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LIMIT THEOREMS FOR DISCRETE PARAMETER RANDOM EVOLUTIONS

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ABSTRACT

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Let E_N be a subset of $\mathbb{R}$. For y in $\mathbb{R}$ and $N = 1,2,\ldots$ let $\{X^{N,y}(k): k = 0,1,\ldots\}$ be a Markov chain in E_N with transition $P^{N,y}$. Assume that for every x in E_N and y in $\mathbb{R}$

$$|E_x(X^{N,y}(1)-x)-\alpha_N p(x,y)-\alpha_N^2 r(x,y)| \leq \gamma_N \rho_N(y),$$

$$|E_x(X^{N,y}(1)-x)^2-\alpha_N^2 s(x,y)| \leq \gamma_N \rho_N(y),$$

where $\alpha_N \in \mathbb{R}^+$, $\gamma_N \in \mathbb{R}^+$ and p,r,s and ρ_N are functions satisfying some additional conditions. Let $\{Y(k): k = 0,1,\ldots\}$ be an ergodic Markov chain in $\mathbb{R}$ with transition P and $Z_N(k) = \{(X_N(k), Y(k)): k = 0,1,\ldots\}$ be the Markov chain in $E_N \times \mathbb{R}$ with transition P^N satisfying

$$P^N((x,y),A \times B) = P^{N,y}(x,A)P(y,B).$$

Then under some technical conditions it is shown that, as $\alpha_N \vee (\gamma_N/\alpha_N) \to 0$ $X^N([\cdot/\alpha_N^2])$ converges to a diffusion process that we characterize by its generator in terms of p,r,s and P. We then use this result to obtain a diffusion approximation to the Wright-Fisher model in Markovian environments, among other applications.

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Notation

$\mathcal{P}(E)$ is the set of Borel probability measures on E.

$\mathcal{B}(E)$ is the σ-algebra of Borel sets of E.

$Bor(E)$ is the set of Borel measurable functions on E.

$B(E)$ is the set of bounded Borel functions on E.

If $f \in Bor(E)$ and the integral of f with respect to the positive measure m on E exists we shall write it

$$\int_E f\,dm \quad \text{or} \quad m(f).$$

$C(E)$ is the set of continuous functions on E.

$\hat{C}(E)$ is the set of continuous functions on E vanishing at infinity.

$C_K(E)$ is the set of continuous functions on E with compact support.

$C_K^r(E)$ is the set of r times continuously differentiable functions on E with compact support.

If f is a function on E and E_N is a subset of E we will also denote by f the restriction of f to E_N.

INTRODUCTION

The primary purpose of this work is to prove limit theorems for discrete parameter random evolutions. Before describing the kind of theorems we have in mind we first recall a few definitions and results pertaining to the continuous case. Intuitively a continuous parameter random evolution describes a situation in which a process controls the development of another process. When the controlling (or driving) process is continuous parameter Markov the main questions concerning representation and asymptotic theorems have been answered.

To describe the type of results which have been obtained we consider one of the simplest examples of a continuous parameter random evolution. The model is that of a particle moving on the real line at one of n possible velocities $v_1,\dots,v_n$. It changes velocity at random according to a pure jump Markov process $Y(\cdot)$ in $\{1,2,\dots,n\}$ with generator $\dot{Q}$. Let $X(t)$ and $v_{Y(t)}$ be respectively the position and velocity of the particle at time t. The Markov process $Z(\cdot) = (X(\cdot), Y(\cdot))$ has generator

$$Af(x,i) = v_i \frac{d}{dx} f(x,i) + \sum_{j=1}^{n} q_{ij} f(x,j)$$

for $f \in \hat{C}^{1,0}(\mathbb{R} \times \{1,2,\dots,n\})$ and if for such an f we define u by

$$u(t,x,i) = E_{x,i} f(Z(t)) ,$$

then u is the solution of the hyperbolic system

$$\frac{du}{dt}(t,x,i) = v_i \frac{du}{dx}(t,x,i) + \sum_{j=1}^{n} q_{ij}\, u(t,x,j), \quad 1 \le i \le n.$$

It is a well known fact that for certain elliptic or parabolic equations the solution can be expressed as the expectation of a function of a Markov process. Random evolutions provide an example where this is also true for hyperbolic equations. It is this type of consideration that led Griego and Hersh (1969) to their definition of random evolutions and, as in the elliptic case, a purely probabilistic analysis of random evolutions can lead to new methods of proof for problems arising outside probability.

For continuous parameter random evolutions, asymptotic theorems involve a balance between two limits: one takes the limit of small stochastic disturbances over long periods of time. Depending on the way this scaling is done at least two types of limit theorems have been obtained: one corresponds to the weak law of large numbers (a first order limit theorem), another to the central limit theorem (a second order limit theorem). For instance for the model of the particle moving on the real line considered above, of interest is the limiting behavior, as $\epsilon \to 0$, of the process

$$X_\epsilon(t) = x + \int_0^t v_{Y(s/\epsilon)}\,ds$$

in the first order case and of the process

$$X_\epsilon(t) = x + \int_0^t (1/\epsilon) v_{Y(s/\epsilon^2)} ds$$

in the second order case.

The connection between the first order limit theorem (resp. the second order limit theorem) and the weak law of large numbers (resp. the CLT) can easily be seen from this example. Consider for instance the second order case. Let $\rho_0, \rho_1, \ldots$ be the successive states occupied by $Y(\cdot)$, $\tau_0, \tau_1, \ldots$ the time spent there, $M(t)$ the number of jumps in the time interval $[0,t]$ and for $j = 1,2,\ldots$ let

$$S_0 = 0,$$

$$S_j = \sum_{i=0}^{j-1} \tau_i.$$

Assume $\epsilon = N^{-\frac{1}{2}}$ and write $X_N(t)$ for $X_\epsilon(t)$. Making a change of variables

$$X_N(t) = x + N^{-\frac{1}{2}} \int_0^{Nt} v_{Y(s)} ds$$

$$= x + N^{-\frac{1}{2}} \left(\sum_0^{M(Nt)-1} v_{Y(S_j)} \tau_j + (Nt - S_{M(Nt)}) \, v_{Y(S_{M(Nt)})} \right).$$

Suppose now that instead of being exponentially distributed, for $j = 0,1,\ldots$ τ_j satisfies $P(\tau_j = 1) = 1$ and let $t = 1$ and $x = 0$. We obtain then

$$X_N(1) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} v_{Y(k-1)},$$

and the second order limit theorem, which is now a result about discrete parameter random evolutions, reduces to the CLT for a Markov chain in $\{v_1, \ldots, v_n\}$.

Asymptotic theorems for discrete parameter random evolutions (both driving and driven processes are discrete parameter) have been obtained by several authors. Here are two examples.

For $N = 1,2,\ldots$, let $\{(X^N(k), Y^N(k)): k = 0,1,\ldots\}$ be a homogeneous Markov chain in $\mathbb{R}^m \times \mathbb{R}^n$. Supposing that asymptotically as $N \to \infty$ the "infinitesimal" covariances and means of $X^N([\cdot/\epsilon_N])$ are $a_{ij}(x,y)$ and $b_i(x,y)$ and those of $Y^N([\cdot/\delta_N])$ are 0 and $c_\ell(x,y)$ and assuming $\lim_{N\to\infty} \delta_N = \lim_{N\to\infty} \epsilon_N/\delta_N = 0$ and the zero solution of $\dot{y} = c(x,y)$ is globally asymptotically stable, Ethier and Nagylaki (1980) show that $X^N([\cdot/\epsilon_N])$ converges weakly to a diffusion process with coefficients $a_{ij}(x,0)$ and $b_i(x,0)$. This could be regarded as an example of a limit theorem for a random evolution with feedback: $X^N(\cdot)$ is driven by $Y^N(\cdot)$ which in turn depends on $X^N(\cdot)$.

In a recent article Kushner and Huang (1981) developed a general method for proving weak convergence to a diffusion process of the sequence of appropriately scaled and interpolated solutions to the equation

$$\begin{aligned} X_N(k+1) - X_N(k) = \alpha_N^2\, k_N(X_N(k), Y_N(k)) \\ + \alpha_N g_N(X_N(k), Y_N(k)) + o(\alpha_N^2), \end{aligned}$$

where $Y_N(k)$ are random variables satisfying certain mixing conditions. In this example $X_N(\cdot)$ can be considered as the driven part of the random evolution $(X_N(\cdot), Y_N(\cdot))$. (We were unaware of Kushner and Huang's article while doing this work and there is some overlap between their results and ours.)

The main result of this work is motivated by a problem of Karlin and Levikson (1974). The problem is to derive a diffusion approximation to the Wright-Fisher genetic model with selection coefficients in a random environment. The mathematical formulation is as follows.

Let $0 < \beta_N < \alpha_N$ for each $N \geq 1$, and suppose $\alpha_N \to 0$. Let $\theta \in \mathbb{R}$ and $E_N = \{0, \frac{1}{N}, \ldots, 1\}$. For $N = 1,2,\ldots$ let $\{Z_N(k) = (X_N(k), Y(k)): k = 0,1,\ldots\}$ be a homogeneous Markov chain in $E_N \times \mathbb{R}$ with transition P^N satisfying

$$(0.1) \qquad P^N((x,y), A \times B) = P^{N,y}(x,A)\, P(y,B),$$

where P is the kernel of a stationary Markov chain in $\mathbb{R}$ and for every y in $\mathbb{R}$ $P^{N,y}(x,\cdot)$ is binomial $(N, P_{x,\sigma_N(y)})$ where

$$(0.2) \qquad P_{x,\sigma} = \frac{(1+\sigma)x}{1+\sigma x}$$

and

$$(0.3) \qquad \sigma_N(y) = (\beta_N\theta + \alpha_N y) \vee (-\tfrac{1}{2}).$$

The problem is to obtain a diffusion approximation for the sequence $\{X_N(k): k = 0,1,\ldots\}$ when it has been suitably scaled. In their article Karlin and Levikson study the case $\alpha_N = N^{-\frac{1}{2}}, \beta_N = N^{-1}$, $\{Y(k): k = 0,1,\ldots\}$ i.i.d. with $E(Y(k)) = 0$.

Our main result can be roughly stated as follows (see §2 for the exact formulation). Let E be a closed interval of $\mathbb{R}$, E_N a subset of E, $m \in P(\mathbb{R})$, $\mu_N \in P(E_N)$, $\mu \in P(E)$ and assume $\mu_N \Rightarrow \mu$. For $N = 1,2,\ldots$, let $\{Z_N(k) = (X_N(k), Y(k)): k = 0,1,\ldots\}$ be a homogeneous

Markov chain in $E_N \times \mathbb{R}$ with initial distribution $\mu_N \times m$ and transition P^N satisfying (0.1). Assume that for each y in $\mathbb{R}$, $P^{N,y}$ is the transition of a homogeneous Markov chain $\{X^{N,y}(k): k = 0,1...\}$ in E_N and P is the kernel of an ergodic Markov chain in $\mathbb{R}$ with invariant measure m. Suppose there exist functions p, r_1, r_2, s on $E \times \mathbb{R}$ and ρ_N in $L^{4/3}(dm)$, assumed to satisfy some additional conditions, such that for every x in E_N,

$$(0.4) \qquad |E_x(X^{N,y}(1)-x)-\alpha_N\, p(x,y)-\alpha_N^2\, r_1(x,y)-\beta_N\, r_2(x,y)| \leq \gamma_N\rho_N(y),$$

$$(0.5) \qquad |E_x(X^{N,y}(1)-x)^2-\alpha_N^2\, s(x,y)| \leq \gamma_N\rho_N(y),$$

$$(0.6) \qquad E_x|X^{N,y}(1)-x|^3 \leq \gamma_N\rho_N(y),$$

where $\gamma_N(\beta_N \vee \alpha_N^2)^{-1} \to 0$. Then as $N \to \infty$ the finite dimensional distributions of $X^N([\cdot/(\alpha_N^2 \vee \beta_N)])$ converge to those of a diffusion process $X(\cdot)$, with initial distribution μ, that we characterize by its generator in terms of p, r_1, r_2, s, m and P.

Depending on the relative values of α_N and β_N in (0.3) and on the scaling there are several possible limiting diffusions. The case $\alpha_N^2\, \beta_N^{-1} \to 0$ corresponds to a first order limit theorem and leads to a deterministic process. If $\alpha_N^2 = \beta_N$ (resp. $\alpha_N^2\, \beta_N^{-1} \to \infty$) condition (0.4) takes the form

$$(0.4)' \qquad |E_x(X^{N,y}(1)-x) - \alpha_N\, p(x,y)-\alpha_N^2\, r(x,y)| \leq \gamma_N\, \rho_N(y)$$

where $r = r_1 + r_2$ (resp. $r = r_1$). Replacing (0.4) by (0.4)' the cases $\alpha_N^2 = \beta_N$ and $\alpha_N^2\, \beta_N^{-1} \to \infty$ can, without loss of generality, be treated simultaneously.

Using our theorem we can generalize the result of Karlin and Levikson to Markovian environments. More precisely let $\{Y(k): k = 0,1,\ldots\}$ be a stationary Markov chain with Markov kernel P, Q the linear contraction on $L^1(dm)$ defined by

$$Qf(y) = \int f(z)\, P(y,dz),$$

and assume there exists η in $L^4(dm)$ and λ in $L^2(dm)$ such that

$$(I-Q)\eta = Id_F, \tag{0.7}$$

$$(I-Q)\lambda = \int y^2 dm - y^2. \tag{0.8}$$

Then $X_N([Nt])$ converges to the diffusion process whose generator A, with domain $C^2[0,1]$, is given by

$$A = x(1-x)[\tau(1-2x) + (\theta - xv)]\frac{d}{dx} + \frac{1}{2}x(1-x)[1 + x(1-x)(v + 2\tau)]\frac{d^2}{dx^2} \tag{0.9}$$

where $v = E[Y^2(k)]$ and τ, a constant, can be computed in terms of P. When $\{Y(k): k = 0,1,\ldots\}$ are i.i.d., which is the case studied by Karlin and Levikson, $\tau = 0$.

We can also easily prove a central limit theorem for ergodic Markov chains, another one for chain dependent random variables and obtain diffusion approximations to a certain type of stochastic difference equation. The CLT for Markov chains we have in mind can essentially be stated as follows. Let $\{Y(k): k = 0,1,\ldots\}$ be an ergodic Markov chain satisfying (0.7) and (0.8). Let

$$\sigma^2 = \int (Qn^2 - (Qn)^2)dm. \tag{0.10}$$

Then $\dfrac{Y(1) + \ldots + Y(N)}{\sigma\sqrt{N}} \xrightarrow{L} N(0,1)$.

Conditions (0.7) and (0.8) are clearly very restrictive and limit the applicability of such a result. Nevertheless it must be noticed that we do not impose any mixing condition on $\{Y(k): k = 0,1,\ldots\}$; conditions (0.7) and (0.8) are assumptions on the one step transitions of the Markov chain. In a certain sense this result is a very natural generalization to the Markov case of the CLT for i.i.d. random variables.

Generalizing this CLT we obtain a diffusion approximation for suitably scaled solutions to the equation

$$X_N(k+1) - X_N(k) = G(X_N(k), S_N(k))$$

where for $k = 0,1,\ldots$ $S_N(k) = \alpha_N\theta + \beta_N Y(k)$ and G satisfies some differentiability assumptions. An application of this can be found in Guess and Gillespie (1978). We do not pursue such applications as they have been recently treated by Huang and Kushner (1981).

In O'Brien (1974) a CLT for random variables defined on a countable Markov chain (or chain dependent random variables) is obtained. We prove a similar result without any countability assumptions.

The last result of this work is a CLT for random variables taking their values in a Riemann manifold. This is essentially a discrete parameter analogue of a theorem of Pinsky (1976, 1978).

Two approaches have been particularly successful to prove asymptotic theorems for random evolutions in the continuous case: the semigroup approximation theorems of Kurtz (1969, 1973, 1975) and the martingale problem of Stroock and Varadhan (Papanicalaou, Stroock, Varadhan, 1977). In the discrete parameter case we can adapt either of these methods. We have chosen the semigroup approach; the approximation theorem we use is a discrete parameter analogue of a theorem of Kurtz (1975 Theorem 3.15). Remarking that $Z_N(k) = (X_N(k), Y(k))$ is a Markov chain in $E_N \times \mathbb{R}$ and defining

$$L_N = \{f \in \mathrm{Bor}(E_N \times \mathbb{R}) : \sup_k E|f(Z_N(k))| < \infty\}$$

we show that for every f in a core for the generator A of a Feller semigroup there exists a sequence f_N in L_N such that

$$\sup_k E|f_N(Z_N(k)) - f(X_N(k))| \to 0$$

and

$$\sup_k E|A_N f_N(Z_N(k)) - Af(X_N(k))| \to 0$$

where

$$A_N f(x,y) = \alpha_N^{-2} (E_{x,y} f(Z_N(1)) - f(x,y)).$$

Assuming convergence of the initial distributions this implies that the finite dimensional distributions of $X_N([\cdot/\alpha_N^2])$ converge to those of a Markov process with generator A.

CHAPTER I

AN ASYMPTOTIC THEOREM FOR DISCRETE PARAMETER RANDOM EVOLUTIONS

Throughout this section E is a closed interval of $\mathbb{R}$, possibly unbounded, E_N a Borel subset of E, F a Borel subset of $\mathbb{R}$, $\{Y(k): k = 0,1,\ldots\}$ an ergodic Markov chain in F with transition P and invariant measure m. If $f \in L^1(dm)$ we define the function Qf in $L^1(dm)$ by setting

$$Qf(y) = \int f(z)P(y,dz). \tag{1.1}$$

We note that Q is a contraction in $L^p(dm)$, $1 \le p \le \infty$.

<u>Theorem 1.1</u>: Let k_1,k_2 be two integers, $k_2 > k_1 > 0$. Let p,r and s be in $\mathrm{Bor}(E \times F)$ and of the form

$$p(x,y) = a_0(x)b_0(y), \tag{1.2}$$

$$r(x,y) = \sum_{1}^{k_1} a_i(x)b_i(y), \tag{1.3}$$

$$s(x,y) = \sum_{k_1+1}^{k_2} a_i(x)b_i(y), \tag{1.4}$$

where, for $i = 0,1,\ldots,k_2$, a_i is bounded twice continuously differentiable, $b_0 \in L^4(dm)$ and for $i = 1,\ldots,k_2$, $b_i \in L^2(dm)$. Assume that for $i = 0,1,\ldots,k_2$ there are functions $\eta \in L^4(dm)$ and $\lambda_i \in L^2(dm)$ satisfying

$$(I-Q)\eta = b_0, \tag{1.5}$$

$$(I-Q)\lambda_0 = (b_0\eta - b_0^2)-\tau, \tag{1.6}$$

where

(1.7) $$\tau = \tfrac{1}{2}\{\int (Q\eta^2 - (Q\eta)^2)dm - \int b_0^2\, dm\},$$

and for $i = 1,\ldots,k_2$,

(1.8) $$(I-Q)\lambda_i = b_i - \int b_i dm.$$

Let $\mu \in P(E)$. For $N = 1,2,\ldots$ and $y \in F$ let $\{X^{N,y}(k)\colon\ k = 0,1,\ldots\}$ be a homogeneous Markov chain in E_N with transition denoted by $P^{N,y}$ and initial distribution μ_N. Suppose that for every A in $B(E_N)$ the function $(x,y) \to P^{N,y}(x,A)$ is Borel measurable and $\mu_N \Rightarrow \mu$ as $N \to \infty$. Let α_N be numbers satisfying $0 < \alpha_N \to 0$ as $N \to \infty$ and assume there exist functions ρ_N in $L^{4/3}(dm)$, numbers $\gamma_N > 0$ and $M > 0$ such that, for every x in E_N and y in F,

(1.9) $$|E_x(X^{N,y}(1) - x) - \alpha_N\, p(x,y) - \alpha_N^2\, r(x,y)| \le \gamma_N\rho_N(y),$$

(1.10) $$|E_x(X^{N,y}(1) - x)^2 - \alpha_N^2\, s(x,y)| \le \gamma_N\rho_N(y),$$

(1.11) $$E_x|X^{N,y}(1) - x|^3 \le \gamma_N\rho_N(y),$$

(1.12) $$\int \rho_N^{4/3} dm < M \quad \text{and} \quad \gamma_N\alpha_N^{-2} \to 0 \quad \text{as} \quad N \to \infty .$$

Let $\{Z_N(k) = (X_N(k), Y(k))\colon\ k = 0,1,\ldots\}$ be the Markov chain in $E_N \times F$ with initial distribution $\mu_N \times m$ and transition P^N satisfying, for $A \in B(E_N)$ and $B \in B(F)$,

(1.13) $$P^N((x,y), A \times B) = P^{N,y}(x,A)P(y,B).$$

Let A be the linear operator on $\hat{C}(E)$, with domain $C_K^2(E)$, defined by

(1.14) $\quad A = [\tau a_0 a_0' + \int r(\cdot,y)m(dy)]\,\frac{d}{dx} + [\tau a_0^2 + \frac{1}{2}\int s(\cdot,y)m(dy)]\,\frac{d^2}{dx^2}.$

a) Assume $\bar{A}$ generates a strongly continuous, positive, conservative, contraction semigroup $\{T(t)\}$ on $\hat{C}(E)$. There exists then a Markov process $X(\cdot)$ with sample paths in $C_E[0,\infty)$, semigroup $\{T(t)\}$ and initial distribution μ such that the finite dimensional distributions of $X_N([\cdot/\alpha_N^2])$ converge to those of $X(\cdot)$.

b) Furthermore if $\sup\limits_{i,y} |b_i(y)| < \infty$ and $\sup\limits_{N,y} |\rho_N(y)| < \infty$ (this being essentially the case when F is compact), convergence is in distribution in $D_E[0,\infty)$.

<u>Remark</u>: Since

$$\int (I-Q)\eta\, dm = \int \eta(z)m(dz) - \int m(dy)\int \eta(z)P(y,dz) = 0, \tag{1.14a}$$

a necessary condition for (1.5) to be satisfied is

$$\int b_0 dm = 0. \tag{1.15}$$

For the same reason (1.6) can hold only if

$$\tau = \int (b_0\eta - b_0^2)dm. \tag{1.16}$$

By (1.14a) with η replaced by η^2,

$$\int Q\eta^2 dm = \int \eta^2 dm.$$

Applying then (1.5) we easily see that

$$\int (Q\eta)^2 dm = \int (\eta^2 - 2b_0\eta + b_0^2)dm.$$

So (1.16) follows from (1.7) and (1.16) is thus always true.

In general to find sufficient conditions for (1.5), (1.6) and (1.8) is more difficult. One case where they exist is the following. Assume that for $i = 0,1,\ldots,k_2$ b_i is a polynomial of degree less than or equal to two. (This is typically what happens in the applications we consider.) Let α, α_1 and β_1 be real numbers with α and α_1 not equal to one, $\mu = \int y m(dy)$, $v = \int y^2 m(dy)$ and suppose $\{Y(k): k = 0,1,\ldots\}$ is such that if f_1 and f_2 are defined by $f_1(y) = y$ and $f_2(y) = y^2$ then

$$Qf_1(y) = \alpha y + \mu(1-\alpha) \tag{1.17}$$

and

$$Qf_2(y) = \alpha_1 y^2 + \beta_1 y + v(1-\alpha_1) - \beta_1\mu. \tag{1.18}$$

Let η_0 and λ be defined by

$$\eta_0(y) = \frac{y}{1-\alpha}$$

and

$$\lambda(y) = \frac{y^2}{1-\alpha_1} + \frac{\beta_1 y}{(1-\alpha)(1-\alpha_1)}.$$

Then $(I-Q)\eta_0 = y-\mu$ and $(I-Q)\lambda = y^2-v$.

A case of particular interest is when

$$b_0(y) = \theta_0(y-\mu)$$

and for $i = 1,\ldots k_2$,

$$b_i(y) - \int b_i \, dm = \theta_i(y-\mu) + \gamma_i(y^2-v),$$

where θ_i and γ_i are constants. Since η is then a polynomial of degree one satisfying (1.16) there exists θ and γ in $\mathbb{R}$ such that

$$\tau-(b_0\eta-b_0^2) = \theta(y-\mu) + \gamma(y^2-v).$$

The form of η and λ_i in (1.5), (1.6) and (1.8) follows then easily.

We give now two examples of Markov chains satisfying (1.17) and (1.18).

Example 1. Let $\{Y(k): k = 0,1,\ldots\}$ be the Markov chain in Z_+ with transition $P(\cdot,\cdot)$ given by

$$P(y,\cdot) = \text{bin}(y,p)*\text{Poisson}(\theta),$$

where $0 < p < 1$ and $\theta > 0$.

If $\lambda = \theta/(1-p)$, Poisson(λ) can serve as invariant measure for $\{Y(k): k = 0,1,\ldots\}$. (To prove this one can use generating functions:

$$\begin{aligned}
&\sum_y \Big(\sum_z s^z P(y;z)m(y)\Big)\\
&\quad = \sum_y (q + sp)^y\, e^{\theta(s-1)}\, \frac{\lambda^y e^{-\lambda}}{y!}\\
&\quad = e^{(\theta+p\lambda)(s-1)}\\
&\quad = e^{(\theta/(1-p))(s-1)}.)
\end{aligned}$$

Using generating functions again it is then easy to show that

$$Qf_1(y) = py + \lambda(1-p)$$

and

$$Qf_2(y) = p^2y^2 + p(1-p)(2\lambda + 1)y + \lambda(1-p)[\lambda(1-p) + 1]$$

Conditions (1.17) and (1.18) are thus satisfied here.

<u>Example 2</u>. Assume $Y(0)$ is $N(0,1)$ and $X(0), X(1),\ldots$ are i.i.d. $N(0,1)$ and independent of $Y(0)$. Let

$$Y(n+1) = \rho Y(n) + \sqrt{1-\rho^2}\, X(n),$$

where $-1 < \rho < 1$. $\{Y(n)\colon\ n = 0,1,\ldots\}$ is a stationary Markov chain with invariant measure the standard normal. Conditions (1.17) and (1.18) are easily seen to be satisfied with $\alpha = \rho$, $\alpha_1 = \rho^2$, $\beta_1 = 0$, $v = 1$. We also note that here $\eta(y) = y/(1-\rho)$ and

$$\begin{aligned}\tau &= \tfrac{1}{2}\{\int (Q\eta^2 - (Q\eta)^2)dm - \int y^2 dm\}\\ &= \tfrac{1}{2}\{\int (\frac{y^2}{(1-\rho)^2} - \frac{\rho^2y^2}{(1-\rho)^2})dm - 1\}\\ &= \rho/(1-\rho).\end{aligned}$$

If we do not want to assume (1.17) and (1.18) the following result (Revuz 1975 Theorem 6.3.10) gives sufficient conditions for (1.5), (1.6) and (1.8). Let

$$B^0(F) = \{f \in B(F)\colon\ m(f) = 0\}.$$

If $\{Y(k)\colon\ k = 0,1,\ldots\}$ is a quasi compact Harris chain then

$$(I-Q)B(F) = B^0(F).$$

We can use this result if for $i = 0,1,\ldots k_2$ b_i is bounded.

Proof of Theorem 1.1: We will prove part a) of this theorem using Theorem A.1.a). To prove part b) one can use Theorem A.1.b).

Let $K = \{g \in B(E): g = \text{constant} + f \text{ with } f \in \hat{C}(E)\}$. Clearly $f \in \hat{C}(E)$ and $g \in K$ implies $f\,g \in K$. The existence of a Markov process with sample paths in $C_E[0,\infty)$ and initial distribution μ corresponding to $\{T(t)\}$ follows from Theorem A.2.

Let $\sim$ be the equivalence relation on $Bor(E_N \times F)$ defined by

$$f \sim g \quad \text{iff} \quad E|f(Z_N(k)) - g(Z_N(k))| = 0 \text{ for every } k \geq 0,$$

and let

$$L_N = \{f \in Bor(E_N \times F)/\sim : \sup_k E|f(Z_N(k))| < \infty\}. \tag{1.19}$$

For $f \in L_N$, $x \in E_N$, $y \in F$ let

$$T_N f(x,y) = \int f(u,v)\, P^N(x,y,du,dv) \tag{1.20}$$

$$A_N f = \alpha_N^{-2} (T_N - I). \tag{1.21}$$

For $f \in B(E)$ define $\pi_N f \in L_N$ by

$$\pi_N f(x,y) = f(x), \tag{1.22}$$

and let $D = C_K^\infty(E)$ be the space of infinitely differentiable functions with compact support. (D is a core for A.) Condition (A.2) of Theorem (A.1) is assumed. We construct a sequence f_N in L_N satisfying (A.7), (A.8), (A.9) and (A.10).

Let $x \in E_N$, $y \in F$ and B be the linear operator on L_N defined by

(1.23) $$Bg(x,y) = \int g(x,v)P(y,dv) - g(x,y).$$

Let f be in D, f_1, f_2, h, h_1, k in L_N and let

$$f_N = \pi_N f + \alpha_N h + \alpha_N^2 k.$$

Using the triangle inequality and denoting $\pi_N f$ by f,

(1.24)
$$\begin{aligned} &\|\alpha_N^{-2}(T_N f_N - f_N) - \pi_N Af\| \\ &\le \alpha_N^{-2} \|T_N f - f - \alpha_N f_1 - \alpha_N^2 f_2\| + \alpha_N^{-1}\|f_1 + Bh\| \\ &+ \alpha_N^{-1}\|T_N h - h - Bh - \alpha_N h_1\| \\ &+ \|h_1 + f_2 + Bk - \pi_N Af\| \\ &+ \|T_N k - k - Bk\|, \end{aligned}$$

where for $f \in L_N$, $\|f\| = \sup_k E|f(Z_N(k))|$.

To finish the proof we find f_1, f_2, h, h_1 and k such that

(1.25) $$\alpha_N^{-2}\|T_N f - f - \alpha_N f_1 - \alpha_N^2 f_2\| \to 0,$$

(1.26) $$f_1 + Bh = 0,$$

(1.27) $$\alpha_N^{-1}\|T_N h - h - Bh - \alpha_N h_1\| \to 0,$$

(1.28) $$h_1 + f_2 + Bk - \pi_N Af = 0 \quad \forall N,$$

(1.29) $$\|T_N k - k - Bk\| \to 0.$$

For $x \in E_N$, $y \in F$, let

(1.30) $$f_1(x,y) = p(x,y)f'(x),$$

and

(1.31) $$f_2(x,y) = r(x,y)f'(x) + \frac{1}{2}\, s(x,y)f''(x).$$

Clearly f_1 and f_2 belong to L_N. Using a Taylor expansion with remainder term of order 3 and (1.20),

$$|T_N f(x,y) - f(x) - f'(x)\int(u-x)P^{N,y}(x,du) - \tfrac{1}{2}\, f''(x)\int(u-x)^2 P^{N,y}(x,du)|$$

$$\le K_1 \int|u-x|^3\, P^{N,y}(x,du),$$

where $K_1 = \sup_x\{|f'''(x)|\}$. By (1.9), (1.10), (1.11) and the triangle inequality,

$$|T_N f(x,y) - f(x) - \alpha_N f_1(x,y) - \alpha_N^2 f_2(x,y)|$$

$$|T_N f(x,y) - f(x) - f'(x)\int(u-x)P^{N,y}(x,du) - \tfrac{1}{2}\, f''(x)\int(u-x)^2 P^{N,y}(x,du)|$$

$$+\ |f'(x)\int(u-x)P^{N,y}(x,du) + \tfrac{1}{2}\, f''(x)\int(u-x)^2 P^{N,y}(x,du) - \alpha_N f_1(x,y) - \alpha_N^2 f_2(x,y)|$$

$$\le 3K_2\, \gamma_N \rho_N(y),$$

where $K_2 = \sup_x\{K_1, |f'(x)|, |f''(x)|\}$. (1.25) follows then from (1.12).

For $x \in E_N$, $y \in F$, let

(1.32) $$h(x,y) = \eta(y)a_0(x)f'(x).$$

(Note that $\|h\| \leq \sup_x |a_0(x)f'(x)|\ \|\eta\|$ implies $h \in L_N$). By (1.23) and (1.5),

$$Bh(x,y) = \int h(x,z)P(y,dz) - h(x,y)$$

$$= a_0(x)f'(x)(\int \eta(z)P(y,dz) - \eta(y))$$

$$= -f_1(x,y).$$

(1.26) is thus proved.

For $x \in E_N$, $g \in F$, let

(1.33) $$h_1(x,y) = p(x,y)\int \frac{\partial h}{\partial x}(x,v)P(y,dv).$$

By (1.32), (1.2) and (1.5),

$$h_1(x,y) = a_0(x)b_0(y)(a_0(x)f'(x))' \int \eta(z)P(y,dz)$$

$$= a_0(x)(a_0(x)f'(x))' \,[b_0(y)\eta(y) - b_0^2(y)].$$

Since b_0 and η are in $L^4(dm)$ and $a_0(x)(a_0(x)f'(x))'$ is bounded, h_1 is in L_N. Using a Taylor expansion in the first variable with remainder term of order two and (1.32),

$$|T_N h(x,y) - h(x,y) - Bh(x,y) - \int(u-x)\frac{\partial h}{\partial x}(x,v)P^N(x,y,du,dv)|$$

$$\leq K_3 \int(u-x)^2|\eta(v)|P^N(x,y,du,dv)$$

where $K_3 = \sup_x\{|(a_0(x)f'(x))'|,\ |(a_0(x)f'(x))''|\}$. By the triangle inequality, (1.13) and (1.33),

$$(1.34)\quad |T_N h(x,y) - h(x,y) - Bh(x,y) - \alpha_N h_1(x,y)|$$

$$\leq |T_N h(x,y) - h(x,y) - Bh(x,y) - \int(u-x)\,\frac{\partial h}{\partial x}(x,v)P^N(x,y\ du,dv)|$$

$$+ |\int(u-x)\,\frac{\partial h}{\partial x}(x,v)P^N(x,y,du,dv) - \alpha_N\, h_1(x,y)|$$

$$\leq K_3 \int(u-x)^2|\eta(v)|P^N(x,y,du,dv)$$

$$+ |\int(u-x)\,\frac{\partial h}{\partial x}(x,v)P^N(x,y,du,dv) - \alpha_N\, h_1(x,y)|$$

$$\leq K_3 \int(u-x)^2 P^{N,y}(x,du) \int |\eta(v)|P(y,dv)$$

$$+ |\int(u-x)P^{N,y}(x,du) \int \frac{\partial h}{\partial x}(x,v)P(y,dv) - \alpha_N p(x,y) \int\frac{\partial h}{\partial x}(x,v)P(y,dv)|.$$

By (1.9) and (1.10) we can find $s_1 \in L^2(dm)$ and $r_1 \in L^2(dm)$ such that

$$(1.35)\qquad \int(u-x)^2 P^{N,y}(x,du) \leq \alpha_N^2\, s_1(y) + \gamma_N\rho_N(y),$$

$$(1.36)\qquad |\int(u-x)P^{N,y}(x,du) - \alpha_N p(x,y)| \leq \alpha_N^2\, r_1(y) + \gamma_N\rho_N(y).$$

Using then the inequality $|\int\frac{\partial h}{\partial x}(x,v)P(y,dv)| \leq K_3 \int |\eta(v)|P(y,dv)$, (1.35) and (1.36),

$$|T_N h(x,y) - h(x,y) - Bh(x,y) - \alpha_N h_1(x,y)|$$

$$\leq K_3(2\gamma_N\rho_N(y) + \alpha_N^2(r_1(y) + s_1(y))) \int |\eta(v)|P(y,dv).$$

Let then $\theta \in L^4(dm)$ be defined by $\theta(y) = \int|\eta(v)|P(y,dv)$. Since $\rho_N \in L^{4/3}(dm)$, the Hölder inequality implies

$$\|T_N h - h - Bh - \alpha_N h_1\| \le K_3(2\gamma_N \|\rho_N \theta\| + \alpha_N^2 \|(r_1 + s_1)\theta\|)$$

$$\le K_3(\gamma_N(\int \rho_N^{4/3} dm)^{3/4}(\int \theta^4 dm)^{1/4} + \alpha_N^2(\int (r_1 + s_1)^2 dm)^{1/2}(\int \theta^2 dm)^{1/2},$$

and (1.27) follows from (1.12).

We now have to find k in L_N such that (1.28) holds. We first rewrite A in a more condensed form. Let $x \in E_N$,

$$c_0(x) = a_0(x)a_0'(x)f'(x) + a_0^2(x)f''(x),$$

$$c_i(x) = \begin{cases} a_i(x)f'(x) & \text{for} \quad 1 \le i \le k_1 \\ \tfrac{1}{2}\, a_i(x)f''(x) & \text{for} \quad k_1 < i \le k_2. \end{cases}$$

Using the definition of A, f_2, h_1 and h we get

$$\pi_N Af(x,y) = \tau c_0(x) + \sum_{i=1}^{k_2} c_i(x) \int b_i(y)m(dy),$$

$$\begin{aligned} f_2(x,y) &= r(x,y)f'(x) + \tfrac{1}{2}\, s(x,y)f''(x) \\ &= (\sum_1^{k_1} a_i(x)b_i(y))f'(x) + \tfrac{1}{2}(\sum_{k_1+1}^{k_2} a_i(x)b_i(y))f''(x) \\ &= \sum_1^{k_2} c_i(x)b_i(y), \end{aligned}$$

$$\begin{aligned} h_1(x,y) &= p(x,y) \int \frac{\partial h}{\partial x}(x,z)P(y,dz) \\ &= a_0(x)b_0(y)[a_0'(x)f'(x) + a_0(x)f''(x)] \int \eta(z)P(y,dz) \end{aligned}$$

$$= c_0(x)[b_0(y)\eta(y) - b_0^2(y)].$$

Let

$$k(x,y) = \sum_0^{k_2} c_i(x)\lambda_i(y).$$

(Note that $\|k\| \leq (\sup_{i,x} |c_i(x)|)\ (\sum_{i=0}^{k_2} \|\lambda_i\|) < \infty$ implies $k \in L_N$.)

By (1.20),

$$\begin{aligned} Bk(x,y) &= \int k(x,z)P(y,dz) - k(x,y) \\ &= \sum_0^{k_2} c_i(x)\ (\int \lambda_i(z)P(y,dz) - \lambda_i(y)). \end{aligned}$$

By (1.5), (1.6), (1.7) and (1.8),

$$\begin{aligned} Bk &= c_0[\tau - (b_0\eta - b_0^2)] + \sum_{i=1}^{k_2} c_i[\int b_i(z)m(dz) - b_i] \\ &= \pi_N Af - h_1 - f_2. \end{aligned}$$

(1.28) follows. Let

$$\Lambda(y) = \sum_{i=0}^{k_2} |\lambda_i(y)|,$$

$$\theta_1(y) = \int \Lambda(z)P(y,dz),$$

$$K_4 = \sup_{x,i} |c_i'(x)|.$$

Since $k(x,y) = \sum_{i=0}^{k_2} c_i(x)\lambda_i(y)$, using the mean value theorem, (1.20) and (1.10),

$$|T_N k(x,y) - k(x,y) - Bk(x,y)| \leq K_4\theta_1(y) \int (u-x)P^{N,y}(x,du)$$

$$\leq K_4\theta_1(y)\ (\int(u-x)^2 P^{N,y}(x,du))^{1/2}$$

$$\leq K_4(\gamma_N\rho_N(y) + \alpha_N^2|s(x,y)|)^{1/2}\ \theta_1(y)$$

$$\leq K_4\ \alpha_N(\gamma_N\alpha_N^{-2}\rho_N(y) + s_1(y))^{1/2}\theta_1(y).$$

Using the fact that $\theta_1 \in L^2(dm)$, $\rho_N^{\bullet} \in L^{4/3}(dm)$, $s_1 \in L^2(dm)$ imply that $\theta_1(\rho_N + s_1)^{1/2} \in L^1(dm)$ and assuming N big enough for $\gamma_N\alpha_N^{-2}$ to be smaller than one, we get

$$\|T_N k - k - Bk\| \leq \alpha_N\ K_4\ \|(\rho_N + s_1)^{1/2}\theta\|.$$

By the Holder inequality

$$\|(\rho_N + s_1)^{1/2}\theta\|^2 = (\int(\rho_N + s_1)^{1/2}\ \theta\ dm)^2$$

$$\leq \int(\rho_N + s_1)dm \int \theta^2 dm.$$

By (1.12) the term on the right side of the inequality is bounded by a number independent of N and (1.29) follows. Since $\{f_N\}$ clearly satisfies (A.7), (A.8) and (A.9) Theorem 2.1 is proved.

<u>Remark</u>. It is possible to give an abstract semigroup version of the previous theorem. For $N = 1,2,\ldots$, let L_N be a Banach space, T_N a linear contraction on L_N, ϵ_N a positive number and put $A_N = (T_n - I)/\epsilon_N$. Let B_N, C_N, D_N be linear operators on L_N. Assume that for every f in $\mathcal{D}(B_N) \cap \mathcal{D}(C_N) \cap \mathcal{D}(D_N)$

$$\|T_N f - f - (B_N f + \sqrt{\epsilon_N}\ C_N f + \epsilon_N\ D_N f)\| = o(\epsilon_N).$$

Let L be a Banach space, $\pi_N\colon\ L \to L_N$ a bounded linear transformation with $\sup_N \|\pi_N\| < \infty$. Let $\{T(t)\}$ be a strongly continuous contraction semigroup on L with generator A. Let Δ be a core for A and assume that for every f in Δ there exists h_N and k_N in $\mathcal{D}(B_N) \cap \mathcal{D}(C_N) \cap \mathcal{D}(\mathcal{D}_N)$ such that $\pi_N f \in \mathcal{D}(B_N) \cap \mathcal{D}(C_N) \cap \mathcal{D}(D_N)$ and

$$B_N h_N = - B_N(\pi_N f), \; C_N \pi_N f = 0, \tag{1.37}$$

$$B_N k_N = \pi_N Af - (C_N h_N + D_N \pi_N f), \tag{1.38}$$

$$\sup_N (\|h_N\|, \|k_N\|, \|C_N h_N\|, \|C_N k_N\|, \|D_N h_N\|, \|D_N k_N\|) < \infty . \tag{1.39}$$

Then, as $\epsilon_N \to 0$, for each $f \in L$, $T_N^{[t/\epsilon_N]} \pi_N f \to T(t)f$ for all $t \geq 0$, uniformly on bounded intervals.

The proof of this result goes as follows. Let $f_N = \pi_N f + \sqrt{\epsilon_N} h_N + \epsilon_N k_N$. Since $\|f_N - \pi_N f\| \to 0$, by Kurtz's approximation theorem for discrete parameter contraction semigroups (Kurtz 1969, Theorem 2.13) it is enough to show that

$$\|A_N f_N - \pi_N Af\| \to 0 \tag{1.40}$$

as $\epsilon_N \to 0$. Denote for simplicity $\pi_N f$ by f. By (1.37)

$$\|T_N f_N - f_N - (B_N g_N + \sqrt{\epsilon_N}\, C_N f_N + \epsilon_N\, D_N f_N)\|$$

$$= \|T_N f_N - f_N - (\sqrt{\epsilon_N} C_N f + \epsilon_N D_N f + \sqrt{\epsilon_N}\, B_N h_N + \epsilon_N h_N + \epsilon_N^{3/2}\, D_N h_N$$

$$+ \epsilon_N B_N k_N + \epsilon_N^{3/2}\, C_N k_N + \epsilon_N^2\, D_N k_N)\|.$$

$$= \|T_N f_N - f_N - \epsilon_N (B_N k_N + C_N h_N + D_N f) + \epsilon_N^{3/2}\, (D_N h_N + C_N k_N) + \epsilon_N^2 D_N k_N\|.$$

Using then (1.38) and (1.39),

$$\|T_N f_N - f_N - \epsilon_N \pi_N A f\| = o(\epsilon_N)$$

and (1.40) follows.

If we go back to the proof of Theorem 2.1,

$$B_N = B$$

and for g, the restriction to $E_N \times F$ of a function on $E \times F$ twice differentiable in x,

$$C_N g(x,y) = p(x,y) \int \frac{\partial g}{\partial x}(x,z)P(y,dz),$$

$$D_N g(x,y) = r(x,y) \int \frac{\partial g}{\partial x}(x,z)P(y,dz) + \tfrac{1}{2}\, s(x,y) \int \frac{\partial^2 g}{\partial x^2}(x,z)P(y,dz),$$

and

$$\Delta = C_K^\infty(E).$$

In this setting (1.5), (1.6) and (1.8) imply (1.37), (1.38) and (1.39).

CHAPTER II

APPLICATIONS

In this chapter we give an application of Theorem 1.1 to the problem of Karlin and Levikson mentioned in the introduction (Theorem 2.1), another one to sums of chain dependent processes and to the central limit theorem for ergodic Markov chains (Theorem 2.2). We discuss then briefly a method for obtaining diffusion approximations to sequences of suitably scaled stochastic difference equations.

Throughout this chapter $\{Y(k): k = 0,1,\ldots\}$ is an ergodic Markov chain in F, a Borel subset of $\mathbb{R}$, with invariant measure m and transition P. We assume $\int ym(dy) = 0$.

<u>Theorem 2.1</u>: Let $E = [0,1]$, $\mu \in P(E)$, $E_N = \{0,\frac{1}{N},\ldots,1\}$, $\mu_N \in P(E_N)$. Assume $\mu_N \Rightarrow \mu$, $\int y^8 m(dy) < \infty$ and for $i = 1,\ldots,4$ there exist functions η and λ_i satisfying (1.5), (1.6) and (1.8) with

$$b_0(y) = y,$$

$$b_1(y) = b_3(y) = 1,$$

$$b_2(y) = b_4(y) = y^2.$$

Let $\alpha_N = N^{-\frac{1}{2}}$, $\beta_N = N^{-1}$ and for $N = 0,1,\ldots$, $\{Z^N(k) = (X^N(k),Y(k)):$ $k = 0,1\ldots\}$ be the Markov chain with transition P^N given by (0.1), (0.2) and (0.3). Then as $N \to \infty$ the finite dimensional distributions of $X_N([Nt])$ converge to those of $X(\cdot)$, the diffusion with initial distribution μ and generator A given by (0.9) where τ is defined by (1.7).

<u>Proof</u>: For y in $\mathbb{R}$ and $N = 0,1,\ldots$ let $\{X^{N,y}(k): k = 0,1,\ldots\}$ be the Markov chain in E_N with initial distribution μ_N and transition $P^{N,y}$. We show first that conditions (1.9) to (1.12) are satisfied with

$$p(x,y) = x(1-x)y,$$

$$r(x,y) = (\theta - xy^2)x(1-x),$$

$$s(x,y) = x(1-x)(1 + x(1-x)y^2).$$

Using the fact that $X^{N,y}(0) = x$ implies $NX^{N,y}(1)$ is binomial $(N, p_{x,\sigma_N(y)})$ with $p_{x,\sigma_N(y)} = (1 + \sigma_N(y))x/(1 + \sigma_N(y)x)$,

$$(2.1) \qquad E_x(X^{N,y}(1) - x) = p_{x,\sigma_N(y)} - x$$

$$= \sigma_N(y)x(1-x) - \sigma_N^2(y)x^2(1-x) + \sigma_N^2(y)x^3\left(\frac{1-x}{1+\sigma_N(y)x}\right).$$

We have the following two relations

$$(2.2) \qquad \sigma_N(y) = (N^{-1}\theta + N^{-1/2}y)\vee(-1/2)$$

$$= (N^{-1}\theta + N^{-1/2}y) - [(N^{-1}\theta + N^{-1/2}y) + 1/2]1_{(-\infty,-1/2]}(\sigma_N(y))$$

and

$$(2.3)\qquad \sigma_N(y)^2 = [(N^{-1}\theta + N^{-\frac{1}{2}}y)\vee(-\tfrac{1}{2})]^2$$

$$= (N^{-1}\theta + N^{-\frac{1}{2}}y)^2 - [(N^{-1}\theta + N^{-\frac{1}{2}}y)^2 - \tfrac{1}{4}]\, 1_{(-\infty,-\frac{1}{2}]}(\sigma_N(y)).$$

From now on we denote $1_{(-\infty,-\frac{1}{2}]}(\sigma_N(y))$ by $\delta_N(y)$. By (2.1), (2.2) and (2.3),

$$E_x(X^{N,y}(1)-x) = N^{-\frac{1}{2}}yx(1-x) + N^{-1}(\theta-y^2x)x(1-x)$$

$$- (N^{-2}\theta^2 + 2N^{-3/2}\theta y)x^2(1-x) - [(N^{-1}\theta + N^{-\frac{1}{2}}y) + \tfrac{1}{2}]x(1-x)\delta_N(y)$$

$$+ (N^{-2}\theta^2 + 2N^{3/2}\theta y - N^{-1}y^2 + \tfrac{1}{4})x^2(1-x)\delta_N(y) + \sigma_N^2(y)x^3\,\frac{(1-x)}{1+\sigma_N(y)x},$$

and we get

$$|E_x(x^{N,y}(1)-x) - N^{-\frac{1}{2}}yx(1-x) - N^{-1}(\theta-y^2x)x(1-x)| \le N^{-3/2}\rho_{N,1}(y),$$

where $\rho_{N,1} = \sum_{k=1}^{8} \psi_k^N$ with

$${}^N_1(y) = N^{-\frac{1}{2}}\theta^2,\ \psi_2^N(y) = 2\theta y,\ \psi_3^N(y) = (\sqrt{N}\theta + N^{3/2})\delta_N(y),$$

$$) = Ny\delta_N(y),\ \psi_5^N(y) = (N^{-\frac{1}{2}}\theta^2)\delta_N(y),$$

$$\psi_6(y) = 2\theta y\delta_N(y),\ \psi_7^N(y) = \sqrt{N}\, y^2\delta_N(y),\ \psi_8^N(y) = N^{3/2}|\sigma_N(y)|^2.$$

To complete the proof of (1.9) we have to show that $\{\int|\rho_{N,1}(y)|^{4/3}m(dy)\}$ is a bounded sequence. By Minkowski's inequality it is enough to show that for $k = 1,2,\ldots,8$, $\int|\psi_k^N|^{4/3}dm$ are bounded sequences. This is clear for $k = 1,2,6$ and 8. We check that the condition is satisfied for $k = 7$. The other cases follow in the same way.

If $\theta > 0$, $\delta_n(y) \leq 1_{(-\infty,-\sqrt{N}/2)}(y)$ and

$$\int|\psi_7^N(y)|^{4/3}m(dy) \leq 2^{4/3}\int_{-\infty}^{-\sqrt{N}/2}[(\sqrt{N}/2)y^2]^{4/3}m(dy) \leq 2^{4/3}\int_{-\infty}^{-\sqrt{N}/2}y^4m(dy)\to 0.$$

The case $\theta \leq 0$ follows by a similar argument,

This completes the proof of (1.9).

We show now that (1.10) is satisfied. Note that

$$E_x(X^{N,y}(1)-x)^2 = E_x(X^{N,y}(1)-p_{x,\sigma_N(y)})^2 + (p_{x,\sigma_N(y)}-x)^2. \tag{2.4}$$

Using a property of the binomial distribution we get

$$E_x(X^{N,y}(1)-p_{x,\sigma_N(y)})^2 = \frac{1}{N}\,p_{x,\sigma_N(y)}(1-p_{x,\sigma_N(y)})$$

$$= \frac{1}{N}[x + \sigma_N(y)x\,\frac{(1-x)}{1+\sigma_N(y)x}][1-x-\sigma_N(y)x\,(\frac{1-x}{1+\sigma_N(y)x})]$$

$$= \frac{1}{N}\,x(1-x) + \frac{1}{N}\,\frac{\sigma_N(y)x(1-x)}{1+\sigma_N(y)x}\,[1-2x-\frac{\sigma_N(y)x(1-x)}{1+\sigma_N(y)x}].$$

This implies

$$|E_x(X^{N,y}(1) - p_{x,\sigma_N(y)}) - \frac{1}{N}x(1-x)| \leq \frac{1}{N}|\sigma_N(y)|(1 + |\sigma_N(y)|). \tag{2.5}$$

We find next a bound for $(p_{x,\sigma_N(y)} - x)^2$. Since

$$(p_{x,\sigma_N(y)} - x) = \sigma_N(y)x(1-x)-\sigma_N^2(y)x^2(\frac{1-x}{1+\sigma_N(y)x}),$$

$$(p_{x,\sigma_N(y)}-x)^2 = \sigma_N^2(y)x^2(1-x)^2 + \psi_1(\sigma_N(y)x), \tag{2.6}$$

where ψ_1 satisfies

$$|\psi_1(\sigma_N(y)x)| \le \sigma_N^4(y) + 2|\sigma_N(y)|^3.$$

Using then (2.3),

$$\sigma_N^2(y) = N^{-1}y^2 + N^{-3/2}\psi_2(\theta,y) + [\psi_3^N(\theta,y)]\delta_N(y),$$

where ψ_2 and ψ_3^N are polynomials of degree 2 in y. We then get

$$|(p_{x,\sigma_N(y)}-x)^2 - N^{-1}y^2x^2(1-x)^2|$$

$$\le N^{-3/2}|\psi_2(\theta,y)| + |\psi_3^N(\theta,y)|\delta_N(y) + |\psi_1(\sigma_N(y)x)|.$$

Using the triangle inequality, (2.4), (2.5) and (2.6),

$$|E_x(x^{N,y}(1)-x)^2 - N^{-1}x(1-x)(1 + x(1-x)y^2)| \le N^{-3/2}\rho_{N,2}(y),$$

where $\rho_{n,2}(y) = N^{3/2}[N^{-1}|\sigma_N(y)|(1 + |\sigma_N(y)|) + N^{-3/2}|\psi_2(\theta,y)|$

$$+ |\psi_3^N(\theta,y)|\delta_N(y) + \psi_1(\sigma_N(y)x)\ ].$$

The fact that $\{\int\rho_{N,2}^{4/3}\,dm\}$ is a bounded sequence can be shown easily and (1.10) is proved.

In the same way it is possible to show that

$$E_x|X^{N,y}(1)-x|^3 \le N^{-3/2}(11 + 2y^4)^{3/2}.$$

If we define $\rho_{N,3}(y) = (11 + 2y^4)^{3/2}$ then $\int\rho_{N,3}^{4/3}\,dm$ is finite, does not depend on N and (1.11) is satisfied. Taking then $\rho_{N1} + \rho_{N2} + \rho_{N3} = \rho_N$ (1.12) follows.

We find now the form of A. Using the notation of Theorem 1.1,

$$a_0(x) = x(1-x),$$

$$\int r(x,y)m(dy) = (\theta - xv)x(1-x),$$

$$\int s(x,y)m(dy) = x(1-x)(1 + x(1-x)v).$$

By (1.14) A is given by (0.9) and by Theorem A.3 $\bar{A}$ is the generator of a strongly continuous positive conservative contraction semigroup on $C[0,1]$. This proves Theorem 2.1.

<u>Definition</u>: Let $\{Y(k): k = 0,1,\ldots\}$ be as in Chapter 1. Random variables $\{X(k): k = 1,2,\ldots\}$ defined on the same space as $\{Y(k): k = 0,1,\ldots\}$ are said to be chain dependant iff for every x in $\mathbb{R}$ and $k = 1,2,\ldots,$

$$P[X(k) \le x | Y(0),X(1),\ldots Y(k-2),X(k-1),Y(k-1)]$$

$$= P[X(k) \le x | Y(k-1)].$$

<u>Theorem 2.2</u>: For y in $\mathbb{R}$ let $P^y \in \mathcal{P}(\mathbb{R})$ be defined by

$$P^y(-\infty,x] = P[X(k) \le x | Y(k-1) = y]$$

and let

$$b_0(y) = \int x P^y(dx),$$

$$b_2(y) = \int x^2 P^y(dx),$$

$$\theta(y) = \int |x|^3 P^y(dx).$$

Assume $b_0 \in L^4(dm)$, $b_2 \in L^2(dm)$ and $\theta \in L^{4/3}(dm)$. Suppose (1.5), (1.6) and (1.8) hold with $k_1 = 1$, $k_2 = 2$, b_0, b_2 as above and $b_1 = 0$. Let

$$\sigma^2 = \int (Qn^2 - (Qn)^2)\,dm.$$

Then $\dfrac{X(1) + \ldots + X(N)}{\sigma\sqrt{N}} \xrightarrow{L} N(0,1)$.

Proof: For y in $\mathbb{R}$ let $\{X^y(k)\colon\ k = 1,2,\ldots\}$ be i.i.d. with distribution P^y and $\{X^{N,y}(k)\colon\ k = 0,1,\ldots\}$ be the Markov chain defined by

$$X^{N,y}(0) \text{ has a given distribution } \mu$$

and

$$X^{N,y}(k) = X^{N,y}(0) + \frac{X^y(1) + \ldots + X^y(k)}{\sigma\sqrt{N}}.$$

Let $P^{N,y}$ be the transition of $X^{N,y}$ and let $Z_N(k) = (X_N(k), Y(k))$ be the Markov chain in $\mathbb{R} \times \mathbb{R}$ with transition P^N satisfying

$$P^N((x,y), A \times B) = P^{N,y}(x,A)P(y,B).$$

(Note that $\{X_N(k)\colon\ k = 0,1,\ldots\}$ is given by

$$X_N(k) = X^{N,y}(0) + \frac{X(1) + \ldots + X(k)}{\sigma\sqrt{N}}.)$$

Conditions (1.9) to (1.11) of Theorem 1.1 take the form

$$|E_x(X^{N,y}(1) - x) - (\sigma\sqrt{N})^{-1} b_0(y)| = 0,$$

$$|E_x(X^{N,y}(1) - x)^2 - (\sigma^2 N)^{-1} b_2(y)| = 0,$$

$$E_x|X^{N,y}(1) - x|^3 \le (\sigma^3 N^{3/2})^{-1} \theta(y).$$

Here $\alpha_N = (\sigma\sqrt{N})^{-1}$, $\beta_N = (\sigma^2 N)^{-1}$, $\gamma_N = (\sigma^3 N^{3/2})^{-1}$ $\rho_N = \theta_3$, $p(x,y) = b_0(y)$, $r(x,y) = 0$ and $s(x,y) = b_2(y)$.

Using then Theorem 1.1 we conclude that the finite dimensional distributions of $X_N([N\cdot])$ converge to those of $X(\cdot)$, the Brownian motion with generator $\frac{1}{2}\frac{d^2}{dx^2}$. This completes the proof.

<u>Remark</u>. If $P^y = \delta(y)$, the Dirac measure at y, Theorem 3.2 is the CLT for ergodic Markov chains mentioned in the introduction.

In the last result of this section we obtain a diffusion approximation for a sequence of adequately scaled stochastic difference equations.

<u>Theorem 2.3</u>: Let $\{\sigma_N(k): k = 0,1,\dots\}$ be defined by

(2.7) $$\sigma_N(k) = \alpha_N^2\theta + \alpha_N Y(k)$$

where $\{Y(k): k = 0,1,\dots\}$ is as in Theorem 1.1. Let $E = [r_0,r_1]$ be a closed interval of $\mathbb{R}$ and let $G \in C(E\times\mathbb{R})$ satisfy the following conditions:

(2.8) $$G(x,0) = 0,$$

(2.9) $$r_0-x \le G(x,y) \le r_1-x$$

for every x in E and y in F, and

(2.10) $$|G(x,y)-yf(x)-y^2g(x)| \le M|y|^3$$

where f and g are continuously differentiable on E, M is a constant and

$$f(r_1) = g(r_1) = f(r_2) = g(r_2) = 0.$$

Assume conditions (1.5), (1.6) and (1.8) are satisfied with b_i as in Theorem 2.1. For $N = 1,2,\ldots$ let $\{X_N(k): \ k = 0,1,\ldots\}$ be defined by

$$X_N(0) \text{ has distribution } \mu \in P(E)$$
$$\vdots$$
$$X_N(k+1) - X_N(k) = G(X_N(k),\sigma_N(k))$$

Let $X(\cdot)$ be the Markov process in E whose generator A, with domain $C^2[0,1]$, is given by

$$A = [\tau ff' + \theta f + \frac{v}{2} g]d/dx + [\tau f^2 + \frac{v}{2} f^2]d^2/dx^2$$

Then, as $N \to \infty$, the finite dimensional distributions of $X_N([\cdot/\alpha_N^2])$ converge to those of $X(\cdot)$.

<u>Proof</u>: We just give an outline of the proof.
Let $\sigma_N = \alpha_N^2\theta + \alpha_N y$ and define $\{X^{N,y}(k): \ k = 0,1,\ldots\}$ by:

$$X^{N,y}(0) \text{ has distribution } \mu,$$

$$X^{N,y}(k+1)-X^{N,y}(k) = G(X_N(k),\sigma_N).$$

(Note that we define like this a deterministic difference equation.) If we call $P^{N,y}$ the transition of $X^{N,y}$ the chain $\{Z_N(k) = (X_N(k), Y(k)): \ k = 0,1,\ldots\}$ has transition $P^{N,y}(x,\cdot)\times P(y,\cdot)$. We can use Theorem 1.1 and here (1.9) and (1.10) take the form

$$E_x[(X^{N,y}(1)-x)-G(x,\sigma_N)] = 0,$$

$$E_x[(X^{N,y}(1)-x)^2-G(x,\sigma_N)^2] = 0.$$

Using (2.10) we obtain

$$p(x,y) = yf(x),$$

$$r(x,y) = \theta f(x) + y^2 g(x),$$

$$s(x,y) = y^2 f(x).$$

Theorem 2.3 follows.

CHAPTER III

A LIMIT THEOREM FOR GEODESIC RANDOM WALKS

Let M be a Riemann manifold of dimension n. The Brownian motion in M is a stochastic process $\{X_t^x: t \geq 0\}$ with continuous sample paths such that $X_0^x = x$ and

$$f(X_t^x) - f(x) + \int_0^t (\Delta f)(X_s^x)ds$$

is a martingale. (For the existence of such a process see Pinsky(1978)). The aim of this section is to approximate the Brownian motion by geodesic random walks. The result we prove is a discrete parameter version of a result of Pinsky (1978).

Let $x \in M$, $T_x(M)$ be the tangent space of M at x, $S_x(M)$ the unit sphere of $T_x(M)$, $S(M)$ the bundle of tangent unit spheres. Let $\xi_x \in S_x(M)$ and $x(\cdot,\xi_x)$ be the unique unit speed geodesic starting from x in the direction ξ_x. Let $\{\tau(k): k = 0,1,\ldots\}$ be i.i.d. in $\mathbb{R}^+$ with distribution P and assume $E[\tau(k)] = E[\tau^2(k)] - 1 = 1$. Let $\{Z(k): k = 0,1,\ldots\}$ be the random variables in $S(M)$ such that $Z(0) = (x_0,\xi_0),\ldots,Z(k) = (x_k,\xi_k)$ where

$$x_k = x_{k-1}(\tau(k-1),\xi_{k-1}),$$

$$\xi_k = \xi_{x_k},$$

and the conditional distribution of ξ_k given $\{(x_0,\xi_0),\ldots,(x_{k-1},\xi_{k-1})\}$ is the uniform distribution on S_{x_k} denoted by μ. The sequence

$\{x_0, x_1, \ldots\}$ is called a geodesic random walk.

Theorem 3.1. For $N = 1,2,\ldots$ let $\{Z_N(k) = (x_k^N, \xi_k)$: $k = 0,1,\ldots\}$ be the random variables such that $x_0^N = x_0$ and

$$x_k^N = x_{k-1}^N \left(\frac{\tau(k-1)}{\sqrt{N}}, \xi_{k-1}\right).$$

Assume the Ricci curvature of M bounded from below. Then for every f in $C(M)$, $\lim_{N\to\infty} E[f(X_{[Nt]}^N)] = E[f(X_{\frac{t}{n}}^x)]$.

Proof. To prove this theorem we need the following Lemma:

Lemma: Denote by $P_\cdot(x,\xi)$ the parallel transport along the geodesic $X(\cdot,\xi)$. Let $f \in C_K^2(S(M))$ and let

$$u(t,x,\xi,\eta) = f(X(t,\xi), P_t(X,\xi)(\eta)),$$

where $x \in M$, $\xi \in S_x(M)$, $\eta \in S_x(M)$. Then

$$\frac{\partial u}{\partial t}(0,x,\xi,\eta) = \xi_i \frac{\partial u}{\partial x_i} - \Gamma^i_{j,k}\xi^j \eta_k \frac{\partial u}{\partial \xi_i}.$$

We will denote $\frac{\partial u}{\partial t}$ by $D_\xi^\eta(u)$ and D_ξ^ξ by D_ξ. We also remark that if $f \in C_K^2(M), D_\xi f = \sum_i \xi_i \frac{\partial f}{\partial x_i}$. We use Theorem A.1b.

In the notation of this theorem $G_N = S(M)$, $E = M$, $D = C_K^2(M)$. For f in $\hat{C}(S(M))$ we define

$$T_N f(x,\xi) = \int f(x(\tfrac{\tau(1)}{\sqrt{N}},\xi), P_{\frac{\tau(1)}{\sqrt{N}}}(x,\xi)(\eta))dP\,\mu(d\eta),$$

$$A_N = N(T_N - I),$$

$$Af = -\frac{1}{n}\Delta f \qquad \text{for} \quad f \in D .$$

Let $f \in D$, $h = D_\xi f$, $k = D_\xi D_\xi f$ and $f_N = f + N^{-1/2} h + N^{-1} k$.

We have to show that $\|A_N f_N - Af\| \to 0$ as $N \to +\infty$. Using the lemma we have the following Taylor expansions for f and h (we replace $\tau(1)$ by τ_1):

(2.1) $|f(x(\tau_1/\sqrt{N},\xi) - f(x) - ((\tau_1/\sqrt{N})\, D_\xi f(x) + (\tau_1^2/N) D_\xi D_\xi f(x))| \leq K\, N^{-3/2}\tau_1^3$,

(3.2) $|h(x(\tau_1/\sqrt{N},\xi),\, P_{\tau_1/\sqrt{N}}(x,\xi)(\eta)) - h(x,\eta) - (\tau_1/\sqrt{N}) D_\xi^\eta h(x,\eta)| \leq KN^{-1}\tau_1^2$.

For $g \in C(S(M))$ let B be defined by

$$Bg(x,\xi) = \int_{S_x(M)} g(x,\eta)\mu(d\eta) - g(x,\xi).$$

By (2.1), (2.2) and the definition of T_N,

(2.3) $\quad N|T_N f(x,\xi) - f(x,\xi) - (\sqrt{N})^{-1} D_\xi f(x) - \tfrac{1}{2} N^{-1} D_\xi D_\xi f(x)| \leq N^{-1/2} K\, E|\tau_1^3|$,

where K is a constant depending on M only,

(2.4) $\sqrt{N}|T_N h(x,\xi) - h(x,\xi) - Bh(x,\xi) - (\sqrt{N})^{-1} \int D_\xi^\eta h(x,\eta)\mu(d\eta)|$,

$$\leq (\sqrt{N})^{-1} K\, E[\tau_1^2] = (\sqrt{N})^{-1} K ,$$

(2.5) $|T_N k(x,\xi) - k(x,\xi) - Bk(x,\xi)| < (\sqrt{N})^{-1} K\, E|\tau_1| = \sqrt{N}^{-1} K$.

Using then a triangle inequality,

$$|A_N f_N(x,\xi) - Af(x)| \leq (\sqrt{N})^{-1} K(E|\tau_1|^3 + 1 + 1)$$

$$+ |Bh(x,\xi) + D_\xi f|\sqrt{N}$$

$$+ |Bk(x,\xi) + D_\xi D_\xi f(x) + \int D_\xi^\eta h(x,\eta) d\mu(\eta) - Af(x)| .$$

To conclude the proof we show the following equalities :

(2.6) $$Bh(x,\xi) + D_\xi f = 0,$$

(2.7) $$\int D_\xi^\eta h(x,\eta)\mu(d\eta) = 0,$$

(2.8) $$Bk(x,\xi) = Af(x) - D_\xi D_\xi f(x).$$

Using the definition of B and the fact that $\int \eta_i \mu(d\eta_i) = 0$,

$$Bh(x,\xi)_i = \int \eta_i \frac{\partial f}{\partial x_i}\mu(d\eta_i) - \xi_i \frac{\partial f}{\partial x_i} = -\xi_i \frac{\partial f}{\partial x_i} = -D_\xi f(x)_i .$$

This proves (2.6). For (2.7),

$$D_\xi^\eta h(x,\eta) = \xi_i \frac{\partial h}{\partial x_i}(x,\eta) - \Gamma^i_{j,k}\xi_j\eta_k \frac{\partial h}{\partial \eta_i}(x,\eta)$$
$$= \xi_i\eta_k \frac{\partial^2 f}{\partial x_i \partial x_k} - \Gamma^i_{j,k}\,\xi_j\eta_k \frac{\partial f}{\partial x_k}.$$

(A.7) follows then from the fact that

$$\int D_\xi^\eta h(x,\eta)\mu(d\eta) = (\xi_i \frac{\partial^2 f}{\partial x_i \partial x_k}(x) - \Gamma^k_{j,k}\xi_j \frac{\partial f}{\partial x_k})\int \eta_k \mu(d\eta_k) = 0 .$$

(A.8) follows essentially as in Pinsky's article. We have to show

$$Bk(x,\xi) = \int D_\xi D_\xi f(x)\mu(d\xi) - D_{\xi\xi} f = Af - D_\xi D_\xi f .$$

This reduces to $\int D_\xi D_\xi f(x)\mu(d\xi) = Af$ (see Pinsky 1978 p. 209).

To conclude we use the following result of Yau.

<u>Lemma:</u> Let M be a complete Riemannian manifold with Ricci curvature bounded from below by a constant. Then the Brownian motion semigroup preserves the class of continuous functions which vanish at infinity.

Remark: In his article Pinsky assumes that for $k = 0,1,2,..$ τ_k is exponential with parameter one. He then introduces the geodesic transport process $X(\cdot)$ defined by

$$x(t) = x_{k-1}(t-(\tau_1+\ldots+\tau_{k-1}), \xi_{k-1})$$

where $\tau_1+\ldots+\tau_{k-1} \leq t < \tau_1+\ldots+\tau_k$. He then shows convergence of a scaled geodesic transport process to Brownian motion.

APPENDIX

In this appendix we give conditions for sequences of discrete parameter processes to converge to Markov processes. Theorem A.1 is a discrete parameter version of Theorem 3.15 in Kurtz (1975). Throughout this section (E,r) is a complete locally compact separable metric space.

Theorem A.1: a) Let K be a Banach subspace of $B(E)$ which contains $\hat{C}(E)$ and such that $f \in \hat{C}(E)$ and $g \in K$ imply $f\,g \in K$. Suppose $\{T(s)\}$ is a strongly continuous contraction semigroup on K with generator A corresponding to a Markov process $X(\cdot)$ with initial distribution μ and sample paths in $D_E[0,\infty)$. For $N = 1,2,\ldots$ let $\{Z_N(k): \; k = 0,1,\ldots\}$ be a sequence of Markov chains with transition P^N and measurable state space G_N. Let ϵ_N be a positive constant, $\epsilon_N \to 0$ as $N \to \infty$, and η_N: $G_N \to E$ a measurable mapping. Let

$$X_N(t) = \eta_N(Z_N([\epsilon_N^{-1}t])) \tag{A.1}$$

and assume

$$\lim_{N\to\infty} E(f(X_N(0))) = E(f(X(0))) \tag{A.2}$$

for every f in K. Let $\sim$ be the equivalence relation on $Bor(G_N)$ defined by

$$f \sim g \quad \text{iff} \quad E|f(Z_N(k)) - g(Z_N(k))| = 0 \text{ for every } k \geq 0$$

and let

(A.3) $$L_N = \{f \in Bor(G_N)/\sim : \ \|f\| = \sup_k E|f(Z_N(k))| < \infty\},$$

(A.4) $$T_N f(x) = \int f(z) P^N(x,dz), \ f \in L_N,$$

(A.5) $$A_N = \epsilon_N^{-1}(T_N - I),$$

and

(A.6) $$\pi_N f(z) = f(\eta_N(z)), \ f \in B(E).$$

Let D be a core for A and assume that for every f in D there is a sequence $\{f_N\}$ in L_N such that

(A.7) $$\sup_N \|f_N\| < \infty ,$$

(A.8) $$\sup_N \|A_N f_N\| < \infty ,$$

(A.9) $$\lim_{N\to\infty} \|f_N - \pi_N f\| = 0 ,$$

(A.10) $$\lim_{N\to\infty} \|A_N f_N - \pi_N A f\| = 0.$$

Then the finite dimensional distributions of $X_N(\cdot)$ converge to those of $X(\cdot)$.

b) If in (A.7) - (A.10) L_N is replaced by $B(G_N)$ with the sup norm, then convergence is in distribution in $D_E[0,\infty)$.

<u>Proof</u>: a) Let V be an independent Poisson process with $E[V(t)] = t$. Remarking that L_N is a Banach space and T_N is a linear contraction on L_N it follows that A is the generator of $Z_N(V[\epsilon_N^{-1}t])$. Let

$Y_N(t) = Z_N(V[\epsilon_N^{-1}t])$. Conditions (A.7) to (A.10) imply conditions (3.17) to (3.20) of Theorem 3.15 in Kurtz(1975), and the finite dimensional distributions of $\eta_N(Y_N(t))$ converge to those of $X(t)$. The convergence of the finite dimensional distributions of $X_N(t)$ to those of $X(t)$ follow then by a standard argument.

The next theorem, due to Blumenthal and Getoor(1968, Theorem I.9.4), gives conditions for a semigroup to be the semigroup associated with a Markov process.

Theorem A.2: Let E be as in Theorem 2.1 and $\{T(t)\}$ be a strongly continuous positive contraction semigroup on $\hat{C}(E)$ whose infinitesimal generator A is conservative in the sense that there exists a sequence $\{f_N\} \subset \mathcal{D}(A)$ such that b. p. lim $f_N = 1$ and b.p. lim $Af_N = 0$, where b.p. lim $f_N = f$ means $f_N(x) \to f(x)$ for every x in E and $\sup_N \|f_N\| < \infty$. Then, for each $\mu \in P(E)$ there exists a Markov process X corresponding to $T(t)$ with initial distribution μ and sample paths in $D_E[0,\infty)$.

Suppose also that the generator A of $\{T(t)\}$ satisfies the following condition: for every $x \in E$ and neighborhood V of x there exists $f \in \mathcal{D}(A)$ and a neighborhood U of x such that $1_U \le f \le 1_V$ and $Af = 0$ on U. Then almost all sample paths of $X(\cdot)$ belong to $C_E[0,\infty)$

The next theorem gives conditions for a one-dimensional diffusion operator to be the generator of a contraction semigroup (Ethier 1978).

<u>Theorem A.3</u>: Let E be a closed interval of $\mathbb{R}$, possibly infinite, with end-points $r_0 < r_1$. Let a and b be continuous functions on E with $a \geq 0$, having bounded devirATIVES a'', b', b''. Suppose

$$a(r_i) = 0 \leq (-1)^i b(r_i) \text{ if } |r_i| < \infty$$

and for $f \in C_K^\infty(E)$ let $Gf = af'' + bf'$. $A = \overline{G}$ generates a semigroup $\{T(t)\}$ satisfying the conditions of Theorem A.2.

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