ell=0]}$$

$$\times \prod_{\ell=0}^{i}\left(\frac{2+m-\ell}{1+m-\ell}\right)^{[d_\ell=0]}\{\tfrac{1}{2}(\varphi^2(U_{(j+1)}\wedge V_{(i+1)}\wedge M)$$

$$- \varphi^2(V_{(i)}\vee U_{(j)}))[V_{(i)}\vee U_{(j)} \le M\wedge U_{(j+1)}][V_{(j+1)} > U_{(j)}]$$

$$+ \varphi\,(U_{(j+1)})(\varphi(V_{(i+1)}\wedge M) - \varphi(V_{(i)} \vee U_{(j+1)}))$$

$$\times\, [U_{(j+1)} \vee V_{(i)} \leq M][V_{(i+1)} > U_{(j+1)}]$$

$$- \varphi\,(U_{(j)})(\varphi(V_{(i+1)}\wedge M) - \varphi\,(V_{(i)} \vee U_{(j)}))$$

$$\times\, [U_{(j)} \vee V_{(i)} \leq M][V_{(i+1)} > U_{(j)}].$$

Therefore,

$$(4.9)\quad \tilde{\Delta}_N(\varphi) = \int_0^M \hat{F}_m\, d\varphi \int_0^M \hat{F}_n\, d\varphi - 2\int_0^M \hat{F}_n(t)\int_0^t \hat{F}_m(s)d\varphi(s)\, d\varphi(t)$$

$$= \frac{1}{mn}\sum_{j=0}^{m-1}\sum_{i=0}^{n-1}(m-j)(n-i)\prod_{\ell=0}^{j}\left(\frac{2+m-\ell}{1+m-\ell}\right)^{[b_\ell=0]}$$

$$\times \prod_{\ell=0}^{i}\left(\frac{2+m-\ell}{1+m-\ell}\right)^{[d_\ell=0]}\{(\varphi(M\wedge U_{(j+1)}) - \varphi(U_{(i)}))[U_{(j)} < M]$$

$$\times\,(\varphi(M \wedge V_{(i+1)}) - \varphi\,(V_{(i)}))[V_{(i)} < M] + 2\,\varphi\,(U_{(j)})$$

$$\times\,(\varphi(V_{(i+1)} \wedge M) - \varphi\,(V_{(i)} \vee U_{(j)}))[U_{(j)} \vee V_{(i)} \leq M]$$
$$[V_{(i+1)} > U_{(j)}]$$

$$- 2\,\varphi\,(U_{(j+1)})(\varphi(V_{(i+1)} \wedge M) - \varphi\,(V_{(i)} \vee U_{(j+1)}))$$

$$\times\, [U_{(j+1)} \vee V_{(i)} \leq M][V_{(i+1)} > U_{(j+1)}]$$

$$- (\varphi^2(U_{(j+1)} \wedge V_{(i+1)} \wedge M) - \varphi^2\,(V_{(i)} \vee U_{(j)}))$$

$$\times\, [V_{(i)} \vee U_{(j)} \leq M\wedge U_{(j+1)}][V_{(j+1)} > U_{(j)}]\}\,.$$

CHAPTER III

THE ESTIMATOR $\tilde{\delta}_N$.

1. Notation and Preliminaries. Let

$$(1.1)\quad \tilde{\delta}_N = \int_0^M \int_0^t \{\hat{F}_m(t)\hat{F}_n(s) - \hat{F}_m(s)\hat{F}_n(t)\} d(\lambda_N \bar{\hat{F}}_m(s) + (1-\lambda_N)\bar{\hat{F}}_n(s)) d(\lambda_N \bar{\hat{F}}_m(t) + (1-\lambda_N)\bar{\hat{F}}_n(t))$$

and

$$(1.2)\quad \delta_N = \int_0^M \int_0^t \{F_1(t)F_2(s) - F_1(s)F_2(t)\} d(\lambda_N \bar{F}_1(s) = (1-\lambda_N)\bar{F}_2(s)) d(\lambda_N \bar{F}_1(t) + (1-\lambda_N)\bar{F}_2(t)).$$

In this Chapter, we obtain the asymptotic distribution of $\sqrt{N}\ (\tilde{\delta}_N - \delta_N)$ and obtain a consistent estimator for its asymptotic null variance.

2. Asymptotic normality of $\sqrt{N}\ (\tilde{\delta}_N - \delta_N)$. Let F_1, F_2, M, G_1, G_2 satisfy the following conditions as $N \to \infty$, and for $i = 1, 2$.

$$(2.1)\quad N^{-2} \int_0^M F_i\ H_i^{-8}\ d\bar{G}_i \to 0$$

$$(2.2)\quad N^{-1} \int_0^M H_i^{-12}\ G_i^{-2}\ d\bar{F}_i \to 0$$

$$(2.3)\quad N^{-1} \int_0^M H_1^{-6}\ H_2^{-6}\ d\bar{F}_i \to 0$$

$$(2.4)\quad N^{-1} \int_0^M H_1^{-6}\ G_1^{-1}\ d\bar{F}_1 \int_0^M H_2^{-6}\ G_1^{-1}\ d\bar{F}_1 \to 0$$

$$N^{-1} \int_0^M H_1^{-6}\ G_2^{-1}\ d\bar{F}_2 \int_0^M H_2^{-6}\ G_2^{-1}\ d\bar{F}_2 \to 0$$

and

$$(2.5)\quad N^{-1}\int_0^M F_2\, G_2^{-1}\int_0^t H_1^{-12}\, G_1\, d\bar{F}_1 d\bar{F}_2 \to 0$$

$$N^{-1}\int_0^M F_1\, G_1^{-1}\int_0^t H_2^{-12}\, G_2\, d\bar{F}_2\, d\bar{F}_1 \to 0.$$

Observe that

$$
\begin{aligned}
(\tilde{\delta}_N - \delta_N) = {} & \lambda_N^2\Big(\int_0^M\int_0^t \{\hat{F}_m(t)\hat{F}_n(s) - \hat{F}_m(s)\hat{F}_n(t)\}d\bar{\hat{F}}_m(s)d\bar{\hat{F}}_m(t) \\
(2.6)\qquad & - \int_0^M\int_0^t D(s,t)d\,\bar{F}_1(s)\; d\,\bar{F}_1(t)\Big) \\
& + \lambda_N(1-\lambda_N)\Big(\int_0^M\int_0^t \{\hat{F}_m(t)\hat{F}_n(s) - \hat{F}_m(s)\hat{F}_n(t)\}d\bar{\hat{F}}_m(s)d\bar{\hat{F}}_n(t) \\
& - \int_0^M\int_0^t D(s,t)d\,F_1(s)\; d\,F_2(t)\Big) \\
& + \lambda_N(1-\lambda_N)\Big(\int_0^M\int_0^t \{\hat{F}_m(t)\hat{F}_n(s) - \hat{F}_m(s)\hat{F}_n(t)\}\; d\,\bar{\hat{F}}_n(s)d\,\bar{\hat{F}}_m(t) \\
& - \int_0^M\int_0^t D(s,t)d\,F_2(s)d\,F_1(t)\Big) \\
& + (1-\lambda_N)^2\Big(\int_0^M\int_0^t \{\hat{F}_m(t)\hat{F}_n(s) - \hat{F}_m(s)\hat{F}_n(t)\}\; d\,\bar{\hat{F}}_n(s)d\,\bar{\hat{F}}_n(t) \\
& - \int_0^M\int_0^t D(s,t)\; d\,F_2(s)\; d\,F_2(t)\Big).
\end{aligned}
$$

But $\sqrt{N}\int_0^M\int_0^t \{\hat{F}_m(t)\hat{F}_n(s) - \hat{F}_m(s)\hat{F}_n(t)\}\; d\,\bar{\hat{F}}_m(s)\; d\,\bar{\hat{F}}_m(t)$

$$
\begin{aligned}
& = \frac{\sqrt{N}}{m^2}\sum_i\sum_j \{\hat{F}_m(U_i)\hat{F}_n(U_j) - \hat{F}_m(U_j)\hat{f}_n(U_i)\}\{\delta_j\; \hat{G}_m^{-1}(U_j) \\
(2.7)\qquad & + (1-\delta_j)\hat{G}_m^{-1}(U_j)(2 + N_m^+(U_j))^{-1}\}\{\delta_i\; \hat{G}_m^{-1}(U_i) \\
& + (1 - \delta_i)\hat{G}_m^{-1}(U_i)(2 + N_m^+(U_i))^{-1}\}[U_i \le U_i \le M].
\end{aligned}
$$

Expanding the right hand side in the usual fashion and using Lemmas 2.2, 2.3, and 2.4, of Chapter 2, we get the following Theorem.

Theorem 2.1: If conditions (2.1) - (2.4) hold then

$$N\, E\{(\int_0^M \int_0^t \{\hat{F}_m(t)\hat{F}_n(s) - \hat{F}_m(s)\hat{F}_n(t)\}\, d\bar{F}_1\, d\bar{F}_1 - \int_0^M \int_0^t D(s,t) d\bar{F}_1\, d\bar{F}_1)$$

$$- (\frac{1}{m^2} \sum_i \sum_j D(U_j, U_i)\, \delta_i\, \delta_j\, G_1^{-1}(U_j)(\hat{G}_1^{-1}(U_i) - G_1(U_i))([U_j \le U_i \le M]$$

$$- [U_i \le U_j \le M])$$

$$+ \frac{1}{m^2} \sum_i \sum_j \{(\hat{F}_m(U_i) - F_1(U_i))F_2(U_j) - (\hat{F}_n(U_i) - F_2(U_i))F_1(U_j)\}\delta_i\, \delta_j$$

$$\times\, G_1^{-1}(U_i) G_1^{-1}(U_j)([U_j \le U_i \le M] - [U_i \le U_j \le M])$$

$$+ \frac{1}{m^2} \sum_i \sum_j D(U_j, U_i)\, \delta_i \delta_j\, G_1^{-1}(U_i) G_1^{-1}(U_j)[U_j \le U_i \le M]$$

$$- \int_0^M \int_0^t D(s,t) d\bar{F}_1\, d\bar{F}_1))\}^2 \to 0.$$

Now proceeding as in section 2 of Chapter 2, we have the following.

Theorem 2.2: If conditions (2.1) - (2.4) hold then

$$\sqrt{N}\, (\int_0^M \int_0^t \{\hat{F}_m(t)\hat{F}_n(s) - \hat{F}_m(s)\hat{F}_n(t)\}\, d\bar{\hat{F}}_m(s) d\bar{\hat{F}}_m(t) - \int_0^M \int_0^t D(s,t) d\bar{F}_1\, d\bar{F}_1)$$

$$\sqrt{N}\, (\frac{1}{m} \sum_i \{\int_0^{U_i \wedge M} G_1^{-1}\, R_{2,M} d\bar{F}_1 - \int_0^M F_1\, R_{2,M} d\bar{F}_1\}$$

$$+ \frac{1}{m} \sum_i (\tfrac{1}{2})\{(1-\delta_i) H_1^{-1}(U_i) \int_{u_i}^M (Q_{1,M} + F_1\, R_{2,M}) d\bar{F}_1 [U_i \le M]$$

$$- \int_0^{U_i \wedge M} H_1^{-2} F_1 \int_x^M (Q_{1,M} + F_1 R_{2,M}) d\bar{F}_1 \, d\bar{G}_1 \}$$

$$+ \frac{1}{m} \sum_i (\tfrac{1}{2}) \{ G_1^{-1}(U_i) \delta_i \, Q_{1,M}(U_i) [U_i \leq M] - 2 \int_0^M \int_0^x D(y, x) d\bar{\bar{F}}_1 \, d\bar{\bar{F}}_1 \}$$

$$- \frac{1}{n} \sum_j (\tfrac{1}{2}) \{ (1-\gamma_j) H_2^{-1}(V_j) \int_{V_j}^M F_2 R_{1,M} d\bar{F}_1 [V_j \leq M]$$

$$- \int_0^{V_j \wedge M} H_2^{-2} F_2 \int_x^M F_2 R_{1,M} d\bar{F}_1 \, d\bar{G}_2 \}$$

$$- \frac{1}{n} \sum_j \{ \int_0^{V_j \wedge M} G_2^{-1} R_{1,M} d\bar{F}_1 - \int_0^M F_2 R_{1,M} d\bar{F}_1 \})$$

converges in L_2 to zero where $R_{i,M}(x) = \int_0^x F_i \, d\bar{F}_1 - \int_x^M F_i d\bar{F}_1$ for $i = 1, 2$, and $Q_{1,M}(x) = \int_0^x D(y,x) d\bar{F}_1 - \int_x^M D(y,x) d\bar{F}_1$

Proceeding as above with each term in (2.6), we get the following theorem.

<u>Theorem 2.3:</u> If (2.1) - (2.5) hold then

$$\sqrt{N} \, (\tilde{\delta}_N - \int_0^M \int_0^t D(s, t) d(\lambda_N \bar{F}_1 + (1-\lambda_N)\bar{F}_2) d(\lambda_N \bar{F}_1 + (1-\lambda_N)\bar{F}_2))$$

$$- \sqrt{N} \, (\lambda_N^2, (1-\lambda_N)^2, \lambda_N(1 - \lambda_N)) \cdot (S_N^{(1)} - S_N^{(2)})$$

converges in L_2 to zero, where $S_N^{(1)}$ and $S_N^{(2)}$ are defined in appendix B.

With the notation as defined in appendix B, we have the following two theorems.

<u>Theorem 2.4:</u> If $\sum^{(1)} < \infty$ and $\sum^{(2)} < \infty$, then

$$\sqrt{N} \, (S_N^{(1)} - S_N^{(2)}) \xrightarrow[D]{} N_3 \, (\underset{\sim}{0}, \frac{\sum^{(1)}}{\lambda} + \frac{\sum^{(2)}}{1-\lambda}) \, .$$

Proof: Since $E(S_N^{(i)}) = \underset{\sim}{0}$ for $i = 1, 2$, $\Sigma_N^{(i)} \to \Sigma^{(i)} < \infty$ for $i = 1, 2$, $\frac{N}{m} \to 1/\lambda$, $\frac{N}{n} \to \frac{1}{1-\lambda}$, and the U_i's, δ_i's, V_j's, and γ_j's are all independent, we have that

$$\sqrt{N}\, a' \cdot (S_N^{(1)} - S_N^{(2)}) \xrightarrow[D]{} N(0, \frac{a' \Sigma^{(1)} a}{\lambda} + \frac{a' \Sigma^{(2)} a}{1-\lambda}) \text{ for all } a \in \mathbb{R}^3 .$$

Thus by the Wold-Carmer device, we may conclude that

$$\sqrt{N}\, (S_N^{(1)} - S_N^{(2)}) \xrightarrow[D]{} N_3\, (\underset{\sim}{0}, \frac{\Sigma^{(1)}}{\lambda} + \frac{\Sigma^{(2)}}{1-\lambda}) . \qquad \square$$

Theorem 2.5: If conditions (2.1) - (2.5) hold and $\Sigma^{(1)} < \infty$, $\Sigma^{(2)} < \infty$ then

$$\sqrt{N}\, (\tilde{\delta}_N - \int_0^M \int_0^t D(s, t) d(\lambda_N \bar{F}_1(s) + (1-\lambda_N)\bar{F}_2(s)) d(\lambda_N \bar{F}_1(t) + (1-\lambda_N)\bar{F}_2(t)))$$

converges in distribution to a normal random variable with mean zero and variance

$$(\lambda^2, (1-\lambda)^2, \lambda(1-\lambda))(\frac{\Sigma^{(1)}}{\lambda} + \frac{\Sigma^{(2)}}{1-\lambda})(\lambda^2, (1-\lambda)^2, \lambda(1-\lambda))' .$$

Proof: This is a result of theorems 2.3 and 2.4. $\square$

Corollory 2.1: Under H_0: $F_1 = F_2 = F$, if conditions (2.1) - (2.5) hold, $\int_0^\infty G_i^{-1} F^2 d\bar{F} < \infty$, and $\int_0^\infty G_i^{-2} F^3 d\bar{G}_i < \infty$ for $i = 1, 2$, then

$$\sqrt{N}\, \tilde{\delta}_N \xrightarrow[D]{} N(0, \sigma_0^2) \text{ where}$$

$$\sigma_0^2 = \frac{1}{64} \{\lambda^{-1} \int_0^\infty G_1^{-1} (7 F^2 - 18 F^4 + 11 F^6) d\bar{F} + (1-\lambda)^{-1} \int_0^\infty G_2^{-1} (7 F^2 - 18 F^4 + 11 F^6) d\bar{F}\}.$$

Proof: Under H_0,

$$\Sigma^{(1)} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{pmatrix} \left(\frac{1}{4}\int_0^\infty G_1^{-1}(F^2 - 3F^4 + 2F^6)d\bar{F} - \frac{3}{64}\int_0^\infty G_1^{-2}(F^3 - 2F^5 + F^7)d\bar{G}_1\right)$$

But $\int_0^\infty G_1^{-2}(F^3 - 2F^5 + F^7)d\bar{G}_1 = \int_0^\infty G_1^{-1}(3F^2 - 10F^4 + 7F^6)d\bar{F}$, so

we have that

$$\Sigma^{(1)} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{pmatrix} \left(\frac{1}{64}\right)\int_0^\infty G_1^{-1}(7F^2 - 18F^4 + 11F^6)d\bar{F} ,$$

and

$$\Sigma^{(2)} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{pmatrix} \left(\frac{1}{64}\right)\int_0^\infty G_2^{-1}(7F^2 - 18F^4 + 11F^6)d\bar{F} .$$

Therefore, by Theorem 2.5, $\sqrt{N}\,\tilde{\delta}_N \xrightarrow[D]{} N(0,\sigma_0^2)$. □

Remark: Notice that if G_1 and G_2 put all their mass at ∞, i.e., there is no censoring, then $\sigma_0^2 = \frac{1}{210\,\lambda(1-\lambda)}$, which is the asymptotic null variance of Kochar's statistic $\sqrt{N}\,\tilde{\delta}_N$.

3. A Consistent Estimator of σ_0^2. Let

(3.1) $$\sigma_{0,i}^2 = \int_0^\infty G_i^{-1}(7F^2 - 18F^4 + 11F^6)d\bar{F} \quad \text{for } i = 1, 2$$

(3.2) $$\hat{\sigma}_{0,m}^2 = \int_0^M \hat{G}_m^{-1}(7\hat{F}_m^2 - 18\hat{F}_m^4 + 11\hat{F}_m^6)d\bar{\hat{F}}_m$$

and

(3.3) $$\hat{\sigma}_{0,m}^2 = \int_0^M \hat{G}_n^{-1}(7\hat{F}_n^2 - 18\hat{F}_n^4 + 11\hat{F}_n^6)d\bar{\hat{F}}_n$$

Theorem 3.1: If $\sigma^2_{0,1} < \infty$, $\sigma^2_{0,2} < \infty$ and conditions (2.1), (2.2) hold then $\hat{\sigma}^2_{0,m} \xrightarrow[p]{} \sigma^2_{0,1}$ and $\hat{\sigma}^2_{0,n} \xrightarrow[p]{} \sigma^2_{0,2}$ under H_0.

Proof: Consider

$$(3.4) \quad \int_0^M \hat{G}_m^{-1} \hat{F}_m^2 \, d\bar{\hat{F}}_m = \frac{1}{m} \sum_{i=1}^m \hat{G}_m^{-2} (U_i) \hat{F}_m^2 (U_i) \, \delta_i \, [U_i \leq M]$$

$$+ \frac{1}{m} \sum_i \hat{G}_m^{-2}(U_i) \hat{F}_m^2 (U_i)(2 + N_m^+ (U_i))^{-1} \, [\delta_i = 0, U_i \leq M].$$

But

$$0 \leq \frac{1}{m} \sum_i \hat{G}_m^{-2} (U_i) \hat{F}_m^2 (U_i)(2 + N_m^+ (U_i))^{-1} \, [\delta_i = 0, U_i \leq M]$$

$$\leq \frac{2}{m} \sum_i (\hat{G}_m^{-1} (U_i) - G_1^{-1} (U_i))^2 \, \hat{F}_m^2 (U_i)(2 + N_m^+ (U_i))^{-1} \, [\delta_i = 0, U_i \leq M]$$

$$+ \frac{2}{m} \sum_i G_1^{-2} (U_i) \hat{F}_m^2 (U_i)(2 + N_m^+ (U_i))^{-1} \, [\delta_i = 0, U_i \leq M] .$$

However,

$$E\{(\hat{G}_m^{-1} (U_i) - G_1^{-1} (U_i))^2 \, \hat{F}_m^2 (U_i)(2 + N_m^+ (U_i))^{-1} [\delta_i = 0, U_i \leq M]\}$$

$$\leq n^{-2} \int_0^M H_1^{-7} \, F \, d\bar{G}_1 \to 0 \quad \text{by (2.1)}$$

and

$$E\{G_1^{-2} (U_i) \, \hat{F}_m^2 (U_i)(2 + N_m^+ (U_i))^{-1} [\delta_i = 0, U_i \leq M]\}$$

$$\leq n^{-1} \int_0^M H_1^{-3} \, F \, d\bar{G}_1 \to 0 \quad \text{by (2.1)}.$$

Therefore, the second term on the R.H.S. of (3.4) converges in probability to zero.

Now observe that

$$(3.5)\quad \frac{1}{m}\sum_i \{\hat{G}_m^{-1}(U_i)\hat{F}_m(U_i) - G_1^{-1}(U_i)F(U_i)\}^2 \delta_i [U_i \leq M]$$

$$= \frac{1}{m}\sum \{(\hat{G}_m^{-1}(U_i) - G_1^{-1}(U_i))\hat{F}_m(U_i) + G_1^{-1}(U_i)(\hat{F}_m(U_i) - F(U_i))\}^2\delta_i$$

$$\times [U_i \leq M]$$

But

$$E\{(\hat{G}_m^{-1}(U_i - G_1^{-1}(U_i))^2 \hat{F}_m^2(U_i)\delta_i[U_i \leq M]\} \leq \frac{1}{m}\int_0^M H_1^{-6} G_1 \, d\bar{F}, \to 0$$

by (2.2) and

$$E\{G_1^{-2}(U_i)(\hat{F}_m(U_i) - F(U_i))^2 \delta_i[U_i \leq M]\} \leq \frac{1}{m}\int_0^M G_1^{-1} H_1^{-6} \, d\bar{F}_1 \to 0$$

by (2.2). Therefore,

$$(3.6)\quad \frac{1}{m}\sum_i \{\hat{G}_m^{-1}(U_i)\hat{F}_m(U_i) - G_1^{-1}(U_i)\}^2 \delta_i[U_i \leq M] \underset{P}{\to} 0.$$

We will now show that

$$(3.7)\quad \frac{1}{m}\sum_{i=1}^m G_1^{-2}(U_i)F^2(U_i)\,\delta_i\,[U_i \leq M] \underset{P}{\to} \int_0^\infty G_1^{-1} F^2 \, d\bar{F}.$$

Since $\int_0^\infty G_1^{-1} F^2 \, d\bar{F} < \infty$, then $\int_0^M G_1^{-1} F^2 \, d\bar{F} \to \int_0^\infty G^{-1} F^2 \, d\bar{F}_1$. Thus to prove (3.7) we need only show that

$$(3.8)\quad \frac{1}{m}\sum_{i=1}^m \{G_1^{-2}(U_i)F^2(U_i)\,\delta_i\,[U_i \leq M] - \int_0^M G_1^{-1} F^2 \, d\bar{F}\} \underset{P}{\to} 0.$$

But $E\{G_1^{-2}(U_i)F^2(U_i)\,\delta_i\,[U_i \leq M]\} = \int_0^M G_1^{-1} F^2 \, d\bar{F}$, and

Var(L.H.S. of 3.8) $\leq n^{-1}\int_0^M G_1^{-3} F^4 \, d\bar{F} \to 0$ by (2.2). Hence statement (3.8) is true which implies that statement (3.7) is true if condition (2.2) hold.

Therefore, by (3.6) and (3.7) we have that the first term on the R.H.S. of (3.4) converges in probability to $\int_0^\infty G_1^{-1} F^2 d\bar{F}_1$.

By similar arguements, we have that

$\int_0^M \hat{G}_m^{-1} \hat{F}_m^4 d\hat{\bar{F}}_m \underset{P}{\to} \int_0^\infty G_1^{-1} F^4 d\bar{F}$ and $\int_0^M \hat{G}_m^{-1} \hat{F}_m^6 d\hat{\bar{F}}_m \underset{P}{\to} \int_0^\infty G_1^{-1} F^6 d\bar{F}$. Therefore, $\hat{\sigma}_{0,m}^2 \underset{p}{\to} \sigma_{0,1}^2$ provided conditions (2.1), (2.2) hold and $\sigma_{0,1}^2 < \infty$. □

Combining this result with Corollory 2.1, we have the following theorem.

<u>Theorem 3.2:</u> Under H_0: $F_1 = F_2 = F$, if $\sigma_{0,1}^2 < \infty$, $\sigma_{0,2}^2 < \infty$ and conditions (2.1) - (2.5) hold, then

$$\frac{8\ \tilde{\delta}_N}{\sqrt{\dfrac{\hat{\sigma}_{0,m}^2}{m} + \dfrac{\hat{\sigma}_{0,n}^2}{n}}} \xrightarrow[D]{} N(0,\ 1)\ .$$

4. Computational form of $\hat{\sigma}_{0,m}^2$ and $\hat{\sigma}_{0,n}^2$.

Using the same notation as in Section 4 of Chapter 2, we have that

$$(4.1)\quad \hat{\sigma}_{0,m}^2 = \frac{1}{m} \sum_i \hat{G}_m^{-2}(U_{(i)})(7\hat{F}_m^2(U_{(i)}) - 11\ \hat{F}_m^4(U_{(i)}) + 18\ \hat{F}_m^6(U_{(i)}))$$

$$\times \{b_i + (1-b_i)(2 + N_m^+(U_{(i)}))^{-1}\}[U_{(i)} \leq M]$$

$$= m^{-3} \sum_i (m-i)^2 \prod_{\ell=1}^{i} \left(\frac{2 + m-i}{1 + m-i}\right)^{4[b_\ell = 0]} (7$$

$$- 11 \left(\frac{m-i}{m}\right)^2 \prod_{\ell=1}^{i} \left(\frac{2 + m-i}{1 + m-i}\right)^{2[b_\ell = 0]}$$

$$+ 18 \left(\frac{m-i}{m}\right)^4 \prod_{\ell=1}^{i} \left(\left(\frac{2 + m-i}{1 + m-i}\right)^{4[b_\ell = 0]}\right)$$

$$\times (b_i + (1-b_i)(2 + m-i)^{-1})[U_{(i)} \leq M],$$

and

$$(4.2) \quad \hat{\sigma}^2_{0,n} = n^{-3} \sum_j (n-j)^2 \prod_{\ell=1}^{j} \left(\frac{2 + n-j}{1 + n-j}\right)^{4[d_\ell = 0]} (7$$

$$- 11 \left(\frac{n-j}{n}\right)^2 \prod_{\ell=1}^{j} \left(\frac{2 + n-j}{1 + n-j}\right)^{2[d_\ell = 0]}$$

$$+ 18 \left(\frac{n-j}{n}\right)^4 \prod_{\ell=1}^{j} \left(\left(\frac{2 + n-j}{1 + n-j}\right)^{4[d_\ell = 0]}\right)$$

$$\times (d_j + (1-d_j)(2 + n-j)^{-1})[V_{(j)} \leq M].$$

APPENDIX A

In this appendix, we want to obtain (2.21) where $2\,\psi_{n,1}$ is defined by (2.20). Observe that

$$4\,\mathrm{Var}(\psi_{n,1}) = \mathrm{Var}\Big(\int_0^{V_1\wedge M} g_{1,M}\, G_2^{-1}\, d\varphi + \mu_{n,M}(V_1)H_2^{-1}(V_1)[\gamma_1 = 0,\ V_1 \le M]$$

$$- \int_0^{V_1\wedge M} \mu_{2,M}H_2^{-2}\, d\tilde{H}_2 + \frac{1}{n}\{\mu_{2,M}(V_1)H_2^{-1}(V_1)[\gamma_1 = 0,\ V, \le M]$$

$$+ \int_0^{V_1\wedge M} \mu_{2,M}H_2^{-2}\, d\tilde{H}_2\}\Big)$$

$$= \mathrm{Var}(A + B - C + D).$$

Now we see that $\mathrm{Var}(D) = O(n^{-2}\, H_2^{-2}(M))$ since $\mu_{2,M}$ is bounded and that

$$\int_0^{V_1\wedge M} H_2^{-2}\, d\tilde{H}_2 \le F_2^{-1}(M)G_2^{-1}(M) = H_2^{-1}(M)$$

where we have used the equality $d\,\tilde{H}_2 = F_2\, d\bar{G}_2$. Consequently,

$$4\,\mathrm{Var}(\psi_{n,1}) = \mathrm{Var}(A + B - C) + O(n^{-2}\, H_2^{-2}(M)) + O(n^{-1}\, H_2^{-1}(M))$$

provided $\mathrm{Var}(A + B - C)$ is bounded for large N. Thus $\sigma_2^2 < \infty$ will imply

$$4\,\mathrm{Var}(\psi_{n,1}) = \int_0^M H_2^{-2}(t)\Big(\int_t^M g_{1,M}\, \underline{F}_2\, d\varphi\Big)^2 d\tilde{\tilde{H}}_2(t) + O(n^{-1}\, H_2^{-2}(M))$$

where $\tilde{\tilde{H}}_2(t) = P(\gamma_1 = 1,\ V_1 \le t)$ if

$$\mathrm{Var}(A + B - C) = \int_0^M H_2^{-2}(t)\ \Big(\int_t^M g_{1,M}\, F_2\, d\varphi\Big)^2\ d\tilde{\tilde{H}}_2(t).$$

Since

$$E(C) = \int_0^\infty \int_0^{t\wedge M} \mu_{2,M}(s)\, H_2^{-2}(s)d\tilde{H}_2(s)d\overline{H}_2(t)$$

$$= \int_0^M \int_t^\infty \mu_{2,M}(s)H_2^{-2}(s)d\overline{H}_2(t)d\tilde{H}_2(s)$$

$$= \int_0^M \mu_{2,M}(s)H_2^{-1}(s)d\tilde{H}_2(s)$$

$$= E(B),$$

we have

$$Var(A + B - C) = Var(A) + E(B^2) + E(C^2) + 2E(AB) - 2E(AC)$$

$$- 2E(BC).$$

But

$$\text{(A1)}\quad Var(A) = \int_0^\infty (\int_0^{t\wedge M} g_{1,M}G_2^{-1}\, d\varphi)^2\, d\overline{H}_2(t) - (\int_0^\infty \int_0^{t\wedge M} g_{1,M}G_2^{-1}\, d\varphi\, d\overline{H}_2(t))^2$$

$$= 2 \int_0^\infty \{\int_0^{t\wedge M} g_{1,M}(u)G_2^{-1} \int_0^u g_{1,M}(v)G_2^{-1}(v)d\varphi(v)d\varphi(u)\}d\overline{H}_2(t)$$

$$- (\int_0^M g_{1,M}(u)G_2^{-1}(u) \int_u^\infty d\overline{H}_2(t)d\varphi(u))^2$$

$$= 2 \int_0^M g_{1,M}(u)F_2(u) \int_0^u g_{1,M}(v)G_2^{-1}(v)d\varphi(v)d\varphi(u)$$

$$- 2 \int_0^M g_{1,M}(u) \int_0^u g_{1,M}(v)F_2(v)d\varphi(v)d\varphi(u)$$

$$\text{(A2)}\quad E(B^2) = \int_0^M \mu_{2,M}^2\, H_2^{-2}\, F_2\, d\overline{G}_2 = \int_0^M \mu_{2,M}^2\, H_2^{-2}\, d\tilde{H}_2$$

(A3) $$E(C^2) = \int_0^\infty \Big(\int_0^{t\wedge M} \mu_{2,M}(u)H_2^{-2}(u)d\tilde{H}_2(u)\Big)^2 \, d\bar{H}_2(t)$$

$$= 2 \int_0^\infty \int_0^{t\wedge M} \mu_{2,M}(u)H_2^{-2}(u) \int_0^u \mu_{2,M}(v)H_2^{-2}(v)d\tilde{H}_2(v)d\tilde{H}_2(u)d\bar{H}_2(t)$$

$$= 2 \int_0^M \mu_{2,n}(u)H_2^{-1}(u) \int_0^u \mu_{2,M}(v)H_2^{-2}(v)d\tilde{H}_2(v)d\tilde{H}_2(u)$$

(A4) $$2E(AB) = 2 \int_0^M \mu_{2,M}(t)H_2^{-1}(t) \int_0^{t\wedge M} g_{1,M}(u)G_2^{-1}(u)d\varphi(u)d\,\tilde{H}_2(t)$$

$$= 2 \int_0^M \Big(\int_t^M g_{1,M}(v)F_2(v)d\varphi(v)\Big) \int_0^{t\wedge M} g_{1,M}(u)G_2^{-1}(u)d\varphi(u)H_2^{-1}(t)d\tilde{H}_2(t)$$

$$= 2 \int_0^M g_{1,M}(u)G_2^{-1}(u)\Big(\int_u^M g_{1,M}(v)F_2(v) \int_u^v H_2^{-1}(t)d\tilde{H}_2(t)d\varphi(v)d\varphi(u)$$

(A5) $$-2E(BC) = -2 \int_0^M \Big(\int_0^{u\wedge M} \mu_{2,M}(v)H_2^{-2}(v)d\tilde{H}_2(v)\Big)\mu_{2,M}(u)H_2^{-1}(u)d\tilde{H}_2(u)$$

$$= -2 \int_0^M \mu_{2,M}(u)H_2^{-1}(u) \int_0^u \mu_{2,M}(v)H_2^{-2}(v)d\tilde{H}_2(v)d\tilde{H}_2(u)$$

(A6) $$-2E(AC) = -2 \int_0^\infty \Big(\int_0^{s\wedge M} g_{1,M}(u)G_2^{-1}(u)d\varphi(u)\Big)\Big(\int_0^{s\wedge M} \mu_{2,M}(t)H_2^{-2}(t)d\tilde{H}_2(t)\Big)d\bar{H}_2(s)$$

$$= -2 \int_0^M \mu_{2,M}(t)H_2^{-2}(t)\Big\{\int_0^t g_{1,M}(u)G_2^{-1}(u) \int_t^\infty d\bar{H}_2(s)d\varphi(u)$$

$$+ \int_t^M g_{1,M}(u)G_2^{-1}(u) \int_u^\infty d\bar{H}_2(s)d\varphi(u)\Big\}d\tilde{H}_2(t)$$

$$= -2 \int_0^M \mu_{2,M}(t)H_2^{-1}(t) \int_0^t g_{1,M}(u)G_2^{-1}(u)d\varphi(u)d\tilde{H}_2(t)$$

$$-2 \int_0^M \mu_{2,M}(t)H_2^{-2}(t) \int_t^M g_{1,M}(u)F_2(u)d\varphi(u)d\tilde{H}_2(t)$$

$$= -2 \int_0^M \Big(\int_t^M g_{1,M}(v)F_2(v)d\varphi(v)\Big)H_2^{-1}(t) \int_0^t g_{1,M}(u)G_2^{-1}(u)d\varphi(u)d\tilde{H}_2(t)$$

$$-2 \int_0^M \mu_{2,M}^2(t)H_2^{-2}(t)d\tilde{H}_2(t)$$

$$= -2 \int_0^M g_{1,M}(u) G_2^{-1}(u) \int_u^M g_{1,M}(v) F_2(v) \left(\int_u^v H_2^{-1}(t)\,d\tilde{H}_2(t)\right) d\varphi(v)\,d\varphi(u)$$

$$-2 \int_0^M \mu_{2,M}^2(t) H_2^{-2}(t)\,d\tilde{H}_2(t).$$

By adding (A1) through (A6), we obtain that

$$\mathrm{Var}(A + B - C) = 2 \int_0^M g_{1,M}(u) F_2(u) \int_0^u g_{1,M}(v) G_2^{-1}(v)\,d\varphi(v)\,d\varphi(u)$$

$$-2 \int_0^M g_{1,M}(u) F_2(u) \int_0^u g_{1,M}(v) F_2(v)\,d\varphi(v)\,d\varphi(u)$$

$$- \int_0^M \mu_{2,M}^2 H_2^{-2}\,d\tilde{H}_2.$$

However,

$$- \int_0^M \mu_{2,M}^2 H_2^{-2}\,d\tilde{H}_2 = - \int_0^M \left(\int_t^M g_{1,M}(u) F_2(u)\,d\varphi(u)\right)^2 H_2^{-2}(t)\,d\tilde{H}_2(t)$$

$$= -2 \int_0^M g_{1,M}(u) F_2(u) \left\{\int_0^u H_2^{-2}(t) \int_t^u g_{1,M}(v) F_2(v)\,d\varphi(v)\,d\tilde{H}_2(t)\right\} d\varphi(u)$$

$$= 2 \int_0^M g_{1,M}(u) F_2(u) \int_0^u g_{1,M}(v) F_2(v) \int_0^v (-H_2^{-2}(t))\,d\tilde{H}_2(t)\,d\varphi(v)\,d\varphi(u)$$

$$= 2 \int_0^M g_{1,M}(u) F_2(u) \int_0^u g_{1,M}(v) F_2(v)\,d\varphi(v)\,d\varphi(u)$$

$$-2 \int_0^M g_{1,M}(u) F_2(u) \int_0^u g_{1,M}(v) G_2^{-1}(v)\,d\varphi(v)\,d\varphi(u)$$

$$+ 2 \int_0^M g_{1,M}(u) F_2(u) \int_0^u g_{1,M}(v) F_2(v) \left(\int_0^v H_2^{-2}(t)\,d\tilde{\tilde{H}}_2(t)\right) d\varphi(v)\,d\varphi(u).$$

Thus

$$\mathrm{Var}(A + B - C) = 2 \int_0^M g_{1,M}(u) F_2(u) \left\{\int_0^u g_{1,M}(v) F_2(v) \int_0^v H_2^{-2}(t)\,d\tilde{\tilde{H}}_2(t)\,d\varphi(v)\right\} d\varphi(u)$$

$$= 2 \int_0^M g_{1,M}(u)F_2(u)\{\int_0^u H_2^{-2}(t) \int_t^u g_{1,M}(v)F_2(v)d\varphi(v)d\tilde{\tilde{H}}_2(t)\}d\varphi(u)$$

$$= 2 \int_0^M H_2^{-2}(t)(\int_t^M g_{1,M}(u)F_2(u) \int_t^u g_{1,M}(v)F_2(v)d\varphi(v)d\varphi(u))d\tilde{\tilde{H}}_2(t)$$

$$= \int_0^M H_2^{-2}(t)(\int_t^M g_{1,M}(u)F_2(u)d\varphi(u))^2 \; d\tilde{\tilde{H}}_2(t)$$

which complets the proof of Theorem (2.2).

APPENDIX B

In this appendix, we present the notation that is used in Chapter 3.

Let $\tilde{R}_{i,M}(x) = \int_0^x F_i d\bar{F}_2 - \int_x^M F_i d\bar{F}_2$, $R_{i,M}(x) = \int_0^x F_i d\bar{F}_1 - \int_x^M F_i d\bar{F}_1$,

$\tilde{Q}_{i,M}(x) = \int_0^x D(x,y)d\bar{F}_i(y) - \int_x^M D(x,y)d\bar{F}_i(y)$, and

$Q_{i,M}(x) = \int_0^x D(y,x)d\bar{F}_i(y) - \int_x^M D(y,c)d\bar{F}_i(y)$ for $i = 1, 2$. Also

let $S_N^{(1)} = (S_{N1}^{(1)}, S_{N2}^{(1)}, S_{N3}^{(1)})'$ and $S_N^{(2)} = (S_{N1}^{(2)}, S_{N2}^{(2)}, S_{N3}^{(2)})'$

where

$$S_{N1}^{(1)} = \frac{1}{m}\sum_i (\tfrac{1}{2})\{(1-\delta_i)H_1^{-1}(U_i)\int_{U_i}^M (Q_{1,M} + F_1\, R_{2,M})d\bar{F}_1\, [U_i \leq M]$$
$$- \int_0^{U_i \wedge M} H_1^{-2}\, F(\int_x^M (Q_{1,M} + F_1\, R_{2,M})d\bar{F}_1)d\bar{G}_1\}$$
$$+ \frac{1}{m}\sum_i \{\int_0^{U_i \wedge M} G_1^{-1}\, R_{2,M}d\bar{F}_1 - \int_0^M F_1\, R_{2,M}d\bar{F}_1\}$$
$$+ \frac{1}{m}\sum_i \tfrac{1}{2}\{G_1^{-1}(U_i)\delta_i\, Q_{1,M}(U_i)[U_i \leq M] - 2\int_0^M\int_0^x D(y,x)d\bar{F}_1 d\bar{F}_1\},$$

$$S_{N,2}^{(1)} = \frac{1}{m}\sum_i \tfrac{1}{2}\{(1-\delta_i)H_1^{-1}(U_i)\int_{U_i}^M F_1\, \tilde{R}_{2,M}d\bar{F}_2\, [U_i \leq M]$$
$$- \int_0^{U_i \wedge M} H_1^{-2}\, F_1 \int_x^M F_1\, \tilde{R}_{2,M}d\bar{F}_2\, d\bar{G}_1\}$$
$$+ \frac{1}{m}\sum_i \{\int_0^{U_i \wedge M} G_1^{-1}\, \tilde{R}_{2,M}\, d\bar{F}_2 - \int_0^M F_1\, \tilde{R}_{2,M}\, d\bar{F}_2\}$$

$$S_{N3}^{(1)} = \frac{1}{m}\sum_i (\tfrac{1}{2})\{(1-\delta_i)H_1^{-1}(U_i)\int_{U_i}^M (Q_{2M} + F_1\, \tilde{R}_{2M})d\bar{F}_1\, [U_i \leq M]$$

$$- \int_0^{U_i \wedge M} H_1^{-2} F_1 \int_x^M (Q_{2,M} + F_1 \tilde{R}_{2\ M}) d\bar{F}_1 \, d\bar{G}_1 \}$$

$$+ \frac{1}{m} \sum_i \{ \int_0^{U_i \wedge M} g_1^{-1} \tilde{R}_{2n} d\bar{F}_1 - \int_0^M F_1 \tilde{R}_{2M} \, d\bar{F}_1 \}$$

$$+ \frac{1}{m} \sum_i \{ \int_0^{U_i \wedge M} G_1^{-1} R_{2M} d\bar{F}_2 - \int_0^M F_1 R_{2M} d\bar{F}_2 \}$$

$$+ \frac{1}{m} \sum_i (\tfrac{1}{2}) \{ (1-\delta_i) H_1^{-1}(U_i) \int_{U_i}^M F_1 R_{2M} d\bar{F}_2 \, [U_i \leq M]$$

$$- \int_0^{U_i \wedge M} H_1^{-2} F_1 \int_x^M F_1 R_{2M} d\bar{F}_2 \, d\bar{G}_1 \}$$

$$+ \frac{1}{m} \sum_i \{ \delta_i G_1^{-1}(U_i) Q_{2,M}(U_i) [U_i \leq M] - 2 \int_0^M \int_0^x D(y,x) d\bar{F}_2 \, d\bar{F}_1 \},$$

$$S_{N1}^{(2)} = \frac{1}{n} \sum_j (\tfrac{1}{2}) \{ (1-\gamma_j) H_2^{-1}(V_j) \int_{V_j}^M F_2 R_{1M} \, d\bar{F}_1 \, [V_j \leq M]$$

$$- \int_0^{V_j \wedge M} H_2^{-2} F_2 \int_x^M F_2 R_{1M} \, d\bar{F}_1 \, d\bar{G}_2 \}$$

$$+ \frac{1}{n} \sum_j \int_0^{V_j \wedge M} G_2^{-1} R_{1M} \, d\bar{F}_1 - \int_0^M F_2 R_{1M} \, d\bar{F}_1 \}$$

$$S_{N2}^{(2)} = \frac{1}{n} \sum_j (\tfrac{1}{2}) \{ (1-\gamma_j) H_2^{-1}(V_j) \int_{V_j}^M (\tilde{Q}_{2M} + F_2 \tilde{R}_{1M}) d\bar{F}_2 \, [V_j \leq M]$$

$$- \int_0^{V_j \wedge M} H_2^{-2} F_2 \int_x^M (\tilde{Q}_{2M} + F_2 \tilde{R}_{1M}) d\bar{F}_2 \, d\bar{G}_2 \}$$

$$+ \frac{1}{n} \sum_j \{ \int_0^{V_j \wedge M} G_2^{-1} \tilde{R}_{1M} \, d\bar{F}_2 - \int_0^M F_2 \tilde{R}_{1M} \, d\bar{F}_2 \}$$

$$+ \frac{1}{n} \sum_j (\tfrac{1}{2}) \{ \gamma_j \, G_2^{-1}(V_j) \tilde{Q}_{2M}(V_j) [V_j \leq M]$$

$$- 2 \int_0^M \int_0^x D(x,y) d\bar{F}_2 \, d\bar{F}_2 \},$$

and

$$S_{N3}^{(2)} = \frac{1}{n}\sum_j (\tfrac{1}{2})\{(1-\gamma_j)H_2^{-1}(V_j)\int_{V_j}^{M} (\tilde{Q}_{1M} + F_2\, R_{1n})d\overline{F}_2\, [V_j \leq M]$$

$$- \int_0^{V_j \wedge M} H_2^{-2}\, F_2 \int_x^M (\tilde{Q}_{1M} + F_2\, R_{1M})d\overline{F}_2\, d\overline{G}_2\}$$

$$+ \frac{1}{n}\sum_j \{\int_0^{V_j \wedge M} G_2^{-1}\, \tilde{R}_{1M}\, d\overline{F}_1 - \int_0^M F_2\, \tilde{R}_{1M}\, d\overline{F}_1\}$$

$$+ \frac{1}{n}\sum_j \{\int_0^{V_j \wedge M} G_2^{-1}\, R_{1M}\, d\overline{F}_2 - \int_0^M F_2\, R_{1M}\, d\overline{F}_2\}$$

$$+ \frac{1}{n}\sum_j (\tfrac{1}{2})\{(1-\gamma_j)H_2^{-1}(V_j)\int_{V_j}^{M} F_2\, \tilde{R}_{1,M} d\overline{F}_1$$

$$- \int_0^{V_j \wedge M} H_2^{-2}\, F_2 \int_x^M F_2\, \tilde{R}_{1M}\, d\overline{F}_1\, d\overline{G}_2\}$$

$$+ \frac{1}{n}\sum_j \{\gamma_j\, G_2^{-1}(V_j)\tilde{Q}_{1M}(V_j)[V_j \leq M] - 2\int_0^M \int_0^x D(x,y)d\overline{F}_1\, d\overline{F}_2\}.$$

Now let $\Sigma_N^{(1)} = (\sigma_{N,i,j}^{(1)})$ $\begin{matrix} i = 1,2,3 \\ j = 1,2,3 \end{matrix}$ where

$$\sigma_{N,1,1}^{(1)} = \frac{1}{4}\{\int_0^M G_1^{-1}\, Q_{1,M}^2\, d\overline{F}_1 - 4\, (\int_0^M \int_0^x D(y,x)d\overline{F}_1\, d\overline{F}_1)^2\}$$

$$+ 2\, \{\int_0^M F_1\, R_{2,M} \int_0^x G_1^{-1}\, R_{2,M}\, d\overline{F}_1\, d\overline{F}_1 - \int_0^M F_1\, R_{2,M} \int_0^x F_1\, R_{2,M}\, d\overline{F}_1\, d\overline{F}_1\}$$

$$+ \{\int_0^M Q_{1,M} \int_0^x G_1^{-1}\, R_{2,M}\, d\overline{F}_1\, d\overline{F}_1$$

$$- 2\int_0^M \int_0^x D(y,x)d\overline{F}_1\, d\overline{F}_1 \int_0^M F_1\, R_{2,M}\, d\overline{F}_1\}$$

$$- \frac{1}{4}\int_0^M H_1^{-2}\, F_1 (\int_x^M Q_{1,M}\, d\overline{F}_1)^2\, d\overline{G}_1$$

$$- 3/4 \int_0^M H_1^{-2}\, F_1\, (\int_x^M F_1\, R_{2,M}\, d\overline{F}_1)^2\, d\overline{G}_1$$

$$- \int_0^M H_1^{-2}\, F_1\, (\int_x^M F_1\, R_{2,M}\, d\overline{F}_1)(\int_x^M Q_{1,M}\, d\overline{F}_1)\, d\overline{G}_1,$$

$$\sigma^{(1)}_{N,2,2} = 2\ \{\int_0^M F_1\ \tilde{R}_{2,M} \int_0^x G_1^{-1}\ \tilde{R}_{2,M}\ d\bar{F}_2\ d\bar{F}_2 - \int_0^M F_1\ \tilde{R}_{2,M} \int_0^x F_1\ \tilde{R}_{2,M}\ d\bar{F}_2\ d\bar{F}_2\}$$

$$- 3/4 \int_0^M H_1^{-2}\ F_1\ (\int_x^M F_1\ \tilde{R}_{2,M}\ d\bar{F}_2)^2\ d\bar{G}_1,$$

$$\sigma^{(1)}_{N,3,3} = 3/4 \int_0^M H_1^{-2}\ F_1\ (\int_x^M (Q_{2,M} + F_1\ \tilde{R}_{2,M})d\bar{F}_1)^2\ d\bar{G}_1$$

$$+ 2\ \{\int_0^M F_1\ \tilde{R}_{2,M} \int_0^x G_1^{-1}\ \tilde{R}_{2,M} d\bar{F}_1\ d\bar{F}_1 - \int_0^M F_1\ \tilde{R}_{2,M} \int_0^x F_1\ \tilde{R}_{2,M}\ d\bar{F}_1\ d\bar{F}_1\}$$

$$+ 2\ \{\int_0^M F_1\ R_{1,M} \int_0^x G_1^{-1}\ R_{1,M} d\bar{F}_2\ d\bar{F}_2 - \int_0^M F_1\ R_{1,M} \int_0^x F_1\ R_{1,M} d\bar{F}_2\ d\bar{F}_2\}$$

$$+ \ \{\int_0^M G_1^{-1}\ Q^2_{2,M} d\bar{F}_1 - 4\ (\int_0^M \int_0^x D(y,x) d\bar{F}_2\ d\bar{F}_1)^2\}$$

$$- 3/4 \int_0^M H_1^{-2}\ F_1\ (\int_x^M F_1\ R_{2,M} d\bar{F}_2)^2\ d\bar{G}_1$$

$$- 3/2 \int_0^M H_1^{-2}\ F_1\ (\int_x^M F_1\ R_{2,M} d\bar{F}_2)(\int_x^M Q_{2,M} d\bar{F}_1) d\bar{G}_1$$

$$- 3/2 \int_0^M H_1^{-2}\ F_1\ (\int_x^M F_1\ R_{2,M} d\bar{F}_2)(\int_x^M F_1\ \tilde{R}_{2,M} d\bar{F}_1) d\bar{G}_1$$

$$+ 2\ \{\int_0^M F_1\ \tilde{R}_{2,M} \int_0^x G_1^{-1}\ R_{2,M} d\bar{F}_2\ d\bar{F}_1 + \int_0^M F_1\ R_{2,M} \int_0^x G_1^{-1}\ \tilde{R}_{2,M} d\bar{F}_1\ d\bar{F}_2$$

$$- \int_0^M F_1\ \tilde{R}_{2,M} d\bar{F}_1 \int_0^M F_1\ R_{2,M} d\bar{F}_2\}$$

$$+ 2\ \{\int_0^M Q_{2,M} \int_0^x G_1^{-1}\ \tilde{R}_{2,M} d\bar{F}_1\ d\bar{F}_1 - 2 \int_0^M F_1\ \tilde{R}_{2,M} d\bar{F}_1 \int_0^M \int_0^x D(y,x) d\bar{F}_2\ d\bar{F}_1\}$$

$$+ 2\ \{\int_0^M Q_{2,M} \int_0^x G_1^{-1}\ R_{2,M} d\bar{F}_2\ d\bar{F}_1 - 2 \int_0^M F_1\ R_{2,M} d\bar{F}_2 \int_0^M \int_0^x D(y,x) d\bar{F}_2\ d\bar{F}_1\},$$

$$\sigma^{(1)}_{N,1,2} = -\ 3/4 \int_0^M H_1^{-2}\ F_1\ (\int_x^M F_1\ R_{2,M} d\bar{F}_1)(\int_x^M F_1\ \tilde{R}_{2,M} d\bar{F}_2) d\bar{G}_1$$

$$- \frac{1}{2} \int_0^M H_1^{-2}\ F_1\ (\int_x^M F_1\ \tilde{R}_{2,M} d\bar{F}_2)(\int_x^M Q_{1,M} d\bar{F}_1) d\bar{G}_1$$

$$+ \{\int_0^M F_1 \tilde{R}_{2,M} \int_0^x G_1^{-1} R_{2,M} d\bar{F}_2\, d\bar{F}_1 + \int_0^M F_1 R_{2,M} \int_0^x G_1^{-1} \tilde{R}_{2,M} d\bar{F}_1\, d\bar{F}_2$$

$$- \int_0^M F_1 \tilde{R}_{2,M} d\bar{F}_1 \int_0^M F_1 R_{2,M} d\bar{F}_2\}$$

$$+ \frac{1}{2} \{\int_0^M Q_{1,M} \int_0^x G_1^{-1} R_{2,M} d\bar{F}_2\, d\bar{F}_1 - 2 \int_0^M F_1 R_{2,M} d\bar{F}_2 \int_0^M \int_0^x D(y,x) d\bar{F}_1\, d\bar{F}_1\},$$

$$\sigma_{N,1,3}^{(1)} = - \frac{1}{2} \int_0^M H_1^{-2} F_1 \left(\int_x^M Q_{1,M} d\bar{F}_1\right)\left(\int_x^M Q_{2,M} d\bar{F}_1\right) d\bar{G}_1$$

$$- 3/4 \int_0^M H_1^{-2} F_1 \left(\int_x^M F_1 \tilde{R}_{2,M} d\bar{F}_1\right)\left(\int_x^M F_1 R_{2,M} d\bar{F}_1\right) d\bar{G}_1$$

$$- 3/4 \int_0^M H_1^{-2} F_1 \left(\int_x^M F_1 R_{2,M} d\bar{F}_1\right)\left(\int_x^M Q_{2,M} d\bar{F}_1\right) d\bar{G}_1$$

$$- \frac{1}{2} \int_0^M H_1^{-2} F_1 \left(\int_x^M Q_{1,M} d\bar{F}_1\right)\left(\int_x^M F_1 \tilde{R}_{2,M} d\bar{F}_1\right) d\bar{G}_1$$

$$- \frac{1}{2} \int_0^M H_1^{-2} F_1 \left(\int_x^M Q_{1,M} d\bar{F}_1\right)\left(\int_x^M F_1 R_{2,M} d\bar{F}_2\right) d\bar{G}_1$$

$$- 3/4 \int_0^M H_1^{-2} F_1 \left(\int_x^M F_1 R_{2,M} d\bar{F}_2\right)\left(\int_x^M F_1 R_{2,M} d\bar{F}_1\right) d\bar{G}_1$$

$$+ \{\int_0^M F_1 R_{2,M} \int_0^x G_1^{-1} \tilde{R}_{2,M} d\bar{F}_1\, d\bar{F}_1 + \int_0^M F_1 \tilde{R}_{2,M} \int_0^x G_1^{-1} R_{2,M} d\bar{F}_1\, d\bar{F}_1$$

$$- \int_0^M F_1 R_{2,M} d\bar{F}_1 \int_0^M F_1 \tilde{R}_{2,M} d\bar{F}_1\}$$

$$+ \{\int_0^M F_1 R_{2,M} \int_0^x G_1^{-1} R_{2,M} d\bar{F}_2\, d\bar{F}_1 + \int_0^M F_1 R_{2,M} \int_0^x G_1^{-1} R_{2,M} d\bar{F}_1\, d\bar{F}_2$$

$$- \int_0^M F_1 R_{2M} d\bar{F}_1 \int_0^M F_1 R_{2,M} d\bar{F}_2\}$$

$$+ \{\int_0^M Q_{2,M} \int_0^x G_1^{-1} R_{2,M} d\bar{F}_1\, d\bar{F}_1 - 2 \int_0^M F_1 R_{2,M} d\bar{F}_1 \int_0^M \int_0^x D(y,x) d\bar{F}_2\, d\bar{F}_1\}$$

$$+ \frac{1}{2} \{\int_0^M Q_{1,M} \int_0^x G_1^{-1} \tilde{R}_{2,M} d\bar{F}_1\, d\bar{F}_1 - 2 \int_0^M F_1 \tilde{R}_{2,M} d\bar{F}_1 \int_0^M \int_0^x D(y,x) d\bar{F}_1\, d\bar{F}_1\}$$

$$+ \frac{1}{4} \{\int_0^M G_2^{-1} \tilde{Q}_{2,M}^2 \, d\bar{F}_2 - 4 (\int_0^M \int_0^x D(x,y) d\bar{F}_2 \, d\bar{F}_1)^2\}$$

$$+ \{\int_0^M \tilde{Q}_{2,M} \int_0^x G_2^{-1} \tilde{R}_{1,M} \, d\bar{F} \, d\bar{F} - 2 \int_0^M F_2 \tilde{R}_{1,M} \, d\bar{F}_2 \int_0^M \int_0^x D(x,y) d\bar{F}_2 \, d\bar{F}_2\}$$

$$- \frac{1}{4} \int_0^M H_2^{-2} F_2 (\int_x^M \tilde{Q}_{2,M} \, d\bar{F}_2)^2 \, d\bar{G}_2$$

$$- 3/4 \int_0^M H_2^{-2} F_2 (\int_x^M F_2 \tilde{R}_{1,M} \, d\bar{F}_2)^2 \, d\bar{G}_2$$

$$- \int_0^M H_2^{-2} F_2 (\int_x^M F_2 \tilde{R}_{1,M} \, d\bar{F}_2)(\int_x^M \tilde{Q}_{2,M} \, d\bar{F}_2) d\bar{G}_2,$$

$$\sigma_{N,3,3}^{(2)} = 2 \{\int_0^M F_2 \tilde{R}_{1,M} \int_0^x G_2^{-1} \tilde{R}_{1,M} \, d\bar{F}_1 \, d\bar{F}_1 - \int_0^M F_2 \tilde{R}_{1,M} \int_0^x F_2 \tilde{R}_{1,M} \, d\bar{F} \, d\bar{F}_1\}$$

$$+ 2 \{\int_0^M F_2 R_{1,M} \int_0^x G_2^{-1} R_{1,M} \, d\bar{F}_2 \, d\bar{F}_2 - \int_0^M F_2 R_{1,M} \int_0^x F_2 R_{1,M} \, d\bar{F}_2 \, d\bar{F}_2\}$$

$$+ \{\int_0^M G_2^{-1} G_{1,M}^2 \, d\bar{F}_2 - 4 (\int_0^M \int_0^x D(x,y) d\bar{F}_1 \, d\bar{F}_2)^2\}$$

$$+ 2 \{\int_0^M F_2 \tilde{R}_{1,M} \int_0^x G_2^{-1} R_{1,M} \, d\bar{F}_2 \, d\bar{F}_1 + \int_0^M F_2 R_{1,M} \int_0^x G_2^{-1} \tilde{R}_{1,M} \, d\bar{F}_1 \, d\bar{F}_2$$

$$- \int_0^M F_2 \tilde{R}_{1,M} \, d\bar{F}_1 \int_0^M F_2 R_{1,M} \, d\bar{F}_2\}$$

$$+ 2 \{\int_0^M \tilde{Q}_{1,M} \int_0^x G_2^{-1} \tilde{R}_{1,M} \, d\bar{F} \, d\bar{F}_2 - 2 \int_0^M F_2 \tilde{R}_{1,M} \, d\bar{F}_1 \int_0^M \int_0^x D(x,y) d\bar{F}_1 \, d\bar{F}_2\}$$

$$+ 2 \{\int_0^M \tilde{Q}_{1,M} \int_0^x G_2^{-1} R_{1,M} \, d\bar{F}_2 d\bar{F}_2 - 2 \int_0^M F_2 R_{1,M} \, d\bar{F}_2 \int_0^M \int_0^x D(x,y) d\bar{F}_1 \, d\bar{F}_2\}$$

$$- 3/4 \int_0^M H_2^{-2} F_2 (\int_x^M \tilde{Q}_{1,M} \, d\bar{F}_2)^2 \, d\bar{G}_2$$

$$- 3/4 \int_0^M H_2^{-2} F_2 (\int_x^M F_2 R_{1,M} \, d\bar{F}_2)^2 \, d\bar{G}_2$$

$$- 3/2 \int_0^M H_2^{-2} F_2 (\int_x^M F_2 R_{1,M} \, d\bar{F}_2)(\int_x^M \tilde{Q}_{1,M} \, d\bar{F}_2) d\bar{G}_2$$

$$- \int_0^M F_2 R_{1,M} d\bar{F}_1 \int_0^M F_2 R_{1,M} d\bar{F}_2\}$$

$$+ \{\int_0^M \tilde{Q}_{1,M} \int_0^x G_2^{-1} R_{1,M} d\bar{F}_1 d\bar{F}_2$$

$$- 2 \int_0^M F_2 R_{1,M} d\bar{F}_2 \int_0^M \int_0^x D(x,y) d\bar{F}_1 d\bar{F}_2\},$$

$$\sigma^{(2)}_{N,2,3} = \frac{1}{2} \{\int_0^M \tilde{Q}_{2,M} \int_0^x G_2^{-1} \tilde{R}_{1,M} d\bar{F}_1 d\bar{F}_2 - 2 \int_0^M \int_0^x D(x,y) d\bar{F}_2 d\bar{F}_2 (\int_0^M F_2 R_{1,M} d\bar{F}_1)\}$$

$$+ \frac{1}{2} \{\int_0^M \tilde{Q}_{2,M} \int_0^x G_2^{-1} R_{1,M} d\bar{F}_2 d\bar{F}_2 - 2 \int_0^M \int_0^x D(x,y) d\bar{F}_2 d\bar{F}_2 (\int_0^M F_2 R_{1,M} d\bar{F}_2)\}$$

$$+ \frac{1}{2} \{\int_0^M G_2^{-1} \tilde{Q}_{2,M} \tilde{Q}_{1,M} d\bar{F}_2 - 4 (\int_0^M \int_0^x D(x,y) d\bar{F}_1 d\bar{F}_2)(\int_0^M \int_0^x D(x,y) d\bar{F}_2 d\bar{F}_2)\}$$

$$+ \{\int_0^M F_2 \tilde{R}_{1,M} \int_0^x G_2^{-1} \tilde{R}_{1,M} d\bar{F}_1 d\bar{F}_2 + \int_0^M F_2 \tilde{R}_{1,M} \int_0^x G_2^{-1} \tilde{R}_{1,M} d\bar{F}_2 d\bar{F}_1$$

$$- (\int_0^M F_2 \tilde{R}_{1,M} d\bar{F}_2)(\int_0^M F_2 \tilde{R}_{1,M} d\bar{F}_1)\}$$

$$+ \{\int_0^M F_2 \tilde{R}_{1,M} \int_0^x G_2^{-1} R_{1,M} d\bar{F}_2 d\bar{F}_2 + \int_0^M F_2 R_{1,M} \int_0^x G_2^{-1} \tilde{R}_{1,M} d\bar{F}_2 d\bar{F}_2$$

$$- (\int_0^M F_2 \tilde{R}_{1,M} d\bar{F}_2) \int_0^M F_2 R_{1,M} d\bar{F}_2\}$$

$$+ \{\int_0^M \tilde{Q}_{1,M} (\int_0^x G_2^{-1} \tilde{R}_{1,M} d\bar{F}_2) d\bar{F}_2 - 2 \int_0^M \int_0^x D(x,y) d\bar{F}_1 d\bar{F}_2 (\int_0^M F_2 \tilde{R}_{1,M} d\bar{F}_2)\}$$

$$- \frac{1}{2} \int_0^M H_2^{-2} F_2 (\int_x^M \tilde{R}_{1,M} d\bar{F}_2)(\int_x^M \tilde{Q}_{1,M} d\bar{F}_2) d\bar{G}_2$$

$$- \frac{1}{2} \int_0^M H_2^{-2} F_2 (\int_x^M F_2 R_{1,M} d\bar{F}_2)(\int_x^M \tilde{Q}_{2,M} d\bar{F}_2) d\bar{G}_2$$

$$- \frac{3}{4} \int_0^M H_2^{-2} F_2 (\int_x^M F_2 \tilde{R}_{1,M} d\bar{F}_2)(\int_x^M \tilde{Q}_{1,M} d\bar{F}_2) d\bar{G}_2$$

$$- \frac{3}{4} \int_0^M H_2^{-2} F_2 \left(\int_x^M F_2 R_{1,M}\, d\bar{F}_2\right)\left(\int_x^M F_2 \tilde{R}_{1,M}\, d\bar{F}_2\right) d\bar{G}_2$$

$$- \frac{1}{2} \int_0^M H_2^{-2} F_2 \left(\int_x^M F_2 \tilde{R}_{1,M}\, d\bar{F}_1\right)\left(\int_x^M \tilde{Q}_{2,M}\, d\bar{F}_2\right) d\bar{G}_2$$

$$- 3/4 \int_0^M H_2^{-2} F_2 \left(\int_x^M F_2 \tilde{R}_{1,M}\, d\bar{F}_2\right)\left(\int_x^M F_2 \tilde{R}_{1,M}\, d\bar{F}_1\right) d\bar{G}_2,$$

$\sigma^{(2)}_{N,2,1} = \sigma^{(2)}_{N,1,2}$, $\sigma^{(2)}_{N,3,1} = \sigma^{(2)}_{N,1,3}$, and $\sigma^{(2)}_{N,3,2} = \sigma^{(2)}_{N,2,3}$.

Define $\Sigma^{(1)}$ and $\Sigma^{(2)}$ similarly except replacing $M = M_N$ with $+\infty$.

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