TCHEBYCHEFF APPROXIMATION BY RECIPROCALS OF POLYNOMIALS ON $[0,\infty)$

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ABSTRACT

TCHEBYCHEFF APPROXIMATION
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In Chapter I we consider the following approximation problem. We let n be a fixed positive integer and use $C_o^+[0,\infty)$ to denote the set of all positive continuous functions on $[0,\infty)$ which vanish at infinity. We define

$$R_n = \left\{ \frac{1}{p} \,\middle|\, p \in \Pi_n,\ \ p(x) > 0 \ \text{ on } \ [0,\infty) \right\}$$

and for a given $f \in C_o^+[0,\infty)$ we search for an element $\frac{1}{p^*}$ from R_n such that

$$\sup_{x\varepsilon[0,\infty)} \left| f(x) - \frac{1}{p^*(x)} \right| = \inf_{\frac{1}{p}\varepsilon R_n} \left\{ \sup_{x\varepsilon[0,\infty)} \left| f(x) - \frac{1}{p(x)} \right| \right\}.$$

In this setting we obtain existence, characterization and uniqueness theorems. Existence follows along somewhat standard lines. The characterization theorem features a two-part alternation condition which states that $\frac{1}{p}$ is a best approximation to f if and only if either $f - \frac{1}{p}$ has an alternating set consisting of at least $n + 2$ points, or $f - \frac{1}{p}$ has an alternating set consisting of exactly $n + 1$ points, $\partial p \leq n - 1$, and $f - \frac{1}{p}$ is positive at the largest point of the alternating set. Uniqueness is then obtained.

In Chapter II we consider approximation with osculatory interpolation in a setting similar to that of Chapter I. Here we obtain a character-

ization theorem similar to that of Chapter I. Although existence fails we have uniqueness of best approximations. If $\frac{1}{p}$ is best with $\partial p = n$, we obtain a characterization in terms of zero in the convex hull of a certain set.

In Chapter III we consider strong uniqueness and continuity of the best approximation operator in the above setting and obtain an existence theorem for approximation with osculatory interpolation for large n.

TCHEBYCHEFF APPROXIMATION

BY RECIPROCALS OF POLYNOMIALS ON $[0,\infty)$

By

Daryl Myron Brink

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INTRODUCTION

This paper will consider the approximation of certain positive continuous functions defined on $[0,\infty)$ by rational functions of the form $\frac{1}{p}$ where p is a polynomial of degree not greater than some fixed integer n. Existence, characterization, uniqueness and continuity of the best approximation operator will be examined. The error in approximating f by $\frac{1}{p}$ will be measured throughout by

$$(0.1) \qquad \left\|f - \frac{1}{p}\right\| = \sup_{x\varepsilon[0,\infty)} \left|f(x) - \frac{1}{p(x)}\right| .$$

Thus a best approximation to a given f is an element $\frac{1}{p^*}$ from the prescribed class such that

$$(0.2) \qquad \left\|f - \frac{1}{p^*}\right\| = \inf_{\frac{1}{p}} \left\|f - \frac{1}{p}\right\| .$$

In Chapter I, we assume that f vanishes at infinity and that p is positive. In this setting, existence and uniqueness theorems paralleling those found in the classical theory of rational Tchebycheff approximation [1], [2], [10], [14], [15] are obtained. The characterization theorem, which involves conditions on the sign alternation of the error function, has non-standard form.

In Chapter II, we further restrict the functions and the approximants and consider the problem of approximating an s-continuously differentiable f by a k-point osculating rational function of the form $\frac{1}{p}$ (precise definitions will be given later). In this setting, we do not necessarily have existence of best approximations; but uniqueness and characterization theorems similar to those in Chapter I are obtained. The results here are generalizations to the infinite interval of work by A. L. Perrie [13] and H. L. Loeb, D. G. Moursund, L. L. Schumaker and G. D. Taylor [9].

Chapter III consists of a discussion of continuity of the best approximation operator for the above approximation problems, along with an existence theorem for large n.

CHAPTER I

TCHEBYCHEFF APPROXIMATIONS WITH RATIONAL FUNCTIONS OF THE FORM $\frac{1}{p}$ ON $[0,\infty)$

Section 1: Introduction

In this chapter we wish to consider the following approximation problem. Let $C_o^+[0,\infty)$ be the set of all positive continuous functions on $[0,\infty)$ which vanish at ∞. Define

$$\|f\| = \sup_{x\varepsilon[0,\infty)} |f(x)| .$$

Then, given an integer n, we wish to approximate f by a function of the form $\frac{1}{p}$ where

$$p(x) = a_o + a_1 x + \ldots + a_n x^n$$

with a_i real. We will use ∂p to denote the degree of the polynomial p.

Let

$$R_n = \{ \frac{1}{p} \mid \partial p \leq n, \ p(x) > 0 \text{ on } [0,\infty) \}$$

denote the family of rational functions from which we wish to approximate f. Thus we search for an element $\frac{1}{p^*}$ from R_n such that

$$\left\|f - \frac{1}{p^*}\right\| = \inf_{\frac{1}{p}\varepsilon R_n} \left\|f - \frac{1}{p}\right\| .$$

Note that, since $f \in C_o^+[0,\infty)$, the constant $c^* \equiv \frac{1}{2}\|f\|$ is always a candidate for a best approximation; moreover, we have $\|f - c^*\| = \frac{1}{2}\|f\|$. Thus

$$(1.1) \qquad \left\|f - \frac{1}{p^*}\right\| \leq \frac{1}{2}\|f\| .$$

Hence p^* has no zeros, and so there is no loss of generality in requiring $p > 0$ on $[0,\infty)$ for $\frac{1}{p} \in R_n$. However, note that $0 \notin R_n$.

The question of whether such an element $\frac{1}{p^*} \in R_n$ can actually be found is answered in Section 2.

Section 2: Existence Theorem

We begin with an important lemma which will have applications throughout this paper.

Lemma 1.1: Let $\{\frac{1}{p_k}\}$ be a sequence of elements from R_n. Suppose there exist constants M_1 and M_2 such that

$$0 < M_1 \leq \left\| \frac{1}{p_k} \right\| \leq M_2 < \infty$$

for all k. Then there exists a subsequence $\{\frac{1}{p_{k_j}}\}$ of $\{\frac{1}{p_k}\}$ which converges uniformly to $\frac{1}{p}$ on any closed interval, with $\frac{1}{p} \in R_n$.

Proof: Since $\left\{\left\| \frac{1}{p_k} \right\|\right\}$ is a bounded infinite sequence, there is a subsequence $\{\frac{1}{p_{k_r}}\}$ such that $\left\{\left\| \frac{1}{p_{k_r}} \right\|\right\}$ converges to some real number L as $k_r \to \infty$. For simplicity we will denote this subsequence by $\{\frac{1}{p_k}\}$.

We write

$$\frac{1}{p_k} = \frac{c_k}{q_k}$$

where $q_k(x) = \sum_{i=0}^{n} a_{i,k}x^i$ with $\sum_{i=0}^{n} a_{i,k}^2 = 1$ and c_k is a constant.

To do this, suppose $p_k(x) = \sum_{i=0}^{n} b_{i,k}x^i$ and take

$$c_k^2 = \sum_{i=0}^{n} b_{i,k}^2.$$

Now $\{(a_{o,k},\ldots,a_{n,k})\}$ is a bounded infinite sequence in E^{n+1} and therefore must have a subsequence $\{(a_{o,k_j},\ldots,a_{n,k_j})\}$ which converges to some $(a_o,\ldots,a_n)$. Define

$$q_{k_j}(x) = \sum_{i=0}^{n} a_{i,k_j}x^i.$$

Then if $q(x) = \sum_{i=0}^{n} a_i x^i$, since $\sum_{i=0}^{n} |a_i - a_{i,k_j}| \to 0$ as $k_j \to \infty$, using a result from Natanson ([11], p. 23) we find that $\{q_{k_j}\}$ converges uniformly to q on any closed interval.

For simplicity, we will write $\{q_j\}$ for $\{q_{k_j}\}$. Consider the associated sequence $\{c_j\}$. Since

$$\left\| \frac{c_j}{q_j} \right\| \leq M_2$$

we have

$$\left| \frac{c_j}{q_j(x)} \right| \leq M_2$$

for any x. Hence

$$\begin{aligned} |c_j| &\leq M_2 \, |q_j(1)| \\ &\leq M_2 \left(\sum_{i=0}^{n} |a_{i,j}| \right) \\ &\leq M_2 \, (n+1) \end{aligned}$$

since $\sum_{i=0}^{n} a_{i,j}^2 = 1$ implies that $|a_{i,j}| \leq 1$. Thus $\{c_j\}$ is a bounded sequence and therefore has a subsequence $\{c_{j_t}\}$ which converges to a real number c.

We wish to show $c > 0$. If $c_{j_t} \to 0$ then the coefficients of p_{j_t} must approach zero, which would force $\left\| \frac{1}{p_{j_t}} \right\|$ to become arbitrarily large. Hence $0 < c < \infty$, and consequently q has no zeros. Thus we let $\frac{c}{q} = \frac{1}{p}$.

Let α be a real number and suppose $x \varepsilon [0,\alpha]$. Then

$$\left| \frac{1}{p(x)} - \frac{1}{p_{j_t}(x)} \right| \leq \left\{ \frac{1}{\displaystyle\min_{x\varepsilon[0,\alpha]} |p(x)p_{j_t}(x)|} \right\} \left| p_{j_t}(x) - p(x) \right| .$$

Since $p_{j_t} \to p$ uniformly on $[0,\alpha]$ with $p > 0$, $p_{j_t} > 0$ on $[0,\alpha]$

it follows that

$$\frac{1}{p_{j_t}} \to \frac{1}{p}$$

uniformly on $[0,\alpha]$. Thus $\{\frac{1}{p_{j_t}}\}$ is the desired subsequence. ∎

With the aid of lemma 1.1 we can prove the existence theorem for R_n.

<u>Theorem 1.1</u>: To each function $f \in C_o^+[0,\infty)$ there corresponds at least one best approximation $\frac{1}{p^*}$ from the class R_n.

<u>Proof</u>: Fix $f \in C_o^+[0,\infty)$ and let

$$E = \inf_{\frac{1}{p} \in R_n} \left\|f - \frac{1}{p}\right\|$$

Then there exists a sequence $\{\frac{1}{p_k}\}$ from R_n such that the sequence $\left\{\left\|f - \frac{1}{p_k}\right\|\right\} \to E$ as $k \to \infty$. It follows from (1.1) that there is no loss of generality in requiring that

$$\left\|f - \frac{1}{p_k}\right\| \leq \frac{1}{2}\|f\|$$

for all k. Thus it follows that

$$\frac{1}{2}\|f\| \leq \left\|\frac{1}{p_k}\right\| \leq \frac{3}{2}\|f\| .$$

We wish to construct an element $\frac{1}{p^*}$ from R_n such that $\left\|f - \frac{1}{p^*}\right\| = E$. By lemma 1.1, we have a subsequence $\{\frac{1}{p_{k_j}}\}$ which converges uniformly on any closed interval to an element $\frac{1}{p^*} \in R_n$.

Let $x \in [0,\infty)$. Then $x \in [0,\alpha]$ for some α, and we have

$$\left|f(x) - \frac{1}{p^*(x)}\right| \leq \max_{x\in[0,\alpha]} \left|f(x) - \frac{1}{p^*(x)}\right|$$

$$\leq \max_{x\in[0,\alpha]} \left|f(x) - \frac{1}{p_{k_j}(x)}\right| + \max_{x\in[0,\alpha]} \left|\frac{1}{p_{k_j}(x)} - \frac{1}{p^*(x)}\right|$$

$$\leq \left\|f - \frac{1}{p_{k_j}}\right\| + \max_{x\in[0,\alpha]} \left|\frac{1}{p_{k_j}(x)} - \frac{1}{p^*(x)}\right|$$

Since $\frac{1}{p_{k_j}} \to \frac{1}{p^*}$ uniformly on $[0,\alpha]$ and $\left\|f - \frac{1}{p_{k_j}}\right\| \to E$ we have, by taking limits on the right hand side of the above inequality, that

$$\left|f(x) - \frac{1}{p^*(x)}\right| \leq E .$$

Hence it follows that

$$\left\|f - \frac{1}{p^*}\right\| \leq E ,$$

and consequently, $\frac{1}{p^*}$ is a best approximation to f. ▮

We remark that the proof given above parallels the proof of the well-known existence theorem for approximation by rational functions on a closed interval. The technical difficulties which arise as a consequence of the use of an infinite interval are somewhat simplified by the assumptions on the function f and the use of a smaller class of rational functions.

Section 3: Characterization of Best Approximations

Our objective here will be to characterize best approximations by means of the sign alternations in the error curve $f - r$ where $r \in R_n$. In order to do so, we will first define the following sets:

(1.2) $X(r) = \{ x \mid |f(x) - r(x)| = \|f - r\| \}$ is called the set of extreme points for $f - r$.

(1.3) A set of N distinct points $0 \leq x_1 < x_2 < \ldots < x_N < \infty$ is called an alternating set for the error $f - r$ if

$$x_j \in X(r), \quad j = 1,\ldots,N$$

and

$$f(x_j) - r(x_j) = - [f(x_{j+1}) - r(x_{j+1})], \quad j = 1,\ldots,N-1.$$

Now for $f \in C[a,b]$ with a,b real, it is well-known that r is a best approximation to f from the set

$$R_n^m[a,b] = \{ \tfrac{p}{q} \mid p \in \Pi_m, q \in \Pi_n, q(x) > 0 \text{ for all } x \in [a,b] \}$$

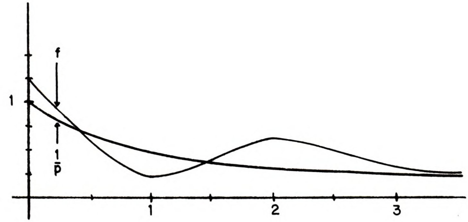
if and only if $f - r$ has an alternating set consisting of

$$N = 2 + \max (n + \partial p, m + \partial q)$$

points [15, p. 122]. Since the set R_n is contained in $R_n^o[0,\infty)$ we might expect a characterization theorem in terms of an alternating set consisting of $N = n + 2$ points. Achieser [1] gives such a theorem for infinite intervals in the case $m \geq n$ with additional conditions. This theorem, however, does not handle the case under consideration here.

The following example shows that some modifications of the usual theorem will have to be made for characterizing best approximations to $f \varepsilon C_o^+[0,\infty)$ from R_n.

Example 1.1: We will consider the approximation $\frac{1}{p(x)} = \frac{1}{x+1}$ to $f \varepsilon C_o^+[0,\infty)$ where f is similar to the function defined as shown, and prove $\frac{1}{p}$ is a best approximation to f from R_2.



We assume f is constructed so that

$$f(0) = \frac{5}{4}$$

$$f(1) = \frac{1}{4}$$

$$f(2) = \frac{7}{12}$$

with $X(\frac{1}{p}) = \{0,1,2\}$, $\|f - \frac{1}{p}\| = \frac{1}{4}$, and $f - \frac{1}{p} < \frac{1}{4}$ for $x > 2$.

So if $\frac{1}{q} \varepsilon R_2$ is such that $\|f - \frac{1}{q}\| < \|f - \frac{1}{p}\|$ we must have

$$\left| f(x) - \frac{1}{q(x)} \right| < \frac{1}{4} \qquad \forall\ x \in X(\frac{1}{p})$$

and

$$\max\{0,\ f(x) - \frac{1}{4}\} < \frac{1}{q(x)} < f(x) + \frac{1}{4}\ .$$

Then at $x = 0$, we must have

$$-\frac{1}{4} < \frac{5}{4} - \frac{1}{q(0)} < \frac{1}{4}$$

which implies that

$$\frac{2}{3} < q(0) < 1.$$

At $x = 1$, we must have

$$-\frac{1}{4} < \frac{1}{4} - \frac{1}{q(1)} < \frac{1}{4}$$

from which it follows that

$$2 < q(1).$$

At $x = 2$, we need

$$-\frac{1}{4} < \frac{7}{12} - \frac{1}{q(2)} < \frac{1}{4}$$

which leads to the inequality

$$\frac{6}{5} < q(0) < 3.$$

Thus if $q(x) = ax + b$ we must have

$$\begin{aligned} \frac{2}{3} &< b < 1 \\ 2 &< a + b \\ \frac{6}{5} &< 2a + b < 3. \end{aligned}$$

But $b < 1$ implies $a > 1$, and hence $a + (a + b) > 1 + 2 = 3$, which cannot happen. So $q(x)$ must be of the form $ax^2 + bx + c$. Thus we have

$$\frac{2}{3} < c < 1$$
$$2 < a + b + c$$
$$\frac{6}{5} < 4a + 2b + c < 3 .$$

In order that $\frac{1}{q} \in R_n$, we must also have $a > 0$. Now $c < 1$ implies $a + b > 1$. So

$$4a + 2b + c = (a + b + c) + (a + b) + 2a > 2 + 1 + 0 = 3 .$$

Thus no such q exists. Hence $\frac{1}{x+1}$ is a best approximation to f from R_2, with an alternating set consisting of only three points. Finally, we note that $\frac{1}{p}$ is also a best approximation to f from R_1.

Since the best constant approximation to $f \in C_o^+[0,\infty)$ is $c^* \equiv \frac{1}{2} \|f\|$, we will assume $n \geq 1$ in the characterization theorem. In this case no best approximation can be constant. To see this, let c^* be the best constant approximation to f from R_n with $n \geq 1$. We will show it is possible to choose a and b so that $\frac{1}{ax+b} \in R_n$ and

$$\left\|f - \frac{1}{ax+b}\right\| < \|f - c^*\| = c^*$$

Since $f(x) > 0$ for all real x, we must have

$$f(x) - c^* > 0$$

for all real $x \in X(c^*)$ and

$$\lim_{x\to\infty} f(x) - c^* = - \|f - c^*\| .$$

Thus there exists a real number α such that $f(x) < c^*$ on (α,∞)

and $f(\alpha) = c^*$. Note that $[0,\alpha)$ contains $X(c^*) \sim \{\infty\}$.

Now let

$$2\varepsilon = \min_{x\varepsilon[0,\alpha]} f(x).$$

Let

$$b = \frac{1}{c^*} - a\alpha$$

and choose $a > 0$ so small that

$$\frac{1}{c^*+\varepsilon} < \frac{1}{c^*} - a\alpha.$$

Then for $x \varepsilon [0,\alpha)$, we have

$$ax + b = a(x - \alpha) + \frac{1}{c^*} < \frac{1}{c^*}$$

and

$$\begin{aligned} ax + b &= ax + \frac{1}{c^*} - a\alpha \\ &\geq \frac{1}{c^*} - a\alpha \\ &> \frac{1}{c^*+\varepsilon} . \end{aligned}$$

Thus

$$c^* < \frac{1}{ax+b} < c^* + \varepsilon$$

for $x \varepsilon [0,\alpha)$. But

$$\frac{1}{a\alpha+b} = c^* = f(\alpha)$$

and hence it follows that

$$\max_{x\varepsilon[0,\alpha]} \left| f(x) - \frac{1}{ax+b} \right| < c^* .$$

Since $\frac{1}{ax+b}$ is decreasing on $[0,\infty)$ we also have

$$\max_{x\varepsilon[\alpha,\infty)} \left| f(x) - \frac{1}{ax+b} \right| < c^* ,$$

and hence

$$\left\| f - \frac{1}{ax+b} \right\| < c^* .$$

This leads us to suggest the following theorem for characterization of best approximations.

<u>Theorem 1.2</u>: Let $f \varepsilon C_o^+[0,\infty)$ and $n \geq 1$. Then $\frac{1}{p}$ is a best approximation to f from the set R_n if and only if either

(1.4) $f - \frac{1}{p}$ has an alternating set consisting of at least $n + 2$ points,

or

(1.5) $f - \frac{1}{p}$ has an alternating set consisting of $n + 1$ points, $\partial p \leq n - 1$ and $f - \frac{1}{p}$ is positive at the largest point of the alternating set.

<u>Proof</u>: We will first show that if we have either (1.4) or (1.5) then $\frac{1}{p}$ is best.

Suppose (1.4) holds. Then $\frac{1}{p}$ cannot be constant. Denote the alternating set by $0 \leq x_1 < x_2 <\ldots< x_{n+2} < \infty$ so that

$$\left|f(x_j) - \frac{1}{p(x_j)}\right| = \|f - \frac{1}{p}\|, \qquad j = 1,\ldots,n+2.$$

If there is an element $\frac{1}{q} \in R_n$ such that

$$(1.6) \qquad \|f - \frac{1}{q}\| < \|f - \frac{1}{p}\|$$

we must have that

$$\frac{1}{q} - \frac{1}{p} = (f - \frac{1}{p}) - (f - \frac{1}{q})$$

alternates in sign on the set $x_1 < x_2 < \ldots < x_{n+2}$, and hence has at least $n + 1$ zeros. But

$$\frac{1}{q} - \frac{1}{p} = \frac{p - q}{pq}$$

with $pq > 0$, and so $p - q$ has at least $n + 1$ zeros. Since $p - q$ is a polynomial of degree at most n, we must have $p - q \equiv 0$. Thus $p \equiv q$ and (1.6) cannot hold.

Now assume we have (1.5). Again $\frac{1}{p}$ cannot be constant, so let the alternating set be $0 \leq x_1 < x_2 < \ldots < x_{n+1} < \infty$, and suppose there is an element $\frac{1}{q} \in R_n$ such that (1.6) holds. By repeating the above argument, we guarantee that there are at least n zeros for $p - q$. So if $\partial q \leq n - 1$, we find that $q \equiv p$ which again contradicts the assumption (1.6).

So we have only to consider the case where $\partial q = n$. Since we assume (1.6) holds, and

$$f(x_{n+1}) - \frac{1}{p(x_{n+1})} > 0$$

we have

$$0 < (f - \frac{1}{p}) - (f - \frac{1}{q}) = \frac{1}{p} - \frac{1}{q} = \frac{p - q}{pq}$$

at x_{n+1}. Hence $p - q > 0$ at x_{n+1}. Since $\frac{1}{q} \varepsilon R_n$, q must have a positive leading coefficient. Here $\partial q > \partial p$ and so $p - q$ has a negative leading coefficient. Thus $p - q \to -\infty$ as $x \to \infty$, which implies that $p - q$ has a zero $x^* > x_{n+1}$. So $p - q$ has $n + 1$ zeros, and again we must have $p \equiv q$. This contradicts the assumption (1.6) and therefore $\frac{1}{p}$ is a best approximation to f.

To complete the proof of the theorem we will show that if $\frac{1}{p}$ is best and (1.4) does not hold, then we must have (1.5). So assume there are $N < n + 2$ points in the alternating set. Then if $\partial p = n$, we can construct a better approximation $\frac{1}{q^*}$ as follows:

Begin in the usual way by constructing N subintervals

$$[0,\xi_1],\ [\xi_1,\xi_2],\ldots,[\xi_{N-1},\infty)$$

in such a way that in each of them in turn one of the following two inequalities is satisfied:

$$(1.7) \qquad - E \le f(x) - \frac{1}{p(x)} < E - \alpha$$

$$(1.8) \qquad - E + \alpha < f(x) - \frac{1}{p(x)} \le E$$

where $E = \left\| f - \frac{1}{p} \right\|$ and α is some positive real number.

Following Achieser [1, p. 55] we consider the function

$$\Phi(x) = (x - \xi_1)(x - \xi_2) \ldots (x - \xi_{N-1})$$

and write $\Phi(x) = kp(x) - \phi(x)$ where $\partial\phi = n$ and k is a constant. Then set

$$\frac{c}{q(x)} = \frac{1 - bk}{p(x) - b\phi(x)} .$$

We will show that there exists $b^* > 0$ such that $\frac{c}{q} \varepsilon R_n$ for all b with $|b| \le b^*$. Then, since

$$f(x) - \frac{c}{q(x)} = f(x) - \frac{1}{p(x)} + \frac{b\Phi(x)}{p(x)[p(x) - b\phi(x)]}$$

and $b\Phi$ changes sign at $\xi_1,\ldots,\xi_{N-1}$, if we show that $p(x)[p(x) - b\phi(x)] > 0$ on $[0,\infty)$ and that

$$\frac{\Phi(x)}{p(x)[p(x) - b\phi(x)]}$$

is bounded, it will follow by taking b small enough with the appropriate sign that $\frac{c}{q}$ is a better approximation to f than $\frac{1}{p}$.

Let $b_o > 0$ be such that $1 - bk > 0$ for all b with $|b| \le b_o$. Let a_n denote the leading coefficient of p. Then $a_n > 0$ so there exists $b_1 > 0$ such that if $|b| \le b_1$, $p - b\phi$ has a positive leading coefficient.

Now observe that there exists $\lambda > 0$ such that $p(x) \ge 2\lambda$ on $[0,\infty)$. Then it is easy to show that there exists $\beta > 0$ such that $p - b\phi \ge \lambda > 0$ on (β,∞) for all b with $|b| \le b_1$.

Now there exists $b_2 > 0$ such that $|b\phi(x)| \le \lambda$ for all $x \in [0,\beta]$ whenever $|b| \le b_2$, and thus $p(x) - b\phi(x) \ge \lambda$ on $[0,\beta]$ whenever $|b| \le b_2$.

Finally let $b^* = \min\{b_o, b_1, b_2\}$. Then if $|b| \le b^*$ we have $\frac{c}{q}$ in R_n, and

$$\frac{b\Phi(x)}{p(x)[p(x) - b\phi(x)]} = \frac{b[k - \frac{\phi(x)}{p(x)}]}{[p(x) - b\phi(x)]}$$

approaches zero as $x \to \infty$. Now we show that by taking b small enough with appropriate sign we can make $\frac{c}{q}$ a better approximation than $\frac{1}{p}$. For x such that (1.7) holds, we have

$$\frac{b\Phi(x)}{p(x)[p(x) - b\phi(x)]} - E \le f(x) - \frac{c}{q(x)} < E + \frac{b\Phi(x)}{p(x)[p(x) - b\phi(x)]} - \alpha$$

so choose b such that $b\Phi > 0$ and

$$\frac{b\Phi(x)}{p(x)[p(x) - b\phi(x)]} < \alpha .$$

Then $b\Phi < 0$ where (1.8) holds, and so if b is so small that we also have

$$\frac{b\Phi(x)}{p(x)[p(x) - b\phi(x)]} > -\alpha$$

where (1.8) holds then $\frac{c}{q}$ is a better approximation to f than $\frac{1}{p}$.

Thus we have shown that if $\frac{1}{p}$ is best, and (1.4) does not hold, we must have $\partial p \le n - 1$.

Now if we have $N' < n + 1$ points in the alternating set, we can again construct a better approximation than $\frac{1}{p}$. If $\partial p = n - 1$, we proceed exactly as above. If $\partial p < n - 1$ we use a slightly modified but simpler approach.

$$\Phi(x) = (x - \xi_1)(x - \xi_2) \cdots (x - \xi_{N'-1})$$

has degree at most $n - 1$. Choose $\varepsilon = 1$ or $\varepsilon = -1$ so that

$$\operatorname{sgn} \varepsilon\Phi(x_i) = -\operatorname{sgn}\left(f(x_i) - \frac{1}{p(x_i)}\right) .$$

Then if the leading coefficient on $\varepsilon\Phi$ is positive, we define $\Phi^* = \varepsilon\Phi$. If not, we select $\xi_{N'} > x_{N'-1}$ such that $X(\frac{1}{p}) \subset [0,\xi_{N'})$ and set

$$\Phi^*(x) = \varepsilon\Phi(x)\ (\xi_{N'} - x).$$

So in either case, $b\Phi^*$ has a positive leading coefficient for any $b > 0$.

Now there exists $\beta > 0$ such that

$$p(x) + b\Phi^*(x) > p(x) > \frac{2}{E}$$

and $|f(x)| < \frac{E}{2}$ on (β,∞). Thus for any $b > 0$ we have

$$\left|f(x) - \frac{1}{p(x) + b\Phi^*(x)}\right| \leq |f(x)| + \left|\frac{1}{p(x) + b\Phi^*(x)}\right|$$

$$< \frac{E}{2} + \frac{E}{2} = E.$$

on (β,∞). As before we can select b so that

$$\left|f(x) - \frac{1}{p(x) + b\Phi^*(x)}\right| < E$$

on $[0,\beta]$; hence $\frac{1}{p}$ is not best.

So if (1.4) does not hold, we have exactly $n + 1$ points in the alternating set.

Finally we must show that if $\frac{1}{p}$ is best and (1.4) does not hold, we have

$$f(x_{n+1}) - \frac{1}{p(x_{n+1})} > 0.$$

where x_{n+1} is the largest point of the alternating set. So assume

$$f(x_{n+1}) - \frac{1}{p(x_{n+1})} < 0.$$

We know $f - \frac{1}{p}$ changes sign at at least n points z_i with $x_i < z_i < x_{i+1}$, $i = 1,\ldots,n$. Let

$$q(x) = \delta \prod_{i=1}^{n} (x - z_i)$$

where δ is either $+1$ or -1 and is chosen so that

$$\operatorname{sgn} q(x_i) = -\operatorname{sgn}\left(f(x_i) - \frac{1}{p(x_i)} \right).$$

Then $q(x_{n+1}) > 0$ and $q(x) \to \infty$ as $x \to \infty$ since all zeros of q appear before x_{n+1}. Thus there exists $\varepsilon_o > 0$ so that $\frac{1}{p + \varepsilon q}$ is in R_n for any ε with $0 < \varepsilon < \varepsilon_o$, and so we will construct an element of this form which is a better approximation to f than $\frac{1}{p}$.

Choose $\beta > 0$ so that both $f(x) < \frac{E}{2}$ and $\frac{1}{p(x)} < \frac{E}{2}$ on (β,∞). Then $[0,\beta]$ contains the alternating set, and $p + \varepsilon q > p$ on (β,∞) for any $\varepsilon > 0$, so it follows that

$$\left| f - \frac{1}{p + \varepsilon q} \right| < E$$

on (β,∞) for any $\varepsilon > 0$.

We must show that $\varepsilon > 0$ can be chosen so that

$$\left| f - \frac{1}{p + \varepsilon q} \right| < E$$

on $[0,\beta]$. To do so, we note that there exists some $\alpha > 0$ such that (1.7) holds on (z_n,β), (z_{n-2}, z_{n-1}), ... and (1.8) holds on (z_{n-1}, z_n), (z_{n-3}, z_{n-2}), ...

On the intervals where (1.7) is satisfied, $q(x) > 0$ so for $\varepsilon > 0$, $p + \varepsilon q > p$ and thus

$$f - \frac{1}{p + \varepsilon q} > f - \frac{1}{p}.$$

We need to choose $\varepsilon > 0$ small enough so that

$$f - \frac{1}{p + \varepsilon q} < E - \frac{\alpha}{2}.$$

This will be the case if

$$1 > pf + p\,\frac{\alpha}{2} - pE + \varepsilon q(f + \frac{\alpha}{2} - E).$$

Since $f - \frac{1}{p} < E - \alpha$ we have $1 > pf + p\alpha - pE.$ So we need

$$p\,\frac{\alpha}{2} > \varepsilon q(f + \frac{\alpha}{2} - E).$$

Since $p(x) \geq \lambda$ for all x, and both q and f assume their maximum value on any closed interval, there exists some $\varepsilon_1 > 0$ such that the above inequality holds when $0 < \varepsilon < \varepsilon_1$. Thus we have

$$-E < f - \frac{1}{p + \varepsilon q} < E - \frac{\alpha}{2}$$

on (z_n,β), (z_{n-2}, z_{n-1}), ... when $\varepsilon > 0$ is less than $\min\,\{\varepsilon_o,\varepsilon_1\}$. Similarly we find that there exists $\varepsilon_2 > 0$ such that

$$\frac{\alpha}{2} - E < f - \frac{1}{p + \varepsilon q} < E$$

on (z_{n-1}, z_n), (z_{n-3}, z_{n-2}), ... when $\varepsilon > 0$ is less than $\min\,\{\varepsilon_o,\varepsilon_2\}$. So if we choose ε^* with $0 < \varepsilon^* < \min\,\{\varepsilon_o,\varepsilon_1,\varepsilon_2\}$, we have

$$\left\|f - \frac{1}{p + \varepsilon^* q}\right\| < \left\|f - \frac{1}{p}\right\|$$

which contradicts the fact that $\frac{1}{p}$ is best. Hence we must have

$$f(x_{n+1}) - \frac{1}{p(x_{n+1})} > 0. \blacksquare$$

Section 4: Uniqueness of Best Approximations

Retaining the setting of the previous sections, we obtain the following uniqueness theorem.

<u>Theorem 1.3</u>: If $\frac{1}{p}$ is a best approximation to $f \in C_o^+[0,\infty)$ from R_n, then it is unique; that is, if $\frac{1}{q} \in R_n$ and $\frac{1}{q} \neq \frac{1}{p}$, then

$$\|f - \frac{1}{q}\| > \|f - \frac{1}{p}\| .$$

This theorem is an immediate consequence of theorem 2.3 in Chapter II, and hence will not be proved here.

CHAPTER II

TCHEBYCHEFF APPROXIMATION BY RATIONAL FUNCTIONS OF THE FORM $\frac{1}{p}$ ON $[0,\infty)$ WITH OSCULATORY INTERPOLATION

Section 1: Introduction

The problem we wish to consider in this chapter is that of approximating a function f by a k-point osculating function of the form $\frac{1}{p}$. In order to be more specific, we retain the notation of Chapter I, and introduce some additional terminology.

Let $\{y_1,\ldots,y_k\}$ be a fixed set of k points from $[0,\infty)$ and $\{m_1,\ldots,m_k\}$ a set of positive integers. Set

$$m = \sum_{i=1}^{k} m_i$$

and

$$s = \max_i \{m_i - 1\}.$$

In addition, we will assume that $m < n + 1$ and use $\tilde{C}^s[0,\infty)$ to denote the set of functions from $C_o^+[0,\infty)$ which have at least s continuous derivatives.

Recall that a function $f \varepsilon \tilde{C}^s[0,\infty)$ is said to have a zero of order $\nu \leq s$ at z if $f(z) = \ldots = f^{(\nu-1)}(z) = 0$, $f^{(\nu)}(z) \neq 0$.

We will adopt essentially the same approach as that of Perrie [13], and hence for a given $f \varepsilon \tilde{C}^s[0,\infty)$, we define the set $K_n(f)$ by

$$K_n(f) = \left\{ \frac{1}{p} \varepsilon R_n \;\middle|\; \left(\frac{1}{p}\right)^{(j)}(y_i) = f^{(j)}(y_i),\ j = 0,\ldots,m_i-1;\ i = 1,\ldots,k \right\}$$

where $f(y_i) \neq f(y_j)$ for some i,j.

Then we will be interested in finding a best approximation to f from the set $K_n(f)$. Such a best approximation may not exist. Gilormini [6] claimed existence for rational approximation with interpolation on a compact interval, but Loeb [8] published a short counterexample to this claim. Here we will postpone further discussion of the existence question until later, when we will obtain results in special cases, and turn instead to characterization of best approximations.

An alternation theorem similar to that of Chapter I can be obtained, with the modification that, roughly speaking, the order of interpolation at the point y_i may "use up" a certain number of alternations of the error curve. Theorems of this type have been given in [9] by Loeb, Moursund, Schumaker and Taylor in the case where $f \in C(X)$, X a compact subset of $[a,b]$, with approximants from a subset of an n-dimensional extended Haar system of order ν, and where the error of approximation was measured with a generalized weight function. Their work generalized results of Paszkowski [12] and Deutsch [5]. Perrie [13] gives similar results for approximation to $f \in C^s(X)$, X a compact subset of $[a,b]$, by functions from a subset of

$$\{ \frac{p}{q} \mid p \in P,\ q \in Q,\ q > 0 \ \text{ on } \ X \}$$

with P and Q finite dimensional subspaces of $C^s(X)$.

<u>Section 2</u>: Characterization Theorems

The following lemma due to Salzer [16] will be used. We assume $f \in \tilde{C}^s[0,\infty)$ is given with $f(y_i) \neq f(y_j)$ for some i,j.

Lemma 2.1: At the points where $q(y_i) \neq 0$, the system

$$\left(\frac{p}{q}\right)^{(j)}(y_i) = f^{(j)}(y_i), \quad \begin{array}{l} j = 0,\ldots,m_i-1 \\ i = 1,\ldots,k \end{array}$$

is equivalent to

$$p^{(j)}(y_i) = (qf)^{(j)}(y_i), \quad \begin{array}{l} j = 0,\ldots,m_i-1 \\ i = 1,\ldots,k. \end{array}$$

We note that this equivalence does not require p or q to be a polynomial or even a linear combination of given functions.

The analogue of theorem 1.2 for characterization of best approximations can now be proved.

Theorem 2.1: Let $f \in \tilde{C}^s[0,\infty)$ and $n \geq 1$. Then $\frac{1}{p}$ is a best approximation to f from the set $K_n(f)$ if and only if either

(2.1) there exist at least $n - m + 2$ consecutive points x_i such that

(a) $$\left|f(x_i) - \frac{1}{p(x_i)}\right| = \left\|f - \frac{1}{p}\right\|$$

and

(b) $$\operatorname{sgn}\left[\left(f(x_i) - \frac{1}{p(x_i)}\right) \Pi(x_i)\right] = (-1)^{i+1} \operatorname{sgn}\left[\left(f(x_1) - \frac{1}{p(x_1)}\right) \Pi(x_1)\right]$$

for $i = 1,2,\ldots,n-m+2$ where

$$\Pi(t) = (y_1 - t)^{m_1} \ldots (y_k - t)^{m_k},$$

or

(2.2) there exist exactly $n - m + 1$ consecutive points x_i such that (a) and (b) hold for $i = 1,2,\ldots,n-m+1$; $\partial p \leq n - 1$ and

$$\left(f(x_{n-m+1}) - \frac{1}{p(x_{n-m+1})} \right) \Pi^* > 0$$

with

$$\Pi^* = \prod_{\nu\in\Omega} (x_{n-m+1} - y_\nu)^{m_\nu}$$

where $\{y_\nu\}_{\nu\in\Omega}$ is the set of interpolation points which are larger than x_{n-m+1}. (If $\{y_\nu\}_{\nu\in\Omega} = \emptyset$, let $\Pi^* \equiv 1$).

<u>Proof</u>: We first show that if we have either (2.1) or (2.2) then $\frac{1}{p}$ is best. Suppose we have (2.1) and that there exists $\frac{1}{q} \in K_n(f)$ such that

$$\|f - \tfrac{1}{q}\| < \|f - \tfrac{1}{p}\| .$$

Since $\frac{1}{p} \in K_n(f)$, by applying lemma 2.1, we see that

$$\left(\frac{1}{p} \right)^{(j)}(y_i) = f^{(j)}(y_i), \qquad \begin{array}{l} j = 0,\ldots,m_i-1 \\ i = 1,\ldots,k \end{array}$$

is equivalent to

$$p^{(j)}(y_i) = \left(\frac{1}{f} \right)^{(j)}(y_i), \qquad \begin{array}{l} j = 0,\ldots,m_i-1 \\ i = 1,\ldots,k. \end{array}$$

But $\frac{1}{q} \in K_n(f)$ and so

$$p^{(j)}(y_i) \;=\; q^{(j)}(y_i), \qquad \begin{array}{l} j = 0,\dots,m_i-1 \\ i = 1,\dots,k \end{array}$$

which implies that $p - q$ has a zero of order at least m_j at each y_j. Thus we have

$$m \;=\; \sum_{j=1}^{k} m_j$$

zeros for $p - q$.

Now consider the relation

$$\left(f(x) - \frac{1}{p(x)} \right) \Pi(x) + \left(\frac{1}{q(x)} - f(x) \right) \Pi(x) \;=\; \left(\frac{1}{q(x)} - \frac{1}{p(x)} \right) \Pi(x)$$

$$= \left(\frac{p(x) - q(x)}{(pq)(x)} \right) \Pi(x) .$$

Since $pq > 0$ and $\left|f(x_i) - \frac{1}{q(x_i)}\right| < \left|f(x_i) - \frac{1}{p(x_i)}\right|$, we find that $\big(p(x) - q(x) \big) \Pi(x)$ alternates in sign at $n - m + 2$ points x_i.

It is given that $x_i \neq y_j$ for any i,j. If the sum of the m_j's such that $y_j \in (x_i, x_{i+1})$ is even, then

$$\Pi(x_i)\Pi(x_{i+1}) \;>\; 0$$

so that

$$[p(x_i) - q(x_i)][p(x_{i+1}) - q(x_{i+1})] \;<\; 0 .$$

Thus $p - q$ either has a zero in $(x_i, x_{i+1}) \sim \bigcup_{i=1}^{k} \{y_i\}$, or $p - q$ has a zero of order $m_j + 1$ at some y_j in (x_i, x_{i+1}). A similar argument is used if the sum is odd. Since there are $n - m + 1$ intervals of the form (x_i, x_{i+1}), we have at least $n - m + 1$ additional zeros for $p - q$. Thus $p - q$ has at least $n + 1$ zeros, which implies $p \equiv q$, and this is a contradiction. Hence (2.1) implies $\frac{1}{p}$ is best.

Now if (2.2) holds, and there exists $\frac{1}{q} \varepsilon K_n(f)$ with

$$\|f - \frac{1}{q}\| < \|f - \frac{1}{p}\|$$

we find that $p - q$ has n zeros by arguing as above. So if $\partial q \leq n - 1$, we find $p \equiv q$ and again reach a contradiction. Thus, we need only consider the case where $\partial q = n$.

If $\Pi^* \equiv 1$, then we have

$$0 < (f - \frac{1}{p}) - (f - \frac{1}{q}) = \frac{p - q}{pq}$$

at x_{n-m+1}, and since $pq > 0$ we find that $p - q > 0$ at x_{n-m+1}. But $\partial q > \partial p$ and q has a positive leading coefficient, and so $p - q \to -\infty$ as $x \to \infty$. Thus $p - q$ has a zero $x^* > x_{n-m+1}$. Hence we have $n + 1$ zeros for $p - q$, which again leads to a contradiction.

Now suppose Π^* is non-trivial. At x_{n-m+1} we have

$$0 < [(f - \frac{1}{p}) - (f - \frac{1}{q})]\, \Pi^* = (\frac{p - q}{pq})\Pi^*$$

and hence $(p - q)\Pi^* > 0$ at x_{n-m+1}.

If the sum of the m_ν is odd, then $\Pi^* < 0$ and so $p - q < 0$ at x_{n-m+1}. Let y_k be the largest member of $\{y_\nu\}$. Then $p - q$ has a zero of order at least m_ν at each y_ν in $(x_{n-m+1}, y_k]$. If the order of the zero y_ν is exactly m_ν for all ν, and if there are no other zeros of $p - q$ in $(x_{n-m+1}, y_k]$ we must have $p(x) - q(x) > 0$ for $x > y_k$. But then since $p - q \to -\infty$ as $x \to \infty$, $p - q$ must have an additional zero $x^* > y_k$. In any case, we find $p - q$ has $n + 1$

zeros, which leads to a contradiction.

If the sum of the m_ν is even, then $\Pi^* > 0$ and so $p - q > 0$ at x_{n-m+1}. Arguing as above leads to a contradiction since if there are no additional zeros in $(x_{n-m+1}, y_k]$, again we have $p(x) - q(x) > 0$ for $x > y_k$.

Thus we have shown that either (2.1) or (2.2) implies that $\frac{1}{p}$ is a best approximation.

Suppose now that $\frac{1}{p}$ is best. We will show that if (2.1) does not hold, we must have (2.2). So assume we have $N \leq n - m + 1$ consecutive points x_i such that (a) and (b) hold. If $\partial p = n$, we can construct a polynomial q such that $\frac{1}{p + \varepsilon q} \in K_n(f)$ and

$$\left\| f - \frac{1}{p + \varepsilon q} \right\| < \left\| f - \frac{1}{p} \right\|$$

for some $\varepsilon > 0$.

In order to do so, we first introduce the following sets:

(1) $\{z_i\}$ is the set of zeros of $f - \frac{1}{p}$.

(2) $\{w_i\}$ is used to denote the set $X(\frac{1}{p}) \sim \{x_i\}$.

(3) $\{s_i\}$ is the set of points which are different from $\{y_i\}$ where sign changes of $f - \frac{1}{p}$ occur.

Notice that $\{s_i\} \subset \{z_i\}$ and $\{x_i\} \cap \{w_i\} = \emptyset$.

Immediately we make the following modifications to the sets above:

(1) If any of the sets contain an interval, we remove the interval from the set and replace it by a single point from the interval.

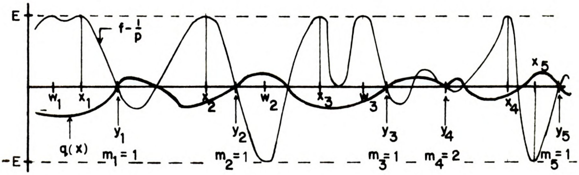
(2) Let c be chosen so large that $X(\frac{1}{p}) \cup \{y_i\}$ is contained in $[0,c)$, with the additional restriction that if there are any s_i which lie to the right of $X(\frac{1}{p}) \cup \{y_i\}$, at least one of these s_i is included in $[0,c)$.

(3) If there are consecutive w_i with no other points from $\{x_i\} \cup \{y_i\} \cup \{z_i\}$ between them, remove all but one of these w_i.

(4) Repeat (3) with w_i replaced by s_i and $\{x_i\} \cup \{y_i\} \cup \{z_i\}$ replaced by $\{x_i\} \cup \{y_i\} \cup (\{z_i\} \sim \{s_i\})$.

Using the same notation for the modified sets, we find that $\{x_i\}$, $\{y_i\}$, $\{z_i\}$ and $\{w_i\}$ are all finite.

We wish to construct a polynomial $q(x)$ with behavior similar to that shown below:



Let $I_{x_i} = (a_i, b_i)$ be the largest open interval containing x_i with all other points of $\{x_i\} \cup \{y_i\} \cup \{s_i\}$ exterior to I_{x_i}. Then between I_{x_i} and $I_{x_{i+1}}$ there must be a y_i or an s_i. In fact, the end points of $[a_i, b_i]$ and $[a_{i+1}, b_{i+1}]$ belong to $\{y_i\} \cup \{s_i\}$.

For convenience we let

$$E(x) = f(x) - \frac{1}{p(x)}$$

and consider

<u>Case I</u>: Suppose there are no points of interpolation y_i in $[b_i, a_{i+1}]$. Then the only points from $\{w_j\}$ which can be contained in $[b_i, a_{i+1}]$ are those at which

$$\text{sgn } E(x_i) = \text{sgn } E(w_j).$$

Hence $q(x)$ need not change sign on $[a_i, a_{i+1}]$ and thus need have no zeros in (a_i, a_{i+1}). So with this x_i we associate a factor $(x - a_{i+1})$ and include this factor in q.

For I_{x_1} we include $\delta = \pm 1$ in q so that

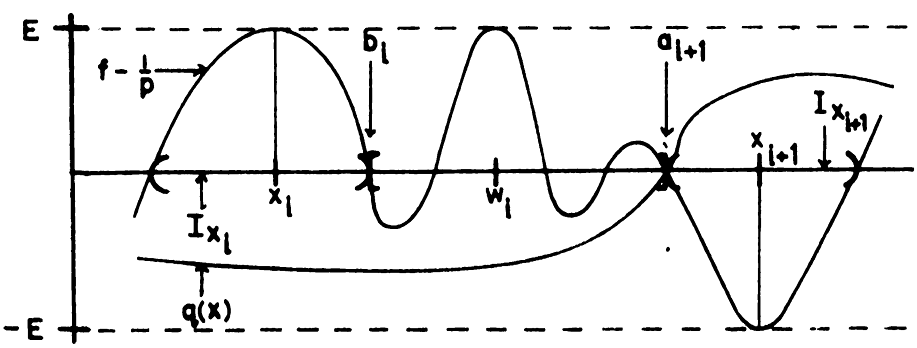
$$\text{sgn } E(x_1) = -\text{sgn } q(x_1);$$

thus q will be constructed so that

$$\text{sgn } E(x_i) = -\text{sgn } q(x_i), \qquad i = 1,\ldots,n-m$$

in either case.

A typical situation for Case I is shown below.



<u>Case II</u>: Suppose we have at least one y_i in $[b_i, a_{i+1}]$. Beginning at b_i, we can list the points of the set

$$(\{y_i\} \cup \{s_i\} \cup \{w_i\}) \cap [b_i, a_{i+1}]$$

in increasing order. We need not consider any s_i before the smallest y_i in the list, since there can be no w_j with

$$\operatorname{sgn} E(x_i) = -\operatorname{sgn} E(w_j)$$

before the smallest y_i. (If there were such a w_j, it would have to be an alternation point.) In order for q to behave as required, it may or may not have to change sign at the y_i's and s_i's, so we need the following scheme to decide what type of factors to include in q. We consider the y_i's and s_i's in the list starting at the first y_i.

(*) At y_i, we either

(1) put a factor $(x - y_i)^{m_i}$ into q

or

(2) put a factor $(x - y_i)^{m_i+1}$ into q

according to the following instructions.

(i) If the next point is a w_i, then m_i is even if $\operatorname{sgn} E(w_i) = \operatorname{sgn} E(x_i)$, and m_i is odd if $\operatorname{sgn} E(w_i) \neq \operatorname{sgn} E(x_i)$. In either case we need only a factor $(x - y_i)^{m_i}$ in order to insure q has the correct sign here.

(ii) If the next point is x_{i+1} then m_i is even if $\operatorname{sgn} E(x_i) \neq \operatorname{sgn} E(x_{i+1})$ and m_i is odd otherwise. In either case q will have the correct sign if we include a factor of the form $(x - y_i)^{m_i+1}$.

(iii) If the next point is y_{i+1}, put a factor of $(x - y_i)^{m_i}$ into q and then repeat the analysis for y_{i+1} going back to (*).

(iv) If the next point is an s_i we proceed to the following point in the list unless a w_i or x_{i+1} appears directly after s_i.

If we have a w_i directly after s_i, and the sum of the previous m_j's associated with the y_j's that occur from x_i up to this point is even, then we will have $\operatorname{sgn} E(w_i) = \operatorname{sgn} E(x_i)$ and q has sign opposite that of $E(w_i)$ here. If the sum is odd, then $\operatorname{sgn} E(x_i) \neq \operatorname{sgn} E(w_i)$ and again q already has sign opposite that of $E(w_i)$. Hence, q need not be altered here.

If x_{i+1} appears directly after s_i then we may have to include a factor $(x - s_i)$ in q.

Thus we proceed to y_{i+1}, and repeat the procedure beginning again at (*), if necessary, until we finish with a_{i+1}. Thus we arrive at x_{i+1} and then continue by going back to Case I.

From the above analysis we conclude that the only time it is necessary to include a factor beside those of the form $(x - y_i)^{m_i}$ in q is directly before x_{i+1} $(i \neq 0)$. Letting q be the product of all the factors $(x - y_i)^{m_i}$ and including only necessary additional factors we find that

$$\partial q \le \sum_{i=1}^{k} m_i + N - 1 = m + N - 1$$

and q has opposite sign from $E(x)$ at all points of $X(\frac{1}{p})$.

Furthermore,

$$(p + \varepsilon q)^{(j)}(x_i) = p^{(j)}(x_i) + (\varepsilon q)^{(j)}(x_i)$$

$$= p^{(j)}(x_i), \qquad j = 0,\ldots,m_i-1; \quad i = 1,\ldots,k.$$

Thus for ε constant, we find that $p + \varepsilon q$ has the interpolation properties specified for p.

Then as in the proof of theorem 1.2 we can show there exists ε such that $\frac{1}{p + \varepsilon q} \in K_n(f)$ and

$$\left\| f - \frac{1}{p + \varepsilon q} \right\| < \left\| f - \frac{1}{p} \right\| .$$

So if (2.1) does not hold, we must have $\partial p \le n - 1$. Then if $N \le n - m$ we have $\partial q \le n - 1$ and again we can construct a better approximation than $\frac{1}{p}$.

It remains to show that

$$\left(f(x_{n-m+1}) - \frac{1}{p(x_{n-m+1})} \right) \Pi^* > 0.$$

If $\Pi^* \equiv 1$, we proceed as in the proof of theorem 1.2. So assume that Π^* is non-trivial, and that

$$\left(f(x_{n-m+1}) - \frac{1}{p(x_{n-m+1})} \right) \Pi^* < 0.$$

Then if Σm_ν is even, $\Pi^* > 0$ and so $f - \frac{1}{p} < 0$ at x_{n-m+1}. Since

$$\operatorname{sgn}\left(f(x_i) - \frac{1}{p(x_i)}\right) = -\operatorname{sgn} q(x_i)$$

we have $q(x_{n-m+1}) > 0$. The only factors included in q after x_{n-m+1} in the earlier analysis are those of the form $(x - y_\nu)^{m_\nu}$, and hence $q(x) > 0$ for $x > y_k$. Thus q has a positive leading coefficient and, as in the proof of theorem 1.2, we can show that there exists $\varepsilon > 0$ so that $\frac{1}{p + \varepsilon q}$ is a better approximation than $\frac{1}{p}$.

If Σm_ν is odd, $\Pi^* < 0$, $f - \frac{1}{p} > 0$ at x_{n-m+1} and $q(x_{n-m+1}) < 0$. Then arguing as above, we see that $q(x) > 0$ for $x > y_k$ and again we can find $\varepsilon > 0$ so that $\frac{1}{p + \varepsilon q}$ is a better approximation than $\frac{1}{p}$. ∎

It is useful to obtain a slightly different characterization theorem, which states that under certain conditions $\frac{1}{p}$ is a best approximation to f if and only if zero is an element of the convex hull of a certain set. In the case of generalized rational approximation, a theorem of this type was proved by Cheney [2]. For ordinary interpolation, Gilormini [6] gave a similar result. Perrie [13] proves such a theorem in the previously mentioned setting.

We will need some preliminary results. Recall f is a given function from $\tilde{C}^s[0,\infty)$. Let $\frac{1}{p^*} \in K_n(f)$ and define

$$S(\tfrac{1}{p^*}) = \{\ 1 - \tfrac{q}{p^*}\ |\ q \in \Pi_n,\ q^{(j)}(y_i) = (\tfrac{1}{f})^{(j)}(y_i),\ \ j = 0,\dots,m_i-1;\ i = 1,\dots,k\ \}$$

Note that since $f > 0$ on $[0,\infty)$ we must have $q(y_i) > 0$ for all i.

<u>Lemma 2.2</u>: $S(\frac{1}{p^*})$ is a subspace of $\{\ \frac{q}{p^*}\ |\ q \in \Pi_n\ \}$

<u>Proof</u>: Let $1 - \frac{p}{p^*}$ and $1 - \frac{q}{p^*}$ be elements of $S(\frac{1}{p^*})$, and let c be a real number. Then

$$c(1 - \frac{p}{p^*}) + (1 - \frac{q}{p^*}) \quad = \quad 1 - \frac{c(p - p^*) + q}{p^*} \quad .$$

Now $c(p - p^*) + q \in \Pi_n$ and

$$[c(p - p^*) + q]^{(j)}(y_i) \quad = \quad c\left(p^{(j)}(y_i) - p^{*(j)}(y_i)\right) \quad + q^{(j)}(y_i)$$

$$= \quad (\frac{1}{f})^{(j)}(y_i) \quad .$$

Thus $c(1 - \frac{p}{p^*}) + (1 - \frac{q}{q^*}) \in S(\frac{1}{p^*})$. ∎

Lemma 2.3: Dim $S(\frac{1}{p^*}) \quad = \quad n - m + 1$.

Proof: Let $N = n - m + 1$. Choose N distinct points $x_1,\ldots,x_N$ which are different from the interpolation points $y_1,\ldots,y_k$. For each t, $t = 1,\ldots,N$ we will show that there exists $g_t \in S(\frac{1}{p^*})$ such that

$$g_t(x_j) \quad = \quad \delta_{tj} \quad = \quad \begin{cases} 0 & t \neq j \\ 1 & t = j \end{cases}$$

for $j = 1,\ldots,N$. So let h^* be any element of $S(\frac{1}{p^*})$, and write h^* in the form $\frac{p^* - p}{p^*}$.

For each t, there exists a polynomial q_t of degree at most n with the following properties:

(1) $\quad q_t^{(j)}(y_i) = 0, \qquad j = 0,\ldots,m_i-1; \quad i = 1,\ldots,k$

(2) $\quad q_t(x_i) = p^*(x_i) - p(x_i), \qquad i + 1,\ldots,t-1,t+1,\ldots,N$

(3) $\quad q_t(x_t) = -p(x_t)$.

We set

$$g_t = 1 - \frac{(p + q_t)}{p^*} .$$

Since $p + q_t \in \Pi_n$ and $(p + q_t)^{(j)}(y_i) = p^{(j)}(y_i)$ we have $g_t \in S(\frac{1}{p^*})$. In addition,

$$g_t(x_j) = \frac{p^*(x_j) - \left(p(x_j) + q_t(x_j) \right)}{p^*(x_j)} = \begin{cases} 0 & t \neq j \\ 1 & t = j \end{cases} .$$

Now suppose h is an arbitrary element of $S(\frac{1}{p^*})$. Define

$$g = \sum_{i=1}^{N} h(x_i) g_i .$$

Then $g \in S(\frac{1}{p^*})$, and

$$\begin{aligned} g(x_j) - h(x_j) &= \sum_{i=1}^{N} h(x_i) g_i(x_j) - h(x_j) \\ &= h(x_j) - h(x_j) \\ &= 0 . \end{aligned}$$

Through the use of the interpolation properties of elements of $S(\frac{1}{p^*})$ we can show

$$\begin{aligned} (g - h)^{(j)}(y_i) &= g^{(j)}(y_i) - h^{(j)}(y_i) \\ &= 0 , \qquad j = 0,\ldots,m_i-1; \quad i = 1,\ldots,k. \end{aligned}$$

Thus $g - h$ has $N + \sum_{i=1}^{k} m_i = n + 1$ zeros, and $g \equiv h$. Hence, any element of $S(\frac{1}{p^*})$ can be written as a linear combination of $\{g,\ldots,g_N\}$. These functions are linearly independent, since

$g_t(x_j) = \delta_{tj}$. So we have a basis for $S(\frac{1}{p^*})$ consisting of $N = n - m + 1$ elements. ∎

Let $\{g_1,\ldots,g_N\}$ be a basis for $S(\frac{1}{p^*})$, $\hat{x} = (g_1(x),\ldots,g_N(x))$, and $H\{\sigma(x)\hat{x} \mid x \in X(\frac{1}{p^*})\}$ denote the convex hull of the vectors $\sigma(x)\hat{x}$ in E^N for $x \in X(\frac{1}{p^*})$ where

$$\sigma(x) = \operatorname{sgn}\left(f(x) - \frac{1}{p^*(x)}\right).$$

Then we can prove

<u>Theorem 2.2</u>: Let $f \in \tilde{C}^s[0,\infty)$ and $\frac{1}{p^*} \in K_n(f)$ with $\partial p^* = n$. Then $\frac{1}{p^*}$ is a best approximation to f if and only if

$$0 \in H\{\sigma(x)\hat{x} \mid x \in X(\frac{1}{p^*})\}.$$

<u>Proof</u>: Suppose $\frac{1}{p^*}$ is not a best approximation to f from $K_n(f)$. Then there exists $\frac{1}{p} \in K_n(f)$ such that

$$\left\|f - \frac{1}{p}\right\| < \left\|f - \frac{1}{p^*}\right\|.$$

If $x \in X(\frac{1}{p^*})$ we have

$$\begin{aligned}
\sigma(x)\left(f(x) - \frac{1}{p(x)}\right) &\leq \left|f(x) - \frac{1}{p(x)}\right| \\
&\leq \left\|f - \frac{1}{p}\right\| \\
&< \left\|f - \frac{1}{p^*}\right\| \\
&= \left|f(x) - \frac{1}{p^*(x)}\right| \\
&= \sigma(x)\left(f(x) - \frac{1}{p^*(x)}\right).
\end{aligned}$$

Thus

$$0 < \sigma(x)\left(\frac{1}{p(x)} - \frac{1}{p^*(x)}\right) = \sigma(x)\left(\frac{p^*(x) - p(x)}{p(x)p^*(x)}\right)$$

Since $p(x) > 0$ we must have

$$\sigma(x)\left(1 - \frac{p(x)}{p^*(x)}\right) > 0 ,$$

and so if we let $h = 1 - \frac{p}{p^*}$ we have $h \in S(\frac{1}{p^*})$ and $\sigma(x)h(x) > 0$ for all $x \in X(\frac{1}{p^*})$.

Now suppose $0 \in H\{\sigma(x)\hat{x} \mid x \in X(\frac{1}{p^*})\}$; that is,

$0 = \sum_{i=1}^{r} \lambda_i \sigma(x_i)\hat{x}_i$ with λ_i such that $\Sigma\lambda_i = 1$, $\lambda_i \geq 0$ for all i, and $\sigma(x_i)\hat{x}_i \in H$. Thus

$$0 = \left(\sum_{i=1}^{r} \lambda_i \sigma(x_i) g_1(x_i), \ldots, \sum_{i=1}^{r} \lambda_i \sigma(x_i) g_N(x_i)\right) .$$

But $\sigma(x_i)h(x_i) > 0$ for each i, and not all λ_i are zero so we have

$$\begin{aligned}
0 &< \sum_{i=1}^{r} \lambda_i \sigma(x_i) h(x_i) \\
&= \sum_{i=1}^{r} \lambda_i \sigma(x_i) \sum_{k=1}^{N} \alpha_k g_k(x_i) \quad \text{for some } \alpha_k,\ k = 1,\ldots,N \\
&= \sum_{i=1}^{r} \sum_{k=1}^{N} \lambda_i \alpha_k \sigma(x_i) g_k(x_i) \\
&= \sum_{k=1}^{N} \sum_{i=1}^{r} \lambda_i \alpha_k \sigma(x_i) g_k(x_i) \\
&= \sum_{k=1}^{N} \alpha_k \sum_{i=1}^{r} \lambda_i \sigma(x_i) g_k(x_i)
\end{aligned}$$

which must be zero since $\sum_{i=1}^{r} \lambda_i \sigma(x_i) g_k(x_i) = 0$ for all k. Hence we have reached a contradiction, and thus it follows that

$$(2.3) \qquad 0 \notin H\{\sigma(x)\hat{x} \mid x \in X(\tfrac{1}{p^*})\} .$$

To complete the proof of the theorem, we assume (2.3) holds and show $\frac{1}{p^*}$ is not best.

The set $X(\frac{1}{p^*})$ is closed and bounded in $[0,\infty)$; hence it is compact. Both σ and $\hat{x}$ are continuous on $X(\frac{1}{p^*})$; hence $\sigma(x)\hat{x}$ is, which implies that $\{\sigma(x)\hat{x} \mid x \in X(\frac{1}{p^*})\}$ is compact in E^N. Therefore $H\{\sigma(x)\hat{x} \mid x \in X(\frac{1}{p^*})\}$ is compact.

Thus by the theorem on linear inequalities (Cheney [2], p. 19) there exists $c = (c_1,\ldots,c_N)$ such that

$$\begin{aligned} 0 < \left\langle \sigma(x)\hat{x}, c \right\rangle &= \sigma(x)c_1g_1(x) + \ldots + \sigma(x)c_Ng_N(x) \\ &= \sigma(x)h(x) \end{aligned}$$

where $h \in S(\frac{1}{p^*})$. We write $h = \frac{p}{p^*} - 1$.

In order to construct a better approximation to f than $\frac{1}{p^*}$, let

$$r_\lambda = \frac{1+\lambda}{p^* + \lambda p} = \frac{1}{(\frac{1}{1+\lambda})p^* + (\frac{\lambda}{1+\lambda})p} .$$

We will first show that there exists a λ_o such that $p^* + \lambda p > 0$ on $[0,\infty)$ for any λ with $|\lambda| \leq \lambda_o$.

Since $p^* > 0$ on $[0,\infty)$, it has a positive leading coefficient. Thus there exists λ_o such that $p^* + \lambda p$ has a positive leading coefficient for all λ with $|\lambda| \leq \lambda_o$. Then we can find $\beta > 0$ such that the inequalities

$$p^*(x) > 1$$

$$p^*(x) + \lambda_o p(x) > 1$$

$$p^*(x) - \lambda_o p(x) > 1$$

hold on (β,∞). Then as in the proof of theorem 1.2 it is not difficult to show that $p^* + \lambda p > 0$ on (β,∞) for any λ with $|\lambda| \leq \lambda_o$. By choosing $\lambda_o < 1$ we guarantee that r_λ is a member of the set $K_n(f)$.

Further, we can find $\varepsilon > 0$ and $\delta > 0$ such that $p^*(x) \geq \varepsilon$ and $|p(x)| \leq \delta$ for all x in $[0,\beta]$. So if we choose λ_1 so small that

$$|\lambda p(x)| \leq |\lambda|\delta < \frac{\varepsilon}{2} \qquad \forall\ x \in [0,\beta]$$

for all λ with $|\lambda| \leq \lambda_1$, it follows that

$$0 < \frac{\varepsilon}{2} < p^*(x) + \lambda p(x)$$

for all $x \in [0,\beta]$. Thus if

$$|\lambda| \leq \lambda_o^* = \min \{\lambda_o,\lambda_1\}$$

we have $p^* + \lambda p > 0$ on $[0,\infty)$.

Then

$$\frac{1}{p^*} - r_\lambda = \frac{1}{p^*} - \frac{1+\lambda}{p^* + \lambda p}$$

$$= \frac{\lambda p^* \left(\frac{p}{p^*} - 1 \right)}{p^*(p^* + \lambda p)}$$

$$= \frac{\lambda h}{p^* + \lambda p}$$

Now let $\alpha = \min \{|h(x)| \mid x \in X(\frac{1}{p^*})\}$, and define

$$X_1 = \left\{ x \;\middle|\; \sigma(x)h(x) > \tfrac{1}{2}\alpha \text{ and } \left|f(x) - \frac{1}{p^*(x)}\right| > \frac{1}{2}\left\|f - \frac{1}{p^*}\right\| \right\}$$

$$X_2 = [0,\infty) \sim X_1 .$$

Then X_1 is open and bounded, and X_2 is closed with $X(\frac{1}{p^*}) \subset X_1$; $X_2 \cap X(\frac{1}{p^*}) = \emptyset$. Hence there exists $\mu > 0$ such that $\mu < \left\|f - \frac{1}{p^*}\right\|$ and $\left|f(x) - \frac{1}{p^*(x)}\right| \leq \mu$ for all $x \in X_2$, since $\left|f(x) - \frac{1}{p^*(x)}\right| \to 0$ as $x \to \infty$.

Next we will show that we can choose $\lambda_1^* > 0$ with $\lambda_1^* \leq \lambda_o^*$ so that for all λ with $|\lambda| \leq \lambda_1^*$ we have

$$\left\|\frac{1}{p^*} - r_\lambda\right\| < \min \left\{ \left\|f - \frac{1}{p^*}\right\| - \mu,\; \frac{1}{2}\left\|f - \frac{1}{p^*}\right\| \right\} .$$

Let K_o denote the right hand side of this inequality. Then if $|\lambda| \leq \lambda_o^*$ we have $p^* + \lambda p \geq \varepsilon > 0$ and hence

$$\left|\frac{1}{p^*} - r_\lambda\right| = \frac{|\lambda| \left|\frac{p}{p^*} - 1\right|}{|p^* + \lambda p|} \leq \frac{|\lambda|}{\varepsilon} \left|\frac{p}{p^*} - 1\right| .$$

From the fact that $\partial p^* = n$, it follows that

$$\left|\frac{p}{p^*} - 1\right| < K_1$$

for some constant K_1. Then we choose $\lambda_1^* \leq \lambda_o^*$ such that

$$\frac{|\lambda| K_1}{\varepsilon} < K_o$$

whenever $|\lambda| \leq \lambda_1^*$.

Finally consider any λ with $|\lambda| \leq \lambda_1^*$. Then if $x \varepsilon X_2$, we have

$$\begin{aligned}
\left|f(x) - r_\lambda(x)\right| &\leq \left|f(x) - \frac{1}{p^*(x)}\right| + \left|\frac{1}{p^*(x)} - r_\lambda(x)\right| \\
&\leq \mu + \left|\frac{1}{p^*(x)} - r_\lambda(x)\right| \\
&\leq \mu + \left(\left\|f - \frac{1}{p^*}\right\| - \mu\right) \\
&= \left\|f - \frac{1}{p^*}\right\| .
\end{aligned}$$

If $x \varepsilon X_1$, we note that

$$\operatorname{sgn}\left(f(x) - \frac{1}{p^*(x)}\right) = \operatorname{sgn}\left(f(x) - r_\lambda(x)\right)$$

since

$$\left|f(x) - \frac{1}{p^*(x)}\right| > \frac{1}{2}\left\|f - \frac{1}{p^*}\right\| ,$$

$$\left\|\frac{1}{p^*} - r_\lambda\right\| < \frac{1}{2}\left\|f - \frac{1}{p^*}\right\|$$

and

$$f - r_\lambda = \left(f - \frac{1}{p^*}\right) + \left(\frac{1}{p^*} - r_\lambda\right).$$

Thus if $x \varepsilon X_1$, and we take λ negative,

$$
\begin{aligned}
\left| f(x) - r_\lambda(x) \right| &= \sigma(x) \; (f(x) - r_\lambda(x)) \\
&= \sigma(x) \left(f(x) - \frac{1}{p^*(x)} \right) + \sigma(x) \left(\frac{1}{p^*(x)} - r_\lambda(x) \right) \\
&\leq \left\| f - \frac{1}{p^*} \right\| + \frac{\lambda\sigma(x)h(x)}{p^*(x) + \lambda p(x)} \\
&\leq \left\| f - \frac{1}{p^*} \right\| + \frac{\lambda\sigma(x)h(x)}{\sup\limits_{x \varepsilon X_1} (p^*(x) + \lambda p(x))} \\
&\leq \left\| f - \frac{1}{p^*} \right\| + \frac{1}{2}\alpha \left[\frac{\lambda}{\sup\limits_{x \varepsilon X_1} (p^*(x) + \lambda p(x))} \right] \\
&< \left\| f - \frac{1}{p^*} \right\| .
\end{aligned}
$$

Hence for all $x \varepsilon [0,\infty)$ and suitable negative values of λ we have

$$\| f - r_\lambda \| < \left\| f - \frac{1}{p^*} \right\| . \blacksquare$$

Under the assumptions of theorem 2.2, we have the following lemma, which will be used in the proof of theorem 3.1.

<u>Lemma 2.4</u>: $0 \varepsilon H\{\sigma(x)\hat{x} \mid x \varepsilon X(\frac{1}{p^*})\}$ if and only if $h(x)\sigma(x) \geq 0$ $\forall\; x \varepsilon X(\frac{1}{p^*})$ and $h \varepsilon S(\frac{1}{p^*})$ imply $h \equiv 0$.

<u>Proof</u>: Suppose $0 \varepsilon H\{\sigma(x)\hat{x} \mid x \varepsilon X(\frac{1}{p^*})\}$ and we have $h \varepsilon S(\frac{1}{p^*})$ with $\sigma(x)h(x) \geq 0$ for all $x \varepsilon X(\frac{1}{p^*})$. We will assume $h \not\equiv 0$ and derive a contradiction.

Let $z_1,\ldots,z_t$ denote the zeros of h which are contained in $X(\frac{1}{p^*})$. Then we must have $z_i \neq y_j$ for any i,j since

$f(y_i) - \frac{1}{p^*(y_i)} = 0$ and $\|f - \frac{1}{p^*}\| > 0$. Observe that $t \leq n$.

We can find $g \in S(\frac{1}{p^*})$ with $g(z_i) = \sigma(z_i)$, $i = 1,\dots,t$; for example we could take

$$g = \sum_{i=1}^{t} \sigma(z_i) g_i$$

where $g_i(z_j) = \delta_{ij}$, $g_i \in S(\frac{1}{p^*})$ as in the proof of lemma 2.3. Then we can show there exists $\lambda > 0$ such that

$$\sigma(x) \, (\, h(x) + \lambda g(x) \,) > 0$$

for all $x \in X(\frac{1}{p^*})$.

To do so, let I_{z_i} be an open interval containing z_i such that $\sigma(x)g(x) > 0$ for all $x \in I_{z_i}$ with $I_{z_i} \cap I_{z_j} = \emptyset$, $i \neq j$. Let

$$Y = X(\frac{1}{p^*}) \sim \bigcup_{i=1}^{t} I_{z_i} .$$

Then Y is closed. Since $X(\frac{1}{p^*}) \subset [0,\beta]$ for some $\beta > 0$, Y is also bounded and hence compact. Since we have removed the zeros of h from Y, we have $\sigma(x)h(x) > 0$ on Y. Let

$$m = \inf \{\sigma(x)h(x) \mid x \in Y\}$$

$$M = \inf \{\sigma(x)g(x) \mid x \in Y\} .$$

Then $m > 0$ and $M > -\infty$. Thus we can find $\lambda > 0$ such that $m + \lambda M > 0$.

Then $h + \lambda g \in S(\frac{1}{p^*})$ and $\sigma(x)\left(h(x) + \lambda g(x)\right) > 0$ on $X(\frac{1}{p^*})$.

By the theorem on linear inequalities, we cannot have

$0 \in H\{\sigma(x)\hat{x} \mid x \in X(\frac{1}{p^*})\}$.

Now assume that $h \in S(\frac{1}{p^*})$ and $\sigma(x)h(x) \geq 0$ for all $x \in X(\frac{1}{p^*})$ imply $h \equiv 0$. If $0 \notin H\{\sigma(x)\hat{x} \mid x \in X(\frac{1}{p^*})\}$, then there exists $h \in S(\frac{1}{p^*})$ such that $\sigma(x)h(x) > 0$ for all $x \in X(\frac{1}{p^*})$. But then $\sigma(x)h(x) \geq 0$ on $X(\frac{1}{p^*})$ and so $h \equiv 0$ which is a contradiction. ∎

Section 3: Uniqueness of Best Approximation

Theorem 2.3: If $\frac{1}{p^*}$ is a best approximation to $f \in \tilde{C}^s[0,\infty)$ from $K_n(f)$, then $\frac{1}{p^*}$ is unique.

Proof: Suppose there exists $\frac{1}{p} \in K_n(f)$ with

$$\left\|f - \frac{1}{p}\right\| = \left\|f - \frac{1}{p^*}\right\| \tag{2.4}$$

and let $x \in X(\frac{1}{p^*})$. Then

$$\sigma(x)\left(f(x) - \frac{1}{p^*(x)}\right) \geq \sigma(x)\left(f(x) - \frac{1}{p(x)}\right)$$

or

$$0 \leq \sigma(x)\left(\frac{1}{p(x)} - \frac{1}{p^*(x)}\right) = \sigma(x)\left(\frac{p^*(x) - p(x)}{p(x)p^*(x)}\right)$$

which implies that $\sigma(x)\left(p^*(x) - p(x)\right) \geq 0$ on $X(\frac{1}{p^*})$.

Now suppose (2.1) holds. Then $p^* - p$ has $m = \sum_{j=1}^{k} m_j$ zeros because of the interpolation conditions. If both $(p^* - p)(x_i)$ and $(p^* - p)(x_{i+1})$ are non-zero, then we can produce an additional zero for $p^* - p$ in (x_i, x_{i+1}) as in the proof of theorem 2.1. If

$$(p^* - p)(x_i) \neq 0,\ (p^* - p)(x_{i+1}) = \ldots = (p^* - p)(x_{i+r}) = 0,\ (p^* - p)(x_{i+r+1}) \neq 0$$

for some $r > 0$ then we have r additional zeros for $p^* - p$ in (x_i, x_{i+r+1}). Thus we have to show that $p^* - p$ has at least one more zero in (x_i, x_{i+r+1}).

To do so, let $\{y_t\}_{t\varepsilon T}$ denote the set of interpolation points in (x_i, x_{i+r+1}). First suppose $\{y_t\}$ is empty. Then

$$\operatorname{sgn} \sigma(x_i) = (-1)^{1+r} \operatorname{sgn} \sigma(x_{i+r+1})$$

and so if r is even, $\sigma(x_i)$ and $\sigma(x_{i+r+1})$ have opposite sign and thus $(p^* - p)(x_i)$ and $(p^* - p)(x_{i+r+1})$ have opposite sign. This implies that $p^* - p$ has at least one additional zero in (x_i, x_{i+r+1}). If r is odd, the argument is similar.

If $\{y_t\}$ is not empty, we note that

$$(2.5) \qquad \operatorname{sgn} \sigma(x_i) = \left[(-1)^{1+r+\sum_{t\varepsilon T} m_t}\right] \operatorname{sgn} \sigma(x_{i+r+1}) \ ;$$

and so if $r + \Sigma m_t$ is even, $\sigma(x_i)$ and $\sigma(x_{i+r+1})$ have opposite sign. Hence $(p^* - p)(x_i)$ and $(p^* - p)(x_{i+r+1})$ have opposite sign and again we must have another zero for $p^* - p$ in (x_i, x_{i+r+1}). The case $r + \Sigma m_t$ odd is similar. Hence we have shown that if (2.1) holds, then (2.4) implies that $p \equiv p^*$.

Now if (2.2) holds, then $\partial p^* \leq n - 1$ and we have $n - m + 1$ alternation points, so we can produce at least n zeros for $p^* - p$ by arguing as above. Thus if $\partial p \leq n - 1$ we have $p^* \equiv p$. Thus we need only show that $p^* - p$ has an additional zero when $\partial p = n$.

Since

$$\left(f(x_{n-m+1}) - \frac{1}{p(x_{n-m+1})} \right) \Pi^* > 0$$

we have $(p^* - p)\Pi^* \geq 0$ at x_{n-m+1}. When $(p^* - p)\Pi^* > 0$ at x_{n-m+1} we can show $p^* - p$ has at least one additional zero by arguing as in the proof of theorem 1.2. If $(p^* - p)\Pi^* = 0$ at x_{n-m+1}, and $p \not\equiv p^*$, there must be some r such that

$$(p^* - p)(x_{n-m+1-r}) \neq 0 .$$

Then using arguments similar to those above along with (2.5) we can show that $p^* - p$ must have an additional zero in $(x_{n-m+1-r}, \infty)$.

Thus, in any case, $p^* \equiv p$ and hence $\frac{1}{p^*}$ is unique. ∎

CHAPTER III

CONTINUITY OF THE BEST APPROXIMATION OPERATOR

Section 1: Introduction

We wish to improve the uniqueness theorem for R_n by obtaining some specific information about how fast $\|f - \frac{1}{p}\|$ increases as $\frac{1}{p}$ recedes from the best approximation. Thus we will prove a generalization of the Strong Uniqueness Theorem which is given by Cheney [2] in the case of generalized rational approximation. The result was extended by Gilormini [7] in the case of ordinary interpolation, and by Perrie [13] for osculating interpolation on $[a,b]$.

Section 2: Continuity of the Best Approximation Operator for R_n.

We begin with the Strong Uniqueness Theorem for R_n.

Theorem 3.1: Let $f \varepsilon C_o^+[0,\infty) \sim R_n$ and suppose $\frac{1}{p*}$ is a best approximation to f from R_n with $\partial p* = n$. Then there exists $\delta > 0$ such that for all $\frac{1}{p} \varepsilon R_n$ we have

$$(3.1) \qquad \|f - \frac{1}{p}\| \geq \delta\|\frac{1}{p*} - \frac{1}{p}\| + \|f - \frac{1}{p*}\| .$$

Proof: We may assume $\frac{1}{p} \neq \frac{1}{p*}$, since if not, take $\delta = 1$. Define

$$\delta(\frac{1}{p}) = \frac{\|f - \frac{1}{p}\| - \|f - \frac{1}{p*}\|}{\|\frac{1}{p*} - \frac{1}{p}\|}$$

Then $\delta(\frac{1}{p}) \geq 0$ since $\frac{1}{p*}$ is best. Let

$$\delta \equiv \inf \{ \delta(\tfrac{1}{p}) \mid \tfrac{1}{p} \varepsilon R_n, \quad p \neq p^* \}.$$

If we can show $\delta > 0$, the theorem will follow.

So assume $\delta = 0$. Then there exists a sequence

$$\{ \frac{1}{p_k} \} \tag{3.2}$$

from $R_n \sim \{ \frac{1}{p^*} \}$ such that

$$\delta(\frac{1}{p_k}) \to 0 \quad \text{as} \quad k \to \infty.$$

We will show that there exist $M_1 > 0$, $M_2 < \infty$ such that

$$M_1 \leq \| \frac{1}{p_k} \| \leq M_2$$

for all k, in order to apply lemma 1.1. Note that

$$\delta(\frac{1}{p_k}) \geq \frac{\| \frac{1}{p_k} \| - \|f\| - \|f - \frac{1}{p^*}\|}{\| \frac{1}{p^*} \| + \| \frac{1}{p_k} \|}$$

$$= \frac{1}{\frac{\| \frac{1}{p^*} \|}{\| \frac{1}{p_k} \|} + 1} - \frac{\left(\|f\| + \|f - \frac{1}{p^*}\| \right)}{\| \frac{1}{p^*} \| + \| \frac{1}{p_k} \|} .$$

Hence if $\left\{ \| \frac{1}{p_k} \| \right\}$ has a subsequence which approaches ∞, $\delta(\frac{1}{p_k}) \not\to 0$. Thus there exists $M_2 < \infty$ such that $\| \frac{1}{p_k} \| \leq M_2$ for all k, and from this it follows that the sequence $\left\{ \|\frac{1}{p^*} - \frac{1}{p_k}\| \right\}$ is bounded.

Suppose $\left\{ \| \frac{1}{p_k} \| \right\}$ has a subsequence which approaches zero as $k \to \infty$. Denote this subsequence again by $\left\{ \| \frac{1}{p_k} \| \right\}$. Then, since

$$\left\| \frac{1}{p^*} - \frac{1}{p_k} \right\| \leq \left\| \frac{1}{p^*} \right\| + \left\| \frac{1}{p_k} \right\|$$

we can find N_1 so that

$$\left\| \frac{1}{p^*} - \frac{1}{p_k} \right\| \leq \frac{3}{2} \left\| \frac{1}{p^*} \right\| \qquad \forall\, k \geq N_1$$

or

$$\frac{1}{\frac{3}{2} \left\| \frac{1}{p^*} \right\|} \leq \frac{1}{\left\| \frac{1}{p^*} - \frac{1}{p_k} \right\|}$$

so that

$$\frac{\frac{2}{3} \left\| f - \frac{1}{p_k} \right\|}{\left\| \frac{1}{p^*} \right\|} \leq \frac{\left\| f - \frac{1}{p_k} \right\|}{\left\| \frac{1}{p^*} - \frac{1}{p_k} \right\|} .$$

Hence we obtain

$$\text{(3.3)} \qquad \frac{\frac{2}{3} \left\| f - \frac{1}{p_k} \right\|}{\left\| \frac{1}{p^*} \right\|} - \frac{\left\| f - \frac{1}{p^*} \right\|}{\left\| \frac{1}{p^*} - \frac{1}{p_k} \right\|} \leq \delta\left(\frac{1}{p_k} \right) \qquad \forall\, k \geq N_1 .$$

We will show the left hand side is positive for large k. Since

$$\left\| \frac{1}{p^*} \right\|^2 - \left\| \frac{1}{p^*} \right\| \left\| \frac{1}{p_k} \right\| \leq \left\| \frac{1}{p^*} \right\| \left\| \frac{1}{p^*} - \frac{1}{p_k} \right\|$$

and $\left\{ \| \frac{1}{p_k} \| \right\} \to 0$ as $k \to \infty$, we can find N_2 such that for all $k \geq N_2$ we have

$$0 < \frac{1}{2} \left\| \frac{1}{p^*} \right\|^2 \leq \left\| \frac{1}{p^*} \right\| \left\| \frac{1}{p^*} - \frac{1}{p_k} \right\| .$$

Thus the common denominator on the left hand side in (3.3) is positive for $k \geq N_2$. Thus we need only show that there exists some $N \geq \max\{N_1, N_2\}$ such that for all $k \geq N$ we have

$$\frac{2}{3} \left\|f - \frac{1}{p_k}\right\| \left\|\frac{1}{p^*} - \frac{1}{p_k}\right\| - \left\|f - \frac{1}{p^*}\right\| \left\| \frac{1}{p^*} \right\| \geq K > 0$$

where K is some constant. To do so, we will make use of the fact that in this case we have

$$\left\|f - \frac{1}{p^*}\right\| \leq \frac{1}{2} \|f\|$$

since $\frac{1}{p^*}$ must be a better approximation than the best constant. So

$$\frac{2}{3} \left\|f - \frac{1}{p_k}\right\| \left\|\frac{1}{p^*} - \frac{1}{p_k}\right\| - \left\|f - \frac{1}{p^*}\right\| \left\| \frac{1}{p^*} \right\|$$

$$\geq \frac{2}{3} \left(\|f\| - \left\| \frac{1}{p_k} \right\| \right) \left(\left\| \frac{1}{p^*} \right\| - \left\| \frac{1}{p_k} \right\| \right) - \frac{1}{2} \left\| \frac{1}{p^*} \right\| \|f\|$$

$$= \frac{2}{3} \|f\| \left\| \frac{1}{p^*} \right\| - \frac{2}{3} \left\| \frac{1}{p_k} \right\| \left\| \frac{1}{p^*} \right\| - \frac{2}{3} \|f\| \left\| \frac{1}{p_k} \right\| + \frac{2}{3} \left\| \frac{1}{p_k} \right\|^2 - \frac{1}{2} \left\| \frac{1}{p^*} \right\| \|f\|$$

$$> \frac{1}{6} \|f\| \left\| \frac{1}{p^*} \right\| - \frac{2}{3} \left\| \frac{1}{p_k} \right\| \left(\left\| \frac{1}{p^*} \right\| + \|f\| \right)$$

$$\geq \frac{1}{12} \|f\| \left\| \frac{1}{p^*} \right\|$$

for all $k \geq N$ if we take $N \geq \max\{N_1, N_2\}$ large enough so that

$$\frac{2}{3} \left\| \frac{1}{p_k} \right\| \left(\left\| \frac{1}{p^*} \right\| + \|f\| \right) < \frac{1}{12} \|f\| \left\| \frac{1}{p^*} \right\|$$

whenever $k \geq N$. Thus $\delta\left(\frac{1}{p_k}\right) \not\to 0$, which is a contradiction, and so the assumption $\left\{ \left\| \frac{1}{p_k} \right\| \right\} \to 0$ is incorrect. Hence there exist constants M_1 and M_2 such that

$$0 < M_1 \leq \left\| \frac{1}{p_k} \right\| \leq M_2 < \infty$$

for all k. From lemma 1.1 we conclude that there is a subsequence $\{ \frac{1}{p_{k_j}} \}$ which converges uniformly to $\frac{1}{p} \varepsilon R_n$ on any closed interval; for simplicity denote this subsequence by $\{ \frac{1}{p_k} \}$.

Suppose $x \varepsilon X(\frac{1}{p^*})$. Then

$$\begin{aligned} \delta\left(\frac{1}{p_k}\right) \left\|\frac{1}{p_k} - \frac{1}{p^*}\right\| &= \left\|f - \frac{1}{p_k}\right\| - \left\|f - \frac{1}{p^*}\right\| \\ &\geq \sigma(x)\left(f(x) - \frac{1}{p_k(x)} \right) - \sigma(x)\left(f(x) - \frac{1}{p^*(x)} \right) \\ &= \sigma(x) \left(\frac{1}{p^*(x)} - \frac{1}{p_k(x)} \right) . \end{aligned}$$

Define c^* by the relation

$$c^* \equiv \inf \left\{ \max_{x \varepsilon X(\frac{1}{p^*})} \{\sigma(x)h(x)\} \;\middle|\; h \varepsilon S\left(\frac{1}{p^*}\right),\ \|h\| = 1 \right\} .$$

Using continuity and compactness along with lemma 2.4 we can show $c^* > 0$.

The assumption that $\partial p^* = n$ implies that $\left\|\frac{p_k}{p^*} - 1\right\| < \infty$, and hence

$$\left\| \frac{\frac{p_k}{p^*} - 1}{\left\|\frac{p_k}{p^*} - 1\right\|} \right\| = 1 .$$

Thus for any k we can find an $x_k \in X(\frac{1}{p^*})$ such that

$$\sigma(x_k) \left(\frac{\frac{p_k(x)_k}{p^*(x_k)} - 1}{\left\|\frac{p_k}{p^*} - 1\right\|} \right) \geq c^*$$

or

$$\sigma(x_k) \left(\frac{p_k(x_k)}{p^*(x_k)} - 1 \right) \geq c^* \left\|\frac{p_k}{p^*} - 1\right\|$$

$$= c^* \left\| p_k\left(\frac{1}{p^*} - \frac{1}{p_k} \right) \right\|$$

$$= c^* \max_x \left\{ |p_k(x)| \left| \frac{1}{p^*(x)} - \frac{1}{p_k(x)} \right| \right\}$$

$$\geq \frac{c^*}{M_2} \left\|\frac{1}{p^*} - \frac{1}{p_k}\right\|$$

for each k since $\min_x |p_k(x)| \geq \frac{1}{M_2}$. Thus we have

$$\delta\left(\frac{1}{p_k} \right) (p_k(x_k)) \left\|\frac{1}{p^*} - \frac{1}{p_k}\right\| \geq \frac{c^*}{M_2} \left\|\frac{1}{p^*} - \frac{1}{p_k}\right\|$$

or

$$\delta(\frac{1}{p_k}) \geq \frac{c^*}{M_2 p_k(x_k)} .$$

Since $\{\frac{1}{p_k}\}$ converges uniformly to $\frac{1}{p} > 0$ on any closed interval, there exists $K > 0$ such that for all k,

$$\frac{1}{p_k(x)} \geq K$$

for all $x \in X(\frac{1}{p^*})$. Thus

$$\delta(\frac{1}{p_k}) \geq \frac{Kc^*}{M_2}$$

which contradicts the assumption that $\delta(\frac{1}{p_k}) \to 0$ as $k \to \infty$. ∎

The assumption $\partial p^* = n$ is crucial for the argument above since if $\partial p^* < n$ we have elements in $S(\frac{1}{p^*})$ of the form $1 - \frac{p}{p^*}$ with $\partial p = n$ and consequently

$$\left\|1 - \frac{p}{p^*}\right\| = \infty .$$

We remark that more stringent assumptions than those used in the Strong Uniqueness Theorem for $R_m^n[a,b]$ given by Cheney [2] are necessary, since if we consider his theorem in the case where $n = 0$, we find that any best approximation $\frac{p}{q} = \frac{1}{p^*} \in R_m$ satisfies the hypothesis

$$\min \{0-\partial p, m-\partial q\} = 0,$$

but the following theorem shows that the inequality (3.1) in the Strong Uniqueness Theorem will not hold if $\partial p^* \leq m - 2$.

<u>Theorem 3.2</u>: Let $\frac{1}{p^*}$ be a best approximation to $f \varepsilon C_o^+[0,\infty)$ from R_n with $0 < \partial p^* \leq n - 2$. Then the inequality (3.1) from the Strong Uniqueness Theorem cannot hold for all $\frac{1}{p} \varepsilon R_n$.

<u>Proof</u>: Assume $0 < \partial p^* \leq n - 2$. Let $E = \left\| f - \frac{1}{p^*} \right\|$. We will construct a sequence $\{ \frac{1}{p_k} \}$ of elements from R_n such that $\left\{ \left\| f - \frac{1}{p_k} \right\| \right\} \to E$ as $k \to \infty$ but $\left\{ \left\| \frac{1}{p_k} - \frac{1}{p^*} \right\| \right\} \not\to 0$ as $k \to \infty$.

Let $e = \frac{1}{E}$ and $a_k = \frac{e}{p^*(k) - e}$. Then $a_k \to 0$ as $k \to \infty$, and $a_k > 0$ for all k larger than some integer N_o. Define

$$\frac{1}{p_k(x)} = \frac{1}{e + [p^*(x) - e] \left\{ \frac{(x - k)^2}{k^2} + a_k \right\}}$$

Then if we choose $\beta > 0$ so large that $p^*(x) > 2e$ on $[\beta,\infty)$ and p^* is increasing on $[\beta,\infty)$, we have $\frac{1}{p_k(x)} > 0$ on $[\beta,\infty)$. In addition, we can specify $f < \frac{E}{2}$ on $[\beta,\infty)$.

We will prove that $\{ \frac{1}{p_k} \} \to \frac{1}{p^*}$ uniformly on $[0,\beta]$, which will guarantee that $\frac{1}{p_k}(x) > 0$ on $[0,\beta]$ if k is sufficiently large and hence that $\frac{1}{p_k} \varepsilon R_n$ if $k \geq N_1$ where N_1 is some integer.

First we observe that

$$\frac{1}{p_k(k)} = \frac{1}{e + [p^*(k) - e] \left\{ \frac{e}{p^*(k) - e} \right\}} = \frac{1}{2e} = \frac{E}{2}$$

when $k \geq N_o$. In addition, if x is fixed, then

$$\lim_{k\to\infty} \frac{1}{p_k(x)} = \frac{1}{e + [p^*(x) - e]\left\{\lim_{k\to\infty}\left(\frac{(x-k)^2}{k^2} + a_k\right)\right\}} = \frac{1}{p^*(x)}$$

In fact, to show $\{\frac{1}{p_k}\} \to \frac{1}{p^*}$ uniformly on $[0,\beta]$, let $x \varepsilon [0,\beta]$ and consider

$$\left|\frac{1}{p_k(x)} - \frac{1}{p^*(x)}\right| = \left|\frac{1}{p^*(x)}\right| \left|\frac{p^*(x)}{p_k(x)} - 1\right|$$

$$\leq K \left|\frac{p^*(x)}{e + [p^*(x) - e]\left\{\frac{(x-k)^2}{k^2} + a_k\right\}} - 1\right|$$

$$= K \left|\frac{p^*(x)}{p^*(x) + [p^*(x) - e]\left\{\frac{x^2}{k^2} - \frac{2x}{k} + a_k\right\}} - 1\right|$$

$$= K \left|\frac{1}{1 + \left(1 - \frac{e}{p^*(x)}\right)\left\{\frac{x^2}{k^2} - \frac{2x}{k} + a_k\right\}} - 1\right|$$

where $K = \max_{x\varepsilon[0,\beta]} \left|\frac{1}{p^*(x)}\right|$. If we let $M = \max_{x\varepsilon[0,\beta]} \left|1 - \frac{e}{p^*(x)}\right|$,

we have, for given $\varepsilon > 0$, that

$$\left|1 - \frac{e}{p^*(x)}\right| \left|\frac{x^2}{k^2} - \frac{2x}{k} + a_k\right| \leq M\left(\frac{x^2}{k^2} + \frac{2x}{k} + a_k\right)$$

$$\leq M \left(\frac{\beta^2}{k^2} + \frac{2\beta}{k} + a_k \right)$$

$$< \frac{\varepsilon}{2K}$$

if k is sufficiently large. Then if $\frac{\varepsilon}{2K} < \frac{1}{2}$,

$$\left| \frac{1}{p_k(x)} - \frac{1}{p^*(x)} \right| \leq \frac{K \left| - \left(1 - \frac{e}{p^*(x)}\right) \left\{ \frac{(x-k)^2}{k^2} + a_k \right\} \right|}{1 - \left|1 - \frac{e}{p^*(x)}\right| \left|\frac{(x-k)^2}{k^2} + a_k\right|}$$

$$\leq \frac{K\left(\frac{\varepsilon}{2K}\right)}{\frac{1}{2}} = \varepsilon .$$

Thus $\{ \frac{1}{p_k} \} \to \frac{1}{p^*}$ uniformly on $[0,\beta]$.

Since $\frac{1}{p^*} \varepsilon R_n$ we must have $\frac{1}{p^*}$ decreasing on $[k,\infty)$ for large k, and since

$$\frac{d}{dx}\left(\frac{1}{p_k(x)}\right) = \frac{-\left[p^{*\prime}(x)\left(\frac{(x-k)^2}{k^2} + a_k\right) + (p^*(x) - e)\left(\frac{2(x-k)}{k^2}\right) \right]}{\left(e + [p^*(x) - e] \left\{ \frac{(x-k)^2}{k^2} + a_k \right\} \right)^2}$$

is negative on $[k,\infty)$ for large k, we have $\frac{1}{p_k}$ decreasing on $[k,\infty)$.

Thus

$$\left|\frac{1}{p_k(x)} - \frac{1}{p^*(x)}\right| \le \left|\frac{1}{p_k(x)}\right| + \left|\frac{1}{p^*(x)}\right|$$

$$\le \frac{1}{p_k(k)} + \frac{1}{p^*(k)}$$

$$= \frac{E}{2} + \frac{1}{p^*(k)}$$

$$< E$$

for large k. Similarly, we have

$$\left|\frac{1}{p_k(x)} - f(x)\right| \le \left|\frac{1}{p_k(k)}\right| + |f(x)|$$

$$< \frac{E}{2} + \frac{E}{2} = E$$

when k is large.

We will now obtain similar estimates on $[\beta,k]$ for large k. Let $x \in [\beta,k]$. Then $p^*(x) > 2e$ and $a_k > 0$ so

$$e + [p^*(x) - e]\left\{\frac{(x-k)^2}{k^2} + a_k\right\} \ge e$$

and hence

$$0 < \frac{1}{p_k(x)} \le \frac{1}{e} = E$$

Since $0 < \dfrac{1}{p^*(x)} < \dfrac{E}{2}$ for $x > \beta$, it follows that

$$\left|\frac{1}{p_k(x)} - \frac{1}{p^*(x)}\right| \le E$$

on $[\beta,k]$ for large k; similarly we obtain

$$\left|\frac{1}{p_k(x)} - f(x)\right| \leq E$$

on $[\beta,k]$.

Combining the above facts, we see that

$$\left\{ \left\|f - \frac{1}{p_k}\right\| \right\} \to E \quad \text{as} \quad k \to \infty$$

and

$$\left\|\frac{1}{p_k} - \frac{1}{p^*}\right\| = \max_{x\varepsilon[0,\infty)} \left|\frac{1}{p_k(x)} - \frac{1}{p^*(x)}\right|$$

$$\geq \left|\frac{1}{p_k(k)} - \frac{1}{p^*(k)}\right|$$

$$\geq \frac{E}{2} - \left|\frac{1}{p^*(k)}\right| .$$

Since $\left\{\frac{1}{p^*(k)}\right\} \to 0$ as $k \to \infty$, there exists an integer N such that

$$\left\|\frac{1}{p_k} - \frac{1}{p^*}\right\| \geq \frac{E}{4}$$

for all $k \geq N$. So if δ is fixed, there exists k_o such that

$$\left\|f - \frac{1}{p_{k_o}}\right\| - \left\|f - \frac{1}{p^*}\right\| < \frac{\delta E}{4}$$

and (3.1) cannot hold for all elements of R_n. ∎

In case $\partial p^* = n$, we obtain continuity of the best approximation operator in the usual way [2]. We will use Tf to denote the best approximation to f.

<u>Theorem 3.3</u>: Let $f_o \in C_o^+[0,\infty) \sim R_n$ and suppose $\frac{1}{p^*}$ is a best approximation to f_o from R_n with $\partial p^* = n$. Then there exists λ_o such that for all $f \in C_o^+[0,\infty)$ we have

$$\|Tf - Tf_o\| \leq \lambda_o \|f - f_o\| .$$

<u>Proof</u>: From the Strong Uniqueness Theorem we obtain the existence of a constant δ depending only on f_o such that

$$\|f_o - \frac{1}{p}\| \geq \|f_o - \frac{1}{p^*}\| + \delta\|\frac{1}{p} - \frac{1}{p^*}\|$$

for any $\frac{1}{p} \in R_n$. Thus

$$\|f_o - Tf\| \geq \|f_o - Tf_o\| + \delta\|Tf - Tf_o\|$$

or

$$\begin{aligned}
\delta\|Tf - Tf_o\| &\leq \|f_o - Tf\| - \|f_o - Tf_o\| \\
&\leq \|f_o - f\| + \|Tf - f\| - \|f_o - Tf_o\| \\
&\leq \|f_o - f\| + \|Tf_o - f\| - \|f_o - Tf_o\| \\
&\leq \|f_o - f\| + \|f - f_o\| + \|Tf_o - f_o\| - \|f_o - Tf_o\| \\
&= 2\,\|f - f_o\| .
\end{aligned}$$

Hence $\|Tf - Tf_o\| \leq \frac{2}{\delta}\|f - f_o\|$. ∎

If $f \in R_n$, then $Tf = f$. In this case we can show T is continuous without making any additional assumptions concerning the degree of the denominator. The method of proof follows that used by Werner [17] to establish a similar result for $R_m^n[a,b]$.

Theorem 3.4: Let $f \in R_n$. Then T is continuous at f.

Proof: Suppose $\{ f_k \}$ is a sequence from $C_o^+[0,\infty)$ with $\left\{ \|f_k - f\| \right\} \to 0$ as $k \to \infty$. Then

$$\|Tf_k - f_k\| \leq \|Tf - f_k\| = \|f - f_k\|$$

and thus

$$\begin{aligned} \|Tf_k - Tf\| &\leq \|Tf_k - f_k\| + \|Tf - f_k\| \\ &\leq \|f - f_k\| + \|f - f_k\| \\ &= 2\,\|f - f_k\| \end{aligned}$$

from whence continuity follows. ∎

When $\partial p^* \leq n - 1$, the question of continuity for T is open. Attempts to modify the proof of theorem 3.1 when $\partial p^* = n - 1$ were not successful, and construction of a counterexample using techniques similar to those in the proof of theorem 3.2 apparently was not possible.

Section 3: Results for $K_n(f)$.

Remark: Lemma 1.1 holds if R_n is replaced by $K_n(f)$. The proof is virtually unchanged except for the verification that $\frac{1}{p} \in K_n(f)$.

This is not difficult; it requires using lemma 2.1 along with the following lemma.

Lemma 3.2: Let $\{p_k\}$ be a sequence of polynomials. Then if $\{p_k\}$ converges uniformly to a polynomial p on $[a,b]$, $\{ p_k^{(j)} \}$ converges uniformly to $p^{(j)}$ on $[a,b]$.

The result follows routinely from the fact that uniform convergence is equivalent to convergence of the respective coefficients.

In addition to lemma 1.1, the proof of the Strong Uniqueness Theorem for R_n depends in part on the existence of constants M_1 and M_2 such that

$$0 < M_1 \leq \left\| \frac{1}{p_k} \right\| \leq M_2 < \infty$$

where $\{ \frac{1}{p_k} \}$ was defined by (3.2). The inequality

$$(1.1) \qquad \left\|f - \frac{1}{p^*}\right\| \leq \frac{1}{2} \|f\|$$

was essential in the proof.

Obviously, for approximations by functions from $K_n(f)$, the inequality (1.1) cannot always hold. However, the interpolation properties allow simplification of this part of the proof and the inequality (1.1) is not necessary.

Theorem 3.5: Let $f \in \tilde{C}^s[0,\infty) \sim K_n(f)$ and suppose $\frac{1}{p^*}$ is the best approximation to f from $K_n(f)$ with $\partial p^* = n$. Then there exists $\delta > 0$ such that for all $\frac{1}{p} \in K_n(f)$ we have

$$(3.1) \qquad \left\|f - \frac{1}{p}\right\| \geq \delta\left\|\frac{1}{p^*} - \frac{1}{p}\right\| + \left\|f - \frac{1}{p^*}\right\| .$$

Proof: Assume $p \neq p^*$ and define

$$\delta\left(\frac{1}{p}\right) = \frac{\|f - \frac{1}{p}\| - \|f - \frac{1}{p^*}\|}{\|\frac{1}{p^*} - \frac{1}{p}\|} .$$

Then $\delta\left(\frac{1}{p}\right) \geq 0$ since $\frac{1}{p^*}$ is best. Let

$$\delta = \inf \left\{ \delta\left(\frac{1}{p}\right) \;\middle|\; \frac{1}{p} \in K_n(f),\ p \neq p^* \right\} .$$

We will show $\delta > 0$.

So assume $\delta = 0$. Then there exists a sequence $\{ \frac{1}{p_k} \}$ from $K_n(f) \sim \{ \frac{1}{p^*} \}$ such that

$$\delta\left(\frac{1}{p_k}\right) \to 0 \quad \text{as} \quad k \to \infty.$$

Again we wish to show that there exist constants M_1 and M_2 such that

$$0 < M_1 \leq \left\| \frac{1}{p_k} \right\| \leq M_2 < \infty$$

for all k. The existence of M_2 is shown as in the proof of theorem 3.1. Furthermore, we have

$$\frac{1}{p_k(y_1)} = f(y_1) > 0$$

for all k, and hence

$$\left\| \frac{1}{p_k} \right\| = \max_{x \in [0,\infty)} \left| \frac{1}{p_k(x)} \right| \geq f(y_1) .$$

So let $M_1 = f(y_1)$.

The remainder of the proof proceeds as in theorem 3.1. ∎

We previously noted that every $f \in \tilde{C}^s[0,\infty)$ does not have a best approximation from $K_n(f)$. Thus the continuity theorem for the best approximation operator must include an existence statement.

<u>Theorem 3.6</u>: Let $f_o \in \tilde{C}^s[0,\infty) \sim K_n(f_o)$ and suppose $\frac{1}{p^*}$ is a best approximation to f_o from $K_n(f_o)$ with $\partial p^* = n$. Let Tf denote the set of best approximations to $f \in \tilde{C}^s[0,\infty)$ from $K_n(f)$ and set

$$G = \left\{ f \in \tilde{C}^s[0,\infty) \;\middle|\; f^{(j)}(y_i) = f_o^{(j)}(y_i), \quad j = 0,\ldots,m_i-1; \quad i = 1,\ldots,k \right\}.$$

Then there exists a neighborhood N of f_o such that Tf is non-empty for all $f \in N \cap G$.

Further, T is continuous in the sense that there exists $\beta > 0$ such that if $f \in N \cap G$, then

$$\|Tf - Tf_o\| \leq \beta \|f - f_o\| .$$

<u>Proof</u>: Suppose $f \in G$ and consider those $\frac{1}{p} \in K_n(f)$ such that

$$\left\|\frac{1}{p} - f\right\| \leq \left\|\frac{1}{p^*} - f\right\| .$$

By theorem 3.5 there exists $\delta > 0$ such that

$$\begin{aligned}
\delta \left\|\frac{1}{p} - \frac{1}{p^*}\right\| &\leq \left\|f_o - \frac{1}{p}\right\| - \left\|f_o - \frac{1}{p^*}\right\| \\
&\leq \|f_o - f\| + \left\|f - \frac{1}{p}\right\| - \left\|f_o - \frac{1}{p^*}\right\| \\
&\leq \|f_o - f\| + \left\|f - \frac{1}{p^*}\right\| - \left\|f_o - \frac{1}{p^*}\right\|
\end{aligned}$$

$$\leq 2 \|f_o - f\| .$$

Thus $\left\|\frac{1}{p} - \frac{1}{p^*}\right\| \leq \frac{2}{\delta} \|f_o - f\|$ whenever $\left\|\frac{1}{p} - f\right\| \leq \left\|\frac{1}{p^*} - f\right\|$.

Suppose $\|f - f_o\| < \frac{\delta}{2}$. Then we have $\left\|\frac{1}{p} - \frac{1}{p^*}\right\| < 1$ for all $\frac{1}{p} \varepsilon K_n(f)$ which satisfy $\left\|f - \frac{1}{p}\right\| \leq \left\|f - \frac{1}{p^*}\right\|$. We will show that f has a best approximation from the set

$$S = \left\{ \frac{1}{p} \varepsilon K_n(f) \;\middle|\; \left\|\frac{1}{p} - \frac{1}{p^*}\right\| < 1 \right\} ,$$

and hence N can be taken to be

$$\left\{ f \varepsilon G \;\middle|\; \|f - f_o\| < \frac{\delta}{2} \right\} .$$

Let

$$E = E(f) = \inf \left\{ \left\|f - \frac{1}{p}\right\| \;\middle|\; \frac{1}{p} \varepsilon S \right\} .$$

Then there exists a sequence $\{ \frac{1}{p_k} \} \subset K_n(f)$ such that

$$\left\{ \left\|f - \frac{1}{p_k}\right\| \right\} \to E \text{ as } k \to \infty,$$

with $\left\|f - \frac{1}{p^*}\right\| \geq \left\|f - \frac{1}{p_k}\right\| \geq E$. Thus for all k

$$\left\| \frac{1}{p_k} \right\| \leq \left\|f - \frac{1}{p^*}\right\| + \|f\|$$

and

$$f(y_1) = \frac{1}{p_k(y_1)} \leq \left\| \frac{1}{p_k} \right\|$$

since $\{\frac{1}{p_k}\} \subset K_n(f)$. Thus by the remark beginning Section 3 we have the existence of a subsequence $\{\frac{1}{p_{k_j}}\}$ and an element $\frac{1}{p} \varepsilon K_n(f)$ such that $\{\frac{1}{p_{k_j}}\} \to \frac{1}{p}$ uniformly on any closed interval $[0,\alpha]$. Denote the subsequence by $\{\frac{1}{p_k}\}$.

Then for any $x \varepsilon [0,\infty)$, there exists α such that $x \varepsilon [0,\alpha]$, and

$$\left|f(x) - \frac{1}{p(x)}\right| \leq \left|f(x) - \frac{1}{p_k(x)}\right| + \left|\frac{1}{p_k(x)} - \frac{1}{p(x)}\right|$$

$$\leq \left\|f - \frac{1}{p_k}\right\| + \max_{x\varepsilon[0,\alpha]} \left|\frac{1}{p_k(x)} - \frac{1}{p(x)}\right| .$$

Since $\left\{\left\|f - \frac{1}{p_k}\right\|\right\} \to E$ and $\{\frac{1}{p_k}\} \to \frac{1}{p}$ uniformly on $[0,\alpha]$ we have

$$\left|f(x) - \frac{1}{p(x)}\right| \leq E .$$

Hence it follows that $\left\|f - \frac{1}{p}\right\| \leq E$ and $\frac{1}{p}$ is a best approximation to f. ∎

Techniques similar to those above can be used to prove the following existence theorem in case n is sufficiently large.

<u>Theorem 3.7</u>: Suppose $n \geq 2m$. Then each $f \varepsilon \tilde{C}^s[0,\infty)$ has a best approximation from $K_n(f)$.

<u>Proof</u>: We will produce an element $\frac{1}{q^*} \varepsilon K_n(f)$ and then restrict the search for a best approximation to the set

$$S = \left\{ \frac{1}{p} \in K_n(f) \;\middle|\; \left\| f - \frac{1}{p} \right\| \leq \left\| f - \frac{1}{q^*} \right\| \right\}.$$

The system

$$\left(\frac{1}{p} \right)^{(j)}(y_i) = f^{(j)}(y_i), \qquad \begin{array}{l} j = 0,\ldots,m_i-1 \\ i = 1,\ldots,k \end{array}$$

is equivalent to the system

$$p^{(j)}(y_i) = \left(\frac{1}{f} \right)^{(j)}(y_i), \qquad \begin{array}{l} j = 0,\ldots,m_i-1 \\ i = 1,\ldots,k\ . \end{array}$$

By the theory of Hermite interpolation, there exists a polynomial p_o with $\partial p_o \leq 2m - 1$ which satisfies the latter set of interpolation constraints. However, p_o is not necessarily positive on $[0,\infty)$, and so we will construct $q^* = p_o + \varepsilon q$ so that $q^* > 0$ and also satisfies the constraints.

Let

$$q(x) = \prod_{i=1}^{k} (x - y_i)^{2m_i} .$$

Then q has the following properties:

(i) $q(x) \geq 0$ on $[0,\infty)$

(ii) $q^{(j)}(y_i) = 0, \quad j = 0,\ldots,m_i-1; \quad i = 1,\ldots,k$

(iii) $\partial q = \Sigma 2m_i = 2m \leq n$.

Thus for any $\varepsilon > 0$, $(p_o + \varepsilon q)^{(j)}(y_i) = \left(\frac{1}{f} \right)^{(j)}(y_i)$.

Now $\partial q > \partial p_o$ and thus $q^* = p_o + \varepsilon q$ has a positive leading coefficient. Hence there exists $\beta > 0$ so that $q^*(x) > 1$ on (β,∞) with $y_1,\ldots,y_k$ interior points of $[0,\beta]$.

In addition, there exists a constant $M > 0$ such that $|p_o(x)| \leq M$ on $[0,\beta]$, and since p_o interpolates positive values at each y_i, there is an interval I_i about y_i with $p_o(x) > 0$ on I_i for all i. So if we choose $\varepsilon > 0$ so that

$$\min_{x\varepsilon[0,\beta]-\cup I_i} \left[\varepsilon q(x)\right] > M$$

then $q^*(x) > 0$ on $[0,\infty)$, and $\frac{1}{q^*} \varepsilon K_n(f)$.

To show f has a best approximation from S repeat the argument used in the proof of theorem 3.6. ∎

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