SIMILARITY RULES IN MAGNETOHYDRODYNAMICS BASED ON MULTI-FLUID THEORY

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ABSTRACT

SIMILARITY RULES IN MAGNETOHYDRODYNAMICS BASED ON MULTI-FLUID THEORY

by Joab Jacob Blech

In the present work similarity rules are derived in magnetohydrodynamics, in the physical space, based on a multi-fluid theory. The basic hypothesis of the multi-fluid theory is that the fluid consists of a number of fluid components, each with its own intrinsic properties (such as molecular mass, charge, etc.). Each of the species is assumed to be inviscid and non-heat conducting. The postulated fundamental equations for each fluid component are: equation of state, first law of thermodynamics, conservation of mass and conservation of momentum. In addition two Maxwell's vector equations, describing the electromagnetic field, are inserted into the system of equations. The flow is assumed to be multi-diabatic, i.e., there is injection of momentum as well as energy (heat) by means of sources from outside into the various fluid components.

We assume a steady flow which depends only on two spatial coordinates; but in the present multi-fluid theory such a flow is not a two-dimensional flow in the usual sense. The velocity, the electric and the magnetic fields have three components. After a reduction of the number of fundamental equations is made, the system of

equations is linearized, i.e., it is assumed that a first order small perturbation theory describes adequately the flow field.

In the present case of MHD a mere linearization of the system of equations seems to be insufficient for obtaining similarity rules. An additional procedure, i.e., some sort of a smoothing process, is applied in which a criterion for neglecting very small terms is introduced, thus leading to a simplified system of equations which governs the flow. A correlation between this simplified system of equations of the compressible flow and corresponding systems of equations of the incompressible flow is established for the case of aligned fields, in which the velocity and the magnetic fields in the undisturbed stream are parallel, and for the case of crossed fields, where the velocity and the magnetic fields in the undisturbed stream are perpendicular. Pressure coefficients for the individual species and for the gross fluid are calculated and correlated. As a special numerical case, a fully ionized plasma is considered and the ion pressure coefficient is plotted vs. the ion free stream Mach number for various orientations of the magnetic field.

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BASED ON MULTI-FLUID THEORY

By

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INTRODUCTION

In the past, many formulas for the similarity relations between incompressible and compressible fluids in an isentropic gas flow, were derived. A collection of such rules in the physical space is given in (8). The Kármán-Tsien technique which employs the hodograph method can be found, for example, in (7, pp. 336-340). Similarly, some attempts to derive similarity rules in diabatic flow, i.e., a flow with heat addition by means of sources, were made (4). In recent years there has been a tendency to derive various similarity rules in magnetohydrodynamics.

From various attempts in the past we may quote and discuss briefly the following:

(i) v. Krzywoblocki and Nutant (5) derive a similarity rule for an inviscid, non-heat conducting, diabatic flow which takes place in an electromagnetic field, with excess electric charge equal to zero, following the technique of Kármán-Tsien in the hodograph plane and assuming a simplified pressure-density-entropy relation. Although a similarity rule is derived, the disadvantage of this procedure is that equations of the electromagnetic nature are not transformed into the hodograph plane. The electric and magnetic fields are treated as known functions. Moreover, the correlation of corresponding coefficients in both stream-function equations of the compressible and incompressible flow requires that there be a certain relation between the vorticity distributions of both flows. A special relation is also imposed on the Jacobians of the transformations in the physical spaces.

(ii) McCune and Ressler (6) treat the two-dimensional steady case of a highly electrically conducting, inviscid, non-heat conducting, isentropic flow passing over a thin body. A single partial differential equation for the current field is used to study the flow. This differential equation is derived from the linearized fundamental system of equations of the hydrodynamic and electromagnetic nature. The discussion is separated into three main cases, depending on the orientation of the externally applied magnetic field: The case of aligned fields, where the magnetic field is parallel to the velocity field of the uniform undisturbed flow, the case of the crossed fields where the magnetic field is perpendicular to the free stream velocity, and the case of an arbitrary field angle. In each case, procedures are developed for the solutions of the magnetoaerodynamic problems involved and the compressibility effects can be studied through the solutions.

In the works mentioned previously the ionized gas is treated as a single fluid. It was pointed out in (2) that if one seeks a single fluid magnetohydrodynamic formulation in which the current density is considered as an unknown, a counting of variables and equations shows the necessity of an additional vector equation which is usually taken as the generalized Ohm's law. The influence of gasdynamic effects on the electric current density has been completely neglected. One way to improve the description of the mechanism which governs the electric current density is to use multi-fluid theory. In this formulation the fluid is assumed to consist of several fluid components, each with its own intrinsic properties (such as molecular mass, charge, etc.) and with its own thermodynamic state variables. Conservation equations from the macroscopic point of view are then postulated for each fluid component. There is no necessity for Ohm's law since the electric current density is defined by means of the velocities and charge densities of the various species, and the former are governed by the conservation equations of the individual fluid components. Thus the effect of the various forces on the electric current density through the velocity vectors and momentum equations can now be treated exactly from the macroscopic point of view.

The main purpose of the present work is to derive similarity rules, avoiding the pitfalls appearing in the works of previous authors. A multi-fluid theory was employed. Each fluid component was assumed to be inviscid and non-heat conducting. In Chapter I, section 1, the dependent variables and a list of the fundamental equations are introduced. In section 2, equations of state, first law of thermodynamics, conservation of mass and of momentum for each fluid component were postulated and the passage from the species equations to a gross fluid formulation is discussed. In order to obtain correct momentum and energy equations for the gross fluid, it was assumed that the fluid component flows may be multi-diabatic, i.e., there may be injection of momentum, as well as energy (heat) by means of sources from outside into the various species. Equations of state and for the internal energy of each component are allowed to deviate from perfect gas equations, so that it is possible to derive perfect gas equations of state and for the internal energy of the gross fluid. Various results pertaining to the charge and current equations and to Ohm's law, which were obtained in (1, 2) are summarized in section 3. The fundamental electromagnetic equations, i.e., Maxwell's equations, are introduced in section 4, and following (9) two possible formulations of the final system of equations are given in section 5.

In Chapter II, section 1, a steady three-dimensional flow which depends only on two spatial coordinates is assumed. In the multi-fluid theory, one cannot assume that such a flow is two-dimensional in the usual sense. The velocity and the magnetic field have three components which have to be calculated from the fundamental equations. The electric field component in the third direction is shown, from the fundamental equations, to be a constant. The electromagnetic system of equations is further reduced into two differential equations for properly chosen functions, the spatial derivatives of which give us the remaining two components of the electric and magnetic fields. In section 2, the single quasi-stream function* and potential equations for each species are derived. Those functions are associated only with two velocity components. The third velocity component is governed by the momentum equation in the corresponding direction. Generalized Crocco and Bernoulli equations are derived in section 3. In section 4, a summary of the governing system of equations is given. The quasi-stream function and potential equations for the incompressible flow are given in section 5, and the final system for the incompressible flow is discussed. Section 6 merely introduces a non-dimensionalization of the various quantities and equations of the flow, after which a linearization procedure is carried out in section 7, i.e., it is assumed that a first order small perturbation theory describes the flow field.

In Chapter III an analogy between the compressible and incompressible flow is obtained. For this analogy it is necessary to simplify further the linearized system of governing equations. Thus in section 1, following a criterion similar to the one given in (11), very small terms are neglected in the linearized system of equations. Certain pairs of coefficients in the linearized quasi-stream function

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* A three-dimensional steady flow has actually two stream functions (3).

equations are approximated by their weighted mean. In section 2, a correlation between the simplified linear system of equations of the compressible flow and corresponding systems of equations of the incompressible flow is established by means of linear transformations of the coordinates and of properly chosen relations between corresponding quantities of both flows. It is necessary here to distinguish between the two separate cases of aligned fields and of crossed fields. Finally, in section 3, pressure coefficients for the individual species and for the gross fluid are calculated in both flows and the relation between them is given. A special case is chosen in section 4, and the compressible pressure coefficient is plotted vs. the free stream Mach number with the externally applied magnetic field components as parameters.

In ordinary isentropic irrotational flow it is sufficient to linearize the equations in the physical space in order to obtain similarity rules. In a rotational flow of such a character, some additional assumptions must be made to take care of the vorticity effects. It seems that in the present case of MHD the procedure known from the classical gas dynamics is absolutely insufficient for obtaining reasonable similarity rules. The additional procedure which is applied in the present work is some sort of a smoothing process after linearization which is actually equivalent to neglecting some terms of smaller order, and taking mean values. It seems as if this procedure can be considered as a first approximation in a chain of a successive approximation process as applied to the similarity rules. Higher approximations are then obtained from a more accurate system of equations of the flow and the results of the first approximation. In the present work only the first approximation is considered.

CHAPTER I

FUNDAMENTAL EQUATIONS AND CLASSIFICATION OF FLOWS

I.1. Fundamental Concepts

We consider the plasma, on the base of a continuous medium, as a mixture of n fluid components, each with its own intrinsic properties (such as molecular mass, charge, etc.) and each with its own thermodynamic state variables. From a macroscopic point of view, the following quantities are sought:

T_s = temperature of the s-th species;
p_s = pressure of the s-th species;
ρ_s = density of the s-th species;
u_s^i = i-th component of the velocity field of the s-th species;
T = temperature of the gross fluid;
p = pressure of the gross fluid;
ρ = density of the gross fluid;
u^i = i-th component of the velocity field of the gross fluid;
E^i = i-th component of the electric field;
H^i = i-th component of the magnetic field;

where s=1,2,...,n and i=1,2, or 3. Counting the unknowns we get 6n+6 quantities to be determined*.

The following gas-dynamics equations are postulated for each fluid component: Equation of state, first law of thermodynamics, equation of continuity and equation of momentum.

In addition, two Maxwell's vector equations, describing the electromagnetic field, are inserted into the system of equations.

Counting the equations we get 6n+6 relations.

I.2. Fundamental equations of the Gasdynamic Subsystem

The following relations are postulated for each of the fluid components, assuming that each component behaves like an inviscid, non-heat conducting fluid:

Equation of state:

$$p_s = R_s \rho_s T_s \ , \qquad \text{(I.2.1)}$$

where R_s is the gas coefficient of the s-th component. The density, ρ_s , is assumed to be given by:

$$\rho_s = m_s \nu_s \ , \qquad \text{(I.2.2)}$$

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* The quantities T, p, ρ, u^i depend on T_s , p_s , ρ_s , u^i_s and, therefore, will not be counted.

where m_s is the molecular mass of the s-th component and ν_s is its number density.

First law of thermodynamics in a form:

$$dQ_s = dU_s + p_s d(\rho_s^{-1}), \tag{I.2.3}$$

where dQ_s is the energy addition per unit mass into the s-th component and U_s is its internal energy per unit mass. With the aid of Eq. (I.2.1), Eq. (I.2.3) can also be written in the well-known form:

$$dQ_s = dI_s - \rho_s^{-1} dp_s, \tag{I.2.4}$$

where I_s is given by:

$$I_s = U_s + R_s T_s. \tag{I.2.5}$$

Equation of continuity*:

$$\frac{\partial \rho_s}{\partial t} + \frac{\partial}{\partial x^j}(\rho_s u_s^j) = \sigma_s, \tag{I.2.6}$$

where σ_s is the mass source, per unit volume, of the s-th component. From the conservation of mass it follows that:

$$\sum_{s=1}^{n} \sigma_s = 0, \tag{I.2.7}$$

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* Summation convention is used for repeated tensorial indices but not for subscripts distinguishing between the fluid components.

which implies that there is no addition of mass from outside.

Equation of momentum is postulated (1, p. 8, Eq. (1)):

$$\frac{\partial}{\partial t}\left(\rho_s u_s^i\right)+\frac{\partial}{\partial x^j}\left(\rho_s u_s^i u_s^j\right)=-\frac{\partial p_s}{\partial x^i}+F_s^i+\sigma_s Z_s^i-\frac{\partial X_s^{ij}}{\partial x^j}, \qquad (I.2.8)$$

where i=1,2, or 3 and F_s^i is the i-th component of the body force acting on the s-th species. This force may be written as:

$$F_s^i=F_{gs}^i+F_{es}^i+F_{os}^i, \qquad (I.2.9)$$

where F_{gs}^i is the i-th component of the non-electromagnetic body force such as gravitation force, etc., F_{es}^i is the i-th component of the electromagnetic force and F_{os}^i is the i-th component of the interaction force, i.e., the force on the s-th component due to all the other kinds of species in the fluid. By Newton's third law of motion, we have:

$$\sum_{s=1}^{n} F_{os}^i=0; \quad (i=1,2,3). \qquad (I.2.10)$$

$\sigma_s Z_s^i$ is a momentum source associated with the mass source σ_s. Following (1, p. 9) we require that:

$$\sum_{s=1}^{n} \sigma_s Z_s^i=0; \quad (i=1,2,3). \qquad (I.2.11)$$

X_s^{ij} is a momentum transfer tensor, associated with the s-th fluid component, the significance of which will be explained later.

Next, we derive an energy equation analogous to the one given in (1, p. 8, Eq. (1)). Assuming that Q_s is a differentiable function, then from Eq. (I.2.4):

$$\frac{\partial Q_s}{\partial x^j} = \frac{\partial I_s}{\partial x^j} - \rho_s^{-1} \frac{\partial p_s}{\partial x^j} \; ; \qquad (j=1,2,3), \qquad \text{(I.2.12)}$$

$$\frac{\partial Q_s}{\partial t} = \frac{\partial I_s}{\partial t} - \rho_s^{-1} \frac{\partial p_s}{\partial t} \; . \qquad \text{(I.2.13)}$$

Multiplying Eq. (I.2.8) by u_s^j and using Eqs. (I.2.1), (I.2.5), (I.2.6), (I.2.12), (I.2.13) we get after a few manipulations:

$$\frac{\partial}{\partial t}\left(\bar{e}_s - \rho_s Q_s\right) + \frac{\partial}{\partial x^j}\left(\bar{e}_s u_s^j - \rho_s Q_s u_s^j + p_s u_s^j\right) = u_s^i F_s^i - u_s^i \frac{\partial X^{ij}}{\partial x^j} +$$

$$+ \sigma_s\left(I_s - Q_s - \frac{1}{2} u_s^i u_s^i + u_s^i Z_s^i\right), \qquad \text{(I.2.14)}$$

where $\bar{e}_s$ is given by:

$$\bar{e}_s = \frac{1}{2} \rho_s u_s^i u_s^i + \rho_s U_s \, . \qquad \text{(I.2.15)}$$

Equation (I.2.14) is the energy equation, analogous to the one obtained in (1).

Define the following gross fluid variables:

$$\rho = \sum_{s=1}^{n} \rho_s \quad , \qquad \text{(I.2.16)}$$

$$\rho u^i = \sum_{s=1}^{n} \rho_s u_s^i \;, \tag{I.2.17}$$

$$F^i = \sum_{s=1}^{n} F_s^i \;, \tag{I.2.18}$$

$$X^{ij} = -\rho u^i u^j - p\delta^{ij} + \sum_{s=1}^{n} (X_s^{ij} + \rho_s u_s^i u_s^j + p_s \delta^{ij}), \tag{I.2.19}$$

$$\bar{e} = \tfrac{1}{2}\rho u^i u^i + \rho U = \sum_{s=1}^{n} \left(\tfrac{1}{2}\rho_s u_s^i u_s^i + \rho_s U_s\right) = \sum_{s=1}^{n} \bar{e}_s \,, \tag{I.2.20}$$

$$\frac{\partial}{\partial t}(\rho Q) + \frac{\partial}{\partial x^j}(\rho Q u^j) = -u^j F^j + \frac{\partial}{\partial x^j}(\bar{e} u^j + p u^j) + u^i \frac{\partial X^{ij}}{\partial x^j} +$$
$$+ \sum_{s=1}^{n} \Big[u_s^i F_s^i - \frac{\partial}{\partial x^j}(\bar{e}_s u_s^j + p_s u_s^j) + \frac{\partial}{\partial t}(\rho_s Q_s) + \frac{\partial}{\partial x^j}(\rho_s Q_s u_s^j) -$$
$$- u_s^i \frac{\partial X_s^{ij}}{\partial x^j} + \sigma_s\left(I_s - Q_s - \tfrac{1}{2} u_s^i u_s^i + u_s^i Z_s^i\right)\Big] \,, \tag{I.2.21}$$

where δ^{ij} is the Kronecker delta. The diffusion velocities, v_s^i, are defined by:

$$v_s^i = u_s^i - u^i \;. \tag{I.2.22}$$

It is evident from Eqs. (I.2.16), (I.2.17), (I.2.22) that:

$$\sum_{s=1}^{n} \rho_s v_s^i = 0 \;. \tag{I.2.23}$$

The pressure, p, is defined in (1, p. 9) to be* :

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* The pressure dealt with in this work is the gas-dynamic pressure. The radiation pressure is neglected (see 9, p. 12).

$$p = \sum_{s=1}^{n} p_s + \frac{1}{3} \rho_s v_s^i v_s^i \ , \qquad (I.2.24)$$

Another possible definition is (9, p. 10):

$$p = \sum_{s=1}^{n} p_s \ . \qquad (I.2.25)$$

We will adopt the latter proposition for the pressure. Using Eqs. (I.2.11), (I.2.16), (I.2.17), (I.2.18), (I.2.22), (I.2.23), (I.2.25) in Eqs. (I.2.19) to (I.2.21) it can be shown that:

$$X^{ij} = \sum_{s=1}^{n} \left(X_s^{ij} + \rho_s v_s^i v_s^j \right), \qquad (I.2.26)$$

$$\rho U = \sum_{s=1}^{n} \left(\rho_s U_s + \frac{1}{2} \rho_s v_s^i v_s^i \right), \qquad (I.2.27)$$

$$\frac{\partial}{\partial t}(\rho Q) + \frac{\partial}{\partial x^j}(\rho Q u^j) = \sum_{s=1}^{n} \Big[v_s^i F_s^i - \frac{\partial}{\partial x^j}\left(\bar{e}_s v_s^j + p_s v_s^j \right) +$$

$$+ \frac{\partial}{\partial t}(\rho_s Q_s) + \frac{\partial}{\partial x^j}(\rho_s Q_s u_s^j) - v_s^i \frac{\partial X_s^{ij}}{\partial x^j} + \sigma_s \Big(I_s - Q_s - \frac{1}{2} u_s^i u_s^i +$$

$$+ v_s^i Z_s^i \Big) + u^i \frac{\partial}{\partial x^j}\left(\rho_s v_s^i v_s^j \right) \Big] . \qquad (I.2.28)$$

Adding each of the individual Eqs. (I.2.6), (I.2.8), (I.2.14) over s, and making use of Eqs. (I.2.7), (I.2.11), (I.2.16) to (I.2.21), (I.2.25) we obtain:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^j}(\rho u^j) = 0 \ , \qquad (I.2.29)$$

$$\frac{\partial}{\partial t}\left(\rho u^i\right) + \frac{\partial}{\partial x^j}\left(\rho u^i u^j\right) = -\frac{\partial p}{\partial x^i} + F^i - \frac{\partial X^{ij}}{\partial x^j}\,, \qquad \text{(I.2.30)}$$

$$\frac{\partial}{\partial t}\left(\bar{e} - \rho Q\right) + \frac{\partial}{\partial x^j}\left(\bar{e} u^j - \rho Q u^j + p u^j\right) =$$

$$= u^i F^i - u^i \frac{\partial X^{ij}}{\partial x^j}\,. \qquad \text{(I.2.31)}$$

Equations (I.2.29), (I.2.30), (I.2.31) are the continuity, momentum and energy equations for the gross fluid.

In the equations of the single plasma components we introduced terms representing momentum and energy flux into the s-th component from outside $\left(\frac{\partial X_s^{ij}}{\partial x^j}\,,\ \frac{\partial}{\partial t}\left(\rho_s Q_s\right) + \right.$ $\left. + \frac{\partial}{\partial x^j}\left(\rho_s Q_s u_s^j\right)\right)$. The inclusion of these terms made it possible to obtain correct momentum and energy equations for the gross fluid. A correct continuity equation for the gross fluid was obtained by the requirement that the density of the gross fluid be equal to the sum of the densities of the component fluids, and that the mass flux in the gross fluid be equal to the sum of the mass fluxes of the component fluids. We will define a "perfect gross fluid flow" as one for which Eulerian formulation is obeyed, i.e., as in our case, $X^{ij} \equiv 0$, due to non-existence of a momentum transfer into the gross fluid. A perfect gross fluid flow will be called "adiabatic" or "diabatic", depending on whether Q does or does not vanish.

In a diabatic fluid flow we assumed that the energy is added into the particle with no viscosity and heat conductivity present; similarly, in our case of perfect

gross fluid flow, we observe, by inspection of Eqs. (I.2.26), (I.2.28), that in general there exists an injection or subtraction of momentum and energy into a particle ($X_s^{ij} \neq 0$, $Q_s \neq 0$), hence the species flow is diabatic not only in an energy sense, but also in a momentum sense. We, therefore, propose the name "multi-diabatic flow" for this model which is more general than the diabatic flow.

From Eq. (I.2.26) we get in the case of a perfect gross fluid flow:

$$\sum_{s=1}^{n} \left(X_s^{ij} + \rho_s v_s^i v_s^j \right) = 0 \; . \qquad \text{(I.2.32)}$$

One way to satisfy Eq. (I.2.32) is by choosing X_s^{ij} in the form:

$$X_s^{ij} = -\rho_s v_s^i v_s^j \; . \qquad \text{(I.2.33)}$$

In the case of an adiabatic perfect gross fluid flow the left-hand side of Eq. (I.2.28) vanishes, and therefore:

$$\sum_{s=1}^{n} \Big[v_s^i F_s^i - \frac{\partial}{\partial x^j}\left(\bar{e}_s v_s^j + p_s v_s^j \right) + \frac{\partial}{\partial t}\left(\rho_s Q_s \right) + \frac{\partial}{\partial x^j}\left(\rho_s Q_s u_s^j \right) -$$

$$- v_s^i \frac{\partial X_s^{ij}}{\partial x^j} + \sigma_s \left(I_s - Q_s - \frac{1}{2} u_s^i u_s^i + v_s^i Z_s^i \right) +$$

$$+ u^i \frac{\partial}{\partial x^j}\left(\rho_s v_s^i v_s^j \right) \Big] = 0 \; . \qquad \text{(I.2.34)}$$

One way to satisfy Eq. (I.2.34) is by choosing Q_s such that:

$$\frac{\partial}{\partial t}(\rho_s Q_s) + \frac{\partial}{\partial x^j}(\rho_s Q_s u_s^j) = -v_s^i F_s^i + \frac{\partial}{\partial x^j}(\bar{e}_s v_s^j + p_s v_s^j) + v_s^i \frac{\partial X_s^{ij}}{\partial x^j} -$$

$$- \sigma_s\left(I_s - Q_s - \frac{1}{2} u_s^i u_s^i + v_s^i Z_s^i\right) - u^i \frac{\partial}{\partial x^j}(\rho_s v_s^i v_s^j), \qquad \text{(I.2.35)}$$

and if X_s^{ij} is given by Eq. (I.2.33) we get, with the aid of Eq. (I.2.22):

$$\frac{\partial}{\partial t}(\rho_s Q_s) + \frac{\partial}{\partial x^j}(\rho_s Q_s u_s^j) = -v_s^i F_s^i + \frac{\partial}{\partial x^j}(\bar{e}_s v_s^j + p_s v_s^j) -$$

$$- \sigma_s\left(I_s - Q_s - \frac{1}{2} u_s^i u_s^i + v_s^i Z_s^i\right) - u_s^i \frac{\partial}{\partial x^j}(\rho_s v_s^i v_s^j). \qquad \text{(I.2.36)}$$

Eqs. (I.2.33), (I.2.36), if taken into account, enable one to eliminate X_s^{ij}, Q_s from the momentum and energy equations of each fluid component.

Defining the operator $\frac{d}{dt}$ by:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u^j \frac{\partial}{\partial x^j}, \qquad \text{(I.2.37)}$$

it can be shown that the equation of energy (I.2.31) can be remodelled with use of Eqs. (I.2.29), (I.2.30) into a form which may express a gross fluid first law of thermodynamics:

$$\frac{dQ}{dt} = \frac{dU}{dt} + p\frac{d}{dt}(\rho^{-1}). \qquad \text{(I.2.38)}$$

It is sometimes convenient to consider another form of the momentum equation for each fluid component which is obtained from Eqs. (I.2.6), (I.2.8):

$$\rho_s\left(\frac{\partial u_s^i}{\partial t} + u_s^j \frac{\partial u_s^i}{\partial x^j}\right) = -\frac{\partial p_s}{\partial x^i} + F_s^i + \sigma_s\left(Z_s^i - u_s^i\right) - \frac{\partial X_s^{ij}}{\partial x^j}; \quad \text{(I.2.39)}$$

where i=1,2, or 3, and for the gross fluid, using Eqs. (I.2.29), (I.2.30):

$$\rho\left(\frac{\partial u^i}{\partial t} + u^j \frac{\partial u^i}{\partial x^j}\right) = -\frac{\partial p}{\partial x^i} + F^i - \frac{\partial X^{ij}}{\partial x^j}; \qquad (i=1,2,3). \quad \text{(I.2.40)}$$

Dividing Eq. (I.2.39) by ρ_s, Eq. (I.2.40) by ρ, subtracting the second equation from the first, and using Eq. (I.2.22) leads to the so-called difference momentum equations:

$$\frac{\partial v_s^i}{\partial t} + u_s^j \frac{\partial u_s^i}{\partial x^j} - u^j \frac{\partial u^i}{\partial x^j} = \rho^{-1}\frac{\partial p}{\partial x^i} - \rho_s^{-1}\frac{\partial p_s}{\partial x^i} + \rho_s^{-1} F_s^i - \rho^{-1} F^i -$$
$$-\rho_s^{-1}\frac{\partial X_s^{ij}}{\partial x^j} + \rho^{-1}\frac{\partial X^{ij}}{\partial x^j} + \sigma_s \rho_s^{-1}\left(Z_s^i - u_s^i\right). \quad \text{(I.2.41)}$$

For the derivation of an equation of state for the gross fluid, we define the gross fluid temperature as (9, p. 11):

$$T = \nu^{-1}\sum_{s=1}^{n} \nu_s T_s , \quad \text{(I.2.42)}$$

where ν is given by:

$$\nu = \sum_{s=1}^{n} \nu_s \quad . \tag{I.2.43}$$

The gas coefficient for the s-th fluid component, R_s , is assumed to be given by:

$$R_s = R_A m_s^{-1}(1 + W_s) = R_{ps}(1 + W_s) \quad , \tag{I.2.44}$$

where R_A is a universal constant and W_s is a function representing the deviation of the state of the s-th species from the state of a perfect gas. We observe that if $W_s = 0$ then the s-th component has a perfect gas equation of state since equation (I.2.1) becomes, using Eqs. (I.2.2), (I.2.44):

$$p_s = R_A \nu_s T_s \quad , \tag{I.2.45}$$

which is a perfect gas equation of state. R_{ps} is given by:

$$R_{ps} = R_A m_s^{-1} \quad . \tag{I.2.46}$$

Inserting Eq. (I.2.44) into Eq. (I.2.1) and using Eq. (I.2.2) furnishes:

$$p_s = R_{ps}(1 + W_s)\rho_s T_s = R_A(1 + W_s)\nu_s T_s \; . \tag{I.2.47}$$

Summing each of the individual Eqs. (I.2.47) over s and making use of Eqs. (I.2.25), (I.2.45) we get:

$$p = R_A(1+W)\nu T \,, \tag{I.2.48}$$

where W is defined by:

$$W = (\nu T)^{-1} \sum_{s=1}^{n} W_s \nu_s T_s \,. \tag{I.2.49}$$

Equation (I.2.49) can also be written in the form:

$$p = R \rho T \,, \tag{I.2.50}$$

where R is defined by:

$$R = R_A m^{-1}(1+W) = R_p(1+W) \,; \quad R_p = R_A m^{-1} \,, \tag{I.2.51}$$

$$m = \rho \nu^{-1} = \left(\sum_{s=1}^{n} \rho_s\right)\left(\sum_{s=1}^{n} \nu_s\right)^{-1} = \left(\sum_{s=1}^{n} m_s \nu_s\right)\left(\sum_{s=1}^{n} \nu_s\right)^{-1} . \tag{I.2.52}$$

The symbol m is referred to as the mean mass of a particle in the gross fluid.

In order that Eq. (I.2.50) be a perfect gas equation of state it is necessary that R be a constant. We observe, from Eqs. (I.2.51), (I.2.52) that the condition W=0 is not sufficient to make R a constant since the mean mass of a particle in the gross fluid, m, given by Eq. (I.2.52) is, in general, not a constant. If one assumes however that m = constant (and W=0), then the gross fluid has a perfect gas equation of state.

Next, a specifying equation for each fluid component is derived. Introduce the entropy per unit mass of the s-th species, S_s, by means of the following relation:

$$T_s \, dS_s = dQ_s \, . \tag{I.2.53}$$

It is assumed that the specific internal energy of the s-th component is given by:

$$U_s = c_{vs}(1 + A_s) T_s \, , \tag{I.2.54}$$

where A_s is a function which represents the deviation of the specific internal energy of the s-th component from a perfect gas specific internal energy, i.e., the imperfections of the gas. c_{vs} is the heat capacity of the s-th fluid component at constant volume if it would have been a perfect gas, and will be assumed to be a constant.

Inserting Eqs. (I.2.53), (I.2.54) into Eq. (I.2.3), deviding the resulting equation by T_s and using Eq. (I.2.47) furnishes:

$$dS_s = c_{vs}(1 + A_s) T_s^{-1} dT_s + c_{vs} dA_s - R_{ps}(1 + W_s) \rho_s^{-1} d\rho_s . \tag{I.2.55}$$

Integrating Eq. (I.2.55) from a zero subscript initial state to some end state gives after the elimination of the logarithms by means of exponential functions:

$$T_s T_{s0}^{-1} = G_s \left(\rho_s \rho_{s0}^{-1}\right)^{K_s - 1} \exp\left[(S_s - S_{s0}) c_{vs}^{-1}\right] , \tag{I.2.56}$$

where G_s , K_s are given by:

$$G_s = \exp\left[-(A_s - A_{s0}) - \int_0 A_s T_s^{-1} dT_s + (K_s - 1)\int_0 W_s \rho_s^{-1} d\rho_s\right], \quad (I.2.57)$$

$$K_s = 1 + R_{ps} c_{vs}^{-1} . \quad (I.2.58)$$

Using Eq. (I.2.47) in Eq. (I.2.56) we get:

$$p_s = C_s \rho_s^{K_s} \exp(S_s c_{vs}^{-1}), \quad (I.2.59)$$

where the function C_s is given by:

$$C_s = p_{s0} \rho_{s0}^{-K_s} (1 + W_s)(1 + W_{s0})^{-1} G_s \exp(-S_{s0} c_{vs}^{-1}). \quad (I.2.60)$$

Eq. (I.2.59) is the specifying equation (generalized pressure-density-entropy relation) for the s-th fluid component.

Similarly, we introduce the specific entropy, S , and internal energy, U , for the gross fluid by the relations:

$$T dS = dQ , \quad (I.2.61)$$

$$U = c_v (1 + A) T , \quad (I.2.62)$$

where A, c_v have a similar meaning for the gross fluid as A_s , c_{vs} have for a fluid component.

It can be shown, by similar operations which were performed to obtain Eq. (I.2.59) and with the additional assumption m = constant, that:

$$p = C\rho^{K} \exp(S c_v^{-1}), \tag{I.2.63}$$

where:

$$C = p_0 \rho_0^{-K} (1+W)(1+W_0)^{-1} G \exp(-S_0 c_v^{-1}), \tag{I.2.64}$$

$$G = \exp\left[-(A-A_0) - \int_0 A T^{-1} dT + (K-1)\int_0 W \rho^{-1} d\rho\right], \tag{I.2.65}$$

$$K = 1 + R_p c_v^{-1}. \tag{I.2.66}$$

Equation (I.2.63) is the specifying equation for the gross fluid*.

It may be worthwhile to notice that the n+1 functions $A, A_1, A_2, \ldots, A_n$ are related by Eq. (I.2.27), remodelled with the use of Eqs. (I.2.54), (I.2.62):

$$\rho c_v (1+A) T = \sum_{s=1}^{n} \left[\rho_s c_{vs} (1+A_s) T_s + \frac{1}{2} \rho_s v_s^i v_s^i\right]. \tag{I.2.67}$$

One cannot assume, in general, that:

$$A \equiv A_1 \equiv A_2 \equiv \ldots \equiv A_n \equiv 0, \tag{I.2.68}$$

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* In the case where the mass m is assumed to be a

since we would get one more relation between ρ, T, ρ_s, T_s, v_s^i , thus overspecifying our system of equations. If, however, one assumes that Eq. (I.2.68) is true, then the gross fluid as well as all the component fluids have perfect gas specific internal energies.

Next, we calculate the value of $\left(\frac{\partial p_s}{\partial \rho_s}\right)_{S_s}$. Using Eq. (I.2.47) in differential form and Eq. (I.2.58) in Eq. (I.2.55) we get, after rearranging:

$$p_s c_{vs}^{-1}(1+A_s)dS_s = dp_s - K_s p_s \rho_s^{-1}(1+N_s)d\rho_s - p_s(1+W_s)^{-1}dW_s + p_s(1+A_s)^{-1}dA_s, \quad (I.2.69)$$

where N_s is given by:

$$N_s = (K_s-1)K_s^{-1}(W_s-A_s)(1+A_s)^{-1}. \quad (I.2.70)$$

Assuming ρ_s and S_s independent, also assuming that p_s, A_s, W_s , are, possibly, functions of ρ_s and some other variables independent of ρ_s, we get, equating coefficients of $d\rho_s$ in Eq. (I.2.69) and rearranging:

$$(\alpha_s)^2 = \left(\frac{\partial p_s}{\partial \rho_s}\right)_{S_s} = K_s p_s \rho_s^{-1}(1+N_s) + p_s\left[\frac{\partial}{\partial \rho_s}\log(1+W_s)(1+A_s)^{-1}\right]_{S_s}. \quad (I.2.71)$$

Similarly, assuming m = constant, ρ and S as independent variables and p, A, W as functions of ρ and, possibly, of

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variable, we can decompose it into two parts: m = mean constant + variable perturbation. This will give us in the final Eqs. (I.2.63) to (I.2.65) an additional term which will represent a deviation from their present forms.

some more variables, independent of ρ , we have:

$$(\alpha)^2 = \left(\frac{\partial p}{\partial \rho}\right)_S = \kappa p\rho^{-1}(1+N) + p\left[\frac{\partial}{\partial \rho}\log(1+W)(1+A)^{-1}\right]_S , \qquad \text{(I.2.72)}$$

where N is given by:

$$N = (\kappa - 1)\kappa^{-1}(W-A)(1+A). \qquad \text{(I.2.73)}$$

I.3. Charge and Current Equations, Ohm's Law

Following (1, pp. 11, 12) we derive a charge and a current equation.

Let ρ_{es} be the charge density of the s-th fluid component, given by:

$$\rho_{es} = e_s \nu_s , \qquad \text{(I.3.1)}$$

where e_s is the charge of a particle of the s-th species. From Eqs. (I.2.2), (I.3.1) we have:

$$\rho_{es} = \gamma_s \rho_s \quad ; \quad \gamma_s = e_s m_s^{-1} \qquad \text{(I.3.2)}$$

Assuming γ_s= consonant, we have, multiplying Eq. (I.2.6) by γ_s and making use of Eq. (I.3.2):

$$\frac{\partial \rho_{es}}{\partial t} + \frac{\partial}{\partial x^i}\left(\rho_{es} u_s^i\right) = \sigma_s \gamma_s . \qquad \text{(I.3.3)}$$

Equation (I.3.3) is the equation of conservation of electrical charge for the s-th fluid component.

Requiring that there be no input of electric charge sources from outside into the fluid, we have:

$$\sum_{s=1}^{n} \sigma_s \gamma_s = 0 . \qquad (I.3.4)$$

The gross fluid excess charge density, ρ_e, is given by:

$$\rho_e = \sum_{s=1}^{n} \rho_{es} , \qquad (I.3.5)$$

and the electrical current density, J^i, is given by:

$$J^i = \sum_{s=1}^{n} \rho_{es} u_s^i = J_u^i + \rho_e u^i , \qquad (I.3.6)$$

where J_u^i is given by, using Eqs. (I.2.22), (I.3.5):

$$J_u^i = \sum_{s=1}^{n} \rho_{es} v_s^i . \qquad (I.3.7)$$

J^i is the current observed in a fixed system of coordinates while J_u^i is the current observed as "moving with the gross fluid". The term $\rho_e u^i$ is called the convection current.

Adding the individual Eqs. (I.3.3) over s and using Eqs. (I.3.4), (I.3.5), (I.3.6) leads to the equation of conservation of electrical charge of the gross fluid:

$$\frac{\partial \rho_e}{\partial t} + \frac{\partial}{\partial x^j}\left(J^j\right) = 0. \qquad (I.3.8)$$

It may be worthwhile to notice that in the case of a fully ionized plasma (n = 2), Eqs. (I.2.7), (I.3.4) lead to the immediate conclusion:

$$\sigma_1 \equiv \sigma_2 \equiv 0. \qquad (I.3.9)$$

which means that there is no mass interchange between the fluids.

Multiplying Eq. (I.2.8) by γ_s and using Eq. (I.3.2), we get the individual current equations:

$$\frac{\partial}{\partial t}\left(\rho_{es}u_s^i\right) + \frac{\partial}{\partial x^j}\left(\rho_{es}u_s^i u_s^j + \gamma_s X_s^{ij}\right) = -\gamma_s \frac{\partial p_s}{\partial x^i} + \gamma_s F_s^i + \gamma_s \sigma_s Z_s^i. \qquad (I.3.10)$$

The total current equation is obtained by summing the individual Eqs. (I.3.8) over s and making use of Eq. (I.3.6):

$$\frac{\partial J^i}{\partial t} + \sum_{s=1}^{n} \frac{\partial}{\partial x^j}\left(\rho_{es}u_s^i u_s^j + \gamma_s X_s^{ij} + \gamma_s p_s \delta^{ij}\right) = \sum_{s=1}^{n} \gamma_s\left(F_s^i + \sigma_s Z_s^i\right). \qquad (I.3.11)$$

Multiplying Eq. (I.2.40) by $\rho_{es}\rho^{-1}$, summing the resulting equations over s, subtracting the resulting equations from Eq. (I.3.11), and using Eqs. (I.3.22), (I.3.5) to (I.3.8), gives, after some operations:

$$\frac{\partial J_u^i}{\partial t} + \sum_{s=1}^{n}\left[\frac{\partial}{\partial x^j}\left(\rho_{es} v_s^i v_s^j\right)\right] + J_u^j \frac{\partial u^i}{\partial x^j} + \frac{\partial}{\partial x^j}\left(u^j J_u^i\right) +$$

$$+\sum_{s=1}^{n}\left[\gamma_s \frac{\partial}{\partial x^j}\left(p_s\delta^{ij} + X_s^{ij}\right)\right] - \rho_e \rho^{-1}\frac{\partial}{\partial x^j}\left(p\delta^{ij} + X^{ij}\right) =$$

$$= \sum_{s=1}^{n}\left[\gamma_s\left(F_s^i + \sigma_s Z_s^i\right)\right] - \rho_e\rho^{-1}F^i . \tag{I.3.12}$$

Either Eq. (I.3.11) or Eq. (I.3.12) are the gross fluid current equations (1, p. 12).

Next we state some results pertaining to Ohm's law. For a more complete discussion, the reader is referred to (2).

It was shown in (2) that a generalized Ohm's law is contained in the difference momentum equation, (I.2.41), as a certain limiting case. If we assume that all terms in the difference momentum equation can be neglected compared to electromagnetic and interaction forces, then Eq. (I.2.41), reduces to*:

$$\rho_s^{-1}\vec{F}_s = \rho^{-1}\vec{F} , \tag{I.3.13}$$

where the force, $\vec{F}_s$, is given by, according to Eq. (I.2.9):

$$\vec{F}_s = \vec{F}_{es} + \vec{F}_{os} . \tag{I.3.14}$$

The electromagnetic force, $\vec{F}_{es}$, is given by:

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* Vector notation is used in the following derivation.

$$\vec{F}_{es} = \rho_{es}(\vec{E} + \vec{u}_s \times \vec{B}) , \qquad (I.3.15)$$

where $\vec{B}$ is the magnetic flux density. The interaction force, $\vec{F}_{os}$, is assumed to have the form:

$$\vec{F}_{os} = \sum_{t=1}^{n} \alpha_{st}(\vec{u}_t - \vec{u}_s) , \qquad (I.3.16)$$

where α_{st} are assumed to be constants. Substituting Eq. (I.3.14) into Eq. (I.2.18) and making use of Eqs. (I.2.10), (I.3.5), (I.3.6) and (I.3.15) we get:

$$\vec{F} = \rho_e \vec{E} + \vec{J}_u \times \vec{B} + \rho_e \vec{u} \times \vec{B} . \qquad (I.3.17)$$

Inserting Eqs. (I.3.14) to (I.3.17) into Eq. (I.3.13), eliminating $\vec{u}_s$ by means of Eq. (I.2.22), we have:

$$(\rho \rho_{es} - \rho_s \rho_e) \vec{E}_u + (\rho \rho_{es} \vec{v}_s - \rho_s \vec{J}_u) \times \vec{B} -$$

$$- \sum_{S=1}^{n} \rho \alpha_{st}(\vec{v}_s - \vec{v}_t) = 0 \; ; \quad s = 1, 2, \ldots, n , \qquad (I.3.18)$$

where $\vec{E}_u$ is given by:

$$\vec{E}_u = \vec{E} + \vec{u} \times \vec{B} . \qquad (I.3.19)$$

If we consider ρ_s, ρ_{es}, $\vec{E}_u$, $\vec{B}$, α_{st} as given, Eq. (I.3.18) is a system of n algebraic linear equations for the velocities $\vec{v}_s$ (also contained implicitly in

$\vec{J}_u = \sum_{s=1}^{n} \rho_{es}\vec{v}_s$). It is shown in (2) that $\vec{J}_u$ can be written in the form:

$$\vec{J}_u = a\vec{E}_u + b\vec{E}_u\times\vec{B} + c(\vec{E}_u\times\vec{B})\times\vec{B}, \qquad \text{(I.3.20)}$$

where a,b and c, obtainable from the solution of the system (I.3.18), are scalar functions of ρ_s, ρ_{es}, α_{st} and B. Equation (I.3.20), if inverted, yields:

$$\vec{E}_u = R_{\parallel}\vec{J}_u + \xi\,\vec{J}_u\times\vec{B} + (R_{\parallel}-R_{\perp})B^{-2}(\vec{J}_u\times\vec{B})\times\vec{B}, \qquad \text{(I.3.21)}$$

where $R_{\parallel}$, $R_{\perp}$ and ξ can be written explicitly in terms of a, b and c, as follows:

$$R_{\parallel} = a^{-1} \;;\quad \xi = -b\left[(a-cB^2)^2 + b^2B^2\right]^{-1};$$

$$R_{\perp} = (a-cB^2)\left[(a-cB^2)^2 + b^2B^2\right]^{-1};\quad a = R_{\parallel}^{-1}\;;$$

$$b = -\xi\left(R_{\perp}^2 + \xi^2B^2\right)^{-1};\quad c = B^{-2}\left[R_{\parallel}^{-1} - R_{\perp}(R_{\perp}^2+\xi^2B^2)^{-1}\right]. \qquad \text{(I.3.22)}$$

For the case of the fully ionized plasma (n = 2), Eq. (I.3.21) becomes (See 2):

$$\vec{E}_u = \sigma^{-1}\vec{J}_u + \xi\vec{J}_u\times\vec{B}, \qquad \text{(I.3.23)}$$

where ξ, σ are given by:

$$\xi = -\left(\gamma_1 \rho_1^{-1} + \gamma_2 \rho_2^{-1}\right)\left(\gamma_1 - \gamma_2\right)^{-2} ; \quad \sigma = \alpha^{-1}\left(\gamma_1 - \gamma_2\right)^{2}\left(\rho_1^{-1} + \rho_2^{-1}\right)^{-2} , \qquad \text{(I.3.24)}$$

and $\alpha = \alpha_{12} = \alpha_{21}$. The inversion of Eq. (I.3.23) can be put in the form:

$$\vec{J}_u = \sigma \vec{E}_u^{\parallel} + \sigma^{-1}\left(\sigma^{-2} + \xi^2 B^2\right)^{-1} \vec{E}_u^{\perp} - \xi\left(\sigma^{-2} + \xi^2 B^2\right)^{-1} \vec{E}_u \times \vec{B} , \qquad \text{(I.3.25)}$$

where $\vec{E}_u^{\parallel}$, $\vec{E}_u^{\perp}$ are given by:

$$\vec{E}_u^{\parallel} = (\vec{B} \cdot \vec{E}_u) B^{-2} \vec{B} \ ; \ \vec{E}_u^{\perp} = B^{-2} \vec{B} \times (\vec{E}_u \times \vec{B}) ; \ \vec{E}_u = \vec{E}_u^{\parallel} + \vec{E}_u^{\perp} . \qquad \text{(I.3.26)}$$

By inspection of Eq. (I.3.23) one is tempted to call σ the electrical conductivity and $\xi \times \vec{J}_u \times \vec{B}$, the "Hall effect". When the Hall effect is negligible, Eq. (I.3.23) reduces to the generalized Ohm's law:

$$\vec{J}_u = \sigma \vec{E}_u = \sigma(\vec{E} + \vec{u} \times \vec{B}) . \qquad \text{(I.3.27)}$$

In a similar manner one may decide to keep more terms in Eq. (I.2.41), in addition to the electromagnetic and interaction terms. Ohm's law will take then different forms than the one stated above (2, p. 17, Eqs. (22), (23)). The reader is also referred to (13) where the derivation of the generalized Ohm's law in a three component plasma is dealt with.

I.4. Fundamental Electromagnetic Equations

It is assumed that the gross fluid, as a whole, is subjected to one electric field, $\vec{E}$, and one magnetic field, $\vec{H}$. The equations governing the electric and magnetic fields are the Maxwell equations (12, p. 2):

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 , \qquad \text{(I.4.1)}$$

$$\nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J} , \qquad \text{(I.4.2)}$$

where $\vec{B}$, $\vec{D}$ are given by:

$\vec{D} = \epsilon \vec{E} =$ dielectric displacement vector, (I.4.3)

$\vec{B} = \mu_e \vec{H} =$ magnetic flux density vector, (I.4.4)

ϵ is the inductive capacity and μ_e, the magnetic permeability.

Following (12) we take the divergence of Eq. (I.4.1). The first term is identically zero. Assuming that the operators $\nabla \cdot$ and $\frac{\partial}{\partial t}$ are commutable for $\vec{B}$ we get:

$$\frac{\partial}{\partial t}\left(\nabla \cdot \vec{B}\right) = 0 , \qquad \text{(I.4.5)}$$

$\vec{B}$ is, therefore, a function of position only. Assuming that ever in its past history the field has vanished, one concludes that:

$$\nabla\cdot\vec{B} = 0. \tag{I.4.6}$$

Similarly, taking the divergence of Eq. (I.4.2), using Eq. (I.3.8), commuting the operations $\nabla\cdot$ and $\frac{\partial}{\partial t}$ on $\vec{D}$, we get:

$$\frac{\partial}{\partial t}\left(\nabla\cdot\vec{D} - \rho_e\right) = 0\ , \tag{I.4.7}$$

which becomes:

$$\nabla\cdot\vec{D} = \rho_e\ , \tag{I.4.8}$$

by a similar reasoning which led to Eq. (I.4.6).

I.5. The Final System of Fundamental Equations

Following (9, p. 14) we state two ways to describe the final system of equations:

(a) Equations of state, continuity, momentum and energy for each fluid component and two Maxwell's vector equations, i.e., Eqs. (I.2.1), (I.2.6), (I.2.8), (I.2.14), (I.4.1), (I.4.2) where s runs over all n fluid components. These are 6n+6 equations for the 6n+6 unknowns (T_s, p_s, ρ_s, u_s^i, E^i, H^i).

(b) Equations of state, continuity, momentum and energy for n-1 fluid components and for the gross fluid, and two

Maxwell's vector equations, i.e., Eqs. (I.2.1), (I.2.6), (I.2.8), (I.2.14), (I.2.29), (I.2.30), (I.2.31), (I.2.50), (I.4.1), (I.4.2) where s runs over n-1 fluid components. These are 6n+6 equations for the 6n+6 unknowns (T_s, p_s, ρ_s, u_s^i, T, p, ρ, u^i, E^i, H^i).

Additionally let us consider a system of equations for a single fluid. Most of the magnetohydrodynamical problems which were dealt with up till now consider the plasma as a single fluid. The electromagnetic phenomena due to the existence of differently charged particles and a relative velocity between them in the plasma, was taken into account through the introduction of additional variables, like the excess charge density, ρ_e, and the electric current density, $\vec{J}$. The final system of equations is supplemented by the equation of conservation of charge, (I.3.8), and by Ohm's law, (I.3.27). In this formulation no injection of momentum and no mass sources are present.

The final system is:

$$p = R\rho T\,, \tag{I.5.1}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^j}\left(\rho u^j\right) = 0\,, \tag{I.5.2}$$

$$\frac{\partial}{\partial t}\left(\rho u^i\right) + \frac{\partial}{\partial x^j}\left(\rho u^i u^j\right) = -\frac{\partial p}{\partial x^i} + F^i\,, \tag{I.5.3}$$

$$\frac{\partial}{\partial t}\left(\bar{e} - \rho Q\right) + \frac{\partial}{\partial x^j}\left(\bar{e} u^j + p u^j - \rho Q u^j\right) = u^j F^j\,, \tag{I.5.4}$$

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0, \qquad (I.5.5)$$

$$\nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = 0, \qquad (I.5.6)$$

$$\frac{\partial \rho_e}{\partial t} + \frac{\partial}{\partial x^j}(J^j) = 0, \qquad (I.5.7)$$

$$\vec{J} = \sigma(\vec{E} + \vec{u} \times \vec{B}) + \rho_e \vec{u}, \qquad (I.5.8)$$

These are 16 equations for the 16 unknowns (T, p, ρ, u^i, E^i, H^i, ρ_e, J^i), with σ being known.

CHAPTER II

QUASI-THREE-DIMENSIONAL STEADY FLOW

II.1. The Governing System of Equations

From this chapter and on we deal with the quasi-three-dimensional steady flow of each component fluid. The basic hypothesis for such a flow is that all quantities are independent of x^3 and t. The equations valid for this type of motion can be obtained from the governing system of equations given in Chapter I.

Eqs. (I.2.6), (I.2.8), (I.2.14), (I.4.1), (I.4.2) become:

$$\frac{\partial}{\partial x^1}\left(\rho_s u_s^1\right) + \frac{\partial}{\partial x^2}\left(\rho_s u_s^2\right) = \sigma_s , \qquad \text{(II.1.1)}$$

$$\frac{\partial}{\partial x^1}\left(\rho_s u_s^1 u_s^1\right) + \frac{\partial}{\partial x^2}\left(\rho_s u_s^1 u_s^2\right) = -\frac{\partial p_s}{\partial x^1} + F_s^1 + \sigma_s Z_s^1 - X_s^1 , \qquad \text{(II.1.2)}$$

$$\frac{\partial}{\partial x^1}\left(\rho_s u_s^2 u_s^1\right) + \frac{\partial}{\partial x^2}\left(\rho_s u_s^2 u_s^2\right) = -\frac{\partial p_s}{\partial x^2} + F_s^2 + \sigma_s Z_s^2 - X_s^2 , \qquad \text{(II.1.3)}$$

$$\frac{\partial}{\partial x^1}\left(\rho_s u_s^3 u_s^1\right) + \frac{\partial}{\partial x^2}\left(\rho_s u_s^3 u_s^2\right) = F_s^3 + \sigma_s Z_s^3 - X_s^2 , \qquad \text{(II.1.4)}$$

$$\frac{\partial}{\partial x^1}\left(\bar{e}_s u_s^1 - \rho_s Q_s u_s^1 + p_s u_s^1\right) + \frac{\partial}{\partial x^2}\left(\bar{e}_s u_s^2 - \rho_s Q_s u_s^2 + p_s u_s^2\right) =$$

$$= u_s^1 F_s^1 + u_s^2 F_s^2 + u_s^3 F_s^3 - u_s^1 X_s^1 - u_s^2 X_s^2 - u_s^3 X_s^3 +$$

$$+ \sigma_s\left[I_s - Q_s - \frac{1}{2}(u_s)^2 + u_s^1 Z_s^1 + u_s^2 Z_s^2 + u_s^3 Z_s^3\right], \qquad \text{(II.1.5)}$$

$$\frac{\partial E^3}{\partial x^2} = 0 \; ; \quad \frac{\partial E^3}{\partial x^1} = 0 \; ; \quad \frac{\partial E^2}{\partial x^1} - \frac{\partial E^1}{\partial x^2} = 0 , \qquad \text{(II.1.6)}$$

$$\frac{\partial H^3}{\partial x^2} = J^1 \; ; \quad \frac{\partial H^3}{\partial x^1} = -J^2 \; ; \quad \frac{\partial H^2}{\partial x^1} - \frac{\partial H^1}{\partial x^2} = J^3 , \qquad \text{(II.1.7)}$$

where X_s^i, $(i=1,2,3)$, is defined by the equation:

$$X_s^i = \frac{\partial X_s^{i1}}{\partial x^1} + \frac{\partial X_s^{i2}}{\partial x^2} \; . \qquad \text{(II.1.8)}$$

From the first two Eqs. (II.1.6) it follows that:

$$E^3 = \text{constant} \qquad \text{(II.1.9)}$$

To determine H^3, the first two Eqs. (II.1.7) are used. From Eq. (I.3.8) we have:

$$\frac{\partial J^1}{\partial x^1} = -\frac{\partial J^2}{\partial x^2} \; , \qquad \text{(II.1.10)}$$

which, together with the first two Eqs. (II.1.7) is a necessary and sufficient condition that $-J^2 dx^1 + J^1 dx^2$ be the total differential of the function H^3:

$$dH^3 = J^1 dx^2 - J^2 dx^1 . \qquad \text{(II.1.11)}$$

Eq. (II.1.11), when integrated from a 0 subscripted initial point to some end point along an arbitrary path gives:

$$H^3 = H^3_0 + \int_0 \left(J^1 dx^2 - J^2 dx^1\right). \qquad \text{(II.1.12)}$$

It should be noticed that the first two Eqs. (II.1.6) determine E^3 only, and the first two Eqs. (II.1.7) determine H^3 alone. By having four equations determining only two unknowns our system of equations becomes underspecified, i.e., the number of unknowns exceeds the number of equations, unless two additional equations can be supplied to the system. We note that the two electromagnetic equations (I.4.6), (I.4.8) which were derived from Maxwell's equations (I.4.1), (I.4.2) cannot be derived from those equations after one puts $\frac{\partial}{\partial t} = 0$ there. In our case we will take them into account, thus making our total system of equations specified by having the two additional equations, using Eqs. (I.4.3), (I.4.4):

$$\frac{\partial}{\partial x^1}\left(\mu_e H^1\right) + \frac{\partial}{\partial x^2}\left(\mu_e H^2\right) = 0 , \qquad \text{(II.1.13)}$$

$$\frac{\partial}{\partial x^1}\left(\epsilon E^1\right) + \frac{\partial}{\partial x^2}\left(\epsilon E^2\right) = \rho_e . \qquad \text{(II.1.14)}$$

The system of governing equations consists, therefore, of Eqs. (I.2.1), (II.1,1) to (II.1.5), the third Eqs. (II.1.6), (II.1.7) and Eqs. (II.1.12) to (II.1.14), which are 6n+5 equations for the 6n+5 unknowns (T_s, p_s, ρ_s, u_s^1, u_s^2, u_s^3, E^1, E^2, H^1, H^2, H^3).

We note that the third Eq. (II.1.6) is automatically satisfied by the choice of a function $\eta(x^1, x^2)$ such that:

$$E^1 = \frac{\partial \eta}{\partial x^1} \; ; \; E^2 = \frac{\partial \eta}{\partial x^2} \, . \qquad \text{(II.1.15)}$$

Inserting Eq. (II.1.15) into Eq. (II.1.14) and assuming ϵ = constant, we get:

$$\frac{\partial^2 \eta}{(\partial x^1)^2} + \frac{\partial^2 \eta}{(\partial x^2)^2} = \rho_e \epsilon^{-1} \, . \qquad \text{(II.1.16)}$$

Similarly, assume μ_e= constant and introduce a function $\xi(x^1, x^2)$ such that:

$$H^1 = -\frac{\partial \xi}{\partial x^2} \; ; \; H^2 = \frac{\partial \xi}{\partial x^1} \, , \qquad \text{(II.1.17)}$$

then Eq. (II.1.13) is automatically satisfied. The third Eq. (II.1.7) becomes, using Eq. (II.1.17):

$$\frac{\partial^2 \xi}{(\partial x^1)^2} + \frac{\partial^2 \xi}{(\partial x^2)^2} = J^3 \, . \qquad \text{(II.1.18)}$$

Eqs. (II.1.16), (II.1.18) will be used to determine the functions $\eta(x^1, x^2)$, $\xi(x^1, x^2)$ respectively.

In the case where ϵ is not assumed to be a constant

Eq. (II.1.16) is replaced by:

$$\frac{\partial}{\partial x^1}\left(\epsilon \frac{\partial \eta}{\partial x^1}\right) + \frac{\partial}{\partial x^2}\left(\epsilon \frac{\partial \eta}{\partial x^2}\right) = \rho_e \ . \qquad \text{(II.1.19)}$$

When μ_e is not assumed to be a constant, the function ξ is assumed to be such that:

$$\mu_e H^1 = -\frac{\partial \xi}{\partial x^2} \ ; \quad \mu_e H^2 = \frac{\partial \xi}{\partial x^1} \ , \qquad \text{(II.1.20)}$$

and Eq. (II.1.18) is replaced by:

$$\frac{\partial}{\partial x^1}\left(\mu_e^{-1} \frac{\partial \xi}{\partial x^1}\right) + \frac{\partial}{\partial x^2}\left(\mu_e^{-1} \frac{\partial \xi}{\partial x^2}\right) = J^3 . \qquad \text{(II.1.21)}$$

We will assume in the present work that ϵ, μ_e are constants.

II.2. Quasi-Stream Function and Potential Equations

In Eq. (II.1.1) the functions σ_s are known and given. Suppose that there exist functions B_s, by means of which the functions σ_s may be expressed in conjunction with ρ_s and u_s^i, by the formula:

$$\sigma_s = \frac{\partial}{\partial x^1}\left(\rho_s u_s^1 B_s\right) + \frac{\partial}{\partial x^2}\left(\rho_s u_s^2 B_s\right) \ , \qquad \text{(II.2.1)}$$

with the condition that when there is no mass source into the s-th fluid component, i.e., $\sigma_s \equiv 0$, the trivial solution

$B_e = 0$ should be taken. Inserting Eq. (II.2.1) into Eq. (II.1.1), we have:

$$\frac{\partial}{\partial x^1}\left(\bar{\rho}_s u_s^1\right) + \frac{\partial}{\partial x^2}\left(\bar{\rho}_s u_s^2\right) = 0 , \tag{II.2.2}$$

where $\bar{\rho}_s$ is given by:

$$\bar{\rho}_s = \rho_s(1 - B_s) . \tag{II.2.3}$$

We introduce the quasi-stream function $\psi_s(x^1, x^2)$ such that:

$$\bar{\rho}_s u_s^1 = \frac{\partial \psi_s}{\partial x^2} \quad ; \quad \bar{\rho}_s u_s^2 = -\frac{\partial \psi_s}{\partial x^1} , \tag{II.2.4}$$

then Eq. (II.2.2) is automatically satisfied by this choice of ψ_s.

Eqs. (II.1.2), (II.1.3), remodelled, using Eq. (II.1.1), yield:

$$u_s^1 \frac{\partial u_s^j}{\partial x^1} + u_s^2 \frac{\partial u_s^j}{\partial x^2} = -\rho_s^{-1}\frac{\partial p_s}{\partial x^j} + \rho_s^{-1} F_s^j + Y_s^j \; ; \; (j=1,2) , \tag{II.2.5}$$

where Y_s^j is given by:

$$Y_s^j = \rho_s^{-1}\left[\sigma_s\left(Z_s^j - u_s^j\right) - X_s^j\right]. \tag{II.2.6}$$

Setting $j = 1$ in Eq. (I.2.5) and multiplying the equation by $-u_s^2$, then setting $j = 2$, and multiplying by u_s^1, adding the resulting equations, furnishes, after a few manipulations:

$$\bar{u}_s\left(u_s^1 \frac{\partial \bar{u}_s}{\partial x^2} - u_s^2 \frac{\partial \bar{u}_s}{\partial x^1}\right) = -(\bar{u}_s)^2 \omega_s^3 + \rho_s^{-1}\left(-u_s^1 \frac{\partial p_s}{\partial x^2} + u_s^2 \frac{\partial p_s}{\partial x^1} + u_s^1 F_s^2 - \right.$$

$$\left. - u_s^2 F_s^1\right) + u_s^1 Y_s^2 - u_s^2 Y_s^1 , \qquad \text{(II.2.7)}$$

where $\bar{u}_s$, ω_s^3 are given by:

$$(\bar{u}_s)^2 = (u_s^1)^2 + (u_s^2)^2 , \qquad \text{(II.2.8)}$$

$$\omega_s^3 = \frac{\partial u_s^2}{\partial x^1} - \frac{\partial u_s^1}{\partial x^2} . \qquad \text{(II.2.9)}$$

The term ω_s^3 is the x^3- component of the vorticity of the s-th species, $\vec{\omega}_s$, given by:

$$\vec{\omega}_s = \nabla \times \vec{u}_s . \qquad \text{(II.2.10)}$$

Next, we represent the term $\rho_s^{-1} \frac{\partial p_s}{\partial x^i}$, (i=j,2), in a different form. The enthalpy of the s-th fluid component, I_s, can be written, using Eqs. (I.2.5), (I.2.44), (I.2.54), in the form:

$$I_s = c_{ps}(1 + D_s) T_s , \qquad \text{(II.2.11)}$$

where c_{ps}, D_s are given by, using Eq. (I.2.58):

$$c_{ps} = c_{vs} + R_{ps} = K_s c_{vs} , \qquad \text{(II.2.12)}$$

$$D_s = K_s^{-1}\left[A_s + (K_s - 1)W_s\right]. \qquad \text{(II.2.13)}$$

Substituting an expression for T_s from Eq. (I.2.47) into Eq. (II.2.11), and inserting the resulting equation into Eq. (I.2.12), differentiating the product, using Eqs. (I.2.70), (II.2.12), (II.2.13) and solving for $\rho_s^{-1}\frac{\partial p_s}{\partial x^j}$, furnishes:

$$\rho_s^{-1}\frac{\partial p_s}{\partial x^j} = (K_s-1)(1+W_s)(1+A_s)^{-1}\frac{\partial Q_s}{\partial x^j} + p_s\rho_s^{-1}\frac{\partial}{\partial x^j}\left[\log(1+W_s)(1+A_s)^{-1}\right] +$$

$$+ K_s p_s \rho_s^{-2}\left(1+N_s\right)\frac{\partial \rho_s}{\partial x^j}. \qquad \text{(II.2.14)}$$

In Chapter I it was already assumed that A_s, W_s are, possibly, functions of ρ_s and some other variables independent of ρ_s. Let us denote those variables by the symbol ξ_s^i $(i=1,2,\ldots,k_s)$, then we have:

$$\frac{\partial}{\partial x^j}\left[\log(1+W_s)(1+A_s)^{-1}\right] = \frac{\partial}{\partial \rho_s}\left[\log(1+W_s)(1+A_s)^{-1}\right]\frac{\partial \rho_s}{\partial x^j} +$$

$$+ \sum_{i=1}^{k_s}\frac{\partial}{\partial \xi_s^i}\left[\log(1+W_s)(1+A_s)^{-1}\right]\frac{\partial \xi_s^i}{\partial x^j}. \qquad \text{(II.2.15)}$$

Substituting Eq. (II.2.15) into Eq. (II.2.14) and using Eq. (I.2.71), we get:

$$\rho_s^{-1}\frac{\partial p_s}{\partial x^j} = (K_s-1)(1+W_s)(1+A_s)^{-1}\frac{\partial Q_s}{\partial x^j} + \rho_s^{-1}(\alpha_s)^2\frac{\partial \rho_s}{\partial x^j} + L_s^j, \qquad \text{(II.2.16)}$$

where L_s^j is given by:

$$L_s^j = p_s \bar{\rho}_s^{-1} \sum_{i=1}^{k_s} \frac{\partial}{\partial \xi_s^i} \left[\log (1+W_s)(1+A_s)^{-1} \right] \frac{\partial \xi_s^i}{\partial x^j} . \qquad (II.2.17)$$

Eq. (II.2.16), remodelled with the use of Eq. (II.2.3) gives:

$$\rho_s^{-1} \frac{\partial p_s}{\partial x^j} = (K_s - 1)(1+W_s)(1+A_s)^{-1} \frac{\partial Q_s}{\partial x^j} + \bar{\rho}_s^{-1} (\alpha_s)^2 \frac{\partial \bar{\rho}_s}{\partial x^j} -$$

$$- (\alpha_s)^2 \frac{\partial}{\partial x^j} \left[\log (1 - B_s) \right] + L_s^j . \qquad (II.2.18)$$

From Eqs. (II.2.4), (II.2.8) we have:

$$(\bar{\rho}_s \bar{u}_s)^2 = \left(\frac{\partial \psi_s}{\partial x^1} \right)^2 + \left(\frac{\partial \psi_s}{\partial x^2} \right)^2 . \qquad (II.2.19)$$

Differentiating Eq. (II.2.19) with respect to x^1 and multiplying the result by $-u_s^2$, then with respect to x^2 and multiplying the result by u_s^1, adding the resulting equations and using Eqs. (II.2.4), (II.2.8), we get:

$$(u_s^2)^2 \frac{\partial^2 \psi_s}{(\partial x^1)^2} - 2 u_s^1 u_s^2 \frac{\partial^2 \psi_s}{\partial x^1 \partial x^2} + (u_s^1)^2 \frac{\partial^2 \psi_s}{(\partial x^2)^2} = (\bar{u}_s)^2 \left(u_s^1 \frac{\partial \bar{\rho}_s}{\partial x^2} - u_s^2 \frac{\partial \bar{\rho}_s}{\partial x^1} \right) +$$

$$+ \bar{\rho}_s \bar{u}_s \left(u_s^1 \frac{\partial \bar{u}_s}{\partial x^2} - u_s^2 \frac{\partial \bar{u}_s}{\partial x^1} \right) . \qquad (II.2.20)$$

Solving Eqs. (II.2.4) for u_s^1, u_s^2 and inserting the results into Eq. (II.2.9) it can be shown that:

$$u_s^1 \frac{\partial \bar{\rho}_s}{\partial x^2} - u_s^2 \frac{\partial \bar{\rho}_s}{\partial x^1} = \bar{\rho}_s \omega_s^3 + \frac{\partial^2 \psi_s}{(\partial x^1)^2} + \frac{\partial^2 \psi_s}{(\partial x^2)^2} . \qquad (II.2.21)$$

Inserting Eq. (II.2.18) into Eq. (II.2.7), substituting the resulting equation into Eq. (II.2.20) by elimination of the term $u_s^1 \frac{\partial \bar{u}_s}{\partial x^2} - u_s^2 \frac{\partial \bar{u}_s}{\partial x^1}$, and using Eq. (II.2.21) to substitute for the term $u_s^1 \frac{\partial \bar{p}_s}{\partial x^2} - u_s^2 \frac{\partial \bar{p}_s}{\partial x^1}$, we get, after rearranging and using Eqs. (II.2.4), (II.2.6):

$$\left[(\alpha_s)^2 - (u_s^1)^2\right] \frac{\partial^2 \psi_s}{(\partial x^1)^2} - 2 u_s^1 u_s^2 \frac{\partial^2 \psi_s}{\partial x^1 \partial x^2} + \left[(\alpha_s)^2 - (u_s^2)^2\right] \frac{\partial^2 \psi_s}{(\partial x^2)^2} =$$

$$= -\bar{\rho}_s (\alpha_s)^2 \omega_s^3 + \sum_{j=1}^{2} \left(\frac{\partial \psi_s}{\partial x^j}\right) \Big\{ (K_s - 1)(1 + W_s)(1 + A_s)^{-1} \frac{\partial Q_s}{\partial x^j} - L_s^j +$$

$$+ (\alpha_s)^2 \frac{\partial}{\partial x^j}\left[\log(1 - B_s)\right] + \rho_s^{-1}\left[F_s^j + \sigma_s Z_s^j - X_s^j\right]\Big\} . \qquad \text{(II.2.22)}$$

Eq. (II.2.22) is one possible form of the quasi-stream function equation.

Next, we derive a quasi-potential equation by introducing the two functions $\varphi_s(x^1, x^2)$, $g_s(x^1, x^2)$ such that

$$u_s^j = \frac{\partial \varphi_s}{\partial x^j} + g_s \quad ; \quad (j = 1, 2). \qquad \text{(II.2.23)}$$

It is readily seen from Eqs. (II.2.9), (II.2.23) that g_s must satisfy the equation:

$$\frac{\partial g_s}{\partial x^1} - \frac{\partial g_s}{\partial x^2} = \omega_s^3 \qquad \text{(II.2.24)}$$

We add the condition that if $\omega_s^3 \equiv 0$, the trivial solution $g_s = 0$ should be taken.

Setting j=1 in Eq. (II.2.5) and multiplying the equation by $-u_s^1$, then setting j=2 and multiplying by $-u_s^2$, adding the resulting equations and using Eqs. (II.1.1), (II.2.2), (II.2.3), (II.2.18), (II.2.23), we get, after rearranging:

$$\left[(\alpha_s)^2-(u_s^1)^2\right]\left[\frac{\partial^2\varphi_s}{(\partial x^1)^2}+\frac{\partial g_s}{\partial x^1}\right]-u_s^1u_s^2\left(2\frac{\partial^2\varphi_s}{\partial x^1\partial x^2}+\frac{\partial g_s}{\partial x^1}+\frac{\partial g_s}{\partial x^2}\right)+$$

$$+\left[(\alpha_s)^2-(u_s^2)^2\right]\left[\frac{\partial^2\varphi_s}{(\partial x^2)^2}+\frac{\partial g_s}{\partial x^2}\right]=\sigma_s\rho_s^{-1}(\alpha_s)^2+$$

$$+\sum_{j=1}^{2}(u_s^j)\left[(K_s-1)(1+W_s)(1+A_s)^{-1}\frac{\partial Q_s}{\partial x^j}+L_s^j-\rho_s^{-1}F_s-Y_s\right]. \qquad \text{(II.2.25)}$$

Eq. (II.2.25) is one possible form of the quasi-potential equation.

We assume that the body force, $\vec{F}_s$, consists only of the electromagnetic force, $\vec{F}_{es}$, which is given by:

$$\vec{F}_s=\vec{F}_{es}=\rho_{es}\left[\vec{E}+\mu_e(\vec{u}_s\times\vec{H})\right]. \qquad \text{(II.2.26)}$$

Inserting the corresponding components of Eq. (II.2.26) into Eq. (II.2.22), using Eqs. (I.3.2), (II.1.15), (II.1.17), (II.2.4) and dividing the resulting equation by $(\alpha_s)^2$ leads to the final form of the quasi-stream function equation:

$$\left[1-(u_s^1\alpha_s^{-1})^2\right]\frac{\partial^2\psi_s}{(\partial x^1)^2} - 2u_s^1u_s^2(\alpha_s)^{-2}\frac{\partial^2\psi_s}{\partial x^1\partial x^2} + \left[1-(u_s^2\alpha_s^{-1})^2\right]\frac{\partial^2\psi_s}{(\partial x^2)^2} =$$

$$= -\bar{\rho}_s\omega_s^3 + \sum_{j=1}^{2}\left(\frac{\partial\psi_s}{\partial x^j}\right)\Big\{\frac{\partial}{\partial x^j}\left[\log(1-B_s)\right] + (\alpha_s)^{-2}\Big[(K_s-1)(1+W_s)(1+A_s)^{-1}\frac{\partial Q_s}{\partial x^j} +$$

$$+ \gamma_s\left(\frac{\partial\eta}{\partial x^j} - \mu_e u_s^3\frac{\partial\xi}{\partial x^j} - \bar{\rho}_s^{-1}\mu_e H^3\frac{\partial\psi_s}{\partial x^j}\right) +$$

$$+\rho_s^{-1}(\sigma_s Z_s^j - X_s^j) - L_s^j\Big]\Big\}. \qquad \text{(II.2.27)}$$

Inserting the corresponding components of Eq. (II.2.26) into Eq. (II.2.25), using Eqs. (I.3.2), (II.1.15), (II.1.17), (II.2.23) and dividing the resulting equation by $(\alpha_s)^2$ leads to the final form of the quasi-potential equation:

$$\left[1-(u_s^1\alpha_s^{-1})^2\right]\left[\frac{\partial^2\varphi_s}{(\partial x^1)^2} + \frac{\partial g_s}{\partial x^1}\right] - u_s^1u_s^2(\alpha_s)^{-2}\left(2\frac{\partial^2\varphi_s}{\partial x^1\partial x^2} + \frac{\partial g_s}{\partial x^1} + \frac{\partial g_s}{\partial x^2}\right) +$$

$$+\left[1-(u_s^2\alpha_s^{-1})^2\right]\left[\frac{\partial^2\varphi_s}{(\partial x^2)^2} + \frac{\partial g_s}{\partial x^2}\right] = \sigma_s\rho_s^{-1} +$$

$$+(\alpha_s)^{-2}\sum_{j=1}^{2}\left(\frac{\partial\varphi_s}{\partial x^j} + g_s\right)\Big\{(K_s-1)(1+W_s)(1+A_s)^{-1}\frac{\partial Q_s}{\partial x^j} + \gamma_s\left(\frac{\partial\eta}{\partial x^j} - \mu_e u_s^3\frac{\partial\xi}{\partial x^j}\right) +$$

$$+\rho_s^{-1}\left[\sigma_s Z_s^j - X_s^j - \sigma_s\left(\frac{\partial\varphi_s}{\partial x^j} + g_s\right)\right] + L_s^j\Big\}. \qquad \text{(II.2.28)}$$

II.3. Generalized Bernoulli and Crocco Equations

Eqs. (II.1.1) to (II.1.4) combined with Eq. (I.2.12), and written in vector form, give:

$$\frac{d\vec{u}_s}{dt} = \nabla Q_s - \nabla I_s + \rho_s^{-1}\vec{F}_s + \vec{Y}_s\,, \qquad \text{(II.3.1)}$$

where the components Y_s^j, $(j=1,2,3)$, of the vector $\vec{Y}_s$ are given by Eq. (II.2.6), and the operator $\frac{d}{dt}$ operating on an s-subscripted quantity denotes:

$$\frac{d}{dt} = u_s^1 \frac{\partial}{\partial x^1} + u_s^2 \frac{\partial}{\partial x^2} = \vec{u}_s \cdot \nabla \quad . \qquad \text{(II.3.2)}$$

Let the position vector of a particle of the s-th fluid component be denoted by $\vec{r}_s$, then the velocity of this particle, $\vec{u}_s$, is given by:

$$\vec{u}_s = \frac{d\vec{r}_s}{dt} \quad . \qquad \text{(II.3.3)}$$

Taking the vector dot product of Eq. (II.3.1) by $\vec{u}_s$, using Eqs. (I.3.2), (II.2.26), (II.3.2), (II.3.3), multiplying the resulting equation by dt and integrating along a streamline of the s-th fluid component from a zero subscripted state to some end state, we have:

$$\frac{1}{2}(u_s)^2 + I_s - \int_0 dQ_s - \gamma_s \int_0 \vec{E}\cdot d\vec{r}_s - \int_0 \vec{Y}_s \cdot d\vec{r}_s = H_{s0}\,, \qquad \text{(II.3.4)}$$

where H_{s0} is given by:

$$H_{s0} = \frac{1}{2}(u_{s0})^2 + I_{s0} \qquad \text{(II.3.5)}$$

Eq. (II.3.4) is the generalized Bernoulli equation.

From Eqs. (I.2.70), (I.2.71), (II.2.11) to (II.2.13) we have:

$$I_s(\alpha_s)^{-2} = (1+P_s)(K_s-1)^{-1}\ , \tag{II.3.6}$$

where P_s is given by:

$$P_s = \left\{(1+W_s)(1+A_s)^{-1} + \rho_s K_s^{-1}(1+N_s)^{-1}\frac{\partial}{\partial\rho_s}\left[(1+W_s)(1+A_s)^{-1}\right]\right\}^{-1} - 1. \tag{II.3.7}$$

Inserting Eq. (II.3.6) into Eq. (II.3.4) we have:

$$\frac{1}{2}(u_s)^2 + (\alpha_s)^2(1+P_s)(K_s-1)^{-1} - \int_0 dQ_s - \gamma_s\int_0 \vec{E}\cdot d\vec{r} - \int_0 \vec{Y}_s\cdot d\vec{r}_s = H_{s0}, \tag{II.3.8}$$

where H_{s0} is given by:

$$H_{s0} = \frac{1}{2}(u_{s0})^2 + (\alpha_{s0})^2(1+P_{s0})(K_s-1)^{-1}. \tag{II.3.9}$$

Eq. (II.3.8) is another form of the generalized Bernoulli equation.

Let Λ_s be a streamline of the s-th fluid component which intersects the x^1-x^2 plane at the point $A_{s0}(x^1_{s0}, x^2_{s0}, 0)$. this streamline satisfies the equation of differential type:

$$dx^1_s : dx^2_s : dx^3_s = u^1_s : u^2_s : u^3_s\ . \tag{II.3.10}$$

Let dr_s be an arc element of Λ_s and let $d\Gamma_s$ be an element of arc length taken along the curve Γ_s which is the projection of Λ_s on the x^1-x^2 plane. We note, from Eqs. (II.2.4), (II.3.10) that Γ_s is a ψ_s = constant curve.

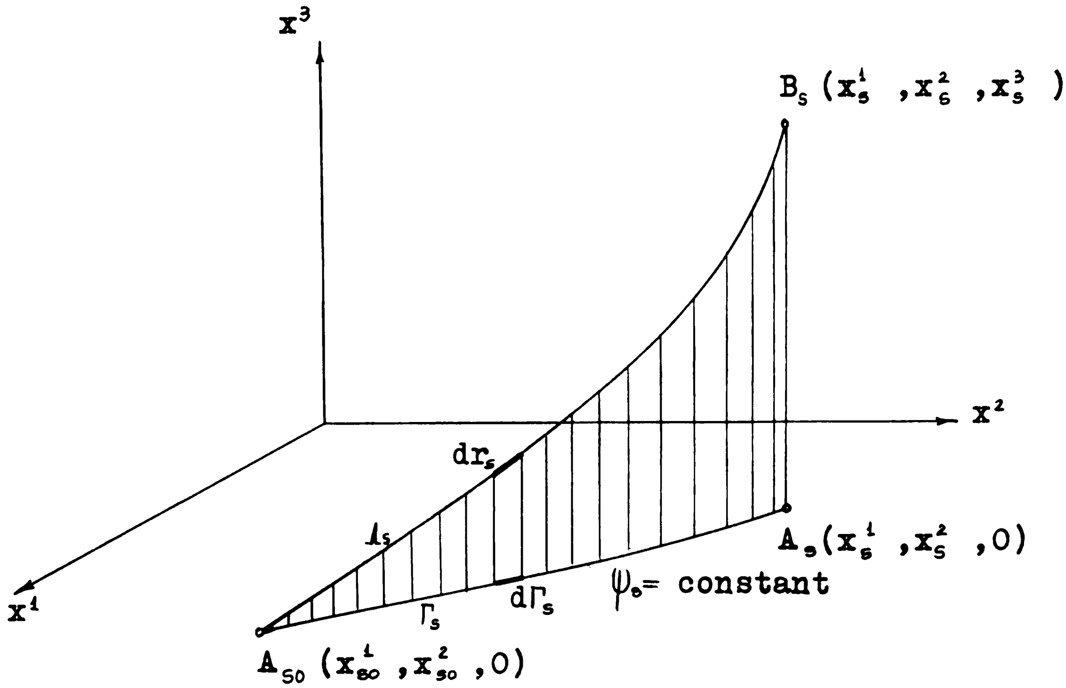

Fig. 1. Streamline Geometry

Let $B_s(x_s^1,\ x_s^2,\ x_s^3)$ be some end point on Λ_s, and let $A_s(x_s^1,\ x_s^2,\ 0)$ be the projection of B_s on the x^1-x^2 plane. Let $\vec{\mu}_s$ be any vector function defined in the three-dimensional space and depending on the coordinates $(x^1,\ x^2)$ only, i.e., for a pair $(x^1,\ x^2)$, $\vec{\mu}_s$ is the same for any x^3. Denote the two unit vectors, tangential and normal to the Γ_s curve, in the x^1-x^2 plane, by $\hat{s}_s$, $\hat{n}_s$ respectively, and let $\hat{k}$ be a unit vector in the x^3 direction. The components of $\vec{\mu}_s$ in the above mentioned directions are denoted by $(\mu_s^s, \mu_s^n, \mu_s^3)$, and the components of the velocity vector $\vec{u}_s$ in those directions are $(\bar{u}_s, 0, u_s^3)$. Using Eq. (II.3.3), the relation $\bar{u}_s = \frac{d\Gamma_s}{dt}$ and the fact that all functions depend only on the $(x^1 - x^2)$ coordinates, we have:

$$\int_{\substack{A_{s0} \\ \Lambda_s}}^{B_s} \vec{\mu}_s \cdot d\vec{r}_s = \int_{\substack{A_{s0} \\ \Lambda_s}}^{B_s} (\vec{\mu}_s \cdot \vec{u}_s)\, \bar{u}_s^{-1}\, d\Gamma_s = \int_{\substack{A_{s0} \\ \Gamma_s}}^{A_s} (\vec{\mu}_s \cdot \vec{u}_s)\, \bar{u}_s^{-1}\, d\Gamma_s\,, \quad \text{(II.3.11)}$$

and

$$\nabla\left(\int_{\substack{A_{s0} \\ \Lambda_s}}^{B_s} \vec{\mu}_s \cdot d\vec{r}_s \right) = \nabla\left(\int_{\substack{A_{s0} \\ \Gamma_s}}^{A_s} (\vec{\mu}_s \cdot \vec{u}_s)\, \bar{u}_s^{-1}\, d\Gamma_s \right) = (\vec{\mu}_s \cdot \vec{u}_s)\, \bar{u}_s^{-1}\, \hat{s}_s +$$

$$+ \frac{\partial}{\partial n_s}\left[\int_{\substack{A_{s0} \\ \Gamma_s}}^{A_s} (\vec{\mu}_s \cdot \vec{u}_s)\, u_s^{-1}\, d\Gamma_s \right] \hat{n}_s = (\mu_s^s + \mu_s^3 u_s^3 \bar{u}_s^{-1})\hat{s}_s + \frac{\partial}{\partial n_s}\left[\int_{\substack{A_{s0} \\ \Gamma_s}}^{A_s} (\vec{\mu}_s \cdot \vec{u}_s)\, \bar{u}_s^{-1}\, d\Gamma_s \right] \hat{n}_s. \quad \text{(II.3.12)}$$

Subtracting $\nabla\left[\frac{1}{2}(u_s)^2\right]$ from both sides of Eq. (II.3.1), using Eqs. (I.3.2), (II.2.26), (II.3.4) and the identity for steady flow: $\frac{d\vec{u}_s}{dt} - \nabla\left[\frac{1}{2}(u_s)^2\right] = \vec{\omega}_s \times \vec{u}_s$, we get:

$$\vec{\omega}_s \times \vec{u}_s = -\nabla H_{s0} + \nabla Q_{s0} - \gamma_s \nabla\left(\int_{\substack{0 \\ \Lambda_s}} \vec{E} \cdot d\vec{r}_s \right) - \nabla\left(\int_{\substack{0 \\ \Lambda_s}} \vec{Y}_s \cdot d\vec{r}_s \right) +$$

$$+ \vec{Y}_s + \gamma_s\left[\vec{E} + \mu_e(\vec{u}_s \times \vec{H}) \right]. \quad \text{(II.3.13)}$$

Substituting for $\vec{E}$ in Eq. (II.3.13) the expression $\nabla\eta + E^3\hat{k}$ which can be obtained when using Eq. (II.1.15), noting that $\int_{\substack{0 \\ \Lambda_s}} \nabla\eta \cdot d\vec{r}_s = \eta - \eta_0$, using Eqs. (II.1.9), (II.3.11) where $\vec{\mu}_s$ is taken to be equal to $\vec{k}$, and using Eq. (II.3.12) where $\vec{Y}_s$ is substituted for $\vec{\mu}_s$, we get in Eq. (II.3.13):

$$\omega_s \times u_s = -\nabla H_{s0} + \nabla Q_{s0} + \gamma_s \nabla \eta_0 - \gamma_s E^3 \nabla\left(\int_{\Gamma_s^0} u_s^3 \bar{u}_s^{-1} d\Gamma_s\right) -$$

$$- Y_s^3 u_s^3 \bar{u}_s^{-1} \hat{s}_s - \frac{\partial}{\partial n_s}\left[\int_{\Gamma_s^0} (\vec{Y}_s \cdot \vec{u}_s) \bar{u}_s^{-1} d\Gamma_s\right] \hat{n}_s + Y_s^n \hat{n}_s + Y_s^3 \hat{k} +$$

$$+ \gamma_s E^3 \hat{k} + \gamma_s \mu_e (\vec{u}_s \times \vec{H}) . \qquad \text{(II.3.14)}$$

It may be worthwhile to notice that the zero subscripted functions are constant along streamlines and, in addition, are not functions of x^3, thus, zero subscripted functions are constant on the cylindrical surfaces ψ_s = constant.

The component of Eq. (II.3.14) in the direction $\hat{n}_s$ is:

$$\omega_s^3 \bar{u}_s - u_s^3 \frac{\partial u_s^3}{\partial n_s} = -\frac{\partial H_{s0}}{\partial n_s} + \frac{\partial Q_{s0}}{\partial n_s} + \gamma_s \frac{\partial \eta_0}{\partial n_s} - \gamma_s E^3 \frac{\partial}{\partial n_s}\left(\int_{\Gamma_s^0} u_s^3 \bar{u}_s^{-1} d\Gamma_s\right) -$$

$$- \frac{\partial}{\partial n_s}\left[\int_{\Gamma_s^0} (\vec{Y}_s \cdot \vec{u}_s) \bar{u}_s^{-1} d\Gamma_s\right] + Y_s^n + \gamma_s \mu_e (\vec{u}_s \times \vec{H})^n , \qquad \text{(II.3.15)}$$

where $(\vec{u}_s \times \vec{H})^n$ is the component of $\vec{u}_s \times \vec{H}$ in the $\hat{n}_s$ direction. Solving Eq. (II.3.15) for ω_s^3 we have:

$$\omega_s^3 = \bar{u}_s^{-1} \left\{ -\frac{\partial H_{s0}}{\partial n_s} + \frac{\partial Q_{s0}}{\partial n_s} + \gamma_s \frac{\partial \eta_0}{\partial n_s} - \gamma_s E^3 \frac{\partial}{\partial n_s}\left(\int_{\Gamma_s^0} u_s^3 \bar{u}_s^{-1} d\Gamma_s\right) - \right.$$

$$\left. - \frac{\partial}{\partial n_s}\left[\int_{\Gamma_s^0} (\vec{Y}_s \cdot \vec{u}_s) \bar{u}_s^{-1} d\Gamma_s\right] + Y_s^n + \gamma_s \mu_e (\vec{u}_s \times \vec{H}) + u_s^3 \frac{\partial u_s^3}{\partial n_s} \right\} . \qquad \text{(II.3.16)}$$

Transforming from (Γ_s, n_s) variables to (Γ_s, ψ_s) variables, we have, using Eqs. (II.2.4), (II.2.8) and the relations: $\frac{\partial x^1}{\partial n_s} = -u_s^2 \bar{u}_s^{-1}$, $\frac{\partial x^2}{\partial n_s} = u_s^1 \bar{u}_s^{-1}$:

$$\frac{\partial \psi_s}{\partial n_s} = \frac{\partial \psi_s}{\partial x^1}\frac{\partial x^1}{\partial n_s} + \frac{\partial \psi_s}{\partial x^2}\frac{\partial x^2}{\partial n_s} = \bar{\rho}_s \bar{u}_s . \qquad \text{(II.3.17)}$$

Using Eq. (II.3.17) we have:

$$\frac{\partial}{\partial n_s} = \frac{\partial}{\partial \psi_s}\frac{\partial \psi_s}{\partial n_s} = \bar{\rho}_s \bar{u}_s \frac{\partial}{\partial \psi_s} \quad . \qquad \text{(II.3.18)}$$

Eq. (II.3.16) becomes, using Eq. (II.3.18):

$$\omega_s^3 = \bar{\rho}_s \Big\{ -\frac{dH_{s0}}{d\psi_s} + \frac{dQ_{s0}}{d\psi_s} + \gamma_s \frac{d\eta_0}{d\psi_s} - \gamma_s E^3 \frac{\partial}{\partial \psi_s}\Big(\int_{\Gamma_0}^{\Gamma_s} u_s^3 \bar{u}_s^{-1} d\Gamma_s\Big) -$$

$$- \frac{\partial}{\partial \psi_s}\Big[\int_{\Gamma_0}^{\Gamma_s} (\vec{Y}_s \cdot \vec{u}_s)\bar{u}_s^{-1}\, d\Gamma_s\Big] + \bar{\rho}_s^{-1}\bar{u}_s^{-1}\Big[Y_s^n + \gamma_s \mu_e (\vec{u}_s \times \vec{H})^n\Big] +$$

$$+ u_s^3 \frac{\partial u_s^3}{\partial \psi_s} \Big\} . \qquad \text{(II.3.19)}$$

Eqs. (II.3.16), (II.3.19) are two possible forms of the generalized Crocco equation. Transformong from (Γ_s, n_s) variables to (x^1, x^2) variables, we have, using Eqs. (II.2.4), (II.2.8):

$$\frac{\partial}{\partial n_s} = \frac{\partial x^1}{\partial n_s}\frac{\partial}{\partial x^1} + \frac{\partial x^2}{\partial n_s}\frac{\partial}{\partial x^2} = \bar{\rho}_s^{-1}\bar{u}_s^{-1}\Big(\frac{\partial \psi_s}{\partial x^1}\frac{\partial}{\partial x^1} + \frac{\partial \psi_s}{\partial x^2}\frac{\partial}{\partial x^2}\Big) , \qquad \text{(II.3.20)}$$

and for the component μ_s^n of an arbitrary vector, $\vec{\mu}_s$:

$$\mu_s^n = \bar{\rho}_s^{-1}\bar{u}_s^{-1}\Big(\frac{\partial \psi_s}{\partial x^1}\mu_s^1 + \frac{\partial \psi_s}{\partial x^2}\mu_s^2\Big) , \qquad \text{(II.3.21)}$$

where μ_s^1, μ_s^2 are the components of $\vec{\mu}_s$, in the directions x^1, x^2 respectively. Using Eqs. (II.1.17), (II.2.4), (II.3.20), (II.3.21) in Eq. (II.3.19) we get:

$$\omega_s^3 = \bar{u}_s^{-1}\left(-\frac{\partial H_{s0}}{\partial n_s} + \frac{\partial Q_{s0}}{\partial n_s} + \gamma_s \frac{\partial \eta_0}{\partial n_s}\right) + \bar{\rho}_s^{-1}(\bar{u}_s)^{-2} \sum_{j=1}^{2} \left(\frac{\partial \psi_s}{\partial x^j}\right)\Big\{-\gamma_s E^3 \cdot$$

$$\cdot \frac{\partial}{\partial x^j}\left(\int_{0}^{\Gamma_s} u_s^3 \bar{u}_s^{-1} \, d\Gamma_s\right) - \frac{\partial}{\partial x^j}\left[\int_{0}^{\Gamma_s} (\vec{Y}_s \cdot \vec{u}_s)\bar{u}_s^{-1} \, d\Gamma_s\right] + Y_s^j -$$

$$- \gamma_s \mu_e \left(\bar{\rho}_s^{-1} H^3 \frac{\partial \psi_s}{\partial x^j} + u_s^3 \frac{\partial \xi}{\partial x^j}\right) + u_s^3 \frac{\partial u_s^3}{\partial x^j} \Big\} . \qquad \text{(II.3.22)}$$

Eq. (II.3.22) is the form of the generalized Crocco equation which will be used in the present work.

II.4. The Final Quasi-Three-Dimensional System

There are two ways in which the governing system of equations can be formulated, depending on whether one chooses the quasi-stream-function or the quasi-potential function as the unknown to be solved for. In the present work, the first formulation is given and dealt with.

The quasi-stream function equation, (II.2.7), becomes, using Eq. (II.3.22) and rearranging terms:

$$\left[1 - (u_s^1 \alpha_s^{-1})^2\right] \frac{\partial^2 \psi_s}{(\partial x^1)^2} - 2 u_s^1 u_s^2 \alpha_s^{-2} \frac{\partial^2 \psi_s}{\partial x^1 \partial x^2} + \left[1 - (u_s^1 \alpha_s^{-1})\right] \frac{\partial^2 \psi_s}{(\partial x^2)^2} =$$

$$- \bar{\rho}_s \bar{u}_s^{-1}\left(-\frac{\partial H_{s0}}{\partial n_s} + \frac{\partial Q_{s0}}{\partial n_s}\right) + \sum_{j=1}^{2} \left(\frac{\partial \psi_s}{\partial x^j}\right)\Big\{ \frac{\partial}{\partial x^j}\left[\log(1 - B_s)\right] +$$

$$+ (\alpha_s)^{-2}\left[(K_s - 1)(1 + W_s)(1 + A_s)^{-1} \frac{\partial Q_s}{\partial x^j} + \gamma_s \frac{\partial \eta}{\partial x^j} - L_s^j\right] + (\bar{u}_s)^{-2} \Big[\!\Big[\gamma_s \frac{\partial \eta_0}{\partial x^j} +$$

$$+\gamma_s E^3 \frac{\partial}{\partial x^j}\left(\int_{\Gamma_s}^{0} u_s^3 \bar{u}_s^{-1} d\Gamma_s\right) + \sigma_s \rho_s^{-1} u_s^j - u_s^3 \frac{\partial u_s^3}{\partial x^j} +$$

$$+\left[1-(\bar{u}_s \alpha_s^{-1})^2\right]\left(\gamma_s \mu_e u_s^3 \frac{\partial \xi}{\partial x^j} + \gamma_s \mu_e \bar{\rho}_s^{-1} H^3 \frac{\partial \psi_s}{\partial x^j} -\right.$$

$$\left.\left.\left. -\rho_s^{-1} \sigma_s Z_s^j + \rho_s^{-1} X_s^j\right)\right]\right\} . \qquad \text{(II.4.1)}$$

Multiplying Eq. (II.1.4) by $1-B_s$ and using Eqs. (I.3.2), (II.1.1), (II.1.17), (II.2.3), (II.2.4), (II.2.26), we get:

$$\frac{\partial \psi_s}{\partial x^2}\frac{\partial u_s^3}{\partial x^1} - \frac{\partial \psi_s}{\partial x^1}\frac{\partial u_s^3}{\partial x^2} = \bar{\rho}_s \gamma_s E^3 + \gamma_s\left(\frac{\partial \psi_s}{\partial x^2}\frac{\partial \xi}{\partial x^1} - \frac{\partial \psi_s}{\partial x^1}\frac{\partial \xi}{\partial x^2}\right) +$$

$$+(1-B_s)\left[\sigma_s(Z_s^3 - u_s^3) - X_s^3\right] . \qquad \text{(II.4.2)}$$

Eq. (II.1.12) becomes, using Eqs. (I.3.2), (I.3.6), (II.2.3), (II.2.4) and the relation $d\psi_s = \frac{\partial \psi_s}{\partial x^1}dx^1 + \frac{\partial \psi_s}{\partial x^2}dx^2$:

$$H^3 = H_0^3 + \sum_{s=1}^{n} \gamma_s \int_0 (1-B_s)^{-1} d\psi_s . \qquad \text{(II.4.3)}$$

It may be worthwhile to notice that in the case of a fully ionized plasma ($n=2$, $B_s \equiv 0$) Eq. (II.4.3) can be integrated and one gets:

$$H^3 = H_0^3 + \sum_{s=1}^{2} \gamma_s(\psi_s - \psi_{s0}) . \qquad \text{(II.4.4)}$$

Using Eqs. (I.3.2), (I.3.5) in Eq. (II.1.16) we get:

$$\frac{\partial^2 \eta}{(\partial x^1)^2} + \frac{\partial^2 \eta}{(\partial x^2)^2} = \epsilon^{-1} \sum_{s=1}^{n} \gamma_s \rho_s \; . \qquad \text{(II.4.5)}$$

Eq. (II.1.18) becomes, using Eqs. (I.3.2), (I.3.6):

$$\frac{\partial^2 \xi}{(\partial x^1)^2} + \frac{\partial^2 \xi}{(\partial x^2)^2} = \sum_{s=1}^{n} \gamma_s \rho_s u_s^3 \; . \qquad \text{(II.4.6)}$$

Eqs. (I.2.1), (I.2.59), (II.3.4), (II.4.1), (II.4.2), (II.4.4), (II.4.5), (II.4.6), are considered as a system of 5n+3 equations for the 5n+3 unknowns T_s, p_s, ρ_s, ψ_s, u_s^3, H^3, η, ξ.

II.5. Incompressible Flow

We distinguish between the following two cases of incompressible flows:

(a) Gross fluid is incompressible (ρ = constant). In this case the component fluids' densities, ρ_s, need not be constant, only their sum is a constant, since $\rho = \sum_{s=1}^{n} \rho_s$.

(b) Each fluid component is incompressible (ρ_s = constant; s = 1,2,...,n). In this case the gross fluid is incompressible. In the present work only case (b) is considered.

To derive a quasi-stream function equation for this type of flow we start from Eq. (II.2.21)*, and using Eqs.

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* Note that $\bar{\rho}_s$ is not a constant in the incompressible flow due to existence of mass sources or sinks, σ_s, in this type of flow.

(II.2.3), (II.2.4) and the condition ρ_s=constant, we have:

$$\frac{\partial^2\psi_s}{(\partial x^1)^2} + \frac{\partial^2\psi_s}{(\partial x^2)^2} = -\bar{\rho}_s\omega_s^3 + \sum_{j=1}^{2}\left(\frac{\partial\psi_s}{\partial x^j}\right)\frac{\partial}{\partial x^j}\left[\log\left(1-B_s\right)\right]. \qquad \text{(II.5.1)}$$

Equation (II.5.1) is the quasi-stream function equation for the incompressible flow. ω_s^3 will be calculated from the generalized Crocco equation which was derived in Section 3.

The quasi-potential equation is obtained from Eq. (II.1.1), using the condition ρ_s=constant and Eq. (II.2.23):

$$\frac{\partial^2\varphi_s}{(\partial x^1)^2} + \frac{\partial^2\varphi_s}{(\partial x^2)^2} + \frac{\partial g_s}{\partial x^1} + \frac{\partial g_s}{\partial x^2} = \sigma_s\rho_s^{-1} \quad , \qquad \text{(II.5.2)}$$

where g_s is calculated through Eq. (II.2.24) and one of the forms of the generalized Crocco equation which was given before.

In the single fluid formulation it is possible to obtain the stream function equation and the potential equation for the incompressible flow, from the stream function equation and the potential equation for the compressible flow, by setting the velocity of sound in each of those equations to be ∞. this result carries on to the present type of flow, for if we set $\alpha_s=\infty$ (s=1,2,...n) in Eqs. (II.2.27), (II.2.28) we obtain Eqs. (II.5.1), (II.5.2) respectively.

When a formulation with the quasi-stream function as an unknown is chosen, in addition to the quasi - stream function equation, (II.5.1), one has to consider in the incompressible flow Eqs. (II.4.2), (II.4.3), (II.4.5), (II.4.6), and n Bernoulli's equations, with the accompanying condition ρ_s = constant. In order that the system of equations for the incompressible flow be specified, there is a need for n additional equations. Those must be supplied by means of n given relations between the pressures, p_s, and the temperatures, T_s, for each fluid component, given, for example, in the form of the equations:

$$f_s(p_s,T_s) = 0; \qquad s=1,2,\ldots,n. \qquad \text{(II.5.3)}$$

Note that Eq. (II.4.5) in the present case contains a known constant on the right hand side, thus, if proper boundary conditions are given for η, this equation "decouples" from the system of equations and one may solve Eq. (II.4.5) for η first, and use the result in the quasi-stream function equation where η appears.

II.6. Non-dimensionalization

We introduce the following non-dimensional quantities, denoted by the symbol $\sim$:

$$x^i = \tilde{x}^i L \ ; \quad n_s = \tilde{n}_s L \ ; \quad \Gamma_s = \tilde{\Gamma}_s L \ , \tag{II.6.1}$$

$$u_s^i = \tilde{u}_s^i u_{s\infty} \ ; \ \bar{u}_s = \tilde{\bar{u}}_s u_{s\infty} \ , \tag{II.6.2}$$

$$\xi = \tilde{\xi} H_\infty L \ ; \ H^i = \tilde{H}^i H_\infty \ , \tag{II.6.3}$$

$$\eta = \tilde{\eta} E_\infty L \ ; \ E^i = \tilde{E}^i E_\infty \ , \tag{II.6.4}$$

$$\rho_s = \tilde{\rho}_s \rho_{s\infty} \ ; \quad p_s = \tilde{p}_s \rho_{s\infty} (u_{s\infty})^2 \ ; \quad T_s = \tilde{T}_s T_{s\infty} \ ; \ \bar{\rho}_s = \tilde{\bar{\rho}}_s \rho_{s\infty} \ , \tag{II.6.5}$$

$$\psi_s = \tilde{\psi}_s \rho_{s\infty} u_{s\infty} L \ ; \quad \varphi_s = \tilde{\varphi}_s u_{s\infty} L \ ; \ g_s = \tilde{g}_s u_{s\infty} \ , \tag{II.6.6}$$

$$Q_s = \tilde{Q}_s (u_{s\infty})^2 \ ; \quad H_s = H_{s\infty} (u_{s\infty})^2 \ , \tag{II.6.7}$$

$$\omega_s^3 = \tilde{\omega}_s^3 u_{s\infty} L^{-1} \ , \tag{II.6.8}$$

$$\sigma_s = \tilde{\sigma}_s \rho_{s\infty} u_{s\infty} L^{-1} \ ; \ X_s^i = \tilde{X}_s^i \rho_{s\infty} (u_{s\infty})^2 L^{-1} \ ;$$

$$Z_s^i = \tilde{Z}_s^i u_{s\infty} \ ; \quad Y_s^i = \tilde{Y}_s^i \rho_{s\infty} (u_{s\infty})^2 L^{-1} \ , \tag{II.6.9}$$

$$\alpha_s = \tilde{\alpha}_s \alpha_{s\infty} \ ; \quad L_s^i = \tilde{L}_s^i (u_{s\infty})^2 L^{-1} \ , \tag{II.6.10}$$

where $T_{s\infty}$, $\rho_{s\infty}$, $p_{s\infty}$, $u_{s\infty}$, H_∞ , E_∞ , $\alpha_{s\infty}$, L are some standard temperatures, densities, pressures, velocities, magnetic and electric fields, velocities of sound and length respectively.

Using Eqs. (II.6.1) to (II.6.10) in Eqs. (II.4.1), (II.4.2), (II.4.3), (II.4.5), (II.4.6) we get:

$$\left[1-(\tilde{u}_s^1\tilde{\alpha}_s^{-1}M_{s\infty})^2\right]\frac{\partial^2\psi_s}{(\partial\tilde{x}^1)^2} - 2\tilde{u}_s^1\tilde{u}_s^2\left(\tilde{\alpha}_s^{-1}M_{s\infty}\right)^2\frac{\partial^2\psi_s}{\partial\tilde{x}^1\partial\tilde{x}^2} +$$

$$+\left[1-(\tilde{u}_s^2\tilde{\alpha}_s^{-1}M_{s\infty})^2\right]\frac{\partial^2\psi_s}{(\partial\tilde{x}^2)^2} = \tilde{\rho}_s(\tilde{u}_s)^{-1}\left(-\frac{\partial\tilde{H}_{s0}}{\partial\tilde{n}_s}+\frac{\partial\tilde{Q}_{s0}}{\partial\tilde{n}_s}\right) +$$

$$+\sum_{j=1}^{2}\left(\frac{\partial\tilde{\psi}_s}{\partial\tilde{x}^j}\right)\Bigg\{\frac{\partial}{\partial\tilde{x}^j}\left[\log(1-B_s)\right] + \left(\tilde{\alpha}_s^{-1}M_{s\infty}\right)^2\Big[(K_s-1)(1+W_s)(1+A_s)^{-1}\frac{\partial\tilde{Q}_s}{\partial\tilde{x}^j} +$$

$$+R_{ms}R_{H_s}R_{E_s}\frac{\partial\tilde{\eta}}{\partial\tilde{x}^j} - \tilde{L}_s^j\Big] + (\tilde{u}_s)^{-2}\Big[\!\Big[R_{ms}R_{H_s}R_{E_s}\frac{\partial\tilde{\eta}_0}{\partial\tilde{x}^j} +$$

$$+R_{ms}R_{H_s}R_{E_s}^3\frac{\partial}{\partial\tilde{x}^j}\left(\int_{\tilde{\Gamma}_s}^{0}\tilde{u}_s^3\tilde{u}_s^{-1}\,d\tilde{\Gamma}_s\right) + \frac{\partial}{\partial\tilde{x}^j}\left[\int_{\tilde{\Gamma}_s}^{0}\left(\sum_{i=1}^{3}\tilde{Y}_s^i\tilde{u}_s^i\right)\tilde{u}_s^{-1}\,d\tilde{\Gamma}_s\right] +$$

$$+\tilde{\sigma}_s\tilde{\rho}_s^{-1}\tilde{u}_s^j - \tilde{u}_s^3\frac{\partial\tilde{u}_s^3}{\partial\tilde{x}^j} + \left[1-(\tilde{u}_s\tilde{\alpha}_s^{-1}M_{s\infty})^2\right]\Big(R_{ms}R_{H_s}\tilde{u}_s^3\frac{\partial\tilde{\xi}}{\partial\tilde{x}^j} +$$

$$+R_{ms}R_{H_s}\tilde{\rho}_s^{-1}\tilde{H}^3\frac{\partial\tilde{\psi}_s}{\partial\tilde{x}^j} - \tilde{\rho}_s^{-1}\tilde{\sigma}_s\tilde{Z}_s^j + \tilde{\rho}_s^{-1}\tilde{X}_s^j\Big)\Big]\!\Big]\Bigg\}, \qquad \text{(II.6.11)}$$

$$\frac{\partial\tilde{\psi}_s}{\partial\tilde{x}^2}\frac{\partial\tilde{u}_s^3}{\partial\tilde{x}^1} - \frac{\partial\tilde{\psi}_s}{\partial\tilde{x}^1}\frac{\partial\tilde{u}_s^3}{\partial\tilde{x}^2} = R_{ms}R_{H_s}R_{E_s}^3\tilde{\rho}_s + R_{ms}R_{H_s}\left(\frac{\partial\tilde{\psi}_s}{\partial\tilde{x}^2}\frac{\partial\tilde{\xi}}{\partial\tilde{x}^1} -\right.$$

$$\left.-\frac{\partial\tilde{\psi}_s}{\partial\tilde{x}^1}\frac{\partial\tilde{\xi}}{\partial\tilde{x}^2}\right) + (1-B_s)\left[\tilde{\sigma}_s(\tilde{Z}_s^3-\tilde{u}_s^3) - \tilde{X}_s^3\right], \qquad \text{(II.6.12)}$$

$$\tilde{H}^3 = \tilde{H}_0^3 + \sum_{s=1}^{n}R_{ms}\int_0(1-B_s)^{-1}\,d\tilde{\psi}_s, \qquad \text{(II.6.13)}$$

$$\nabla^2\tilde{\eta} = \sum_{s=1}^{n}R_{ms}R_{E_s}^{-1}R_{cs}^{-1}\tilde{\rho}_s; \quad \nabla^2 = \frac{\partial^2}{(\partial\tilde{x}^1)^2}+\frac{\partial^2}{(\partial\tilde{x}^2)^2}, \qquad \text{(II.6.14)}$$

$$\nabla^2\tilde{\xi} = \sum_{s=1}^{n}R_{ms}\tilde{\rho}_s\tilde{u}_s^3, \qquad \text{(II.6.15)}$$

where $M_{s\infty}$, R_{ms}, R_{H_s}, R_{cs}, R_{E_s}, are given by:

$$M_{s\infty} = u_{s\infty} \alpha_{s\infty}^{-1}, \qquad \text{(Mach number)} \tag{II.6.16}$$

$$R_{ms} = e_s \nu_{s\infty} u_{s\infty} L H_\infty^{-1}, \qquad \text{(modified magnetic Reynolds number)} \tag{II.6.17}$$

$$R_{H_s} = \mu_e (H_\infty)^2 \rho_{s\infty}^{-1} u_{s\infty}^{-2}, \qquad \text{(magnetic pressure number)} \tag{II.6.18}$$

$$R_{cs} = \left(u_{s\infty} c^{-1}\right)^2 = \mu_e \epsilon \left(u_{s\infty}\right)^2, \qquad \text{(relativity parameter)} \tag{II.6.19}$$

$$R_{E_s} = E_\infty \mu_e^{-1} u_{s\infty}^{-1} H_\infty^{-1}, \qquad \text{(electrical field parameter)} \tag{II.6.20}$$

$$R_{E_s^3} = R_{E_s} E_\infty^3 E_\infty^{-1}, \tag{II.6.21}$$

where c is the velocity of light, and E_∞^i are the components of the standard electric field vector, $\vec{E}_\infty$. Using Eq. (II.1.9) we choose $E^3 = E_\infty^3$.

The corresponding incompressible equations to Eqs. (II.6.11) to (II.6.15) are obtained by setting $M_{s\infty} = 0$, $\tilde{\rho}_s = 1$ in those equations.

II.7. Linearization

We shall treat the problem of flow under uniformly applied electric and magnetic fields such that a first order small perturbation theory is sufficient to describe the flow field in the neighbourhood of the origin of the disturbances in question.

The velocity vector of each one of the fluid components is assumed to have the following components:

$$u_s^1 = u_{s\infty}^1 + u_{sp}^1 \; ; \quad u_s^2 = u_{sp}^2 \; ; \quad u_s^3 = u_{sp}^3 \, , \qquad \text{(II.7.1)}$$

where $u_{s\infty}^1$ is the velocity of the undisturbed uniform flow of the s-th fluid component and u_{sp}^i (i=1,2,3) are the perturbed x^i components of the velocity respectively. We assume $u_{s\infty}^1 \gg u_{sp}^i$.

The externally applied magnetic field has the following components:

$$H^i \equiv H_\infty^i = \text{constant} \; ; \quad (i=1,2,3). \qquad \text{(II.7.2)}$$

Hence the resultant magnetic field has the components:

$$H^i = H_\infty^i + H_p^i \; ; \quad H_\infty^i \gg H_p^i \, . \qquad \text{(II.7.3)}$$

Similarly we have for the electric field*:

$$E^i = E_\infty^i + E_p^i \; ; \quad E_\infty^i \gg E_p^i \, . \qquad \text{(II.7.4)}$$

We also assume:

$$\rho_s = \rho_{s\infty} + \rho_{sp} \; ; \quad \rho_{s\infty} \gg \rho_{sp} \, . \qquad \text{(II.7.5)}$$

---

* From Eq. (II.1.9) we have $E_p^3 = 0$.

Eqs. (II.7.1), (II.7.3), (II.7.4), (II.7.5), written in non-dimensional form, become, using Eqs. (II.6.2) to (II.6.5):

$$\tilde{u}_s^1 = 1 + \tilde{u}_{sp}^1 \; ; \quad \tilde{u}_s^2 = \tilde{u}_{sp}^2 \; ; \quad \tilde{u}_s^3 = \tilde{u}_{sp}^3 \; , \tag{II.7.6}$$

$$\tilde{H}^i = \tilde{H}_\infty^i + \tilde{H}_p^i \; ; \quad \tilde{E}^i = \tilde{E}_\infty^i + \tilde{E}_p^i \; , \tag{II.7.7}$$

$$\tilde{\rho}_s = 1 + \tilde{\rho}_{sp} \; , \tag{II.7.8}$$

where $\tilde{u}_{sp}^i$, $\tilde{\rho}_{sp}$, $\tilde{H}_p^i$, $\tilde{E}_p^i$, $\tilde{H}_\infty^i$, $\tilde{E}_\infty^i$, are given by:

$$\tilde{u}_{sp}^i = u_{sp}^i u_{s\infty}^{-1} \; ; \tag{II.7.9}$$

$$\tilde{H}_p^i = H_p^i H_\infty^{-1} \; ; \quad \tilde{E}_p^i = E_p^i E_\infty^{-1} \; , \tag{II.7.10}$$

$$\tilde{H}_\infty^i = H_\infty^i H_\infty^{-1} \; ; \quad \tilde{E}_\infty^i = E_\infty^i E_\infty^{-1} \; . \tag{II.7.11}$$

The function $\tilde{\eta}$ can be split into the form:

$$\tilde{\eta} = \tilde{E}_\infty^1 \tilde{x}^1 + \tilde{E}_\infty^2 \tilde{x}^2 + \tilde{\eta}_p \; , \tag{II.7.12}$$

where $\tilde{\eta}_p$ represents the perturbation part for the electric field components in the (x^1, x^2) plane. Using Eqs. (II.1.15), (II.6.1), (II.6.4), (II.7.4), (II.7.7), (II.7.8), (II.7.11), (II.7.12) we get:

$$\tilde{E}^i_p = \frac{\partial \tilde{\eta}_p}{\partial \tilde{X}^i} \; ; \qquad (i=1,2). \qquad \text{(II.7.13)}$$

In a similar manner we split $\tilde{\xi}$:

$$\tilde{\xi} = -\tilde{H}^1_\infty \tilde{X}^2 + \tilde{H}^2_\infty \tilde{X}^1 + \tilde{\xi}_p \, , \qquad \text{(II.7.14)}$$

and get, using Eqs. (II.1.17), (II.6.1), (II.6.3), (II.7.3), (II.7.7), (II.7.10), (II.7.11), (II.7.14):

$$\tilde{H}^1_p = -\frac{\partial \tilde{\xi}_p}{\partial \tilde{X}^2} \; ; \quad \tilde{H}^2_p = \frac{\partial \tilde{\xi}_p}{\partial \tilde{X}^1} \, . \qquad \text{(II.7.15)}$$

We will assume conservation of mass of each fluid component in the undisturbed stream, i.e., no mass sources or sinks ($\sigma_{s\infty} = 0$ and, therefore, $B_{s\infty} = 0$). If we choose the non-dimensional quasi-stream function, $\tilde{\psi}_s$, in the form:

$$\tilde{\psi}_s = \tilde{X}^2 + \tilde{\psi}_{sp} \, , \qquad \text{(II.7.16)}$$

we get, using Eqs. (II.2.3), (II.2.4), (II.6.1), (II.6.2), (II.6.5), (II.6.6), (II.7.1), (II.7.6), (II.7.8), (II.7.16):

$$\frac{\partial \tilde{\psi}_{sp}}{\partial \tilde{X}^1} = -(1-B_s)\tilde{u}^2_{sp} \; ; \quad \frac{\partial \tilde{\psi}_{sp}}{\partial \tilde{X}^2} = (1-B_s)(\tilde{u}^1_{sp} + \tilde{\rho}_{sp}) - B_s . \qquad \text{(II.7.17)}$$

It may be assumed, since we are dealing with a first order small perturbation system, that the non-dimensional mass source of the s-th fluid component, $\tilde{\sigma}_s$, is small at least to the first order. It can be shown* that B_s is of the same order as $\tilde{\delta}_s$. Thus, neglecting second order terms in Eq. (II.7.17), we have:

$$\frac{\partial \tilde{\psi}_{sp}}{\partial \bar{x}^1} = -\tilde{u}^2_{sp} \; ; \qquad \frac{\partial \tilde{\psi}_{sp}}{\partial \bar{x}^2} = \tilde{u}^1_{sp} + \tilde{\rho}_{sp} - B_s . \qquad \text{(II.7.18)}$$

In order that the electromagnetic forces, $\vec{F}_{es}$, be small at least to the first order we assume that those forces vanish in the undisturbed stream. Using Eq. (II.2.26) we have:

$$\vec{E}_\infty + \mu_e(\vec{u}_{s\infty} \times \vec{H}_\infty) = 0 . \qquad \text{(II.7.19)}$$

The components of the velocity vectors in the undisturbed stream are ($u^1_{s\infty}$, 0, 0). The three component equations of Eq. (II.7.19) are, therefore:

$$E^1_\infty = 0 \; ; \quad E^2_\infty = \mu_e u^1_{s\infty} H^3_\infty \; ; \quad E^3_\infty = -\mu_e u^1_{s\infty} H^2_\infty ; \; s=1,2,\ldots,n. \qquad \text{(II.7.20)}$$

It follows from the last two equations given in Eq. (II.7.20) that either:

$$E^2_\infty \equiv H^3_\infty \equiv E^3_\infty \equiv H^2_\infty \equiv 0 , \qquad \text{(II.7.21)}$$

or:

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* Using Eq. (A.4) which is derived in Appendix A.

$$u_{1\infty} = u_{2\infty} = \ldots = u_{n\infty} = u_\infty \; , \qquad \text{(II.7.22)}$$

i.e., the velocities of the n fluid components in the undisturbed stream are equal.

Summarizing, from Eqs. (II.7.20) to (II.7.22) we have one of the following two possible cases:

(a) $u_{1\infty} = u_{2\infty} = \ldots = u_{n\infty} = u_\infty \; ; \quad E^1_\infty = 0 \; ; \quad E^2_\infty = \mu_e u_\infty H^3_\infty \; ;$

$$E^3_\infty = -\mu_e u_\infty H^2_\infty \; . \qquad \text{(II.7.23)}$$

(b) $E^1_\infty \equiv E^2_\infty \equiv E^3_\infty \equiv H^2_\infty \equiv H^3_\infty \equiv 0 \; .$ (II.7.24)

In the present work the case (a) is dealt with.

Using Eqs. (II.7.11), (II.7.23) in Eqs. (II.6.19), (II.6.20) we get:

$$R_{E_s} = \left[(\tilde{H}^2_\infty)^2 + (\tilde{H}^3_\infty)^2 \right]^{\frac{1}{2}} = \left[1 - (\tilde{H}^1_\infty)^2 \right]^{\frac{1}{2}} ; \quad R_{E^3_s} = -\tilde{H}^2_\infty . \qquad \text{(II.7.25)}$$

We assume that the equations of state and of the internal energy of each one of the n fluid components follow perfect gas laws in the undisturbed stream, i.e., $W_{s\infty} = A_{s\infty} = 0$. We also assume, since we are dealing with a first order small perturbation system, that W_s, A_s, are small quantities, at least to the first order, throughout the disturbed flow field. Similarly, it is assumed that $\tilde{X}^i_{s\infty} = 0$, and that $\tilde{X}^i_s$, $\tilde{\mathfrak{z}}_s$ $\tilde{Z}^i_s$ (i=1,2,3), $\frac{\partial Q_s}{\partial x^i}$

($j=1,2$), are small quantities, at least to the first order.

Inserting Eqs. (II.2.6), (II.3.9), (II.7.1) into Eq. (II.3.8) and using Eqs. (II.6.1), (II.6.2), (II.6.4), (II.6.7), (II.6.9), (II.6.10) we get:

$$\frac{1}{2}(\tilde{u}_{sp})^2 + \tilde{u}^1_{sp} + (\tilde{\alpha}_s M_{s\infty})^2(1+P_s)(K_s-1)^{-1} - \mathcal{J}_s =$$

$$= (M_{s\infty})^{-2}(K_s-1)^{-1}, \qquad \text{(II.7.26)}$$

where $\tilde{u}_{sp}$, $\mathcal{J}_s$ are given by:

$$(\tilde{u}_{sp})^2 = \sum_{j=1}^{3}(\tilde{u}^j_{sp})^2 , \qquad \text{(II.7.27)}$$

$$\mathcal{J}_s = \int_\infty d\tilde{Q}_s + R_{ms}R_{E_s}\int_\infty \sum_{i=1}^{3} \tilde{E}^i d\tilde{x}^i + \int_\infty \sum_{i=1}^{3} \tilde{\rho}_s^{-1}\left[\tilde{X}^i_s + \tilde{\sigma}_s(\tilde{Z}^i_s - \tilde{u}^i_s)\right]d\tilde{x}^i. \qquad \text{(II.7.28)}$$

Note that P_s, which is given in Eq. (II.3.7), vanishes in the undisturbed stream, since W_s, A_s do. Solving Eq. (II.7.26) for $(\tilde{\alpha}_s)^{-2}$ we have:

$$(\tilde{\alpha}_s)^{-2} = (1+P_s)\left\{1+(K_s-1)(M_{s\infty})^2\left[\mathcal{J}_s - \tilde{u}^1_{sp} - \frac{1}{2}(\tilde{u}_{sp})^2\right]\right\}^{-1}. \qquad \text{(II.7.29)}$$

We assume that $(K_s-1)(M_{s\infty})^2\,\tilde{u}^1_{sp}$, $\frac{1}{2}(K_s-1)(M_{s\infty})^2(\tilde{u}_{sp})^2 \ll 1$, thus neglecting them in Eq. (II.7.29) we get:

$$(\tilde{\alpha}_s)^{-2} = (1+P_s)\left[1+(K_s-1)(M_{s\infty})^2\mathcal{J}_s\right]^{-1}. \qquad \text{(II.7.30)}$$

Furthermore, we will approximate P_s, $\mathcal{J}_s$ by chosen

constant mean values $\bar{P}_s$, $\bar{\mathcal{J}}_s$ respectively, thus having in Eq. (II.7.30):

$$(\tilde{\alpha}_s)^{-2} \approx (1+\bar{P}_s)\left[1+(K_s-1)(M_{s\infty})^2 \bar{\mathcal{J}}_s\right]^{-1} = \text{constant.} \qquad \text{(II.7.31)}$$

Using Eqs. (I.3.2), (I.3.6), (II.1.17), (II.2.3), (II.2.4), (II.6.5), (II.6.6), (II.6.16) to (II.6.18), (II.6.20), (II.6.21), (II.7.1), (II.7.5) to (II.7.8), (II.7.11), (II.7.12), (II.7.14), (II.7.16), (II.7.17), (II.7.23), (II.7.25) in Eqs. (II.6.11) to (II.6.15), noting that $Q_{s\infty}$, $H_{s\infty}$ are universally constants, i.e., $\frac{\partial Q_{s\infty}}{\partial n_s} = \frac{\partial H_{s\infty}}{\partial n_s} = 0$, neglecting second and higher order terms, assuming $\frac{\partial \tilde{\eta}_\infty}{\partial \tilde{x}^1}$ is small at least to the first order, $\frac{\partial \tilde{\eta}_\infty}{\partial \tilde{x}^2} = \tilde{E}^2_\infty$, and making the following changes in notation: $\tilde{x}^1 = x$, $\tilde{x}^2 = y$, $\tilde{\psi}_{sp} = \psi_s$, $\tilde{u}^3_{sp} = w_s$, $\tilde{H}^3_p = H_z$, $\tilde{\eta}_p = \eta$, $\tilde{\xi}_p = \xi$, $\tilde{\rho}_{sp} = \rho_s$, $\tilde{\sigma}_s = \sigma_s$, $\tilde{Z}^1_s = Z_{sx}$, $\tilde{Z}^2_s = Z_{sy}$, $\tilde{Z}^3_s = Z_{sz}$, $\tilde{X}^1_s = X_{sx}$, $\tilde{X}^2_s = X_{sy}$, $\tilde{X}^3_s = X_{sz}$, $\tilde{Q}_s = Q_s$, $\tilde{L}^1_s = L_{sx}$, $\tilde{L}^2_s = L_{sy}$, we get*:

$$\beta^2_{s\infty}\frac{\partial^2\psi_s}{\partial x^2} + \frac{\partial^2\psi_s}{\partial y^2} = \beta^2_{s\infty} R_{ms} R_{H^3_s}\frac{\partial\psi_s}{\partial y} - \beta^2_{s\infty} R_{ms} R_{H^1_s} w_s - R_{ms} R_{H^2_s}\frac{\partial}{\partial y}\left(\int_{\infty}^{r_s} w_s\,dx\right) +$$

$$+ \beta^2_{s\infty} R_{ms} R_{H_s} H_z + R_{ms} R_{H_s} R_{E_s}(1-\beta^2_{s\infty})\frac{\partial\eta}{\partial y} + G_s, \qquad \text{(II.7.32)}$$

$$\frac{\partial w_s}{\partial x} = R_{ms} R_{H^1_s}\frac{\partial\psi_s}{\partial x} + R_{ms} R_{H^2_s}\frac{\partial\psi_s}{\partial y} + R_{ms} R_{H_s}\frac{\partial\xi}{\partial x} + R_{ms} R_{H^2_s} B_s \; +$$

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* In order that the equations which were derived above comply with the linearized case, the 0 subscripted state is taken at the undisturbed stream, i.e., the subscripts 0 and ∞ are assumed to be equivalent.

$$+\sigma_s Z_{sz} - X_{sz} \,, \tag{II.7.33}$$

$$H_z = \sum_{r=1}^{n} R_{mr}\left[\psi_r + \int_\infty (1-B_r)^{-1} dy\right]; \quad \frac{\partial H_z}{\partial x} = \sum_{r=1}^{n} R_{mr} \frac{\partial \psi_r}{\partial x} \,, \tag{II.7.34}$$

$$\nabla^2 \eta = \sum_{r=1}^{n} R_{mr} R_{E_r}^{-1} R_{cr}^{-1} (1+\rho_r) \,, \tag{II.7.35}$$

$$\nabla^2 \xi = \sum_{r=1}^{n} R_{mr} w_r \,, \tag{II.7.36}$$

where $\beta_{s\infty}$, $R_{H_s^i}$, G_s are given by:

$$\beta_{s\infty}^2 = 1 - (\tilde{\alpha}_s^{-1} M_{s\infty})^2 \,, \tag{II.7.37}$$

$$R_{H_s^i} = R_{H_s} \tilde{H}_\infty^i = \mu_e H_\infty H_\infty^i \rho_{s\infty}^{-1} (u_{s\infty})^{-2} \,, \tag{II.7.38}$$

$$G_s = -\frac{\partial B_s}{\partial y} + \beta_{s\infty}^2 R_{ms} R_{H_s^3} B_s - \beta_{s\infty}^2 (\sigma_s Z_{sy} - X_{sy}) + \frac{\partial}{\partial y}\left\{\int_{\infty, \Gamma_s} [\sigma_s(Z_{sx} - 1) - X_{sx}]\, dx\right\} + (1-\beta_{s\infty}^2)\left[(K_s - 1)(1+W_s)(1+A_s)^{-1} \frac{\partial Q_s}{\partial y} - L_{sy}\right]. \tag{II.7.39}$$

In the incompressible case $\rho_r \equiv 0$, $(r=1,2,\ldots,n)$. Taking the derivative of Eq. (II.7.35) once with respect to x and then with respect to y, and commuting differentiation signs we have:

$$\nabla^2\left(\frac{\partial \eta}{\partial x}\right) = 0 \;; \qquad \nabla^2\left(\frac{\partial \eta}{\partial y}\right) = 0 \,. \tag{II.7.40}$$

Assuming that the perturbed electric field, E_p^i, vanishes at large distances from the origin of disturbances, we obtain, using Eqs. (II.7.10), (II.7.13), $\frac{\partial\eta}{\partial x}, \frac{\partial\eta}{\partial y} = 0$ at large distances. Using this result and Eq. (II.7.40) we have:

$$\frac{\partial\eta}{\partial x} = \frac{\partial\eta}{\partial y} = 0 \quad \text{everywhere,} \qquad \text{(II.7.41)}$$

i.e., in the incompressible case, the perturbed electric field vanishes identically everywhere. We assume in the compressible case, that the right hand side of Eq. (II.7.35) is approximately a constant and, therefore, the same result is obtained for this case.

CHAPTER III

SIMILARITY OF FLOWS

III.1. Approximate Governing System of Equations

To obtain similarity rules we apply, additionally to the linearization, a procedure which could be called an equalization of the order of terms, i.e., we will further simplify the governing system of equations neglecting very small terms.

Let $m_\alpha = \min\limits_{r=1,\dots,n} (m_r)$ and assume furthermore that*:

$$m_\alpha m_r^{-1} \ll 1 \; ; \quad r=1,2,\dots,n \; ; \quad r \neq \alpha \; ; \quad e_\alpha \neq 0 \, . \qquad \text{(III.1.1)}$$

In addition we assume:

$$R_{m\alpha}^{-2} R_{H_\alpha}^{-1} = c^2 \Omega_\alpha^{-2} L^{-2} \ll 1 \, , \qquad \text{(III.1.2)}$$

where Ω_α is obtained by using Eqs. (II.6.17), (II.6.18) in Eq. (III.1.2):

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* The α-th fluid component may be the electron fluid.

$$\Omega_\alpha = e_\alpha \left(\nu_{\alpha\infty} m_\alpha^{-1} \epsilon^{-1} \right)^{\frac{1}{2}} . \tag{III.1.3}$$

Ω_α is called the plasma frequency of the α-th species*.

Assuming that the number density of each fluid component at ∞ is the same, we have, using Eqs. (II.6.17), (II.6.18), (II.7.38):

$$R_{m\tau} R_{m\alpha}^{-1} = e_\tau e_\alpha^{-1} \; ; \quad R_{H_\tau} R_{H_\alpha}^{-1} = m_\alpha m_\tau^{-1} \; ;$$

$$R_{H_\tau^i} R_{H_\alpha}^{-1} = m_\alpha m_\tau^{-1} H_\infty^i H_\infty^{-1} \; ; \quad (i = 1, 2, 3 \; ; \; \tau = 1, 2, \ldots, n). \tag{III.1.4}$$

Taking the derivative of Eq. (II.7.36) with respect to x, substituting Eq. (II.7.33) into the resulting equation, dividing by $R_{m\alpha}^2 R_{H_\alpha}$, using Eq. (III.1.4), commuting the operators ∇^2 and $\frac{\partial}{\partial x}$ and rearranging, we get:

$$R_{m\alpha}^{-2} R_{H_\alpha}^{-1} \nabla^2 \left(\frac{\partial \xi}{\partial x} \right) - \left(\frac{\partial \xi}{\partial x} \right) \sum_{\tau=1}^{n} \left(e_\tau^2 e_\alpha^{-2} m_\alpha m_\tau^{-1} \right) =$$

$$= \sum_{\tau=1}^{n} \Big\{ e_\tau^2 e_\alpha^{-2} m_\alpha m_\tau^{-1} \Big[H_\infty^1 H_\infty^{-1} \frac{\partial \psi_\tau}{\partial x} + H_\infty^2 H_\infty^{-1} \frac{\partial \psi_\tau}{\partial y} + H_\infty^2 H_\infty^{-1} B_\tau \Big] +$$

$$+ e_\tau e_\alpha^{-1} R_{m\alpha} R_{H_\alpha} \left(\sigma_\tau Z_{\tau z} - X_{\tau z} \right) \Big\} . \tag{III.1.5}$$

Neglecting small terms in Eq. (III.1.5), following the assumptions made in Eqs. (III.1.1), (III.1.2), we have:

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* Similar assumptions to the ones above can be found in (10, 11).

$$\frac{\partial \xi}{\partial x} = -H^{1}_{\infty} H^{-1}_{\infty} \frac{\partial \psi_{\alpha}}{\partial x} - H^{2}_{\infty} H^{-2}_{\infty} \frac{\partial \psi_{\alpha}}{\partial y} - H^{2}_{\infty} H^{-1}_{\infty} B_{\alpha} -$$

$$-\sum_{t=1}^{n} e_t e_{\alpha}^{-1} R_{m\alpha} R_{H_{\alpha}} \left(\sigma_t Z_{tz} - X_{tz} \right). \qquad \text{(III.1.6)}$$

Inserting Eq. (III.1.6) into Eq. (II.7.33), using Eq. (II.7.38) and rearranging furnishes:

$$\frac{\partial w_s}{\partial x} = R_{ms} R_{H^1_s} \left(\frac{\partial \psi_s}{\partial x} - \frac{\partial \psi_{\alpha}}{\partial x} \right) + R_{ms} R_{H^2_s} \left(\frac{\partial \psi_s}{\partial y} - \frac{\partial \psi_{\alpha}}{\partial y} \right) + R_{ms} R_{H^2_s} \left(B_s - B_{\alpha} \right) -$$

$$- e_s e_{\alpha}^{-2} m_{\alpha} m_s^{-1} \sum_{t=1}^{n} \left[e_t \left(\sigma_t Z_{tz} - X_{tz} \right) \right] + \sigma_s Z_{sz} - X_{sz}. \qquad \text{(III.1.7)}$$

Taking the derivative of Eq. (II.7.32) with respect to x, substituting Eq. (III.1.7) into the resulting equation and dividing by $R^{3}_{m\alpha} R^{2}_{H_{\alpha}}$, we have, using Eq. (III.1.4):

$$R_{m\alpha}^{-3} R_{H_{\alpha}}^{-2} \left(\beta^{2}_{s\infty} \frac{\partial^3 \psi_s}{\partial x^3} + \frac{\partial^3 \psi_s}{\partial x \, \partial y^2} \right) = \beta^{2}_{s\infty} e_s e_{\alpha}^{-1} H^{3}_{\infty} H^{-1}_{\infty} m_{\alpha} m_s^{-1} R_{m\alpha}^{-2} R_{H_{\alpha}}^{-1} \frac{\partial^2 \psi_s}{\partial x \, \partial y} -$$

$$- \beta^{2}_{s\infty} e_s^{2} e_{\alpha}^{-2} H^{1}_{\infty} H^{-1}_{\infty} m_{\alpha}^{2} m_s^{-2} \Big\{ H^{1}_{\infty} H^{-1}_{\infty} \left(\frac{\partial \psi_s}{\partial x} - \frac{\partial \psi_{\alpha}}{\partial x} \right) +$$

$$+ H^{2}_{\infty} H^{-1}_{\infty} \left(\frac{\partial \psi_s}{\partial y} - \frac{\partial \psi_{\alpha}}{\partial y} \right) + H^{2}_{\infty} H^{-1}_{\infty} \left(B_s - B_{\alpha} \right) -$$

$$- R_{m\alpha}^{-1} R_{H_{\alpha}}^{-1} \sum_{t=1}^{n} \left[e_t e_{\alpha}^{-1} \left(\sigma_t Z_{tz} - X_{tz} \right) \right] + R_{ms}^{-1} R_{H_s}^{-1} \left(\sigma_s Z_{sz} - X_{sz} \right) \Big\} -$$

$$- e_s e_{\alpha}^{-1} m_{\alpha} m_s^{-1} H^{2}_{\infty} H^{-1}_{\infty} R_{m\alpha}^{-2} R_{H_{\alpha}}^{-1} \frac{\partial^2}{\partial x \, \partial y} \left(\int_{\Gamma_s}^{\infty} w_s \, dx \right) +$$

$$+ \beta^{2}_{s\infty} e_s e_{\alpha}^{-2} m_{\alpha} m_s^{-1} R_{m\alpha}^{-1} R_{H_{\alpha}}^{-1} \sum_{t=1}^{n} \left(e_t \frac{\partial \psi_t}{\partial x} \right) +$$

$$+R_{m\alpha}^{-3} R_{H_\alpha}^{-2} \frac{\partial G_s}{\partial x} . \tag{III.1.8}$$

Neglecting small terms in Eq. (III.1.8), following the assumption made in Eq. (III.1.2) we have:

$$-\beta_{s\infty}^{2} R_{m\alpha}^{-1} e_s^{2} e_\alpha^{-2} H_\infty^{1} H_\infty^{-1} m_\alpha^{2} m_s^{-2} \Big\{ H_\infty^{1} H_\infty^{-1} \Big(\frac{\partial \psi_s}{\partial x} - \frac{\partial \psi_\alpha}{\partial x} \Big) +$$

$$+ H_\infty^{2} H_\infty^{-1} \Big(\frac{\partial \psi_s}{\partial y} - \frac{\partial \psi_\alpha}{\partial y} \Big) + H_\infty^{2} H_\infty^{-1} (B_s - B_\alpha) -$$

$$- R_{m\alpha}^{-1} R_{H_\alpha}^{-1} \sum_{t=1}^{n} \Big[e_t e_\alpha^{-1} (\sigma_t Z_{1t2} - X_{1t2}) \Big] + R_{ms}^{-1} R_{H_s}^{-1} (\sigma_t Z_{1t2} - X_{1t2}) \Big\} +$$

$$+ \beta_{s\infty}^{2} e_s e_\alpha^{-2} m_\alpha m_s^{-1} R_{m\alpha}^{-1} R_{H_\alpha}^{-1} \sum_{t=1}^{n} \Big(e_t \frac{\partial \psi_t}{\partial x} \Big) = 0 . \tag{III.1.9}$$

For $s \neq \alpha$, Eq. (III.1.9) becomes, neglecting small terms in view of Eq. (III.1.1):

$$\sum_{t=1}^{n} e_t \frac{\partial \psi_t}{\partial x} = 0 , \tag{III.1.10}$$

or, using Eqs. (II.6.17), (II.7.34), (III.1.10) we get:

$$\frac{\partial H_z}{\partial x} = 0 , \tag{III.1.11}$$

and since $H_z = 0$ in the undisturbed stream we conclude that:

$$H_z = 0 \quad \text{everywhere.} \tag{III.1.12}$$

For s=α, using Eq. (III.1.10) in Eq. (III.1.9) we get:

$$\sum_{t=1}^{n} e_t\left(\sigma_t Z_{tz} - X_{tz}\right) = e_\alpha\left(\sigma_\alpha Z_{\alpha z} - X_{\alpha z}\right). \qquad \text{(III.1.13)}$$

Inserting Eq. (III.1.13) into Eqs. (III.1.6), (III.1.7) we have:

$$\frac{\partial \xi}{\partial x} = -H_\infty^1 H_\infty^{-1} \frac{\partial \psi_\alpha}{\partial x} - H_\infty^2 H_\infty^{-1} \frac{\partial \psi_\alpha}{\partial y} - H_\infty^2 H_\infty^{-1} B_\alpha -$$

$$-R_{m\alpha}^{-1} R_{H_\alpha}^{-1}\left(\sigma_\alpha Z_{\alpha z} - X_{\alpha z}\right), \qquad \text{(III.1.14)}$$

$$\frac{\partial w_s}{\partial x} = R_{ms} R_{H_s^1}\left(\frac{\partial \psi_s}{\partial x} - \frac{\partial \psi_\alpha}{\partial x}\right) + R_{ms} R_{H_s^2}\left(\frac{\partial \psi_s}{\partial y} - \frac{\partial \psi_\alpha}{\partial y}\right) + R_{ms} R_{H_s^2}\left(B_s - B_\alpha\right) +$$

$$+\sigma_s Z_{sz} - X_{sz} \quad \text{for} \quad s \neq \alpha \ ; \quad w_\alpha = 0 \ . \qquad \text{(III.1.15)}$$

Eq. (II.7.32) becomes, using Eqs. (II.7.41), (III.1.12):

$$\beta_{s\infty}^2 \frac{\partial^2 \psi_s}{\partial x^2} + \frac{\partial^2 \psi_s}{\partial y^2} = \beta_{s\infty}^2 R_{ms} R_{H_s^3} \frac{\partial \psi_s}{\partial y} - \beta_{s\infty}^2 R_{ms} R_{H_s^1} w_s -$$

$$-R_{ms} R_{H_s^2} \frac{\partial}{\partial y}\left(\int_{\infty}^{\Gamma_s} w_s \, dx\right) + G_s \ . \qquad \text{(III.1.16)}$$

Eqs. (III.1.14) to (III.1.16) will be considered as a system of equations governing approximately the n-component fluid flow. The equations of this system can be solved separately by solving first Eq. (III.1.16) for ψ_α, after which Eq. (III.1.14) is solved for ξ, and Eqs. (III.1.15), (III.1.16) can be solved for w_s, ψ_s (s$\neq\alpha$).

III.2. Correlation of Flows

The system of equations governing approximately the compressible flow of an n-component fluid will be correlated to n systems of equations governing approximately the incompressible flow of a corresponding n-component fluid. By correlation it is meant that there will be derived simple relations between corresponding functions of the compressible and those of the incompressible flow. This procedure can serve, then, as a first approximation in the correlation of the more general system of Eqs. (II.7.32) to (II.7.36). Having this first approximation one can try to derive, by an itterative procedure, a higher order correlation, using the system of Eqs. (II.7.32) to (II.7.36).

The equations for the α-th fluid component and each of the s-th fluid components separately are coupled together and will be taken as one system when compared to the incompressible flow. We will denote functions and coordinates in the incompressible flow by primes. The equations for the s-th and α-th fluid components in the incompressible flow are, from Eqs. (II.7.39), (III.7.39), (III.1.15), (III.1.16), (here is $\beta_{s\infty}^2 = 1$, $\beta_{\alpha\infty}^2 = 1$):

$$\frac{\partial^2 \psi_s'}{\partial x'^2} + \frac{\partial^2 \psi_s'}{\partial y'^2} = R_{ms}' R_{H_s^3}' \frac{\partial \psi_s'}{\partial y'} - R_{ms}' R_{H_s^1}' w_s' -$$

$$-R'_{ms} R'_{H_s^2} \frac{\partial}{\partial y'}\left(\int_{\Gamma'_s}^{\infty} w'_s \, dx'\right) + G'_s, \qquad \text{(III.2.1)}$$

$$\frac{\partial w'_s}{\partial x'} = R'_{ms} R'_{H_s^1}\left(\frac{\partial \psi'_s}{\partial x'} - \frac{\partial \psi'_\alpha}{\partial x'}\right) + R'_{ms} R'_{H_s^2}\left(\frac{\partial \psi'_s}{\partial y'} - \frac{\partial \psi'_\alpha}{\partial y'}\right) + R'_{ms} R'_{H_s^2}\left(B'_s - B'_\infty\right) +$$

$$+ \sigma'_s Z'_{sz} - X'_{sz}, \qquad \text{(III.2.2)}$$

$$\frac{\partial^2 \psi'_\alpha}{\partial x'^2} + \frac{\partial^2 \psi'_\alpha}{\partial y'^2} = R'_{m\alpha} R'_{H_\alpha^3} \frac{\partial \psi'_\alpha}{\partial y'} + G'_\alpha, \qquad \text{(III.2.3)}$$

$$G'_t = -\frac{\partial B'_t}{\partial y'} + R'_{mt} R'_{H_t^3} B'_t - \sigma'_t Z'_{ty} + X'_{ty} +$$

$$+ \frac{\partial}{\partial y'}\left\{\int_{\Gamma'_t}^{\infty}\left[\sigma'_t\left(Z'_{tx} - 1\right) - X'_{tz}\right] dx'\right\}; \quad (t = s, \alpha). \qquad \text{(III.2.4)}$$

The equations for ψ'_s, ψ'_α have both Laplacian operators on the left hand side. If we want to correlate the equations for ψ_s, ψ_α of the compressible flow to the corresponding equations in the incompressible flow by means of a linear transformation between the (x, y) and the (x', y') coordinates we have to introduce one further approximation by introducing a weighted mean value $\bar{\beta}^2_{s\infty}$:

$$\bar{\beta}^2_{s\infty} = \epsilon_s \beta^2_{s\infty} + (1 - \epsilon_s)\beta^2_{\alpha\infty}; \quad 0 \leq \epsilon_s \leq 1, \qquad \text{(III.2.5)}$$

where ϵ_s is a weight factor, the value of which must be known. We approximate in Eqs. (II.7.39), (III.1.15), (III.1.16) for the s-th and the α-th fluid components the $\beta^2_{s\infty}$ and the $\beta^2_{\alpha\infty}$ by $\bar{\beta}^2_{s\infty}$, and get:

$$\bar{\beta}_{s\infty}^{2}\frac{\partial^2\psi_s}{\partial x^2}+\frac{\partial^2\psi_s}{\partial y^2}=\bar{\beta}_{s\infty}^{2}R_{ms}R_{H_s^3}\frac{\partial\psi_s}{\partial y}-\bar{\beta}_{s\infty}^{2}R_{ms}R_{H_s^1}w_s-$$

$$-R_{ms}R_{H_s^2}\frac{\partial}{\partial y}\left(\int_{\Gamma_s}^{\infty}w_s\,dx\right)+G_s\,, \qquad \text{(III.2.6)}$$

$$\frac{\partial w_s}{\partial x}=R_{ms}R_{H_s^1}\left(\frac{\partial\psi_s}{\partial x}-\frac{\partial\psi_\alpha}{\partial x}\right)+R_{ms}R_{H_s^2}\left(\frac{\partial\psi_s}{\partial y}-\frac{\partial\psi_\alpha}{\partial y}\right)+R_{ms}R_{H_s^2}(B_s-B_\alpha)+$$

$$+\sigma_s Z_{sz}-X_{sz}\,, \qquad \text{(III.2.7)}$$

$$\bar{\beta}_{s\infty}^{2}\frac{\partial^2\psi_\alpha}{\partial x^2}+\frac{\partial^2\psi_\alpha}{\partial y^2}=\bar{\beta}_{s\infty}^{2}R_{m\alpha}R_{H_\alpha^3}\frac{\partial\psi_\alpha}{\partial y}+G_\alpha\,, \qquad \text{(III.2.8)}$$

$$G_t=-\frac{\partial B_t}{\partial y}+\bar{\beta}_{s\infty}^{2}R_{mt}R_{H_t^3}B_t-\bar{\beta}_{s\infty}^{2}(\sigma_t Z_{ty}-X_{ty})+\frac{\partial}{\partial y}\left\{\int_{\Gamma_t}^{\infty}\left[\sigma_t(Z_{tx}-1)-\right.\right.$$

$$\left.\left.-X_{tx}\right]dx\right\}+(1-\beta_{t\infty}^{2})\left[(K_t-1)(1+W_t)(1+A_t)^{-1}\frac{\partial Q_t}{\partial y}-L_{ty}\right];\ (t=s,\alpha). \qquad \text{(III.2.9)}$$

We correlate the system of Eqs. (III.2.6) to (III.2.9) with the system of Eqs. (III.2.1) to (III.2.4). Introduce the following linear transformation:

$$x'=x\ ;\quad y'=\bar{\beta}_{s\infty}y\,, \qquad \text{(III.2.10)}$$

and assume that the dependent variables in the two flows are related by:

$$\psi_s=a_s\psi_s'\ ;\quad w_s=b_s w_s'\ ;\quad \psi_\alpha=a_\alpha\psi_\alpha'\,, \qquad \text{(III.2.11)}$$

where a_s, b_s, a_α are constants which will be determined.

Using Eqs. (III.2.10), (III.2.11) in Eqs. (III.2.6)

to (III.2.9) we have:

$$\frac{\partial^2\psi_s'}{\partial x'^2} + \frac{\partial^2\psi_s'}{\partial y'^2} = \bar{\beta}_{s\infty} R_{ms} R_{H_s^3} \frac{\partial\psi_s'}{\partial y'} - b_s a_s^{-1} R_{ms} R_{H_s^1} w_s' -$$

$$- b_s a_s^{-1} \bar{\beta}_{s\infty} R_{ms} R_{H_s^2} \frac{\partial}{\partial y'}\left(\int_{\Gamma_s}^{\infty} w_s'\, dx'\right) + a_s^{-1} \bar{\beta}_{s\infty}^{-2} G_s, \qquad \text{(III.2.12)}$$

$$\frac{\partial w_s'}{\partial x'} = b_s^{-1} R_{ms} R_{H_s^1}\left(a_s \frac{\partial\psi_s'}{\partial x'} - a_\alpha \frac{\partial\psi_\alpha'}{\partial x'}\right) + \bar{\beta}_{s\infty} b_s^{-1} R_{ms} R_{H_s^2}\left(a_s \frac{\partial\psi_s'}{\partial y'} - a_\alpha \frac{\partial\psi_\alpha'}{\partial y'}\right) +$$

$$+ b_s^{-1} R_{ms} R_{H_s^2}(B_s - B_\alpha) + b_s^{-1}(\sigma_s Z_{sz} - X_{sz}), \qquad \text{(III.2.13)}$$

$$\frac{\partial^2\psi_\alpha'}{\partial x'^2} + \frac{\partial^2\psi_\alpha'}{\partial y'^2} = \bar{\beta}_{s\infty} R_{ms} R_{H_\alpha^3} \frac{\partial\psi_\alpha'}{\partial y'} + a_\alpha^{-1} \bar{\beta}_{s\infty}^{2} G_\alpha, \qquad \text{(III.2.14)}$$

$$a_t^{-1} \bar{\beta}_{s\infty}^{2} G_t = - a_t^{-1} \bar{\beta}_{s\infty}^{-1} \frac{\partial B_t}{\partial y'} + a_t^{-1} R_{mt} R_{H_t^3} B_t - a_t^{-1}(\sigma_t Z_{ty} - X_{ty}) +$$

$$+ a_t^{-1} \bar{\beta}_{s\infty}^{-1} \frac{\partial}{\partial y'}\left\{\int_{\Gamma_t}^{\infty} [\sigma_t(Z_{tx} - 1) - X_{tx}]\, dx'\right\} +$$

$$+ a_t^{-1} \bar{\beta}_{s\infty}^{2}(1 - \beta_{s\infty}^{2})\left[(K_t - 1)(1 + W_t)(1 + A_t)^{-1} \frac{\partial Q_t}{\partial y} - L_{ty}\right]; \quad (t = s, \alpha). \qquad \text{(III.2.15)}$$

In order to correlate Eqs. (III.2.12) to (III.2.15) with Eqs. (III.2.1) to (III.2.4) we require the following relations to exist:

$$b_s a_s^{-1} R_{ms} R_{H_s^1} = R_{ms}' R_{H_s^1}' \; ; \quad a_t b_s^{-1} R_{ms} R_{H_s^1} = R_{ms}' R_{H_s^1}', \qquad \text{(III.2.16)}$$

$$b_s a_s^{-1} \bar{\beta}_{s\infty}^{-1} R_{ms} R_{H_s^2} = R_{ms}' R_{H_s^2}' \; ; \quad a_t b_s^{-1} R_{ms} R_{H_s^2} = R_{ms}' R_{H_s^2}', \qquad \text{(III.2.17)}$$

$$\bar{\beta}_{s\infty} R_{ms} R_{H_t^3} = R_{mt}' R_{H_t^3}' \; ; \quad \bar{\beta}_{s\infty}^{-2} a_t^{-1} G_t = G_t', \qquad \text{(III.2.18)}$$

$$R_{ms}R_{H_s^2}(B_s-B_\alpha)=R'_{ms}R'_{H_s^2}(B'_s-B'_\alpha);\ b_s^{-1}(\sigma_s Z_{sz}-X_{sz})=\sigma'_s Z'_{sz}-X'_{sz}, \quad \text{(III.2.19)}$$

where r = s, α. In addition we require that the curve Γ_s be transformed into the curve Γ'_s, i.e., projected streamlines in the compressible flow transform into projected streamlines in the incompressible flow. This leads to the so-called "streamline analogy" (8, pp. 180, 181), which requires that ψ_s be transformed in the same way as y, thus, using Eqs. (III.2.10), (III.2.11) we have:

$$a_s = \bar{\beta}_{s\infty}^{-1}. \quad \text{(III.2.20)}$$

From either the second Eq. (III.2.16) or the second Eq. (III.2.17) we get:

$$a_\alpha = a_s. \quad \text{(III.2.21)}$$

From Eq. (III.2.16) we have:

$$b_s a_s^{-1} = 1 \quad \text{unless} \quad R_{ms}R_{H_s^1} \equiv R'_{ms}R'_{H_s^1} \equiv 0. \quad \text{(III.2.22)}$$

From Eq. (III.2.17) it can be seen that:

$$b_s a_s^{-1} = \bar{\beta}_{s\infty} \quad \text{unless} \quad R_{ms}R_{H_s^2} \equiv R'_{ms}R'_{H_s^2} \equiv 0. \quad \text{(III.2.23)}$$

Since the first Eqs. (III.2.22), (III.2.23) are conflicting each other we will limit our correlation to the following two special cases:

Case (a): $H^1_\infty \equiv H^{1'}_\infty \equiv 0$; $\left(R_{ms}R_{H^1_s} \equiv R'_{ms}R'_{H^1_s} \equiv 0\right)$, $b_s a_s^{-1} = \bar{\beta}_{s\infty}$. (III.2.24)

Case (b): $H^2_\infty \equiv H^{2'}_\infty \equiv 0$; $\left(R_{ms}R_{H^2_s} \equiv R'_{ms}R'_{H^2_s} \equiv 0\right)$, $b_s a_s^{-1} = 1$. (III.2.25)

Using Eq. (III.2.20) in Eqs. (III.2.24), (III.2.25) we have:

$b_s = 1$ in case (a) ; $b_s = \bar{\beta}^{-1}_{s\infty}$ in case (b). (III.2.26)

We assume the following relations:

$$m_+ = m'_+ \;;\quad e_+ = e'_+ \quad (+ = s, \alpha) \;;\quad u_\infty = u'_\infty \;;$$

$$\nu_\infty = \nu'_\infty \;;\quad L = L' . \tag{III.2.27}$$

Thus, in order that the relations in the first Eqs. (III.2.18) be established, we assume, using Eqs. (II.6.17), (II.6.18), (III.2.27):

$$\bar{\beta}_{s\infty} H^3_\infty = H^{3'}_\infty , \tag{III.2.28}$$

and, in order that Eqs. (III.2.16), (III.2.17) be satisfied we choose, using Eqs. (II.6.17), (II.6.18), (III.2.27):

$$H^1_\infty = H^{1'}_\infty \;;\quad H^2_\infty = H^{2'}_\infty . \tag{III.2.29}$$

To satisfy the second Eq. (III.2.18) we assume, using Eqs. (III.2.4), (III.2.15), the first Eq. (III.2.18), and Eq. (III.2.20):

$$B_{+} = B'_{+} \; ; \quad \sigma_{+} = \sigma'_{+} \; ; \quad \sigma_{+} Z_{+x} = \sigma'_{+} Z'_{+x} \; ; \quad \bar{\beta}_{s\infty} \sigma_{+} Z_{+y} = \sigma'_{+} Z'_{+y} \; ;$$

$$X_{+x} = X'_{+x} \; ; \quad \bar{\beta}_{s\infty} X_{+y} = X'_{+y} \; ;$$

$$\bar{\beta}_{s\infty}^{-2}(1-\beta_{+\infty}^{2})\left[(K_{+}-1)(1+W_{+})(1+A_{+})^{-1}\frac{\partial Q_{+}}{\partial y} - L_{+y}\right] \approx 0 \; ; \quad (+ = s, \alpha). \qquad \text{(III.2.30)}$$

The expression on the left hand side of the last Eq. (III.2.30) is assumed to be negligibly small since $1 - \beta_{+\infty}^{2} = 1 - 1 + (\tilde{\alpha}_{+}^{-1} M_{+\infty})^{2} = (\tilde{\alpha}_{+}^{-1} M_{+\infty})^{2}$ is assumed to be a small quantity.

The first Eq. (III.2.19) is automatically satisfied when using Eqs. (III.2.17), (III.2.24) and the first Eq. (III.2.30). In order satisfy the second Eq. (III.2.19), we assume, using Eq. (III.2.26):

$$\sigma_{s} Z_{sz} = \sigma'_{s} Z'_{sz} \; ; \quad X_{sz} = X'_{sz} \qquad \text{in case (a),} \qquad \text{(III.2.31)}$$

$$\bar{\beta}_{s\infty} \sigma_{s} Z_{sz} = \sigma'_{s} Z'_{sz} \; ; \quad \bar{\beta}_{s\infty} X_{sz} = X'_{sz} \qquad \text{in case (b).} \qquad \text{(III.2.32)}$$

Denoting the non-dimensional perturbation velocities in the y direction by v_{s}, we have, using Eqs. (III.7.18), (III.2.10), (III.2.11), (III.2.20), (III.2.21):

$$\bar{\beta}_{s\infty} v_r = v_r' \; ; \qquad (r = s, \alpha). \tag{III.2.33}$$

A summary of the relations between corresponding quantities of the compressible and the incompressible flow, when the s-th fluid component is correlated, is given below in a table form:

Table 1. Correlation for the s-th fluid component quantities

Compressible	Incompressible
$m_r, e_r, \mu_\infty, \nu_\infty, H^1_\infty, H^2_\infty, H^3_\infty$ $E^1_\infty = 0, E^2_\infty, E^3_\infty, L$	$m_r', e_r', \mu_\infty', \nu_\infty', H^{1'}_\infty, H^{2'}_\infty, \bar{\beta}^{-1}_{s\infty} H^{3'}_\infty$ $E^{1'}_\infty = 0, \bar{\beta}_{s\infty} E^{2'}_\infty, E^{3'}_\infty, L'$
$\psi_r, v_r, B_r, \sigma_r, \sigma_r Z_{rx},$ $\sigma_r Z_{ry}, X_{rx}, X_{ry}.$ Case (a): $w_s, \sigma_s Z_{sz}, X_{sz}$ Case (b): $w_s, \sigma_s Z_{sz}, X_{sz}$	$\bar{\beta}^{-1}_{s\infty} \psi_r', \bar{\beta}^{-1}_{s\infty} v_r', B_r', \sigma_r', \sigma_r' Z_{rx}',$ $\bar{\beta}^{-1}_{s\infty} \sigma_r' Z_{ry}', X_{rx}', \bar{\beta}^{-1}_{s\infty} X_{ry}'$ $w_s', \sigma_s' Z_{sz}', X_{sz}'$ $\bar{\beta}^{-1}_{s\infty} w_s', \bar{\beta}^{-1}_{s\infty} \sigma_s' Z_{sz}', \bar{\beta}^{-1}_{s\infty} X_{sz}'$

where $r=s, \alpha$.

For the correlation of the α-th fluid component we consider Eq. (III.1.16) for $s=\alpha$ combined with the second Eq. (III.1.15). We also consider Eq. (III.1.14).

Introducing the linear transformation:

$$x' = x \; ; \quad y' = \beta_{\alpha\infty} y , \qquad \text{(III.2.34)}$$

and assuming:

$$\psi_\alpha = a_\alpha \psi'_\alpha \; ; \quad \xi = b \xi' , \qquad \text{(III.2.35)}$$

where a_α, b are constants which will be determined, we get, using Eqs. (II.7.41), (III.1.15), (III.2.34), (III.2.35), in Eqs. (II.7.39), (III.1.14), (III.1.16) for $s = \alpha$:

$$\frac{\partial^2 \psi'_\alpha}{\partial x'^2} + \frac{\partial^2 \psi'_\alpha}{\partial y'^2} = \beta_{\alpha\infty} R_{m\alpha} R_{H_\alpha^3} \frac{\partial \psi'_\alpha}{\partial y'} - a_\alpha^{-1} \beta_{\alpha\infty}^{-2} G_\alpha , \qquad \text{(III.2.36)}$$

$$a_\alpha^{-1} \beta_{\alpha\infty}^{-2} G_\alpha = - a_\alpha^{-1} \beta_{\alpha\infty}^{-1} \frac{\partial B_\alpha}{\partial y'} + a_\alpha^{-1} R_{m\alpha} R_{H_\alpha^3} B_\alpha - a_\alpha^{-1} \left(\sigma_\alpha Z_{\alpha y} - X_{\alpha y} \right) +$$

$$+ a_\alpha^{-1} \beta_{\alpha\infty} \frac{\partial}{\partial y'} \left\{ \int_{\Gamma_\alpha}^{\infty} \left[\sigma_\alpha (Z_{\alpha x} - 1) - X_{\alpha x} \right] dx' \right\} +$$

$$+ a_\alpha^{-1} \beta_{\alpha\infty}^{-2} \left[(K_\alpha - 1)(1 + W_\alpha)(1 + A_\alpha)^{-1} \frac{\partial Q_\alpha}{\partial y} + L_{\alpha y} \right] , \qquad \text{(III.2.37)}$$

$$\frac{\partial \xi'}{\partial x'} = - a_\alpha b^{-1} H_\infty^1 H_\infty^{-1} \frac{\partial \psi'_\alpha}{\partial x'} - a_\alpha b^{-1} H_\infty^2 H_\infty^{-1} \frac{\partial \psi'_\alpha}{\partial y'} -$$

$$- b^{-1} H_\infty^2 H_\infty^{-1} B_\alpha - b^{-1} R_{m\alpha}^{-1} R_{H_\alpha}^{-1} \left(\sigma_\alpha Z_{\alpha z} - X_{\alpha z} \right) . \qquad \text{(III.2.38)}$$

Requiring a streamline analogy we have:

$$a_\alpha = \beta_{\alpha\infty}^{-1} \qquad \text{(III.2.39)}$$

In order to correlate Eqs. (III.2.36) to (III.2.39) with the corresponding equations for the incompressible flow we will require the following relations to exist, using Eq. (III.2.39):

$$\beta_{\alpha\infty} R_{m\alpha} R_{H^3_\alpha} = R'_{m\alpha} R'_{H^3_\alpha} \quad ; \qquad \beta^{-1}_{\alpha\infty} G_\alpha = G'_\alpha \,, \tag{III.2.40}$$

$$b^{-1}\beta^{-1}_{\alpha\infty} H^1_\infty H^{-1}_\infty = H^{1'}_\infty H'^{-1}_\infty \quad ; \qquad b^{-1}\beta^{-1}_{\alpha\infty} H^2_\infty H^{-1}_\infty = H^{2'}_\infty H'^{-1}_\infty \,, \tag{III.2.41}$$

$$b^{-1} R^{-1}_{m\alpha} R^{-1}_{H_\alpha} \sigma_\alpha Z_{\alpha z} = R'^{-1}_{m\alpha} R'^{-1}_{H_\alpha} \sigma'_\alpha Z'_{\alpha z} \; ; \; b^{-1} R^{-1}_{m\alpha} R^{-1}_{H_\alpha} X_{\alpha z} = R'^{-1}_{m\alpha} R'^{-1}_{H_\alpha} X'_{\alpha z} . \quad \text{(III.2.42}$$

We assume the following relations:

$$m_\alpha = m'_\alpha \; ; \; e_\alpha = e'_\alpha \; ; \; u_\infty = u'_\infty \; ; \; \nu_\infty = \nu'_\infty \; ; \; L = L' \; ;$$

$$\beta_{\alpha\infty} H^3_\infty = H^{3'}_\infty \; ; \quad H^1_\infty = H^{1'}_\infty \; ; \quad H^2_\infty = H^{2'}_\infty \tag{III.2.43}$$

From the last three Eqs. (III.2.43) we have:

$$H_\infty = \left[\left(H^1_\infty\right)^2 + \left(H^2_\infty\right)^2 + \left(H^3_\infty\right)^2 \right]^{\frac{1}{2}} =$$

$$= H'_\infty \left[1 + \beta^{-2}_{\alpha\infty} \left(1 - \beta^2_{\alpha\infty}\right) \left(H^{3'}_\infty H'^{-1}_\infty \right)^2 \right]^{\frac{1}{2}} . \tag{III.2.44}$$

In order that Eq. (III.2.41) be completely satisfied we use Eqs. (III.2.43), (III.2.44) and choose:

$$b = \beta^{-1}_{\alpha\infty} \left[1 + \beta^{-2}_{\alpha\infty} \left(1 - \beta^2_{\alpha\infty}\right) \left(H^{3'}_\infty H'^{-1}_\infty \right)^2 \right]^{-\frac{1}{2}} . \tag{III.2.45}$$

The first Eq. (III.2.40) is satisfied by the assumptions made in the first six Eqs. (III.2.43). In order that the second Eq. (III.2.40) be satisfied we assume, using Eq. (III.2.37) and the first Eq. (III.2.40):

$$B_\alpha = B'_\alpha \; ; \quad \sigma_\alpha = \sigma'_\alpha \; ; \quad \sigma_\alpha Z_{\alpha x} = \sigma'_\alpha Z'_{\alpha x} \; ;$$

$$\beta_{\alpha\infty} \sigma_\alpha Z_{\alpha y} = \sigma'_\alpha Z'_{\alpha y} \; ; \; X_{\alpha x} = X'_{\alpha x} \; ; \quad \beta_{\alpha\infty} X_{\alpha y} = X'_{\alpha y} \; ;$$

$$\beta_{\alpha\infty}^{-1} \left(1 - \beta_{\alpha\infty}^{2}\right) \left[(K_\alpha - 1)(1 + W_\alpha)(1 + A_\alpha)^{-1} \frac{\partial Q_\alpha}{\partial y} + L_{\alpha y} \right] \approx 0. \qquad \text{(III.2.46)}$$

From Eq. (II.6.17), (II.6.18), (III.2.44), (III.2.45) we have:

$$b^{-1} R'_{m\alpha} R'_{H_\alpha} R_{m\alpha}^{-1} R_{H_\alpha}^{-1} = \beta_{\alpha\infty} . \qquad \text{(III.2.47)}$$

Using Eq. (III.2.47) in Eq. (III.2.42) we have:

$$\sigma_\alpha Z_{\alpha z} = \beta_{\alpha\infty}^{-1} \sigma'_\alpha Z'_{\alpha z} \; ; \quad X_{\alpha z} = \beta_{\alpha\infty}^{-1} X'_{\alpha z} . \qquad \text{(III.2.48)}$$

Summarizing the correlations of the α-th fluid component in a table form:

Table 2. Correlation for the α-th fluid component quantities

Compressible	Incompressible
$m_\alpha, e_\alpha, u_\infty, \nu_\infty, H_\infty^1, H_\infty^2, H_\infty^3,$ $E_\infty^1 = 0, E_\infty^2, E_\infty^3, L$	$m'_\alpha, e'_\alpha, u'_\infty, \nu'_\infty, H_\infty^{1'}, H_\infty^{2'}, \beta_{\alpha\infty}^{-1} H_\infty^{3'},$ $E_\infty^{1'} = 0, \beta_{\alpha\infty}^{-1} E_\infty^2, E_\infty^{3'}, L'$
$\psi_\alpha, v_\alpha, \xi$ $B_\alpha, \sigma_\alpha, \sigma_\alpha Z_{\alpha x}, \sigma_\alpha Z_{\alpha y}$ $\sigma_\alpha Z_{\alpha z}, X_{\alpha x}, X_{\alpha y}, X_{\alpha z}$	$\psi'_\alpha, v'_\alpha, \xi' \beta_{\alpha\infty}^{-1}\left[1 + \beta_{\alpha\infty}^{-2}(1 - \beta_{\alpha\infty}^{2})(H_\infty^{3'} H_\infty^{\prime -1})^2\right]^{-\frac{1}{2}}$ $B'_\alpha, \sigma'_\alpha, \sigma'_\alpha Z'_{\alpha x}, \beta_{\alpha\infty}^{-1} \sigma'_\alpha Z'_{\alpha y},$ $\beta_{\alpha\infty}^{-1} \sigma'_\alpha Z'_{\alpha z}, X'_{\alpha x}, \beta_{\alpha\infty}^{-1} X'_{\alpha y}, \beta_{\alpha\infty}^{-1} X'_{\alpha z}$

III.3. Pressure Coefficients

In this section we invalidate the changes in notation which were introduced for the perturbed quantities in Chapter II, Section 7.

The s-th species' pressure coefficient is defined by:

$$C_{ps} = 2(p_s - p_{s\infty}) \rho_{s\infty}^{-1} u_{s\infty}^{-2} . \qquad \text{(III.3.1)}$$

Bernoulli's Eq. (II.3.4) becomes, using Eqs. (II.3.5), (II.6.2), (II.6.4), (II.6.5), (II.6.7), (II.6.9), (II.6.17), (II.6.18), (II.6.20), (II.7.1)

(II.7.4) to (II.7.11), (II.7.23), (II.7.25), (II.7.28), neglecting second order terms and using the result $\vec{E}$ = constant which was obtained in Chapter II, Section 7:

$$\tilde{u}^{1}_{sp} + I_s u_\infty^{-2} - J_s = I_{s\infty} u_\infty^{-2} \, , \tag{III.3.2}$$

where:

$$I_s u_\infty^{-2} = c_{ps}(1+D_s) T_s u_\infty^{-2} \; ; \; I_{s\infty} u_\infty^{-2} = c_{ps} T_s u_\infty^{-2} \, , \tag{III.3.3}$$

$$J_s = J_{s1} + J_{s2} + J_{s3} + J_{s4} \, ; \; J_{s1} = \int_{\infty}^{\tilde{r}_s} d\tilde{Q}_s \; ; \; J_{s2} = -R_{ms} R_{H_s^3} \int_{\infty}^{\tilde{r}_s} d\tilde{x}^2 \, ;$$

$$J_{s3} = R_{ms} R_{H_s^2} \int_{\infty}^{\tilde{r}_s} \tilde{u}^{3}_{sp} \, d\tilde{x}^1 \, ; \; J_{s4} = \int_{\infty}^{\tilde{r}_s} \left[\tilde{\sigma}_s (\tilde{Z}^1_s - 1) + \tilde{X}^1_s \right] d\tilde{x}^1 . \tag{III.3.4}$$

Note that $D_{s\infty} \equiv 0$ since each of the fluid components follows perfect gas laws in the undisturbed stream. From Eqs. (I.2.70), (I.2.71) and the relations $A_{s\infty}, W_{s\infty} \equiv 0$ we also have:

$$(\alpha_{s\infty})^2 = K_s p_{s\infty} \rho_{s\infty}^{-1} \, . \tag{III.3.5}$$

Using the relations $p_{s\infty} \rho_{s\infty}^{-1} = R_{ps} T_{s\infty}$, $R_{ps} = C_{ps} - C_{vs}$, $K_s = C_{ps} C_{vs}^{-1}$, we have:

$$(\alpha_{s\infty})^2 = C_{ps} (K_s - 1) T_s \, . \tag{III.3.6}$$

Using Eqs. (II.6.16), (III.3.5), (III.3.6) in Eq. (III.3.3) we get:

$$I_s u_\infty^{-2} = (1+D_s)(K_s-1)^{-1}(M_{s\infty})^{-2} T_s T_{s\infty}^{-1} \;;$$

$$I_{s\infty} u_\infty^{-2} = (K_s-1)^{-1}(M_{s\infty})^{-2} . \qquad \text{(III.3.7)}$$

Inserting Eq. (III.3.7) into Eq. (III.3.2) and solving for $T_s T_{s\infty}^{-1}$ furnishes:

$$T_s T_{s\infty}^{-1} = (1+D_s)^{-1}\left[1+ (K_s-1)(M_{s\infty})^2(\mathcal{J}_s - \tilde{u}_{sp}^{\,1})\right]. \qquad \text{(III.3.8)}$$

We introduce the following approximations:

$$\int_\infty A_s T_s^{-1}\, dT_s \approx \bar{A}_s \int_\infty T_s^{-1}\, dT_s \;;\; \int_\infty W_s \rho_s^{-1}\, d\rho_s \approx \bar{W}_s \int_\infty \rho_s^{-1}\, d\rho_s , \qquad \text{(III.3.9)}$$

where $\bar{A}_s$, $\bar{W}_s$ are some mean values representing A_s, W_s in those integrals. Using Eq. (III.3.9) in Eq. (I.2.57) we have:

$$G_s = (T_s T_{s\infty}^{-1})^{-\bar{A}_s} (\rho_s \rho_{s\infty}^{-1})^{(K_s-1)\bar{W}_s} \exp(-A_s) . \qquad \text{(III.3.10)}$$

Inserting Eq. (III.3.10) into Eq. (I.2.56), making use of the relation $\rho_s \rho_{s\infty}^{-1} = p_s p_{s\infty}^{-1} T_{s\infty} T_s^{-1} (1+W_s)^{-1}$ which is obtained from the equation of state, and solving for $p_s p_{s\infty}^{-1}$ we get:

$$p_s p_{s\infty}^{-1} = (T_s T_{s\infty}^{-1})^{K_s(K_s-1)^{-1}(1+\bar{D}_s)(1+\bar{W}_s)} (1+W_s) \exp(-A_s + c_{vs}^{-1} \Delta S_s) , \qquad \text{(III.3.11)}$$

where $\bar{D}_s, \Delta S_s$ are given by:

$$\bar{D}_s = K_s^{-1}\left[\bar{A}_s + (K_s - 1)\bar{W}_s\right]; \quad \Delta S_s = S_s - S_{s\infty}. \qquad \text{(III.3.12)}$$

Inserting Eq. (III.3.8) into Eq. (III.3.11) gives:

$$p_s p_{s\infty}^{-1} = \left\{(1+D)^{-1}\left[1 + (K_s-1)(M_{s\infty})^2(\mathcal{J}_s - \tilde{u}_{sp}^{1})\right]\right\}^{K_s(K_s-1)^{-1}(1+\bar{D}_s)(1+\bar{W}_s)^{-1}} \cdot$$

$$\cdot (1+W_s)\exp\left[(A_s - c_{vs}^{-1}\Delta S_s)(K_s-1)^{-1}(1+\bar{W}_s)^{-1}\right]. \qquad \text{(III.3.13)}$$

We make the following approximations by expanding the power and exponential expressions in Eq. (III.3.13), neglecting higher than first order terms and using Eqs. (I.2.53), (II.6.7), (II.6.16), (III.3.4), (III.3.6) and the relation $K_s = c_{ps}c_{vs}^{-1}$:

$$(1+D_s)^{-K_s(K_s-1)^{-1}(1+\bar{D}_s)(1+\bar{W}_s)^{-1}} \approx 1 - K_s(K_s-1)^{-1}D_s, \qquad \text{(III.3.14)}$$

$$\left[1+(K_s-1)(M_{s\infty})^2(\mathcal{J}_s - \tilde{u}_{sp}^{1})\right]^{K_s(K_s-1)^{-1}(1+\bar{D}_s)(1+\bar{W}_s)^{-1}} \approx 1 + K_s(M_{s\infty})^2(\mathcal{J}_s - \tilde{u}_{sp}^{1}), \qquad \text{(III.3.15)}$$

$$\exp\left[(A_s - c_{vs}^{-1}\Delta S_s)(K_s-1)^{-1}(1+\bar{W}_s)^{-1}\right] \approx 1 + (K_s-1)^{-1}A_s -$$

$$- c_{vs}^{-1}(K_s-1)^{-1}\int T_s^{-1}\,dQ_s \approx 1 + (K_s-1)^{-1}A_s - K_s(M_{s\infty})^2\mathcal{J}_{s1}. \qquad \text{(III.3.16)}$$

Inserting Eqs. (III.3.14) to (III.6.16) into Eq. (III.3.13), neglecting higher than first order terms and using Eq. (II.2.13) and the first Eq. (III.3.4) we get:

$$p_s p_{s\infty}^{-1} = 1 + K_s (M_{s\infty})^2 \left(\mathcal{J}_{s2} + \mathcal{J}_{s3} + \mathcal{J}_{s4} - \tilde{u}_{sp}^{1} \right). \qquad \text{(III.3.17)}$$

Dividing Eq. (III.3.1) by $p_{s\infty}$ and using Eqs. (II.6.16), (III.3.5), (III.3.17) leads to:

$$C_{ps} = 2\left[(p_s p_{s\infty}^{-1}) - 1 \right] K_s^{-1} M_{s\infty}^{-2} = 2\left(\mathcal{J}_{s2} + \mathcal{J}_{s3} + \mathcal{J}_{s4} - \tilde{u}_{sp}^{1} \right). \qquad \text{(III.3.18)}$$

Eliminating $T_s T_{s\infty}^{-1}$ in Eq. (III.3.11) by means of the equation of state, using the relation $\rho_s \rho_{s\infty}^{-1} = 1 + \tilde{\rho}_{sp}$ and Eqs. (III.3.12), (III.3.16), neglecting higher than first order terms gives:

$$p_s p_{s\infty}^{-1} \approx 1 + K_s \tilde{\rho}_{sp} + W_s - A_s + K_s (K_s - 1)(M_{s\infty})^2 \mathcal{J}_{s1}. \qquad \text{(III.3.19)}$$

Using Eqs. (III.3.4), (III.3.17) in Eq. (III.3.19) and solving for $\tilde{\rho}_{sp}$ we get:

$$\tilde{\rho}_{sp} = (M_{s\infty})^2 \left(\mathcal{J}_s - K_s \mathcal{J}_{s1} - \tilde{u}_{sp}^{1} \right) - K_s^{-1} (W_s - A_s). \qquad \text{(III.3.20)}$$

Inserting Eq. (III.3.20) into Eq. (II.7.18) we get:

$$\frac{\partial \tilde{\Phi}_{sp}}{\partial \tilde{x}^2} = (\lambda_{s\infty})^2 \tilde{u}_{sp}^{1} + (M_{s\infty})^2 \left(\mathcal{J}_s - K_s \mathcal{J}_{s1} \right) - K_s^{-1} (W_s - A_s) - B_s. \qquad \text{(III.3.21)}$$

where $\lambda_{s\infty}$ is given by:

$$(\lambda_{s\infty})^2 = 1 - (M_{s\infty})^2. \qquad \text{(III.3.22)}$$

In the incompressible case, the corresponding Bernoulli equation to Eq. (III.3.2) is:

$$\tilde{u}_{sp}^{1\,\prime} + p_s' \rho_{s\infty}^{\prime -1} u_{s\infty}^{\prime -2} - \mathcal{J}_s' = p_{s\infty}' \rho_{s\infty}^{\prime -1} u_{s\infty}^{\prime -2} , \qquad \text{(III.3.23)}$$

where $\mathcal{J}_s'$ is given by:

$$\mathcal{J}_s' = \mathcal{J}_{s2}' + \mathcal{J}_{s3}' + \mathcal{J}_{s4}' \; ; \qquad \mathcal{J}_{s2}' = - R_{ms}' R_{H_s^3}' \int_{\infty}^{\tilde{\Gamma}_s'} d\tilde{x}^{2\,\prime} \; ;$$

$$\mathcal{J}_{s3}' = R_{ms}' R_{H_s^2}' \int_{\infty}^{\tilde{\Gamma}_s'} \tilde{u}_{sp}^{3\,\prime} \, d\tilde{x}^{1\,\prime} \; ; \quad \mathcal{J}_{s4}' = \int_{\infty}^{\tilde{\Gamma}_s'} \left[\tilde{\mathcal{Z}}_6' (\tilde{Z}_5^{1\,\prime} - 1) + X_5^{1\,\prime} \right] d\tilde{x}^{1} . \qquad \text{(III.3.24)}$$

The pressure coefficients in the incompressible flow are given by the following relation, using Eq. (III.3.23):

$$C_{ps}' = 2 \, (p_s' - p_{s\infty}') \rho_{s\infty}^{\prime -1} u_{\infty}^{\prime -2} = 2 (\mathcal{J}_s' - \tilde{u}_{sp}^{1\,\prime}) . \qquad \text{(III.3.25)}$$

In the incompressible case the Eq. (II.7.18) becomes:

$$\frac{\partial \tilde{\psi}_{sp}'}{\partial \tilde{x}^{2\,\prime}} = \tilde{u}_{sp}^{1\,\prime} - B_s' \; . \qquad \text{(III.3.26)}$$

From Eqs. (III.2.10), (III.2.11), the first Eq. (III.2.30), (III.3.21), (III.3.26) we have:

$$\tilde{u}_{sp}^{1} = \lambda_{s\infty}^{-2} \tilde{u}_{sp}^{1\,\prime} - (\lambda_{s\infty}^{-1} M_{s\infty})^2 (\mathcal{J}_s - \kappa_s \mathcal{J}_{s1}) + \kappa_s^{-1} \lambda_{s\infty}^{-2} (W_s - A_s) . \qquad \text{(III.3.27)}$$

From Eqs. (III.2.10), (III.2.11), (III.2.17), (III.2.18), (III.2.20), (III.2.24), (III.2.30), (III.3.4), (III.3.24) we have:

$$\mathcal{J}'_{s2} = (\bar{\beta}_{s\infty})^2 \mathcal{J}_{s2} \; ; \quad \mathcal{J}'_{s3} = \mathcal{J}_{s3} \; ; \quad \mathcal{J}'_{s4} = \mathcal{J}_{s4} \, . \tag{III.3.28}$$

Combining Eqs. (III.3.4), (III.3.18), (III.3.24), (III.3.25), (III.3.27), (III.3.28) we get, after a few manipulations:

$$C_{ps} = (\lambda_{s\infty})^{-2} C'_{ps} + 2(\lambda_{s\infty})^{-2} \Big\{ -\left[1-(\bar{\beta}_{s\infty})^2\right] R_{ms} R_{H_s^3} \int_{\infty}^{\tilde{\Gamma}_s} d\tilde{x}^2 +$$

$$+ (K_s - 1)\left[1-(\lambda_{s\infty})^2\right] \int_{\infty}^{\tilde{\Gamma}_s} d\tilde{Q}_s - K_s^{-1}(W_s - A_s) \Big\} . \tag{III.3.29}$$

The pressure coefficient for the gross fluid is defined by:

$$C_p = 2(p - p_\infty)\rho_\infty^{-1} u_\infty^{-2} . \tag{III.3.30}$$

It can be shown, using Eqs. (I.2.25), the first Eq. (II.7.23), (III.3.1), (III.3.30), that:

$$C_p = \sum_{s=1}^{n} \rho_{s\infty} \rho_\infty^{-1} C_{ps} = \sum_{s=1}^{n} m_s (nm)^{-1} C_{ps} , \tag{III.3.31}$$

where nm is given by:

$$nm = \sum_{t=1}^{n} m_t . \tag{III.3.32}$$

Similarly, in the incompressible case, we have:

$$C'_p = \sum_{s=1}^{n} m'_s (n'm')^{-1} C'_{ps} , \tag{III.3.33}$$

where $n'm'$ is given by:

$$n'm' = \sum_{t=1}^{n} m'_t . \tag{III.3.34}$$

From Eqs. (III.3.29), (III.3.31), (III.3.33) and the first Eq. (III.2.27), we have:

$$C_p = \sum_{s=1}^{n} m'_s (n'm') \Big\{ \lambda_{s\infty}^{-2} C'_{ps} + 2\lambda_{s\infty}^{-2} \Big[\!\Big[[1-(\bar\beta_{s\infty})^2] R_{ms} R_{H_s^3} \int_{\tilde\Gamma_s}^{\infty} d\tilde x^2 +$$

$$+ (K_s - 1)[1-(\lambda_{s\infty})^2] \int_{\tilde\Gamma_s}^{\infty} d\tilde Q_s - K_s^{-1}(W_s - A_s) \Big]\!\Big] \Big\} . \tag{III.3.35}$$

Adding and subtracting $\lambda_\infty^{-2} C'_p$ to Eq. (III.3.35) and making use of Eq. (III.3.33) we get:

$$C_p = \lambda_\infty^{-2} C'_p \Big\{ 1 + (n'm' C'_p)^{-1} \sum_{s=1}^{n} m'_s \Big[\!\Big[C'_{ps} [1 - (\lambda_\infty^{-1} \lambda_{s\infty})^2] +$$

$$+ 2 \Big[[1-(\bar\beta_{s\infty})^2] R_{ms} R_{H_s^3} \int_{\tilde\Gamma_s}^{\infty} d\tilde x^2 + (K_s - 1)[1-(\lambda_{s\infty})^2] \int_{\tilde\Gamma_s}^{\infty} d\tilde Q_s -$$

$$- K_s^{-1}(W_s - A_s) \Big]\Big]\!\Big] \Big\} , \tag{III.3.36}$$

where $(\lambda_\infty)^2$ is given by:

$$(\lambda_\infty)^2 = 1 - (M_\infty)^2 , \tag{III.3.37}$$

and $(M_\infty)^2$ is the gross fluid free stream Mach number.

In the case $s=\alpha$, we replace $\bar{\beta}_{s\infty}$ by $\beta_{\alpha\infty}$ in all the formulas.

III.4. Application

Eq. (III.3.29) is applied to the calculation of the pressure coefficients, C_{ps}, in a fully ionized plasma consisting of electrons and singly charged ions (hydrogen ions). The subscript i refers to the values for ions and the subscript e refers to the values for electrons. We assume $W_i = A_i = W_e = A_e = 0$ (and, therefore, using Eq. (II.3.7), $P_i = \bar{P}_i = P_e = \bar{P}_e = 0$). Furthermore, we make the assumption:

$$\rho\, dQ \approx \rho_i\, dQ_i + \rho_e\, dQ_e\,, \tag{III.4.1}$$

which becomes, approximating ρ_i, ρ_e by their values in the undisturbed stream, using Eqs. (I.2.2), (I.2.16) and neglecting m_e in comparison to m_i :

$$dQ = dQ_i \tag{III.4.2}$$

Following (5) we split dQ into the form:

$$dQ = d\bar{Q} + (J)^2 (\rho\sigma u)^{-1}\, dt\,, \tag{III.4.3}$$

where $d\bar{Q}$ is all the heat injected from outside into the gross fluid except the Joule heat, $(J)^2(\rho\sigma u)^{-1}$ is the rate of Joule heat, σ is the electrical conductivity of the gross fluid and dr is an element of arc of the gross fluid streamline. Assuming $d\bar{Q} \equiv 0$, approximating $(J)^2(\rho\sigma u)^{-1}$ by its value in the undisturbed stream and using Eqs. (I.2.2), (I.2.16), (I.2.17), (I.3.2), (I.3.6), (III.4.2), (III.4.3) we have:

$$dQ_i = (e_i)^2 \nu_\infty u_\infty m_i^{-1} \sigma^{-1} . \qquad \text{(III.4.4)}$$

Assuming $\epsilon_i = 1$, $\epsilon_e = 0$ and using Eqs. (I.3.9), (II.7.28), (II.7.31), (II.7.37), (III.2.5), (III.3.4), (III.3.22) in Eq. (III.4.4) we get:

$$C_{pi} = C'_{pi}\left[1-(M_{i\infty})^2\right]^{-1} + 2\left[1-(M_{i\infty})^2\right]^{-1}\Big\{(K_i-1)(M_{i\infty})^2\int_\infty^{\tilde{\Gamma}_i} d\tilde{Q}_i -$$
$$-\Big[(M_{i\infty})^2 R_{mi} R_{H_i^3}\int_\infty^{\tilde{\Gamma}_i} d\tilde{x}^2\Big]\Big[\Big[1+(K_i-1)(M_{i\infty})^2\Big[\int_\infty^{\tilde{\Gamma}_i} d\tilde{Q}_i -$$
$$-R_{mi} R_{H_i^3}\int_\infty^{\tilde{\Gamma}_i} d\tilde{x}^2 + R_{mi} R_{H_i^2}\int_\infty^{\tilde{\Gamma}_i} \tilde{u}^3_{ip}\, d\tilde{x}^1\Big]\Big]\Big]^{-1}\Big\} . \qquad \text{(III.4.5)}$$

Eq. (III.4.5) becomes, using Eqs. (II.6.3), (II.6.7), (II.6.16) to (II.6.18), (II.7.38), (III.4.4):

$$C_{pi} = C'_{pi}\left[1-(M_{i\infty})^2\right]^{-1} + 2M_{i\infty}\left[1-(M_{i\infty})^2\right]^{-1}\Big\{(K_i-1)A_{\sigma_i}\int_\infty^{\tilde{\Gamma}_i} d\tilde{\Gamma}_i -$$
$$-\Big[A_{H_i}\tilde{H}^3_\infty\int_\infty^{\tilde{\Gamma}_i} d\tilde{x}^2\Big]\Big[\Big[1+(K_i-1)M_{i\infty}\Big[A_{\sigma_i}\int d\tilde{\Gamma}_i + A_{H_i}\Big(\tilde{H}^2_\infty\int_\infty^{\tilde{\Gamma}_i} \tilde{u}^3_{ip}\, d\tilde{x}^1 -$$

$$- \tilde{H}_{\infty}^{3} \int_{\tilde{\Gamma}_i}^{\infty} d\tilde{x}^2 \Big) \Big] \Big]^{-1} \Bigg\} . \tag{III.4.6}$$

where A_{σ_i} , A_{H_i} are given by:

$$A_{\sigma_i} = (e_i)^2 \nu_\infty L m_i^{-1} \alpha_{i\infty}^{-1} \sigma^{-1} ; \quad A_{H_i} = e_i \mu_e H_\infty L m_i^{-1} \alpha_{i\infty}^{-1} . \tag{III.4.7}$$

We calculate C_{pi} using the following values*:

$$C'_{pi} = 1 ; \quad K_i = \frac{5}{3} ; \quad \nu_\infty = 10^{18} m^{-3} ; \quad L = 1 m. ; \quad \sigma = 88.5 \text{ mhos-}m^{-1} ;$$

$$\alpha_{i\infty} = 2 \times 10^4 \text{ m-sec}^{-1} ; \quad \int_{\tilde{\Gamma}_i}^{\infty} d\tilde{\Gamma}_i = 1 ; \quad \int_{\tilde{\Gamma}_i}^{\infty} d\tilde{x}^2 = 5 \times 10^{-2} ;$$

$$\int_{\tilde{\Gamma}_i}^{\infty} \tilde{u}_{ip}^3 d\tilde{x}^1 = 5 \times 10^{-2} ; \quad e_i = 1.6 \times 10^{-19} \text{ Cb.} ; \quad m_i = 1.67 \times 10^{-27} \text{ kg.} ;$$

$$\mu_e = 4\pi \times 10^{-7} \text{ kg-m-Cb}^{-2} ; \quad H_\infty = 10^2 \text{ gauss.} \tag{III.4.8}$$

Figs. 2 to 5 show the results compared with the isentropic curve $C_p = C'_p [1 - (M_\infty)^2]^{-1}$ for various orientations of the magnetic field**. The relation between M_∞ and $M_{i\infty}$ is assumed to be (11, p. 10, Eq. (31)):

---

* The value of $\alpha_{i\infty}$ is based on a temperature of 30,000°K approximately. σ is calculated from the formula: $\sigma = 1.56 \times 10^{-4} \times T^{1.5} \times [\log(1.23 \times 10^4 \times T^{1.5} \times n_e^{-0.5})]^{-1} \pm 15\%$, where $T \approx$ 30,000°K = plasma temperature, $n_e \approx 10^{24}$ cm^{-3} = electron number density. This formula is given in (Spitzer, L., Jr., and R. Härm: Transport Phenomena in a Completely Ionized Gas. Phys. Rev., 89, 997, 1953.).

** The calculations were made on a CDC 3600 computer at the Michigan State University computing center.

$$(M_{\infty})^2 = \tfrac{1}{2}(M_{\ell\infty})^2 . \tag{III.4.9}$$

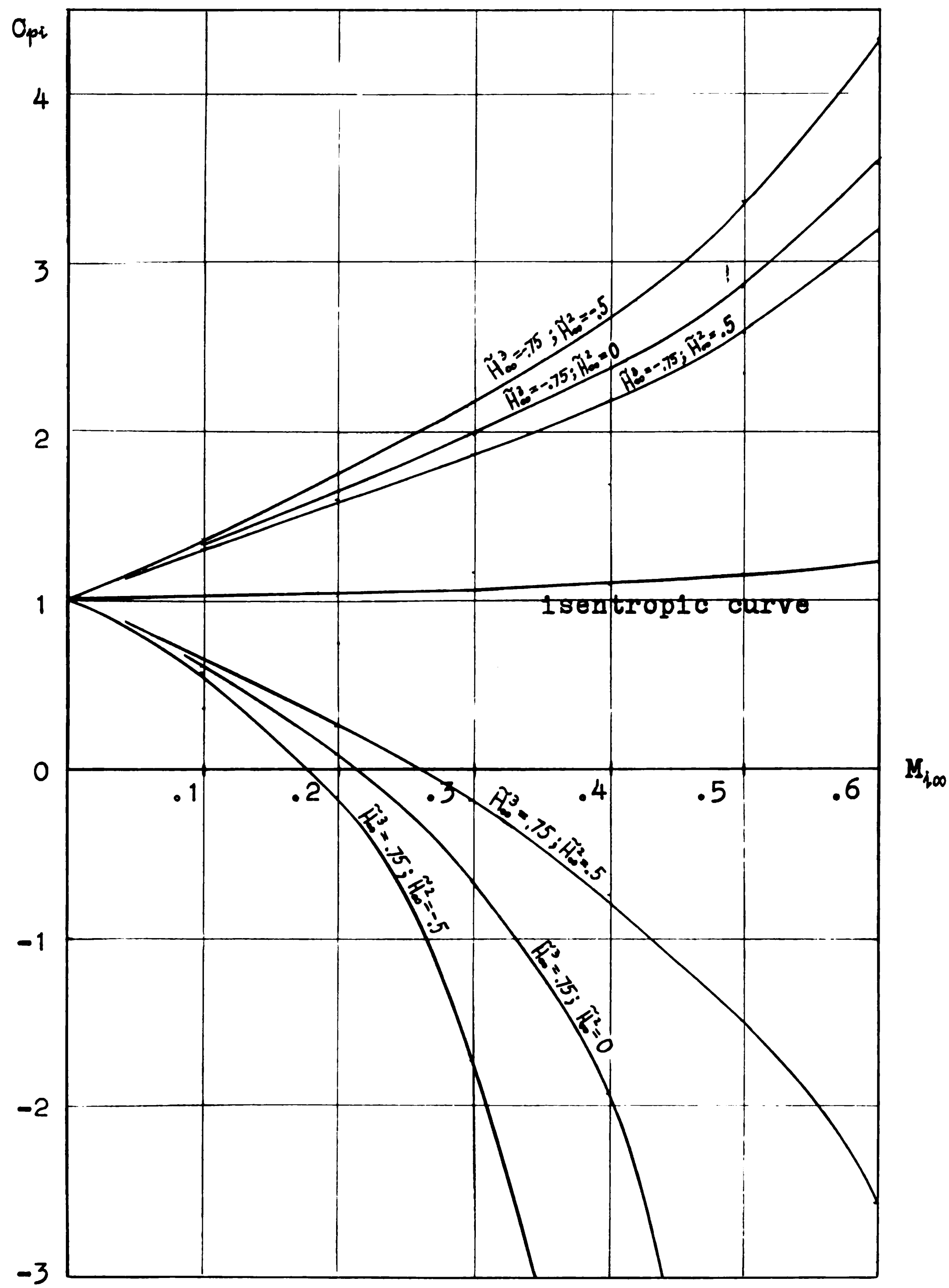

Fig. 2. Ion pressure coefficient for $\tilde{H}^3_\infty = .75$, $\tilde{H}^3_\infty = -.75$.

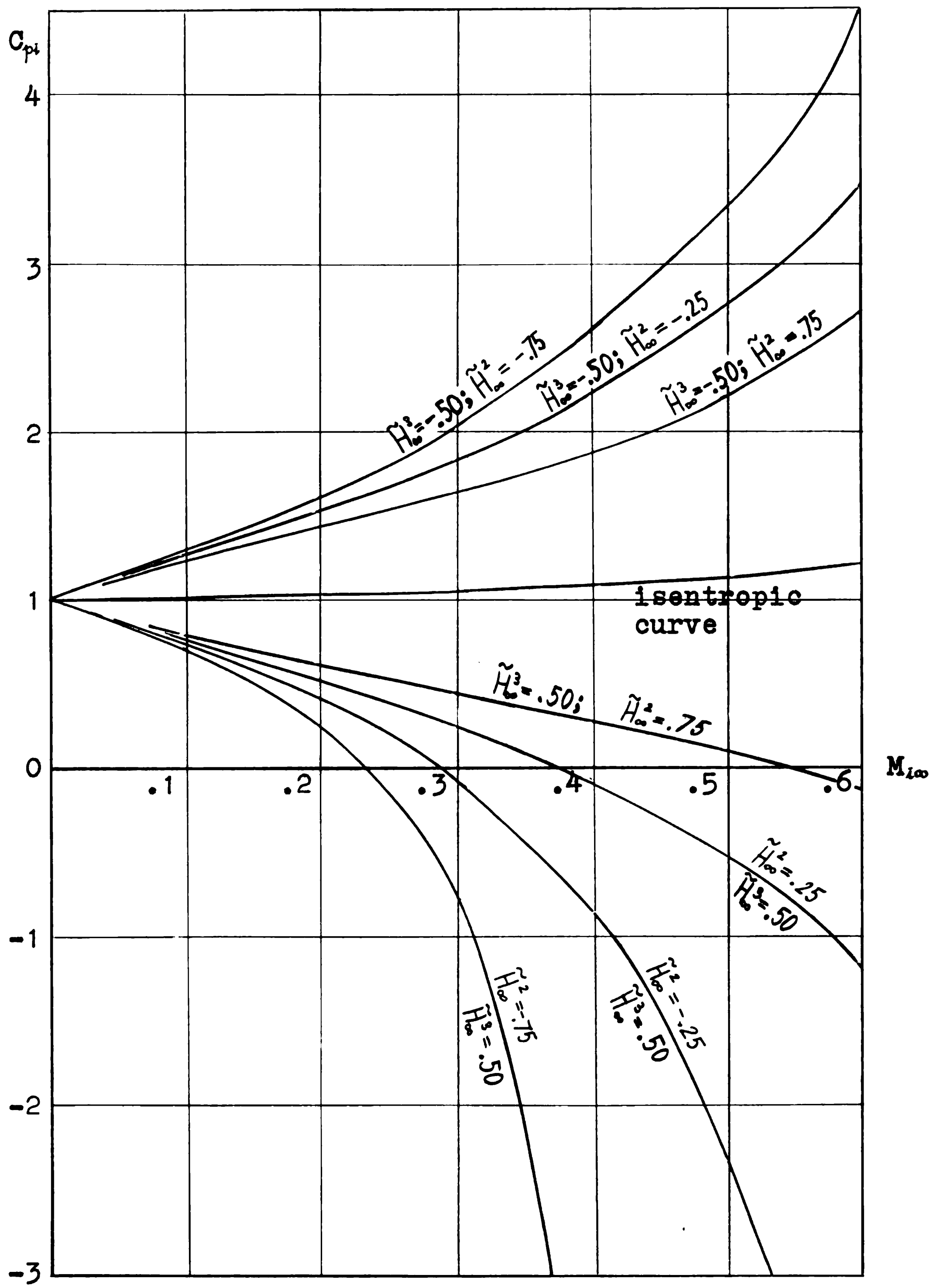

Fig. 3. Ion pressure coefficient for $\tilde{H}^3_\infty = .50$, $\tilde{H}^3_\infty = -.50$.

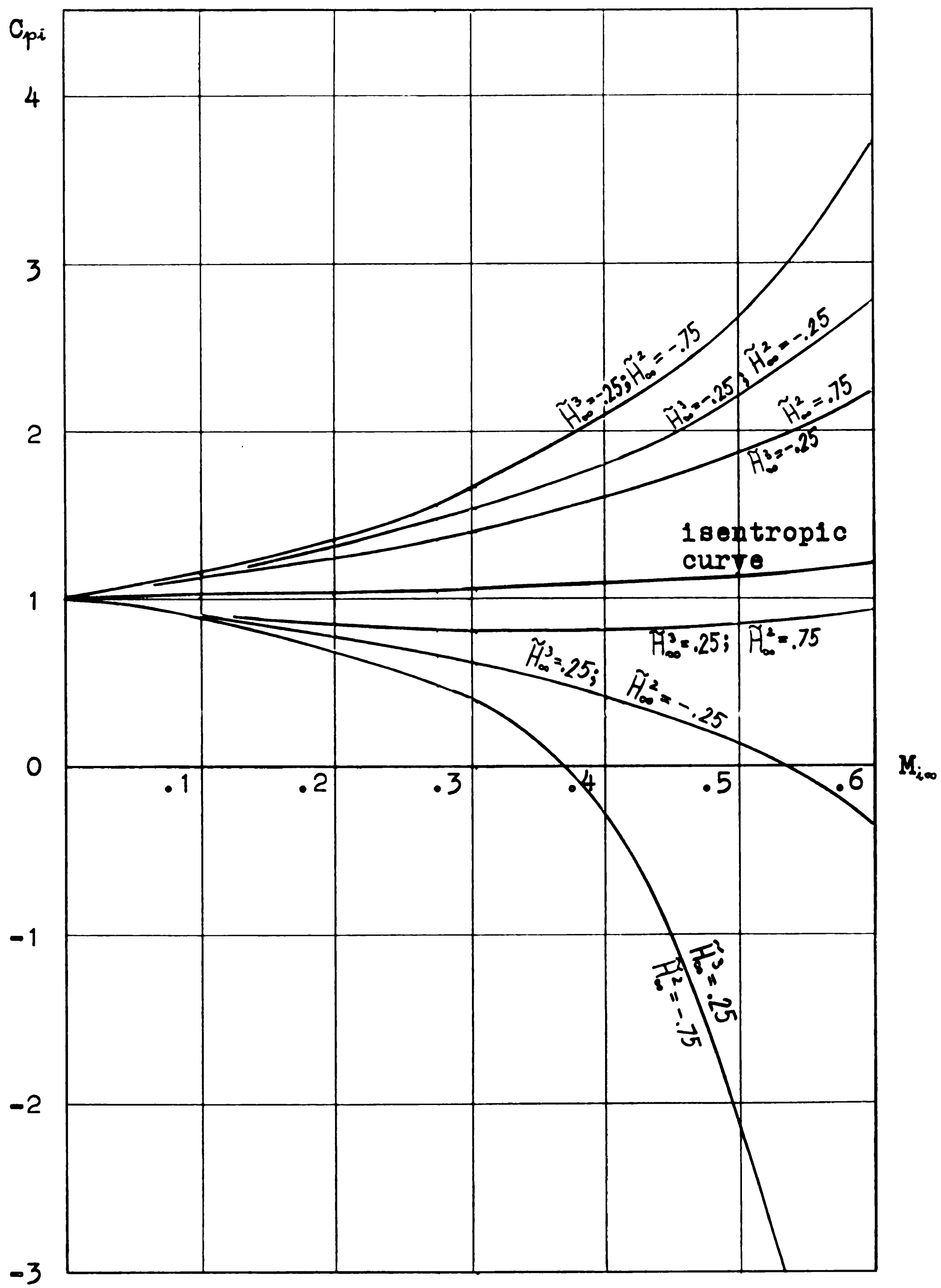

Fig. 4. Ion pressure coefficient for $\tilde{H}^3_\infty = .25$, $\tilde{H}^3_\infty = -.25$

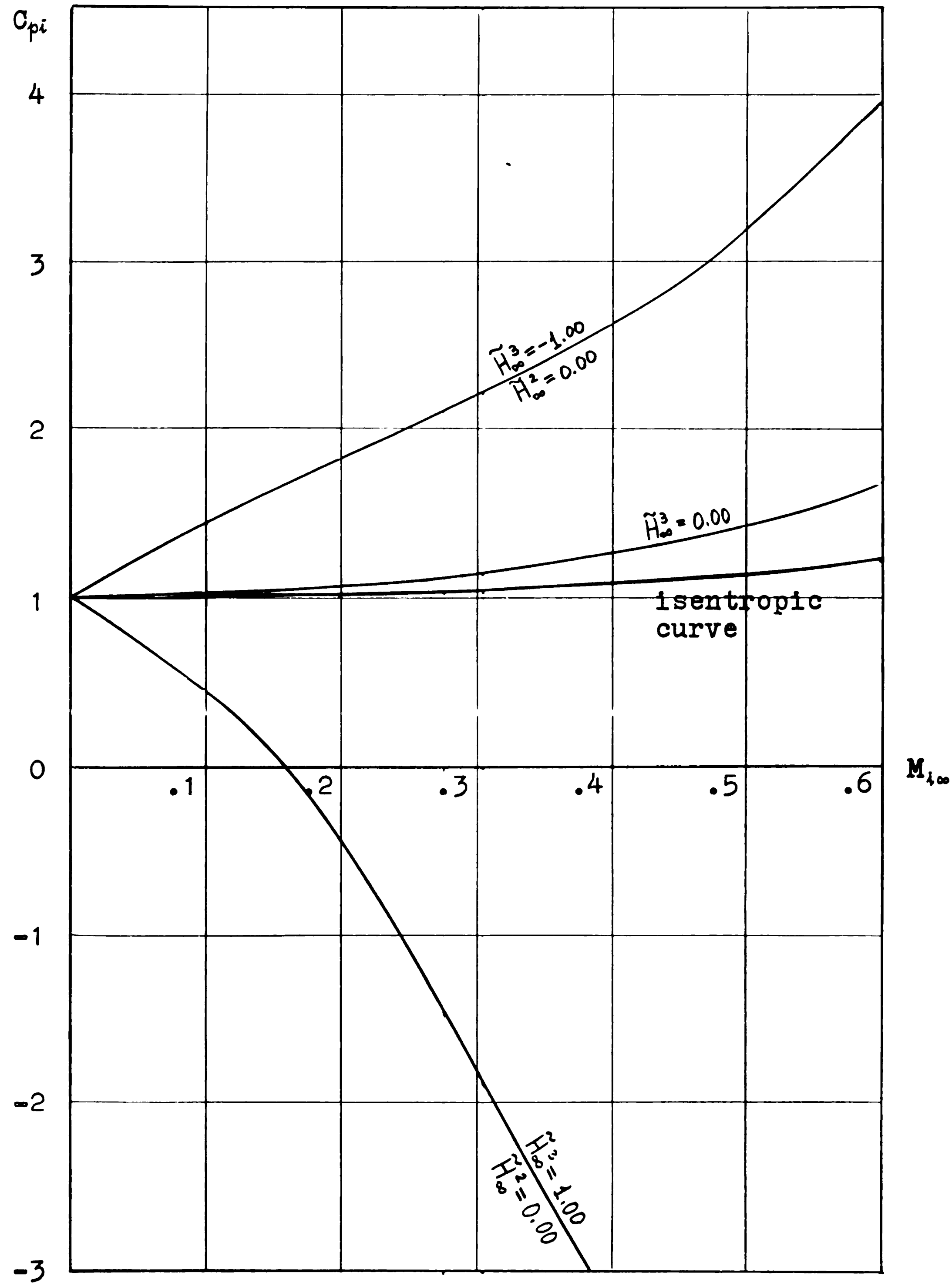

Fig. 5. Ion pressure coefficient for $\tilde{H}^3_\infty = -1.00$, $\tilde{H}^3_\infty = 0.00$ and $\tilde{H}^3_\infty = 1.00$.

APPENDIX A

Some Remarkas on the Functions B_s

Carrying out the differentiation in Eq. (II.2.1) and using Eqs. (II.1.1), (II.3.2) we get, after a few manipulations:

$$\frac{d}{dt}\left[\log\left(1-B_s\right)\right] = -\rho_s^{-1}\sigma_s . \qquad \text{(A.1)}$$

Integrating Eq. (A.2) along a streamline of the s-th fluid component, starting from an ∞ subscripted state to some end state, and using the relation $u_s = \frac{dt_s}{dt}$, we have:

$$1-B_s = \left(1-B_{s\infty}\right)\exp\left(-\int_{\infty}^{\Lambda_s}\rho_s^{-1}u_s^{-1}\sigma_s\, dt_s\right). \qquad \text{(A.2)}$$

In the linearized, steady quasi-three dimensional flow we assume $B_{s\infty}=0$, $\rho_s=\rho_{s\infty}+\rho_{sp}$, $u_s=\left[(u_{s\infty}+u^1_{sp})^2+(u^2_{sp})^2+(u^3_{sp})^2\right]^{\frac{1}{2}}$, $dt_s \approx d\Gamma_s$, where $d\Gamma_s$ is an element of arc taken along the curve Γ_s which is the projection of the streamline, Λ_s, on the (x^1, x^2) plane. Using Eqs. (II.6.1), (II.6.2), (II.6.5), (II.6.9) and the relations above, we get , neglecting higher than first order terms:

$$1-B_s = \exp\left(-\int_{\tilde{\Gamma}_s}^{\infty} \tilde{\sigma}_s \, d\tilde{\Gamma}_s\right) \approx 1-\int_{\tilde{\Gamma}_s}^{\infty} \tilde{\sigma}_s \, d\tilde{\Gamma}_s \quad , \tag{A.3}$$

from which:

$$B_s \approx \int_{\tilde{\Gamma}_s}^{\infty} \tilde{\sigma}_s \, d\tilde{\Gamma}_s \quad . \tag{A.4}$$

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